Anomalous behaviour of solutions to aggregation equation in bounded and unbounded domain

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First problem

First, we consider the following one-dimensional initial value problem

$$u_t = \varepsilon u_{xx} + \big(u\ K' * u\big)_x \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \tag{2}$$

where the initial datum $u_0 \in L^1(\mathbb{R})$ is nonnegative and $\varepsilon \geq 0$.

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Equation (1) arises in study of an animal aggregation as well as in some problems in mechanics of continous media.

Notice, that in particular case, this problem is equivalent to the famous parabolic-elliptic Keller-Segel model describing chemotaxis

$$\begin{aligned} u_t &= \varepsilon u_{xx} - (uv_x)_x, \quad x \in \mathbb{R}, \quad t > 0 \\ -v_{xx} &= u - v, \end{aligned}$$

Indeed, if we take $K(x) = -\frac{1}{2}e^{-|x|}$, which is the fundamental solution of the operator $\partial_x^2 - Id$, then we get $v = -K * u$.

Recent works (existence)

For every $u_0 \in L^1(\mathbb{R})$ such that $u_0 \geq 0$, there exists the unique global-in-time solution u of first problem satisfying

$$u \in C\left([0,+\infty),\ L^1(\mathbb{R})\right) \cap C\left((0,+\infty),\ W^{1,1}(\mathbb{R})\right) \cap C^1\left((0,+\infty),\ L^1(\mathbb{R})\right).$$

In addition, the condition $u_0(x) \geq 0$ implies $u(x,t) \geq 0$ and we have conservation of the L^1-norm of nonnegative solutions:

$$\|u(t)\|_{L^1} = \int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = \|u_0\|_{L^1}.$$

Recent works (asymptotics)

It was shown that if $K' \in L^1(\mathbb{R})$ and if $\int_{\mathbb{R}} K' \, dx = 0$ then **fundamental solution of heat equation** appear in asymptotic expansion as $t \to \infty$ *i.e.*

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$$u(x,t) \sim M(4\pi t)^{-1/2} e^{-\frac{\|x\|^2}{4t}}$$

whereas, if $\int_{\mathbb{R}} K' \, dx \neq 0$ then we get a **nonlinear diffusion wave** (the fundamental solution of the viscous Burgers equation) *i.e.*

$$u(x,t) \sim \mathcal{U}_{M,B}(x,t) = \frac{Bt^{-1/2} \exp\left(-|x|^2/(4t)\right)}{C_{M,B} + \frac{1}{2}\int_0^{x/\sqrt{t}} \exp\left(-\xi^2/4\right) d\xi}.$$

Assumptions to first system

Under our assumptions on interaction kernel $K' = K_x$, our model describe particles under some **repulsive force**.

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Moreover, we assume that $A \in (0, \infty)$ is a constant and the function V satisfy

$$V \in W^{1,1}(\mathbb{R}) \tag{4}$$

$$\|V_x\|_{L^1} < A. \tag{5}$$

Primitive of solution

From now on, we assume that $\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 1$. Now, let us put

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y - \frac{1}{2}, \tag{6}$$

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Then, we show that the large time behaviour of U is described by a self-similar profile, given by a rarefaction wave, namely, the unique entropy solution of the following Riemann problem

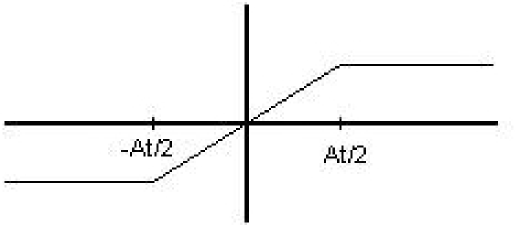
$$W_t^R + AW^R W_x^R = 0 \tag{7}$$

$$W^R(x,0) = \frac{1}{2}\mathrm{sgn}(x). \tag{8}$$

Rarefaction wave

Rarefaction wave is given by explicit formula

$$W^R(x,t) := \begin{cases} -\frac{1}{2} & \text{for} \quad x < -\frac{At}{2}, \\ \frac{x}{At} & \text{for} \quad -\frac{At}{2} < x < \frac{At}{2}, \\ \frac{1}{2} & \text{for} \quad x > \frac{At}{2}. \end{cases} \tag{9}$$



Convergence towards rarefaction waves

Theorem

Assume that $u_0(x) > 0$, $\|u_0\| = 1$ *and* $\varepsilon > 0$. *Suppose also, that*

$$\int_{-\infty}^{x} u_0(y)\,dy \in L^1(-\infty,0), \quad \text{and} \quad \int_{-\infty}^{x} u_0(y)\,dy - 1 \in L^1(0,\infty).$$

Then, for every $t > 0$ *and each* $p \in (1,\infty]$ *the following estimate hold true*

$$\|U(\cdot,t) - W^R(\cdot,t)\|_p \leq Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \left(\log(2+t)\right)^{\frac{1}{2}(1+\frac{1}{p})}.$$

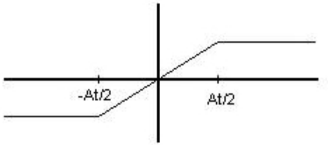
Final result

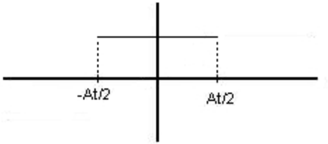
Corollary

For every test function $\varphi \in C_c^\infty(\mathbb{R})$ and each $t_0 > 0$, rescaled solution $u^\lambda(x,t) = \lambda u(\lambda x, \lambda t)$ for $\lambda > 0$, $x \in \mathbb{R}$ and $t > 0$ satisfy

$$\int_{\mathbb{R}} u^\lambda(x,t_0)\varphi(x)\,dx \xrightarrow{\lambda\to\infty} -\int_{\mathbb{R}} W^R(x,t_0)\varphi_x(x)\,dx.$$

In other words, for each $t_0 > 0$, the family of functions $u^\lambda(\cdot, t_0)$ converges weakly as $\lambda \to \infty$ to $\left(W^R\right)_x(\cdot, t_0)$.





Second system

Now, let us consider the following initial value problem

$$u_t = \nabla \cdot (\nabla u - u \nabla \mathcal{K}(u)) \quad \text{for } x \in \Omega \subset \mathbb{R}^d, \;\; t > 0, \tag{10}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, \;\; t > 0 \tag{11}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega, \tag{12}$$

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where the initial datum $u_0 \in L^1(\Omega)$ is nonnegative and the operator $\mathcal{K}$ depends linearly on u via the following integral formula

$$\mathcal{K}(u)(x,t) = \int_\Omega K(x,y) u(y,t) \, \mathrm{d}y \tag{13}$$

for a certain function $K = K(x,y)$ which we call as an *aggregation kernel*.

Main assumptions to the second system

We assume that the aggregation kernel satisfy

$$B = \operatorname{ess\,sup}_{x \in \Omega} \|\nabla_x K(x, \cdot)\|_2 < \infty. \tag{14}$$

$$\frac{\partial K}{\partial n}(\cdot, y) = 0 \quad \text{on } \partial\Omega \quad \text{for all} \quad y \in \Omega, \tag{15}$$

$$\int_\Omega \nabla_x K(x, y)\, dy = 0 \tag{16}$$

Existence

Theorem (Global existence for mildly singular kernels)

Assume that the aggregation kernel is mildly singular. Then for every positive initial condition $u_0 \in L^1(\Omega)$ *and for every* $T > 0$ *problem* (10)-(12) *has a unique mild solution in the space*

$$\mathcal{Y}_T = C([0,T], L^1(\Omega)) \cap \{u : C\big([0,T], L^q(\Omega)\big), \sup_{0 \le t \le T} t^{\frac{d}{2}(1-\frac{1}{q})} \|u\|_q < \infty\}$$

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Theorem (Local existence for strongly singular kernels)

Assume that aggregation kernel is strongly singular. Then for every $u_0 \in L^1(\Omega) \cap L^q(\Omega)$ *there exists* $T = T(u_0, \nabla_x K) > 0$ *and a unique mild solution of problem* (10)–(12) *in the space*

$$\mathcal{X}_T = C([0,T], L^1(\Omega)) \cap C([0,T], L^q(\Omega)).$$

Perturbed problem

We look for solution of the second system in the form

$$u(x,t) = M + \varphi(x,t),$$

where M is an arbitrary constant and φ is a perturbation. Moreover, we assume that

$$\int_\Omega \varphi(x)\,\mathrm{d}x = 0,$$

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and obtain the following problem for the perturbation φ

$$\varphi_t = \Delta\varphi - \nabla\cdot\Big(M\nabla\mathcal{K}(\varphi) + \varphi\nabla\mathcal{K}(\varphi)\Big) \tag{17}$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{for } x\in\partial\Omega,\ \ t>0 \tag{18}$$

$$\varphi(x,0) = \varphi_0(x). \tag{19}$$

Linear stability of constant solutions

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Proposition

If, $\nabla_x K \in L^2(\Omega \times \Omega)$ *and*

$$M \|\nabla_x K\|_{L^2(\Omega\times\Omega)} < \lambda_1,$$

where λ_1 *is the first non-zero eigenvalue of* $-\Delta$ *on* Ω *under the Neumann boundary condition then* M **is linearily asymptotically stable stationary solution** *to the second problem.*

Nonlinear stability of constant solutions

Theorem

Let the assumptions of above Proposition holds true. If moreover

$$\|\nabla_x K\|_{\infty,2} = \operatorname*{ess\,sup}_{x\in\Omega} \|\nabla_x K(x,\cdot)\|_2 < \infty$$

then there exist $\eta > 0$ *such that for every* $\varphi_0 \in L^2(\Omega)$ *satisfying* $\|\varphi_0\|_2 < \eta$ *and* $\int_\Omega \varphi_0(x)\,dx = 0$, *the perturbed (nonlinear) problem* (17)-(19) *has a solution* $\varphi \in C^1((0,\infty), L^2(\Omega))$ *such that* $\int_\Omega \varphi(x,t)\,dx = 0$ *for all* $t > 0$. *Moreover, we have*

$$\|\phi(t)\|_2 \to 0 \quad as \quad t \to \infty.$$

Theorem

Let $w_1 = w_1(x)$ be the eigenfunction of $-\Delta$ on Ω under the Neumann boundary condition corresponding to the first nonzero eigenvalue λ_1 and such that $\|w_1\|_2 = 1$. Assume that $\|\nabla_x K\|_{L^2(\Omega\times\Omega)} < \infty$. If moreover,

$$\int_\Omega \int_\Omega K(x,y) w_1(y) w_1(x)\, dx\, dy = A > 0, \qquad (23)$$

then for $M > 1/A$ the constant solution M of problem (10)-(12) *is* **linearily unstable stationary solution**.

- G. Karch and K. Suzuki, *Spikes and diffusion waves in one dimensional model of chemotaxis*, Nonlinearity (2010).
- G. Karch and K. Suzuki, *Blow-up versus global existence of solutions to aggregation equations*, Appl. Math. (Warsaw) (2011).
- R.C. *Asymptotic behaviour in one dimensional model of interacting particles*, Nonlinear Analysis (2011)
- R.C. *Stability of solutions to aggregation equation in a bounded domain*, in preparation