Asymptotic behaviour of the degenerate doubly nonlinear equation on bounded domains

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Heat Equation(HE): $\boxed{u_t = \Delta u}$

$\Longrightarrow$**Typical nonlinear diffusion models**

- Porous Medium Equation(PME) :

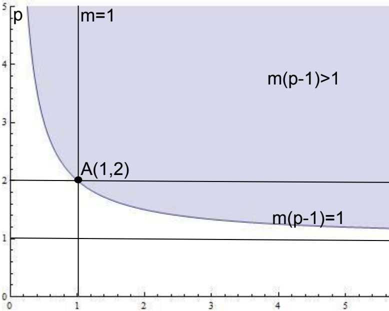
$$u_t = \Delta u^m.$$

- p-Laplacian Equation(PLE):

$$u_t = \Delta_p u = \nabla.(|\nabla u|^{p-2}\nabla u).$$

- Doubly Nonlinear Equation(DNLE):

$$u_t = \Delta_p u^m.$$



Different behavior depending on m and p:

- $0 < m < 1,\ 1 < p < 2,\ m(p-1) < 1$: extinction in finite time.
- $m > 1,\ p > 2,\ m(p-1) > 1$: positivity for all times when $u_0 \geq 0$.
- $m = 1,\ p = 2 \Longrightarrow$ PME=PLE=HE.

Heat Equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x) & \text{for } t > 0 \text{ and } x \in \Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial\Omega. \end{cases}$$

with $u_0 \geq 0$, $u_0 \in L^1(\Omega)$. Then
• Let $0 < \lambda_0 < \lambda_1 \leq \dots$ the eigenvalues values of Δ on Ω with Dirichlet boundary conditions, and $\phi_0 \geq 0$, $\|\phi_0\|_2 = 1$ the eigenfunction (λ_0). Then

$$\|u(t,\cdot)\|_2 \leq K_1 e^{-\lambda_0 t}, \quad t \geq 0,$$

$$\|e^{\lambda_0 t} u(t,\cdot) - C_0\phi_0\|_2 \leq K_2 e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq 0,$$

where

$$C_0 = \int_\Omega u_0 \phi_0 dx$$

The asymptotic profiles $C_0\phi_0$ form a linear, one-parameter family of solutions.

Porous Medium Equation with $m > 1$

$$\text{(PME)} \begin{cases} u_t(t,x) = \Delta u^m(t,x) & \text{for } t > 0 \text{ and } x \in \Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial\Omega. \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $u_0 \geq 0$, $u_0 \in L^1(\Omega)$.

Asymptotic behavior for the PME (J.L.Vázquez, Mon.Math.,2004)

There exists a unique separate variables solution of the PME of the form $U(t,x) = t^{-1/(m-1)} f(x)$ such that

$$\boxed{\lim_{t \to +\infty} t^{1/(m-1)} |u(t,x) - U(t,x)| = \lim_{t \to +\infty} |t^{1/(m-1)} u(t,x) - f(x)| = 0,}$$

unless u is trivial, $u \equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. The asymptotic profile f is the unique non-negative solution of the stationary problem:

$$\boxed{\Delta f^m(x) + \frac{1}{m-1} f(x) = 0, \ x \in \Omega,} \quad f(x) = 0, \ x \in \partial\Omega$$

DNLE with $m(p-1)>1$

$$\begin{cases} u_t(t,x)=\Delta_p u^m(t,x) & \text{for } t>0 \text{ and } x\in\Omega,\\ u(0,x)=u_0(x) & \text{for } x\in\Omega,\\ u(t,x)=0 & \text{for } t>0 \text{ and } x\in\partial\Omega. \end{cases} \qquad (1)$$

where $m(p-1)>1$, $\Omega\in\mathbb{R}^N$ is a bounded domain with regular boundary, $u_0\geq 0, u_0\in L^1(\Omega)$.

Asymptotic behavior for the DNLE

There exists a unique separate variable solution of problem DNLE of the form $U(t,x)=t^{-1/(m(p-1)-1)}f(x)$ s.t.

$$\boxed{\lim_{t\to+\infty} t^{1/(m(p-1)-1)}|u(t,x)-U(t,x)| = \lim_{t\to+\infty}|t^{1/(m(p-1)-1)}u(t,x)-f(x)|=0,}$$

unless $u\equiv 0$. The convergence is uniform in space and monotone non-decreasing in time. The asymptotic profile f is the unique non-negative solution of the stationary problem

$$\boxed{\Delta_p f^m(x)+\frac{1}{m(p-1)-1}f(x)=0,\ \ x\in\Omega,} \quad f(x)=0,\ \ x\in\partial\Omega \qquad (2)$$

Sketch of the proof

Method of rescaling and time transformation: $\boxed{v(\tau,x) = t^{\frac{1}{m(p-1)-1}} u(t,x),\ \ t = e^{\tau}.}$
Rescaled problem:

$$\begin{cases} v_\tau(\tau,x) = \Delta_p v^m(\tau,x) + \frac{1}{m(p-1)-1} v(\tau,x), & \tau \in (-\infty,+\infty),\ x \in \Omega \\ v(0,x) = v_0(x) = u(x,1), & x \in \Omega, \\ v(\tau,x) = 0, & \tau \in \mathbb{R},\ x \in \partial\Omega. \end{cases} \tag{1}$$

• Bounded and regular initial data $v_0(x) = u(x,1)$.
Convergence: the main tools are the a-priori (smoothing effects, Bénilan-Crandall type)estimates

$$u(t,x) \le Ct^{-1/(m(p-1)-1)} \quad \text{and} \quad u_t(t,x) \ge -C\frac{u}{(m(p-1)-1)t}$$

$\Rightarrow \boxed{0 \le v \le C \quad \text{and} \quad v_\tau \ge 0} \Rightarrow \boxed{\lim_{\tau\to\infty} v(\tau,x) = f(x), \forall x \in \Omega}$ monotone non-decreasing $\Rightarrow$ $f(x)$ is nontrivial and bounded
$\Rightarrow$ $v(\tau,\cdot) \to f$ strong in $L^q(\Omega), 1 \le q < \infty$.

The limit is a stationary solution

Test function: $\phi(x) \in C_c^\infty(\Omega)$. Fixe $T_0 > 0$ and let $t_2 = t_1 + T_0$.

$$\int_\Omega (v(t_2) - v(t_1))\phi = -\int_{t_1}^{t_2} \int_\Omega |\nabla v^m|^{p-2} \nabla v^m \nabla \phi + \frac{1}{m(p-1)-1} \int_{t_1}^{t_2} \int_\Omega v\phi.$$

Let $t_1 \to \infty$. Then $\boxed{-\Delta_p f^m(x) = \frac{1}{m(p-1)-1} f(x),\ x \in \Omega.}$

Also we prove the uniqueness of the stationary solution.

Difficult convergence:

$$\int_0^{T_0} \int_\Omega |\nabla v^m(\tau+n,x)|^{p-2} \nabla v^m(\tau+n,x) \nabla\phi dxdt \to T_0 \int_\Omega |\nabla f^m|^{p-2} \nabla f^m \nabla \phi dx.$$

$\to$Convergence in measure of gradients: $\nabla v^m(\tau,\cdot) \to \nabla f^m(\cdot)$, $\tau \to \infty$ in measure.

$\to$ Energy estimate: $\int_\Omega |\nabla v^m(\tau,x)|^p dx \le M, \quad \forall t \in \mathbb{R}.$

Then (1) + (2) $\Longrightarrow$ $\nabla v^m(t,\cdot) \to \nabla f^m(\cdot)$ a.e. in Ω.

Brezis,Cont.Nonl.Funct.An.,1971

Let A be a maximal monotone operator on a Hilbert space H. Let Z_n and W_n be measurable functions from Ω (a finite measure space) into H. Assume $Z_n \to Z$ a.e. on Ω and $W_n \rightharpoonup W$ weakly in $L^1(\Omega; H)$. If $W_n(x) \in A(Z_n(x))$ a.e. on Ω, then $W(x) \in A(Z(x))$ a.e. on Ω.

Our case:

- $\Omega_1 = [0, T_0) \times \Omega$ (finite measure space), $H = \mathbb{R}^N$ (Hilbert space).
- $A : H \to H, \quad A(Z) = |Z|^{p-2} Z$ maximal monotone operator.
- $Z_n(t, x) = \nabla v^m(\tau + n, x) : \Omega_1 \to H,$
- $W_n(t, x) = A(Z_n(t, x)) = |\nabla v^m(\tau + n, x)|^{p-2} \nabla v^m(\tau + n, x) : \Omega_1 \to H.$

$$\boxed{\text{Lemma} \implies W_n(t, x) \rightharpoonup W(t, x) \text{ weakly in } L^1(\Omega_1; H).}$$

Better convergence

Uniform Convergence:

$$\boxed{v(\tau,x) = t^{1/(p-2)}u(t,x) \to f(x), \;\; t = e^{\tau}.}$$

- Idea $\longrightarrow$ Second type of rescaling - fixed rate rescaling:

$$u_{\lambda}(t,x) = \lambda^{\frac{1}{m(p-1)-1}}u(\lambda t), \quad \lambda > 0.$$

- u_λ is still a solution of (DNLE).
- On $\Omega \times (t_1, t_2)$ the family $\{u_\lambda\}_{\lambda>0}$ is equicontinuous (because of the Hölder continuity and the a-priori estimates).
- Ascoli Arzelà Theorem $\Longrightarrow$ uniform convergence on subsequences $(u_{\lambda_j})_j$.
- Remark: $u_\lambda(1,x) = v(\log\lambda, x)$
- $v(\log\lambda_j, x)$ converges uniformly
- The limit $v(\tau,x) \to f$ is unique $\Longrightarrow$ $v(\tau,x) \to f$ uniformly.

The stationary profile

$$\Delta f^m(x) + \frac{1}{m(p-1)-1} f(x) = 0,\ x \in \Omega,\ \ f(x) = 0,\ x \in \partial\Omega$$

- **Existence**: f is obtained as the limit of $v(\tau, \cdot)$ when $\tau \to \infty$.
- **Uniqueness**: also proved by parabolic arguments.
- **Regularity**: $w \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1]$.
- **Behaviour up to the boundary** [Manfredi & Vespri 1994]:

$$|\nabla f| \leq C_1(\mathrm{dist}(x, \partial\Omega))^{1/m-1}, \quad \forall x \in \Omega,$$

$$\boxed{C_2 d(x,\partial\Omega) \leq f^m(x) \leq d(x,\partial\Omega)}, \quad \forall x \in \Omega.$$

that is $w(x) = f^m(x)$ has a linear growth near the boundary.

Rate of convergence for the PME

Hypothesis (H):

1. Ω is a bounded arcwise connected open set with compact closure and regular boundary.
2. u_0 is a nonnegative Lipschitz function defined on $\overline{\Omega}$ such that $u_0 = 0$ on $\partial\Omega$.

Weighted Rate of convergence for the PME (Aronson & Peletier, J.Diff.Eq.1981)

Assume that Ω and u_0 satisfy (H). Then $\exists\ \mathcal{C} \in [0, +\infty)$ which depends only on the data such that

$$|(1+t)^{1/(m-1)} u(t,x) - f(x)| \le \mathcal{C} f(x) (1+t)^{-1} \quad \text{in } \overline{\Omega} \times [0, +\infty).$$

Optimal rate because of the special family of global solutions

$$\mathcal{U}(t,x;s) = (s+t)^{-1/(m-1)} f(x).$$

Idea of the proof: use comparison with self similar solutions of the PME.

Weighted rate of convergence for the DNLE

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded connected domain of class $C^{2,\alpha}$, $\alpha > 0$. Let $u(t,\cdot)$ be a solution to Problem (1), $u_0 \in L^1(\Omega)$, $u_0 \geq 0$. Then for every $t_0 > 0$ there exist $C > 0$ such that

$$\boxed{\left|(1+t)^{\frac{1}{m(p-1)-1}} u(t,x) - f(x)\right| \leq \mathcal{C} f(x) (1+t)^{-1} \quad \text{for all } t \geq t_0, x \in \overline{\Omega},}$$

where C depends only on p, m, N, u_0, Ω and t_0, and f is the solution of

$$\Delta_p f^m(x) + \frac{1}{m(p-1)-1} f(x) = 0, \ x \in \Omega, \quad f(x) = 0, \ x \in \partial\Omega.$$

Recall

- the separate variables solution of the DNLE : $U(\tau, x) = \tau^{-1/(m(p-1)-1)} f(x)$.
- the *rescaled solution* of the DNLE: $v(t,x) = \tau^{1/(m(p-1)-1)} u(\tau, x), \quad \tau = e^t$.

Convergence in relative error

$$\lim_{t\to\infty} \left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} = \lim_{\tau\to\infty} \left\| \frac{v(\tau,\cdot)}{f(\cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0.$$

Self-Similar solutions for the DNLE

$$\boxed{\mathcal{U}(t,x) = (t+s)^{-\alpha} h(\eta), \quad \eta = |x|(t+s)^{-\beta},}$$

where $s \geq 0$ is a constant, α and β are positive parameters related by

$$(m(p-1)-1)\alpha + p\beta = 1,$$

and $g := h^m : [0,\infty) \to \mathbb{R}$ is a function satisfying the differential equation

$$\alpha g^{\frac{1}{m}}(\eta) + \beta\eta \left(g^{\frac{1}{m}}\right)'(\eta) + \frac{N-1}{\eta}|g'(\eta)|^{p-2}g'(\eta) + (p-1)|g'(\eta)|^{p-2}g''(\eta) = 0,\ \eta > 0,$$

Initial conditions

$$h(0) = M^m, \quad h'(0) = 0.$$

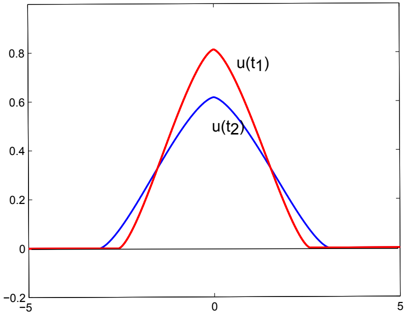
Case $\alpha = \beta N$. Barenblatt solutions

$$\boxed{\mathcal{U}(x,t;a,s) = c(t+s)^{-\alpha}\left(a^{\frac{p}{p-1}} - |x(t+s)^{-\beta}|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m(p-1)-1}}}$$

where s is a fixed positive parameter and

$$\alpha = \alpha_B = \frac{1}{m(p-1)-1+(p/N)}, \ \beta = \beta_B = \frac{\alpha_B}{N}.$$

1. Good point: compactly supported and they propagate in time.
2. Good point: solutions in the whole space.
3. Landing contact is flat.



Case $\alpha > \beta N$. Intermediate self-similar solutions

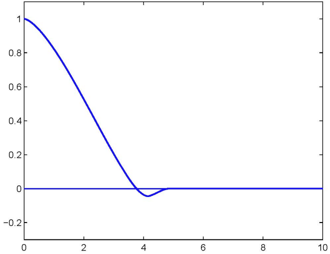
$$\boxed{\mathcal{V}(x,t;M,s) = (t+s)^{-\alpha}[h(\eta;M)]_+, \ \eta = |x|(t+s)^{-\beta}.}$$

Good point: compactly supported and they propagate in time

$$h^m(\eta) = g(\eta) \le \left(M_1 - c_1\eta^{\frac{p}{p-1}}\right)^{\frac{m(p-1)}{m(p-1)-1}}, \ \forall \eta \in (0,a).$$

$$supp\ \mathcal{V}(x,t;M,s) = \{(x,t) : |x| \le a(t+s)^\beta, t \ge 0\}.$$

Transversal cross: $g'(a) = -\left(\frac{\alpha-\beta N}{a^{N-1}}\int_0^a \zeta^{N-1} g^{1/m}(\zeta)d\zeta\right)^{\frac{1}{p-1}} = k < 0.$



Case $\alpha < \beta N$. Self-similar solutions positive for all times

$$\boxed{\mathcal{U}(t,x) = (t+s)^{-\alpha} g^{\frac{1}{m}}(\eta), \quad \eta = |x|(t+s)^{-\beta},}$$

Asymptotic decay: g behaves as $r \to \infty$ like $G_a(r) = ar^{-\gamma}$, $\gamma = \frac{\alpha m}{\beta}$.

1. g and g' satisfy

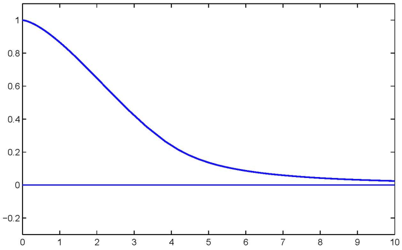
$$g > 0, \quad g' < 0 \quad \text{on } (0,\infty)$$

2. For every $r_0 > 0$ there exists $C = C(r_0) > 0$ such that

$$g(r) \geq C r^{-\frac{\alpha m}{\beta}}, \quad \forall r \geq r_0.$$

3. The limit at infinity is zero

$$\lim_{r\to\infty} g(r) = 0.$$



Improved upper bound for u

Theorem

Assume that Ω and u_0 satisfy hypothesis (H), and let u be the corresponding solution of Problem (1). Then there exists a constant $s_1 > 0$ such that

$$u(t,x) \le (s_1 + t)^{-1/(m(p-1)-1)} f(x), \;\; t \ge 0, x \in \overline{\Omega}. \tag{2}$$

where s_1 depends on p, m, N, u_0 and Ω.

$\longrightarrow$ Use Comparison Principle between u and the separate variable solution

$$U(t,x) := U(t,x;s_1) = (s_1 + t)^{-\frac{1}{m(p-1)-1}} f(x),$$

for an appropriate constant $\tau_1 > 0$ chosen s.t.

$$U(0,x) = s_1 f(x) \ge u_0(x), \quad \forall x \in \overline{\Omega}.$$

Improved lower bound for u

Theorem

There exist two positive constants $s_0 > 0$ and $T_4 > 0$ such that

$$u(t,x) \geq (s_0 + t)^{-\frac{1}{m(p-1)-1}} f(x), \quad \forall x \in \overline{\Omega}, \ \forall t \in [T_4, +\infty).$$

where s_0 and T_4 depend only on m, p, N, Ω and u_0.

Idea:

- Comparison from below with an intermediate self similar solution:

$$u(T_1 + t, x) \geq \mathcal{V}(x - y, t; M, s), \quad \forall x \in \overline{\Omega}, \ \forall t \in [0, T_3].$$

- Linear growth near the boundary:

$$u^m(T_4, x) \geq \omega d(x) \quad \text{for } x \in \Omega_\delta.$$

where $\Omega_\delta = \{x \in \overline{\Omega} : d(x) < \delta\}$.

The quasilinear case $m(p-1)=1$

Let λ the first eigenvalue of the Δ_p. Rescaling:

$$v(t,x)=e^{\lambda t}u(t,x),\ \ t\in[0,\infty),x\in\Omega,$$

The rescaled problem

$$\begin{cases} v_t(t,x)=\Delta_p v^m(t,x)+\lambda v(t,x) & \text{for } t>0 \text{ and } x\in\Omega,\\ v(0,x)=u_0(x) & \text{for } x\in\Omega,\\ v(t,x)=0 & \text{for } t>0 \text{ and } x\in\partial\Omega. \end{cases}$$

The associated stationary problem→ $V=S^m$ →the eigenvalue problem for the Δ_p

$$\begin{cases} \Delta_p S^m(x)+\lambda S(x)=0 & \text{for } x\in\Omega,\\ S(x)=0 & \text{for } x\in\partial\Omega. \end{cases} \quad \begin{cases} \Delta_p V+\lambda V^{p-1}=0 & \text{for } x\in\Omega,\\ V(x)=0 & \text{for } x\in\partial\Omega. \end{cases}$$

Uniform convergence to an unique asymptotic profile S

$$\|v(t,\cdot)-S(\cdot)\|_{L^\infty(\Omega)}\to 0,\quad t\to\infty.$$

$$\lim_{t\to\infty}\left\|\frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)}-1\right\|_{L^\infty(\Omega)}=\lim_{t\to\infty}\left\|\frac{v(t,\cdot)}{S(\cdot)}-1\right\|_{L^\infty(\Omega)}=0.$$

Future work:

1. Extend the result for Fast Diffusion case of the $p-$Laplacian equation:

$$\boxed{u_t = \Delta_p u, \quad 1 < p < 2.}$$

$\longrightarrow$ extinction in finite time, no conservation of mass.

$\longrightarrow$ PME case: [Bonforte-Grillo-Vazquez-2001].

2. Extend the result for the doubly nonlinear equation

$$\boxed{u_t = \Delta_p u^m.}$$

in the fast diffusion case $m(p-1) < 1$. $\longrightarrow$ [Savare-Vespri-1994]

3. Extend the result for equations with variable coefficients

$$\boxed{u_t = \sum_{ij} \partial_i (a_{ij} \partial_j u).}$$

Thank you for your attention!