Stabilization of Periodic Stokesian Hele–Shaw Flows of Ferrofluids

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3-D model
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The Moving Boundary Problem

\[ Qu := - \text{div} \left( \frac{Du}{\mu(|Du|^2)} \right) = 0 \quad \text{in} \ \Omega(f) \]

\[ u = b(f) \quad \text{on} \ \Gamma_- \]

\[ u = -\gamma \kappa_f - \frac{t^2}{(1-f)^2} + g \rho f \quad \text{on} \ \Gamma(f) \]

\[ \partial_t f = - \frac{\sqrt{1 + f'^2}}{\mu(|Du|^2)} \partial_\nu u \quad \text{on} \ \Gamma(f) \]

\[ f(0) = f_0 \quad \text{on} \ \mathcal{S} \quad (P) \]
Little Hölder Spaces

For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $U \subset \mathbb{R}^2$ open
Little Hölder Spaces

For \( k \in \mathbb{N}, \alpha \in (0, 1) \) and \( U \subset \mathbb{R}^2 \) open let

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\text{buc}^{k+\alpha}(U) := \text{BUC}^{\infty}(U) \quad \text{in} \quad \text{BUC}^{k+\alpha}(U),
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Little Hölder Spaces

For \( k \in \mathbb{N}, \alpha \in (0,1) \) and \( U \subset \mathbb{R}^2 \) open let

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h^{k+\alpha}(\mathcal{S}) := \overline{\text{C}}^\infty(\mathcal{S}) \quad \text{in} \quad \text{C}^{k+\alpha}(\mathcal{S}).
\]
Little Hölder Spaces

For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $U \subset \mathbb{R}^2$ open let

$$ \text{buc}^{k+\alpha}(U) := \overline{\text{BUC}^\infty(U)} \quad \text{in} \quad \text{BUC}^{k+\alpha}(U), $$

$$ h^{k+\alpha}(S) := \overline{\text{C}^\infty(S)} \quad \text{in} \quad \text{C}^{k+\alpha}(S). $$

Goal of this choice: Use of strongly continuous analytic semigroups and abstract parabolic theory.
Classical Hölder Solutions

Let

\[ V := \{ f \in h^{4+\alpha}(\mathbb{S}) : \|f\|_\infty < 1 \}. \]

We seek \((u, f)\) satisfying

\[ f \in C([0, T], V) \cap C^1([0, T], h^{1+\alpha}(\mathbb{S})), \]
\[ u(\cdot, t) \in \text{buc}^{2+\alpha}(\Omega(f(t))), \quad 0 \leq t \leq T \]

that fulfill (P) pointwise.
The Wellposedness Result

Theorem
Assume

\[ 0 < c \leq \mu(r) \leq C \quad \text{for all} \quad r \geq 0, \]
\[ 0 < c \leq \mu(r) + 2r\mu'(r) \leq C \quad \text{for all} \quad r \geq 0 \]

hold and let \(|c| < 1\). There is an open neighborhood \(\mathcal{O}\) of \(c\) in \(H^{4+\alpha}(\mathcal{S})\) such that for all \(f_0 \in \mathcal{O}\) problem (P) has a unique, maximal defined classical Hölder solution in \(\mathcal{O}\).
Sketch of the Proof

Transformation on a fixed reference domain $\Omega$. 
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Solve the first three equations of the transformed system.
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Plug this solution in the fourth equation.
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Transformation on a fixed reference domain $\Omega$.

Solve the first three equations of the transformed system.

Plug this solution in the fourth equation.

Study the linearization of the evolution operator.
The Transformation

The Diffeomorphism

\[ \phi_f(x, y) = (x, y + (1 + y)f(x)) \quad \text{for} \quad (x, y) \in \Omega = S \times (-1, 0) \]

straightens the boundary:

**Figure:** Original, time-dependent geometry.
The Transformation

The Diffeomorphism

\[ \phi_f(x, y) = (x, y + (1 + y)f(x)) \quad \text{for} \quad (x, y) \in \Omega = S \times (-1, 0) \]

straightens the boundary:

\[ \Gamma_0 \]

\[ \Omega \]

\[ \Gamma_- \]

\[ 2\pi \]

\[ x \]

\[ 0 \]

\[ 1 \]

\[ y \]

**Figure:** Transformed, fixed geometry.
The Transformation

Push forward and pull back operators

\[ \phi_f^* : \text{buc}^{2+\alpha}(\Omega) \rightarrow \text{buc}^{2+\alpha}(\Omega(f)), \quad v \mapsto v \circ \phi_f^{-1} \]

\[ \phi_f^* : \text{buc}^{2+\alpha}(\Omega(f)) \rightarrow \text{buc}^{2+\alpha}(\Omega), \quad u \mapsto u \circ \phi_f \]
The Transformation

Push forward and pull back operators

\[
\begin{align*}
\phi^f_\ast & : \text{buc}^{2+\alpha}(\Omega) \to \text{buc}^{2+\alpha}(\Omega(f)), \quad \nu \mapsto \nu \circ \phi^{-1}_f \\
\phi^*_f & : \text{buc}^{2+\alpha}(\Omega(f)) \to \text{buc}^{2+\alpha}(\Omega), \quad u \mapsto u \circ \phi_f
\end{align*}
\]

Transformed operators

\[
\begin{align*}
A(f) = \phi^*_f \circ Q \circ \phi^f_\ast & : \text{buc}^{2+\alpha}(\Omega) \to \text{buc}^\alpha(\Omega) \\
B(f, \cdot) = -\text{tr}_0 \phi^*_f \left\langle \frac{D(\phi^f_\ast \cdot)}{\mu(|D(\phi^f_\ast \cdot)|^2)}, n \right\rangle & : \mathcal{V} \times \text{buc}^{2+\alpha}(\Omega) \to h^{1+\alpha}(\mathcal{S})
\end{align*}
\]
The Transformed System

System (P) is equivalent to the transformed system

\[
\mathcal{A}(f)v = 0 \quad \text{in} \quad \Omega
\]
\[
v = b(f) \quad \text{on} \quad \Gamma_-
\]
\[
v = -\gamma \kappa f - \frac{\iota^2}{(1 - f)^2} + g \rho f \quad \text{on} \quad \Gamma_0
\]
\[
\partial_t f = \mathcal{B}(f, v) \quad \text{on} \quad \Gamma_0
\]
\[
f(0) = f_0 \quad \text{on} \quad \mathcal{S}
\]
A first Existence and Uniqueness Result

**Theorem**
Let $f \in \mathcal{V}$. There is a unique solution $T(f) \in \text{buc}^{2+\alpha}(\Omega)$ of the quasilinear Dirichlet problem

$$
A(f)v = 0 \quad \text{in} \quad \Omega, \\
v = b(f) \quad \text{on} \quad \Gamma, \\
v = -\gamma\kappa\Gamma(f) - \frac{\nu^2}{(1 - f)^2} + g\rho f \quad \text{on} \quad \Gamma_0.
$$

The mapping $\mathcal{V} \ni f \mapsto T(f) \in \text{buc}^{2+\alpha}(\Omega)$ is smooth.
The Evolution Equation

Advection equation on variable domain:

\[ \partial_t f = -\frac{\sqrt{1 + f'^2}}{\mu(|Du|^2)} \partial_{\nu} u, \quad f(0) = f_0. \]

Evolution equation on fixed domain:

\[ \partial_t f = \Phi(f), \quad f(0) = f_0 \]

for the non-linear operator \( \Phi(\cdot) = B(\cdot, T(\cdot)) \).
The Generation Result

**Theorem**

(The complexification of) \( \partial \Phi(c) \) generates a strongly continuous analytic semigroup in \( \mathcal{L}(h^{1+\alpha}(\mathbb{S})) \), i.e.,

\[
-\partial \Phi(c) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).
\]
Sketch of the Proof

Equivalent characterization:

$$\lambda - \partial \Phi(c) \in \mathcal{L}is(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \quad |\lambda||R(\lambda, \partial \Phi(c))|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}))} \leq \chi$$

for $h^{4+\alpha}(\mathbb{S}) \xrightarrow{d} h^{1+\alpha}(\mathbb{S})$, some $\chi, \omega > 0$, and all $\text{Re}\, \lambda \geq \omega$
Sketch of the Proof

Equivalent characterization:

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Consider $\lambda - \partial \Phi(c)$ as an operator between Sobolev spaces and apply a Marcinkiewicz multiplier theorem.
Sketch of the Proof

Equivalent characterization:

\[ \lambda - \partial \Phi(c) \in \mathcal{L}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \]

\[ |\lambda| \| R(\lambda, \partial \Phi(c)) \|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}))} \leq \chi \]

for \( h^{4+\alpha}(\mathbb{S}) \xrightarrow{d} h^{1+\alpha}(\mathbb{S}) \), some \( \chi, \omega > 0 \), and all \( \text{Re} \lambda \geq \omega \)

Consider \( \lambda - \partial \Phi(c) \) as an operator between Sobolev spaces and apply a Marcinkiewicz multiplier theorem.

Transfer the result to little Hölder spaces with a density argument.
Proof of the Well-Posedness Result

Apply a perturbation theorem for the class $\mathcal{H}(h^{4+\beta}(S), h^{1+\beta}(S))$ with $0 < \beta < \alpha$. 
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The generation result holds of all $f$ in a neighborhood $O_\beta \subset h^{4+\beta}(\mathbb{S})$ of $c$. 
Proof of the Well-Posedness Result

Apply a perturbation theorem for the class $\mathcal{H}(h^{4+\beta}(S), h^{1+\beta}(S))$ with $0 < \beta < \alpha$.

The generation result holds of all $f$ in a neighborhood $O_\beta \subset h^{4+\beta}(S)$ of $c$.

Use interpolation property of the little Hölder spaces

$$(h^{\theta_1}(S), h^{\theta_2}(S))_\sigma = h^{\theta_1 + \sigma(\theta_2 - \theta_1)}(S)$$

for $0 < \sigma < 1$ and $\theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N}$. 
Proof of the Well-Posedness Result

Apply a perturbation theorem for the class \( \mathcal{H}(h^{4+\beta}(\mathbb{S}), h^{1+\beta}(\mathbb{S})) \) with \( 0 < \beta < \alpha \).

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Use interpolation property of the little Hölder spaces

\[
(h^{\theta_1}(\mathbb{S}), h^{\theta_2}(\mathbb{S}))_\sigma = h^{\theta_1 + \sigma(\theta_2 - \theta_1)}(\mathbb{S})
\]

for \( 0 < \sigma < 1 \) and \( \theta_1 + \sigma(\theta_2 - \theta_1) \notin \mathbb{N} \).

Putting \( \sigma := (\alpha - \beta)/3 \) turns the generation result in a well-posedness result for \( f \in \mathcal{O} := \mathcal{O}_\beta \cap h^{4+\alpha}(\mathbb{S}) \) (cf. Lunardi, 1995).
The Linearization

$$\partial \Phi(c) \left[ \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right] = \sum_{k \in \mathbb{Z}} \lambda_k c_k e^{ikx},$$
The Linearization

\[ \partial \Phi(c) \left[ \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right] = \sum_{k \in \mathbb{Z}} \lambda_k c_k e^{ikx}, \]

\[ \lambda_k = \Gamma_{\mu,c} \left[ -\gamma \coth((1 + c)k)k^3 + \left( b'(c) \frac{1}{\text{sech}((1 + c)k)} \right) \right. \]
\[ \left. - G_c \coth((1 + c)k) \right] k, \quad k \neq 0, \]

\[ \lambda_0 = \Gamma_{\mu,c} \frac{b'(c) - G_c}{1 + c}, \quad G_c = g \rho - 2 \frac{\mu^2}{(1 - c)^2} - \frac{\theta_c}{1 + c}, \]

\[ \theta_c = \partial_y \mathcal{T}(c), \quad \Gamma_{\mu,c} = \frac{1}{\mu \left( \frac{\theta_c}{1 + c} \right)^2} + 2 \frac{\theta_c^2}{(1 + c)^2} \left( \frac{1}{\mu} \right)' \left( \frac{\theta_c}{1 + c} \right)^2 \]

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Periodic Stokesian Hele–Shaw Flows
Stability Conditions

Stability of the flat interface $f = c$ under the condition

$$0 \leq b'(c) < -\frac{2\iota^2}{(1 - c)^2} + g\rho \quad \text{or}$$

$$0 > b'(c), \quad \max\{b'(c), -I\} < -\frac{2\iota^2}{(1 - c)^2} + g\rho,$$

where $I = \inf_{k \geq 1} (\gamma k^2 - b'(c) \sech((1 + c)k))$
Stability Conditions

Instability of the flat interface $f = c$ if the former condition is violated such that

$$0 \leq b'(c), \quad b'(c) > -\frac{2\iota^2}{(1 - c)^2} + g\rho \quad \text{or}$$

$$0 > b'(c), \quad \max\{b'(c), -\iota\} > -\frac{2\iota^2}{(1 - c)^2} + g\rho$$

(2)
The Main Result

Theorem
Let $|c| < 1$ and $\omega_0 := -\sup \sigma(\partial \Phi(c))$.

(i) If (1) holds then the solution to (P) is exponentially stable. More precisely, given $\omega \in (0, \omega_0)$, there exist positive constants $M$ and $\delta$ such that for all $f_0 \in h^{4+\alpha}(S)$ with $\|f_0 - c\|_{C^{4+\alpha}(S)} \leq \delta$ the solution to (P) corresponding to $f_0$ exists in the large and for all $t \geq 0$ it holds that

$$\|f(t) - c\|_{C^{4+\alpha}(S)} + \|f'(t)\|_{C^{1+\alpha}(S)} \leq M e^{-\omega t} \|f_0 - c\|_{C^{4+\alpha}(S)}.$$ 

(ii) If (2) holds then the flat solution $f = c$ is unstable.
The Case $b'(c) > 0$

Stability condition

$$b'(c) < -\frac{2\nu^2}{(1 - c)^2} + g\rho,$$

which means:
The Case $b'(c) > 0$

Stability condition

$$b'(c) < -\frac{2\nu^2}{(1 - c)^2} + g\rho,$$

which means:

$\triangleright \quad b'(c) < g\rho,$
The Case $b'(c) > 0$

Stability condition

$$b'(c) < -\frac{2\nu^2}{(1-c)^2} + g\rho,$$

which means:

- $b'(c) < g\rho$,
- the larger $c$ the lower $\nu$, 
The Case $b'(c) > 0$

Stability condition

$$b'(c) < -\frac{2\ell^2}{(1 - c)^2} + g\rho,$$

which means:

- $b'(c) < g\rho$,
- the larger $c$ the lower $\ell$,
- critical maximum value $\ell_* := \frac{|1 - c|}{2} \sqrt{2(g\rho - b'(c))}$