ON ENTIRE SOLUTIONS OF SOME INHOMOGENEOUS LINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS IN A BANACH SPACE

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1. Introduction

The present paper studies special holomorphic solutions of inhomogeneous linear differential equation

$$w'(z+h) = Aw(z) + f(z). \tag{1}$$

in a Banach space. Here A is a closed linear operator on a Banach space E with the domain of definition $D(A)$ ($D(A)$ is not necessarily dense in E), having a bounded inverse operator. These operators appear under studying some boundary-value problems for parabolic type equations (see [1] - [4]). For example this case is appeared in the problem on the heat conduction on the finite segment $[0,l]$ with zero boundary condition. In this situation we can consider $E = C[0,l]$, and operator $A = \frac{d^2}{dz^2}$ with the domain of definition $D(A) = \left\{u \in C^2[0,l] : u(0) = u(l) = 0\right\}$. Studying Equation (1) we suppose that $f(z)$ is an E-valued function, which is holomorphic in a neighborhood of zero and under a solution of the equation we understand a holomorphic in the neighborhood of zero E-valued function $w(z)$, such that $w(z) \in D(A)$ and Equation (1) is fulfilled in the same neighborhood. The properties of holomorphic and entire solutions of the equation
$w'(z) = Aw(z) + f(z)$ for the case when the operator A is unbounded were studied in numerous works (see, for example, [1], [7] - [9]).

The main result of the paper is the existence proof and the uniqueness one of an entire solution of exponential type in a case when $f(z)$ is an entire function of exponential type (see Theorem 1 and Theorem 3). Let us recall that $f(z)$ is of exponential type if for $f(z)$ the following condition is fulfilled: $\exists\gamma > 0\ \exists C > 0 \forall z \in \mathbb{C} : \|f(z)\| \le Ce^{\gamma|z|}$. The proof of the main theorem is based on studying the implicit differential equation $Tw'(z+h) + g(z) = w(z)$, where $T = A^{-1}$ and $g(z) = -A^{-1}f(z)$. The holomorphic solutions behavior of the implicit equation mentioned above were studied with another technique in [10].

2. Main results

Let E be a Banach space, $T : E \to E$ be an bounded linear operator, $h \in \mathbb{C}$ and $g : \mathbb{C} \longrightarrow E$ be an entire function.

At first consider the inhomogeneous implicit differential-difference equation of the form

$$Tw'(z+h) + g(z) = w(z), \tag{2}$$

Theorem 1. *Let $\rho(T)$ be a spectral radious of operator T, and $g(z)$ is an entire function of exponential type σ. If $\rho(T)\sigma e^{\sigma h} < 1$, then Equation (2) has a unique entire solution of exponential type σ, $w(z) = \sum\limits_{n=0}^{\infty} T^n g^{(n)}(z+nh)$.*

Proof. Let $\sigma_1 > \sigma$, $\rho(T)\sigma_1 e^{\sigma_1 h} < 1$ and $\|g(z)\| \le C_1 e^{\sigma_1|z|}$. As $\|g^{(n)}(z)\| \le 2\sqrt{\pi n} C_1 \sigma_1^n e^{\sigma_1|z|}$, then

$$\|g^{(n)}(z+nh)\| \le 2\sqrt{\pi n} C_1 \sigma_1^n e^{\sigma_1|z|} e^{\sigma_1 nh}$$

. If $|z| \le R$, then

$$\|T^n g^{(n)}(z+nh)\| \le \|T^n\| \|g^{(n)}(z+nh)\| \le 2\|T^n\| \sqrt{\pi n} {\sigma_1}^n C_1 e^{\sigma_1 R} e^{\sigma_1 nh}.$$

As according to the Cauchy sign and Gelfand formula $\sqrt[n]{\|T^n\| \, \sigma_1^n e^{\sigma_1 nh}} \to \rho\,(T)\,\sigma_1 e^{\sigma_1 h} < \ 1$

Hence $\|w(z)\| \le 2C_1 \left(\sum\limits_{n=0}^{\infty} {\sigma_1}^n \|T^n\| \sqrt{\pi n} e^{\sigma_1 nh} \right) e^{\sigma_1 h} = C_2 e^{\sigma_1|z|}$.

Therefore the exponential type of w is not more than σ. Taking into consideration left hand side of Equation 1 the exponential type of w is equal to σ. It is left to show a uniqueness of a solution. Consider the homogeneous equation $Tw'\,(z+h) = w\,(z)$. Then $T^n w^{(n)}\,(z+h) = w\,(z)$. We have

$$\|w(0)| \le \|T^n\| \|w^{(n)}(nh)\| \le 2\|T^n\| \sqrt{\pi n} {\sigma_1}^n C_1 e^{\sigma_1 nh}.$$

So $\lim\limits_{n\to\infty} \left(\sqrt[n]{\|w(0)\|} \right) \le \rho\,(T)\sigma_1 e^{\sigma_1 h} < 1$, i. e. $w(0) = 0$. As the function $w^{(k)}(z)$ satisfies to the homogeneous equation as well, we have $w^{(k)}(0) = 0$ and the uniqueness is proved. □

Corollary 2. *Let g be of zero exponential type (that is $\forall \varepsilon > 0\, \exists C_\varepsilon > 0\, \forall z \in \mathbb{C} : \|f\,(z)\| \le C_\varepsilon e^{\varepsilon|z|}$), T be arbitrary bounded operator and h be arbitrary complex number. Then Equation (2) has a unique entire solution of zero exponential type $w\,(z) = \sum\limits_{n=0}^{\infty} T^n g^{(n)}\,(z+nh)$.*

Theorem 3. *Let A be a closed linear operator on a Banach space (domain of definition $D(A)$ of A is not necessarily dense). Consider differential-difference Equation (1)*

If the operator A has a bounded inverse one and $f(z)$ is an entire function of zero exponential type, then Equation (1) has a unique entire solution of zero exponential type $w(z) = -\sum\limits_{n=0}^{\infty} A^{-(n+1)} f^{(n)}\,(z+nh)$. Moreover the Cauchy problem

$$\begin{cases} w'(z+h) = Aw + f(z) \\ w\,(0) = w_0 \end{cases}$$

has an entire solution of zero exponential type if and only if $w_0 + \sum\limits_{n=0}^{\infty} A^{-(n+1)} f^{(n)}\,(nh) = 0$.

Proof. Let $T = A^{-1}$ and $g(z) = -A^{-1} f(z)$. Then $D(T) = E$, T is bounded, $g(z)$ is an entire function of zero exponential type and Equation (1) is equivalent to Equation (2). According to Theorem 2 Equation (2) has the unique entire solution of zero exponential type
$w\,(z) = \sum\limits_{n=0}^{\infty} T^n g^{(n)}\,(z+nh) = -\sum\limits_{n=0}^{\infty} A^{-(n+1)} f^{(n)}\,(z+nh)$ and $w(0) = -\sum\limits_{n=0}^{\infty} A^{-(n+1)} f^{(n)}\,(nh)$.
Theorem is proved. □

Example 4. Let $E = \mathbb{C}$ and $A = I$. Consider the differential-difference equation
$w'(z+h) = w + f(z)$. If $f(z)$ is an entire function of zero exponential type, then this equation has a unique entire solution of zero exponential type $w\,(z) = -\sum\limits_{n=0}^{\infty} f^{(n)}\,(z+nh)$ and this solution continuously depends on f in the topology of the space E_0 (E_0 is the space of entire E-valued function of zero exponential type).

Corollary 5. *Let E be finite-dimensional space and $f(z)$ be polynomial degree m. Consider the system of differential-difference equations $w'(z+h) = Aw + f(z)$, where A is invertible. Therefore*

this system has a unique polynomial solution of degree m,
$w(z) = -\sum\limits_{n=0}^{m} A^{-(n+1)} f^{(n)}(z+nh)$.

Corollary 6. *Consider the system of differential-difference equations* $w'(z+h) = Aw + f(z)$, *where* A *is a matrix with entire coefficients,* $f(z)$ *is a polynomial with entire coefficients and* $h \in \mathbb{Z}$. *If* $\det A = \pm 1$, *then this system has a unique polynomial solution and this solution is a polynomial with integer coefficients.*

Proof. It is clear from existence inverse matrix A^{-1} with entire coefficients and form using Cramer's rule. □

Example 7. Let $E = C[0,1]$, $A = \frac{d}{dx}$ and $D(A) = \left\{u \in C^1[0,1] : u(0) = 0\right\}$.

Then $\left(A^{-1}h\right)(x) = \int\limits_0^x h(y)\,dy$, $\left(A^{-(n+1)}h\right)(x) = \frac{1}{n!}\int\limits_0^x (x-y)^n h(y)\,dy$ and $\rho\left(A^{-1}\right) = 0$.

By transition to real axes Equation (1) has the form

$$\left\{\begin{array}{ll} \frac{\partial w}{\partial t}(x, t+h) = \frac{\partial w}{\partial x} + f(t,x), & t \in \mathbb{R},\ x \in (0,1) \\ w(t,0) = 0 \end{array}\right. \tag{3}$$

If in the second variable f can be extended to an entire function of exponential type, then in this class of functions Problem (3) has the unique solution

$$w(t,x) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_0^x (x-y)^n \frac{\partial^n f}{\partial t^n}(t+nh, y)\,dy.$$

It is important to note, that Problem (3) has only zero solution for the homogeneous equation even in class of continuously differentiablle functions. In particularly, if $h = 0$ using Tejlor's formula we have

$$w(t,x) = -\sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_0^x (x-y)^n \frac{\partial^n f}{\partial t^n}(t,y)\,dy = -\int\limits_0^x f(t+x-y, y)\,dy.$$

Example 8. Let $E = C[0,\ 1]$, $A = \frac{d^2}{dx^2}$, $D(A) = \left\{u \in C^2[0,\ 1] : u(0) = u(1) = 0\right\}$. Then operator A is invertible, $\left(A^{-1}h\right)(x) = \int\limits_0^1 G(x,\ y)\ h(y)dy$, where G is the Green function of corresponding boundary problem and $\rho\left(A^{-1}\right) = \frac{1}{\pi^2}$. In this case
$\left(A^{-(n+1)}h\right)(x) = \int\limits_0^1 G_{n+1}(x,\ y)\ h(y)\,dy$, where $G_1(x,\ y) = G(x,y)$,
$G_{n+1}(x,\ y) = \int\limits_0^1 G_n(x,\ s)\ G(s,\ y)\,ds$.

In this example by transition to real axes Equation (1) has the form of the heat equation on $(0,\ 1)$ with zero boundary conditions

$$\left\{\begin{array}{ll} \frac{\partial w}{\partial t}(t+h, x) = \frac{\partial^2 w}{\partial x^2} + f(t,x), & t \in \mathbb{R},\ x \in (0,\ 1) \\ w(t,\ 0) = w(t,\ 1) = 0 \end{array}\right. \tag{4}$$

If $f(t,x) = \sum\limits_{n=0}^{\infty} c_n(x)\ t^n$, where $c_n \in C[0,1]$ and $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{n!\,\|c_n\|} < \frac{1}{\pi^2}$, then the problem (4) has the solution $w(t,x) = -\sum\limits_{n=0}^{\infty} \int\limits_0^1 G_{n+1}(x,y)\,\frac{\partial^n f}{\partial\, t^n}(t+nh, y)\,dy$.

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