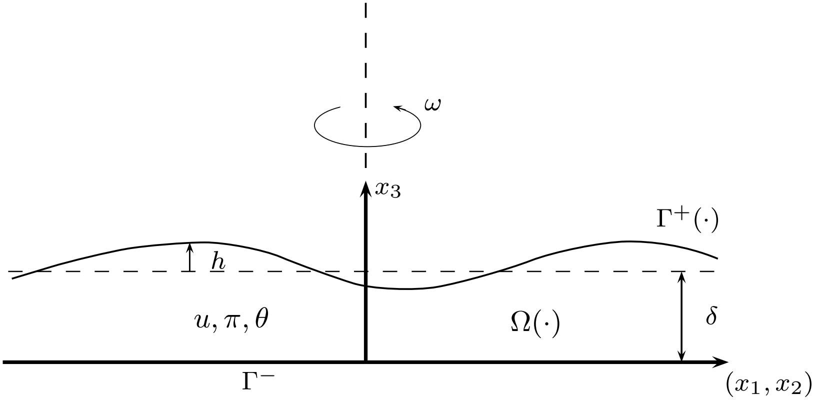
The spin-coating process with convective heat transfer

Lorenz von Below



The model

TECHNISCHE UNIVERSITÄT DARMSTADT

$$
\begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla \pi + 2\omega \times u = -\omega \times \omega \times \chi x - \theta e_3 & \text{in } \Omega(t) \\
\partial_t \theta + u \cdot \nabla \theta = 0 & \text{in } \Omega(t) \\
\operatorname{div} u = 0 & \text{in } \Omega(t) \\
-\mathcal{T} \nu = \sigma \kappa \nu & \text{on } \Gamma^+(t) \\
V = u \cdot \nu & \text{on } \Gamma^+(t) \\
(u^1, u^2) = c(h + \delta)^\alpha \partial_3 (u^1, u^2) & \text{on } \Gamma^-(t) \\
u^3 = 0 & \text{on } \Gamma^-(t) \\
u(0) = u_0 & \text{in } \Omega(0) \\
\theta(0) = \theta_0 & \text{in } \Omega(0) \\
\Omega(0) = \{x \in \mathbb{R}^3_+ : x_3 < \delta + h_0(x_1, x_2)\}. &
\end{cases}
$$

Main result

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem

Let $p > 5$ and $J = (0, T)$. There is $\varepsilon > 0$ such that for $\omega \geq 0$ and initial values satisfying certain compatibility conditions and

$$\omega + \|u_0\|_{W^{2-2/p,p}(\Omega(0))} + \|h_0\|_{W^{3-2/p,p}(\mathbb{R}^2)} + \|\theta_0\|_{W^{1,p}(\Omega(0))} < \varepsilon$$

the spin coating system with heat convection admits a uniquely determined solution

$$u \in \mathbb{E}_u(J) := L^p(J; W^{2,p}(\Omega(\cdot))) \cap W^{1,p}(J; L^p(\Omega(\cdot)))$$

$$\pi \in \mathbb{E}_\pi(J) := \{\pi \in L^p(J; \dot{W}^{1,p}(\Omega(\cdot))) : \gamma\pi \in W^{\frac{p-1}{2p},p}(J; L^p(\Gamma^+(\cdot))) \cap L^p(J; W^{1-\frac{1}{p},p}(\Gamma^+(\cdot)))\}$$

$$h \in \mathbb{E}_h(J) := W^{2-\frac{1}{2p},p}(J; L^p(\Gamma^+(\cdot))) \cap W^{1,p}(J; W^{2-\frac{1}{p},p}(\Gamma^+(\cdot))) \cap L^p(J; W^{3-\frac{1}{p},p}(\Gamma^+(\cdot)))$$

$$\theta \in \mathbb{E}_\theta(J) := L^\infty(J; W^{1,p}(\Omega(\cdot))) \cap W^{1,\infty}(J; L^p(\Omega(\cdot))).$$

Transformation to a fixed domain

Hanzawa-Transform

$$
\left\{
\begin{aligned}
\partial_t u - \Delta u + 2\omega \times u + \nabla \pi &= -\theta e_3 + F_1(u, \pi, h) && \text{in } J \times \Omega \\
\partial_t \theta + V(u, h) \cdot \nabla \theta &= 0 && \text{in } J \times \Omega \\
\operatorname{div} u &= F_d(u, h) && \text{in } J \times \Omega \\
\gamma \mathcal{T}(u, \pi)\nu_D - \sigma \Delta' h \nu_D &= G_+(u, \pi, h) && \text{on } J \times \Gamma^+ \\
\partial_t h - \gamma u^3 &= H(u, h) && \text{on } J \times \Gamma^+ \\
\gamma(u^1, u^2) - \gamma c \delta^\alpha \partial_3(u^1, u^2) &= G_-(u^1, u^2, h) && \text{on } J \times \Gamma^- \\
\gamma u^3 &= 0 && \text{on } J \times \Gamma^- \\
u(0) &= u_0 && \text{in } \Omega \\
\theta(0) &= \theta_0 && \text{in } \Omega \\
h(0) &= h_0 && \text{in } \mathbb{R}^2
\end{aligned}
\right.
$$

The nonlinearities

$$
\begin{aligned}
F_1(u,\pi,h) = {} & \omega \times \omega \times \chi x + \frac{x_3}{h+\delta}\partial_3 u \partial_t h - \frac{h^2 + 2\delta h}{(h+\delta)^2}\partial_3^2 u - 2\frac{x_3}{h+\delta}\nabla' \partial_3 u \nabla' h \\
& + \frac{x_3^2}{(h+\delta)^2}\left|\nabla' h\right|^2 \partial_3 u - \frac{x_3}{h+\delta}\partial_3 u \Delta' h + 2\frac{x_3}{(h+\delta)^2}\partial_3 u \left|\nabla' h\right|^2 \\
& + \frac{x_3}{h+\delta}\partial_3 \pi (\nabla' h, 0)^T - u \cdot (\nabla' u, \frac{\delta}{h+\delta}\partial^3 u) \\
& + \frac{x_3}{h+\delta}\partial_3 u (u^1, u^2) \cdot \nabla' h + \frac{h}{h+\delta}(0, 0, \partial_3 \pi)^T \\
V(u,h) = {} & u - e_3 \frac{1}{h+\delta}\left(x_3 \partial_t h + u \cdot \begin{pmatrix} x_3 \nabla' h \\ h \end{pmatrix} \right) \\
F_d(u,h) = {} & \frac{h}{h+\delta}\partial_3 u^3 + \frac{x_3}{h+\delta}(\partial_3 u^1, \partial_3 u^2) \cdot \nabla' h
\end{aligned}
$$

$\vdots$

Solving the spin-coating system

Denk/Geißert/Hieber/Saal/Sawada 2011

TECHNISCHE UNIVERSITÄT DARMSTADT

There is a Lipschitz continuous operator

$$\Psi\colon U \subset L^p(J; L^p(\Omega)) \to \mathbb{E}_u(J) \times \mathbb{E}_\pi(J) \times \mathbb{E}_h(J), \qquad f_1 \mapsto (u, \pi, h)$$

with

$$\begin{cases} \partial_t u - \Delta u + 2\omega \times u + \nabla\pi = f_1 + F_1(u, \pi, h) & \text{in } J \times \Omega \\ \operatorname{div} u = F_d(u, h) & \text{in } J \times \Omega \\ \gamma\mathcal{T}(u, \pi)\nu_D - \sigma\Delta' h\nu_D = G_+(u, \pi, h) & \text{on } J \times \Gamma^+ \\ \partial_t h - \gamma u^3 = H(u, h) & \text{on } J \times \Gamma^+ \\ \gamma(u^1, u^2) - \gamma c\delta^\alpha \partial_3(u^1, u^2) = G_-(u^1, u^2, h) & \text{on } J \times \Gamma^- \\ \gamma u^3 = 0 & \text{on } J \times \Gamma^- \\ u(0) = u_0 & \text{in } \Omega \\ h(0) = h_0 & \text{in } \mathbb{R}^2. \end{cases}$$

Reduction to a nonlinear transport equation

TECHNISCHE UNIVERSITÄT DARMSTADT

Let

$$\mathcal{V}(\theta) = V(\Psi_1(-\theta e_3), \Psi_3(-\theta e_3)).$$

Then (u, π, h, θ) solves the spin coating system with heat convection if and only if

$$\begin{cases} \partial_t \theta + \mathcal{V}(\theta) \cdot \nabla\theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega \end{cases}$$

and $(u, \pi, h) = \Psi(-\theta e_3)$.

Lemma

There is an open neighbourhood $U \subset L^p(J; L^p(\Omega))$ of zero such that

- $\mathcal{V}\colon U \subset L^p(J; L^p(\Omega)) \to L^1(J; W^{1,\infty}(\Omega))$ *is well defined,*
- $\|\mathcal{V}(\theta) - \mathcal{V}(\tau)\|_{L^1(J;L^\infty(\Omega))} \leq C \|\theta - \tau\|_{L^p(J;L^p(\Omega))}$ *for* $\theta, \tau \in U$,
- $\mathcal{V}(\theta) \cdot \nu = 0$ *on* $\partial\Omega$.

The linear transport equation

TECHNISCHE UNIVERSITÄT DARMSTADT

Let $\theta_0 \in W^{1,p}(\Omega)$, $v \in L^1(J; W^{1,\infty}(\Omega))$ with $v \cdot \nu = 0$ on $\partial\Omega$. Then there is a unique solution $\theta \in \mathbb{E}_\theta(J)$ of

$$\text{(LTE)} \begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega \end{cases}$$

with

$$\|\theta\|_{\mathbb{E}_\theta(J)} \leq \|\theta_0\|_{W^{1,p}(\Omega)} e^{C\|v\|_{L^1(J;W^{1,\infty}(\Omega))}}.$$

The solution $\tau \in \mathbb{E}_\theta(J)$ of (LTE) corresponding to data $\tau_0 \in W^{1,p}(\Omega)$ and $w \in L^1(J; W^{1,\infty}(\Omega))$ with $w \cdot \nu = 0$ satisfies

$$\|\theta - \tau\|_{L^\infty(L^p)} \leq C \left(\|\theta_0 - \tau_0\|_{L^p} + \|v - w\|_{L^1(L^\infty)} \|\tau\|_{\mathbb{E}_\theta(J)} \right) e^{C\|v\|_{L^1(W^{1,\infty})}}.$$

Solving the nonlinear transport equation

Proposition

Let $5 < p < \infty$, $J = (0, T)$ *and* $\mathcal{V}$ *as before. There is* $\varepsilon > 0$ *such that whenever*

$$\|\theta_0\|_{W^{1,p}(\Omega)} \leq \varepsilon$$

is satisfied there is a uniquely determined solution

$$\theta \in L^\infty(J; W^{1,p}(\Omega)) \cap W^{1,\infty}(J; L^p(\Omega))$$

of

$$\text{(NTE)} \begin{cases} \partial_t\theta + \mathcal{V}(\theta) \cdot \nabla\theta = 0 & \text{in } J \times \Omega \\ \theta(0) = \theta_0 & \text{in } \Omega. \end{cases}$$

Proof.

- Let $\mathbb{B}_\theta^R(J)$ denote the closed ball of radius R in $\mathbb{E}_\theta(J)$ around zero,
- $\Phi\colon (\theta_0, v) \mapsto \theta$ the solution operator of the linear transport equation (LTE).
- Define $\Lambda\colon \mathbb{B}_\theta^R(J) \to \mathbb{E}_\theta(J), \quad \theta \mapsto \Phi(\theta_0, \mathcal{V}(\theta))$.
- $\theta = \Lambda(\theta) \quad \Longleftrightarrow \quad \theta$ solves the nonlinear transport equation (NTE).
- $\mathbb{B}_\theta^R(J)$ is weakly-* closed in $\mathbb{E}_\theta(J) \quad \Rightarrow \quad \left(\mathbb{B}_\theta^R(J), \|\cdot\|_{L^\infty(J;L^p(\Omega))}\right)$ is a complete metric space.
- Apply Banach's Fixed Point Theorem

Solving the nonlinear transport equation

TECHNISCHE UNIVERSITÄT DARMSTADT

For $\theta, \tau \in \mathbb{B}_\theta^R(J)$

- $\|\Lambda(\theta)\|_{\mathbb{E}_\theta(J)} \leq \|\theta_0\|_{W^{1,p}(\Omega)} e^{C\|\mathcal{V}(\theta)\|_{L^1(J;W^{1,\infty}(\Omega))}} \leq C\|\theta_0\|_{W^{1,p}(\Omega)} \overset{!}{\leq} R$
- $$\begin{aligned}\|\Lambda(\theta) - \Lambda(\tau)\|_{L^\infty(J;L^p(\Omega))} &\leq C\|\mathcal{V}(\theta) - \mathcal{V}(\tau)\|_{L^1(J;L^\infty(\Omega))} \|\tau\|_{\mathbb{E}_\theta(J)} e^{C\|\mathcal{V}(\tau)\|_{L^1(J;W^{1,\infty}(\Omega))}} \\ &\leq CR\|\theta - \tau\|_{L^\infty(J;L^p(\Omega))}.\end{aligned}$$
- Choose $\|\theta_0\|_{W^{1,p}(\Omega)}$ small enough.

$\square$

Thank you for your attention!