The problem of fluid motion in a deformable porous medium

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Problem

We consider the following quasi-linear system of equations of composite type, which describes the 3-dimensional non-stationary motion of viscous compressible fluid in a deformable viscoelastic rock [1], [2], [3], [4]:

$$\frac{\partial(1-\phi)\rho_s}{\partial t} + div((1-\phi)\rho_s\vec{v}_s) = 0,$$
$$\frac{\partial(\rho_f\phi)}{\partial t} + div(\rho_f\phi\vec{v}_f) = 0,$$
$$\phi(\vec{v}_f - \vec{v}_s) = -\frac{k\phi^n}{\mu}(\nabla p_f + \rho_f\vec{g}),$$
$$\nabla\cdot\vec{v}_s = -\frac{\phi^m}{\eta}p_e - \phi^b\beta_\phi\frac{dp_e}{dt},$$
$$p_{tot} = p_0 - \rho_s g x_3 = \phi p_f + (1-\phi)p_s;\ p_e = (1-\phi)(p_s - p_f);$$

where $\rho_f, \rho_s, \vec{v}_s, \vec{v}_f$ – are the true density and velocity of the phase respectively; ϕ – is the porosity; $\vec{g} = (0,0,-g)$ – is the density of mass forces; k – is the permeability, μ – is the dynamic viscosity of the fluid; η, β_ϕ – are the parameters of the rock; p_f – is the fluid pressure, p_{tot} – is the total pressure, $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}_s\cdot\nabla)$.

Closing relations

In this system of equations of the unknown 10: $1\phi, 3v_s, 3v_f, 1p_e, 1p_f, 1\rho_f$. We consider that ρ_s is constant. The system is closed in the one-dimensional case, if $p_f = p_f(\rho_f)$ or $\rho_f = const$. In general, the system except for the equation of state is added to the equation of conservation of momentum of the system "solid matrix - pore fluid", namely the equation of an incompressible solid skeleton deformation, taking into account the influence of pore fluid pressure

$$\nabla\cdot\sigma + \rho g = 0,$$

where $\rho = (1-\phi)\rho_s + \phi\rho_f$ – is the average density of the medium, $\sigma = (1-\phi)\sigma_s + \phi\sigma_f$ – is the total stress tensor, σ_s, σ_f – are the solid and fluid stress tensors, respectively.

Solvability in the small (1D)

$$\frac{\partial(1-\phi)\rho_s}{\partial t} + \frac{\partial}{\partial x}((1-\phi)\rho_s v_s) = 0, \quad (1)$$
$$\frac{\partial(\rho_f\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho_f\phi v_f) = 0, \quad (2)$$
$$\phi(v_f - v_s) = -\frac{k\phi^n}{\mu}(\frac{\partial p_f}{\partial x} + \rho_f g), \quad (3)$$
$$\frac{\partial v_s}{\partial x} = -\frac{\phi^m}{\eta}p_e - \phi^b\beta_\phi\frac{dp_e}{dt}, \quad (4)$$
$$p_{tot} = p_0 - \rho_s g x = \phi p_f + (1-\phi)p_s;\ p_e = (1-\phi)(p_s - p_f). \quad (5)$$
$$v_s\mid_{\partial Q_T} = 0, \quad v_f\mid_{\partial Q_T} = 0,$$
$$v_s\mid_{t=0} = v_s^0(x), \quad v_f\mid_{t=0} = v_f^0(x), \quad (6)$$
$$\phi\mid_{t=0} = \phi^0(x), \quad \rho_f\mid_{t=0} = \rho^0(x).$$

Definition 1. A classical solution of problem (1) – (4) : $(\phi,\ v_i,\quad p_i,\ \rho_f)$, $i = f, s$, $\phi \in C^{1+\alpha}(Q_T)$, $(v_i, p_i, \rho_f) \in C^{2+\alpha, 1+\alpha/2}(Q_T)$, satisfying equations (1) – (4) and the initial and boundary conditions (5) as continuous functions in $\overline{Q_T}$. **Theorem 1.** Let $\phi^0 \in C^{1+\alpha}(\overline{\Omega})$, $(v_f^0, v_s^0, \rho^0) \in C^{2+\alpha}(\overline{\Omega})$, $v_i^0\mid_{\partial Q_T} = \frac{d\rho^0}{dx}\mid_{\partial Q_T} = 0$,

$$0 < m_0 \le \phi^0(x) \le M_0 < 1,\ 0 < m_1 \le \rho^0(x) \le M_1 < \infty,\ x \in \overline{\Omega},$$

m_0, M_0, m_1, M_1 are positive constants. Then (1) – (5) has a classical local solution , i.e. are exist t_0:

$$\phi(x,t) \in C^{1+\alpha}(\overline{Q_{t_0}}), (v_i(x,t), p_i(x,t), \rho_f^0) \in C^{2+\alpha,1+\alpha/2}(\overline{Q_{t_0}}), i = f, s.$$

More over $0 < \phi(x,t) < 1$, $\rho_f(x,t) > 0$ in $\overline{Q_{t_0}}$.

References

[1] Connolly J.A.D., Podladchikov Y.Y. Compaction-driven fluid flow in viscoelastic rock // Geodin. Acta. 1998. Vol. 11, 55 84.

[2] Fowler A.C., Yang X. Pressure solution and viscous compaction in sedimentary basins // J. Geophys. Res. 1999. Vol. 104.

Local classical solvability of the Cauchy problem

In Lagrange variables equations (1)-(5) reduce to a system

$$\frac{\partial}{\partial t}(\frac{\phi}{1-\phi}\rho_f) = \frac{\partial}{\partial x}(\rho_f k(\phi)((1-\phi)\frac{\partial p_f}{\partial x} + \rho_f g), \quad (7)$$
$$p_f = A\rho_f,$$
$$\frac{1}{(1-\phi)}\frac{\partial(1-\phi)}{\partial t} = a_1(\phi)p_e + a_2(\phi)\frac{\partial p_e}{\partial t}, \quad (8)$$
$$p_e = p_{tot} - p_f.$$

Here $k(\phi) = \frac{k_0}{\mu}\phi^n$, k_0, μ, A – positive constants. Consider conditions for system (6),(7):

$$p_f\mid_{t=0} = p^0(x), \quad \phi\mid_{t=0} = \phi^0(x). \quad (9)$$

Theorem 2 Let $p^0(x), \phi^0(x) \in W_2^l(\Omega), 0 \le \phi^0(x) < 1$, $l > \frac{7}{2}, \frac{\partial p}{\partial \rho_f} > 0$, then (6) – (8) has a uniqueness classical local solution.

Incompressible liquid (self-similar solution, $\xi = x - ct$)

$$\frac{d}{d\xi}((1-\phi)v_s - c(1-\phi)) = 0, \quad (10)$$
$$\frac{d}{d\xi}(\phi v_f - c\phi) = 0, \quad (11)$$
$$\phi(v_f - v_s) = \frac{k\phi^n}{\mu}\frac{dp_e}{d\xi}, \quad (12)$$
$$\frac{dv_s}{d\xi} = -\frac{\phi^m}{\eta}p_e + (c - v_s)\phi^b\beta_\phi\frac{dp_e}{d\xi}, \quad (13)$$
$$p_{tot} = p_0 - \rho_s g z = \phi p_f + (1-\phi)p_s;\ p_e = (1-\phi)(p_s - p_f). \quad (14)$$
$$v_s(0) = v_s^0, v_f(0) = v_f^0, \phi(0) = \phi^0,$$
$$\lim_{\xi\to\infty} v_s(\xi) = u^+,\ \lim_{\xi\to\infty} v_f(\xi) = u^+,\ \lim_{\xi\to\infty}\phi(\xi) = \phi^+, \quad (15)$$

v_s^0, v_f^0, ϕ^0, ϕ^+ are constants: $\phi^0 \ne \phi^+$, $v_s^0 \ne v_f^0$.

Theorem 3. Let $g = 0, \phi^0 \ne \phi^+$, $(\phi^0, \phi^+) \in (0,1)$. Then (10)-(15) has a unique classical self-similar solution $(\phi_i(\xi), v_i(\xi), p_f(\xi)), i = s, f$: $0 < \phi^+ \le \phi \le \phi^0 < 1$.

Generalized solution

In Lagrange variables equations (1)-(5) reduce to a system ($\rho_f = const$)

$$\frac{\partial}{\partial t}\left(\frac{\phi}{1-\phi}\right) = \frac{\partial}{\partial x}\left(\frac{k(\phi)}{a_2}\frac{\partial\phi}{\partial x} + k(\phi)(1-\phi)G(x)\right), \quad (16)$$
$$(\frac{k(\phi)}{a_2}\frac{\partial\phi}{\partial x} + k(\phi)(1-\phi)G(x))\mid_{x=0,x=H} = 0, \quad \phi\mid_{t=0} = \phi^0(x), \quad (17)$$
$$G(x) = -\frac{1}{a_2(1-\phi^0(x))}\frac{\partial\phi^0(x)}{\partial x} + \frac{\partial p^0(x)}{\partial x}.$$

Definition 2. Bounded measurable function $\phi(x,t)$ in Ω_T is generalized solution to problem (3.3.33), (3.3.34), if $0 \le \phi(x,t) \le 1$ almost everywhere in Ω_T, $k^{1/2}(\phi)(1-\phi)^{-1}\frac{\partial\phi}{\partial x} \in L_2(\Omega_T)$ and for any function $\psi(x,t) \in W_2^1(\Omega_T)$, $\psi(x,T) = 0$, $x \in \Omega$ d for almost all $t \in [0,T]$ have the identity

$$-\int_0^H(\frac{\phi(x,t)}{1-\phi(x,t)}\psi(x,t) - \frac{\phi^0(x)}{1-\phi^0(x)}\psi(x,0))dx =$$
$$= \int_0^T\int_0^H(\frac{\partial\psi(x,\tau)}{\partial x}(\frac{k(\phi)}{a_2}\frac{\partial\phi}{\partial x} + k(\phi)(1-\phi)G(x)) - \quad (18)$$
$$-\frac{\phi(x,\tau)}{1-\phi(x,\tau)}\frac{\partial\psi(x,\tau)}{\partial\tau})dxd\tau.$$

Theorem 4. Let

$$c(\phi) = \int_0^\phi(1-\xi)k(\xi)\,d\xi \ge 0, \quad \phi \in [0,1],$$
$$\frac{\partial G(x)}{\partial x} \ge 0, \quad x \in (0,H), \quad G(0) \ge 0, \quad G(H) \le 0.$$

. Then exist at least one generalized solution to problem (16)-(18).

References

[3] Rajagopal K.L., Tao L. Mechanics of mixtures. London: World Scientific Publishing. 1995.

[4] Morency Christina , Huismans R. S., Beaumont C., Fullsack P. A numerical model for coupled fluid flow and matrix deformation with applications to disequilibrium compaction and delta stability // Journal of geophysical research. 2007.