Parabolic equations on domains of wedge type

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Joint work with Jürgen Saal

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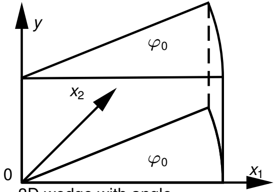

3D wedge with angle φ_0

For $n \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}_0$ let

$$G := \mathbb{R}^m \times C_\Omega$$

be a domain of wedge (or cone) type: Let

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be an unbounded cone in $\mathbb{R}^n$ with vertex in zero.

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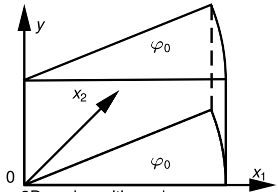

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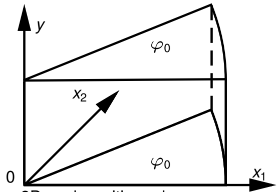

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Notation: $\varphi_0 \in (0, \pi]$: angle of the wedge, $J = [0, T]$: finite time interval, $(t, x) \in J \times G$: time and space variables

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We consider a diffusion equation on a three-dimensional wedge $G \subset \mathbb{R}^3$ ($n = 2, m = 1$):

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } G \times (0, T) \\ \nu \times (\operatorname{curl} u) = 0, \ u_\nu = 0 & \text{on } \partial G \times (0, T) \\ u\big|_{t=0} = 0 & \text{on } G \end{cases} \quad . \tag{1}$$

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Assumption:

- For $1 < p < \infty$ let $f \in L^p(J \times \mathbb{R}; L^p(C_\Omega, \mathbb{R}^3, |x|^\gamma dx))$ with weight $\gamma \in \mathbb{R}$.

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Notation:

- $u = u(t, x) \in \mathbb{R}^3$: unknown function
- $\nu = \nu(x)$: outer normal vector $(x \in \partial G)$
- $u_\nu = u \cdot \nu$: normal projection of u
- $\times$ denotes the vector product in $\mathbb{R}^3$ and $\operatorname{curl} u = \nabla \times u$.

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Operator sums for non-commuting operators

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Approach:

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To this end we need the Labbas Terrini commutator condition for two sectorial operators A and B with spectral angles ϕ_A and ϕ_B:

$$\left\{\begin{array}{c} 0 \in \rho(A). \text{There are constants } c > 0\ , 0 \leq \alpha < \beta < 1, \\ \psi_A > \phi_A, \psi_B > \phi_B, \psi_A + \psi_B < \pi, \\ \text{such that for all } \lambda \in \Sigma_{\pi-\psi_A}\ \mu \in \Sigma_{\pi-\psi_B} \\ |A(\lambda + A)^{-1}[A^{-1}(\mu + B)^{-1} - (\mu + B)^{-1}A^{-1}]| \leq c/((1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}) \end{array}\right. \tag{2}$$

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Results of Prüss and Simonett:

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Theorem 1

Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ *and suppose that (2) holds for some angles* $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^R$ *such that* $\psi_A + \psi_B < \pi$.
Then there is a constant $c_0 > 0$ *such that* $A + B$ *is invertible and sectorial with* $\phi_{A+B} \leq \max\{\psi_A, \psi_B\}$ *whenever* $c < c_0$. *Moreover, if in addition* $B \in \mathcal{RH}^\infty(X)$ *and* $\psi_B > \phi_B^{R\infty}$, *then* $A + B \in \mathcal{H}^\infty(X)$ *and* $\phi_{A+B}^\infty \leq \max\{\psi_A, \psi_B\}$.

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Theorem 1

Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ and suppose that (2) holds for some angles $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^R$ such that $\psi_A + \psi_B < \pi$.
Then there is a constant $c_0 > 0$ such that $A + B$ is invertible and sectorial with $\phi_{A+B} \leq \max\{\psi_A, \psi_B\}$ whenever $c < c_0$. Moreover, if in addition $B \in \mathcal{RH}^\infty(X)$ and $\psi_B > \phi_B^{R\infty}$, then $A + B \in \mathcal{H}^\infty(X)$ and $\phi_{A+B}^\infty \leq \max\{\psi_A, \psi_B\}$.

Corollary 2

Let the assumption of Theorem 1 be satisfied. Then there is $\nu \geq 0$ such that $\nu + A + B$ is sectorial with spectral angle not larger than $\max\{\psi_A, \psi_B\}$. If $B \in \mathcal{RH}^\infty(X)$ and $\psi_B > \phi_B^{R\infty}$, we have $\nu + A + B \in \mathcal{H}^\infty(X)$ as well and $\phi_{\nu+A+B}^\infty \leq \max\{\psi_A, \psi_B\}$.

Sketch of Proceeding

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5. retransformation to obtain maximal regularity spaces for (1)

Sketch of Proceeding

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1. We introduce cylinder coordiantes:

For $r > 0$, $\varphi \in (0, \varphi_0]$ and $y \in \mathbb{R}^m$ ($m = 1$) we have

$$x_1 = r \cos\varphi, \quad x_2 = r \sin\varphi, \quad y = y.$$

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The diffusion operator $\partial_t - \Delta$ transforms into

$$\partial_t - \partial_y^2 - \left[\partial_r^2 + \frac{1}{r}\partial_r\right] - \frac{1}{r^2}\partial_\varphi^2$$

and

$$f \in L^p\left(J \times \mathbb{R} \times (0, \varphi_0); L^p\left(\mathbb{R}_+, \mathbb{R}^3, r^{\gamma+2}\frac{dr}{r}\right)\right).$$

Sketch of Proceeding

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2. Applying Euler transformation $r = e^x$ $(x \in \mathbb{R})$and 3. rescaling

$$u(t, r, \varphi, y) = r^{\beta} v(t, \ln r, \varphi, y), \;\; g(t, x, \varphi, y) = r^{2-\beta} f(t, r, \varphi, y)$$

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yields: $g \in L^p(J \times \mathbb{R} \times (0,\varphi_0) \times \mathbb{R}, \mathbb{R}^3)$ and

$$\left\{ \begin{array}{ccc} e^{2x}(\partial_t - \partial_y^2)v + P(\partial_x)v - \partial_\varphi^2 v = g & \text{in} & (0,T) \times \mathbb{R} \times (0,\varphi_0) \times \mathbb{R} \\ \tilde{v}_\varphi = 0,\ \partial_\varphi \tilde{v}_y = 0,\ \partial_\varphi \tilde{v}_r = 0 & \text{on} & (0,T) \times \mathbb{R} \times \{0,\varphi_0\} \times \mathbb{R} \\ v\big|_{t=0} = 0 & \text{in} & \mathbb{R} \times (0,\varphi_0) \times \mathbb{R} \end{array} \right. , \qquad (3)$$

where $P(\partial_x) := -[\partial_x^2 + 2\beta\partial_x + \beta^2]$, $\beta = 2 - (\gamma + 2)/p$.

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where $P(\partial_x) := -[\partial_x^2 + 2\beta\partial_x + \beta^2]$, $\beta = 2 - (\gamma + 2)/p$.

Now we discuss the obtained operator sum on the layer $\tilde{G} := \mathbb{R} \times (0, \varphi_0) \times \mathbb{R}$ by means of Theorem 1 and Corollary 2.

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- Notice, that we deal with the non-standard differential operators

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- The boundary conditions of (1) transform into boundary conditions for $-\partial_\varphi^2$.
  Since of the Neumann conditions $\partial_\varphi \tilde{v}_y = 0$ and $\partial_\varphi \tilde{v}_r = 0$, we need to handle the eigenvalue 0.
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- The last point can be overcome by projection to the space of L^p functions with vanishing mean value.

After step 4. and 5. we obtain the main result.

Main Result

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Theorem 3

Suppose $1 < p < \infty$ *and that* $\gamma \in \mathbb{R}$ *is subject to the condition*

$$\lambda_1 > -a_0 = \beta(\beta + n - 2) = \left(2 - \frac{n}{p} - \frac{\gamma}{p}\right)\left(n - \frac{n}{p} - \frac{\gamma}{p}\right), \quad (n = 2)$$

where $\lambda_1 = \pi^2/\varphi_0^2$. *Then for each* $g \in L^p(J \times \mathbb{R} \times (0, \varphi_0) \times \mathbb{R}, \mathbb{R}^3)$ *there is a unique solution* v *of (3) in the regularity class*

$$\begin{aligned} &v \in L^p(J \times \mathbb{R}; H^{2,p}((0, \varphi_0) \times \mathbb{R}, \mathbb{R}^3)), \\ &e^{2x} v \in H^{1,p}(J; L^p(\mathbb{R} \times (0, \varphi_0) \times \mathbb{R}, \mathbb{R}^3)) \cap L^p(J; H^{2,p}(\mathbb{R}; L^p((0, \varphi_0) \times \mathbb{R}, \mathbb{R}^3))). \end{aligned}$$

In particular, the map $[v \mapsto g]$ *defines an isomorphism between the corresponding spaces.*

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Corollary 4

Suppose $1 < p < \infty$ *and suppose* $\gamma \in \mathbb{R}$ *is subject to the condition*

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$$u, u/|x|^2, \partial_t u, \nabla^2 u \in L^p(J \times \mathbb{R}; L^p(C_\Omega, \mathbb{R}^3, |x|^\gamma dx))$$

In particular, the map $[u \mapsto f]$ *defines an isomorphism between the corresponding spaces.*

Corollary 5

The result remains valid for the Stokes equation, since $P\Delta = \Delta P$.

Outlook

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Possible future work relying on this method:

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- apply this method to a stationary Stokes equation and also transform ∇p

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- solve parabolic equations subject to other boundary conditions
- apply this method to a stationary Stokes equation and also transform ∇p
- approach to problems resulting from phenomena of contact lines

References

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Thank you !