Stabilization of Periodic Stokesian Hele–Shaw Flows of Ferrofluids

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March 21, 2012

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The Hele–Shaw Cell

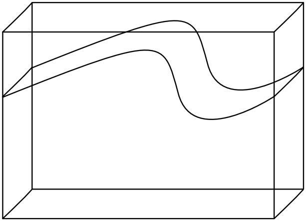
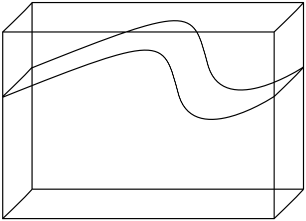


Figure: A vertical Hele-Shaw cell with a fluid's interface.

The Hele–Shaw Cell



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3-D model
Navier–Stokes equations

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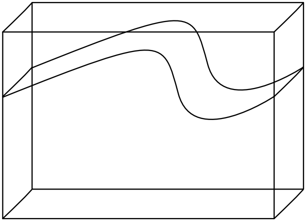

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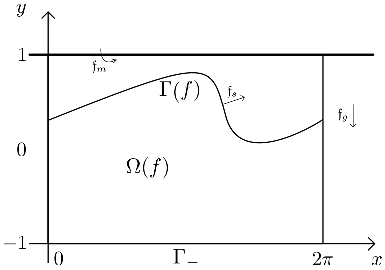

Figure: 2-dimensional profile of the Hele-Shaw cell.

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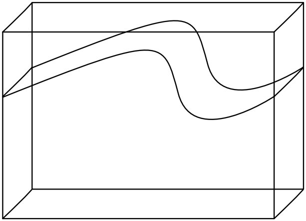

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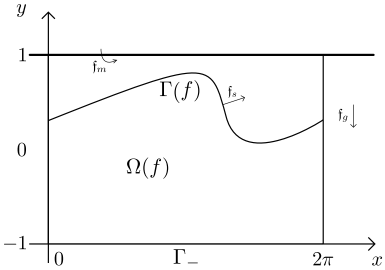

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3-D model
Navier–Stokes equations

2-D, gap-averaged model
Darcy's law
Moving boundary problem

The Moving Boundary Problem

$$
\begin{aligned}
\mathcal{Q}u := -\operatorname{div} \frac{Du}{\overline{\mu}(|Du|^2)} &= 0 && \text{in} \quad \Omega(f) \\
u &= b(f) && \text{on} \quad \Gamma_- \\
u &= -\gamma\kappa_f - \frac{\iota^2}{(1-f)^2} + g\rho f && \text{on} \quad \Gamma(f) \\
\partial_t f &= -\frac{\sqrt{1+f'^2}}{\overline{\mu}(|Du|^2)} \partial_\nu u && \text{on} \quad \Gamma(f) \\
f(0) &= f_0 && \text{on} \quad \mathbb{S}
\end{aligned} \tag{P}
$$

Little Hölder Spaces

For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $U \subset \mathbb{R}^2$ open

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For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $U \subset \mathbb{R}^2$ open let

$$\begin{aligned} \mathrm{buc}^{k+\alpha}(U) &:= \overline{\mathrm{BUC}^\infty(U)} \quad \text{in} \quad \mathrm{BUC}^{k+\alpha}(U), \\ h^{k+\alpha}(\mathbb{S}) &:= \overline{C^\infty(\mathbb{S})} \quad \text{in} \quad C^{k+\alpha}(\mathbb{S}). \end{aligned}$$

Goal of this choice: Use of strongly continuous analytic semigroups and abstract parabolic theory

Classical Hölder Solutions

Let

$$\mathcal{V} := \{f \in h^{4+\alpha}(\mathbb{S}) : \|f\|_\infty < 1\}.$$

We seek (u, f) satisfying

$$f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha}(\mathbb{S})),$$
$$u(\,\cdot\,, t) \in \mathrm{buc}^{2+\alpha}(\Omega(f(t))), \quad 0 \le t \le T$$

that fulfill (P) pointwise.

The Wellposedness Result

Theorem

Assume

$$
\begin{aligned}
&0 < c \leq \mu(r) \leq C && \text{for all} \quad r \geq 0, \\
&0 < c \leq \mu(r) + 2r\mu'(r) \leq C && \text{for all} \quad r \geq 0
\end{aligned}
$$

hold and let $|c| < 1$*. There is an open neighborhood* $\mathcal{O}$ *of* c *in* $h^{4+\alpha}(\mathbb{S})$ *such that for all* $f_0 \in \mathcal{O}$ *problem* (P) *has a unique, maximal defined classical Hölder solution in* $\mathcal{O}$*.*

Sketch of the Proof

Transformation on a fixed reference domain Ω.

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Plug this solution in the fourth equation.

Study the linearization of the evolution operator.

The Transformation

The Diffeomorphism

$$\phi_f(x,y) = (x, y + (1+y)f(x)) \quad \text{for} \quad (x,y) \in \Omega = \mathbb{S} \times (-1,0)$$

straightens the boundary:

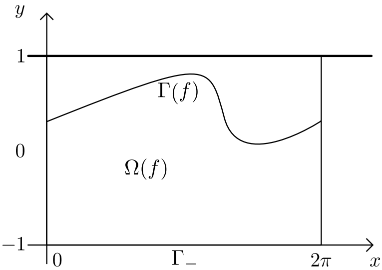

Figure: Original, time-dependent geometry.

The Transformation

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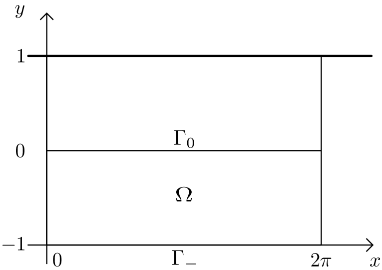

Figure: Transformed, fixed geometry.

The Transformation

Push forward and pull back operators

$$\phi_*^f : \mathrm{buc}^{2+\alpha}(\Omega) \to \mathrm{buc}^{2+\alpha}(\Omega(f)), \quad v \mapsto v \circ \phi_f^{-1}$$
$$\phi_f^* : \mathrm{buc}^{2+\alpha}(\Omega(f)) \to \mathrm{buc}^{2+\alpha}(\Omega), \quad u \mapsto u \circ \phi_f$$

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Transformed operators

$$\mathcal{A}(f) = \phi_f^* \circ \mathcal{Q} \circ \phi_*^f : \mathrm{buc}^{2+\alpha}(\Omega) \to \mathrm{buc}^{\alpha}(\Omega)$$
$$\mathcal{B}(f,\cdot) = -\operatorname{tr}_0 \phi_f^* \left\langle \frac{D(\phi_*^f \cdot)}{\overline{\mu}(|D(\phi_*^f \cdot)|^2)}, n \right\rangle : \mathcal{V} \times \mathrm{buc}^{2+\alpha}(\Omega) \to h^{1+\alpha}(\mathbb{S})$$

The Transformed System

System (P) is equivalent to the transformed system

$$\begin{aligned}
\mathcal{A}(f)v &= 0 && \text{in} \quad \Omega \\
v &= b(f) && \text{on} \quad \Gamma_- \\
v &= -\gamma\kappa_f - \frac{\iota^2}{(1-f)^2} + g\rho f && \text{on} \quad \Gamma_0 \\
\partial_t f &= \mathcal{B}(f,v) && \text{on} \quad \Gamma_0 \\
f(0) &= f_0 && \text{on} \quad \mathbb{S}
\end{aligned}$$

A first Existence and Uniqueness Result

Theorem

Let $f \in \mathcal{V}$. There is a unique solution $\mathcal{T}(f) \in \mathrm{buc}^{2+\alpha}(\Omega)$ of the quasilinear Dirichlet problem

$$\begin{aligned} \mathcal{A}(f)v &= 0 && \text{in} \quad \Omega, \\ v &= b(f) && \text{on} \quad \Gamma_-, \\ v &= -\gamma\kappa_{\Gamma(f)} - \frac{\iota^2}{(1-f)^2} + g\rho f && \text{on} \quad \Gamma_0. \end{aligned}$$

The mapping $\mathcal{V} \ni f \mapsto \mathcal{T}(f) \in \mathrm{buc}^{2+\alpha}(\Omega)$ is smooth.

The Evolution Equation

Advection equation on variable domain:

$$\partial_t f = -\frac{\sqrt{1+f'^2}}{\overline{\mu}(|Du|^2)}\partial_\nu u, \quad f(0) = f_0.$$

Evolution equation on fixed domain:

$$\partial_t f = \Phi(f), \quad f(0) = f_0$$

for the non-linear operator $\Phi(\,\cdot\,) = \mathcal{B}(\,\cdot\,, \mathcal{T}(\,\cdot\,))$.

The Generation Result

Theorem

(The complexification of) $\partial\Phi(c)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}))$, i.e.,

$$-\partial\Phi(c) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Sketch of the Proof

Equivalent characterization:

$$\lambda - \partial\Phi(c) \in \mathcal{L}\mathrm{is}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})),$$
$$|\lambda| \|R(\lambda, \partial\Phi(c))\|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}))} \leq \chi$$

for $h^{4+\alpha}(\mathbb{S}) \overset{d}{\hookrightarrow} h^{1+\alpha}(\mathbb{S})$, some $\chi, \omega > 0$, and all $\mathrm{Re}\,\lambda \geq \omega$

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Consider $\lambda - \partial\Phi(c)$ as an operator between Sobolev spaces and apply a Marcinkiewicz multiplier theorem.

Transfer the result to little Hölder spaces with a density argument.

Proof of the Well-Posedness Result

Apply a perturbation theorem for the class $\mathcal{H}(h^{4+\beta}(\mathbb{S}), h^{1+\beta}(\mathbb{S}))$ with $0 < \beta < \alpha$.

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Use interpolation property of the little Hölder spaces

$$(h^{\theta_1}(\mathbb{S}), h^{\theta_2}(\mathbb{S}))_\sigma = h^{\theta_1+\sigma(\theta_2-\theta_1)}(\mathbb{S})$$

for $0 < \sigma < 1$ and $\theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N}$.

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Putting $\sigma := (\alpha - \beta)/3$ turns the generation result in a well-posedness result for $f \in \mathcal{O} := \mathcal{O}_\beta \cap h^{4+\alpha}(\mathbb{S})$ (cf. Lunardi, 1995).

The Linearization

$$\partial\Phi(c)\left[\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}\right] = \sum_{k\in\mathbb{Z}} \lambda_k c_k \mathrm{e}^{\mathrm{i}kx},$$

The Linearization

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$$\begin{aligned}\lambda_k &= \Gamma_{\overline{\mu},c}\Bigg[-\gamma\coth((1+c)k)k^3 + \Big(b'(c)\frac{1}{\operatorname{sech}((1+c)k)} \\ &\quad - G_c\coth((1+c)k)\Big)k\Bigg], \quad k \neq 0,\\ \lambda_0 &= \Gamma_{\overline{\mu},c}\frac{b'(c)-G_c}{1+c}, \quad G_c = g\rho - 2\frac{\iota^2}{(1-c)^2} - \frac{\theta_c}{1+c},\\ \theta_c &= \partial_y \mathcal{T}(c), \quad \Gamma_{\overline{\mu},c} = \frac{1}{\overline{\mu}((\frac{\theta_c}{1+c})^2)} + 2\frac{\theta_c^2}{(1+c)^2}\Big(\frac{1}{\overline{\mu}}\Big)'((\tfrac{\theta_c}{1+c})^2)\end{aligned}$$

Stability Conditions

Stability of the flat interface $f = c$ under the condition

$$\begin{aligned} &0 \leq b'(c) < -\frac{2\iota^2}{(1-c)^2} + g\rho \quad \text{or} \\ &0 > b'(c), \quad \max\{b'(c), -\mathrm{I}\} < -\frac{2\iota^2}{(1-c)^2} + g\rho, \end{aligned} \tag{1}$$

where $\mathrm{I} = \inf_{k\geq 1}\big(\gamma k^2 - b'(c)\,\mathrm{sech}((1+c)k)\big)$

Stability Conditions

Instability of the flat interface $f = c$ if the former condition is violated such that

$$\begin{aligned} &0 \leq b'(c), \quad b'(c) > -\frac{2\iota^2}{(1-c)^2} + g\rho \quad \text{or} \\ &0 > b'(c), \quad \max\{b'(c), -\mathrm{I}\} > -\frac{2\iota^2}{(1-c)^2} + g\rho \end{aligned} \tag{2}$$

The Main Result

Theorem

Let $|c| < 1$ *and* $\omega_0 := -\sup \sigma\big(\partial\Phi(c)\big)$.

(i) *If* (1) *holds then the solution to* (P) *is exponentially stable. More precisely, given* $\omega \in (0, \omega_0)$, *there exist positive constants* M *and* δ *such that for all* $f_0 \in h^{4+\alpha}(\mathbb{S})$ *with* $\|f_0 - c\|_{C^{4+\alpha}(\mathbb{S})} \leq \delta$ *the solution to* (P) *corresponding to* f_0 *exists in the large and for all* $t \geq 0$ *it holds that*

$$\|f(t) - c\|_{C^{4+\alpha}(\mathbb{S})} + \|f'(t)\|_{C^{1+\alpha}(\mathbb{S})} \leq Me^{-\omega t}\|f_0 - c\|_{C^{4+\alpha}(\mathbb{S})}.$$

(ii) *If* (2) *holds then the flat solution* $f = c$ *is unstable.*

The Case $b'(c) > 0$

Stability condition

$$b'(c) < -\frac{2\iota^2}{(1-c)^2} + g\rho,$$

which means:

The Case $b'(c) > 0$

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- the larger c the lower ι,

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which means:

- $b'(c) < g\rho$,
- the larger c the lower ι,
- critical maximum value $\iota_* := \frac{|1-c|}{2}\sqrt{2(g\rho - b'(c))}$