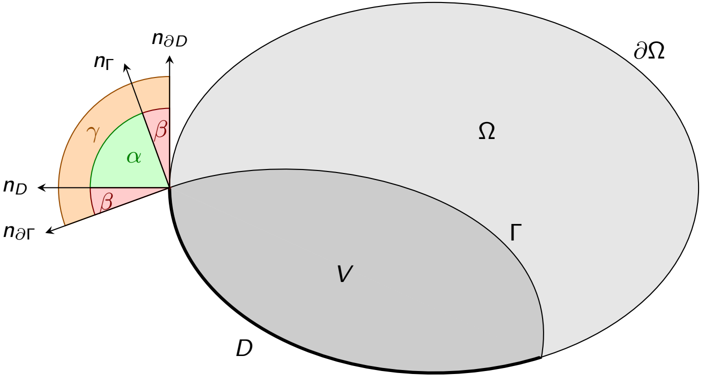
Linearization and local existence of solutions for the volume preserving mean curvature flow with line tension

Lars Müller

Universität Regensburg
Fakultät für Mathematik

January 31, 2012

General situation and notation



Basic technical assumption: $0 < \alpha(p) < \pi$ for all $p \in \partial\Gamma$.

Energy functional

We consider the energy functional

$$E(\Gamma) := \int_\Gamma 1 d\mathcal{H}^2 - a\int_D 1 d\mathcal{H}^2 + b\int_{\partial\Gamma} 1 d\mathcal{H}^1 + \lambda\left(\int_V 1 dx - V_0\right)$$

for $a, b, V_0 \in \mathbb{R}$ with $b \geq 0$.

Energy functional

We consider the energy functional

$$E(\Gamma) := \int_\Gamma 1 d\mathcal{H}^2 - a \int_D 1 d\mathcal{H}^2 + b \int_{\partial\Gamma} 1 d\mathcal{H}^1 + \lambda \left(\int_V 1 dx - V_0 \right)$$

for $a, b, V_0 \in \mathbb{R}$ with $b \geq 0$. Varying the hypersuface by

$$\psi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (t, p) \longmapsto \psi(t, p) := p + t\zeta(p)$$

where $\zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a tangential vectorfield to D and obtaining a family of evolving hypersurfaces by $\Gamma(t) := \psi(t, \Gamma)$, gives the first variation

$$\frac{d}{dt} E(\Gamma(t))\Big|_{t=0} = \int_\Gamma (\lambda - H_\Gamma)(n_\Gamma \cdot \zeta) d\mathcal{H}^2 + \int_{\partial\Gamma} (n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \zeta d\mathcal{H}^1$$

Energy functional

We consider the energy functional

$$E(\Gamma) := \int_\Gamma 1 d\mathcal{H}^2 - a \int_D 1 d\mathcal{H}^2 + b \int_{\partial\Gamma} 1 d\mathcal{H}^1 + \lambda \left(\int_V 1 dx - V_0 \right)$$

for $a, b, V_0 \in \mathbb{R}$ with $b \geq 0$. Varying the hypersuface by

$$\psi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : (t, p) \longmapsto \psi(t, p) := p + t\zeta(p)$$

where $\zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a tangential vectorfield to D and obtaining a family of evolving hypersurfaces by $\Gamma(t) := \psi(t, \Gamma)$, gives the first variation

$$\frac{d}{dt} E(\Gamma(t))\Big|_{t=0} = \int_\Gamma (\lambda - H_\Gamma)(n_\Gamma \cdot \zeta) d\mathcal{H}^2 + \int_{\partial\Gamma} (n_{\partial\Gamma} - a n_{\partial D} - b\vec{\varkappa}) \cdot \zeta d\mathcal{H}^1$$

This shows that a minimizer of that energy necessarily satisfies

$$\begin{aligned} H_\Gamma = \lambda = const. \qquad & \text{on } \Gamma \\ 0 = a + b\varkappa_{\partial D} + \langle n_\Gamma, n_D \rangle \qquad & \text{on } \partial\Gamma \end{aligned}$$

Corresponding flow equations

One possible flow that tends towards a minimizer of the energy E is given by

$$V_\Gamma = H_\Gamma - \overline{H_\Gamma} \qquad \text{in } \Gamma$$

where $\overline{H_\Gamma}$ is the mean of the mean curvature given by

$$\overline{H_\Gamma} := \frac{1}{\int\limits_\Gamma 1 d\mathcal{H}^2} \int_\Gamma H_\Gamma d\mathcal{H}^2$$

Corresponding flow equations

One possible flow that tends towards a minimizer of the energy E is given by

$$V_\Gamma = H_\Gamma - \overline{H_\Gamma} \qquad \text{in } \Gamma$$

where $\overline{H_\Gamma}$ is the mean of the mean curvature given by

$$\overline{H_\Gamma} := \frac{1}{\int_\Gamma 1 d\mathcal{H}^2} \int_\Gamma H_\Gamma d\mathcal{H}^2$$

Additionally there are several reasonable choices of boundary conditions. We will impose the boundary condition

$$v_{\partial D} = a + b\varkappa_{\partial D} + \langle n_\Gamma, n_D \rangle \qquad \text{on } \partial\Gamma$$

The "Curvilinear Coordinates System" $\Psi(q,w)$

We introduce the coordinate system Ψ over a fixed reference hypersurface Γ^* as

$$\Psi : \Gamma^* \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \Omega : (q,w) \longmapsto \Psi(q,w) := q + w n_{\Gamma^*}(q) + t(q,w)T(q)$$

where $T : \Gamma^* \longrightarrow \mathbb{R}^3$ is an arbitrary tangential vectorfield, that coincides with $n_{\partial\Gamma^*}$ on $\partial\Gamma^*$ and vanishes outside a small neighborhood of $\partial\Gamma^*$ and

$$t : \Gamma^* \times (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathbb{R} : (q,w) \longmapsto t(q,w)$$

is some smooth function such that $\Psi(q,0) = q$ for all $q \in \Gamma^*$ and $\Psi(q,w) \in \partial\Omega$ for all $q \in \partial\Gamma^*$ and all $w \in (-\varepsilon_0, \varepsilon_0)$.

Distance function ϱ and the L_2-Gradient flow

With the help of the curvilinear coordinate system Ψ one can write the evolving hypersurface as a familiy of graphs over the fixed hypersurface Γ^*. To this purpose define a distance function

$$\varrho : \mathbb{R}_+ \times \Gamma^* \longrightarrow (-\varepsilon_0, \varepsilon_0) : (t, q) \longmapsto \varrho(t, q)$$

and set $\Gamma_\varrho(t) := \mathrm{Im}(\Psi(\bullet, \varrho(t, \bullet)))$ and observe that by construction one has $\Gamma_{\varrho\equiv 0}(t) = \Gamma^*$ for all $t \in \mathbb{R}_+$.

The considered PDE in terms of ϱ

With this notation the flow from above transforms into

$$\begin{aligned} V_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))) &= H_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))) - \overline{H}(\varrho(t)) && \text{in } \Gamma^* \\ v_{\partial D_{\varrho(t)}}(\Psi(q,\varrho(t,q))) &= a + b\varkappa_{\partial D_{\varrho(t)}}(\Psi(q,\varrho(t,q))) \\ &\quad + \left\langle n_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))), n_{D_{\varrho(t)}}(\Psi(q,\varrho(t,q)))\right\rangle && \text{on } \partial\Gamma^* \end{aligned}$$

which is a non-linear second order PDE in ϱ with second order boundary conditions.

The considered PDE in terms of ϱ

With this notation the flow from above transforms into

$$\begin{aligned} V_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))) &= H_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))) - \overline{H}(\varrho(t)) && \text{in } \Gamma^* \\ v_{\partial D_{\varrho(t)}}(\Psi(q,\varrho(t,q))) &= a + b\varkappa_{\partial D_{\varrho(t)}}(\Psi(q,\varrho(t,q))) \\ &\quad + \left\langle n_{\Gamma_{\varrho(t)}}(\Psi(q,\varrho(t,q))), n_{D_{\varrho(t)}}(\Psi(q,\varrho(t,q)))\right\rangle && \text{on } \partial\Gamma^* \end{aligned}$$

which is a non-linear second order PDE in ϱ with second order boundary conditions.

Problem

This PDE is also non-local due to $\overline{H}(\varrho(t))$!

Linearization

The linearization of the PDE reads as

$$\begin{aligned}
\partial_t \varrho(t) &= \Delta_{\Gamma^*} \varrho(t) + |\sigma^*|^2 \varrho(t) + \left(\nabla_{\Gamma^*} H_{\Gamma^*} \cdot P\left(\partial_w \Psi(0)\right)\right) \varrho(t) \\
&\quad - \fint_{\Gamma^*} \left(\Delta_{\Gamma^*} + |\sigma^*|^2 - H_{\Gamma^*}^2 + \overline{H}(\mathbb{O}) H_{\Gamma^*}\right) \varrho(t) \, d\mathcal{H}^2 \\
&\quad + \frac{1}{\int_{\Gamma^*} 1 \, d\mathcal{H}^2} \int_{\partial\Gamma^*} \left(H_{\Gamma^*} - \overline{H}(\mathbb{O})\right) \cot(\alpha) \varrho(t) \, d\mathcal{H}^1 \qquad \text{on } \Gamma^* \\
\partial_t \varrho(t) &= -\sin(\alpha)^2 (n_{\partial\Gamma^*} \cdot \nabla_{\Gamma^*} \varrho(t)) - \sin(\alpha) II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) \\
&\quad + \sin(\alpha)\cos(\alpha) II_{\Gamma^*}(n_{\partial\Gamma^*}, n_{\partial\Gamma^*}) \varrho(t) + b \sin(\alpha) \varrho_{\sigma\sigma}(t) \\
&\quad + b \sin(\alpha) \varkappa_{D^*} II_{D^*}(n_{\partial D^*}, n_{\partial D^*}) \varrho(t) - b \sin(\alpha) \varkappa_{\partial D^*} \left\langle \vec{\tau}^*, (n_{\partial D^*})_\sigma \right\rangle \varrho(t) \\
&\quad - b \sin(\alpha) \left\langle n_{\partial D^*}, (n_{D^*})_\sigma \right\rangle^2 \varrho(t) \qquad \text{on } \partial\Gamma^* \\
\varrho(0) &= \varrho_0 \qquad \text{on } \Gamma^*
\end{aligned}$$

Highest order terms   Non-local terms