ARITHMETIC HIRZEBRUCH-ZAGIER DIVISORS AND CENTRAL DERIVATIVE VALUES OF RANKIN-SELBERG L-FUNCTIONS

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Abstract. We derive two distinct proofs of the Gross-Zagier formula in terms of sums of automorphic Green's functions realized as regularized theta lifts, including one involving arithmetic Hirzebruch-Zagier divisors on the Hilbert modular surface $X_0(N) \times X_0(N)$. We then describe applications to the refined conjecture of Birch and Swinnerton-Dyer. Through these calculations, we also describe known and conjectural relations of the central derivative values of Rankin-Selberg L-functions that appear to Fourier coefficients of certain half-integral weight forms.

Contents

1. Introduction

The theorem of Gross-Zagier [25, Theorem §I (6.3)] represents one of the most significant advances on the conjecture of Birch and Swinnerton-Dyer to date, and forms the foundation for all progress made on the case of Mordell-Weil rank one through the techniques of Kolyvagin Euler systems and Iwasawa main conjectures.

To recall it, let E be an elliptic curve of conductor N defined over the rationals $\mathbf{Q}$. Hence, E is modular by fundamental work of Wiles [58], Taylor-Wiles [52], and Breuil-Conrad-Diamond-Taylor [5], and consequently parametrized by some cuspidal newform

$$\phi(\tau) = \sum_{m \geq 1} c_\phi(m) e(m\tau) = \sum_{m \geq 1} a_\phi(m) m^{\frac{1}{2}} e(m\tau) \in S_2^{\mathrm{new}}(\Gamma_0(N)), \quad \tau = u + iv \in \mathfrak{H}, e(z) = \exp(2\pi i z).$$

In particular, the Hasse-Weil L-function $L(E,s) = L(E/\mathbf{Q}, s)$ of E over $\mathbf{Q}$ has an analytic continuation $\Lambda(E,s) = L_\infty(E,s)L(E,s)$ to all $s \in \mathbf{C}$ given by a shift of the standard L-function, $\Lambda(s,\phi) = L_\infty(s,\phi)L(s,\phi)$

$$\Lambda(E,s) = L_\infty(E,s)L(E,s) = \Lambda(s-1/2,\phi) = L_\infty(s-1/2,\phi)L(s-1/2,\phi).$$

Here, $L_\infty(s,\phi) = (2\pi)^{-s}\Gamma(s)$ denotes the archimedean local Euler factor, with $L(s,\phi) = \prod_{p<\infty} L(s,\pi(\phi)_p)$ the finite Euler product whose Dirichlet series expansion for $\Re(s) > 1$ is given by

$$L(s,\phi) = \sum_{m \geq 1} a_\phi(m) m^{-s} = \sum_{m \geq 1} c_\phi(m) m^{-(s+1/2)}.$$

Hence, $\Lambda(s,\phi)$ satisfies a symmetric functional equation $\Lambda(s,\phi) = \pm N^{1-2s}\Lambda(1-s,\phi)$ with odd sign or root number $\Lambda(s,\phi) = -N^{1-2s}\Lambda(1-s,\phi)$ if and only if ϕ is invariant under the Fricke involution $w_N\phi = \phi$.

Let k be an imaginary quadratic field of discriminant d_k prime to N and odd quadratic Dirichlet character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Write $C(\mathcal{O}_k) = I(k)/P(k)$ to denote the ideal class group of k, with class number $h_k = \#C(\mathcal{O}_k)$. Let $\chi \in C(\mathcal{O}_k)^\vee$ be any class group character, with

$$\theta(\chi)(\tau) = \sum_{A \in C(\mathcal{O}_k)} \chi(A)\theta_A(\tau) \in M_1(\Gamma_0(|d_k|), \eta_k)$$

the corresponding Hecke theta series of weight 1, level $\Gamma_0(|d_k|)$, and character η_k. By the theory of Rankin-Selberg convolution, the twisted basechange Hasse-Weil L-function $L(E/k,\chi,s)$ of E over K twisted by χ has an analytic continuation $\Lambda(E/k,\chi,s) = L_\infty(E/k,\chi,s)L(E/k,\chi,s)$ given by a shift of the completed Rankin-Selberg L-function $\Lambda(s,\phi\times\theta(\chi)) = L_\infty(s,\phi\times\theta(\chi))L(s,\phi\times\theta(\chi))$,

$$\Lambda(E/k,\chi,1) = \Lambda(s-1/2,\phi\times\theta(\chi)) = L_\infty(s-1/2,\phi\times\theta(\chi))L(s-1/2,\phi\times\theta(\chi)).$$

Here, the archimedean local factor is given by

$$\Lambda_\infty(s,\phi\times\theta(\chi)) = (2\pi)^{-s}\Gamma\left(s-\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right),$$

and does not depend on the choice of class group character $\chi \in C(\mathcal{O}_k)$. As we explain in Proposition 6.1, this L-function has a well-known analytic continuation to $s \in \mathbf{C}$ via the symmetric functional equation

$$\Lambda(s,\phi\times\theta(\chi)) = \eta_k(-N)|d_k N|^{1-2s}\Lambda(1-s,\phi\times\theta(\chi)).$$

In particular, if $\eta_k(-N) = -1$ so that the sign of this symmetric functional equation is odd, then the central value $\Lambda(1/2,\phi\times\theta(\chi)) = 0$ is forced to vanish, and it makes sense to study central derivative values $\Lambda'(1/2,\phi\times\theta(\chi)) = \Lambda'(E/k,\chi,1)$. This happens for instance if the level N is squarefree, and the number of prime divisors $q \mid N$ which remain inert in k (so $(\frac{d_k}{q}) = -1$) is even, or more stringently if N is squarefree and totally split, so that the "Heegner hypothesis" of Gross-Zagier [25] holds. In this latter setting, the compactified modular curve $X_0(N)$ comes equipped with a family of Heegner divisors y of conductor d_k. In brief, there are h_k many Heegner points $z : E \to E'$ of conductor d_k on $X_0(N)(k[1])$, where $k[1]$ denotes the Hilbert class field of k. More precisely, the class group $C(\mathcal{O}_k) \cong \mathrm{Gal}(k[1]/k)$ acts simply transitively on the set of these points, and we denote this natural action by z^A. We obtain from each Heegner point $z \in X_0(N)(k[1])$ the divisor $y = (z) - (\infty)$ in the corresponding jacobian $J_0(N)(k[1])$.

Theorem 1.1 (Gross-Zagier). *Let E be an elliptic curve of conductor N, parametrized by a cuspidal newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$. Let k be an imaginary quadratic field of discriminant d_k prime to $2N$ and odd Dirichlet character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Assume the Heegner hypothesis, so that $\eta_k(-N) = -1$. Then for any character χ of the ideal class group $C(\mathcal{O}_k)$ of k, we have the central derivative value formula*

$$\Lambda'(E/k, \chi, 1) = \Lambda'(1/2, \phi \times \theta(\chi)) = \frac{8\pi^2 ||\phi||^2}{h_k u_k^2} \cdot \left[y_\chi^\phi, y_\chi^\phi \right]_{\mathrm{NT}}.$$

Here, $||\phi||^2 = \langle \phi, \phi \rangle$ denoted the Petersson inner product of ϕ, $h_k = \# C(\mathcal{O}_k)$ the class number, $u_k = w_k/2$ half the number of roots of unity in k, and $\left[y_\chi^\phi, y_\chi^\phi \right]_{\mathrm{NT}}$ the Néron-Tate height of the projection y_χ^ϕ to the ϕ-isotypical component of $J_0(N)(k[1]) \otimes \mathbf{C}$ of the twisted Heegner divisor

$$y_\chi = \sum_{A \in C(\mathcal{O}_k)} \chi(A) y^A$$

We remark that this theorem has been generalized by the various works of Zhang [64], [65], [62], [63] and Yuang-Zhang-Zhang [59] to quaternionic Shimura curves over totally real fields, developing similar ideas with the theta correspondence and the Jacquet-Langlands correspondence [30] [31] in the style of Waldspurger's theorem [57]. Here, we give a distinct proof using regularized theta lifts and arithmetic Hirzebruch-Zagier divisors on the Hilbert modular surface $X_0(N) \times X_0(N)$, developing the main theorems of Bruinier-Yang [13] and Andreatta-Goren-Howard-Madapusi Pera [1] on the Kudla programme for Shimura varieties or orthogonal type for this setting, and describing the integral presentations of the Rankin-Selberg L-functions in more detail. We explain implications for the refined conjecture of Birch and Swinnerton-Dyer through Euler characteristic calculations, after Iwasawa main conjectures. In Appendix A, we explain how the theorem of Bruinier-Yang [13, Theorem 7.3] can be developed to recover the full Gross-Zagier formula. In Appendix B, we describe known and conjectural links between the central derivative values and Fourier coefficients of half-integral weight forms, using the theorem of Bruinier-Funke-Imamoglu [11] on traces and periods of modular functions. Finally, we derive many of the analytic results more generally for real quadratic fields.

1.1. **Main results.** Let us first suppose more generally that (V, Q) is a rational quadratic space of signature $(n, 2)$ for any integer $n \geq 1$. Write $(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2)$ for the corresponding inner product. Let $\mathrm{GSpin}(V)$ denote the corresponding general spin group, which fits into the short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

Let

$$D(V) = D^{\pm}(V) = \{ z \subset V(\mathbf{R}) : \dim(z) = 2, Q|_z > 0 \}$$

denote a fixed connected component of the Grassmannian of oriented negative definite hyperplanes in $V(\mathbf{R})$. Given any maximal lattice $L \subset V$, the adelization $\widehat{L} = L \otimes \widehat{\mathbf{Z}}$ is fixed under the action of $\mathrm{GSpin}(V)(\mathbf{A}_f)$ via conjugation by a uniquely-determined compact open subgroup $K = K_L$ of $\mathrm{GSpin}(V)(\mathbf{A}_f)$. We consider the corresponding Shimura variety X_K with complex points

$$X_K(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q}) \backslash D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K,$$

which determines a quasiprojective variety of dimension n over $\mathbf{Q}$. This Shimura variety X_K is projective if and only if the space (V, Q) is anisotropic, and smooth only if the corresponding level structure $K = K_L$ is neat. We refer to the discussion below for more details. Fixing a set of reprentatives h for the finite set $\mathrm{GSpin}(V)(\mathbf{Q}) \backslash \mathrm{GSpin}(V)(\mathbf{A}_f)/K$ and writing $\Gamma_h = \mathrm{GSpin}(V)(\mathbf{Q}) \cap hKh^{-1}$ for the arithmetic subgroup for each representative, we have the decomposition into geometrically connected components

$$X_K(\mathbf{C}) = \coprod_{h \in \mathrm{GSpin}(V)(\mathbf{Q}) \backslash \mathrm{GSpin}(V)(\mathbf{A}_f)/K} \Gamma_h \backslash D(V).$$

One important feature is that any subspace $(V', Q') = (V', Q|_{V'})$ of signature $(n', 2)$ determines an algebraic cycle of dimension n', given by the corresponding Shimura variety for $\mathrm{GSpin}(V')$ and $D(V')$ (see [39]). In particular, we obtain from this construction special divisors of the following type. Let $L^\vee$ denote the dual lattice of L, and $L^\vee/L$ the corresponding discriminant group. Given any vector $x \in V$ with $Q(x) = m > 0$, we consider the corresponding divisor given by the orthogonal complement $D(V)_x = \{z \in D(V) : (z, x) = 0\}$. Given any coset $\mu \in L^\vee/L$ and positive rational $m > 0$, we consider the divisor $Z(\mu, m) \subset X_K$ defined by

$$Z(\mu, m) = \coprod_{h \in \mathrm{GSpin}\, V(\mathbf{Q}) \backslash \mathrm{GSpin}\, V(\mathbf{A}_f)/K} \Gamma_h \Big\backslash \left(\coprod_{\substack{x \in \mu_h + L_h \\ Q(x) = m}} D(V)_x \right),$$

where $L_h \subset V$ denotes the lattice determined by $\widehat{L}_h = h \cdot \widehat{L}$, and $\mu_h = h \cdot \mu \in L_h^\vee/L_h$. Note that when (V, Q) has signature $(1, 2)$, these special divisors $Z(\mu, m)$ recover summands of classical Heegner divisors or CM divisors on quaternionic Shimura curves. Similarly for (V, Q) of signature $(2, 2)$, the special divisors $Z(\mu, m)$ recover summands of classical Hirzebruch-Zagier divisors on quaternionic Hilbert modular surfaces.

Important theorems of Borcherds [3] and more generally Bruinier [6] give explicit constructions of the Arakelov theoretic automorphic Green's functions for these divisors $Z(\mu, m)$. As we describe in detail below, these are given by the regularized theta lifts $\Phi(F_{\mu,m}, \cdot)$ of certain Maass Poincaré series $F_{\mu,m}$. To be more precise, let $\omega_L : \widetilde{\mathrm{SL}}_2(\mathbf{Z}) \rightarrow \mathfrak{S}_L$ denote the Weil representation associated to the chosen lattice $L \subset V$, and let $\theta_L(\tau, z) : \mathfrak{H} \times D(V) \rightarrow \mathfrak{S}_L^\vee$ denote the corresponding Siegel theta series defined in (15) below. Let $f = f^+ + f^- \in H_l(\omega_L)$ more generally be any harmonic weak Maass form of weight $l = 1 - n/2$ and representation ω_L. Here, we write the Fourier series expansion of the holomorphic part f^+ as

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(\mu, m) e(m\tau) \mathbf{1}_\mu$$

and the Fourier series expansion of the nonholomorphic part f^- as

$$f^-(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_f^-(\mu, m) W_l(2\pi m v) e(m\tau) \mathbf{1}_\mu,$$

where $W_l(a) := \int_{-2a}^\infty e^{-t} t^{-l} dt = \Gamma(1 - l, 2|a|)$ denotes the Whittaker function given by the partial Gamma function, and $\mathbf{1}_\mu = \mathrm{char}(\mu + \widehat{L})$ the characteristic function of $\mu + \widehat{L}$. Write $M_l^!(\omega_L) \subset H_l(\omega_L)$ for the space of weakly holomorphic forms whose poles are supported at the cusps, the subspace of holomorphic forms $M_l(\omega_L) \subset M_l^!(\omega_L)$, and the suspace of holomorphic cusp forms $S_l(\omega_L) \subset M_l(\omega_L) \subset M_l^!(\omega_L) \subset H_l(\omega_L)$. Here, we also have the antilinear differential operator $\xi_l : H_l(\omega_l) \rightarrow S_{2-l}(\overline{\omega}_L)$ of Bruinier-Funke [10] defined in (17) below, which allows us to identify the weakly holomorphic forms as the kernel $\ker(\xi_l) = M_l^!(\omega_L)$. Given harmonic weak Maass forms

$$f(\tau) = \sum_{\mu \in L^\vee/L} f_\mu(\tau) \mathbf{1}_\mu \in H_l(\omega_L) \quad \text{and} \quad g(\tau) = \sum_{\mu \in L^\vee/L} g_\mu(\tau) \mathbf{1}_\mu \in H_l(\omega_L^\vee),$$

we consider the pairing

$$\langle\langle f(\tau), g(\tau) \rangle\rangle = \sum_{\mu \in L^\vee/L} f_\mu(\tau) g_\mu(\tau).$$

Hence, $\langle\langle f, g \rangle\rangle$ determines a scalar-valued Maass form of weight l. We write

$$\mathrm{CT}\langle\langle f, g \rangle\rangle = \sum_{\mu \in L^\vee/L} \sum_{m \in \mathbf{Q}} c_f(\mu, -m) c_g(\mu, m)$$

to denote the constant term in its Fourier series expansion. Let $\mathcal{F} = \{\tau \in \mathfrak{H} : -1/2 \leq \Re(\tau) \leq 1/2, \tau\overline{\tau} \geq 1\}$ denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$. Given any positive real number $T > 0$, we also consider the truncated fundamental domain $\mathcal{F}_T = \{\tau \in \mathfrak{H} : -1/2 \leq \Re(\tau) \leq 1/2, \tau\overline{\tau} \geq 1, \Im(\tau) \leq T\}$. Let $\mu(\tau) = \frac{dudv}{v^2}$ denote the Poincaré measure on $\mathfrak{H}$. We define the regularized theta lift $\Phi(f, z, h)$ for $f \in H_{1-n/2}(\omega_L)$, $z \in D(V)$ and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by the regularized integral

$$\Phi(f, z, h) = \int_{\mathcal{F}}^\star \langle\langle f(\tau), \theta_L(\tau, z, h) \rangle\rangle d\mu(\tau) = \mathrm{CT}_{s=0} \left\{ \lim_{T \rightarrow \infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_L(\tau, z, h) \rangle\rangle v^{-s} d\mu(\tau) \right\}$$

given by the constant term in the Laurent series around $s=0$ of the function

$$\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle\langle f(\tau),\theta_L(\tau,z,h)\rangle\rangle v^{-s}d\mu(\tau).$$

As we explain for Theorem 4.2, the main theorems of [3] and [6] show that regularized theta lift $\Phi(f,\cdot)$ determines an automorphic Green's function in the sense of Arakelov theory for the divisor defined by

$$Z(f)=\sum_{\mu\in L^\vee/L}\sum_{\substack{m\in\mathbf{Q}\\ m>0}}c_f^+(\mu,-m)Z(\mu,m)\subset X_K.$$

1.1.1. *Quadratic summation formulae.* As a preliminary step for our main result, we calculate these Green's functions $\Phi(f,\cdot)$ along zero cycles corresponding to subspaces of signature $(0,2)$ of (V,Q) following the theorems of Bruinier-Yang [13, Theorem 4.7] and Schofer [47], as well as along "geodesic sets" corresponding to Lorenzian subspaces of signature $(1,1)$ in the style of [54]. To be more precise, each rational quadratic subspace (V_0,Q_0) of (V,Q) of signature $(0,2)$ gives rise to a zero cycle $Z(V_0)\subset X_K$ with complex points

$$Z(V_0)(\mathbf{C})=\mathrm{GSpin}(V_0)(\mathbf{Q})\backslash\{z_0^\pm\}\times\mathrm{GSpin}(V_0)(\mathbf{A}_f)/(K\cap\mathrm{GSpin}(V_0)(\mathbf{A}_f)),\quad z_0^\pm=V_0(\mathbf{R})\in D(V)$$

as defined in (6) below. Such as cycle is sometimes called a CM cycle, as it can be associated to an imaginary quadratic field $k=k(V_0)$. We associate to the lattice $L_0=V_0\cap L$, an Eisenstein series $E_{L_0}(\tau,s;1)\in H_1(\omega_{L_0})$ of weight $1=2+(0-2)/2$ and representation ω_{L_0} as defined in (31), as well as its derivative

$$E'_{L_0}(\tau,s;1)=\frac{d}{ds}E_{L_0}(\tau,s;1)=E'^{,+}_{L_0}(\tau,s;1)+E'^{,-}_{L_0}(\tau,s;1)\in H_1(\omega_{L_0}),$$

and we write

$$\mathcal{E}_{L_0}(\tau)=E'^{,+}_{L_0}(\tau,0;1)=\sum_{\mu\in L_0^\vee/L_0}\sum_{m\in\mathbf{Q}}\kappa_{L_0}(\mu,m)e(m\tau)\mathbf{1}_\mu$$

to denote the holomorphic part of this latter Eisenstein series at $s=0$. Similarly, any rational quadratic space $(W,Q_W)=(W,Q|_W)$ of signature $(1,1)$ of (V,Q) gives rise to a geodesic set $G(W)$ with complex points

$$G(W)(\mathbf{C})=\mathrm{GSpin}(W)(\mathbf{Q})\backslash D(W)\times\mathrm{GSpin}(W)(\mathbf{A}_f)/(K\cap\mathrm{GSpin}(W)(\mathbf{A}_f))$$

as defined in (7) below, where $D(W)=\{z\in W(\mathbf{R}):\dim(z)=1,Q_W|_z<0\}$ denotes the corresponding domain of oriented negative definite lines in $W(\mathbf{R})$ This space can be associated to a real quadratic field $k=k(W)$. We associate to the lattice $L_W=W\cap L$, an Eisenstein series $E_{L_W}(\tau,s;2)\in H_2(\omega_{L_W})$ of weight $2=2+(1-1)/2$ and representation ω_{L_W} as defined in (38), as well as its derivative

$$E'_{L_W}(\tau,s;2)=\frac{d}{ds}E_{L_W}(\tau,s;2)=E'^{,+}_{L_W}(\tau,s;2)+E'^{,-}_{L_W}(\tau,s;2)\in H_2(\omega_{L_W}),$$

and we write

$$\mathcal{E}_{L_W}(\tau)=E'^{,+}_{L_W}(\tau,0;2)=\sum_{\mu\in L_W^\vee/L_W}\sum_{m\in\mathbf{Q}}\kappa_{L_W}(\mu,m)e(m\tau)\mathbf{1}_\mu$$

to denote the holomorphic part of this latter Eisenstein series at $s=0$. Using the functional equations and behaviour under Maass lowering operators of these Eisenstein series, we compute the sum

$$\Phi(f,Z(V_0))=\sum_{(z_0^\pm,h)\in Z(V_0)}\Phi(f,z_0^\pm,h)$$

of the Green's function $\Phi(f,\cdot)$ along the CM cycle $Z(V_0)\subset X_K$ corresponding to an imaginary quadratic field $k(V_0)$, by a minor variation of the arguments of [13, Theorem 4.7] and [47] (cf. [1, Theorem 5.7.1]). We also use such properties to compute the sum

$$\Phi(f,G(W))=\sum_{(z_W,h)\in G(W)}\Phi(f,z_W,h)$$

of the Green's function $\Phi(f,\cdot)$ along the geodesic set $G(W)$ corresponding to a real quadratic field $k(W)$. Writing $U=V_0,W\subset V$ to denote either of these subspaces of dimension 2 with $k(U)$ the corresponding quadratic field, we fix Tamagawa measures on the special orthogonal group $\mathrm{SO}(U)$ as follows: We fix the

Tamagawa measure on $\mathrm{SO}(U)(\mathbf{A})$ for which $\mathrm{vol}(\mathrm{SO}(U)(\mathbf{R})) = 1$ and $\mathrm{vol}(\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A}_f)) = 2$. We also fix the Haar measure on $\mathbf{A}_f^\times$ with $\mathrm{vol}(\mathbf{Z}_p^\times) = 1$ for each prime p so that $\mathrm{vol}(\widehat{\mathbf{Z}}) = 1$ and $\mathrm{vol}(\mathbf{A}_f^\times/\mathbf{Q}^\times) = \frac{1}{2}$. This determines Haar a measures on $\mathbf{A}_{k(U)}^\times$ via the exact sequence $1 \to \mathbf{A}_f^\times \to \mathbf{A}_{k(U),f}^\times \to \mathrm{SO}(U)(\mathbf{A}_f) \to 1$.

Theorem 1.2 (Theorem 5.12 and 5.14)**.** *Let (V, Q) be any rational quadratic space of signature $(n, 2)$. Fix a maximal lattice L with corresponding Weil representation ω_L. Let $f \in H_l(\omega_L)$ be any harmonic weak Maass form of weight $l = 1 - n/2$ and representation ω_L, with $g = \xi_l(f) \in S_{2-l}(\overline{\omega}_L)$ its image under the antilinear differential operator $\xi_l : H_l(\omega_l) \to S_{2-l}(\overline{\omega}_L)$.*

(i) *Let (V_0, Q_0) be a rational quadratic subspace of signature $(0, 2)$ with sublattices $L_0 = V_0 \cap L$, $L_0^\perp \subset V$, and $L_0 \oplus L_0^\perp \subset L$. Write $k = k(V_0)$ for the imaginary quadratic field determined by the space. Let*

$$\theta_{L_0^\perp}(\tau) = \theta_{L_0^\perp}(\tau, 1, 1)$$

denote the holomorphic Siegel theta series associated to the positive definite lattice $L_0^\perp$ of signature $(n, 0)$, defined via restriction of $\theta_L(\tau, z, h)$ as in Lemma 5.7 and (48)*. Let*

$$L(s, g \times \theta_{L_0^\perp}) = \langle g(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; 1)\rangle$$

denote the Rankin-Selberg L-series defined in (51)*, with $L^\star(g \times \theta_{L_0^\perp}) = \Lambda(s+1, \eta_k)L(s, g \times \theta_{L_0^\perp})$ its completion. Let $\mathrm{vol}(K_0)$ denote the volume of the compact open subgroup $K_0 = K \cap \mathrm{GSpin}(V_0)(A_f)$. Then, we have the summation formula*

$$\begin{aligned}\Phi(f, Z(V_0)) &= -\frac{4}{\mathrm{vol}(K_0)}\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_0^\perp})\right) \\ &= -\deg(Z(V_0))\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_0^\perp})\right).\end{aligned}$$

(ii) *Let (W, Q_W) be a rational quadratic subspace of signature $(1, 1)$ with sublattices $L_W = W \cap L$, $L_W^\perp \subset V$, and $L_W \oplus L_W^\perp \subset L$. Write $k = k(W)$ for the real quadratic field determined by W. Let*

$$\theta_{L_W^\perp}(\tau) = \theta_{L_W^\perp}(\tau, 1, 1)$$

denote the nonholorphic Siegel theta series associated to the Lorenztian lattice $L_W^\perp$ of signature $(n-1, 1)$, defined via restriction of $\theta_L(\tau, z, h)$ as in Lemma 5.7 and (48)*. Let*

$$L(s, g \times \theta_{L_W^\perp}) = \langle g(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, s; 2)\rangle$$

denote the Rankin-Selberg L-series defined in (52)*, with $L^\star(g \times \theta_{L_W^\perp}) = \Lambda(s+1, \eta_k)L(s, g \times \theta_{L_W^\perp})$ its completion. Let $\mathrm{vol}(K_0)$ denote the volume of the compact open subgroup $K_W = K \cap \mathrm{GSpin}(V_W)(A_f)$. Then, we have the summation formula*

$$\Phi(f, G(W)) = -\frac{4}{\mathrm{vol}(K_W)}\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_W^\perp}^+(\tau) \otimes \mathcal{E}_{L_W}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_W^\perp})\right).$$

We note again that Theorem 1.2 (i) (Theorem 5.12) a reproof of Bruinier-Yang [13, Theorem 4.7] and Schofer [47] (cf. [1, Theorem 5.7.1]). On the other hand, Theorem 1.2 (ii) (Theorem 5.14) appears to be new, and generalizes the main calculation of [54].

1.1.2. *Spaces of signature* $(2, 2)$ *associated to quadratic fields and relations to standard Rankin-Selberg L-functions.* We apply these calculations to the following quadratic spaces. Let k be any quadratic field, real or imaginary, of discriminant d_k prime to N. We again write $\eta_k(\cdot) = (\frac{d_k}{\cdot})$ to denote the quadratic Dirichlet character, and $C(\mathcal{O}_k) = I(k)/P(k)$ to denote the ideal class group. We consider he following rational quadratic spaces (V_A, Q_A) attached to each class $A \in C(\mathcal{O}_K)$. Given a class $A \in C(\mathcal{O}_k)$, fix an integer ideal representative $\mathfrak{a} \subset \mathcal{O}_k$, and let $\mathfrak{a}_\mathbf{Q} := \mathfrak{a} \otimes_\mathbf{Z} \mathbf{Q}$ be the corresponding fractional ideal. We write $Q_\mathfrak{a}(z) = \mathbf{N}(z)/\mathbf{N}\mathfrak{a} = \mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{a}$ to denote the corresponding norm form, where $\mathbf{N}(z) = \mathbf{N}_{k/\mathbf{Q}}(z) = zz^\tau$ for $\tau \in \mathrm{Gal}(k/\mathbf{Q})$ the nontrivial automorphism denotes the norm homomorphism. Hence, $Q_\mathfrak{a}$ has signature $(2, 0)$ if k is imaginary quadratic, and signature $(1, 1)$ if k is real quadratic. In either case, we consider the space (V_A, Q_A) defined by $V_A = \mathfrak{a}_\mathbf{Q} \oplus \mathfrak{a}_\mathbf{Q}$ with quadratic form $Q_A(z) = Q_A((z_1, z_2)) = Q_\mathfrak{a}(z_1) - Q_\mathfrak{a}(z_2)$. As we explain in Proposition 2.3 below, we have an exceptional isomorphism of algebraic groups $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$ over $\mathbf{Q}$. As we explain in Corollary 2.4, we can choose a lattice $L_A = L_A(N) = N^{-1}\mathfrak{a} \oplus N^{-1}\mathfrak{a} \subset V_A$ whose adelization $L_A(N) \otimes \mathbf{Z}$ is fixed under the action of $\mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)^2$ via conjugation by the compact open subgroup $K_A = K_{L_A(N)} = K_0(N) \oplus K_0(N)$. In this way, we find that $X_A \cong Y_0(N) \times Y_0(N)$ is isomorphic to two copies of the noncompactified modular curve $Y_0(N)$, and that the corresponding special divisors $Z_A(\mu, m) \subset X_A \cong Y_0(N)^2$ corresponding to summands of classical Hirzebruch-Zagier divisors. In this setting, we deduce from known theorems in the literature such as Strömberg [51, Theorem 5.2] or more directly via the Doi-Naganuma lifting (Theorem 6.4) that each cuspidal holomorphic newform $\phi \in S_l^{\mathrm{new}}(\Gamma_0(N))$ has a unique/canonical lifting to a vector-valued form $g_{\phi,A} \in S_l(\omega_{L_A})$ (Corollary 6.5). We use this to derive the following integral presentation of standard Rankin-Selberg L-functions. Fix a character χ of $C(\mathcal{O}_k)$, and let $\theta(\chi)$ denote the corresponding Hecke theta series. Hence,

$$\theta(\chi) = \sum_{A \in C(\mathcal{O}_k)} \chi(A)\theta_A \in M_{l(k)}(\Gamma(|d_k|), \eta_k), \quad l(k) := \begin{cases} 1 & \text{if } k \text{ is imaginary quadratic} \\ 0 & \text{if } k \text{ is real quadratic} \end{cases}.$$

Let $\Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi))L(s, \phi \times \theta(\chi))$ denote the corresponding standard Rankin-Selberg L-function, whose analytic continuation and functional equation we recall in Proposition 6.1. We first show the following link between these completed Rankin-Selberg L-functions, and the Rankin-Selberg L-functions appearing in Theorem 1.2 for the spaces $(V, Q) = (V_A, Q_A)$.

Theorem 1.3 (Proposition 6.3, Theorem 6.8 and Corollary 6.9)**.** *Fix a holomorphic cuspidal newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2, level $\Gamma_0(N)$, and trivial character. Let $g_{\phi,A} \in S_2(\overline{\omega}_{L_A})$ denote the lifting of ϕ to a vector-valued cusp form of weight 2 and conjugate Weil representation $\overline{\omega}_{L_A}$. Let $f_{0,A} \in H_0(\omega_{L_A})$ be any harmonic weak Maass form of weight zero and representation ω_{L_A} whose image $\xi_0(f_{0,A})$ under the antilinear differential operator $\xi_0 : H_0(\omega_{L_A}) \to S_2(\overline{\omega}_{L_A})$ equals $g_{\phi,A}$. We have the following identifications of completed Rankin-Selberg L-functions.*

(i) *If k is the imaginary quadratic field associated to the negative definite subspace $V_{A,0} \subset V_A$ with $L_{A,0} = L_A \cap V_{A,0}$, then we have the identifications of completed Rankin-Selberg L-functions*

$$L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

for each class $A \in C(\mathcal{O}_k)$, and for each class group character $\chi \in C(\mathcal{O}_k)^\vee$ the identification

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(s - 1/2, \phi \times \theta(\chi)).$$

Hence, if k is imaginary quadratic with $(d_k, N) = 1$ and $\eta_k(-N) = -\eta_k(N) = -1$, then we have

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -\frac{2\pi h_k}{w_k} \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\left(\frac{w_k}{4h_k} \right) \Phi(f_{0,A}, Z(V_{A,0})) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle \right].$$

Here, $h_k = \# C(\mathcal{O}_k)$ denotes the class number, and $w_k = \# \mathcal{O}_k^\times$ the number of roots of unity in k.

(ii) *If k is the real quadratic field associated to the Lorentzian subspace $W_A \subset V_A$ with $L_{A,W} \cap W_A \cap L_A$, then we have the identifications of completed Rankin-Selberg L-functions*

$$L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

for each class $A \in C(\mathcal{O}_k)$, and for each class group character $\chi \in C(\mathcal{O}_k)^\vee$ that

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s - 1/2, \phi \times \theta(\chi)).$$

Hence, if k is real quadratic with $(d_k, N) = 1$ and $\eta_k(-N) = \eta_k(N) = -1$, then we have

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -2\ln(\varepsilon_k) h_k \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\left(\frac{w_k \ln(\varepsilon_k)}{4h_k} \right) \Phi(f_{0,A}, G(W_A)) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle \right].$$

Here, $h_k = \# C(\mathcal{O}_k)$ denotes the class number, and ε_k the fundamental unit of $\mathcal{O}_k^\times \cong \mu(k) \times \langle \varepsilon_k \rangle$.

1.1.3. *Relations to arithmetic Hirzebruch-Zagier divisors, Birch-Swinnerton-Dyer constants, and periods.* To derive applications to arithmetic heights in the first setting (i) with $k = k(V_{A,0})$ an imaginary quadratic field, we first recall the arithmetic height formula implied by the combined works of Bruinier-Yang [13] and Andreatta-Goren-Howard-Madapusi Pera [1] for the general setup we consider above for Theorem 1.2. Hence, let us again take (V, Q) to be any rational quadratic space of signature $(n, 2)$. Since we know that the regularized theta lift $\Phi(f, \cdot)$ gives the automorphic Green's function for the divisor $Z(f) \subset X_K$, we have a supply of arithmetic divisors $\widehat{Z}(f) = (Z(f), \Phi(f, \cdot))$ in the corresponding arithmetic Chow group of codimension one cycles on X_K. It then makes sense to consider the arithmetic/Faltings height $[\widehat{Z}(f), Z(V_0)]$ of such a divisor $\widehat{Z}(f)$ along the a CM cycle $Z(V_0)$. As we explain for Theorem 7.5 below, if d_k is odd, then the combined works of Bruinier-Yang [13, Theorem 4.7] and Andreatta-Goren-Howard-Madapusi-Pera [1, Theorem A] imply that we have the arithmetic height formula

$$\left[\widehat{\mathcal{Z}}(f), \mathcal{Z}(V_0)\right] = \left[\widehat{\mathcal{Z}}(f), \mathcal{Z}(V_0)\right]_{\mathrm{Fal}} = -\frac{\deg(\mathcal{Z}(V_0))}{2} \cdot \left(c_f^+(0,0) \cdot \kappa_{L_0}(0,0) + L'(0, \xi_{1-n/2}(f), \theta_{L_0^\perp})\right).$$

Here, we suppress the discussion of the extensions $\mathcal{Z}(f)$ of the divisors $Z(f)$ and $\mathcal{Z}(V_0)$ of the CM cycles $Z(V_0)$ to the integral model $\mathcal{X} = \mathcal{X}_K$ of X_K, and refer to the discussion below for more details. Writing $Z_A(\mu, m) \in X_A \cong Y_0(N)^2$ for the arithmetic special divisors for the spaces (V_A, Q_A) we consider above, with $Z_A^c(\mu, m) \subset X_0(N)^2$ their extensions to the compactification $X_A^\star \cong X_0(N) \times X_0(N)$, we obtain the following consequence. Here, we write $\mathcal{Z}_A(\mu, m)$ to denote the extension of $Z_A(\mu, m)$ to the integral model $\mathcal{X} = \mathcal{X}_{K_A} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$, and $\mathcal{Z}_A^c(\mu, m)$ to denote the extension of $Z_A^c(\mu, m)$ to the integral model $\mathcal{X}^\star = \mathcal{X}_{K_A}^\times \cong \mathcal{X}_0(N) \times \mathcal{X}_0(N)$.

Theorem 1.4 (Theorem 7.8, Corollary 7.9, Corollary A.10)**.** *Retain the setup of Theorem 1.3 (i). Then, for any class group character $\chi \in C(\mathcal{O}_k)^\vee$, we have the central derivative value formula*

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right]$$

for the completed Rankin-Selberg L-function $\Lambda(s, \phi \times \theta(\chi))$ of ϕ times the Hecke theta series $\theta(\chi)$, where each term on the right-hand side denotes the arithmetic height of the arithmetic special divisor

$$\widehat{\mathcal{Z}}_A(f_{0,A}) = \sum_{\mu \in L_A^\vee/L_A} \sum_{\substack{m \in \mathbf{Q} \\ m>0}} c_{f_{0,A}}^+(\mu, -m) \mathcal{Z}_A(\mu, m)$$

on the integral model $\mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$ of the Hilbert modular surface $X = Y_0(N) \times Y_0(N)$ evaluated along the corresponding CM cycle $\mathcal{Z}(V_{A,0}) \subset \mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$. Here, each $\mathcal{Z}_A(\mu, m)$ is the arithmetic Hirzebruch-Zagier divisor $\widehat{\mathcal{Z}}_A(\mu, m) = (\mathcal{Z}_A(\mu, m), \Phi_{\mu,m}^{L_A})$ on $\mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$ for the theta lift $\Phi_{\mu,m}^{L_A}$ of the corresponding Poincaré series $F_{\mu,m}^{L_A}$ described below. We can also extend arithmetic divisors to the compactification $\mathcal{X}^\star \cong \mathcal{X}_0(N) \times \mathcal{X}_0(N)$ as described in (75) *to derive the corresponding formula*

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A^c(f_{0,A}) : \mathcal{Z}(V_{A,0})\right]$$

In particular, if the newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizes an elliptic curve $E/\mathbf{Q}$, then we have the central derivative value formula

$$\Lambda'(E/K, \chi, 1) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right]$$

for the Hasse-Weil L-function $\Lambda(E/K, \chi, s) = \Lambda(s - 1/2, \phi \times \theta(\chi))$ of E over K twisted by χ in terms of arithmetic divisors on the Hilbert modular surface $\mathcal{Y}_0(N) \times \mathcal{Y}_0(N) \longrightarrow \mathrm{Spec}(\mathbf{Z})$. Extending to the compactification $\mathcal{X}_0(N) \times \mathcal{X}_0(N) \longrightarrow \mathrm{Spec}(\mathbf{Z})$, we derive the central derivative value formula

$$\Lambda'(E/K, \chi, 1) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A^c(f_{0,A}) : \mathcal{Z}(V_{A,0})\right].$$

By comparison with the Gross-Zagier formula (Theorem 1.1) for $L'(E/K,\chi,1)$, we also derive the relation

$$2\pi \cdot \widehat{h}_k(y_\chi^\phi) = \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2} \cdot \Lambda'(1/2,\phi\times\theta(\chi)) = -2\pi \sum_{A\in C(\mathcal{O}_k)} \chi(A)\left[\widehat{Z}^c(f_{0,A}) : \mathcal{Z}(V_{A,0})\right],$$

between heights of Heegner divisors on the modular curve $X_0(N)$ and arithmetic heights of arithmetic Hirzeburch-Zagier divisors on the Hilbert modular surface $X_0(N)\times X_0(N)$.

Here, in both cases on the quadratic field k, we also use Theorem 1.2 to derive some results towards the refined conjecture of Birch and Swinnerton-Dyer for the principal class group character $\chi_0 = \mathbf{1} \in C(\mathcal{O}_k)^\vee$ and Euler characteristic calculations, using known results on the Iwasawa main conjectures in the setting of Mordell-Weil rank one. To fix ideas, let E be an elliptic curve of conductor N defined over $\mathbf{Q}$, parametrized via modularity by a cuspidal newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$ as above. Let k be a quadratic field with discriminant d_k and character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$. We consider the Mordell-Weil group $E(k)\cong \mathbf{Z}^{r_E(k)}\oplus E(k)_{\mathrm{tors}}$, along with that of the quadratic twist $E^{(d_k)}(\mathbf{Q})\cong\mathbf{Z}^{r_{E^{(d_k)}}(\mathbf{Q})}\oplus E^{(d_k)}(\mathbf{Q})_{\mathrm{tors}}$ and $E(\mathbf{Q})\cong\mathbf{Z}^{r_E(\mathbf{Q})}\oplus E(\mathbf{Q})_{\mathrm{tors}}$ over $\mathbf{Q}$. Recall that the conjecture of Birch and Swinnerton-Dyer predicts that the completed L-function

$$\begin{aligned}\Lambda(E/K,s) &= \Lambda(s-1/2,\phi\times\theta(\mathbf{1})) = \Lambda(s-1/2,\Pi(\phi))\\ &= \Lambda(s-1/2,\phi)\Lambda(s-1/2,\phi\otimes\eta_k) = \Lambda(E,s)\Lambda(E^{(d_k)},s)\end{aligned}$$

has order of vanishing $\mathrm{ord}_{s=1}\Lambda(E/K,s) = r_E(k)$. Moreover, the leading term in the Taylor series expansion around $s=1$ of this function is expected to be given by the corresponding Birch-Swinnerton-Dyer constant $\kappa_E(k)$, which is defined more generally as follows. For any number field K, we put

$$\kappa_E(K) := \frac{\#\text{Ш}(E/K)\cdot T(E/K)\cdot R(E/K)\cdot\Omega_\infty(E/K)}{|d_K|^{\frac{1}{2}}|E(K)_{\mathrm{tors}}|^2}. \tag{1}$$

Here, $\#\text{Ш}(E/K)$ denotes the cardinality of the conjecturally finite Tate-Shafarevich group

$$\text{Ш}(E/K) = \ker\left(H^1(K,E)\longrightarrow\prod_w H^1(K_w,E)\right).$$

We write $R(E/K)$ to denote the regulator, defined for any basis $\{e_j\}_j$ of $E(K)/E(K)_{\mathrm{tors}}$ by the determinant of the corresponding height matrix $([e_i,e_j]_{\mathrm{NT}})_{i,j}$,

$$R(E/K) = \det\left([e_i,e_j]_{\mathrm{NT}}\right)_{i,j}.$$

We write $T(E/K)$ to denote the product over the local Tamagawa factors,

$$T(E/K) = \prod_{\substack{v<\infty\\ v\subset\mathcal{O}_k\text{ prime}}} [E(K_v):E_0(K_v)]\cdot\left|\frac{\omega}{\omega_v^*}\right|_v,$$

where ω denotes a fixed invariant differential for E/k, and each ω_v^* denotes the local Néron differential at v. We then define the corresponding archimedean local periods

$$\Omega_\infty(E/K) = \prod_{\substack{v|\infty\\ v:k\hookrightarrow\mathbf{R}\\ \text{real}}}\int_{E(k_v)\cong E(\mathbf{R})}|\omega|\cdot\prod_{\substack{v|\infty\\ v,\overline{v}:k\hookrightarrow\mathbf{C}\\ \text{complex}}} 2\int_{E(k_v)\cong E(\mathbf{C})}\omega\wedge\overline{\omega}.$$

Theorem 1.5 (Theorem 7.10 and Corollary 7.11)**.** *Let $E/\mathbf{Q}$ be an elliptic curve parametrized by a newform $\phi\in S_2^{\mathrm{new}}(\Gamma_0(N))$. Let k be a quadratic field of discriminant d_k prime to N and character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$. Assume E has semistable reduction, hence that N squarefree. Assume that the completed L-function*

$$\Lambda(E/K,s) = \Lambda(E,s)\Lambda(E^{(d_k)},s) = \Lambda(s-1/2,\phi)\Lambda(s-1/2,\phi\otimes\eta_k)$$

has order of vanishing $\mathrm{ord}_{s=1}\Lambda(E/K,s)=1$, so that exactly one of the central values $\Lambda(E,1)=\Lambda(1/2,\phi)$ or $\Lambda(E^{(d_k)},1)=\Lambda(1/2,\phi\otimes\eta_k)$ vanishes. Write $[e,e]$ to denote either the regulator $R(E/\mathbf{Q})$ or the regulator $R(E^{(d_k)}/\mathbf{Q})$ according to which factor vanishes. Let us also assume for each prime $p\geq 5$ that

- *The residual Galois representations $E[p]$ and $E^{(d_k)}[p]$ are irreducible.*
- *There exists a prime $l \mid N$ distinct from p where $E[p]$ is ramified, and a prime $q \mid N$ distinct from p where $E^{(d_k)}[p]$ is ramified.*

Then, up to powers of 2 and 3, we have the following unconditional identifications for the constant(s)

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) = \frac{\#\text{Ш}(E/\mathbf{Q}) \cdot \#\text{Ш}(E^{(d_k)}/\mathbf{Q}) \cdot [e,e] \cdot T(E/\mathbf{Q}) \cdot T(E^{(d_k)}\mathbf{Q}) \cdot \Omega_\infty(E/\mathbf{Q}) \cdot \Omega_\infty(E^{(d_k)}/\mathbf{Q})}{\#E(\mathbf{Q})^2_{\text{tors}} \cdot \#E^{(d_k)}(\mathbf{Q})^2_{\text{tors}}}.$$

(i) *If k is imaginary quadratic with $\eta_k(-N) = -\eta_k(N) = -1$, then*

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) \approx \Lambda'(E/k,1) = -\frac{\pi}{2} \sum_{A \in C(\mathcal{O}_k)} \Phi(f_{0,A}, Z(V_{A,0})) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \left[\widehat{\mathcal{Z}}_A^c(f_{0,A}) : \mathcal{Z}(V_{A,0}) \right].$$

(ii) *If k is real quadratic with $\eta_k(-N) = \eta_k(N) = -1$, then*

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) \approx \Lambda'(E/k,1)$$
$$= -2\ln(\varepsilon_k) h_k \sum_{A \in C(\mathcal{O}_k)} \left[\left(\frac{w_k \ln(\varepsilon_k)}{4h_k} \right) \Phi(f_{0,A}, G(W_A)) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle \right].$$

Here, in either case, we write $\approx$ to denote equality up to powers of 2 and 3. Moreover, in each case, the central derivative value $\Lambda'(E/K,1)$ lies in the ring of periods $\mathcal{P}$ described by Kontsevich-Zagier [38].

1.1.4. *Relation to quadratic spaces of signature $(1,2)$ and Fourier coefficients of half-integral weight forms.* Finally, we explain in Appendix A how to develop similar ideas via the rational quadratic space

$$(V,Q) = (\mathrm{Mat}_{2\times 2}^{\mathrm{tr}=0}(\mathbf{Q}), N\det(\cdot)) \tag{2}$$

of signature $(1,1)$ from Bruinier-Yang [13, Theorem 1.5, Theorem 7.7, § 7] to reprove the full Gross-Zagier formula (Theorem 1.1). We refer to Theorem A.9 for details.

In both this case and the case of the signature $(2,2)$ spaces (V_A, Q_A) described above, we explain in the second Appendix B the known and conjectural links between the central derivative values and Fourier coefficients of half-integral weight Maass forms. We refer to Theorem B.5 for the main result derived in Appendix B, using the main theorem of Bruinier-Funke-Imamoglu [11] for quadratic spaces of the form (2) to interpret the sums $\Phi(f, Z(V_0))$ and $\Phi(f, G(W))$ of Theorem 1.2 as the Fourier coefficients of some harmonic weak Maass form of weight 1/2. To describe this in some more detail, let k be any quadratic field of discriminant $d_k = D$ prime to N, and let $\mathcal{Q}_{d_k}$ denote the class group of binary quadratic forms $q_{a,b,c}(x,y) = ax^2 + bxy + cy^2$ of discriminant $d_k = b^2 - 4ac$. We write $[q_{a,b,c}] = [a,b,c] \in \mathcal{Q}_{d_k}$ to denote the corresponding class. We have a well-known identification of class groups

$$\varphi : \mathcal{Q}_{d_k} \cong C(\mathcal{O}_k), \quad [a,b,c] \longmapsto [a, (-b+\sqrt{d_k})/2].$$

For each class $A \in C(\mathcal{O}_k)$, we fix an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_k$, so that the norm form $[Q_\mathfrak{a}] \in \mathcal{Q}_{d_k}$ represents the corresponding class, i.e. $\varphi([Q_\mathfrak{a}]) = A \in C(\mathcal{O}_k)$. We then define $\mathcal{L}_A = \mathcal{L}_A(N) \subset V$ to be the lattice of the space (2) given by

$$\mathcal{L}_A = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : a,b,c \in \mathbf{Z}, \quad N\det \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \Big|_{L_{A,U}} \equiv -q_{a,b,c} \text{ for } \varphi([a,b,c]) = A \right\},$$

with dual lattice

$$\mathcal{L}_A^\vee = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : a,b,c \in \mathbf{Z}, \quad N\det \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \Big|_{L_{A,U}} \equiv -q_{a,b,c} \text{ for } \varphi([a,b,c]) = A \right\}.$$

As explained in Lemma A.2 (cf. [13, Lemma 7.4] or [12, Lemma 7.3]), if we assume $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$ is invariant under the Fricke involution w_N, then there exists both a vector-valued lift $g_A = g_{\phi,A} \in S_{3/2}^{\mathrm{old}}(\overline{\omega}_{\mathcal{L}_A})$ of the Shimura lift of ϕ, as well as a harmonic weak Maass form $f_{1/2,A} \in H_{1/2}(\overline{\omega}_{\mathcal{L}_A})$ such that

- We have the relation $\xi_{1/2}(f_{1/2,A}) = g_A/||g_A||^2$.
- The Fourier coefficients $c^{+}_{f_{1/2,A}}(\mu, m)$ lie in the Hecke field $\mathbf{Q}(\phi) = \mathbf{Q}$ of the newform ϕ.
- The constant Fourier coefficient $c^{+}_{f_{1/2,A}}(0,0)$ vanishes.

When k is an imaginary quadratic field, the map $\mathcal{S}_{\mu_0,m_0}$ of Gross-Kohnen-Zagier [26, § II.4] can be used to relate the L-series $L(s, g_A \times \theta_{\mathcal{L}_{A,0}^{\perp}})$ to $L(s+1/2, \mathcal{S}_{\mu,m}(g))$ (see Lemma A.1). We derive the following relations.

Theorem 1.6 (Theorem B.5)**.** *We have via Theorem 5.12 and Theorem 5.14 for the quadratic space (V, Q) of signature $(1, 2)$ described above the following identification of central derivative values of L-functions as Fourier coefficients of half-integral weight forms.*

(i) *Let k be an imaginary quadratic field of discriminant $D = d_k < 0$. Assume as in Lemma A.1 that $m = -D/4N$ for $D = -4Nm$ with $D \equiv r^2 \bmod 4N$, and take $\mu = \mu_r$. Then, for χ and character of the ideal class group $C(\mathcal{O}_k)$, we have the relation*

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) \cdot c_{g_A}(\mu, m) \cdot \mathrm{tr}_{\mu,m}\left(\Phi(f_{A,1/2})\right) = -\frac{|D|^{\frac{1}{2}}}{16\pi^2 ||\phi||^2} \cdot L'(1/2, \phi \times \theta(\chi)).$$

(ii) *Let k be a real quadratic field of discriminant $d_k > 0$, and $x \in \Omega_{A,\mu,m}(\mathbf{Q}) \subset \mathcal{K}_A$ a positive norm vector with orthogonal complement $W_A = W_A(x) := x^{\perp} \subset V$ as in Proposition B.4 (ii). We have for each class $A \in C(\mathcal{O}_k)$ the relation*

$$\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) = -\frac{4h_k}{w_k \ln(\varepsilon_k)} \cdot L'(0, g_A \times \theta_{\mathcal{L}_{A,W}^{\perp}}).$$

As we describe in Appendix B, it should be possible to extend these calculations in the real quadratic case (Theorem 1.2 (ii)) to derive an affirmative answer to the conjecture posed implicitly in Bruinier-Ono [12, Theorem 1.1 (2)], relating Fourier coefficients of the holomorphic part of the vector-valued Shimura lift to the nonvanishing central derivative values of the real quadratic twisted L-function. We speculate that such relations via Green's functions along CM cycles or geodesics to singular moduli should be possible to establish for Hilbert Maass cusp forms in the setup of signature $(2, 2)$ described in Theorem 1.2 and Theorem 1.3 above; see Conjecture B.2. We plan to return to this in a subsequent work, however describe it here as it fits organically into the topic of proving the formula of Gross-Zagier (Theorem 1.1) via sums of automorphic Green's functions along CM cycles of spin Shimura varieties.

2. Quadratic spaces and spin groups

Let $k = \mathbf{Q}(\sqrt{d})$ be a quadratic field of discriminant

$$d_k = \begin{cases} d & \text{if } d \equiv 1 \bmod 4 \\ 4d & \text{if } d \equiv 2, 3 \bmod 4. \end{cases}$$

We write $\mathcal{O}_k$ for its ring of integers, $C(O_k)$ for its ideal class group, and $h_k = \#C(\mathcal{O}_k)$ its class number. We also write $w_k = \#\mu(k)$ to denote the number of roots of unity in k.

2.1. Quadratic spaces associated to class groups of quadratic fields.

Fix an ideal class $A \in C(\mathcal{O}_k)$. Let $\mathfrak{a} \subset \mathcal{O}_k$ be an integral ideal representative. Consider the corresponding fractional ideal $\mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{Q}$, which when equipped with the norm form $Q_{\mathfrak{a}}(\lambda) := \lambda \mapsto \mathbf{N}_{k/\mathbf{Q}}(\lambda)/\mathbf{N}\mathfrak{a}$ can be viewed as a rational quadratic space over $\mathbf{Q}$. That is, $(\mathfrak{a}_{\mathbf{Q}}, Q_{\mathfrak{a}})$ determines a rational quadratic space of signature

$$\begin{cases} (2,0) & \text{if } d < 0 \text{ so that } k = \mathbf{Q}(\sqrt{d}) \text{ is imaginary quadratic} \\ (1,1) & \text{if } d > 0 \text{ so that } k = \mathbf{Q}(\sqrt{d}) \text{ is real quadratic.} \end{cases}$$

On the other hand, we can also consider the corresponding isomorphic quadratic space $(\mathfrak{a}_{\mathbf{Q}}, -Q_{\mathfrak{a}})$ of signature

$$\begin{cases} (0,2) & \text{if } d < 0 \text{ so that } k = \mathbf{Q}(\sqrt{d}) \text{ is imaginary quadratic} \\ (1,1) & \text{if } d > 0 \text{ so that } k = \mathbf{Q}(\sqrt{d}) \text{ is real quadratic.} \end{cases}$$

Thus, we obtain for each class $A = [\mathfrak{a}] \in C(\mathcal{O}_k)$ a rational quadratic space of signature (2,2) defined by

$$(V_A, Q_A), \quad V_A := \mathfrak{a}_{\mathbf{Q}} + \mathfrak{a}_{\mathbf{Q}}, \quad Q_A(z) = Q_A((z_1, z_2)) := Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2). \tag{3}$$

2.1.1. *Anisotopic subspaces.* Henceforth, we consider the isotropic quadratic space (V_A, Q_A) of signature $(2,2)$ for each class $A \in C(\mathcal{O}_k)$. We then consider the corresponding isotropic subspaces $(V_{A,1}, Q_{A,1}) = (V_{\mathfrak{a}}, Q_{\mathfrak{a}})$ and $(V_{A,2}, Q_{A,2}) = (\mathfrak{a}_{\mathbf{Q}}, -Q_{\mathfrak{a}})$ of respective signatures $(2,0)$ and $(0,2)$ when $k = \mathbf{Q}(\sqrt{d})$ is imaginary quadratic $(d < 0)$, and signature $(1,1)$ when $k = \mathbf{Q}(\sqrt{d})$ is real quadratic $(d > 0)$.

2.2. **Spin groups.** Let $V = (V, Q)$ be a rational quadratic space of signature $(n, 2)$ for any $n \in \mathbf{Z}_{\geq 1}$. We consider the reductive group $\mathrm{GSpin}(V)$ over $\mathbf{Q}$, which fits into the short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

2.2.1. *General characterization.* Given R a commutative ring with identity, we consider a quadratic space (V, Q) over R. Hence, V is a projective R-module of finite rank equipped with a homogeneous function $Q : V \to R$ of degree two for which the corresponding symmetric pairing $(x, y) = Q(x + y) - Q(x) - Q(y)$ is R-bilinear. We call the space (V, Q) *self-dual* if this pairing induces an isomorphism $V \cong \mathrm{Hom}(V, R)$. We call the space (V, Q) *non-degenerate* if its orthogonal complement $V^{\perp} = \{x \in V : (x, y) = 0 \ \forall y \in V\}$ is $\{0\}$.

Let $C(V) = T(V)/I(V)$ denote the Clifford algebra of V, given by the quotient of the tensor algebra

$$T(V) = \bigoplus_{m=0}^{\infty} V^{\otimes} = R \oplus V \oplus (V \otimes_R V) \oplus \cdots$$

of V by the two-sided ideal $I(V)$ generated by elements of the form $v \otimes v - Q(v)$ for $v \in V$. Note that we have canonical embeddings of R and V into $C(V)$. In this way, we see that R-algebra $C(V)$ is generated by the image of the natural injection $V \to C(V)$, and that the grading on $T(V)$ induces a $\mathbf{Z}/2\mathbf{Z}$ grading

$$C(V) = C^0(V) \oplus C^1(V).$$

Concretely, $C^0(V)$ is the R-subalgebra of $C(V)$ generated by products of an even number of basis vectors, and $C^1(V)$ is the R-subalgebra of $C(V)$ generated by products of an odd number of basis vectors. We call $C^0(V)$ the *even (or second) Clifford algebra of* V. For simplicity, we shall write $v_1 \cdots v_m$ to denote the element of $C(V)$ represented by $v_1 \otimes \cdots \otimes v_m$ (for $v_1, \ldots, v_m \in V$). Observe that for $v_1, v_2 \in V \subset C(V)$, we have $v_i^2 = Q(v_i)$ (for $i = 1, 2$) and $v_1 v_2 + v_2 v_1 = (v_1, v_2)$. In particular, we have that $v_1 v_2 = -v_2 v_1$ if and only if v_1 and v_2 are orthogonal.

As explained in [7, §2.2], multiplication by -1 defines an isometry on V, which by the universal property of $C(V)$ (e.g. [7, Proposition 2.3]) induces an algebra automorphism $J : C(V) \to C(V)$ known as the *canonical automorphism.* If 2 is invertible in R, then the even Clifford algebra can be characterized equivalently as

$$C^0(V) = \{v \in C(V) : J(v) = v\}.$$

We also consider the anti-automorphism defined by ${}^tC_V \longrightarrow C_V, (x_1 \otimes \cdots \otimes x_m)^t := x_m \otimes \cdots \otimes x_1$, better known as *canonical involution* on $C(V)$. This is the identity on $R \oplus V$, and gives rise to the *Clifford norm*

$$N_{C(V)} : C(V) \longrightarrow C(V), \quad N_{C(V)}(x) := {}^t x x.$$

On vectors $x \in V$, this reduces to $N_{C(V)}(x) = Q(x)$, and so $N_{C(V)}$ can be viewed as an extension of the quadratic form Q. Note that $N_{C(V)}$ is not generally multiplicative. We have the following classical results.

Proposition 2.1. *Let (V, Q) be a non-degenerate quadratic space over a field F of characteristic* $\mathrm{char}(F) \neq 2$. *Fix an orthogonal basis $v_1, \ldots, v_m$ of V, and let $\delta(V) := v_1 \cdots v_m \in C(V)$. Let $d(V)$ denote the discriminant of the space (V, Q), given by the determinant of the Gram matrix $((v_i, v_j))_{i,j}$ (for any basis $v_1, \cdots v_m$ of V).*

(i) *We have that*

$$\delta(V)^2 = \begin{cases} (-1)^{\frac{m}{2}} 2^{-m} d(V) \in F^{\times}/(F^{\times})^2 & \text{if } m \equiv 0 \bmod 2 \\ (-1)^{\frac{m-1}{2}} 2^{-m} d(V) \in F^{\times}/(F^{\times})^2 & \text{if } m \equiv 1 \bmod 2 \end{cases}.$$

(ii) *The centre $Z(C(V))$ of $C(V)$ is given by*

$$Z(C(V)) = \begin{cases} F & \text{if } m \equiv 0 \bmod 2 \\ F + F\delta(V) & \text{if } m \equiv 1 \bmod 2 \end{cases},$$

and the centre $Z(C^0(V))$ of $C^0(V)$ is given by

$$Z(C^0(V)) = \begin{cases} F + \delta(V)F & \text{if } m \equiv 0 \bmod 2 \\ F & \text{if } m \equiv 1 \bmod 2 \end{cases},$$

Proof. For (i), see [7, Remark 2.5]. For (ii), see [7, Theorem 2.6]. □

Let us now consider the Clifford group of $C(V)$,

$$G_{C(V)} = \left\{ x \in C(V) : \ x \text{ is invertible and } xVJ(x)^{-1} = V \right\}.$$

We can then define the general spin group $\mathrm{GSpin}(V)$ as the intersection $\mathrm{GSpin}(V) = G_{C(V)} \cap C^0(V)$, and the spin group as the subgroup $\mathrm{Spin}(V) = \left\{ x \in \mathrm{GSpin}(V) : N_{C(V)}(x) = 1 \right\}$ of elements of Clifford norm one.

Lemma 2.2. *If $m = \dim_F(V) \leq 4$, then we have identifications*

$$\mathrm{GSpin}(V) \cong \left\{ x \in C^0(V) : N_{C(V)}(x) \in F^\times \right\} \text{ and } \mathrm{Spin}(V) \cong \left\{ x \in C^0(V) : N_{C(V)}(x) = 1 \right\}.$$

Proof. See [7, Lemma 2.14]. □

2.2.2. *Exceptional isomorphisms.* Let $(V_A, Q_A) = (\mathfrak{a}_{\mathbf{Q}} + \mathfrak{a}_{\mathbf{Q}}, Q_{\mathfrak{a}} - Q_{\mathfrak{a}})$ be any of the rational quadratic spaces of signature $(2, 2)$ considered above. Hence, $\dim_{\mathbf{Q}}(V_A) = 4$, $\dim_{\mathbf{Q}} C(V_A) = 2^4 = 16$, and $\dim_{\mathbf{Q}} C^0(V_A) = 8$.

Proposition 2.3. *Let k be a quadratic field with class group $C(\mathcal{O}_k)$. Fix any class $A \in C(\mathcal{O}_k)$, together with an integer ideal representative $\mathfrak{a} \subset \mathcal{O}_k$, and write $Q_{\mathfrak{a}}(z) = \mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ to denote the corresponding norm form. Consider the corresponding rational quadratic space $(V_A, Q_A) = (\mathfrak{a}_{\mathbf{Q}} + \mathfrak{a}_{\mathbf{Q}}, Q_{\mathfrak{a}} - Q_{\mathfrak{a}})$ of signature $(2, 2)$, with Clifford algebra $C(V_A)$ and even Clifford subalgebra $C^0(V_A) \subset C(V_A)$. We have the identification $C^0(V_A) \cong M_2(\mathbf{Q})^2$, from which we derive exceptional isomorphisms* $\mathrm{Spin}(V_A) \cong \mathrm{SL}_2^2$ *and* $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$ *of algebraic groups over $\mathbf{Q}$.*

Proof. Fix a $\mathbf{Z}$-basis $[\alpha_{\mathfrak{a}}, z_{\mathfrak{a}}]$ of the chosen integral ideal representative $\mathfrak{a} \subset \mathcal{O}_k$ for each class A. We also let $Q_{\mathfrak{a}}(z) = N_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ denote the norm form. Here, writing $\tau \in \mathrm{Gal}(k/\mathbf{Q})$ to denote the nontrivial automorphism, we define the norm $\mathbf{N}_{k/\mathbf{Q}}(z) = zz^\tau$ and the trace $\mathrm{Tr}_{k/\mathbf{Q}}(z) = z + z^\tau$. Consider the basis

$$v_1 = (\alpha_{\mathfrak{a}}, 0), \quad v_2 = (z_{\mathfrak{a}}, 0), \quad v_3 = (0, \alpha_{\mathfrak{a}}), \quad v_4 = (0, z_{\mathfrak{a}}).$$

We compute the inner products

$$\begin{aligned}
(v_1, v_1)_A &= Q_{\mathfrak{a}}(2\alpha_{\mathfrak{a}}) - 2Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1} 2\mathbf{N}_{k/\mathbf{Q}}(\alpha_{\mathfrak{a}}) \\
(v_1, v_2)_A &= Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}} + z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1} \mathrm{Tr}_{k/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau) = (v_2, v_1)_A \\
(v_1, v_3)_A &= Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) + Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) = 0 = (v_3, v_1)_A \\
(v_1, v_4)_A &= Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = 0 = (v_4, v_1)_A \\
(v_2, v_2)_A &= Q_{\mathfrak{a}}(2z_{\mathfrak{a}}) - 2Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1}(2\mathbf{N}_{k/\mathbf{Q}}(z_{\mathfrak{a}})) \\
(v_2, v_3)_A &= Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) = 0 = (v_3, v_2)_A \\
(v_2, v_4)_A &= (Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}})) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = 0 = (v_4, v_2)_A \\
(v_3, v_3)_A &= -Q_{\mathfrak{a}}(2\alpha_{\mathfrak{a}}) + 2Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) = -\mathbf{N}\mathfrak{a}^{-1} 2\mathbf{N}_{k/\mathbf{Q}}(\alpha_{\mathfrak{a}}) = -(v_1, v_1)_A \\
(v_3, v_4)_A &= -Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}} + z_{\mathfrak{a}}) + Q_{\mathfrak{a}}(\alpha_{\mathfrak{a}}) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = -\mathbf{N}\mathfrak{a}^{-1} \mathrm{Tr}_{k/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau) = (v_4, v_3)_A = -(v_1, v_2)_A \\
(v_4, v_4)_A &= -Q_{\mathfrak{a}}(2z_{\mathfrak{a}}) + 2Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = -\mathbf{N}\mathfrak{a}^{-1} 2\mathbf{N}_{k/\mathbf{Q}}(z_{\mathfrak{a}}) = -(v_2, v_2)_A.
\end{aligned}$$

We then compute the determinant of the corresponding Gram matrix

$$d(V_A) = \det((v_i, v_j)_A)_{i,j} = \det \begin{pmatrix} \frac{2\mathbf{N}_{k/\mathbf{Q}}(\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & \frac{2\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ 0 & 0 & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} \\ 0 & 0 & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} \end{pmatrix}$$

$$= \frac{2\mathbf{N}_{k/\mathbf{Q}}(\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} \begin{vmatrix} \frac{2\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ 0 & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} \\ 0 & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} \end{vmatrix}$$

$$- \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} \begin{vmatrix} \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ 0 & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} \\ 0 & -\frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)}{\mathbf{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a})}{\mathbf{N}\mathfrak{a}} \end{vmatrix}$$

$$= \frac{4\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}^2} \left(\frac{4\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right) - \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \left(\frac{4\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right)$$

$$= \left(\frac{4\mathbf{N}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{k/\mathbf{Q}}(z_\mathfrak{a}\alpha_\mathfrak{a}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right)^2 \equiv 1 \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2.$$

That is, we find that $d(V_A) \in (\mathbf{Q}^\times)^2$ is a nonzero rational square, and hence trivial. Using the relation $\delta(V_A)^2 = 2^{-4}d(V_A)$ of Proposition 2.1 (i), we deduce that the volume form $\delta(V_A) \in \mathbf{Q}^\times$ must be rational. We then deduce from Proposition 2.1 (ii) that $Z(C^0(V_A)) = \mathbf{Q} + \delta(V_A)\mathbf{Q} = \mathbf{Q}$, and hence that the even Clifford algebra $C^0(V_A)$ of $\dim_\mathbf{Q}(C^0(V_A)) = 8$ must be a direct sum of two isomorphic copies of a quaternion algebra B over $\mathbf{Q}$. Using the classifications of Clifford algebras over $\mathbf{R}$, we see that $C(V_A \otimes \mathbf{R}) \cong C_{2,2}(\mathbf{R}) \cong M_4(\mathbf{R})$ and $C^0(V_A \otimes \mathbf{R}) \cong C^0_{2,2}(\mathbf{R}) \cong M_2(\mathbf{R}) \oplus M_2(\mathbf{R})$. Hence, B must be indefinite. Since the discriminant $d(V_A) = 1$ is trivial, we deduce that B must be the matrix algebra $M_2(\mathbf{Q})$, with the Clifford norm corresponding to the reduced norm homomorphism $\mathrm{nrd} : B \to \mathbf{Q}$, which for $B = M_2(\mathbf{Q})$ is simply the determinant $\det = \mathrm{nrd}$. The claimed isomorphisms for the spin groups then follow from the characterization given in Lemma 2.2. □

Corollary 2.4. *Fix $N \in \mathbf{Z}_{\geq 1}$. Let $L_A = L_A(N) \subset V_A$ denote the lattice whose adelization $L_A \otimes \widehat{\mathbf{Z}}$ is stabilized under the action via conjugation by $\mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)^2$ by the compact open subgroup $K_0(N) \oplus K_0(N)$, where $K_0(N) \subset \mathrm{GL}_2(\widehat{\mathbf{Z}}) \subset \mathrm{GL}_2(\mathbf{A}_f)$ denotes the congruence subgroup defined by*

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : c \equiv 0 \bmod N \right\}.$$

(i) *The lattice is $L_A = L_A(N) = N^{-1}\mathfrak{a} \oplus N^{-1}\mathfrak{a}$, with dual lattice $L_A^\vee = L_A(n)^\vee = \mathfrak{d}_k^{-1}N^{-1}\mathfrak{a} \oplus \mathfrak{d}_k^{-1}\mathfrak{a}$*
(ii) *The level of the lattice is $N = \{\min a \in \mathbf{Z}_{\geq 1}: aQ_A(\lambda) \in \mathbf{Z} \forall \gamma \in L_A^\vee\}$.*
(iii) *The discriminant $d(L_A) = d(L_A(N))$ of the lattice is 1.*

Proof. Consider $\mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)^2$ acting on $V_A = \mathfrak{a}_\mathbf{Q} \oplus \mathfrak{a}_\mathbf{Q} \cong [\alpha_\mathfrak{a}, z_\mathfrak{a}]\mathbf{Q} \oplus [\alpha_\mathfrak{a}, z_\mathfrak{a}]\mathbf{Q}$ by conjugation. Here, we use the canonical embedding $V_A \to C(V_A)$ and the identification $C^0(V_A) \cong M_2(\mathbf{Q}) \oplus M_2(\mathbf{Q})$. Writing

$$R(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod N \right\} \subset M_2(\mathbf{Q})$$

to denote the Eichler order of level N, we can characterize $L_A = L_A(N)$ as the lattice stabilized under conjugation by invertible elements of $R(N) \oplus R(N)$. We claim that the conjugation action $g \cdot v = gvg^{-1}$ for $g = (g_1, g_2) \in \mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)^2$ and $v = (v_1, v_2) \in \mathfrak{a}_{\mathbf{A}_f} \oplus \mathfrak{a}_{\mathbf{A}_f}$ takes the simpler form $(g_1, g_2) \cdot (v_1, v_2) = (g_1 v_1 g_1^{-1}, g_2 v_2 g_2^{-1})$, for $g_i \in \mathrm{GL}_2(\mathbf{A}_f)$ and $v_i \in \mathfrak{a}_{\mathbf{A}_f} = [\alpha_\mathfrak{a}, z_\mathfrak{a}]\mathbf{A}_f$ for $i = 1, 2$. We can then see by inspection that $L_A = L_A(N) = N^{-1}\mathfrak{a} \oplus N^{-1}\mathfrak{a}$ is the stabilized lattice, with dual lattice $L_A^\vee = L_A(N)^\vee = \mathfrak{d}_k^{-1}N^{-1}\mathfrak{a} \oplus \mathfrak{d}_k^{-1}N^{-1}\mathfrak{a}$, and that this lattice has level N. A minor variation of the calculation given in Proposition 2.3, replacing $\alpha_\mathfrak{a}$ with $\alpha_\mathfrak{a}/N$ and $z_\mathfrak{a}$ with $z_\mathfrak{a}/N$ to get the basis

$$v_1 = (\alpha_\mathfrak{a}/N, 0), \quad v_2 = (z_\mathfrak{a}/N, 0), \quad v_3 = (0, \alpha_\mathfrak{a}/N), \quad v_4 = (0, z_\mathfrak{a}/N),$$

shows that the discriminant $d(L_A) = \det((v_i, v_j)_A) = 1$.

□

3. GSpin Shimura varieties

We now describe the GSpin Shimura varieties and special cycles that appear, starting with the general setting, then describing the Hilbert modular surfaces corresponding to the rational quadratic spaces (V_A, Q_A) of signature $(2,2)$ introduced above.

3.1. **Complex Shimura varieties.** Let (V, Q) be any rational quadratic space of signature $(n, 2)$ with bilinear form $(x, y) = Q(x+y) - Q(x) - Q(y)$. Write $\mathrm{GSpin}(V)$ for the corresponding general spin group. We consider the Grassmannian $D(V) = D^{\pm}(V)$ of oriented[1] negative two-planes in $V(\mathbf{R})$,

$$D(V) = \{z \subset V_{\mathbf{R}} : \dim(z) = 2, Q|_z < 0\}.$$

Extending the bilinear form $(\cdot,\cdot)$ to $V_{\mathbf{C}}$, we see that this real manifold $D(V)$ is isomorphic to the complex manifold of dimension n defined by the quadric

$$\mathcal{Q}(V) = \{w \in V_{\mathbf{C}} \backslash \{0\} : (w, w) = 0, (w, \overline{w}) < 0\} / \mathbf{C}^{\times} \subset \mathbf{P}(V(\mathbf{C})),$$

from which $D(V)$ acquires the structure of a complex manifold. Here, the isomorphism sends an oriented hyperplane $z = [x, y]$ with basis $[x, y]$ such that $Q(x) = Q(y)$ and $(x, y) = 0$ to $w = x + iy \in V_{\mathbf{C}}$.

We now explain how $(\mathrm{GSpin}(V), D(V))$ determines a Shimura datum. We have a natural embeddings of $\mathbf{R}$-algebras $\mathbf{C} \to C(z) \to C(V_{\mathbf{R}})$ for any hyperplane $z = [x, y] \subset V_{\mathbf{R}}$, with the first induced by the map

$$i \longmapsto \frac{xy}{\sqrt{Q(x)Q(y)}}.$$

The induced map $\mathbf{C}^{\times} \to C(V_{\mathbf{R}})^{\times}$ takes values in $\mathrm{GSpin}(V)(\mathbf{R})$, and arises from a morphism of real algebraic groups $\alpha_z : \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m \to \mathrm{GSpin}(V)(\mathbf{R})$. In this way, we can identify $D(V)$ with a conjugacy class in $\mathrm{Hom}(\mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m, \mathrm{GSpin}(V)(\mathbf{R}))$. Hence, we can associate a Shimura variety to $(\mathrm{GSpin}(V), D(V))$.

Any choice maximal lattice $L \subset V$ determines a compact open subgroup

$$K = K_L := \mathrm{GSpin}(V)(\mathbf{A}_f) \cap C(L) \subset \mathrm{GSpin}(V)(\mathbf{A}_f), \quad \widehat{L} = L_{\widehat{\mathbf{Z}}}.$$

We write $L^{\vee} = \{x \in V : (v, L) \subset \mathbf{Z}\}$ for the dual lattice, and $L^{\vee}/L \cong \widehat{L}^{\vee}/\widehat{L}$ for the discriminant group. Note that $K = K_L$ acts trivially on $\widehat{L}$. Fixing such a choice, we consider the corresponding Shimura variety

$$X_K(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q}) \backslash D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K \cong \coprod_{h \in \mathrm{GSpin}\, V(\mathbf{Q}) \backslash \mathrm{GSpin}\, V(\mathbf{A}_f)/K} \Gamma_h \backslash D(V) \tag{4}$$

for arithmetic subgroups $\Gamma_h = \mathrm{GSpin}(V)(\mathbf{Q}) \cap hKh^{-1}$. This complex orbifold $X_K(\mathbf{C})$ has the structure of a quasiprojective variety X_K of dimension n over $\mathbf{Q}$ which is projective if and only if V is anisotropic. It is smooth if $K = K_L$ is neat. We refer to [1, §2], [39], and [40, §1] for more background.

3.2. **Special divisors.** Given a vector $x \in V$ with $Q(x) > 0$, we define a divisor

$$D(V)_x = \{z \in D(V) : z \perp x\}.$$

For each $\mu \in L^{\vee}/L$ and $m \in \mathbf{Q}_{>0}$, we consider the divisor $Z(\mu, m)$ on X_K given by the complex orbifold

$$Z(\mu, m)(\mathbf{C}) = \coprod_{h \in \mathrm{GSpin}\, V(\mathbf{Q}) \backslash \mathrm{GSpin}\, V(\mathbf{A}_f)/K} \Gamma_h \Bigg\backslash \left(\coprod_{\substack{x \in \mu_h + L_h \\ Q(x) = m}} D(V)_x \right). \tag{5}$$

Here, for any element $h \in \mathrm{GSpin}\, V(\mathbf{A}_f)$, we write $L_h \subset V$ for the lattice determined by $\widehat{L}_h = h \cdot \widehat{L}$, and $\mu_h = h \cdot \mu \in L_h^{\vee}/L_h$. As explained in [1, §2] (cf. [39], [40, §1]), these $Z(\mu, m)(\mathbf{C}) \to X_K(\mathbf{C})$ determine effective Cartier divisors, and admit a moduli description given in terms of the Kuga-Satake abelian scheme over X_K.

[1]Although we drop it from the notation henceforth, we write $D^{+}(V)$ to denote the hyperplanes with positive orientation, and $D^{-}(V)$ the hyperplanes with negative orientation, so that $D^{\pm}(V)$ denotes one of these choices – which we fix consistently.

3.3. **CM cycles and geodesic sets.** Let $V_0 \subset V$ be any rational quadratic subspace of signature $(0, 2)$ with corresponding lattice $L_0 = V_0 \cap L$. The Clifford algebra $C(L_0)$ then determines an order in a quaternion algebra over $\mathbf{Q}$, and its even part $C^0(L_0)$ an order in some imaginary quadratic field $k(V_0)$ determined by V_0. The corresponding spin group $\mathrm{GSpin}(V_0) \cong \mathrm{Res}_{k(V_0)/\mathbf{Q}} \mathbf{G}_m$ forms a rank-two torus $T(V_0)$ in $\mathrm{GSpin}(V)$. Fixing an embedding $k(V_0) \subset \mathbf{C}$, the left multiplication in $V_0(\mathbf{R})$ gives $V_0(\mathbf{R})$ the structure of a complex vector space, and determines an orientation. In this way, we see that each of the two oriented negative definite subspaces $z_0^{\pm} = V_0(\mathbf{R})$ determines a point in $D(V) = D^{\pm}(V)$, and $(T(V_0), z_0^{\pm})$ a Shimura datum associated to the zero-dimensional complex orbifold

$$Z(V_0)(\mathbf{C}) = T(V_0)(\mathbf{Q}) \backslash \left\{ z_0^{\pm} \right\} \times T(V_0)(\mathbf{A}_f)/K_0, \quad K_0 = K_{L_0} = T(V_0)(\mathbf{A}_f) \cap K_L. \tag{6}$$

We call the corresponding zero cycle $Z(V_0) \subset X_K$ the *CM cycle associated to* V_0.

As explained in [1], if we assume that $C^0(V_0) \cong \mathcal{O}_{k(V_0)}$ is the maximal order, then the $\mathbf{Z}/2\mathbf{Z}$-grading on $C(L_0)$ takes the form $C(L_0) \cong \mathcal{O}_{k(V_0)} \oplus L_0$, where L_0 is both a left and right $\mathcal{O}_{k(V_0)}$-module. In this case, there exists a proper fractional $\mathcal{O}_{k(V_0)}$-ideal $\mathfrak{b}$ and left $\mathcal{O}_{k(V_0)}$-module isomorphism $L_0 \cong \mathfrak{b}$ which identifies the corresponding quadratic form $Q_0 = Q|_{V_0}$ with the norm form $Q_0(\cdot) = \mathbf{N}_{k(V_0)/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{b}$. The dual lattice $L_0^{\vee}$ is then identified with $\mathfrak{d}_{k(V_0)}^{-1} L_0$ for $\mathfrak{d}_{k(V_0)}^{-1}$ the inverse different of $k(V_0)$. In this setting, the zero cycle $Z(V_0)$ can be reinterpreted as the moduli stack of elliptic curves with complex multiplication by $\mathcal{O}_{k(V_0)}$. We refer to [1, §4] for details, and note that there appears to be some subtlety in extending this discussion to the setting where $C^0(V_0) \cong \mathcal{O}$ is a nonmaximal order (which at the time of writing remains an open problem).

Motivated by the study of real quadratic fields, we consider sets attached to rational quadratic subspaces $W \subset V$ of signature $(1, 1)$. Writing $D(W) = D^{\pm}(W) = \{z \subset W(\mathbf{R}) : \dim(z) = 1, \text{orientation } \pm, Q_W|_z < 0\}$ to denote the corresponding domain of oriented lines in $W(\mathbf{R})$, we consider the sets defined

$$G(W)(\mathbf{C}) = \mathrm{GSpin}(W)(\mathbf{Q}) \backslash D(W) \times \mathrm{GSpin}(W)(\mathbf{A}_f)/(K_L \cap \mathrm{GSpin}(W)(\mathbf{A}_f)). \tag{7}$$

We call $G(W)$ the *geodesic set associated to* W.

3.4. **Classical description as Hilbert modular surfaces.** Fix any class A in $C(\mathcal{O}_k)$, together with any integral ideal representative $\mathfrak{a}$, and consider the corresponding rational quadratic space (V_A, Q_A) of signature (2,2) introduced in (3). By Proposition 2.3, we have an accidental isomorphism

$$\zeta : \mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2 \tag{8}$$

of algebraic groups over $\mathbf{Q}$. Write $L_A \subset V_A$ for the maximal lattice whose corresponding compact open subgroup $K_A = K_{L_A} \subset \mathrm{GSpin}(V_A)(\mathbf{A}_f)$ given by $K_A = \prod_{p<\infty} K_{A,p} = \prod_{p<\infty} K_{\Lambda_A,p}$ has the property that each $K_{A,p} \subset \mathrm{GSpin}(V_A)(\mathbf{Q}_p)$ corresponds under (8) to the Cartesian product of congruence subgroups

$$\zeta(K_{A,p}) = K_{0,p}(N)^2, \quad K_{0,p}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) : c \in N\mathbf{Z}_p \right\} \subseteq \mathrm{GL}_2(\mathbf{Z}_p) \tag{9}$$

for some fixed integer $N \geq 1$. That is, we assume that K_A is identified under (8) with the Cartesian self-product $K_0(N)^2$ of the congruence subgroup $K_0(N) = \prod_{p<\infty} K_{0,p}(N)$ of $\mathrm{GL}_2(\widehat{\mathbf{Z}}) \subset \mathrm{GL}_2(\mathbf{A}_f)$.

3.4.1. *Hermitian symmetric domains.* Recall that we consider the Grassmannian

$$D(V_A) = D^{\pm}(V_A) = \{z \subset V_A(\mathbf{R}) : \dim(z) = 2, \text{ orientation } \pm, Q_A|_z < 0\}$$

of oriented negative definite hyperplanes in $V_A(\mathbf{R}) \cong \mathfrak{a}_{\mathbf{R}} + \mathfrak{a}_{\mathbf{R}}$. Extending the bilinear pairing $(\cdot, \cdot)_A$ to $\mathbf{C}$, we saw that the real manifold $D(V_A)$ is isomorphic to the complex surface defined by the quadric

$$\mathcal{Q}(V_A) = \{w \in V_A(\mathbf{C}) : (w, w)_A = 0, (w, \overline{w})_A < 0\} / \mathbf{C}^{\times} \subset \mathbf{P}(V_A(\mathbf{C}))$$

via the isomorphism sending a properly oriented hyperplane z with standard basis $z = [x, y] \in D(V_A)$ such that $(x, x)_A = (y, y)_A = -1$ and $(x, y)_A = 0$ to the complex point $w = w(z) := x + iy \in \mathcal{Q}(V_A)$. Here, we remark that the quadric $\mathcal{Q}(V_A)$ determines a complex surface with two connected components $\mathcal{Q}^{\pm}(V_A)$. Our choice of orientation $D^{\pm}(V_A)$ determines one of these, so that we have the corresponding identification $D^{\pm}(V_A) \cong \mathcal{Q}^{\pm}(V_A)$. This identification is sometimes referred to as the *projective model* for $D(V_A) = D^{\pm}(V_A)$. It is useful for identifying the complex structure on $D(V_A)$, which makes it a hermitian symmetric domain.

We also have the following equivalent description. Fix a Witt decomposition $V_A = W_A \oplus \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$, with nonzero isotropic basis vectors e_1 and e_2 chosen so that $(e_1, e_1)_A = (e_2, e_2)_A = 0$ and $(e_1, e_2)_A = 1$.

Hence, $W_A \subset V_A$ denotes the Lorentzian rational quadratic subspace of signature $(1,1)$ determined by the intersection $W_A = V_A \cap e_1^\perp \cap e_2^\perp$. We can then identify $D(V_A) \cong \mathcal{Q}(V_A)$ with the corresponding tube domain

$$\mathcal{H}(V_A) = \{\mathfrak{z} \in W_A(\mathbf{C}) : Q_A(\Im(\mathfrak{z})) < 0\}.$$

To be more precise, given an element $w \in V_A(\mathbf{C}) = W_A(\mathbf{C}) \oplus \mathbf{C}e_1 \oplus \mathbf{C}e_2$, let us write its corresponding Witt decomposition $w = \mathfrak{z} + ae_1 + be_2$ for $\mathfrak{z} \in W_A(\mathbf{C})$ and $a, b \in \mathbf{C}$ as $w = (\mathfrak{z}, a, b)$. Given an element $w \in V_A(\mathbf{C})$, we also write $[w]$ to denotes its image in $\mathcal{Q}(V_A) \subset \mathbf{P}(V_A(\mathbf{C}))$. We have a biholomorphic map

$$\mathcal{H}(V_A) \cong \mathcal{Q}(V_A), \quad \mathfrak{z} \longmapsto [(\mathfrak{z}, 1, -Q_A(\mathfrak{z}) - Q_A(e_2))] = [\mathfrak{z} + e_1 - Q_A(\mathfrak{z})e_2 + Q_A(e_2)e_2].$$

We refer to [7, Lemma 2.18] (for instance) for more details. The domain $\mathcal{H}(V_A) \subset W_A(\mathbf{C}) \cong \mathbf{C}^2$ has two connected components $\mathcal{H}^\pm(V_A)$ corresponding to the two cones of negative norm vectors in the Lorenzian subspace $W_A(\mathbf{R})$, and we have natural identifications $\mathcal{H}^\pm(V_A) \cong \mathfrak{H}^2$ with products of two copies of the Poincaré upper-half plane $\mathfrak{H} = \{\tau \in \mathbf{C}, \Im(\tau) > 0\}$. The corresponding identification $D^\pm(V_A) \cong \mathcal{H}^\pm(V_A) \cong \mathfrak{H}^2$ is sometimes referred to as the *tube domain model* for $D(V_A) = D^\pm(V_A)$.

3.4.2. *Hilbert modular surfaces.* Taking for granted the various identifications of domains

$$D^\pm(V_A) \cong \mathcal{Q}^\pm(V_A) \cong \mathcal{H}^\pm(V_A) \cong \mathfrak{H}^2$$

with the accidental isomorphism (8) and the choice of level structure (9), we obtain the identifications

$$\begin{aligned} X_{K_A}(\mathbf{C}) &= \mathrm{GSpin}(V_A)(\mathbf{Q})\backslash D^\pm(V_A) \times \mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_A \\ &\cong \mathrm{GL}_2(\mathbf{Q})^2\backslash\mathfrak{H}^2 \times \mathrm{GL}_2(\mathbf{A}_f)^2/\zeta(K_A) = Y_0(N) \times Y_0(N), \end{aligned} \tag{10}$$

where

$$Y_0(N) = \Gamma_0(N)\backslash\mathfrak{H} \cong \mathrm{GL}_2(\mathbf{Q})\backslash\mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}_f)/K_0(N), \quad K_0(N) := \prod_{p<\infty} K_{0,p}(N)$$

denotes the noncompactified modular curve of level $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$. Hence, we can identify each spin Shimura surface $X_{K_A}(\mathbf{C}) = X_{K_{L_A}}(\mathbf{C})$ with the Hilbert modular surface $Y_0(N) \times Y_0(N)$.

3.4.3. *Hirzebruch-Zagier divisors.* We see from (8), (9), and (10) that each special divisor $Z(\mu, m) = Z_A(\mu, m)$ as defined in (5) above for $\mu \in L_A^\vee/L_A$ and $m \in \mathbf{Q}_{>0}$ is given more explicitly by the analytic divisor

$$\begin{aligned} Z_A(\mu, m)(\mathbf{C}) &= \coprod_{h \in \mathrm{GSpin}(V_A)(\mathbf{Q})\backslash \mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_A} \Gamma_h \Big\backslash \left(\coprod_{\substack{x \in \mu_h + L_{A\,h} \\ Q_A(x) = m}} D(V_A)_x \right) \\ &\cong \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} D(V_A)_x = \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} \left\{ z \in D^\pm(V_A) : (z, x)_A = 0 \right\} \\ &\cong \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} \left\{ z = (z_1, z_2) \in \mathfrak{H}^2 : Q_A(z + x) - Q_A(z) = m \right\} \subset Y_0(N)(\mathbf{C}) \times Y_0(N)(\mathbf{C}). \end{aligned}$$

Note that these divisors can be viewed as embeddings of modular curves into $Y_0(N) \times Y_0(N)$. This is apparent from the description above, as well as their more intrinsic characterization as analytic divisors in [39, §2]. That is, we choose a positive norm vector $x \in V_A$, or more precisely, an element of the quadratic

$$\Omega_{A,\mu,m}(\mathbf{Q}) = \{x \in \mu + L_A : Q_A(x) = m\}.$$

We consider the corresponding one-dimensional subspace $V_{A,+} := \mathbf{Q}x \subset V_A$, with its orthogonal complement $U_A := V_{A,+}^\perp \subset V_A$. Hence, $U_A \subset V_A$ determines a subspace of signature $(1,2)$. Its spin group $\mathrm{GSpin}(U_A)$ is isomorphic to the stabilizer of $V_{A,+}$ in $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$ ([39, Lemma 2.1]). The natural subspace embedding $U_A \subset V_A$ gives rise to an embedding of reductive algebraic groups $\mathrm{GSpin}(U_A) \to \mathrm{GSpin}(V_A)$. Writing $D(U_A) = D^\pm(U_A) = \{z \subset U_A(\mathbf{R}) : \dim(z) = 2, Q_A|_z < 0\}$ for the corresponding Grassmannian, and

$K_{A,U} = K_A \cap \mathrm{GSpin}(U_A)(\mathbf{A}_f)$ the corresponding compact open subgroup, we can identity $Z_A(\mu, m)$ with the Shimura subcurve

$$Z_A(\mu,m)(\mathbf{C}) = \mathrm{GSpin}(U_A)(\mathbf{Q})\backslash D(U_A) \times \mathrm{GSpin}(U_A)(\mathbf{A}_f)/K_{A,U} \longrightarrow X_A(\mathbf{C}) \cong Y_0(N) \times Y_0(N)$$
$$\mathrm{GSpin}(U_A)(\mathbf{Q})(z,h)K_{A,U} \longmapsto \mathrm{GSpin}(V_A)(\mathbf{Q})(z,h)K_A.$$

To be more precise, we know from the discussion above that $Z_A(\mu, m)$ can be identified with the modular curve $\Gamma(U_A)\backslash D(U_A)$, where $D(U_A) = D^{\pm}(U_A) \cong \mathfrak{H}$ and $\Gamma(U_A) = \mathrm{GSpin}(U_A)(\mathbf{Q}) \cap K_{A,U} \subseteq \Gamma_0(N)^2$ is a congruence subgroup. As can be seen through this description, the sums over cosets $\mu \in L_A^\vee/L_A$ of these divisors $Z_A(\mu, m)$ give the classical Hirzebruch-Zagier divisors of the forms described in [29] and [7, §2]. We shall return to this relation to the classical Hirzebruch-Zagier divisors on $Y_0(N) \times Y_0(N)$ given in terms of the moduli discussion (see e.g. [29]) in our discussion of arithmetic heights below.

3.4.4. *CM cycles.* Let $V_{0,A} \subset V_A$ be any rational quadratic subspace of signature $(0, 2)$, with corresponding lattice $L_{0,A} = V_{0,A} \cap V_A$ and quadratic form $Q_{A,0} = Q_A|_{V_{A,0}}$. The even Clifford algebra $C^0(L_{A,0}) \subset C^0(V_{A,0})$ determines an order in the imaginary quadratic field $k(V_{A,0})$ determined by $V_{A,0}$. Recall that any such subspace $V_{A,0} \subset V_A$ determines a rank-two torus $\mathrm{GSpin}(V_{0,A}) \cong \mathrm{Res}_{k(V_{A,0})/\mathbf{Q}}\, \mathbf{G}_m$ and the even Clifford algebra $C^0(L_{A,0}) \cong \mathcal{O}$ an order in $\mathcal{O}_{k(V_{A,0})}$. Again, if $C^0(L_{A,0}) \cong \mathcal{O}_{k(V_{A,0})}$ is maximal, then the $\mathbf{Z}/2\mathbf{Z}$-grading on $C(L_{A,0})$ takes the form $C(L_{A,0}) \cong \mathcal{O}_{k(V_{A,0})} \oplus L_{A,0}$, with $L_{A,0}$ being both a left and right $\mathcal{O}_{k(V_{A,0})}$-module, and there exists a fractional $\mathcal{O}_{k(V_{A,0})}$-ideal $\mathfrak{b}$ and an isomorphism $L_{A,0} \cong \mathfrak{b}$ of left $\mathcal{O}_{k(V_{A,0})}$-modules which identifies the quadratic form $Q_{A,0}$ on $L_{A,0}$ with the norm form $-\mathbf{N}_{k(V_{A,0})/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{b}$ and $L_{A,0}^\vee \cong \mathfrak{d}_{k(V_{A,0})}^{-1} L_{A,0}$.

Remark 3.1. If we start with $k = \mathbf{Q}(\sqrt{d})$ an imaginary quadratic field in our setup, each of the anisotropic subspaces $(V_{A,0}, Q_{A,0}) = (V_{A,1}, -Q_{A,1}) = (V_{\mathfrak{a}}, -Q_{\mathfrak{a}})$ is a rational quadratic subspace of signature $(0, 2)$ associated to the imaginary quadratic field $k(V_{A,1}) = k$. In the same way, each of the anisotropic subspaces $(V_{A,0}, Q_{A,0}) = (V_{A,2}, Q_{A,2}) = (V_{\mathfrak{a}}, Q_{\mathfrak{a}})$ determines a rational quadratic subspace of signature $(0, 2)$ associated to the same imaginary quadratic field $k(V_{A,2}) = k$.

If on the other hand we start with $k = \mathbf{Q}(\sqrt{d})$ a real quadratic field, any subspace $(V_{A,0}, Q_{A,0}) \subset (V_A, Q_A)$ of signature $(0, 2)$ will also determine an imaginary quadratic field $k(V_{A,0})$. For instance, writing τ to denote the nontrivial automorphism of $\mathrm{Gal}(k/\mathbf{Q})$, we see that $U_{A,0} := V_{A,1}^\tau \oplus (V_{A,2}/V_{A,2}^\tau)$ with $Q_A|_{U_{A,0}}$ is a rational quadratic subspace (V_A, Q_A) of signature $(2, 0)$, and that $V_{A,0} = U_{A,0}$ with $Q_{A,0} = -Q_A|_{U_{A,0}}$ is a rational quadratic subspace V_A of signature $(0, 2)$ which determines an imaginary quadratic field $k(U_{A,0})$.

Each sublattice $L_{A,0} \subset L_A$ of signature $(0, 2)$ gives rise to a group scheme T_A over $\mathbf{Z}$ with functor of points $T_A(R) = (C^0(L_{A,0}) \otimes_{\mathbf{Z}} R)^\times$ for any $\mathbf{Z}$-algebra R. This gives a rank-two torus $T_A \otimes_{\mathbf{Z}} \mathbf{Q} = \mathrm{GSpin}(V_{A,0})$ which appears as a maximal subgroup of $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$. Writing $\Lambda_{A,0} = L_{A,0}^\perp \subset L_A$ for the complement of the lattice $L_{A,0}$, this maximal subgroup acts trivially on the corresponding subspace $V_{A,0}^\perp = \Lambda_{A,0} \otimes_{\mathbf{Z}} \mathbf{Q} \subset V_A$. Let $K_{A,0} = T_A(\mathbf{A}_f) \cap K_{L_A}$ denote the corresponding compact open subgroup of $T_A(\mathbf{A}_f) = \mathrm{GSpin}(V_{A,0})(\mathbf{A}_f)$.

Fixing an embedding $\mathcal{O}_{k(V_{A,0})} \subset \mathbf{C}$, we can view $V_{A,0}(\mathbf{R}) = V_{A,0} \otimes_{\mathbf{Q}} \mathbf{R}$ as an oriented hyperplane of $\mathbf{C}$, and hence as a point $z_{A,0} = V_{A,0}(\mathbf{R}) \subset V_A(\mathbf{R})$ in $D(V_A) \cong \mathcal{Q}(V_A) \cong \mathcal{H}(V_A) \cong \mathfrak{H}^2$. This makes $(T_A \otimes_{\mathbf{Z}} \mathbf{Q}, z_{A,0}) = (\mathrm{GSpin}(V_{A,0}), z_{A,0})$ a Shimura datum with reflex field $k(V_{A,0})$. The corresponding orbifold

$$\begin{aligned} Z(V_{A,0})(\mathbf{C}) &= T_A(\mathbf{Q})\backslash \{z_{A,0}\} \times T_A(\mathbf{A}_f)/K_{A,0} \\ &= \mathrm{GSpin}(V_{A,0})(\mathbf{Q})\backslash \{V_{0,A}(\mathbf{R})\} \times \mathrm{GSpin}(V_{A,0})(\mathbf{A}_f)/\left(\mathrm{GSpin}(V_{A,0})(\mathbf{A}_f) \cap K_{L_A}\right) \end{aligned}$$

can be viewed as the complex points of a zero-dimensional Shimura variety $Z(V_{A,0}) \longrightarrow \mathrm{Spec}(k(V_{A,0}))$, or a complex fibre on the moduli stack of elliptic curves with complex multiplication by $\mathcal{O} \cong C^0(L_{A,0}) \subset \mathcal{O}_{k(V_{A,0})}$ and $\Gamma_0(N)$-level structure.

3.4.5. *Geodesic sets.* Let $W_A \subset V_A$ be any Lorentzian quadratic subspace of signature $(1, 1)$, with lattice $M_A = W_A \cap L_A$. Note that the complement $N_A = M_A^\perp \subset L_A$, also determines a Lorentzian subspace $U_A = N_A \otimes_{\mathbf{Z}} \mathbf{Q} \subset V_A$ of signature $(1, 1)$. We consider the corresponding domain

$$D(W_A) = D^{\pm}(W_A) = \{\mathfrak{y} = [\alpha, \beta] \subset W_A(\mathbf{R}) : \dim(\mathfrak{y}) = 1, \text{orientation } \pm, Q_A|_{W_A}(\mathfrak{y}) < 0\}$$

of oriented hyperbolic lines $\mathfrak{y} = [\alpha, \beta] \equiv [\alpha : \beta] \in \mathbf{P}^1(\mathbf{R})$, given equivalently as a space of projective lines

$$D(W_A) = D^{\pm}(W_A) = \left\{\mathfrak{y} = [\alpha : \beta] \subset \mathbf{P}^1(\mathbf{R}) : \text{orientation } \pm, Q_A|_{W_A}(\alpha, \beta) < 0\right\}.$$

Recall that after fixing an oriented basis $z = [x, y]$ of each negative definite hyperplane $z \subset V_A(\mathbf{R})$, and fixing a Witt decomposition $V_A = W_A \oplus \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$ corresponding to W_A, we have identifications

$$D^{\pm}(V_A) \cong \mathcal{Q}^{\pm}(V_A) \cong \mathcal{H}^{\pm}(V_A), \quad z = [x, y] \mapsto [w(z) = x + iy] \mapsto \mathfrak{z}(w) = \mathfrak{z}(w(z)).$$

Here, we write the corresponding Witt decomposition of a point $w \in V_A(\mathbf{C})$ as $w = \mathfrak{z}(w) + a(w)e_1 + b(w)e_1$. Note that while the point $w(\mathfrak{y}) = \alpha + i\beta \in \mathbf{C}$ determined by a hyperbolic line $\mathfrak{y} = [\alpha : \beta] \in \mathbf{P}^1(\mathbf{R})$ does not lie in the upper-half plane $\mathfrak{H}$, the roots of the quadratic polynomial $\mathrm{Q}_A|_{W_A}(X, 1) = 0$ (or $Q_A|_{W_A}(1, Y) = 0$) determine endpoints of a geodesic arc in $\mathfrak{H}$. For this reason, we shall sometimes call the corresponding sets

$$\mathcal{G}(W_A) := \mathrm{GSpin}(W_A)(\mathbf{Q})\backslash D^{\pm}(W_A) \times \mathrm{GSpin}(W_A)(\mathbf{A}_f)/\overline{K}_A, \quad \overline{K}_A := K_A \cap \mathrm{GSpin}(W_A)(\mathbf{A}_f)$$

geodesic sets associated to the Hilbert modular surface $X_A \cong Y_0(N) \times Y_0(N)$. We shall later see that the summation of automorphic Green's functions associated to certain linear combinations of the special divisors $Z_A(\mu, m)$ gives information about central derivative values of certain Rankin-Selberg L-functions. Hence, we have reason to believe that these geodesic sets are related to (Hirzebruch-Zagier) special arithmetic divisors on $X_A \cong Y_0(N) \times Y_0(N)$. In fact, clarifying such a relation forms the main motivation for this note.

4. Green's functions for special divisors

We describe the automorphic Green's functions that can be constructed from regularized theta lifts for the special divisors $Z(\mu, m)$. We start with the general setting, following [1], [40], [7], [13], and [54], then specialize to the case of Hilbert modular surfaces parametrized by the rational quadratic spaces V_A of signature $(2, 2)$.

4.1. Siegel theta functions.

Fix (V, Q) a rational quadratic space of signature $(n, 2)$, with maximal lattice $L \subset V$. We write $L^{\vee}/L$ for the discriminant group, and $\mathfrak{S}_L$ the finite-dimensional space of $\mathbf{C}$-valued functions on $L^{\vee}/L$. Writing $\widetilde{\mathrm{SL}}_2$ for the two-fold metaplectic cover of SL_2, we consider the Weil representation

$$\omega_L : \widetilde{\mathrm{SL}}_2(\mathbf{Z}) \longrightarrow \mathfrak{S}_L,$$

which for $n \geq 1$ even factors through $\mathrm{SL}_2(\mathbf{Z})$ as $\omega_L : \mathrm{SL}_2(\mathbf{Z}) \to \mathfrak{S}_L$. We define the conjugate action $\overline{\omega}_L$ by $\overline{\omega}_L(\gamma)\Phi = \overline{\omega_L(\gamma)\overline{\Phi}}$, and write $\omega_L^{\vee}$ for the contragredient action of $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ on the complex linear dual $\mathfrak{S}_L^{\vee}$.

We now describe how for each $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)/K_L$, we can use ω_L to construct a Siegel theta function

$$\theta_L(\tau, z) : \mathfrak{H} \times D(V) \longrightarrow \mathfrak{S}_L^{\vee},$$

which in the variable $z \in D(V) = D^{\pm}(V)$ is Γ_h-invariant, and in the variable $\tau = u + iv \in \mathfrak{H}$ transforms as a nonholomorphic modular form of weight $\frac{n}{2} - 1$ and representation $\omega_L^{\vee}$. We give the precise definition in (15).

4.1.1. *Theta kernels.*

To give a more precise account of the Weil representation for later constructions of theta series, let $\psi = \otimes_v \psi_v$ denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, which has archimedean component $\psi_\infty(x) = e(x) = \exp(2\pi i x)$ for $x \in \mathbf{R} \cong \mathbf{Q}_\infty$.

Recall that the two-fold metaplectic cover $\widetilde{\mathrm{SL}}_2$ fits into a short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\mathrm{SL}}_2 \longrightarrow \mathrm{SL}_2 \longrightarrow 1,$$

and that the general spin group $\mathrm{GSpin}(V)$ fits into a short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

Both groups act on the space of Schwartz-Bruhat functions $\Phi = \otimes_v \Phi_v \in \mathcal{S}(V(\mathbf{A}_f))$ by the Weil representation

$$\omega_L = \omega_{L,\psi} : \widetilde{\mathrm{SL}}_2(\mathbf{A}) \times \mathrm{GSpin}(V)(\mathbf{A}) \longrightarrow \mathcal{S}(V(\mathbf{A})).$$

This gives a natural theta kernel, defined on $g \in \widetilde{\mathrm{SL}}_2(\mathbf{A})$, $h \in \mathrm{GSpin}(V)(\mathbf{A})$, and $\Phi = \otimes_v \Phi_v \in \mathcal{S}(V(\mathbf{A}))$ by

$$\vartheta_L(g, h; \Phi) = \sum_{x \in V(\mathbf{Q})} (\omega_L(g, h)\Phi)(x). \tag{11}$$

This function $\vartheta_L(g, h; \Phi)$ is seen by inspection to be left $\mathrm{GSpin}(V)(\mathbf{Q})$-invariant, and by Poisson summation to be left $\widetilde{\mathrm{SL}}_2(\mathbf{Q})$-invariant. It is referred to as the *theta kernel associated to the Weil representation* ω_L.

4.1.2. *Choice of local Schwartz functions.* Following [3], [6], and [13, §2], we choose the following decomposable Schwartz functions $\Phi = \otimes_v \Phi_v \in \mathcal{S}(V(\mathbf{A}))$ to construct Siegel theta functions from the theta kernel (11).

We first define the following Gaussian function $\Phi_\infty \in \mathcal{S}(V(\mathbf{R}))$. Given a hyperplane $z \in D(V) = D^{\pm}(V)$, we define the corresponding majorant $(x,x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z)$, which can be viewed as a positive definite quadratic form on $V(\mathbf{R})$. We then define the Gaussian function

$$\Phi_\infty(x,z) = \exp\left(-(x,x)_z\right), \quad z \in D(V) = D^{\pm}(V),\ x \in V(\mathbf{R}). \tag{12}$$

As a function $x \in V(\mathbf{R})$, this determines an archimedean local Schwartz function $\Phi_\infty \in \mathcal{S}(V(\mathbf{R}))$. It satisfies the transformation property $\Phi_\infty(hx, hz) = \Phi_\infty(x,z)$ for all $h \in \mathrm{GSpin}(V)(\mathbf{R})$. It has weight $\frac{n}{2} - 1$ under the action of the maximal compact subgroup of $\widetilde{\mathrm{SL}}_2(\mathbf{R})$.

For the remaining finite part $\Phi_f = \otimes_{v<\infty} \Phi_v \in \mathcal{S}(V(\mathbf{A})_f)$, we shall later take the characteristic functions

$$\Phi_f = \mathbf{1}_\mu := \mathrm{char}\left(\mu + L \otimes \widehat{\mathbf{Z}}\right) \quad \text{for a coset } \mu \in L^\vee/L.$$

4.1.3. *Construction of Siegel theta functions.* Fix a basepoint $z_0 \in D(V) = D^{\pm}(V)$. For any finite archimedean Schwartz function $\Phi_f = \otimes_{v<\infty}\Phi_v \in \mathcal{S}(V(\mathbf{A})_f)$, we can define from (11) the theta function

$$\theta_L(g,h;\Phi_f) := \vartheta_L(g,h;\Phi_\infty(\cdot, z_0) \otimes \Phi_f(\cdot)). \tag{13}$$

We obtain a classical Siegel theta series on $\mathfrak{H} \times D(V)$ from this as follows. Given any oriented hyperplane $z = D(V) = D^{\pm}(V)$, we choose an element $h_z \in \mathrm{GSpin}(V)(\mathbf{R})$ for which $h_z z_0 = z$. Note that

$$\omega_L(h_z)\Phi_\infty(\cdot, z_0) = \Phi_\infty(\cdot, z).$$

Choosing $i \in \mathfrak{H}$ as the basepoint, let us for any $\tau = u + iv \in \mathfrak{H}$ write g_τ to denote the matrix

$$g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & \\ & v^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}),$$

and $\widetilde{g}_\tau = (g_\tau, 1)$ its image in $\widetilde{\mathrm{SL}}_2(\mathbf{R})$. Note that $\widetilde{g}_\tau \cdot i = \tau$. Via (13), we can then define the Siegel theta series

$$\theta_L(\tau, z, h_f; \Phi_f) = v^{-\frac{n}{4}+\frac{1}{2}} \vartheta_L(\widetilde{g}_\tau, h_z h_f; \Phi_\infty(\cdot, z_0) \otimes \Phi_f(\cdot)) = v^{-\frac{n}{4}+\frac{1}{2}} \sum_{x \in V(\mathbf{Q})} \omega_L(\widetilde{g}_\tau)\left(\Phi_\infty(\cdot, z) \otimes \omega(h_f)\Phi_f\right)(x)$$

for $\tau = u + iv \in \mathfrak{H}$, $z \in D(V) = D^{\pm}(V)$, $h_f \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, and $\Phi_f = \otimes_{v<\infty}\Phi_v \in \mathcal{S}(V(\mathbf{A}_f))$. Since

$$v^{-\frac{n}{4}+\frac{1}{2}} \omega_L(\widetilde{g}_\tau)\left(\Phi_\infty(\cdot, z)\right)(x) = ve\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right),$$

we have the more explicit expansion

$$\theta_L(\tau, z, h_f; \Phi_f) = v \sum_{x \in V(\mathbf{Q})} e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right) \otimes \Phi_f(h_f^{-1}x). \tag{14}$$

As explained for [13, (2.5)], this theta series satisfies a transformation property for $\widetilde{\mathrm{SL}}_2(\mathbf{Q})$. Viewing $\theta_L(\tau, z, h_f; \cdot)$ as a function on $\tau \in \mathfrak{H}$ taking values in the dual space $\mathcal{S}(V(\mathbf{A}_f))^\vee$ of $\mathcal{S}(V(\mathbf{A}_f))$, we see that $\theta_L(\tau, z, h_f; \cdot)$ determines a nonholomorphic modular form of weight $\frac{n}{2} - 1$ and representation $\omega_L^\vee$. In fact, it determines a harmonic weak Maass form $\theta_L(\tau, \cdot) \in H_{\frac{n}{2}-1}(\omega_L^\vee)$ in the sense of the definition given below.

Let $\mathfrak{S}_L$ denote the subspace of $\mathcal{S}(V(\mathbf{A}_f))$ which are supported on $L^\vee \otimes \widehat{\mathbf{Z}}$, and constant on cosets of $L \otimes \widehat{\mathbf{Z}}$. For instance, S_L contains the characteristic function $\mathbf{1}_\mu = \mathrm{char}(\mu + L \otimes \widehat{\mathbf{Z}})$ for a given coset $\mu \in L^\mu/L$. In fact, these functions form a basis for the space, and we have the decompositon

$$\mathfrak{S}_L = \bigoplus_{\mu \in L^\vee/L} \mathbf{C}\mathbf{1}_\mu \subset \mathcal{S}(V(\mathbf{A}_f)).$$

In particular, it follows that $\dim_{\mathbf{C}} \mathfrak{S}_L = |L^\vee/L|$ is finite. Writing $\mathfrak{e}_\mu$ for the standard basis element in $\mathbf{C}[L^\vee/L]$, we also have a natural identification $\mathfrak{S}_L \cong \mathbf{C}[L^\vee/L], \mathbf{1}_\mu \mapsto \mathfrak{e}_\mu$. This space $\mathfrak{S}_L$ is stable under the image of $\mathrm{SL}_2(\widehat{\mathbf{Z}})$ in $\widetilde{\mathrm{SL}}_2(\mathbf{A}_f)$. We define from (14) the corresponding $\mathfrak{S}_L$-valued Siegel theta series

$$\theta_L(\tau, z, h_f) = \sum_{\mu \in L^\vee/L} \theta_L(\tau, z, h_f; \mathbf{1}_\mu)\mathbf{1}_\mu \tag{15}$$

We refer to [13, §2] for more on these theta series, which coincide with those considered by Borcherds in [3].

4.2. **Harmonic weak Maass forms.** Fix a half-integer $l \in \frac{1}{2}\mathbf{Z}$. Recall that a twice-differentiable function $f : \mathfrak{H} \longrightarrow \mathfrak{S}_L$ is said to be a *harmonic weak Maass form of weight l and representation ω_L* if

(i) $f|_{l,\omega_L}\gamma = f$ for all $\gamma \in \Gamma = \mathrm{SL}_2\,\mathbf{Z}$, where $|_{l,\omega_L}$ denotes the Petersson weight-l operator.

(ii) There exists an $\mathfrak{S}_L$-value Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \geq 0} c_f^+(\mu, m) e(m\tau) \mathbf{1}_\mu, \quad \mathbf{1}_\mu := \mathrm{char}(\mu + L \otimes \widehat{\mathbf{Z}})$$

known as the *principal part of f* for which $f(\tau) = P_f(\tau) + O(e^{-\varepsilon v})$ for some $\varepsilon > 0$ as $v = \Im(\tau) \to \infty$.

(iii) The function is harmonic: $\Delta_l f = 0$ for Δ_l the hyperbolic Laplacian of weight l defined by

$$\Delta_l := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + il \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right), \quad \tau = u + iv \in \mathfrak{H}.$$

We write $H_l(\omega_L)$ to denote the $\mathbf{C}$-vector space of harmonic weak Maass forms of weight l and representation. Each harmonic weak Maass form $f \in H_l(\omega_L)$ has a unique decomposition $f = f^+ + f^-$ where

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(\mu, m) e(m\tau) \mathbf{1}_\mu$$

and

$$f^-(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_f^-(\mu, m) W_l(2\pi m v) e(m\tau) \mathbf{1}_\mu,$$

where $W_l(a) := \int_{-2a}^{\infty} e^{-t} t^{-l} dt = \Gamma(1-l, 2|a|)$ denotes the Whittaker function given by the partial Gamma function, and $e(\tau) = \exp(2\pi i \tau)$ for $\tau = u + iv \in \mathfrak{H}$. We call f^+ the *holomorphic part of f* and f^- the *non-holomorphic part of f*. We consider the subspace $M_l^!(\omega_L) \subset H_l(\omega_L)$ of *weakly holomorphic forms* whose poles are supported at the cusps, as well as the suspace of holomorphic forms $M_l(\omega_L) \subset M_l^!(\omega_L)$, and the subspace of holomorphic cusp forms $S_l(\omega_L) \subset M_l(\omega_L) \subset M_L^!(\omega_L) \subset H_l(\omega_L)$.

Recall that we have the Maass weight lowering operator L_l and the Maass weight raising operator R_l,

$$L_l := -2iv^2 \cdot \frac{\partial}{\partial \overline{\tau}}, \quad R_l := 2i \cdot \frac{\partial}{\partial \tau} + l \cdot v^{-1}. \tag{16}$$

Bruinier and Funke [10] define an antilinear differential operator

$$\xi_l : H_l(\omega_L) \longrightarrow S_{2-l}(\overline{\omega}_L), \quad f(\tau) \longmapsto v^{l-2}\overline{L_l f(\tau)}, \tag{17}$$

and show that it sits in a short exact sequence

$$0 \longrightarrow M_l^!(\omega_L) \longrightarrow H_l(\omega) \xrightarrow{\xi_l} S_{2-l}(\overline{\omega}_L) \longrightarrow 0$$

so that $\ker(\xi_l) = M_l^!(\omega_L)$. We refer to [10] and [13, §3] for more details and basic properties.

4.2.1. *Definition of the divisor $Z(f)$.* Given $f \in H_{1-\frac{n}{2}}(\omega_L)$, we define the corresponding divisor

$$Z(f) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f^+(\mu, -m) Z(\mu, m) \tag{18}$$

on $X_K = X_{K_L}$.

In the special case of the quadratic spaces (V_A, Q_A) of signature $(2, 2)$ with maximal lattices $L_A \subset V_A$, we consider for any $f_0 = f_{0,A} \in H_0(\omega_{L_A})$ the corresponding divisors on $X_A = X_{K_A} \cong Y_0(N)^2$ defined by

$$Z_A(f_0) = \sum_{\mu \in L_A^\vee/L_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_0}^+(\mu, -m) Z_A(\mu, m). \tag{19}$$

4.3. **Regularized theta lifts.** We describe the regularized theta lifts $\Phi(f, z, h)$ associated to $f \in H_{1-\frac{n}{2}}(\omega_L)$.

4.3.1. *The tautological pairing.* Let $\langle\langle\cdot,\cdot\rangle\rangle : \mathfrak{S}_L \times \mathfrak{S}_L^\vee \longrightarrow \mathbf{C}$ denote the tautological pairing. Hence, given

$$f(\tau) = \sum_{\mu \in L^\vee/L} f_\mu(\tau)\mathbf{1}_\mu \in H_l(\omega_L) \quad \text{and} \quad g(\tau) = \sum_{\mu \in L^\vee/L} g_\mu(\tau)\mathbf{1}_\mu \in H_l(\omega_L^\vee),$$

we have

$$\langle\langle f(\tau), g(\tau)\rangle\rangle = \sum_{\mu \in L^\vee/L} f_\mu(\tau)g_\mu(\tau).$$

4.3.2. *Regularized theta integrals.* Given $f \in H_{1-\frac{n}{2}}(\omega_L)$ a harmonic weak Maass form of weight $1 - \frac{n}{2}$ and representation ω_L, we define the corresponding regularized theta lift $\Phi(f, z, h)$ for $z \in D(V) = D^\pm(V)$ and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by the regularized theta integral

$$\Phi(f, z, h) = \int_{\mathcal{F}}^\star \langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle d\mu(\tau) = \mathrm{CT}_{s=0}\left\{\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right\}.$$

Here, we write $\mathcal{F} = \{\tau \in \mathfrak{H} : -1/2 \leq \Re(\tau) \leq 1/2, \tau\overline{\tau} \geq 1\}$ to denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$, and $\mu(\tau) = \frac{dudv}{v^2}$ the Poincaré measure on $\mathfrak{H}$. The regularized theta integral $\Phi(f, z, h)$ is given by the constant term in the Laurent series around $s = 0$ of the function

$$\lim_{T\to\infty}\int_{\mathcal{F}_T}\langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau),$$

where the limit is taken over truncated domains $\mathcal{F}_T = \{\tau \in \mathfrak{H} : -1/2 \leq \Re(\tau) \leq 1/2, \tau\overline{\tau} \geq 1, \Im(\tau) \leq T\}$.

4.3.3. *Arithmetic automorphic forms and Petersson norms.* Let $\mathcal{L}_{D(V)}$ be the restriction to $D(V) \cong \mathcal{Q}(V)$ of the tautological bundle on $\mathbf{P}(V(\mathbf{C}))$. The natural action of $\mathrm{O}(V)(\mathbf{R})$ on $V(\mathbf{C})$ induces one of the connected component of the identity $\mathrm{GSpin}(V)(\mathbf{R})^0 \subset \mathrm{GSpin}(V)(\mathbf{R})$ on $\mathcal{L}_{D(V)}$. This gives a holomorphic line bundle

$$\mathcal{L} = \mathrm{GSpin}(V)(\mathbf{Q})\backslash\mathcal{L}_{D(V)} \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K \longrightarrow X_K,$$

which is known to have a canonical model defined over $\mathbf{Q}$ by work of Harris [27], cf. [40, §1], [28]. Note that on the component $\Gamma_h\backslash D(V)$, it takes the form $\Gamma_h\backslash\mathcal{L}_{D(V)}$. We define a hermitian metric $h_{\mathcal{L}_{D(V)}}$ on $\mathcal{L}_{D(V)}$ by

$$h_{\mathcal{L}_{D(V)}}(w_1, w_2) = \frac{1}{2}\cdot(w_1, \overline{w}_2).$$

Observe that this metric is fixed by the action of $\mathrm{O}(V)(\mathbf{R})$, and hence descends to $\mathcal{L}$.

We now describe the Petersson inner product on sections of $\mathcal{L}^{\otimes l}$ for $l \in \mathbf{Z}$ any integer. Fix a Witt decomposition $V = W \oplus \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$ for basis vectors e_1, e_2 satisfying $(e_1, e_2) = 1$ and $(e_1, e_1) = (e_2, e_2) = 0$, so that $W = V \cap e_1^\perp \cap e_2^\perp$ determines a rational quadratic subspace of signature $(n-1, 1)$. Given any vector $w \in V(\mathbf{C})$, we then write the corresponding decomposition for the Witt decomposition as $w = \mathfrak{z} + ae_1 + be_2$. Note that $D(V) \cong \mathcal{Q}(V)$ is isomorphic to the tube domain

$$\mathcal{H}(V) = \left\{\mathfrak{z} \in W(\mathbf{C}) : \Im(\mathfrak{z}) \in C^-(V)\right\}, \quad C^-(V) := \{y \in W : (y, y) < 0\}$$

via

$$\mathcal{H}(V) \longrightarrow V(\mathbf{C}) = W(\mathbf{C}) + \mathbf{C}e_1 + \mathbf{C}e_2 \longrightarrow \mathcal{Q}(V), \quad \mathfrak{z} \longmapsto w(\mathfrak{z}) := \mathfrak{z} + e_1 - Q(\mathfrak{z})e_2 \longmapsto [w(\mathfrak{z})].$$

The map $\mathfrak{z} \longmapsto w(\mathfrak{z}) := \mathfrak{z} + e_1 - Q(\mathfrak{z})_2$ can be viewed as a nowhere vanishing holomorphic section of $\mathcal{L}_{D(V)}$. Observe that this section has norm for the hermitian metric $h_{\mathcal{L}_{D(V)}}$ given by

$$||w(\mathfrak{z})||^2 = -\frac{1}{2}\cdot(w(\mathfrak{z}), w(\overline{\mathfrak{z}})) = -\left(\Im(\mathfrak{z}), \Im(\mathfrak{z})\right) = -(y, y) =: |y|^2.$$

Let us now write $z = [x, y] \in D(V) = D^\pm(V)$ for the basis $[x, y]$ of an oriented hyperplane $z \subset V(\mathbf{R})$ $w(z) := x + iy$, and $[w(z)]$ its image in $\mathcal{Q}(V) = \mathcal{Q}^\pm(V) \cong D^\pm(V)$. Given $h \in \mathrm{GSpin}(V)(\mathbf{R})$, we have

$$h \cdot w(z) = w(hz)j(h, z)$$

for a holomorphy factor

$$j : \mathrm{GSpin}(V)(\mathbf{R}) \times D(V) \longrightarrow \mathbf{C}.$$

In this way, we can identify the holomorphic sections of $\mathcal{L}^{\otimes l}$ with functions

$$\Psi : D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f) \longrightarrow \mathbf{C}$$

satisfying the transformation properties

- $\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$
- $\Psi(\gamma z, \gamma h) = j(\gamma, z)^l \Psi(z, h)$ for all $\gamma \in \mathrm{GSpin}(V)(\mathbf{Q})$.

The norm of the section

$$(z, h) \longmapsto \Psi(z, h) \cdot w(z)^{\otimes l}$$

corresponding to any such function Ψ is given by

$$||\Psi(z, h)||^2 = |\Psi(z, h)|^2 |y|^{2l},$$

and referred to as the *Petersson norm of* Ψ.

4.3.4. *Borcherd's products and automorphic Green's functions.* We now summarize several important results.

Theorem 4.1 (Borcherds)**.** *Let $f \in M^!_{1-\frac{n}{2}}(\omega_L)$ be a weakly holomorphic form with Fourier series expansion*

$$f(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f(\mu, m) e(m\tau) \mathbf{1}_\mu, \quad c_f(\mu, m) \in \mathbf{Z}.$$

Then,

$$\Phi(f, z, h) = -2 \log |\Psi(f, z, h)|^2 - c_f^+(0, 0) \cdot (2 \log |y| + \Gamma'(1)) \tag{20}$$

for $\Psi(f, z, h)$ a meromorphic modular form on $D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)$ of weight $\frac{1}{2} \cdot c_f^+(0, 0)$ with divisor

$$\mathrm{Div}\left(\Psi(f, \cdot)^2\right) = Z(f) := \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_f^+(\mu, m) Z(\mu, m).$$

Proof. See [3, Theorem 13.3] with [40, Theorems 1.2 and 1.3] and relevant discussions in Bruiner [6], [7]. □

Theorem 4.2 (Borcherds/Bruinier)**.** *Let $f \in H_{1-\frac{n}{2}}(\omega_L)$ be any harmonic weak Maass form of weight $1 - \frac{n}{2}$ and representation ω_L. The regularized theta lift $\Phi(f, \cdot)$ is an automorphic Green's function in the sense of Arakelov theory for the divisor $Z(f)$ on X_K. That is, $\Phi(f, z, h)$ satifies the following characterizing properties:*

(i) *$\Phi(f, z, h)$ is a smooth function on $X_K \backslash Z(f)$ with a logarithmic singularity along $-2 \log Z(f)$.*

(ii) *The $(1, 1)$-form $dd^c \Phi(f, z, h)$ has a smooth extension to all of X_K, and satisfies the Green's current equation $dd^c[\Phi(f, z, h)] = \delta_{Z(f)} + [dd^c \Phi(f, z, h)]$ for $\delta_{Z(f)}$ the Dirac current for $Z(f)$.*

(iii) *$\Phi(f, z, h)$ is an eigenvector for the generalized Laplacian Δ_z on $z \in D(V)$, and more precisely*

$$\Delta_z \Phi(f, z, h) = \frac{n}{4} \cdot c_f^+(0, 0) \cdot \Phi(f, z, h).$$

Proof. See [13, Theorem 4.3] and more generally [6]. □

As explained in [28, §1.1], the Shimura variety $X_K = X_{K_L}$ comes equipped with a metrized line bundle

$$\mathcal{L} \in \widehat{\mathrm{Pic}}(X_K)$$

of weight one modular forms, which has an extension to the integral model $\widehat{\mathrm{Pic}}(\mathcal{X}_K)$.

Theorem 4.3 (Borcherds/Howard-Madapusi Pera)**.** *Let $f \in M^!_{1-\frac{n}{2}}(\omega_L)$ be a weakly holomorphic form whose principal part has integral Fourier coefficients,*

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(\mu, m) e(m\tau) \mathbf{1}_\mu, \quad c_f^+(\mu, m) \in \mathbf{Z}.$$

Replacing f by a suitable integer multiple if needed, there exists a rational section $\Psi(f)$ of the line bundle $\mathcal{L}^{c_f^+(0,0)}$ on $\mathcal{X}_K$ whose norm under the metric defined by

$$||z|| = \frac{(z, \overline{z})}{4\pi e^{\gamma}}, \quad (z, \overline{z}) = Q(z, \overline{z}) - Q(z) - Q(\overline{z})$$

satisfies the relation

$$-2\log||\Psi(f)|| = \Phi(f)$$

and hence

$$\mathrm{Div}(\Psi(f)) = \sum_{\mu \in L^{\vee}/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f^+(\mu, -m) Z(\mu, m).$$

In particular, the Borcherds product $\Phi(f)$ is defined over $\mathbf{Q}$, and takes algebraic values. To be more precise, $\Phi(f, z, h)$ takes values in the algebraic number field to which the point $(z, h) \in X_K$ belongs.

Proof. See [28, Theorem 9.1.1], which refines the original theorem of Borcherds [3] (cf. [28, Theorem 5.2.2]). □

4.4. **Hejhal Poincaré series and Green's functions of special divisors.** For future reference, we now describe the automorphic Green's functions $\Phi_{\mu,m}(z, h) = \Phi(F_{\mu,m}, z, h)$ for each of the special divisors $Z(\mu, m)$ on $X = X_K$ following [6] (cf. [9, §4]). This appears in the discussion leading to Corollary 4.7 below. We then describe the setup more explicitly for the case of Hilbert modular surfaces corresponding to $n = 2$, leading to classical higher Green's functions on $X_0(1) \times X_0(1)$ and more generally $X_0(N) \times X_0(N)$.

4.4.1. *Hejhal Maass-Poincaré series.* We follow the discussion in [6, §1.3]. Hence, for complex numbers $\alpha, \beta, z \in \mathbf{C}$, we consider the standard Whittaker functions $W_{\alpha,\beta}(z)$ and $M_{\alpha,\beta}(z)$ giving linearly independent solutions of the Whittaker differential equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\alpha}{z} - \frac{\beta^2 - 1/4}{z^2}\right) w = 0.$$

Note that these functions are related by

$$W_{\alpha,\beta}(z) = \frac{\Gamma(-2\beta)}{\Gamma(\frac{1}{2} - \beta - \alpha)} \cdot M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma(\frac{1}{2} + \beta - \alpha)} \cdot M_{\alpha,-\beta}(z),$$

from which it follows by inspection $W_{\alpha,\beta}(z) = W_{\alpha,-\beta}(z)$. As $z \to 0$, these functions behave as

$$\begin{aligned} M_{\alpha,\beta}(z) &\sim z^{\beta + \frac{1}{2}} & \beta &\notin -\frac{1}{2}\mathbf{Z}_{>0} \\ W_{\alpha,\beta}(z) &\sim \frac{\Gamma(2\beta)}{\Gamma(\beta - \alpha + \frac{1}{2})} \cdot z^{-\beta + \frac{1}{2}} & \beta &\geq \frac{1}{2}. \end{aligned}$$

As $y = \Im(z) \to \infty$, they behave as

$$\begin{aligned} M_{\alpha,\beta}(y) &= \frac{\Gamma(1 + 2\beta)}{\Gamma(\beta - \alpha + \frac{1}{2})} \cdot e^{\frac{y}{2}} \cdot y^{-\alpha} \left(1 + O(y^{-1})\right) \\ W_{\alpha,\beta}(y) &= e^{-\frac{y}{2}} \cdot y^{\alpha} \left(1 + O(y^{-1})\right). \end{aligned}$$

Let $l = 1 - n/2$. Given $s \in \mathbf{C}$ and $y \in \mathbf{R}_{>0}$, we define the normalized functions

$$\mathcal{M}_s(y) = y^{-\frac{l}{2}} M_{-\frac{l}{2}, s - \frac{1}{2}}(y), \quad \mathcal{W}_s(|y|) = |y|^{-\frac{l}{2}} W_{\frac{l}{2}\,\mathrm{sgn}(y), s - \frac{1}{2}}(|y|).$$

Note that these functions are both holomorphic in s, and also that we have the identities

$$\mathcal{M}_{\frac{l}{2}}(y) = y^{-\frac{l}{2}} M_{-\frac{l}{2}, \frac{l}{2} - \frac{1}{2}}(y) = e^{\frac{y}{2}}, \quad \mathcal{W}_{1 - \frac{l}{2}}(y) = y^{-\frac{l}{2}} W_{\frac{l}{2}, \frac{1}{2} - \frac{l}{2}}(y) = e^{-\frac{y}{2}}.$$

Recall that we consider the weight l hyperbolic Laplacian operator Δ_l, defined on $\tau = u + iv \in \mathfrak{H}$ by

$$\Delta_l = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ilv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

This operator acts on smooth functions $f : \mathfrak{H} \longrightarrow \mathbf{C}[L^\vee/L]$ component-wise. Recall as well that we have the Petersson slash operator $|_l$ acting on such functions. To be more precise, for any generic element (M, ζ) in the two-fold metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ of $\mathrm{SL}_2(\mathbf{Z})$, this operator acts on such a function f via the rule

$$(f|_l(M,\zeta))(\tau) = \zeta(\tau)^{-2l}\omega_L(M,\zeta)^{-1}f(M\tau). \tag{21}$$

Note that the actions of Δ_l and $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ via (21) on any smooth function $f : \mathfrak{H} \to \mathbf{C}[L^\vee/L]$ commute in that

$$\Delta_l(f|_l(M,\zeta)) = (\Delta_l f)|_l(M,\zeta). \tag{22}$$

Given a coset $\mu \in L^\vee/L$ with characteristic function $\mathbf{1}_\mu$ as above, and a negative integer $m \in \mathbf{Z} + Q(\mu)$, is it not hard to check that the function defined on $\tau = u + iv \in \mathfrak{H}$ by

$$\mathcal{M}_s(4\pi|m|v)\mathbf{1}_\mu \tag{23}$$

is invariant under the action of the unipotent generator

$$T = \left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, 1 \right) \in \widetilde{\mathrm{SL}}_2(\mathbf{Z})$$

via the slash operator (21). Remarkably, this function (23) is an eigenfunction of the Laplacian Δ_l, with eigenvalue $s(1-s) + (l^2 - 2l)/4$. We can take sums of these functions to obtain the following Poincaré series.

Definition 4.4 (Hejhal Maass Poincaré series)**.** *Fix an even quadratic lattice $L = (L, Q)$ of signature $(n, 2)$, and let $l = 1 - n/2$. Given any complex number $s \in \mathbf{C}$, coset $\mu \in L^\vee/L$, and integer $m \in \mathbf{Z} + Q(\mu)$, let $F_{\mu,m}(\tau, s)$ denote the Poincaré series defined on $\tau = u + iv \in \mathfrak{H}$ by the summation*

$$F_{\mu,m}(\tau,s) = F^L_{\mu,m}(\tau,s) = \frac{1}{\Gamma(2s)} \sum_{(M,\zeta)\in\widetilde{\Gamma}_\infty\backslash\widetilde{\mathrm{SL}}_2(\mathbf{Z})} \left[\mathcal{M}_s(4\pi|m|v)\mathbf{1}_\mu e(mu)\right]|_l(M,\zeta),$$

where $\widetilde{\Gamma}_\infty = (\Gamma_\infty, 1) \subset \widetilde{\mathrm{Sp}}_2(\mathbf{Z})$ denotes the image of the unipotent subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbf{Z} \right\} \in \mathrm{SL}_2(\mathbf{Z})$$

in the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$. This series converges normally for $\tau = u + iv \in \mathfrak{H}$ and $s = \sigma + it \in \mathbf{C}$ with $\sigma > 1$. Via the commutativity of the actions of Δ_l and $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ described in (22), we deduce that the Poincaré series $F_{\mu,m}(\tau, s)$ is an eigenfunction for the Laplacian Δ_l, with

$$\Delta_l F_{\mu,m}(\tau,s) = \left(s(1-s) + (l^2-2l)/4\right) F_{\mu,m}(\tau,s).$$

In particular, $F_{\mu,m}(\tau, 1 - l/2) = F_{\mu,m}(\tau, s)|_{s=1-l/2}$ determines an eigenfunction of eigenvalue zero for Δ_l.

Note that the Fourier series expansion of each $F_{\mu,m}(\tau, s)$ is computed in [6, Theorem 1.9], with simplifications for the special case of $F_{\mu,m}(\tau, 1 - l/2)$ detailed in [6, Proposition 1.10]. Here, we note that the Fourier series expansion of each of the latter functions $F_{\mu,m}(\tau, 1 - l/2)$ can be described crudely for our purposes as

$$F_{\mu,m}(\tau, 1-l/2) = \mathbf{1}_\mu e(m\tau) + \mathbf{1}_{-\mu} e(m\tau) + O(1). \tag{24}$$

4.4.2. *Decompositions of cuspidal harmonic weak Maass forms.* We now explain how the harmonic weak Maass forms we consider above can be decomposed into linear combinations of Hejhal Maass Poincaré series.

Proposition 4.5. *Let $L = (L, Q)$ be a even lattice of signature $(n, 2)$, and put $l = 1-n/2$. Let $f \in H_l^{\mathrm{cusp}}(\overline{\omega}_L)$ be any cuspidal harmonic weak Maass form of weight l and representation $\overline{\omega}_L$, with holomorphic part*

$$f^+(\tau) = \sum_{\mu\in L^\vee/L} \sum_{\substack{m\in\mathbf{Z}+Q(\mu)\\ m<0}} c_f^+(\mu,m)e(mu)\mathbf{1}_\mu.$$

Then, we have the decomposition

$$f(\tau) = \frac{1}{2} \sum_{\mu\in L^\vee/L} \sum_{\substack{m\in\mathbf{Z}+Q(\mu)\\ m<0}} c_f^+(\mu,m)F_{\mu,m}(\tau,1-l/2).$$

Proof. We give a minor variation of [6, Proposition 1.12]. Consider the form $g(\tau)$ defined by the difference

$$g(\tau) = f(\tau) - \frac{1}{2} \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Z}+Q(\mu) \\ m<0}} c_f^+(\mu, m) F_{\mu,m}(\tau, 1 - l/2).$$

Note that this form $g(\tau)$ is invariant under the action of

$$Z = \left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, i \right) \in \widetilde{\mathrm{SL}}_2(\mathbf{Z}),$$

and hence $c_f^+(\mu, m) = c_f^+(-\mu, m)$ for all $\mu \in L^\vee/L$ and $m \in \mathbf{Z} + Q(\mu)$. Using these properties together with the expansion (24), we deduce that $g(\tau)$ must be bounded as $\Im(\tau) \to \infty$. On the other hand, it is apparent that $g|_l(M, \zeta) = g$ for all $(M, \zeta) \in \widetilde{\mathrm{SL}}_2(\mathbf{Z})$ with $\Delta_l g = 0$. It follows that $g(\tau)$ must be identically zero. □

4.4.3. *Unfolding against the Maass-Poincaré series.* We now explain how to compute the regularized theta lifts $\Phi_{\mu,m}(z, h)$ associated to the Poincaré series $F_{\mu,m}(\tau, s)$ against the Siegel theta series $\theta_L(\tau, z)$ defined above[2]. Recall we start with $L = (L, Q)$ an even full-rank lattice in V of signature $(n, 2)$. We consider the Siegel theta series $\theta_L(\tau, z) \in M_{\frac{n}{2}-1}(\omega_L)$, defined for $z \in D(V)$, $\tau = u + iv \in \mathfrak{H}$, and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by

$$\theta_L(\tau, z, h) = v \sum_{\lambda \in L^\vee} e\left(\tau Q(\lambda_z) + \overline{\tau} Q(\lambda_{z^\perp})\right) \mathbf{1}_{h^{-1}(\lambda + L^\vee)},$$

which we decompose into coset components as

$$\theta_L(\tau, z, h) = \sum_{\mu \in L^\vee/L} \theta_\mu(\tau, z, h) \mathbf{1}_\mu, \quad \theta_\mu(\tau, z, h) := \sum_{\gamma \in h(\lambda + L)} e\left(\tau Q(\lambda_z) + \overline{\tau} Q(\lambda_{z^\perp})\right).$$

Here, we write λ_z to denote the orthogonal projection of λ to z, with $\lambda_{z^\perp}$ the orthogonal projection to the complement $z^\perp \subset D(V)$. Note that the Fourier series expansion of each constituent theta series is given by

$$\theta_\mu(\tau, z, h) = \sum_{\lambda \in h(\mu + L)} e\left(-2\pi Q(\lambda_z) + 2\pi v Q(\lambda_{z^\perp})\right) e(Q(\lambda) u),$$

which via the elementary identity

$$Q_0(\gamma_z) + Q_0(\gamma_{z^\perp}) = Q(\gamma) \implies 2\pi v Q(\gamma_{z^\perp}) = 2\pi v Q(\gamma) - 2\pi Q(\gamma_z)$$

is the same as

$$\begin{aligned} \theta_\mu(\tau, z, h) &= \sum_{\lambda \in h(\mu + L)} \exp\left(2\pi v Q_1(\lambda_1)\right) \exp\left(-4\pi v Q(\lambda_z) + 2\pi v Q(\lambda)\right) e(Q(\lambda) u) \\ &= \sum_{\lambda \in h(\mu + L)} \exp\left(2\pi v Q(\lambda) - 4\pi v Q(\lambda_z)\right) e(Q(\lambda) u) \end{aligned}$$

and hence

$$\overline{\theta_\mu(\tau, z)} = \sum_{\lambda \in h(\mu + L)} \exp\left(2\pi v Q(\lambda) - 4\pi v Q_0(\lambda_z)\right) e(-Q(\lambda) u). \tag{25}$$

Theorem 4.6. *Let $L = (L, Q)$ be an even lattice of signature $(n, 2)$. Fix any coset $\mu \in L^\vee/L$ and negative integer $m \in \mathbf{Z} + Q(\mu)$. Then, for any point $z \in D(V)$ which is not contained in the set*

$$Z(\mu, m) = \bigcup_{\substack{\lambda \in \mu + L \\ Q(\lambda) = m}} \lambda^\perp$$

[2] Note that Bruinier [6, §2.2] considers the isomorphic lattice $-L = (L, -Q)$ of signature $(2, n)$ for this discussion, and hence that we have to alter signs of weights accordingly to work with the quadratic lattice (L, Q) directly. In particular, his signature $(2, n) = (b^+, b^-)$ will correspond to our signature $(n, 2)$ in the unfolding calculations below.

and element $h \in \mathrm{Spin}(V)(\mathbf{A}_f)$ together with any $s, w \in \mathbf{C}$ satisfying $\Re(s) > -l/2 + \Re(w) = n/4 + 1/2 + \Re(w)$, the regularized theta lift defined[3] *according to* [6, Proposition 2.11] *by the limit of truncated integrals*

$$\Phi_{\mu,m}(z,s,h;w) := \lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle\langle F_{\mu,m}(\tau,s), \overline{\theta_L(\tau,z,h)}\rangle\rangle v^w \frac{dudv}{v^2}$$

is given by the explicit (absolutely convergent) formula

$$\Phi_{\mu,m}(z,s,h;w) = \frac{2(4\pi m)^{s+\frac{n}{4}-\frac{1}{2}}\Gamma\left(\frac{n}{4}-\frac{1}{2}+s+w\right)}{\Gamma(2s)} \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \frac{{}_2F_1\left(s+\frac{n}{4}-\frac{1}{2}+w, s-\frac{n}{4}+\frac{1}{2}, 2s; \frac{m}{Q(\lambda_{z_0^\perp})}\right)}{(4\pi Q(\lambda_{z^\perp}))^{s+\frac{n}{4}-\frac{1}{2}+w}}.$$

In particular, at $w=0$, we obtain for $\Re(s) > n/4 + 1/2$ the formula

$$\Phi_{\mu,m}(z,s,h;0) = \frac{2\Gamma\left(s+\frac{n}{4}-\frac{1}{2}\right)}{\Gamma(2s)} \sum_{\substack{\lambda\in h(\mu+L)\\ Q(\lambda)=m}} \left(\frac{m}{Q(\lambda_{z^\perp})}\right)^{s+\frac{n}{4}-\frac{1}{2}} {}_2F_1\left(s+\frac{n}{4}-\frac{1}{2}, s-\frac{n}{4}+\frac{1}{2}, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right).$$

Proof. See Bruinier [6, Theorem 2.14]. We present a slight generalization to help make our later deductions more explicit. Fix $s \in \mathbf{C}$ with $\Re(s) > n/4 + 1/2 + \Re(w)$. Fix a point $z \in D(V) \backslash Z(\mu, m)$. We have that

$$\Phi_{\mu,m}(z,s,h;w) = \lim_{T\to\infty} I_T(\mu,m,s,h;w),$$

where

$$\begin{aligned} I_T(\mu,m,s,h;w) &= \int\limits_{\mathcal{F}_T} \langle\langle F_{\mu,m}(\tau,s), \theta_L(\tau,z,h)\rangle\rangle v^w \frac{dudv}{v^2} \\ &= \frac{1}{\Gamma(2s)} \int\limits_{\mathcal{F}_T} \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbf{Z})} \langle\langle [\mathcal{M}_s(4\pi|m|v)\mathbf{1}_\mu e(mu)]\,|_l(M,1), \overline{\theta_L(\tau,z,h)}\rangle\rangle v^w \frac{dvdu}{v^2}, \end{aligned}$$

which via the transformation of the theta series is the same as

$$\begin{aligned} I_T(\mu,m,s,h;w) &= \frac{1}{\Gamma(2s)} \int\limits_{\mathcal{F}_T} \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbf{Z})} \mathcal{M}_s(4\pi|m|\Im(M\tau))\Im(M\tau)^w \langle\langle \mathbf{1}_\mu e(m\Re(M\tau)), \overline{\theta_L(\tau,z,h)}\rangle\rangle \frac{dudv}{v^2} \\ &= \frac{2}{\Gamma(2s)} \int\limits_{\mathcal{F}_T} \mathcal{M}_s(4\pi|m|v)v^w e(mu)\overline{\theta_\mu(\tau,z,h)}\frac{dudv}{v^2} \\ &+ \frac{1}{\Gamma(2s)} \int\limits_{\mathcal{F}_T} \sum_{M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbf{Z}), c\neq 0} \mathcal{M}_s(4\pi|m|\Im(M\tau))\Im(M\tau)^w e(m\Re(M\tau))\overline{\theta_\mu(\tau,z,h)}\frac{dudv}{v^2}. \end{aligned}$$

Using a standard unfolding argument, we can evaluate the second integral in the latter expression as

$$\frac{2}{\Gamma(2s)} \int\limits_{\mathcal{G}_T} \mathcal{M}_s(4\pi|m|v)v^w \overline{\theta_\mu(\tau,z)}\frac{dudv}{v^2},$$

where $\mathcal{G}_\mathcal{T} = \{\tau = u + iv \in \mathfrak{H} : |u| \leq 1/2, u^2 + v^2 \leq 1, v \leq T\}$ denotes the truncated fundamental domain for the action of Γ_∞ on $\bigcup_{M-\Gamma_\infty} M\mathcal{F}$. In this way, we compute

$$I_T(\mu,m,s,h;w) = \frac{2}{\Gamma(2s)} \int\limits_{\mathcal{F}_T} \mathcal{M}_s(4\pi|m|v)v^w e(mu)\overline{\theta_\mu(\tau,z,h)}\frac{dudv}{v^2} + \frac{2}{\Gamma(2s)} \int\limits_{\mathcal{G}_T} \mathcal{M}_s(4\pi|m|v)v^w \overline{\theta_\mu(\tau,z,h)}\frac{dudv}{v^2}$$

$$= \frac{2}{2\Gamma(2s)} \int\limits_{v=0}^{T} \int\limits_{u=0}^{1} \mathcal{M}_s(4\pi|m|v)v^w \overline{\theta_\mu(u+iv,z,h)}\frac{dudv}{v^2},$$

[3] A priori, $\Phi_{\mu,m}(z,h)$ is defined as in [6, Definition 2.10] to be the constant term in the Laurent series expansion around $s = 1 - l/2 = n/4 + 1/2$ of the analytic continuation of $\Phi_{\mu,m}(z,h,s) = \Phi_{\mu,m}(z,h,s;0)$. However, by to [6, Proposition 2.11], we can define it equivalently as the stated limit of truncated theta integrals when $s \neq 1 - l/2 = n/4 + 1/2$.

which after opening the Fourier series expansion (25) of $\overline{\theta_\mu(u+iv,z,h)}$, switching the order of summation, and evaluating the unipotent integral via the orthogonality of characters on $\mathbf{R}/\mathbf{Z} \cong [0,1]$ is the same as

$$\begin{aligned}
I_T(\mu,m,s,h;w) &= \frac{2}{\Gamma(2s)} \int_{v=0}^{T} \sum_{\lambda \in h(\mu+L)} \mathcal{M}_s(4\pi|m|v) v^w \exp\left(2\pi v Q(\lambda) - 4\pi v Q(\lambda_z)\right) \int_{u=0}^{1} e(mu - Q(\lambda)u) du dv \\
&= \frac{2}{\Gamma(2s)} \int_{v=0}^{T} \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \mathcal{M}_s(4\pi|m|v) v^w \exp\left(-4\pi v Q(\lambda_z) + 2\pi v m\right) dv \\
&= \frac{2(4\pi|m|)^{-\frac{l}{2}}}{\Gamma(2s)} \int_{v=0}^{T} \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} M_{-\frac{l}{2}, s-\frac{1}{2}}(4\pi|m|v) v^{w-\frac{l}{2}-2} \exp\left(-4\pi v Q(\lambda_z) + 2\pi v m\right) dv \\
&= \frac{2(4\pi|m|)^{\frac{n}{4}-\frac{1}{2}}}{\Gamma(2s)} \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \int_{v=0}^{T} M_{\frac{n}{4}-\frac{1}{2}, s-\frac{1}{2}}(4\pi|m|v) v^{w+\frac{n}{4}-\frac{1}{2}-2} \exp\left(-4\pi v Q(\lambda_z) + 2\pi v m\right) dv.
\end{aligned}$$

Here, we view the latter sum as one over lattice points $\lambda \in h(\mu+L)$ for which

$$Q(\lambda) = Q(\lambda_z) + Q(\lambda_{z^\perp}) = m,$$

where $z^\perp$ denotes the orthogonal complement of z in D(V). Taking the limit with $T \to \infty$, we obtain

(26)

$$\Phi_{\mu,m}(z,s,h;w) = \frac{2(4\pi|m|)^{\frac{n}{4}-\frac{1}{2}}}{\Gamma(2s)} \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \int_{v=0}^{\infty} M_{\frac{n}{4}-\frac{1}{2}, s-\frac{1}{2}}(4\pi|m|v) v^{\left(w+\frac{n}{4}-\frac{1}{2}-1\right)-1} \exp\left(-4\pi v Q(\lambda_z) + 2\pi m v\right) dv.$$

Now, for each $\lambda \in h(\mu+L)$ with $\lambda_z \neq 0$, we view the corresponding integral in (26) as a Laplace transform of the type computed in [21, p. 215 (11)], with $\nu = w - l/2 - 1 = w + (n/4 - 1/2) - 1$, $\kappa = -l/2 = -1/2 + n/4$, $\varpi = s - 1/2$, $\alpha = 4\pi|m|$, and $p = (4\pi Q_0(\lambda_z) - 2\pi m) = (4\pi Q(\lambda_z) + 2\pi|m|)$ to obtain[4]

$$\begin{aligned}
&\int_{v=0}^{\infty} M_{\frac{n}{4}-\frac{1}{2}, s-\frac{1}{2}}(4\pi|m|v) v^{\left(w+\frac{n}{4}-\frac{1}{2}-1\right)-1} e^{-v(4\pi Q(\lambda_z) - 2\pi m)} dv \\
&= \alpha^{\mu+1/2} \Gamma\left(\mu+\nu+1/2\right) \cdot \frac{{}_2F_1\left(\varpi+\nu+1/2, \varpi-\kappa+1/2, 2\varpi+1; \alpha/(p+\alpha/2)\right)}{(p+\alpha/2)^{\varpi+\nu+1/2}} \qquad (27) \\
&= (4\pi|m|)^s \Gamma\left(\frac{n}{4} - \frac{1}{2} + s + w\right) \cdot \frac{{}_2F_1\left(s+\frac{n}{4}-\frac{1}{2}+w, s-\frac{n}{4}+\frac{1}{2}, 2s; \frac{|m|}{Q(\lambda_{z_0})}\right)}{(4\pi Q(\lambda_z))^{s+\frac{n}{4}-\frac{1}{2}+w}}.
\end{aligned}$$

We treat the remaining contributions from $\lambda_z = 0$ the same way, using that $Q(\lambda_z) = m - Q(\lambda_{z^\perp})$ and hence $4\pi Q(\lambda_z) - 2\pi m + 2\pi|m| = -4\pi Q(\lambda_{z^\perp}) + 2\pi m + 2\pi|m| = -4\pi Q(\lambda_{z^\perp})$. Here, note that $m < 0$ implies $2\pi m + 2\pi|m| = 0$. We can and do express all contributions λ via this latter substitution as in [6, Theorem 2.14]. In this way, we deduce the stated formula from (26) via (27). □

4.4.4. *Regularized theta lifts $\Phi_{\mu,m}$ of the Hejhal-Maass Poincaré series $F_{\mu,m}$.* Note that the function of Theorem 4.6 above, defined for $\Re(s) > u = u(n) := 1 - l/2 = n/4 + 1/2$ by the absolutely convergent series

$$\begin{aligned}
\Phi_{\mu,m}(z,h,s) = \Phi_{\mu,m}(z,h,s,0) &= \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle F_{\mu,m}(\tau,s), \overline{\theta_L(\tau,h,z)} \rangle\rangle \frac{dudv}{v^2} \\
&= 2 \cdot \frac{\Gamma\left(s+\frac{n}{4}-\frac{1}{2}\right)}{\Gamma(2s)} \sum_{\substack{\lambda \in g(\mu+L) \\ Q(\lambda)=m}} \left(\frac{m}{Q(\lambda_z^\perp)}\right)^{s+\frac{n}{4}-\frac{1}{2}} {}_2F_1\left(s+\frac{n}{4}-\frac{1}{2}, s-\frac{n}{4}+\frac{1}{2}, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right), \qquad (28)
\end{aligned}$$

[4]Noting that $Q(\lambda_z) + |m| = Q(\lambda_{z^\perp})$ to simplify the expression for p

determines a real analytic function on $z \in D(V)$, with logarithmic singularities along the special divisors

$$Z(\mu, m) = \bigcup_{\substack{\lambda \in \mu + L \\ Q(\lambda) = m}} \lambda^{\perp}.$$

Defined a priori for $\Re(s) > u$, this function $\Phi_{\mu,m}(z,h,s)$ has an analytic continuation to all $s \in \mathbf{C}$, with a simple pole at u. Hence, the sum defining (28) converges normally, and $\Phi_{\mu,m}$ determines a smooth function on $X \backslash Z(\mu, m)$ with logarithmic singularities along $Z(\mu, m)$, and an analytic continuation to a meromorphic function in $s \in \mathbf{C}$. It is also an eigenfunction for the Laplacian Δ_z described in Theorem 4.2, satisfying

$$\Delta_z \Phi_{\mu,m}(z,h,s) = \frac{1}{2} \cdot (s-u) \cdot (s+u-1) \cdot \Phi_{\mu,m}(z,h,s). \tag{29}$$

It can also be viewed as a square integrable function. We refer to the discussion of [9, §4] for more details. At the point $u = n/4 + 1/2$, it determines the automorphic Green's function of the special divisor $Z(\mu, m)$:

Corollary 4.7. *The regularized theta lift $\Phi_{\mu,m}(z,h,n/4+1/2) = \Phi(F_{\mu,m}, z, h)$ of the harmonic weak Maass form $F_{\mu,m}(\tau) = F_{\mu,m}(\tau,(2-l)/2) = F_{\mu,m}(\tau, n/4+1/2) \in H_{1-n/2}(\omega_L)$ is the automorphic Green's function for the special divisor $Z(\mu,m)$ on the spin Shimura variety $X = X_K = X_{K_L}$.*

Proof. Cf. [10, Remark 3.10] with [6, Definiton 1.8 and Proposition 1.10], [9, §4], and [1, §5]. □

4.4.5. *Extension to compactifications.* Fix $f \in H_{1-n/2}(\omega_L)$ a harmonic weak Maass form whose holomorphic part f^+ has integral Fourier coefficients. Fix a compactification $X^\star$ of the Shimura variety $X = X_K$. For the divisor $Z(f) \subset X$ defined in (18) above, there exists a divisor $C(f)$ supported on the boundary $\partial X^\star = X^\star \backslash X$ such that $\Phi(f, \cdot)$ is the automorphic Green's function in the sense of Theorem 4.2 for the corresponding divisor

$$Z^c(f) = Z(f) + C(f) \tag{30}$$

of degree zero on X. For a more precise description of this in the setting of the modular curve, with the quadratic space given in Example 7.2 below, we refer to the discussion in [13, §7.3].

4.4.6. *The special case of Hilbert modular surfaces.* Fix k any quadratic field with class group $C(\mathcal{O}_K)$, and $A = [\mathfrak{a}] \in C(\mathcal{O}_K)$ any ideal class. We now explain how to specialize the definition (28) to the setup we consider with the spaces (V_A, Q_A) of signature $(2,2)$ with lattices $L_A \subset V_A$ giving rise to the Hilbert modular surface $X_{K_A} \cong Y_0(N)^2$. Here, we adapt the discussion of [9, §6.1] as follows. As explained above, we have a natural identification $D(V_A) = D^{\pm}(V_A) \cong \mathfrak{H}^2$, so can view the variable $z \in D(V_A)$ as a pair $z = (z_1, z_2) \in \mathfrak{H}^2$. Let $\mathcal{Q}_{s-1}$ denote the classical Legendre function of the second kind

$$\mathcal{Q}_{s-1}(t) = \int_0^\infty \left(t + \sqrt{t^2-1}\cosh(u)\right)^{-s} du = \frac{\Gamma(s)^2}{2\Gamma(2s)} \left(\frac{2}{1+t}\right)^s {}_2F_1\left(s, s, 2s; \frac{2}{1+t}\right).$$

Hence, we can rewrite the corresponding theta lift (28) in this setting as

$$\begin{aligned}
\Phi_{\mu,m}(z,s) = \Phi_{\mu,m}(z,1,s) &= 2 \cdot \frac{\Gamma\left(s + \frac{2}{4} - \frac{1}{2}\right)}{\Gamma(2s)} \sum_{\substack{\lambda \in \mu + L_A \\ Q_A(\lambda) = m}} \left(\frac{m}{Q_A(\lambda_z^{\perp})}\right)^s {}_2F_1\left(s, s, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right) \\
&= \frac{4}{\Gamma(s)} \sum_{\substack{\lambda \in \mu + L_A \\ Q_A(\lambda) = m}} \mathcal{Q}_{s-1}\left(1 - \frac{2Q_A(\lambda_z)}{m}\right) \\
&= \frac{4}{\Gamma(s)} \sum_{\substack{\lambda = (\lambda_1, \lambda_2) \in \mu + L_A = \mathfrak{a} + N^{-1}\mathfrak{a} \\ Q_A(\lambda) = m}} \mathcal{Q}_{s-1}\left(1 - \frac{2}{m\mathbf{N}\mathfrak{a}}\left(\lambda_{1,z}\lambda_{1,z}^{\tau} - \lambda_{2,z}\lambda_{2,z}^{\tau}\right)\right).
\end{aligned}$$

Here again, we write $\tau \in \mathrm{Gal}(k/\mathbf{Q})$ to denote the nontrivial automorphism, with $\lambda = (\lambda_1, \lambda_2) \in L_A$ a vector, and $\lambda_{j,z} = \lambda_{j,(z_1,z_2)}$ the corresponding projections to $z = (z_1, z_2) \in \mathfrak{H}^2$. In this way, we see that

$$\Phi_m(z_1, z_2, s) = \Phi_m^{L_A}(z_1, z_2, s) := \sum_{\mu \in L_A^\vee / L_A} \Phi_{\mu,m}(z_1, z_2, s) = -\frac{2}{\Gamma(s)} G_s^{\Gamma_0(N), m}(z_1, z_2),$$

where $G_s^{\Gamma_0(N),m}(z_1, z_2, s)$ denotes a translate the higher Green's function characterized in [25, §II.2, (2.3)] and [64, §3.4], cf. [9, §6.1]. To be more precise, we see from (29) that $\Phi_m(z, s) = \Phi_m^{L_A}(z_1, z_2, s)$ is an eigenvector for the hyperbolic Laplacian operator $\Delta_z = \Delta_{(z_1,z_2)} = \Delta_{z_1} \times \Delta_{z_2}$ on $X_0(N) \times X_0(N)$ satisfying

$$\Delta_z \Phi_m(z_1, z_2, s) = \frac{s}{2}(s-1)\Phi_m(z_1, z_2, s).$$

On the other hand, there is a unique resolvent kernel function $G_s^{\Gamma_0(N)}(z_1, z_2)$ on $\mathfrak{H} \times \mathfrak{H}$ characterized by:

(i) $G_s^{\Gamma_0(N)}(z_1, z_2)$ is smooth $\mathfrak{H} \times \mathfrak{H} \backslash \{(\tau, \gamma\tau) : \gamma \in \Gamma_0(N)\}$

(ii) $G_s^{\Gamma_0(N)}(\gamma_1 z_1, \gamma_2 z_2) = G_s^{\Gamma_0(N)}(z_1, z_2)$ for all $\gamma_1, \gamma_2 \in \Gamma_0(N)$

(iii) $\Delta_{z_j} G_s^{\Gamma_0(N)}(z_1, z_2) = s(s-1)G_s^{\Gamma_0(N)}(z_1, z_2)$ for each of $j = 1, 2$, where $\Delta_{z_j} - y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right)$ denotes the hyperbolic Laplacian of weight zero in either variable $z_j = x_j + iy_j \in \mathfrak{H}$

(iv) $G_s^{\Gamma_0(N)}(z_1, z_2) = e_{z_2} \log |z_1 - z_2|^2 + O(1)$ as $z_1 \to z_2$, where $e_{z_2} = \# \operatorname{Stab}_{\Gamma_0(N)}(z_2)$ is the order of the stabilizer of z_2 in $\Gamma_0(N)$

(v) In a neighbourhood of a cusp $\alpha^{-1}\infty$, the function $\Im(\alpha z_1)^{s-1} G_s^{\Gamma_0(N)}(z_1, z_2)$ extends to a continuous function.

The existence of such a function follows from the construction given in [25, §II.2]. That properties (iii)-(v) characterize it uniquely for $\Re(s) > 1$ is shown in [64, §3.4]. We deduce that the function $-\frac{\Gamma(s)}{2}\Phi_m(z_1, z_1, s)$ at $m = 1$ satisfies these properties, and that the corresponding "higher" Green's functions $-\frac{\Gamma(s)}{2}\Phi_m(z_1, z_2, s)$ for $m > 1$ can be viewed as translates of this resolvent kernel function $G_s^{\Gamma_0(N)}(z_1, z_2)$.

5. Summation along isotropic quadratic subspaces

We now compute the regularized theta lifts $\Phi(f, z, h)$ along CM cycles $Z(V_0)$ and geodesic sets $G(W)$.

5.1. Eisenstein series and Siegel-Weil formulae.

Fix $V_0 \subset V$ any quadratic subspace of signature $(0, 2)$, writing $L_0 = V_0 \cap L$ for the corresponding lattice and $Q_0 = Q|_{V_0}$ the corresponding quadratic form. We also fix $W \subset V$ any rational quadratic space of signature $(1, 1)$, writing $L_W = W \cap L$ for the corresponding lattice and $Q_W = Q|_W$ the corresponding quadratic form. We now describe the Eisenstein series associated with these quadratic spaces.

5.1.1. *Langlands Eisenstein series and the Siegel-Weil formula.* We first describe the construction in more general terms. Let (U, Q) be any anistropic rational quadratic space of even dimension $\dim(U)$ and signature $(p(U), q(U))$. Fix a lattice $L \subset U$, and consider the corresponding Weil representation $\omega_L : \mathrm{SL}_2(\mathbf{Z}) \longrightarrow \mathfrak{S}_L$.

Let us write $P = MN \subset \mathrm{SL}_2$ to denote the parabolic group of upper-triangular matrices, with Levi subgroup M and unipotent radical N parametrized with the standard shorthand notations

$$M = \{m(a) : a \in \mathbf{G}_m\}, \qquad m(a) := \begin{pmatrix} a^{\frac{1}{2}} & \\ & a^{-\frac{1}{2}} \end{pmatrix}$$

$$N = \{N(b) : b \in \mathbf{G}_a\}, \qquad n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}.$$

Hence, writing $K_\infty = \mathrm{SO}_2(\mathbf{R})$ and $K = \mathrm{SL}_2(\widehat{\mathbf{Z}})$, we have the Iwasawa decomposition $\mathrm{SL}_2(\mathbf{A}) = N(\mathbf{A})M(\mathbf{A})K_\infty K$.

Let χ_U denote the quadratic idele class character of $\mathbf{Q}$ given on $x \in \mathbf{A}^\times$ by

$$\chi_U(x) = \left(x, (-1)^{\frac{\dim(U)}{2}} \det(U)\right)_{\mathbf{A}},$$

where $(\cdot, \cdot)_{\mathbf{A}}$ denotes the Hilbert symbol on $\mathbf{A}^\times$, and $\det(U)$ the Gram determinant of U. Given $s \in \mathbf{C}$, let $I(s, \chi_U)$ denote the principal series representation of $\mathrm{SL}_2(\mathbf{A})$ induced by the quasicharacter $\chi_U(\cdot)|\cdot|^s$. Hence,

$I(s,\chi_U)$ consists of all smooth functions $\phi(g,s)$ on $g \in \mathrm{SL}_2(\mathbf{A})$ satisfying the transformation property

$$\phi\left(n(b)m(a)g,s\right) = \chi_U(a)|a|^{s+1}\phi(g,s)$$

for all $a \in \mathbf{A}^\times$ and $b \in \mathbf{A}$, with $\mathrm{SL}_2(\mathbf{A})$ acting by right translation. There is a $\mathrm{SL}_2(\mathbf{A})$-intertwining map

$$\lambda: \mathcal{S}(U(\mathbf{A})) \longrightarrow I(s_0(U),\chi_U), \quad \lambda(\phi)(g) := \left(\omega_L(g)\phi\right)(0) \quad \text{for } s_0(U) := \frac{\dim(U)}{2} - 1.$$

Recall that a section $\phi(s) \in I(s,\chi_U)$ is *standard* if its restriction to the maximal compact subgroup $K_\infty K \subset \mathrm{SL}_2(\mathbf{A})$ does not depend on s. Via the Iwasawa decomposition, we see that any $\lambda(\phi) \in I(s_0(U),\chi_U)$ has a unique extension to a standard section $\lambda(\phi)(s) \in I(s,\chi_U)$ such that $\lambda(\phi)(s_0(U)) = \lambda(\phi)$. We consider the following standard sections. Let us for any $l \in \mathbf{Z}$ write χ_l to denote the character of K_∞ defined by

$$\chi_l(k_\theta) = e^{il\theta} = \exp(il\theta), \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

Let $\phi_\infty^l(s) \in I(s,\chi_U)$ be the unique standard section for which $\phi_\infty^l(k_\theta,s) = \chi_l(k_\theta) = e^{il\theta}$. In terms of the Iwasawa decomposition, this section can be characterized by the transformation property

$$\phi_\infty^l(n(b)m(a)k_\infty,s) = \chi_U(a)|a|^{s+1}e^{il\theta}.$$

Now, recall we defined the Gaussian $\Phi_\infty \in \mathcal{S}(U(\mathbf{R}))$ in (12), at least for (U,Q) of signature $(n,2)$. More generally[5], writing $D(U) = \{z \subset U(\mathbf{R}) : \dim(u) = p(U), Q|_z < 0\}$ for the corresponding domain, and defining for a given $z \in D(U)$ the corresponding majorant $(x,x)_z = (x_{z^\perp},x_{z^\perp}) - (x_z,x_z)$ on $x \in U(\mathbf{R})$, we let

$$\Phi_\infty(x,z) = \exp\left(-(x,x)_z\right).$$

Again, we see that $\Phi_\infty(hx,hz) = \Phi_\infty(x,z)$ for all $h \in \mathrm{GSpin}(U)(\mathbf{R})$. Viewed as a function of $x \in U(\mathbf{R})$, we obtain an archimedean local Schwartz function $\Phi_\infty \in \mathcal{S}(U(\mathbf{R}))$. Through the Weil representation ω_L, we also know that K_∞ acts on $\Phi_\infty(x,z)$ with weight $\frac{p(U)-q(U)}{2}$. Hence, we see in general that

$$\lambda_\infty(\Phi_\infty(\cdot,z)) = \phi_\infty^{\frac{p(U)-q(U)}{2}}(s_0(U)).$$

Given any standard section $\phi(s) \in I(s,\chi_U)$, we define the corresponding Eisenstein series

$$E_L(g,s;\phi) = \sum_{\gamma \in P(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{Q})} \phi(\gamma g,s)$$

on $g \in \mathrm{SL}_2(\mathbf{A})$, first for $\Re(s) > 1$. This sum $E_L(g,s;\phi)$ has a meromorphic continuation to all $s \in \mathbf{C}$ via the Langlands functional equation, which relates $E_L(g,s;\phi)$ to $E_L(g,-s;M(s)\phi)$ for the corresponding intertwining operator $M(s)$. This Langlands Eisenstein series $E_L(g,s;\phi)$ determines an automorphic form on $g \in \mathrm{SL}_2(\mathbf{A})$. Its value at $s_0(U)$ is known classically to be holomorphic, as it is given as an average of the theta series $\vartheta_L(g,h;\Phi)$ we considered above by the following well-known result.

Theorem 5.1 (Siegel-Weil). *Let (U,Q) be any anisotropic quadratic space of signature $(p(U),q(U))$ and even dimension $p(U)+q(U)$. Let $L \subset U$ be any maximal lattice, and $\Phi \in \mathcal{S}(U(\mathbf{A}))$ any Schwartz function. The Eisenstein series $E_L(g,s;\Phi)$ is holomorphic at $s_0(U) = (p(U)+q(U))/2 - 1$, and given by the formula*

$$\frac{\alpha}{2} \int_{\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A})} \vartheta_L(g,h;\Phi)dh = E_L(g,s_0(U);\lambda(\Phi)), \quad \alpha := \begin{cases} 2 & \text{if } p(U) = 0 \\ 1 & \text{if } p(U) > 1 \end{cases}.$$

Here, we write dh to denote the Tamagawa measure on $\mathrm{SO}(U)(\mathbf{A})$.

Proof. See e.g. [40, Theorem 4.1] or [13, Theorem 2.1]. □

[5] We can assume without loss of generality for our later discussion that $(p(U),q(U)) = (0,2)$ or $(p(U),q(U)) = (1,1)$, so that we only need to consider the case of $(p(U),q(U)) = (1,1)$ separately.

5.1.2. *The CM case.* We now consider the special case of the negative definite subspace (L_0, Q_0). Consider for each integer $l \in \mathbf{Z}$ the $\mathfrak{S}_{L_0}$-valued Langlands Eisenstein series on $\tau = u + iv \in \mathfrak{H}$ and $s \in \mathbf{C}$ defined by

$$E_{L_0}(\tau, s; l) := v^{-\frac{l}{2}} \sum_{\mu \in L_0^\vee / L_0} E_{L_0}(g_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) \mathbf{1}_\mu. \tag{31}$$

We can then describe Theorem 5.1 in terms of the Siegel theta series (15) as

$$\int_{\mathrm{SO}(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \theta_{L_0}(\tau, z_0, h_f) dh = E_{L_0}(\tau, 0; -1). \tag{32}$$

Here (cf. [13, Proposition 2.2]), we write $z_0 \in D(V_0) = D^{\pm}(V_0)$ to denote the oriented hypersurface determined by $V_0(\mathbf{R})$. We also normalize the measure on $\mathrm{SO}(V_0)(\mathbf{R}) \cong \mathrm{SO}_2(\mathbf{R})$ so that $\mathrm{vol}(\mathrm{SO}(V_0)(\mathbf{R})) = 1$. This determines a normalization of the measure on $\mathrm{SO}(V_0)(\mathbf{A}_f)$ so that $\mathrm{vol}\left(\mathrm{SO}(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)\right) = 2$.

As explained in [13, §2.2], the Langlands Eisenstein series (31) has the following classical description. Writing $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and $\Gamma_\infty = P(\mathbf{Q}) \cap \Gamma = \{n(b) : b \in \mathbf{Z}\}$, we see that $P(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{Q}) = \Gamma_\infty \backslash \Gamma$. Using the Iwasawa decomposition, we can write the action of each matrix in the sum as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \Longrightarrow \quad \gamma g_\tau = n(\beta) m(\alpha) k_\infty \quad \text{for some } \alpha \in \mathbf{R}_{>0} \text{ and } \beta, \theta \in \mathbf{R}.$$

A direct computation reveals that

$$\alpha = v^{\frac{1}{2}} |c\tau + d|^{-1} \quad \text{and} \quad e^{i\theta} = \frac{c\overline{\tau} + d}{|c\tau + d|}$$

and hence

$$\phi_\infty^l(\gamma g_\tau) = v^{\frac{s}{2} + \frac{1}{2}} (c\tau + d)^{-l} |c\tau + d|^{l - s - 1},$$

so that

$$\begin{aligned} E_{L_0}(g_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-l} \cdot \frac{v^{\frac{s}{2} + \frac{1}{2}}}{|c\tau + d|^{s+1-l}} \cdot \lambda_f(\mathbf{1}_\mu)(\gamma) \\ &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-l} \cdot \frac{v^{\frac{s}{2} + \frac{1}{2}}}{|c\tau + d|^{s+1-l}} \cdot \langle \mathbf{1}_\mu, \left(\omega_{L_0}^{-1}(\gamma)\right) \mathbf{1}_0 \rangle. \end{aligned}$$

It follows that

$$E_{L_0}(\tau, s; l) := \sum_{\mu \in L_0^\vee / L_0} E_{L_0}(g_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{s+1-l}{2}} \mathbf{1}_0 \right] \Big|_{l, \omega_{L_0}} \gamma. \tag{33}$$

Recall that we consider the Maass weight raising and lowering operators R_l and L_l, as defined in (16). A simple computation with the series expansion on the right-hand side of (33) reveals that

$$\begin{aligned} L_l E_{L_0}(\tau, s; l) &= \frac{1}{2}(s + 1 - l) \cdot E_{L_0}(\tau, s; l - 2) \\ R_l E_{L_0}(\tau, s; l) &= \frac{1}{2}(s + 1 + l) \cdot E_{L_0}(\tau, s; l + 2), \end{aligned}$$

and in particular

$$L_1 E_{L_0}(\tau, s; 1) = \frac{s}{2} \cdot E_{L_0}(\tau, s; -1). \tag{34}$$

Since the Eisenstein series $E_{L_0}(\tau, s; -1)$ on the right-hand side of (34) is holomorphic at $s = s_0(V_0) = 0$ by the Siegel-Weil formula (32), we deduce that the Eisenstein series $E_{L_0}(\tau, s; 1)$ appearing on the left-hand side must vanish at its central point $s = 0$ for its functional equation. We also obtain from (34) the relation

$$L_1 E'_{L_0}(\tau, 0; 1) = \frac{1}{2} E_{L_0}(\tau, 0; -1) \tag{35}$$

for the derivative $E'_{L_0}(\tau, 0; 1) = \frac{d}{ds} E_{L_0}(\tau, s; 0)\big|_{s=0}$ at $s = 0$. Writing ∂ and $\overline{\partial}$ to denote the Dolbeault operators, so that the exterior derivative on differential forms on $\mathfrak{H}$ is given by $d = \partial + \overline{\partial}$, and again $d\mu(\tau) = \frac{dudv}{v^2}$ for $\tau = u + iv \in \mathfrak{H}$, we can express the relation (35) in terms of differential forms as

$$-2\overline{\partial}\left(E'_{L_0}(\tau, 0; 1) d\tau\right) = E_{L_0}(\tau, 0; -1) d\mu(\tau). \tag{36}$$

More generally, we have the following useful description of the operator L_l.

Lemma 5.2. *The Maass weight-lowering operator L_l can be described in terms of differential forms as*

$$\overline{\partial}(f d\tau) = -v^{2-l} \xi_l(f) d\mu(\tau) = -L_l f d\mu(\tau).$$

Proof. See [22, Lemma 2.5] and [13, Lemma 2.3]. □

Let us now consider the Fourier series expansion of the Eisenstein series $E_{L_0}(\tau, s; 1)$, which we write as

$$E_{L_0}(\tau, s; 1) = \sum_{\mu \in L_0^\vee / L_0} \sum_{m \in \mathbf{Q}} A_{L_0}(s, \mu, m, v) e(m\tau) \mathbf{1}_\mu.$$

Following the discussion of Kudla [40, Theorem 2.12] (cf. [13, §2.2]), and using the fact that $E_{L_0}(\tau, 0; 1) = 0$ by (34), we compute the Laurent series expansions of each of the coefficients $A_{L_0}(s, \mu, m, v)$ around $s = 0$ as

$$A_{L_0}(s, \mu, m, v) = b_{L_0}(\mu, m, v) s + O(s^2).$$

We deduce from this that $E'_{L_0}(\tau, 0; 1)$ has the Fourier series expansion

$$E'_{L_0}(\tau, 0; 1) = \sum_{\mu \in L_0^\vee / L_0} \sum_{m \in \mathbf{Q}} b_{L_0}(\mu, m, v) e(m\tau) \mathbf{1}_\mu.$$

Viewing $E'_{L_0}(\tau, 0; 1) = E'^{+}_{L_0}(\tau, 0; 1) + E'^{-}_{L_0}(\tau, 0; 1) \in H_1(\omega_{L_0}^\vee)$ as a harmonic weak Maass form of weight 1 and representation $\omega_{L_0}^\vee$, we also use the general calculation of Kudla [40, Theorem 2.12] to compute the Fourier series expansion of the principal/holomorphic part $\mathcal{E}_{L_0}(\tau) := E'^{+}_{L_0}(\tau, 0; 1)$ as

$$\mathcal{E}_{L_0}(\tau) = E'^{+}_{L_0}(\tau, 0; 1) = \sum_{\mu \in L_0^\vee / L_0} \sum_{m \in \mathbf{Q}} \kappa_{L_0}(\mu, m) e(m\tau) \mathbf{1}_\mu, \tag{37}$$

where the coefficients are given explicitly by the convergent limits

$$\kappa_{L_0}(\mu, m) = \begin{cases} \lim_{v \to \infty} b_{L_0}(\mu, m, v) & \text{if } \mu \neq 0 \text{ or } m \neq 0 \\ \lim_{v \to \infty} b_{L_0}(0, 0, v) - \log(v) & \text{if } \mu = 0 \text{ and } m = 0. \end{cases}$$

Let us now specialize the setting we consider below, where the negative definite space (L_0, Q_0) is *incoherent* in the sense that it is constructed from an ideal $L_0 = \mathfrak{a} \subset \mathcal{O}_k$ in an imaginary quadratic field $k = k(V_0)$, with its positive definite norm form $Q_\mathfrak{a}(\cdot) := \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a}$, but we take $(V_0, Q_0) = (\mathfrak{a}_\mathbf{Q}, -Q_\mathfrak{a})$ to get a negative definite space of signature $(0, 2)$. This construction amounts to taking the positive definite quadratic space $(\mathfrak{a}_\mathbf{Q}, Q_\mathfrak{a})$ at all of the finite places, but then switching invariants at the real place in replacing by $(\mathfrak{a}_\mathbf{Q}, -Q_\mathfrak{a})$. Such a choice of $(V_0, Q_0) = (\mathfrak{a}_\mathbf{Q}, -Q_\mathfrak{a})$ makes sense locally at each place of $\mathbf{Q}$, but does not correspond globally to any quadratic number field – hence the name "incoherent".

Proposition 5.3. *Suppose $(L_0, Q_0) = (\mathfrak{a}, -Q_\mathfrak{a})$ for $\mathfrak{a} \subset \mathcal{O}_k$ a nonzero integral ideal of an imaginary quadratic field $k = k(V_0)$ of discriminant d_k and odd quadratic Dirichlet character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$, with $Q_\mathfrak{a}(\cdot) = \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a}$ the corresponding positive definite norm form. Let $E_{L_0}(\tau, s) = E_{L_0}(\tau; s; 1)$ denote the corresponding Eisenstein series of weight $l = 1$ defined in* (31) *and* (33) *above. Writing*

$$\Lambda(s, \eta_k) = |d_k|^{\frac{s}{2}} \Gamma_\mathbf{R}(s+1) L(s, \eta_k), \quad \Gamma_\mathbf{R}(s) := \pi^{-\frac{s}{2}} \Gamma(s)$$

to denote the completed Dirichlet L-function $L(s, \eta_k)$ of the character η_k, the completed Eisenstein series

$$E^\star_{L_0}(\tau, s) := \Lambda(s+1, \eta_k) E_{L_0}(\tau, s) = \Lambda(s+1, \eta_k) E_{L_0}(\tau, 0; 1)$$

satisfies the odd, symmetric functional equation

$$E^\star_{L_0}(\tau, s) = -E^\star_{L_0}(\tau, -s).$$

Proof. See [13, Proposition 2.5]. The functional equation is deduced from the Langlands functional equation of each of the constituent incoherent Eisenstein series $E_{L_0}(g_\tau, s; \phi_\infty^1 \otimes \lambda_f(\mathbf{1}_\mu))$, where switching invariants at infinity as we describe above leads to switching the corresponding archimedean local sign to -1. $\square$

5.1.3. *The geodesic case.* We summarize the discussion of [54, §4.6] for the subspace (L_W, Q_W) of signature $(1,1)$. Consider for each $l \in \mathbf{Z}$ the $\mathfrak{S}_{L_W}$-valued Eisenstein series on $\tau = u + iv \in \mathfrak{H}$ and $s \in \mathbf{C}$ defined by

$$E_{L_W}(\tau, s; l) := v^{-\frac{l}{2}} \sum_{\mu \in L_W^\vee / L_W} E_{L_W}(g_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu))\mathbf{1}_\mu. \tag{38}$$

We can then describe Theorem 5.1 in terms of the Siegel theta series (15) as

$$\int_{\mathrm{SO}(W)(\mathbf{Q})\backslash \mathrm{SO}(W)(\mathbf{A}_f)} \theta_{L_W}(\tau, z_W, h_f) dh = E_{L_W}(\tau, 0; 0). \tag{39}$$

Here, we fix an oriented hyperbolic line $z_W \in D(W) = D^\pm(W)$, and again normalize the measure on $\mathrm{SO}(V)(\mathbf{R})$ so that $\mathrm{vol}(\mathrm{SO}(W)(\mathbf{R})) = 1$ and $\mathrm{vol}(\mathrm{SO}(W)(\mathbf{Q})\backslash \mathrm{SO}(W)(\mathbf{A}_f)) = 2$.

In the same way as for (31), we can describe the Langlands Eisenstein series (38) in classical terms as

$$E_{L_W}(\tau, s; l) := \sum_{\mu \in L_W^\vee / L_W} E_{L_W}(g_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{s+1-l}{2}} \mathbf{1}_0 \right] \Big|_{l, \omega_{L_W}} \gamma. \tag{40}$$

Let us now specialize immediately to the setting we consider later on, where the quadratic space (L_W, Q_W) is given by a nonzero integer ideal $L_W = \mathfrak{a} \subset \mathcal{O}_k$ in a real quadratic field $k = k(W)$, with indefinite quadratic form Q_W given by the norm form $Q_W(\cdot) = Q_\mathfrak{a}(\cdot) = \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a}$. Unlike in the CM setup above with the negative definite subspaces constructed from the positive definite subspaces attached to imaginary quadratic fields, these quadratic spaces $(L_W, Q_W) = (\mathfrak{a}, Q_\mathfrak{a})$ are *coherent* in that they correspond globally to integral ideals in some real quadratic number field $k = k(W)$. This has the following consequences for the corresponding Eisenstein series $E_{L_W}(\tau, s) = E_{L_W}(\tau, s; 0)$.

Proposition 5.4. *Suppose $(L_W, Q_W) = (\mathfrak{a}, Q_\mathfrak{a})$ for $\mathfrak{a} \subset \mathcal{O}_k$ a nonzero integral ideal of a real quadratic field $k = k(W)$ of discriminant d_k and even quadratic Dirichlet character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$, with $Q_\mathfrak{a}(\cdot) = \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a}$ the corresponding indefinite norm form. Let $E_{L_W}(\tau, s) = E_{L_W}(\tau; s; 0)$ denote the corresponding Eisenstein series of weight $l = 0$ defined in* (38) *and* (40) *above. Writing*

$$\Lambda(s, \eta_k) = |d_k|^{\frac{s}{2}} \Gamma_\mathbf{R}(s+1) L(s, \eta_k), \quad \Gamma_\mathbf{R}(s) := \pi^{-\frac{s}{2}} \Gamma(s)$$

to denote the completed Dirichlet L-function $L(s, \eta_k)$ of the character η_k, the completed Eisenstein series

$$E_{L_W}^\star(\tau, s) := \Lambda(s+1, \eta_k) E_{L_W}(\tau, s) = \Lambda(s+1, \eta_k) E_{L_W}(\tau, 0; 0)$$

satisfies the even, symmetric functional equation

$$E_{L_W}^\star(\tau, s) = E_{L_W}^\star(\tau, -s).$$

Proof. See [54, Proposition 4.10], this can be deduced as a direct consequence of the Langlands functional equation for the Eisenstein series on the right-hand side of (38) corresponding to the coherent quadratic space $(L_W, Q_W) = (\mathfrak{a}, Q_\mathfrak{a})$ of signature $(1,1)$. $\square$

Fixing such a coherent choice of Lorentzian quadratic space (L_W, Q_W) henceforth, we consider the images of the Eisenstein series $E_{L_W}(\tau, s; l)$ under the Maass raising and lowering operators (16). Here, we see by inspection of the series expansion on the right-hand side of (40) that

$$L_l E_{L_W}(\tau, s; l) = \frac{1}{2}(s + 1 - l) \cdot E_{L_W}(\tau, s; l-2)$$
$$R_l E_{L_W}(\tau, s; l) = \frac{1}{2}(s + 1 + l) \cdot E_{L_W}(\tau, s; l+2),$$

and in particular

$$L_2 E_{L_W}(\tau, s; 2) = \frac{1}{2} \cdot (s-1) \cdot E_{L_W}(\tau, s; 0). \tag{41}$$

Now, the Eisenstein series $E_{L_W}(\tau, s; 0)$ on the right-hand side of (41) is holomorphic at $s = s_0(W) = 0$, and so we can evaluate this relation at $s = 0$ to obtain the identity

$$L_2 E_{L_W}(\tau, 0; 2) = -\frac{1}{2} \cdot E_{L_W}(\tau, 0; 0).$$

On the other hand, we can differentiate each side of (41) with respect to s to obtain the relation

$$L_2 E'_{L_W}(\tau, s; 2) = \frac{1}{2} \cdot (s-1) \cdot E'_{L_W}(\tau, s; 0) + \frac{1}{2} \cdot E_{L_W}(\tau, s; 0),$$

then evaluate this latter relation at $s = 0$ to obtain the identity

$$L_2 E'_{L_W}(\tau, 0; 2) = \frac{1}{2} \cdot E_{L_W}(\tau, 0; 0) - \frac{1}{2} \cdot E'_{L_W}(\tau, 0; 0),$$

equivalently

$$2L_2 E'_{L_W}(\tau, 0; 2) = E_{L_W}(\tau, 0; 0) - E'_{L_W}(\tau, 0; 0). \tag{42}$$

We now use the even functional equation of $E^{\star}_{L_W}(\tau, s)$ (Proposition 5.4) to deduce that $E'_{L_W}(\tau, 0; 0) = 0$.

Corollary 5.5. *We have $E'_{L_W}(\tau, 0; 0) = 0$, and hence via* (41) *the relation $-2L_2 E'_{L_W}(\tau, 0; 2) = -E_{L_W}(\tau, 0; 0)$. Expressed in terms of differential forms according to Lemma 5.2, we obtain the relation*

$$-2L_2 E'_{L_W}(\tau, 0; 2) d\mu(\tau) = 2\overline{\partial}\left(E'_{L_W}(\tau, 0; 2) d\tau\right) = -E_{L_W}(\tau, 0; 0),$$

equivalently

$$E_{L_W}(\tau, 0; 0) d\mu(\tau) = -2\overline{\partial}\left(E'_{L_W}(\tau, 0; 2) d\tau\right). \tag{43}$$

Proof. See [54, Proposition 4.12]. We know by the Siegel-Weil formula that $E_{L_W}(\tau, s; 0)$ is analytic at $s = 0$. Hence, $E_{L_W}(\tau, s; 0)$ and all of its derivatives with respect to s are analytic at $s = 0$. In particular, both values $E_{L_W}(\tau, 0; 0)$ and $E'_{L_W}(\tau, 0; 0)$ are defined (finite), and we can expand $E_{L_2}(\tau, s; 0)$ into its Taylor series expansion around $s = 0$. Now, we know by Proposition 5.4 that the completed Eisenstein series $E^{\star}_{L_W}(\tau, s; 0)$ satisfies the even, symmetric functional equation $E^{\star}_{L_W}(\tau, s; 0) = E^{\star}_{L_W}(\tau, -s; 0)$. Comparing the Taylor series expansions around $s = 0$ as we may, we then derive for any $s \in \mathbf{C}$ with $0 \leq \Re(s) < 1$ the relation

$$E^{\star}_{L_W}(\tau, 0; 0) + E^{\star\prime}_{L_W}(\tau, 0; 0)s + O(s^2) = E^{\star}_{L_W}(\tau, 0; 0) - E^{\star\prime}_{L_W}(\tau, 0; 0)s + O(s^2),$$

equivalently

$$E^{\star\prime}_{L_W}(\tau, 0; 0)s + O(s^2) = -E^{\star\prime}_{L_W}(\tau, 0; 0)s + O(s^2). \tag{44}$$

Taking the limit as $\Re(s) \to 0$ of (44), we see that $E^{\star\prime}_{L_W}(\tau, 0; 0)$ must vanish, and hence $E'_{L_W}(\tau, 0; 0) = 0$. □

Remark 5.6. Observe that we could also have considered the Lorentzian lattice $(L_W, -Q_W)$ of signature $(1,1)$, denoted by $-L_W$, together with the corresponding incoherent Eisenstein series $E_{-L_W}(\tau, s; 0)$. Writing $k = k(W)$ again to denote the real quadratic field attached to the genuine space $(L_W, Q_W) = (\mathfrak{a}, Q_{\mathfrak{a}})$, with $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$ its character, a minor variation of the argument for [13, Proposition 2.5] shows that the completed Eisenstein series $E^{\star}_{-L_W}(\tau, s) := \Lambda(s+1, \eta_k) E_{-L_W}(\tau, s; 0)$ satisfies an odd, symmetric functional equation

$$E^{\star}_{-L_W}(\tau, s) = -E^{\star}_{-L_W}(\tau, -s),$$

and hence that $E^{\star}_{-L_W}(\tau, s)$ vanishes at the central point $s = 0$. In particular, $E_{-L_W}(\tau, 0; 0) = 0$. Deriving the corresponding identities (41) and (41) in the same way for the incoherent Eisenstein series $E_{-L_W}(\tau, s; 0)$, the vanishing $E_{-L_W}(\tau, 0; 0) = 0$ at the central point implies the corresponding functional identity

$$2L_2 E'_{-L_W}(\tau, 0; 2) = -E'_{-L_W}(\tau, 0; 0). \tag{45}$$

While this latter identity (45) describes the true analogue of the CM setup for the incoherent Eisenstein series $E_{-L_W}(\tau, s; 0)$, it is unfortunately not useful for the derivation of integral presentations as we describe below (cf. [13, Theorem 4.7]), in particular as it is the vanishing central value $E_{-L_W}(\tau, 0; 0) = 0$ which appears in the corresponding average over theta series $\theta_{-L_W}(\tau, z, h)$ according to Siegel-Weil (Theorem 5.1). It could be of independent interest to investigate vanishing averages of regularized theta integrals of this type.

Let us now consider the Fourier series expansion of the Eisenstein series $E_{L_W}(\tau, s; 2)$,

$$E_{L_W}(\tau, s; 2) = \sum_{\mu \in L_W^\vee / L_W} \sum_{m \in \mathbf{Q}} A_{L_W}(s, \mu, m, v) e(m\tau) \mathbf{1}_\mu.$$

Following [40, Theorem 2.12], we write the Laurent series expansions around $s = 0$ of the coefficients as

$$A_{L_W}(s, \mu, m, v) = a_{L_W}(\mu, m, v) + b_{L_W}(\mu, m, v)s + O(s^2),$$

and deduce that the derivative Eisenstein series $E'_{L_W}(\tau, 0; 2)$ at $s = 0$ has the Fourier series expansion

$$E'_{L_W}(\tau, 0; 2) = \sum_{\mu \in L_W^\vee / L_W} \sum_{m \in \mathbf{Q}} b_{L_W}(\mu, m, v) e(m\tau) \mathbf{1}_\mu.$$

Viewing this derivative Eisenstein series as a harmonic weak Maass form

$$E'_{L_W}(\tau, 0; 2) = E'^{+}_{L_W}(\tau, 0; 2) + E'^{-}_{L_W}(\tau, 0; 2) \in H_2(\omega_{L_W}^\vee)$$

of weight 2 and representation $\omega_{L_W}^\vee$, we consider the principal/holomorphic part $\mathcal{E}_{L_W}(\tau) := E'^{+}_{L_W}(\tau, 0; 2)$. Using the argument of [40, Theorem 2.12] again, we can compute the coefficients in its Fourier series expansion

$$\mathcal{E}_{L_W}(\tau) = E'^{+}_{L_W}(\tau, 0; 2) = \sum_{\mu \in L_W^\vee / L_W} \sum_{m \in \mathbf{Q}} \kappa_{L_W}(\mu, m) e(m\tau) \mathbf{1}_\mu \tag{46}$$

as the convergent limits

$$\kappa_{L_W}(\mu, m) = \begin{cases} \lim_{v \to \infty} b_{L_W}(\mu, m, v) & \text{if } \mu \neq 0 \text{ or } m \neq 0 \\ \lim_{v \to \infty} b_{L_W}(0, 0, v) - \log(v) & \text{if } \mu = 0 \text{ and } m = 0. \end{cases}$$

5.2. **Summation formulae.** Fix $f \in H_{1-\frac{n}{2}}(\omega_L)$. Write $l = 1 - \frac{n}{2}$ for simplicity.

5.2.1. *Decompositions of theta series.* We first justify how to decompose the Siegel theta series $\theta_L(\tau, z, h)$ for later calculation. Let L_j for $j = 1, 2$ be any pair of even lattices, with corresponding Weil representations

$$\omega_{L_j} : \widetilde{\mathrm{SL}}_2(\mathbf{Z}) \longrightarrow \mathbf{C}[L_j^\vee / L_j].$$

The Weil representation of the direct sum $L_1 \oplus L_2$ is given by the tensor product $\omega_{L_1} \otimes \omega_{L_2}$. Given

$$f(\tau) = \sum_{\mu \in L_1^\vee / L_1} f_\mu(\tau) \mathbf{1}_\mu \in H_{l_1}(\omega_{L_1})$$

and

$$g(\tau) = \sum_{\nu \in L_2^\vee / L_2} g_\nu(\tau) \mathbf{1}_\nu \in H_{l_2}(\omega_{L_2})$$

harmonic weak Maass forms of weights l_j and representations ω_{L_j}, the corresponding tensor product

$$f(\tau) \otimes g(\tau) = \sum_{\substack{\mu \in L_1^\vee / L_1 \\ \nu \in L_2^\vee / L_2}} f_\mu(\tau) g_\nu(\tau) \mathbf{1}_{\mu + \nu} \in H_{l_1 + l_2}(\omega_{L_1 \oplus L_2}) = H_{l_1 + l_2}(\omega_{L_1} \otimes \omega_{L_2})$$

determines a harmonic weak Maass form of weight $l_1 + l_2$ and representation $\omega_{L_1 \oplus L_2} = \omega_{L_1} \otimes \omega_{L_2}$.

Suppose now that $M \subset L$ is any sublattice of finite index. Observe that we have inclusions

$$M \subset L \subset L^\vee \subset M^\vee \quad \Longrightarrow \quad L/M \subset L^\vee / M \subset M^\vee / M,$$

and hence an inclusion of spaces $H_l(\omega_L) \subset H_l(\omega_M)$ for any weight $l \in \frac{1}{2}\mathbf{Z}$. Consider the natural map

$$L^\vee / M \longrightarrow L^\vee / L, \quad \mu \longmapsto \overline{\mu}.$$

Lemma 5.7. *Let $M \subset L$ be any sublattice of finite index. We have natural restriction and trace maps*

$$\mathrm{res}_{L/M} : H_l(\omega_L) \longrightarrow H_l(\omega_M), \quad f(\tau) = \sum_{\overline{\mu} \in L^\vee/L} f_{\overline{\mu}}(\tau)\mathbf{1}_{\overline{\mu}} \longmapsto f_M(\tau) = \sum_{\mu \in M^\vee/M} f_{M,\mu}(\tau)\mathbf{1}_\mu$$

and

$$\mathrm{tr}_{L/M} : H_l(\omega_M) \longrightarrow H_l(\omega_L), \quad g(\tau) = \sum_{\mu \in M^\vee/M} g_\mu(\tau)\mathbf{1}_\mu \longmapsto g^L(\tau) = \sum_{\overline{\mu} \in L^\vee/L} g^L_{\overline{\mu}}(\tau)\mathbf{1}_{\overline{\mu}}$$

such that for any pair of vector-valued forms $f \in H_l(\omega_L)$ and $g \in H_l(\omega_M)$, we have

$$\langle\langle f(\tau), g^L(\tau)\rangle\rangle = \langle\langle f_M(\tau), g(\tau)\rangle\rangle.$$

Explicitly, the restriction map is given for any $\mu \in M^\vee/M$ and $f \in H_l(\omega_L)$ by

$$f_{M,\mu}(\tau) = \begin{cases} f_{\overline{\mu}}(\tau) & \text{if } \mu \in L^\vee/M \\ 0 & \text{if } \mu \notin L^\vee/M \end{cases}$$

The trace map is given for any $\overline{\mu} \in L^\vee/L$ with fixed preimage $\mu \in L^\vee/M$ and $g \in H_l(\omega_M)$ by

$$g^L_{\overline{\mu}}(\tau) = \sum_{\nu \in L/M} g_{\nu+\mu}(\tau).$$

Proof. See [13, Lemma 3.1]. □

As explained in [13, Remark 3.2], we have for the Siegel theta series we consider the relation

$$\theta_L = (\theta_M)^L. \tag{47}$$

We shall use this relation (73) for the finite-index subgroups $M = L_0 \oplus L_0^\perp \subset L$ and $M = L_W \oplus L_W^\perp \subset L$. In particular, we obtain from (73) the relations

$$\theta_{L_0 \oplus L_0^\perp} = \theta_{L_0} \otimes \theta_{L_0^\perp} \quad \Longrightarrow \quad \theta_L = (\theta_{L_0 \oplus L_0^\perp})^L$$
$$\theta_{L_W \oplus L_W^\perp} = \theta_{L_W} \otimes \theta_{L_W^\perp} \quad \Longrightarrow \quad \theta_L = (\theta_{L_W \oplus L_W^\perp})^L$$

which via Lemma (5.7) imply the corresponding relations

$$\begin{aligned} \langle\langle f, \theta_L\rangle\rangle &= \langle\langle f, (\theta_{L_0 \oplus L_0^\perp})^L\rangle\rangle = \langle\langle f_{L_0 \oplus L_0^\perp}, \theta_{L_0 \oplus L_0^\perp}\rangle\rangle = \langle\langle f_{L_0 \oplus L_0^\perp}, \theta_{L_0} \otimes \theta_{L_0^\perp}\rangle\rangle \\ \langle\langle f, \theta_L\rangle\rangle &= \langle\langle f, (\theta_{L_W \oplus L_W^\perp})^L\rangle\rangle = \langle\langle f_{L_W \oplus L_W^\perp}, \theta_{L_W \oplus L_W^\perp}\rangle\rangle = \langle\langle f_{L_W \oplus L_W^\perp}, \theta_{L_W} \otimes \theta_{L_W^\perp}\rangle\rangle. \end{aligned} \tag{48}$$

We shall take these relations (48) for granted in what follows, and drop various subscripts and bars from the notations for simpler reading. That is, we shall simply write $\langle\langle f, \theta_{L_0} \otimes \theta_{L_0^\perp}\rangle\rangle = \langle\langle f, \theta_{L_W} \otimes \theta_{L_W^\perp}\rangle\rangle$ to denote the right-hand side(s) $\langle\langle f_{L_0 \oplus L_0^\perp}, \theta_{L_0} \otimes \theta_{L_0^\perp}\rangle\rangle = \langle\langle f_{L_W \oplus L_W^\perp}, \theta_{L_W} \otimes \theta_{L_W^\perp}\rangle\rangle$ of (48) from now on.

5.2.2. *Preliminary calculations.* We first relate our sums to the integrals appearing in the Siegel-Weil formula (Theorem 5.1), (32) and (39). Recall we consider the CM cycle $Z(V_0)$ on X_K with complex points as described in (6), as well as the geodesic set $G(W)$ with complex points as described in (7). Let us write

$$T_0 = T(V_0) := \mathrm{GSpin}(V_0) \cong \mathrm{Res}_{k(V_0)/\mathbf{Q}}\,\mathbf{G}_m \qquad [k(V_0) : \mathbf{Q}] = 2 \text{ imaginary quadratic}$$
$$T_W = T(W) := \mathrm{GSpin}(W) \cong \mathrm{Res}_{k(W)/\mathbf{Q}}\,\mathbf{G}_m \qquad [k(W) : \mathbf{Q}] = 2 \text{ real quadratic}$$

for the corresponding maximal tori in $\mathrm{GSpin}(V)$ and quadratic number fields attached to these spaces.

Let us again write U to denote either of these subspaces $V_0, W \subset V$, with $T_U = \mathrm{GSpin}(U) = \mathrm{Res}_{k(U)/\mathbf{Q}}\,\mathbf{G}_m$ for $k(U)$ the corresponding quadratic field. We have in each case a short exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(U) \longrightarrow \mathrm{SO}(U) \longrightarrow 1, \tag{49}$$

which after taking adelic points modulo rational points recovers the Hilbert exact sequence for $k = k(U)$,

$$1 \longrightarrow \mathbf{Q}^\times \backslash \mathbf{A}^\times \longrightarrow k^\times \backslash \mathbf{A}_k^\times \longrightarrow k^1 \backslash \mathbf{A}_k^1 \longrightarrow 1.$$

Recall that we fix the Haar measure on $\mathrm{SO}(U)(\mathbf{A})$ so that $\mathrm{vol}(\mathrm{SO}(U)(\mathbf{R})) = 1$ and

$$\mathrm{vol}(\mathrm{SO}(U)(\mathbf{Q}) \backslash \mathrm{SO}(U)(\mathbf{A}_f)) = 2.$$

We fix the standard Haar measure on $\mathbf{A}^\times$ with $\mathrm{vol}(\mathbf{Z}_p^\times) = 1$ for each prime p so that $\mathrm{vol}(\widehat{\mathbf{Z}}) = 1$, as well as $\mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}^\times) = 2$, and $\mathrm{vol}(\mathbf{A}_f^\times / \mathbf{Q}^\times) = \frac{1}{2}$. This determines a measure on $T_U(\mathbf{A}) \cong \mathbf{A}_{k(U)}^\times$ via (49), with

$$\mathrm{vol}(k^\times \backslash \mathbf{A}_{k,f}^\times) = \mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}_f^\times) \cdot \mathrm{vol}(k^1 \backslash \mathbf{A}_{k,f}^1) = \frac{1}{2} \cdot \mathrm{vol}\left(\mathrm{SO}(U)(\mathbf{Q}) \backslash \mathrm{SO}(U)(\mathbf{A}_f)\right) = 1$$

and

$$\mathrm{vol}(k^\times \backslash \mathbf{A}_k^\times) = \mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}^\times) \cdot \mathrm{vol}(k^1 \backslash \mathbf{A}_k^1) = 2 \cdot \mathrm{vol}\left(\mathrm{SO}(U)(\mathbf{Q}) \backslash \mathrm{SO}(U)(\mathbf{A}_f)\right) = 2.$$

Hence, we derive the following result for future use.

Lemma 5.8. *We have $\mathrm{vol}(T_U(\mathbf{Q}) \backslash T_U(\mathbf{A}_f)) = 1$ for either choice of quadratic space $U = V_0, W$. Writing $k = k(U)$ for either choice of quadratic field, let $w_k = \#\mu(k)$ denote the number of roots of unity in k. Hence, by Dirichlet's unit theorem, we know that $\#\mathcal{O}_k = w_k$ when $k = k(U) = k(V_0)$ is an imaginary quadratic field. When $k = k(U) = k(W)$ is a real quadratic field, we know that $\mathcal{O}_k^\times = \langle \varepsilon_k \rangle \times \mu(k)$ for ε_k the fundamental unit, so the solution $\varepsilon_k = \frac{1}{2}(t + u\sqrt{d_k})$ with u minimal to Pell's equation $t^2 - d_k u^2 = 4$. We have that*

$$\mathrm{vol}(\widehat{\mathcal{O}}_k^\times) = \frac{w_k}{h_k},$$

and that

$$\mathrm{vol}(k_\infty^\times \widehat{\mathcal{O}}_k^\times) = \begin{cases} \frac{2w_k}{h_k} & \text{if } k \text{ is imaginary quadratic} \\ \frac{2w_k \ln(\varepsilon_k)}{h_k} & \text{if } k \text{ is real quadratic.} \end{cases}$$

Proof. Cf. [13, Lemma 6.3]. The first claim follows from the discussion above. Now, observe that

$$1 = \int_{k^\times \backslash \mathbf{A}_{k,f}^\times} d^\times x = \int_{k^\times \backslash \mathbf{A}_{k,f}^\times / \widehat{\mathcal{O}}_k^\times} \int_{\mathcal{O}_k^\times \backslash \widehat{\mathcal{O}}_k^\times} d^\times x = \frac{h_k}{w_k} \cdot \mathrm{vol}(\widehat{\mathcal{O}}_k^\times).$$

More generally, we have that

$$2 = \int_{k^\times \backslash \mathbf{A}_k^\times} d^\times x = \int_{k^\times \backslash \mathbf{A}_k^\times / k_\infty^\times \widehat{\mathcal{O}}_k^\times} \int_{\mathcal{O}_k^\times \backslash k_\infty^\times \widehat{\mathcal{O}}_k^\times} d^\times x = \begin{cases} \frac{h_k}{w_k} \cdot \mathrm{vol}(k_\infty^\times \widehat{\mathcal{O}}_k^\times) & \text{if } k \text{ is imaginary quadratic} \\ \frac{h_k}{w_k \ln(\varepsilon_k)} \cdot \mathrm{vol}(k_\infty^\times \widehat{\mathcal{O}}_k^\times) & \text{if } k \text{ is real quadratic.} \end{cases}$$

□

Let us now consider the sums we wish to compute, which we now denote by

$$\Phi(f, Z(V_0)) := \sum_{(z_0, h) \in Z(V_0)(\mathbf{C})} \Phi(f, z_0, h)$$

(50)

$$\Phi(f, G(W)) := \sum_{(z_W, h) \in G(W)(\mathbf{C})} \Phi(f, z_W, h).$$

Note that we have two orientations $z_0^\pm \in D(V_0)$ and $z_W^\pm \in D(W)$ to consider in each case, but that we drop this from the notation henceforth for simplicity.

Lemma 5.9. *We have the following expressions for the sums* (50) *in terms of integrals over the corresponding adelic quotients of orthogonal groups* $\mathrm{SO}(V_0)$ *and* $\mathrm{SO}(W)$.

(i) *If V_0 is a rational quadratic space of signature $(0, 2)$, then we have*

$$\begin{aligned} \Phi(f, Z(V_0)) &= \frac{1}{\mathrm{vol}(K_0)} \int_{h \in SO(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \Phi(f, z_0, h) dh \\ &= \frac{\deg(Z(V_0))}{2} \int_{h \in SO(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \Phi(f, z_0, h) dh, \end{aligned}$$

where

$$\deg(Z(V_0)) = \frac{4}{\mathrm{vol}(K_0)}.$$

(ii) *If W is a rational quadratic space of signature $(1,1)$, then we have*

$$\Phi(f, G(W)) = \frac{2}{\mathrm{vol}(K_W)} \int_{(z_W,h)\in \mathrm{SO}(W)(\mathbf{Q})\backslash \mathrm{SO}(W)(\mathbf{A})} \Phi(f, z_W, h)dh.$$

Proof. See [47, Lemma 2.13]. Write $U = V_0, W \subset V$ for either of the quadratic subspaces we consider, with corresponding torus $T_U = \mathrm{GSpin}(U) \cong \mathrm{Res}_{k(U)/\mathbf{Q}}\mathbf{G}_m$, quadratic field $k(U)$, and compact open subgroup $K_U = T_U(\mathbf{A}_f) \cap K_L$. Via the exact sequence (49), we have an identification of spaces

$$\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A}) = \mathrm{GSpin}(U)(\mathbf{Q})\backslash \mathrm{GSpin}(U)(\mathbf{A})/\mathbf{A}^\times \cong k(U)^\times\backslash \mathbf{A}_{k(U)}^\times/\mathbf{A}^\times$$

and

$$\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A}_f) = \mathrm{GSpin}(U)(\mathbf{Q})\backslash \mathrm{GSpin}(U)(\mathbf{A}_f)/\mathbf{A}_f^\times \cong k(U)^\times\backslash \mathbf{A}_{f,k(U)}^\times/\mathbf{A}_f^\times.$$

Both spaces are compact, and modding out by the compact open subgroup $K_U \subset T_U(\mathbf{A}_f)$ gives finite quotients

$$\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A})/K_U = \mathrm{GSpin}(U)(\mathbf{Q})\backslash \mathrm{GSpin}(U)(\mathbf{A})/\mathbf{A}^\times K_U \cong k(U)^\times\backslash \mathbf{A}_{k(U)}^\times/\mathbf{A}^\times K_U$$

and

$$\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A}_f)/K_U = \mathrm{GSpin}(U)(\mathbf{Q})\backslash \mathrm{GSpin}(U)(\mathbf{A}_f)/\mathbf{A}_f^\times K_U \cong k(U)^\times\backslash \mathbf{A}_{f,k(U)}^\times/\mathbf{A}_f^\times K_U.$$

Given B any function of $h \in T_U(\mathbf{A}_f)$ which depends only on the image of $h \in \mathrm{SO}(U)(\mathbf{A}_f)$ and is both left-$T_U(\mathbf{Q})$-invariant and right K_U-invariant, the argument of [47, Lemma 2.13] shows that

$$\int_{\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A}_f)} B(h)dh = \mathrm{vol}(K_U) \sum_{h\in T_U(\mathbf{Q})\backslash T_U(\mathbf{A}_f)/K_U} B(h).$$

We apply this to the function $B(h) = \Phi(f, z_0, h)$ to obtain the relation

$$\Phi(f, Z(V_0)) = \frac{1}{\mathrm{vol}(K_0)} \int_{h\in \mathrm{SO}(V_0)(\mathbf{Q})\backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \Phi(f, z_0, h)dh,$$

and to the constant function $B(h) = 1$ to obtain the relation

$$\deg(Z(V_0)) = \sum_{z\in \mathrm{supp}(Z(V_0))} 1 = \frac{1}{\mathrm{vol}(K_0)} \int_{h\in \mathrm{SO}(V_0)(\mathbf{Q})\backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} dh = \frac{2}{\mathrm{vol}(K_0)}.$$

For the function $B(h) = \Phi(f, z_W, h)$, we also obtain

$$\int_{(z_W,h)\in \mathrm{SO}(W)(\mathbf{Q})\backslash \mathrm{SO}(W)(\mathbf{A})} \Phi(f, z_W, h)dh = \mathrm{vol}(K_W) \cdot \Phi(f, G(W)).$$

□

We now give the following more convenient expression for $\Phi(f, z, h)$.

Proposition 5.10 (Kudla)**.** *We have the following expressions for the regularized theta integral $\Phi(z, h)$ as limits of truncated sums of integrals. Here, we take for granted the relation of scalar products (48).*

(i) *In the CM case with negative definite lattice $L_0 \subset L$, we have for any $(z_0, h) \in D(V_0) \times T_0(\mathbf{A}_f)$ that*

$$\Phi(f, z_0, h) = \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \theta_{L_0}(\tau, z_0, h)\rangle\rangle d\mu(\tau) - A_0 \log(T) \right],$$

where

$$A_0 = \mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{0+L_0}\rangle\rangle.$$

Here, we write $\theta_{L_0^\perp}(\tau) = \theta_{L_0^\perp}(\tau, 1, 1)$, *and note that the underlying theta series* $\theta_{L_0^\perp}(\tau, z_0, h)$ *for the positive definite lattice* $L_0^\perp$ *of signature* $(n, 0)$ *is holomorphic in the variable* $\tau \in \mathfrak{H}$.

(ii) *In the case with the signature* $(1,1)$ *lattice* $L_W \subset L$, *we have for any* $(z_W, h) \in D(W) \times T_0(\mathbf{A}_f)$ *that*

$$\Phi(f, z_W, h) = \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes \theta_{L_W}(\tau, z_W, h) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right],$$

where

$$A_0 = \mathrm{CT}\langle\langle f^+(\tau), \theta^+_{L_W^\perp}(\tau) \otimes \mathbf{1}_{0+L_W} \rangle\rangle.$$

Here, we write $\theta_{L_W^\perp}(\tau) = \theta_{L_W^\perp}(\tau, 1, 1)$, *and note that the underlying theta series* $\theta_{L_W^\perp}(\tau, z_W, h)$ *for the Lorenztian lattice* $L_W^\perp$ *of signature* $(n-1, 1)$ *is nonholomorphic in the variable* $\tau \in \mathfrak{H}$. *We write* $\theta^+_{L_W^\perp}(\tau, z_W, h)$ *to denote its holomorphic/principal part.*

Proof. See [40, Proposition 2.5], with [13, Lemma 4.5] and [54, Lemma 4.18]. Let L_U denote either of the lattices $L_0, L_W \subset L$, with $M = L_U \oplus L_U^\perp \subset L$ the corresponding finite index lattice for (48) above, which again we express simply as $\langle\langle f, \theta_L \rangle\rangle = \langle\langle f, \theta_{L_U^\perp} \otimes \theta_{L_U} \rangle\rangle$. Hence, for any $z_U \in D(U)$ and $h \in T_U(\mathbf{A}_f)$, we have

$$\Phi(f, z_U, h) = \int_{\mathcal{F}}^\star \langle\langle f(\tau), \theta_L(\tau, z_U, h) \rangle\rangle d\mu(\tau) = \int_{\mathcal{F}}^\star \langle\langle f(\tau), \theta_{L_U^\perp}(\tau, 1, 1) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau).$$

Write $\theta_{L_U^\perp}(\tau) = \theta_{L_U^\perp}(\tau, 1, 1)$. Splitting the regularized integral on the right-hand side into parts according to the decomposition $f(\tau) = f^+(\tau) + f^-(\tau)$, we obtain

$$\begin{aligned} &\Phi(f, z_U, h) \\ &= \int_{\mathcal{F}}^\star \langle\langle f^+(\tau), \theta_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau) + \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f^-(\tau), \theta_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau), \end{aligned}$$

where the second integral is absolutely convergent. Let us now decompose $\theta_{L_U^\perp}(\tau) = \theta^+_{L_U^\perp}(\tau) + \theta^-_{L_U^\perp}(\tau)$ in the same way, noting that we do not need to do this when $U = V_0$ has signature $(0,2)$ so that the complement $L_U^\vee$ is positive definite and consequently the theta series $\theta_{L_U^\perp}(\tau)$ is holomorphic. We then get

$$\begin{aligned} &\int_{\mathcal{F}}^\star \langle\langle f^+(\tau), \theta_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau) \\ &= \int_{\mathcal{F}}^\star \langle\langle f^+(\tau), \theta^+_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau) + \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f^+(\tau), \theta^-_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau), \end{aligned}$$

where the second integral is again absolutely convergent. We then use [40, Proposition 2.5] to evaluate the first integral on the right-hand side of the latter identity as

$$\int_{\mathcal{F}}^\star \langle\langle f^+(\tau), \theta^+_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau) = \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle\langle f^+(\tau), \theta^+_{L_U^\perp}(\tau) \otimes \theta_{L_U}(\tau, z_U, h) \rangle\rangle d\mu(\tau) - A_0 \log T \right],$$

where

$$A_0 = \mathrm{CT}\langle\langle f^+(\tau), \theta^+_{L_U^\perp}(\tau) \otimes \mathbf{1}_{0+L_U} \rangle\rangle.$$

Putting the pieces back together, we get the stated formulae. □

Corollary 5.11. *We have the following preliminary expressions for the sums* (50).

(i) *In the CM case with the negative definite lattice* $L_0 \subset L$ *of signature* $(0,2)$, *we have*

$$\Phi(f, Z(V_0)) = \lim_{T \to \infty} \left[\frac{2}{\mathrm{vol}(K_0)} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0, -1) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right].$$

(ii) *In the geodesic case with the Lorenzian lattice* $L_W \subset L$ *of signature* $(1,1)$, *we have*

$$\Phi(f, G(W)) = \lim_{T \to \infty} \left[\frac{2}{\mathrm{vol}(K_W)} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right].$$

Proof. We start with the expressions of Lemma 5.9, evaluating the regularized theta integrals according to Proposition 5.10. Switching the order of summation and applying the Siegel-Weil formula (Theorem 5.1), so (32) for (i) and (39) for (ii), we obtain the stated formulae. □

5.2.3. *Summation along CM cycles.* We now evaluate $\Phi(f, Z(V_0))$. Let $g(\tau) = g_f(\tau) := \xi_l(f) = \xi_{1-\frac{n}{2}}(f)(\tau)$ be the holomorphic modular form of weight $2 - l = 1 + \frac{n}{2}$ obtained by applying the antilinear differential operator ξ_l to the initial harmonic weak Maass form $f \in H_l(\omega_L)$. We consider the Rankin-Selberg L-function

$$L(s, g \times \theta_{L_0^\perp}) := \langle g(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; 1)\rangle,$$

as well as its completion

$$L^\star(s, g \times \theta_{L_0^\perp}) := \Lambda(s+1, \eta_k)\langle g(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; 1)\rangle = \langle g(\tau), E^\star_{L_0}(\tau, s)\rangle.$$

Here, we write $k = k(V_0)$ for the imaginary quadratic field attached to the (incoherent) quadratic space $(V_0, Q_0) = (\mathfrak{a}, -Q_\mathfrak{a})$, with discriminant d_k and character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$, and $E_{L_0}(s, \tau; 1)$ for the corresponding (incoherent) Eisenstein series with completion $E^\star_{L_0}(s, \tau) := \Lambda(s+1, \eta_k)E_{L_0}(s, \tau; 1) = -E^\star_{L_0}(-s, \tau)$. Writing

$$g(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_g(\mu, m) e(m\tau) \mathbf{1}_\mu$$

and

$$\theta_{L_0^\perp}(\tau) = \sum_{\mu \in (L_0^\perp)^\vee/L_0^\perp} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{\theta_{L_0^\perp}}(\mu, m) e(m\tau) \mathbf{1}_\mu = \sum_{\mu \in (L_0^\perp)^\vee/L_0^\perp} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} r_{L_0^\perp}(\mu, m) e(m\tau) \mathbf{1}_\mu$$

for the Fourier series expansions of the holomorphic forms $g(\tau) \in S_{2-l}(\omega_L)$ and $\theta_{L_0^\perp}(\tau) \in H_{\frac{n}{2}}(\omega^\vee_{L_0})$, the L-function $L(s, g \times \theta_{L_0^\perp}) = \langle g, \theta_{L_0^\perp} \otimes E_{L_0}(\cdot, s; 1)\rangle$ has for $\Re(s) \gg 1$ the Dirichlet series expansion

$$L(s, g \times \theta_{L_0^\perp}) = (4\pi)^{-\left(\frac{s+n}{2}\right)} \Gamma\left(\frac{s+n}{2}\right) \sum_{\mu \in (L_0^\perp)^\vee/L_0^\perp} \sum_{m \geq 1} c_g(\mu, m) r_{L_0^\perp}(\mu, m) m^{-\left(\frac{s+n}{2}\right)}. \tag{51}$$

Theorem 5.12 (Bruinier-Yang)**.** *We have that*

$$\begin{aligned}\Phi(f, Z(V_0)) &= -\frac{4}{\mathrm{vol}(K_0)}\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_0^\perp})\right) \\ &= -\deg(Z(V_0))\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_0^\perp})\right).\end{aligned}$$

Proof. See [13, Theorem 4.7]. As there seems to be at least one sign error in their formula[6], we supply a detailed proof. We know from Corollary 5.11 that we have

$$\Phi(f, Z(V_0)) = \lim_{T \to \infty}\left[\frac{2}{\mathrm{vol}(K_0)} \cdot I_T(f) - A_0 \log(T)\right],$$

where

$$I_T(f) = \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1)\rangle\rangle d\mu(\tau)$$

and

$$A_0 = \mathrm{CT}\langle\langle f^+(\tau), \theta^+_{L_0^\perp}(\tau) \otimes \mathbf{1}_{0+L_0}\rangle\rangle.$$

[6]See also [1, Theorem 5.7.1], where the same sign error for the contribution of $L'(0, \xi_{1-n/2}(f) \times \theta_{L_0^\perp})$ in [13, Theorem 4.7] is acknowledged. That is, the integral in the last line of [13, p. 654] should be evaluated using the differential forms identity $\overline{\partial}(f\tau) = -v^{2-l}\overline{\xi_l(f)}d\mu(\tau) = -L_l f d\mu(\tau)$, and the substitution made implicity for the first identity in [13, p. 655] misses the sign change. Moreover, the application of Stokes' theorem for the remaining integral does not involve a change of sign after identifying the boundary $\partial\mathcal{F}_T$ with the interval $[iT, 1+iT]$.

To evaluate the integral $I_T(f)$, we first use the identity (36) and the relation $d = \partial + \overline{\partial}$ to compute

$$I_T(f) = -2 \int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\vee} \otimes \overline{\partial} E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau = -2 \int\limits_{\mathcal{F}_T} \langle\langle \partial f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau$$
$$= -2 \int\limits_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau + 2 \int\limits_{\mathcal{F}_T} \langle\langle \overline{\partial} f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau,$$

which after using Lemma 5.2 to compute the second integral in the latter expression becomes

$$I_T(f) = -2 \int\limits_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - 2 \int\limits_{\mathcal{F}_T} \langle\langle \xi_l(f)(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle v^{2-l} d\mu(\tau),$$

and which after using Stokes' theorem to evaluate the first integral becomes

$$I_T(f) = -2 \int\limits_{\partial\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - 2 \int\limits_{\mathcal{F}_T} \langle\langle \xi_l(f)(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle v^{2-l} d\mu(\tau)$$
$$= -2 \int\limits_{\tau = iT}^{1+iT} \langle\langle f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - 2 \int\limits_{\mathcal{F}_T} \langle\langle \xi_{1-\frac{n}{2}}(f)(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle v^{1+\frac{n}{2}} d\mu(\tau).$$

Inserting this back into the initial formula, we obtain

$$\Phi(f, Z(V)_0) = -\frac{4}{\mathrm{vol}(K_0)} \cdot \langle g(\tau), \theta_{L_0^\perp}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle$$
$$- \lim_{T \to \infty} \left[\frac{4}{\mathrm{vol}(K_0)} \int\limits_{\tau = iT}^{1+iT} \langle\langle f(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - A_0 \log(T) \right].$$

To evaluate the limiting term in this latter expression, we first split the integral into parts according to the decomposition $f(\tau) = f^+(\tau) + f^-(\tau)$ of $f(\tau)$ into principal/holomorphic and nonholomorphic parts as

$$\lim_{T \to \infty} \int\limits_{\tau = iT}^{1+iT} \langle\langle f(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau$$
$$= \lim_{T \to \infty} \int\limits_{\tau = iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau + \lim_{T \to \infty} \int\limits_{\tau = iT}^{1+iT} \langle\langle f^-(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau.$$

We argue that the second integral on the right of this latter expression vanishes (cf. [22, Theorem 3.5]). To be more precise, let us write the Fourier series expansion as

$$\langle\langle f^-(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle = \sum_{m \in \mathbf{Z}} a(m, iv) e(m\tau), \quad \tau = u + iv \in \mathfrak{H}.$$

Using the orthogonality of additive characters, we find that

$$\int_{\tau = iT}^{1+iT} \langle\langle f^-(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau = \int_0^1 \langle\langle f^-(u + iT), \theta_{L_0^\vee}(u + iT) \otimes E'_{L_0}(u + iT, 0; 1)\rangle\rangle du$$
$$= \sum_{m \in \mathbf{Z}} a(m, iT) \int_0^1 e(mu) du = a(0, iT).$$

Here,

$$a(0, iT) = \sum_{\mu \in L^\vee / L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f^-(\mu, -m) W_l(-2\pi m v) c_F(\mu, m, v)$$

denotes the constant coefficient of the scalar-valued form $\langle\langle f^-(\tau), \theta_{L_0^\perp}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle$, and we write

$$F(\tau) := \theta_{L_0^\perp}(\tau) \otimes E'_{L_0}(\tau, 0; 1) = \sum_{\mu \in (L_0^\perp \oplus L_0)^\vee/(L_0^\perp \oplus L_0)} \sum_{m \in \mathbf{Q}} c_F(\mu, m, v) e(m\tau) \mathbf{1}_\mu.$$

Recall that the Whittaker coefficients $W_l(y) := \int_{-2y}^\infty e^{-t} t^{-l} dt = \Gamma(1 - l, 2|y|)$ decay rapidly for $y \to -\infty$. Using this together with standard bounds for the Fourier coefficients of $f^-(\tau)$ and $F(\tau)$, we deduce that for some integer $M > 0$ and constant $C > 0$ we have for each integer $m \geq M$ the bounds

$$c_f^-(\mu, -m) W_l(-2\pi m v) c_F(\mu, m, v) = O(e^{-mCv}).$$

Hence, via geometric series, we derive the bound

$$a(0, iT) = O\left(\frac{e^{-CT}}{(1 - e^{-CT})}\right).$$

It is then apparent that

$$\lim_{T\to\infty} a(0, iT) = \lim_{T\to\infty} \int_{\tau=iT}^{1+iT} \langle\langle f^-(\tau), \theta_{L_0^\vee}(\tau) \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau = 0.$$

Thus, it remains to evaluate the streamlined expression

$$\lim_{T\to\infty} \left[\frac{4}{\mathrm{vol}(K_0)} \int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - A_0 \log(T)\right].$$

Here, we first use the calculation of coefficients (37) to see that

$$\begin{aligned}
&\lim_{T\to\infty} \left[\int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau - A_0 \log(T)\right] \\
&= \lim_{T\to\infty} \int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \sum_{\mu \in L_0^\vee/L_0} \sum_{m \in \mathbf{Q}} \left(b_{L_0}(\mu, m, v) - \delta_{\mu,0}\delta_{m,0} \log(v)\right) e(m\tau) \mathbf{1}_\mu\rangle\rangle d\tau \\
&= \mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle.
\end{aligned}$$

To compute the remaining integral

$$\lim_{T\to\infty} \left(\frac{4}{\mathrm{vol}(K_0)} - 1\right) \int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee} \otimes E'_{L_0}(\tau, 0; 1)\rangle\rangle d\tau,$$

we decompose the Eisenstein series $E'_{L_0}(\tau) \in H_1(\omega_{L_0})$ into principal/holomorphic nonholomorphic parts $E'_{L_0}(\tau) = E'^{+}_{L_0}(\tau) + E'^{-}_{L_0}(\tau)$ to get the corresponding decomposition of integrals. Again, we argue that the contributions from the nonholomorphic parts vanish. To be more precise, we claim here that

$$\lim_{T\to\infty} \int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee} \otimes E'^{-}_{L_0}(\tau, 0; 1)\rangle\rangle d\tau = 0.$$

To see this, we again open up Fourier series expansions and use the orthogonality of characters to find that

$$\begin{aligned}
&\lim_{T\to\infty} \int_{\tau=iT}^{1+iT} \langle\langle f^+(\tau), \theta_{L_0^\vee}(\tau)\otimes E_{L_0}'^{-}(\tau,0;1)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \int_0^1 \langle\langle f^+(u+iT), \theta_{L_0^\vee}(u+iT)\otimes E_{L_0}'^{-}(u+iT,0;1)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \sum_{\mu\in(L_0^\perp+L_0)^\vee/(L_0^\perp+L_0)} \sum_{\substack{m\in\mathbf{Q}\\ m>0}} c_f^+(\mu,m) c_{\theta_{L_0^\perp}\otimes E_{L_0}'^{-}}(-\mu,-m) W_1(-2\pi m T) \\
&= \lim_{T\to\infty} \sum_{\mu\in(L_0^\perp+L_0)^\vee/(L_0^\perp+L_0)} \sum_{\substack{m\in\mathbf{Q}\\ m>0}} c_f^+(\mu,m) \sum_{\substack{\mu_1\in(L_0^\perp)^\vee/L_0^\perp\\ \mu_2\in L_0^\vee/L_0\\ \mu_1+\mu_2\equiv-\mu \bmod (L_0^\perp+L_0)}} \sum_{\substack{m_1\in\mathbf{Q}_{\geq 0}\\ m_2\in\mathbf{Q}_{<0}\\ m_1+m_2=-m}} c_{\theta_{L_0^\perp}}(\mu_1,m_1) c^-_{E'_{L_0}}(\mu_2,m_2) W_1(-2\pi m_2 T).
\end{aligned}$$

Again, we use the rapid decay of the Whittaker function $W_1(y) := \int_{-2y}^\infty e^{-t}t^{-1}dt = \Gamma(-1,2|y|)$ with $y\to-\infty$ to see that each inner sum tends to zero with $T\to\infty$. In this way, we derive the stated formula

$$\Phi(f, Z(V_0)) = -\frac{4}{\mathrm{vol}(K_0)}\left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau)\otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, g\times\theta_{L_0^\perp})\right).$$

□

5.2.4. *Summation along geodesic sets.* We now evaluate $\Phi(f, G(W))$. Again, we consider $g(\tau) = \xi_{1-\frac{n}{2}}(f)(\tau)$ the holomorphic modular form of weight $2-l = 1+\frac{n}{2}$ obtained by applying ξ_l to $f\in H_l(\omega_L)$. We consider the Rankin-Selberg L-function

$$L(s, g\times\theta_{L_W^\perp}) := \langle g(\tau), \theta_{L_W^\perp}(\tau)\otimes E_{L_W}(\tau,s;2)\rangle,$$

as well as its completion

$$L^\star(s, g\times\theta_{L_W^\perp}) := \Lambda(s+1,\eta_k)\langle g(\tau), \theta_{L_W^\perp}(\tau)\otimes E_{L_W}(\tau,s;2)\rangle = \langle g(\tau), E^\star_{L_W}(\tau,s;2)\rangle.$$

Here, we write $k=k(W)$ for the real quadratic field attached to the quadratic space $(W,Q_W) = (\mathfrak{a}, Q_\mathfrak{a})$, with discriminant d_k and character $\eta_k(\cdot) = \left(\frac{d_k}{\cdot}\right)$, and $E_{L_W}(s,\tau;2)$ for the corresponding (coherent) Eisenstein series of weight $l=2$ with completion $E^\star_{L_W}(s,\tau;2) := \Lambda(s+1,\eta_k)E_{L_W}(s,\tau;2)$. Notice that by Corollary 5.5, the image under the Maass weight lowering operator L_2 of the first derivative $E^{\star\prime}_{L_W}(\tau,s;2) = \frac{d}{ds}E^\star_{L_W}(\tau,s;2)$ of this Eisenstein series $E^\star_{L_W}(\tau,s;2)$ at the central point $s=0$ satisfies the functional identity

$$L_2 E^{\star\prime}_{L_W}(\tau,0;2) = E^\star_{L_W}(\tau,0;0).$$

Here, $E^\star_{L_W}(\tau,s;0) := \Lambda(s+1,\eta_k)E_{L_W}(\tau,s;0)$ is the coherent Eisenstein series associated to the quadratic space $(\mathfrak{a}, Q_\mathfrak{a})$ satisfying the even, symmetric Langlands functional equation $E^\star_{L_W}(\tau,s;0) = E^\star_{L_W}(\tau,-s;0)$.

Remark 5.13. Notice that while the latter Eisenstein series $E_{L_W}(\tau,s;0)$ of weight 0 (at $s=0$) appears in the Siegel-Weil formula (39) for the average over theta series $\theta_{L_W}(\tau,s,h)$, it is rather the Eisenstein series $E_{L_W}(\tau,s;2)$ of weight 2 that appears in integral presentation for the Rankin-Selberg L-function $L(s, g\times\theta_{L_W^\perp})$.

To describe the Dirichlet series expansion of $L(s, g\times\theta_{L_W^\perp})$, let us write the Fourier series expansion of the holomorphic/principal part $\theta^+_{L_W^\perp}(\tau)$ of the theta series $\theta_{L_W^\perp}(\tau) = \theta^+_{L_W^\perp}(\tau) + \theta^-_{L_W^\perp}(\tau) \in H_{\frac{n-1}{2}}(\omega^\vee_{L_W^\perp})$ as

$$\theta^+_{L_W^\perp}(\tau) = \sum_{\mu\in(L_W^\perp)^\vee/L_W^\perp} \sum_{\substack{m\in\mathbf{Q}\\ m\gg-\infty}} c^+_{\theta_{L_W^\perp}}(\mu,m)e(m\tau)\mathbf{1}_\mu$$

with positive coefficients denoted by $r_{L_W^\perp}(\mu,m) := c^+_{\theta_{L_W^\perp}}(\mu,m)$ for $m>0$. We then have the expansion

$$L(s, g\times\theta_{L_W^\perp}) = (4\pi)^{-\left(\frac{s+n}{2}\right)}\Gamma\left(\frac{s+n}{2}\right) \sum_{\mu\in(L_W^\perp)^\vee/L_W^\perp} \sum_{m\geq1} c_g(\mu,m) r_{L_W^\perp}(\mu,m) m^{-\left(\frac{s+n}{2}\right)}. \quad (52)$$

Theorem 5.14. *We have that*

$$\Phi(f, G(W)) = -\frac{4}{\mathrm{vol}(K_W)} \left(\mathrm{CT}\langle\langle f^+(\tau), \theta^+_{L_W^\perp}(\tau) \otimes \mathcal{E}_{L_W}(\tau)\rangle\rangle + L'(0, g \times \theta_{L_W^\perp}) \right).$$

Proof. See [54, Theorem 4.16]. A minor generalization of the same argument works here, or that of Theorem 5.12 using the identity of differential forms (43) in lieu of (36). To be sure, we know from Corollary 5.11 that

$$\Phi(f, G(W)) = \lim_{T\to\infty} \left[\frac{2}{\mathrm{vol}(K_W)} \cdot I_T(f) - A_0 \log(T) \right],$$

where

$$I_T(f) = \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0)\rangle\rangle d\mu(\tau)$$

and

$$A_0 = \mathrm{CT}\langle\langle f^+(\tau), \theta^+_{L_W^\perp}(\tau) \otimes \mathbf{1}_{0+L_W}\rangle\rangle.$$

To evaluate the integral $I_T(f)$, we first use the identity (43) and the relation $d = \partial + \overline{\partial}$ to compute

$$\begin{aligned} I_T(f) &= -2 \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\vee} \otimes \overline{\partial} E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau \\ &= -2 \int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau + 2 \int_{\mathcal{F}_T} \langle\langle \overline{\partial} f(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau, \end{aligned}$$

which after using Lemma 5.2 to compute the second integral in the latter expression becomes

$$I_T(f) = -2 \int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau - 2 \int_{\mathcal{F}_T} \langle\langle \xi_l(f)(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle v^{2-l} d\mu(\tau),$$

and which after using Stokes' theorem to evaluate the first integral becomes

$$I_T(f) = -2 \int_{\tau=iT}^{1+iT} \langle\langle f(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau - 2 \int_{\mathcal{F}_T} \langle\langle \xi_{1-\frac{n}{2}}(f)(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle v^{1+\frac{n}{2}} d\mu(\tau).$$

Inserting this back into the initial formula, we obtain the preliminary formula

$$\begin{aligned} \Phi(f, G(W)) &= -\frac{4}{\mathrm{vol}(K_W)} \cdot \langle g(\tau), \theta_{L_W^\perp}(\tau) \otimes E'_{L_W}(\tau, 0; 2)\rangle \\ &\quad - \lim_{T\to\infty} \left[\frac{4}{\mathrm{vol}(K_W)} \int_{\tau=iT}^{1+iT} \langle\langle f(\tau), \theta_{L_W^\vee} \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau - A_0 \log(T) \right]. \end{aligned}$$

As in the proof of Theorem 5.12, we argue that the limiting constant coefficient integral depends only on the holomorphic parts. The only difference is that the theta series $\theta_{L_W}(\tau)$ is not holomorphic. Hence, we decompose it into holomorphic/principal and nonholomorphic parts $\theta_{L_W}(\tau) = \theta^+_{L_W}(\tau) + \theta^-_{L_W}(\tau)$. That is, we split the constant coefficient term in this preliminary expression into three parts

$$\begin{aligned} &\lim_{T\to\infty} \int_{\tau=iT}^{iT+1} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau \\ &= \lim_{T\to\infty} \int_{\tau=iT}^{iT+1} \langle\langle f^+(\tau), \theta^+_{L_W^\perp}(\tau) \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau \\ &+ \lim_{T\to\infty} \int_{\tau=iT}^{iT+1} \langle\langle f^+(\tau), \theta^-_{L_W^\perp}(\tau) \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau \\ &+ \lim_{T\to\infty} \int_{\tau=iT}^{iT+1} \langle\langle f^-(\tau), \theta_{L_W^\perp}(\tau) \otimes E'_{L_W}(\tau, 0; 2)\rangle\rangle d\tau. \end{aligned}$$

We argue in the same way as for the proof of Theorem 5.12 that the third integral on the right-hand side vanishes. The same argument also shows that the second integral in this expression vanishes. Hence, only the first integral contributes. To evaluate its contribution to the initial expression

$$\lim_{T\to\infty}\left[\frac{4}{\mathrm{vol}(K_W)}\int\limits_{\tau=iT}^{1+iT}\langle\langle f^+(\tau),\theta^+_{L_W^\vee}\otimes E'_{L_W}(\tau,0;2)\rangle\rangle d\tau - A_0\log(T)\right],$$

we again use the calculations of (46) to find that

$$\begin{aligned}&\lim_{T\to\infty}\left[\int\limits_{\tau=iT}^{1+iT}\langle\langle f^+(\tau),\theta_{L_W^\vee}\otimes E'_{L_W}(\tau,0;2)\rangle\rangle d\tau - A_0\log(T)\right]\\&=\lim_{T\to\infty}\int_{\tau=iT}^{1+iT}\langle\langle f^+(\tau),\theta_{L_W^\perp}(\tau)\otimes\sum_{\mu\in L_W^\vee/L_W}\sum_{m\in\mathbf{Q}}\left(b_{L_W}(\mu,m,v)-\delta_{\mu,0}\delta_{m,0}\log(v)\right)e(m\tau)\mathbf{1}_\mu\rangle\rangle d\tau\\&=\mathrm{CT}\langle\langle f^+(\tau),\theta_{L_W^\perp}(\tau)\otimes\mathcal{E}_{L_W}(\tau)\rangle\rangle,\end{aligned}$$

and argue the same way via the rapid decay of the Whittaker functions $W_2(y)$ that

$$\lim_{T\to\infty}\int\limits_{\tau=iT}^{1+iT}\langle\langle f^+(\tau),\theta_{L_W^\vee}\otimes E'^{-}_{L_W}(\tau,0;2)\rangle\rangle d\tau = 0.$$

In this way, we see that

$$\begin{aligned}&-\lim_{T\to\infty}\left[\frac{4}{\mathrm{vol}(K_W)}\int\limits_{\tau=iT}^{1+iT}\langle\langle f(\tau),\theta_{L_W^\vee}\otimes E'_{L_W}(\tau,0;2)\rangle\rangle d\tau - A_0\log(T)\right]\\&=-\frac{4}{\mathrm{vol}(K_W)}\cdot\mathrm{CT}\langle\langle f^+(\tau),\theta^+_{L_W^\perp}(\tau)\otimes\mathcal{E}_{L_W}(\tau)\rangle\rangle.\end{aligned}$$

The stated formula now follows from the preliminary formula. □

6. Integral presentations of Rankin-Selberg L-functions

We now explain how to identify the Rankin-Selberg L-functions $L^\star(s,\xi_{1-n/2}(f)\times\theta_{L_U^\perp})$ appearing in Theorems 5.12 and 5.14 with standard Rankin-Selberg L-functions for $\mathrm{GL}_2(\mathbf{A})\times\mathrm{GL}_2(\mathbf{A})$.

Let k be any (real or imaginary) quadratic field of discriminant d_k and corresponding Dirichlet character $\eta_k(\cdot)=\left(\frac{d_k}{\cdot}\right)$. We consider the ideal class group[7] $C(\mathcal{O}_k)$ of k. Recall that we fix an integer ideal representative $\mathfrak{a}\subset\mathcal{O}_k$ for each class $A=[\mathfrak{a}]\in C(\mathcal{O}_k)$, and write $Q_\mathfrak{a}(z)=\mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ for the corresponding norm form. Again, each space $(\mathfrak{a},Q_\mathfrak{a})$ has signature $(2,0)$ when k is an imaginary quadratic field, and signature $(1,1)$ when k is a real quadratic field. We consider for each class $A\in C(\mathcal{O}_k)$ the rational quadratic space (V_A,Q_A) of signature $(2,2)$ given by $V_A=\mathfrak{a}_\mathbf{Q}\oplus\mathfrak{a}_\mathbf{Q}$ and quadratic form $Q_A(z)=Q_A(z_1,z_2)=Q_\mathfrak{a}(z_1)-Q_\mathfrak{a}(z_2)$. We fix a level N prime to d_k, and consider the lattice $L_A\subset V_A$ whose adelization corresponds to the compact open subgroup K_A of $\mathrm{GSpin}(V_A)(\mathbf{A}_f)\cong\mathrm{GL}_2(\mathbf{A}_f)^2$ given by $K_0(N)^2$, as in Proposition 2.4. Hence, the corresponding spin Shimura variety X_{K_A} can be identified with $Y_0(N)\times Y_0(N)$. In this setting, we explain two ways to associate with a cuspidal newform $\phi\in S_l(\Gamma_0(N))$ a vector-valued cusp form $g_\phi=g_{\phi,A}\in S_l(\omega_{L_A})$. As we explain below, we can use the Doi-Naganuma lift (see e.g. [7, §3.1], [61]) to show the existence of such a form. We can also use the theorem of Strömberg [51, Theorem 5.4] – see also Scheithauer [47, Theorem 3.1], Zhang [60, Theorem 4.15], and Bruinier-Bundschuh [8] – to construct such a form more explicity. We then show that we have identifications of completed Rankin-Selberg L-functions

$$L^\star(2s-s,g_{\phi,A}\times\theta_A)=\Lambda(s-1/2,\phi\times\theta_A), \tag{53}$$

[7]More generally, we could consider the ring class group $C(\mathcal{O})$ of any order $\mathcal{O}\subset\mathcal{O}_k$ for all of the analytic/archimedean discussion here. However, since the discussion of integral models and arithmetic heights in [1] is so far only understood for the maximal order $\mathcal{O}_k$, we stick to this case for simplicity.

where θ_A denotes the Hecke theta series associated to the class $A \in C(\mathcal{O}_k)$, and hence that

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s - s, g_{\phi,A} \times \theta_A) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) \Lambda(s - 1/2, \phi \times \theta_A) = \Lambda(s - 1/2, \pi \times \theta(\chi)). \tag{54}$$

We then use this to reinterpret the calculations of Theorems 5.12 and 5.14 in terms of $\Lambda(s, \phi \times \theta(\chi))$.

6.1. **Equivalences of L-functions.** We now show the identifications (53) and (54).

6.1.1. *Hecke theta series associated to class group characters of quadratic fields.* Given a quadratic field k as above and a class group character $\chi : C(\mathcal{O}_k) \longrightarrow \mathbf{C}^\times$, we consider the corresponding Hecke theta series

$$\theta(\chi)(\tau) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) \theta_A(\tau).$$

Here, each $\theta_A(\tau)$ denotes the theta series associated to the class $A \in C(\mathcal{O}_k)$ and quadratic space $(\mathfrak{a}, Q_\mathfrak{a})$. Hence, when k is an imaginary quadratic field, this theta series has the explicit expansion

$$\theta_A(\tau) = \frac{1}{w_k} \sum_{\lambda \in \mathfrak{a}} e\left(Q_\mathfrak{a}(\lambda)\tau\right) = \sum_{m \in \mathbf{Z}_{\geq 0}} r_A(m) e(m\tau),$$

where $w_k = \#\mu(k)/2$ denotes half the number of roots of unity in k, and $r_A(m)$ the counting function

$$r_A(m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mathfrak{a} : Q_\mathfrak{a}(\lambda) = m\right\}.$$

A classical theorem of Hecke shows that this theta series $\theta_A \in M_1(\Gamma_0(|d_k|), \eta_k)$ is a modular form of weight $1 = (2-0)/2$, level $\Gamma_0(|d_k|)$, and character η_k. Hence, $\theta(\chi) \in M_1(\Gamma_0(|d_k|), \eta_k)$ when k is imaginary quadratic. When k is a real quadratic field, the unit group $\mathcal{O}_k^\times \cong \mathbf{Z} \times \mu(k) = \langle \varepsilon_k \rangle \times \mu(k)$ is no longer torsion, and we must fix a fundamental domain $\mathfrak{a}^\star$ for the action of $\mathcal{O}_k^\times/\mu(k) = \langle \varepsilon_k \rangle$ on the lattice $\mathfrak{a} \subset k_\infty \cong \mathbf{R}^2$. We can then describe the corresponding theta series via the explicit expansion

$$\theta_A(\tau) = \frac{1}{w_k} \sum_{\lambda \in \mathfrak{a}^\star} e\left(Q_\mathfrak{a}(\lambda)\tau\right) = \sum_{m \in \mathbf{Z}_{\geq 0}} r_A(m) e(m\tau),$$

where $w_k = \#\mu(k)/2$ again denotes half the number of roots of unity in k, and $r_A(m)$ the counting function

$$r_A(m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mathfrak{a}^\star : Q_\mathfrak{a}(\lambda) = m\right\}.$$

The theorem of Hecke shows that this $\theta_A \in M_0(\Gamma_0(d_k), \eta_k)$ is a modular form of weight $0 = (1-1)/2$, level $\Gamma_0(|d_k|)$, and character η_k. Hence, $\theta(\chi) \in M_0(\Gamma_0(d_k), \eta_k)$ when k is real quadratic.

Let us henceforth fix such a Hecke theta series

$$\theta(\chi) \in M_{l(k)}(\Gamma(|d_k|), \eta_k), \quad l(k) := \begin{cases} 1 & \text{if } k \text{ is imaginary quadratic} \\ 0 & \text{if } k \text{ is real quadratic} \end{cases}.$$

6.1.2. *Rankin-Selberg L-functions.* Let $\phi \in S_{l(\phi)}(\Gamma_0(N))$ be a holomorphic cusp form of weight $l(\phi)$ on $\Gamma_0(N)$. We write the Fourier series expansion as

$$\phi(\tau) = \sum_{m \geq 1} c_\phi(m) e(m\tau) = \sum_{m \geq 1} a_\phi(m) m^{\frac{l(\phi)-1}{2}} e(m\tau).$$

so that the finite part $L(s, \phi)$ of the standard L-function $\Lambda(s, \phi) = L_\infty(s, \phi) L(s, \phi)$ has Dirichlet series expansion for $\Re(s) > 1$ given by $L(s, f) = \sum_{m \geq 1} a_\phi(m) m^{-s} = \sum_{m \geq 1} c_\phi(m) m^{-(s+1/2)}$. Let us also write the Fourier series expansion of the theta series $\theta(\chi) \in M_{l(k)}(\Gamma_0(|d_k|), \eta_k)$ as

$$\theta(\chi)(\tau) = \sum_{m \geq 1} c_{\theta(\chi)}(m) e(m\tau) = \sum_{m \geq 1} a_{\theta(\chi)}(m) m^{\frac{l(k)-1}{2}} e(m\tau).$$

We look at the corresponding Rankin-Selberg L-functions

$$\begin{aligned} L(s, \phi \times \theta(\chi)) &= L(2s, \eta_k) \sum_{m \geq 1} c_\phi(m) c_{\theta(\chi)}(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)} \\ &= L(2s, \eta_k) \sum_{m \geq 1} a_\phi(m) a_{\theta(\chi)}(m) m^{-s}. \end{aligned}$$

That is, we consider the corresponding partial Rankin-Selberg L-functions, defined first for $\Re(s) > 1$ by

$$L(s, \phi \times \theta_A) = L(2s, \eta_k) \sum_{m \geq 1} c_\phi(m) r_A(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}$$

whose χ-twisted linear combinations give the χ-twisted Rankin-Selberg L-function

$$L(s, \phi \times \theta(\chi)) = L(2s, \eta_k) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \sum_{m \geq 1} c_\phi(m) r_A(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}.$$

We can also consider the quadratic twist $\phi \otimes \eta_k \in S_{l(\phi)}(\Gamma_0(d_k^2 N), \eta_k)$, with Fourier series expansion

$$\phi \otimes \eta_k(\tau) = \sum_{m \geq 1} c_\phi(m) \eta_k(m) e(m\tau) = \sum_{m \geq 1} a_\chi(m) m^{\frac{l(\phi)-1}{2}} \eta_k(m) e(m\tau).$$

Here, the corresponding partial Rankin-Selberg L-functions

$$L(s, \phi \otimes \eta_k \times \theta_A) = \langle \phi, \theta_A E_A(\cdot, s; l(\phi) + l(k)) \rangle = L(2s, \eta_k^2) \sum_{m \geq 1} c_\phi(m) \eta_k(m) r_A(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}$$

give rise to the corresponding χ-twisted Rankin-Selberg L-function

$$L(s, \phi \times \theta(\chi)) = \Lambda(2s, \eta_k^2) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \sum_{m \geq 1} c_\phi(m) \eta_k(m) r_A(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}.$$

Note (cf. [25, §V.1]) that we have the integral presentation

$$\frac{\Gamma\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}{(4\pi)^{s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1}} \sum_{m \geq 1} c_\phi(m) c_{\theta(\chi)}(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)} = \langle \phi, \theta_A E_{A,\eta_k}(\cdot, s; l(\phi) + l(k)) \rangle \tag{55}$$

for $E_{A,\eta_k}(\tau, s; l(\phi) - l(k)) \in M_{l(\phi)-l(k)}(\Gamma_0(\mathrm{lcm}(d_k, N)), \eta_k)$ some uniquely-determined Eisenstein series of weight $l(\phi) - l(k)$, level $\Gamma_0(\mathrm{lcm}(d_k, N))$, and character η_k. Similarly, we have the integral presentation

$$\frac{\Gamma\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)}{(4\pi)^{s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1}} \sum_{m \geq 1} c_{\phi \otimes \eta_k}(m) c_{\theta(\chi)}(m) m^{-\left(s + \left\{\frac{l(\phi)+l(k)}{2}\right\} - 1\right)} = \langle \phi, \theta_A E_A(\cdot, s; l(\phi) - l(k)) \rangle \tag{56}$$

for $E_A(\tau, s; l(\phi) - l(k)) \in M_{l(\phi)-l(k)}(\Gamma_0(d_k^2 N))$ some Eisenstein series of weight $l(\phi) - l(k)$, level $\Gamma_0(d_k^2 N)$, and trivial character $\eta_k^2 = \mathbf{1}$. The classical theory of Rankin-Selberg convolution shows that these Rankin-Selberg L-functions have analytic continuations given by a functional equation inherited from the Eisenstein series appearing in these integral presentations (55) and (56). Here, we have the following more precise result.

Proposition 6.1. *Let $\phi \in S_{l(\phi)}(\Gamma_0(N))$ be a normalized newform. Assume that $(N, d_k) = 1$ and that $l(\phi) > l(k)$, where $l(k) = \{0, 1\}$ denotes the weight of the Hecke theta series $\theta(\chi) \in M_{l(k)}(\Gamma_0(|d_k|), \eta_k)$. Put*

$$L_\infty(s, \phi \times \theta(\chi)) = (2\pi)^{-2s} \Gamma\left(s - \left\{\frac{l(\phi) - l(k)}{2}\right\}\right) \Gamma\left(s + \left\{\frac{l(\phi) + l(k)}{2}\right\} - 1\right).$$

Then, the completed L-function

$$\Lambda(s, \phi \times \theta(\chi)) := L_\infty(s, \phi \times \theta(\chi)) L(s, \phi \times \theta(\chi))$$

satisfies the symmetric functional equation

$$\Lambda(s, \phi \times \theta(\chi)) = \eta_k(-N) |d_k N|^{1-2s} \Lambda(1 - s, \phi \times \theta(\chi)).$$

Proof. The proof is well-known for the more general setup of $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-functions given by Jacquet [30] and Jacquet-Langlands [31]. Here, we give the more explicit classical calculation of Li [43, Theorem 2.2, Example 2]. □

Remark 6.2. Note that the class group characters χ are wide ray class characters, and hence the archimedean factor $L_\infty(s, \phi \times \theta(\chi))$ does not depend on the choice of χ, although it does depend on the choice of quadratic field k. This is also apparent by inspection of the formula for $L_\infty(s, \phi \times \theta(\chi))$ given in Proposition 6.1.

6.1.3. *Quadratic basechange equivalences.* Fix a cuspidal newform $\phi \in S_{l(\phi)}(\Gamma_0(N))$ of weight $l(\phi) > l(k)$ on $\Gamma_0(N)$ of trivial central character as in Proposition 6.1. Let $\pi(\phi) = \otimes_v \pi(\phi)_v$ denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ determined by ϕ, with $\Lambda(s, \pi(\phi)) = \prod_{v \leq \infty} L(s, \pi(\phi)_v)$ its standard L-function. Note that this coincides with the corresponding completed L-function $\Lambda(s, \phi) = L_\infty(s, \phi) L(s, \phi)$ of ϕ. Let us also write $\pi(\chi)$ to denote the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ determined by the class group character $\chi \in C(\mathcal{O}_k)^\vee$, determined by the corresponding theta series $\theta(\chi) \in M_{l(k)}(\Gamma_0(d_k), \eta_k)$. Hence, we consider the corresponding Rankin-Selberg L-function

$$\Lambda(s, \pi(\phi) \times \pi(\chi)) = \prod_{v \leq \infty} L(s, \pi(\phi)_v \times \pi(\chi)_v) = \Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi)) L(s, \phi \times \theta(\chi)).$$

Let us now consider the quadratic basechange lifting

$$\Pi(\phi) = \otimes_w \Pi(\phi)_w = \mathrm{BC}_{k/\mathbf{Q}}(\pi(\phi))$$

of $\pi(\phi)$ to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_k)$. Such a lifting exists by the theta lifting construction of Shintani (c.f. [7, §2.7]), and more generally for any $\mathrm{GL}_2(\mathbf{A})$-automorphic representation by Langlands [42], and for any $\mathrm{GL}_n(\mathbf{A})$-automorphic representation by Arthur-Clozel [2]. We refer to the article [24] for more background on these quadratic basechange liftings and their L-functions. In brief, writing $\Lambda(s, \Pi(\phi)) = \prod_{w \leq \infty} L(s, \Pi(\phi)_w)$ to denote the corresponding completed standard L-function of $\Pi(\chi)$, we have a equivalences of standard L-functions

$$\Lambda(s, \Pi(\phi)) = \Lambda(s, \pi(\phi)) \Lambda(s, \pi(\phi) \otimes \eta_k) = \Lambda(s, \phi) \Lambda(s, \phi \otimes \eta_k) \tag{57}$$

and

$$\Lambda(s, \Pi(\phi) \otimes \chi) = \Lambda(s, \pi(\phi) \times \pi(\chi)) = \Lambda(s, \phi \times \theta(\chi)). \tag{58}$$

To be clear, the identity (58) relates the $\mathrm{GL}_2(\mathbf{A}_k) \times \mathrm{GL}_1(\mathbf{A}_k)$ automorphic L-function $\Lambda(s, \Pi(\phi) \otimes \chi)$ to the $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-function $\Lambda(s, \pi(\phi) \times \pi(\chi)) = \Lambda(s, \phi \times \theta(\chi))$. Although we do not use it, we can derive from these basechange equivalences of L-functions the following consequence.

Lemma 6.3. *Let $\phi \in S_{l(\phi)}(\Gamma_0(N))$ be any cuspidal form, with quadratic twist $\phi \otimes \eta_k \in S_{l(\phi)}(\Gamma_0(d_k^2 N), \eta_k)$. We have an equivalence of Rankin-Selberg L-functions*

$$\Lambda(s, \phi \otimes \eta_k \times \theta(\chi)) = \Lambda(s, \phi \times \theta(\chi)).$$

Proof. Replacing the cusp form ϕ with its quadratic twist $\phi \otimes \eta_k \in S_{l(\phi)}(\Gamma_0(d_k^2 N), \eta_k)$ in the discussion above, we consider the corresponding $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(\phi \otimes \eta_k) \cong \pi(\phi) \otimes \eta_k$ and its quadratic basechange lifting $\Pi(\phi \otimes \eta_k)$ to $\mathrm{GL}_2(\mathbf{A}_k)$. We have via (57) the equivalences of standard L-functions

$$\Lambda(s, \Pi(\phi \otimes \eta_k)) = \Lambda(s, \pi(\phi \otimes \eta_k)) \Lambda(s, \pi(\phi \otimes \eta_k) \otimes \eta_k) = \Lambda(s, \pi(\phi) \otimes \eta_k) \Lambda(s, \pi(\phi)) = \Lambda(s, \Pi).$$

Consequently, for any character χ of $\mathbf{A}_k^\times / k^\times$, we have $\Lambda(s, \Pi(\phi \otimes \eta_k) \otimes \chi) = \Lambda(s, \Pi(\phi) \otimes \chi)$. We then obtain the stated identification of Rankin-Selberg L-functions from the corresponding basechange equivalence (58). □

6.1.4. *Vector-valued lifts of cuspidal eigenforms via the Doi-Naganuma lift.* Let us return to the quadratic spaces (L_A, Q_A) of signature $(2, 2)$ described in Proposition 2.1 and Corollary 2.3. Hence, we fix an integer $N \geq 1$ prime to the discriminant d_k of the quadratic field k. We then consider the lattice

$$L_A = L_A(N) = N^{-1}\mathfrak{a} \oplus N^{-1}\mathfrak{a} \subset V_A$$

of level N and trivial discriminant $d(L_A) = 1$ whose adelization $L_A \otimes \widehat{\mathbf{Z}}$ is fixed under the conjugation action of $\mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A})^2$ by the compact open subgroup $K_0(N) \oplus K_0(N)$. Here, we introduce the Doi-Naganuma lift (see e.g. [7, §3.1]) to describe how to construct from the scalar-valued cusp form

$\phi \in S_{l(\phi)}(\Gamma_0(N))$ and the classical scalar-valued Siegel theta series $\Theta_{L_A, l(\phi)}(\tau, z)$ associated to (L_A, Q_A) a Hilbert modular form $F_{\phi, L_A}(z)$ of parallel weight $l(\phi)$ on $X_A(\mathbf{C}) \cong Y_0(N) \times Y_0(N)$. As we explain, this construction allows us to find a unique vector-valued cusp form $g_\phi = g_{\phi, A} \in H_{l(\phi)}(\omega_{L_A})$ which lifts ϕ, and consequently for which the corresponding Rankin-Selberg L-function $L^\star(s, g_\phi \times \theta_{L_{A,U}^\vee})$ describes the partial completed Rankin-Selberg L-function $\Lambda(s, \phi \times \theta_A)$.

Let us first describe the Siegel theta series $\Theta_{L_A, l}(\tau, z)$ of weight $l \in \mathbf{Z}$ associated to the signature $(2,2)$ lattice $L_A \subset V_A$, as described in Corollary 2.3 above. We refer to [7, §2.6] for more background. Hence, this lattice $L_A = L_A(N)$ has level N and discriminant $d(V_A) = 1$, so that the corresponding Dirichlet character $\eta_{d(V_A)}(\cdot) = (\frac{d(V_A)}{\cdot})$ can be identified with the trivial/principal character modulo N. Given $z \in D(V_A)$ and $\lambda \in V_A(\mathbf{R})$, we have a unique decomposition $\lambda = \lambda_z + \lambda_{z^\perp}$, where λ_z and $\lambda_{z^\perp}$ denote the corresponding projections to z and $z^\perp$. Let us write $Q_A(\lambda)_z := Q_A(\lambda_z) - Q_A(\lambda_{z^\perp})$ for the corresponding majorant. We consider the Siegel theta function $\Theta_{L_A, l} : \mathfrak{H} \times D(V_A) \rightarrow \mathbf{C}$ of weight l associated to the lattice L_A, defined on $\tau = u + iv \in \mathfrak{H}$ and $z \in D(V_A)$ by the series

$$\Theta_{L_A, l}(\tau, z) = v \sum_{\lambda \in L_A^\vee} \frac{(\lambda, z)_A^l}{(z, \overline{z})_A^l} \cdot e\left(Q_A(\lambda_z) N \tau + Q_A(\lambda_{z^\perp}) N \overline{\tau}\right).$$

This series converges normally, is nonholomorphic in both variables, and satisfies the transformation property

$$\Theta_{L_A, l}(\gamma z, z) = \eta_{d(V_A)}(d)(c\tau + d)^l \Theta_{L_A, l}(\tau, z) = (c\tau + d)^l \Theta_{L_A, l}(\tau, z) \qquad \forall \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Theorem 6.4 (Doi-Naganuma). *Let $\Theta_{L_A, l}(\tau, z)$ denote the Siegel theta series of weight l associated to the signature $(2,2)$ lattice $L_A \subset V_A$ chosen according to Corollary 2.4, whose adelization $L_A \otimes \widehat{\mathbf{Z}}$ is fixed by the compact open subgroup subgroup $K_0(N) \oplus K_0(N)$ of $\mathrm{GL}_2(\widehat{\mathbf{Z}})^2 \subset \mathrm{GL}_2(\mathbf{A}_f)^2 \cong \mathrm{GSpin}(V_A)(\mathbf{A}_f)$. Hence, we identify this function $\Theta_{L_A, l}(\tau, \cdot)$ in the variable $\tau \in \mathfrak{H}$ as a nonholomorphic modular form of weight l, level $\Gamma_0(N)$, and trivial character. Let $\varphi \in S_l^{\mathrm{new}}(\Gamma_0(N))$ be a cuspidal holomorphic newform of the same weight, level, and character. Assume that φ lies in the corresponding Kohnen plus space*

$$S_l^+(\Gamma_0(N)) = \{f \in S_l(\Gamma_0(N)) : c_f(n) = -1 \implies c_f(n) = 0\} \subset S_l^{\mathrm{new}}(\Gamma_0(N)),$$

and hence that φ is invariant under the Fricke involution W_N, equivalently that the corresponding standard L-function $\Lambda(s, \varphi) = L_\infty(s, \varphi) L(s, \varphi)$ has odd, symmetric functional equation $\Lambda(s, \varphi) = -\Lambda(1 - s, \varphi)$. Then, the theta lift defined on $z \in D(V_A) = D^\pm(V_A) \cong \mathfrak{H}^2$ by the convergent integral

$$F_{\varphi, L_A}(z) = \int_{\mathcal{F}} \varphi(\tau) \Theta_{L_A, l}(\tau, z) v^l \frac{du dv}{v^2}$$

determines a cuspidal eigenform of parallel weight l on the Hilbert modular surface $X_0(N) \times X_0(N)$. Here again, we write $\mathcal{F}$ to denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$.

Proof. This is a special case of the Doi-Naganuma lifting for the setup we consider above for Proposition 2.3, leading to the identifications (8) and (9) with Remark 2.4. See [7, §3.1], and more generally the relevant discussions in Doi-Naganuma [20], Naganuma [44], van der Geer [53, §4], and Zagier [61] for more background. □

Let us now return to the setup of Theorems 5.12 and 5.14 above with vector-valued forms for $L_A \subset V_A$. We now write $\theta_{L_A, l}(\tau, z) = \theta_{L_A, l}(\tau, z)$ to denote the Siegel theta function $\theta_{L_A, l} : \mathfrak{H} \times D(V_A) \longrightarrow \mathfrak{S}_{L_A}^\vee$ of weight l constructed from the Weil representation $\omega_{L_A} : \mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathfrak{S}_{L_A}^\vee$ in the same way as (13) above. To be more precise, we make the following modification to the choice of Gaussian archimedean Schwartz function $\Phi_\infty(x, z) := \exp(-(x, x)_{A, z})$ for $z \in D(V_A) \cong \mathfrak{H}^2$ and $x \in V_A(\mathbf{R})$ in (12). Let $\mathcal{P}_l(x, z)$ be a weight l harmonic polynomial, so that $\omega_{L_A}(k_\theta) \mathcal{P}_l(x, z) = e^{il\theta} \mathcal{P}_l(x, z)$ for all $k_\theta \in \mathrm{SO}_2(\mathbf{R})$. We then define the corresponding function $\Phi_\infty^{(l)}(x, z) = \mathcal{P}_l(x, z) \Phi_\infty(x, z)$. Hence, in the variable $x \in V(\mathbf{R})$, we obtain an archimedean local Schwartz function $\Phi_\infty(x, \cdot) \in \mathcal{S}(V(\mathbf{R}))$ which transforms with weight l under the action of the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R})$. Using the same conventions and notations as above with the Iwasawa decomposition, we then define the corresponding theta series

$$\theta_{L_A,l}(\tau,z)=\theta_{L_A,l}(\tau,z,1)=\sum_{\mu\in L_A^\vee/L_A}\vartheta_{L_A}(g_\tau,1;\Phi_\infty^{(l)}(\cdot,z_0)\otimes\mathbf{1}_\mu)\mathbf{1}_\mu$$

from the theta kernel

$$\vartheta_{L_A}(g,h;\Phi)=\sum_{x\in V_A(\mathbf{Q})}\left(\omega_{L_A}(g,h)\Phi\right)(x).$$

Here again, we fix a basepoint $z_0\in D(V_A)=D^{\pm}(V_A)\cong\mathfrak{H}^2$, and we write $g\in\mathrm{SL}_2(\mathbf{A})$, $h\in\mathrm{GSpin}(V_A)(\mathbf{A})$, and $\Phi=\otimes_v\Phi_v\in\mathcal{S}(V_A(\mathbf{A}))$ to denote generic elements. We also write $\overline{\theta}_{L_A,l}(\tau,z)$ to denote the theta series obtained from the conjugate Weil representation $\overline{\omega}_{L_A}$.

Corollary 6.5. *Fix a holomorphic cuspidal newform $\phi\in S_l(\Gamma_0(N))$. Assume ϕ lies in the Kohnen plus space $S_l^+(\Gamma_0(N))\subset S_l(\Gamma_0(N))$, hence that ϕ is invariant under the Fricke involution W_N, equivalently that ϕ equivalently that the corresponding standard L-function $\Lambda(s,\phi)=L_\infty(s,\phi)L(s,\phi)$ has odd, symmetric functional equation $\Lambda(s,\phi)=-\Lambda(1-s,\phi)$. There exists a unique $g_\phi=g_{\phi,A}\in H_l(\overline{\omega}_{L_A})$ for which*

$$\langle\langle g_\phi(\tau),\overline{\theta}_{L_A,l}(\tau,z)\rangle\rangle=\Theta_{L_A,l}(\tau,z)\phi\otimes\eta_k(\tau),\tag{59}$$

so that the Doi-Naganuma lifting $F_{\phi,L_A}(z)$ can be characterized equivalently as the theta integral

$$F_{\phi,L_A}(z)=\int_{\mathcal{F}}\langle\langle g_\phi(\tau),\overline{\theta}_{L_A,l}(\tau,z)\rangle\rangle v^l\frac{dudv}{v^2}.$$

Remark 6.6. The lifting $g_\phi\in H_l(\overline{\omega}_{L_A})$ of $\phi\in S_l(\Gamma_0((N))$ can be described explicitly in special cases by Zhang [60, Theorem 4.15] and Scheithauer [46, Theorem 3.1]. In the special case of prime discriminant p, Bruinier-Bundschuh [8, Theorem 5] shows that the plus space $S_l^+(\Gamma_0(p),(\frac{p}{\cdot}))\subset S_l(\Gamma_0(p),(\frac{p}{\cdot}))$ is isomorphic to the corresponding space of holomorphic vector-valued cusp forms $S_l(\omega_L)$ for any even lattice L with discriminant group $L^\vee/L\cong\mathbf{F}_p$. More generally, the theorem of Strömberg[8] [51, Theorems 5.2 and 5.4] allow us to construct such a lift of any modular form $\phi\in M_l(\Gamma_0(M(L)),\eta_{|d(L)|})$ of level $M(L)$ equal to that of the lattice L and quadratic character $\eta_{|d(L)|}(\cdot)=(\frac{|d(L)|}{\cdot})$ with $d=d(L)$ the discrimnant of the lattice to a vector-valued form $g_\phi\in M_l(\omega_L)$ via the expansion

$$g_\phi(\tau)=\sum_{M\in\Gamma_0(M(L))\backslash\mathrm{SL}_2(\mathbf{Z})}\omega_L(M)^{-1}\mathbf{1}_0\phi|_lM(\tau).$$

6.1.5. *Equivalences of Rankin-Selberg L-functions.* We now return to Theorems 5.12 and 5.14 for the special case of the quadratic space (V_A,Q_A) of signature $(2,2)$ with lattice L_A corresponding to the congruence subgroup $K_0(N)\subset\mathrm{GL}_2(\mathbf{A}_f)$, as described in Corollary 2.4.

Proposition 6.7. *Fix a holomorphic cuspidal newform $\phi\in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2, level $\Gamma_0(N)$, and trivial character. Let $g_{\phi,A}\in S_2(\overline{\omega}_{L_A})$ denote the lifting of ϕ to a vector-valued cusp form of weight 2 and conjugate Weil representation $\overline{\omega}_{L_A}$. We have the following identifications of completed Rankin-Selberg L-functions.*

(i) *If k is the imaginary quadratic field associated to the negative definite subspace $V_{A,0}\subset V_A$ with $L_{A,0}=L_A\cap V_{A,0}$, then we have the identifications of completed Rankin-Selberg L-functions*

$$L^\star(2s-2,g_{\phi,A}\times\theta_{L_{A,0}^\perp})=\Lambda(s-1/2,\phi\times\theta_A)$$

for each class $A\in C(\mathcal{O}_k)$, and for each class group character $\chi\in C(\mathcal{O}_k)^\vee$ the identification

$$\sum_{A\in C(\mathcal{O}_k)}\chi(A)L^\star(2s-2,g_{\phi,A}\times\theta_{L_{A,0}^\perp})=\Lambda(s-1/2,\phi\times\theta(\chi)).$$

---

[8]taking the isotropic subgroup $S_0=\{0\}\subset L^\vee/L$

(ii) *If k is the real quadratic field associated to the Lorentzian subspace $W_A \subset V_A$ with $L_{A,W} \cap W_A \cap L_A$, then we have the identifications of completed Rankin-Selberg L-functions*

$$L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s-1/2, \phi \times \theta_A)$$

for each class $A \in C(\mathcal{O}_k)$, and for each class group character $\chi \in C(\mathcal{O}_k)^\vee$ that

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s-1/2, \phi \times \theta(\chi)).$$

Proof. Cf. [54, Corollary 4.18]. Fix any class $A \in C(\mathcal{O}_k)$. If k is imaginary quadratic as for (i), we write the Fourier series expansions of the corresponding holomorphic vector-valued forms as

$$g_{\phi,A}(\tau) = \sum_{\mu \in L_A^\vee/L_A} \sum_{m>0} c_{\phi,A}(\mu,m) e(m\tau) \mathbf{1}_\mu \in S_2(\overline{\omega}_{L_A})$$

and

$$\theta_{L_{A,0}^\perp}(\tau) = \sum_{\mu \in (L_{A,0}^\perp)^\vee/L_{A,0}^\perp} \sum_{m \geq 0} r_{L_{A,0}^\perp}(\mu,m) e(m\tau) \mathbf{1}_\mu \in M_0(\omega_{L_{A,0}^\perp}),$$

and consider the Dirichlet series of the corresponding Rankin-Selberg L-function for $\Re(s) \gg 1$,

$$L(s, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in (L_{A,0}^\perp)^\vee/L_{A,0}^\perp} \sum_{m \geq 1} \frac{c_{g_{\phi,A}}(\mu,m) r_{L_{A,0}^\perp}(\mu,m)}{m^{\frac{s+2}{2}}}.$$

Here, we can identify the discriminant group as

$$(L_{A,0}^\perp)^\vee/L_{A,0}^\perp \cong \mathfrak{d}_k^{-1} N^{-1}\mathfrak{a}/N^{-1}\mathfrak{a} \cong \mathfrak{d}_k^{-1}\mathcal{O}_k/\mathcal{O}_k,$$

and the counting functions appearing in the Fourier series expansion of the theta series as

$$r_{L_{A,0}^\perp}(\mu,m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mu + L_{A,0}^\perp : Q_A|_{L_{A,0}^\perp}(\lambda) = m\right\} = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mu + N^{-1}\mathfrak{a} : Q_\mathfrak{a}(\lambda) = m\right\}.$$

It is easy to see from this that we have the identification[9] of counting functions

$$\sum_{\mu \in (L_{A,0}^\perp)^\vee/L_{A,0}^\perp} r_{L_{A,0}^\perp}(\mu,m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in N^{-1}\mathfrak{a} : Q_\mathfrak{a}(\lambda) = m\right\} = r_A(m).$$

Similarly, as a consequence of the relation

$$\langle\langle g_{\phi,A}(\tau), \overline{\theta}_{L_A,2}(\tau,z)\rangle\rangle = \Theta_{L_A,2}(\tau,z)\phi(\tau)$$

implied by (59), we deduce that we have the relation of Fourier coefficients

$$\sum_{\mu \in (L_{A,0}^\perp)^\vee/L_{A,0}^\perp} c_{\phi,A}(\mu,m) = c_\phi(m),$$

and more generally, that we have an identification of scalar-valued forms

$$\langle\langle g_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes E_{L_{A,0}}(\tau,s;1)\rangle\rangle = \phi(\tau)\theta_A(\tau)E_A(\tau,s;1),$$

where $E_A(\tau,s;1)$ denotes the Eisenstein series in the Rankin-Selberg integral presentation (56) corresponding to $E_{L_{A,0}}(\tau,s;1) \in H_1(\omega_{L_{A,0}})$. This implies the corresponding identification of Rankin-Selberg products

$$\begin{aligned} L(s, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) &= \langle\langle g_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes E_{L_{A,0}}(\tau,s;1)\rangle = \int_{\mathcal{F}} \langle\langle g_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes E_{L_{A,0}}(\tau,s;1)\rangle\rangle \\ &= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in (L_{A,0}^\perp)^\vee/L_{A,0}^\perp} \sum_{m \geq 1} \frac{c_{g_{\phi,A}}(\mu,m) r_{L_{A,0}^\perp}(\mu,m)}{m^{\frac{s+2}{2}}} \\ &= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{m \geq 1} \frac{c_\phi(m) r_A(m)}{m^{\frac{s+2}{2}}} = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{m \geq 1} \frac{c_\phi(m) r_A(m)}{m^{\frac{s+2}{2}}}. \end{aligned}$$

[9]More formally, we use that $[N^{-1}\mathfrak{a}] = [(N^{-1})\mathfrak{a}] = [\mathfrak{a}] \in C(\mathcal{O}_k) = I(k)/P(k)$.

We then deduce from (56) that Proposition 6.1 that we have the identification of completed L-functions

$$L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(s-1/2, \phi \times \theta_A)$$

and hence that

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \sum_{A \in C(\mathcal{O}_k)} \chi(A)\Lambda(s-1/2, \phi \times \theta_A) = \Lambda(s-1/2, \phi \times \theta(\chi)).$$

If k is real quadratic as for (ii), we again first open up the Dirichlet series expansion (for $\Re(s) \gg 1$)

$$L(s, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in (L_{A,W}^\perp)^\vee / L_{A,W}^\perp} \sum_{m \geq 1} \frac{c_{g_{\phi,A}}(\mu, m) r_{L_{A,W}^\perp}(\mu, m)}{m^{\frac{s+2}{2}}}.$$

Again, we can identify the discriminant group as

$$(L_{A,W}^\perp)^\vee / L_{A,W}^\perp \cong \mathfrak{d}_k^{-1} N^{-1}\mathfrak{a} / N^{-1}\mathfrak{a} \cong \mathfrak{d}_k^{-1}\mathcal{O}_k/\mathcal{O}_k,$$

and the counting functions appearing in the Fourier series expansion of the theta series $\theta_{L_{A,W}^\perp}^+(\tau)$ as

$$r_{L_{A,W}^\perp}(\mu, m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mu + L_{A,W}^\perp/\langle \varepsilon_k \rangle : Q_A|_{L_{A,W}^\perp}(\lambda) = m\right\} = \frac{1}{w_k} \cdot \#\left\{\lambda \in \mu + N^{-1}\mathfrak{a}^\star : Q_\mathfrak{a}(\lambda) = m\right\}$$

so that

$$\sum_{\mu \in (L_{A,W}^\perp)^\vee / L_{A,W}^\perp} r_{L_{A,W}^\perp}(\mu, m) = \frac{1}{w_k} \cdot \#\left\{\lambda \in N^{-1}\mathfrak{a}^\star : Q_\mathfrak{a}(\lambda) = m\right\} = r_A(m).$$

We obtain from the corresponding relation (59) the identification of Fourier coefficients

$$\sum_{\mu \in (L_{A,W}^\perp)^\vee / L_{A,W}^\perp} c_{\phi,A}(\mu, m) = c_\phi(m),$$

and more generally the identification of scalar-valued forms

$$\langle\langle g_{\phi,A}(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes E_{L_{A,W}}(\tau, s; 2)\rangle\rangle = \phi(\tau)\theta_A(\tau)E_A(\tau, s; 2),$$

where $E_A(\tau, s; 2)$ denotes the Eisenstein series in the Rankin-Selberg integral presentation (56) corresponding to $E_{L_{A,W}}(\tau, s; 2) \in H_2(\omega_{L_{A,W}})$. Taking Petersson inner products, we then obtain the same identifications

$$\begin{aligned} L(s, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \langle\langle g_{\phi,A}(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes E_{L_{A,W}}(\tau, s; 2)\rangle &= \int_{\mathcal{F}} \langle\langle g_{\phi,A}(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes E_{L_{A,W}}(\tau, s; 2)\rangle\rangle \\ &= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in (L_{A,W}^\perp)^\vee / L_{A,W}^\perp} \sum_{m \geq 1} \frac{c_{g_{\phi,A}}(\mu, m) r_{L_{A,W}^\perp}(\mu, m)}{m^{\frac{s+2}{2}}} \\ &= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{m \geq 1} \frac{c_\phi(m) r_A(m)}{m^{\frac{s+2}{2}}} = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{m \geq 1} \frac{c_\phi(m) r_A(m)}{m^{\frac{s+2}{2}}} \end{aligned}$$

of the corresponding Rankin-Selberg inner products. We then deduce from (56) that Proposition 6.1 that we have the identification of completed L-functions

$$L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s-1/2, \phi \times \theta_A)$$

and hence that

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^\star(2s-2, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \sum_{A \in C(\mathcal{O}_k)} \chi(A)\Lambda(s-1/2, \phi \times \theta_A) = \Lambda(s-1/2, \phi \times \theta(\chi)).$$

$\square$

6.2. **Relations to sums of Green's functions along anisotropic subspaces.** Putting all of these observations together, we derive the following consequences of Theorems 5.12 and 5.14.

Theorem 6.8. *We retain the setup of Proposition 6.7. For each class $A \in C(\mathcal{O}_k)$, let $f_{0,A} \in H_0(\omega_{L_A})$ be any harmonic weak Maass form whose image under the antilinear differential operator ξ_0 equals $g_{\phi,A}$, so*

$$\xi_0(f_{0,A})(\tau) = g_{\phi,A}(\tau).$$

We have the following integral presentations of completed Rankin-Selberg L-functions, given in terms of sums over CM cycles or geodesic sets as in Theorems 5.12 and 5.14 above respectively.

(i) *If k is imaginary quadratic and $\eta_k(-N) = -\eta_k(N) = -1$, then*

$$\begin{aligned}&\Lambda'(1/2, \phi \times \theta(\chi)) \\ &= -\Lambda(1,\eta_k) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\left(\frac{\mathrm{vol}(K_{A,0})}{4} \right) \Phi(f_{0,A}, Z(V_{A,0})) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle \right] \\ &= -\Lambda(1,\eta_k) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\deg(Z(V_{A,0}))\Phi(f_{0,A}, Z(V_{A,0})) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle \right].\end{aligned}$$

(ii) *If k is real quadratic and $\eta_k(-N) = \eta_k(N) = -1$, then*

$$\begin{aligned}&\Lambda'(1/2, \phi \times \theta(\chi)) \\ &= -\Lambda(1,\eta_k) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\left(\frac{\mathrm{vol}(K_{A,W})}{4} \right) \Phi(f_{0,A}, G(W_A)) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle \right].\end{aligned}$$

Proof. For (i), we have for each class $A \in C(\mathcal{O}_k)$ the relation

$$\Phi(f_{0,A}, Z(V_{A,0})) = -\frac{4}{\mathrm{vol}(K_{A,0})} \left(\mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle + L'(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) \right)$$

and hence

$$-\left(\frac{\mathrm{vol}(K_{A,0})}{4} \right) \Phi(f_{0,A}, Z(V_{A,0})) - \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle = L'(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) \tag{60}$$

by Theorem 5.12. Observe that since $L^\star(s, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = -L^\star(s, g_{\phi,A} \times \theta_{L_{A,0}^\perp})$ by the odd, symmetric functional equation $E_{L_{A,0}}^\star(\tau, s; 1) = -E_{L_{A,0}}^\star(\tau, -s; 1)$ described in Proposition 5.3, we have the vanishing of the central value $L^\star(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = L(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = 0$, and hence that

$$\Lambda^{\star\prime}(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(1,\eta_k) L'(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}). \tag{61}$$

Moreover, observe that by the equivalence of L-functions shown in Proposition 6.7 (i) with the functional equation for $\Lambda(s, \phi \times \theta(\chi))$ described in Proposition 6.1, we are only in the non-degenerate situation when $\eta_k(-N) = -\eta_k(N) = -1$. Hence, we can multiply each side of (60) to obtain the corresponding relation

$$\begin{aligned}&-\Lambda(1,\eta_k) \left[\left(\frac{\mathrm{vol}(K_{A,0})}{4} \right) \Phi(f_{0,A}, Z(V_{A,0})) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle \right] \\ &= L^{\star\prime}(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}).\end{aligned} \tag{62}$$

Taking a twisted linear combination of the L-values on each side of (62) and using Proposition 6.7, we obtain

$$\begin{aligned}&-\Lambda(1,\eta_k) \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\left(\frac{\mathrm{vol}(K_{A,0})}{4} \right) \Phi(f_{0,A}, Z(V_{A,0})) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,0}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,0}}(\tau) \rangle\rangle \right] \\ &= \sum_{A \in C(\mathcal{O}_k)} \chi(A) L^{\star\prime}(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda'(1/2, \phi \times \theta(\chi)) = \Lambda'(1/2, \phi \times \theta(\chi)).\end{aligned} \tag{63}$$

For (ii), we have for each class $A \in C(\mathcal{O}_k)$ the relation

$$\Phi(f_{0,A}, G(W_A)) = -\frac{4}{\mathrm{vol}(K_{A,W})} \left(\mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,W}^\perp}^+(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle + L'(0, g_{\phi,A} \times \theta_{L_{A,W}^\perp}) \right)$$

and hence

$$-\left(\frac{\mathrm{vol}(K_{A,W})}{4}\right)\Phi(f_{0,A},G(W_A))+\mathrm{CT}\langle\langle f_{0,A}^+(\tau),\theta^+_{L^\perp_{A,W}}(\tau)\otimes\mathcal{E}_{L_{A,W}}(\tau)\rangle\rangle = L'(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}}) \tag{64}$$

by Theorem 5.14. Using the identification of completed L-functions of Proposition 6.7 (ii) with the functional equation of Proposition 6.1, we see that $L^\star(s,0,g_{\phi,A}\times\theta_{L^\perp_{A,W}})$ satisfies an odd symmetric functional equation when $\eta_k(-N)=\eta_k(N)=-1$, whence $L^\star(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}})=L(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}})=0$, and

$$\Lambda^{\star\prime}(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}})=\Lambda(1,\eta_k)L'(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}}).$$

Hence, we can multiply each side of (64) to obtain the corresponding relation

$$\begin{aligned}&-\Lambda(1,\eta_k)\left[\left(\frac{\mathrm{vol}(K_{A,W})}{4}\right)\Phi(f_{0,A},G(W_A))+\mathrm{CT}\langle\langle f_{0,A}^+(\tau),\theta_{L^\perp_{A,W}}(\tau)\otimes\mathcal{E}_{L_{A,W}}(\tau)\rangle\rangle\right]\\&=L^{\star\prime}(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}}).\end{aligned} \tag{65}$$

Taking a twisted linear combination of the L-values on each side of (65) and using Proposition 6.7, we obtain

$$\begin{aligned}&-\Lambda(1,\eta_k)\sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[\left(\frac{\mathrm{vol}(K_{A,W})}{4}\right)\Phi(f_{0,A},G(W_A))+\mathrm{CT}\langle\langle f_{0,A}^+(\tau),\theta_{L^\perp_{A,W}}(\tau)\otimes\mathcal{E}_{L_{A,W}}(\tau)\rangle\rangle\right]\\&=\sum_{A\in C(\mathcal{O}_k)}\chi(A)L^{\star\prime}(0,g_{\phi,A}\times\theta_{L^\perp_{A,W}})=\Lambda'(1/2,\phi\times\theta(\chi))=\Lambda'(1/2,\phi\times\theta(\chi)).\end{aligned} \tag{66}$$

□

Now, using the Dirichlet analytic class number formula

$$L(1,\eta_k)=\begin{cases}\frac{2\pi h_k}{w_k\sqrt{|d_k|}} & \text{if } k \text{ is imaginary quadratic}\\ \frac{2\ln(\varepsilon_k)h_k}{\sqrt{d_k}} & \text{if } k \text{ is real quadratic}\end{cases} \tag{67}$$

to evaluate

$$\Lambda(1,\eta_k)=|d_k|^{\frac{1}{2}}\Gamma_{\mathbf{R}}(2)L(1,\eta_k)=\begin{cases}\frac{2\pi h_k}{w_k} & \text{if } k \text{ is imaginary quadratic}\\ 2\ln(\varepsilon_k)h_k & \text{if } k \text{ is real quadratic}\end{cases}, \tag{68}$$

we can simplify the formulae of Theorem 6.8. Here, we observe that each of the compact open subgroups $K_{A,0}\subset T_{A,0}(\mathbf{A}_f)$ and $K_{A,W}\in T_{A,W}(\mathbf{A}_f)$ must be the maximal compact group $\mathcal{O}_k^\times$. We then calculate $\mathrm{vol}(K_{A,0})=w_k/h_k$ and $\mathrm{vol}(K_{A,W})=(w_k\ln(\varepsilon_k))/h_k$ for each class $A\in C(\mathcal{O}_k)$ using Lemma 5.8 to obtain

Corollary 6.9. *We have the following identities for the central derivative value $\Lambda'(1/2,\phi\times\theta(\chi))$.*

(i) *If k is imaginary quadratic and $\eta_k(-N)=-\eta_k(N)=-1$, then*

$$\begin{aligned}&\Lambda'(1/2,\phi\times\theta(\chi))\\&=-\frac{2\pi h_k}{w_k}\sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[\left(\frac{w_k}{4h_k}\right)\Phi(f_{0,A},Z(V_{A,0}))+\mathrm{CT}\langle\langle f_{0,A}^+(\tau),\theta_{L^\perp_{A,0}}(\tau)\otimes\mathcal{E}_{L_{A,0}}(\tau)\rangle\rangle\right].\end{aligned}$$

(ii) *If k is real quadratic and $\eta_k(-N)=\eta_k(N)=-1$, then*

$$\begin{aligned}&\Lambda'(1/2,\phi\times\theta(\chi))\\&=-2\ln(\varepsilon_k)h_k\sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[\left(\frac{w_k\ln(\varepsilon_k)}{4h_k}\right)\Phi(f_{0,A},G(W_A))+\mathrm{CT}\langle\langle f_{0,A}^+(\tau),\theta_{L^\perp_{A,W}}(\tau)\otimes\mathcal{E}_{L_{A,W}}(\tau)\rangle\rangle\right].\end{aligned}$$

7. Arithmetic implications

We now explain how to compute the Faltings heights of arithmetic divisors $Z(f)$ along zero cycles to prove higher Gross-Zagier formulae. We also explain how to derive a new proof/variant of the theorem of Gross-Zagier [25, §I. (6.3)] in terms of arithmetic Hirzebruch-Zagier divisors on the Hilbert modular surface $Y_0(N)\times Y_0(N)$. We also explain some applications to the refined conjecture of Birch-Swinnerton-Dyer.

7.1. **Arithmetic heights and higher Gross-Zagier formulae.** Let us first explain how to derive from the theorems of Bruinier-Yang [13, Theorem 4.7] – as presented in Theorem 5.12 above – and Andreatta-Goren-Howard-Madapusi Pera [1, Theorem A] the following "higher Gross-Zagier formula", relating the central derivative values $L'(0, \xi_{1-\frac{n}{2}}(f) \times \theta_{L_0^\perp})$ of the Rankin-Selberg L-functions

$$L(s, \xi_{1-\frac{n}{2}}(f) \times \theta_{L_0^\perp}) = \langle \xi_{1-\frac{n}{2}}(f)(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; 1) \rangle, \qquad f \in H_{1-\frac{1}{2}}(\omega_L)$$

to Faltings heights of arithmetic divisors $\widehat{Z}(f) = (Z(f), \Phi(f, \cdot))$ along the CM cycles $Z(V_0)$ on the spin Shimura variety $X = X_K = X_{K_L}$ introduced in (4) and (6) above.

7.1.1. *Extension to integral models.* Let us henceforth fix the level structure $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ associated to a choice of lattice $L \subset V$ in the quadratic space (V, Q), and simply write $X = X_K$ for the corresponding spin Shimura variety. Hence, the orbifold $X(\mathbf{C}) = X_K(\mathbf{C})$ describes the set of complex points of a quasi-projective Shimura variety X over $\mathbf{Q}$ of dimension n. As explained in [1], $X(\mathbf{C})$ can be viewed as the space of complex points of an algebraic Mumford-Deligne stack $X \longrightarrow \mathrm{Spec}(\mathbf{Q})$. In general, apart from some cases of small dimension ($n \leq 3$), the Shimura variety X is not of PEL type, and hence does not generally represent a moduli space of abelian variety with PEL structure. it is however of Hodge type, and so the theorems of Kisin [36], Madapusi Pera [41], and Kim-Madapusi Pera [35] apply to show the existence of a regular, flat integral model $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathbf{Z})$.

Recall from (5) that we have for each coset $\mu \in L^\vee/L$ and rational number $m \in \mathbf{Q}$ for which the quadric

$$\Omega_m(\mathbf{Q}) = \{x \in V : Q(x) = m\}$$

is nonempty the special divisor $Z(\mu, m) \to X$ defined by the sum

$$Z(\mu, m) = \sum_{x \in (\mathrm{GSpin}(V)(\mathbf{Q}) \cap K) \backslash \Omega_m(\mathbf{Q})} \mathbf{1}_\mu(x) \, \mathrm{pr}(D(V)_x).$$

As explained in [1], each of these special divisors admits an extension $\mathcal{Z}(\mu, m) \longrightarrow \mathcal{X}$ to the integral model. Roughly speaking, this is obtained as follows through the Kuga-Satake abelian scheme $\mathcal{A} \longrightarrow \mathcal{X}$. That is, the Shimura variety $X = X_K$ comes equipped with a family of Kuga-Satake abelian varieties $A_z \longrightarrow X$ indexed by points $z \in D(V)$. To describe the construction of such an abelian variety $A_z \longrightarrow X$, consider that we have a natural functor

$$\{\text{algebraic representations of } \mathrm{GSpin}(V)\} \longrightarrow \{\text{local systems of } \mathbf{Q}\text{-vectorspaces on } X(\mathbf{C})\}$$
$$(\mathrm{GSpin}(V) \to \mathrm{GL}(W)) \longmapsto (W_{\mathrm{Betti}, \mathbf{Q}} \to X(\mathbf{C}))$$

where

$$W_{\mathrm{Betti}, \mathbf{Q}} := \mathrm{GSpin}(V)(\mathbf{Q}) \backslash (W \times D(V)) \times \mathrm{GSpin}(\mathbf{A}_f)/K.$$

This allows us to associate to each algebraic representation $\mathrm{GSpin}(V) \to \mathrm{GL}(W)$ a pair $(W_{\mathrm{dR}}, \nabla)$ consisting of a locally free $\mathcal{O}_{X(\mathbf{C})}$-module $W_{\mathrm{dR}} = W_{\mathrm{Betti}, \mathbf{Q}} \otimes \mathcal{O}_{X(\mathbf{C})}$ and a connection $\nabla = 1 \otimes d$. Each such pair can be viewed as a vector bundle W_{dR} with integrable connection ∇ such that $W_{\mathrm{dR}}^{\nabla = 0} = W_{\mathrm{Betti}, \mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ and:

(i) For all $z \in D(V)$, the map $h_z : \mathbf{S} \longrightarrow \mathrm{GSpin}(V)(\mathbf{R}) \longrightarrow \mathrm{GL}(W_{\mathbf{R}})$ induces a map $\mathbf{S}(\mathbf{C}) \to \mathrm{GL}(W_{\mathbf{C}})$.

(ii) The fibre $W_{\mathrm{dR}, z}$ at $z \in D(V)$ has a bigradation $W_{\mathrm{dR}, z} = \bigoplus_{p,q} W_{\mathrm{dR}, z}^{p,q}$ induced by the action of $\mathbf{S}(\mathbf{C})$.

(iii) The $\mathcal{O}_{X(\mathbf{C})}$-module W_{dR} is endowed with a decreasing filtration $\mathrm{Fil}_J(W_{\mathrm{dR}}) \subseteq W_{\mathrm{dR}}$ of submodules, defined pointwise by $\mathrm{Fil}^J(W_{\mathrm{dR}, z}) = \bigoplus_{p \geq J} W_{\mathrm{dR}, z}^{p,q}$.

Here, have have natural identifications $\mathbf{S}(\mathbf{C}) = \mathbf{C}^\times \times \mathbf{C}^\times$ and $\mathrm{GL}(W_{\mathbf{C}}) = \mathrm{GL}(W_{\mathbf{Q}} \otimes \mathbf{C})$ Now, consider the representation of the Clifford algebra $C(V)$ on $\mathrm{GSpin}(V)$ induced by the inclusion

$$\mathrm{GSpin}(V) \subset C^0(V)^\times := C^0(V) \backslash \{0\},$$

with action given by left multiplication of $C^0(V)^\times$ on $\mathrm{GSpin}(V)$. The corresponding vector bundle $C(V)_{\mathrm{dR}}$ gives rise to the following variation of Hodge structures: For each $z \in D(V)$, we have

$$C(V)_{\mathrm{dR},z} = C(V)_z^{-1,0} \oplus C(V)_z^{0,-1}. \tag{69}$$

Note that having such a variation of Hodge structures (69) for each $z \in D(V)$ is equivalent to having a complex structure on the Clifford algebra $C(V_{\mathbf{R}}) = C(V_{\mathbf{Q}} \otimes \mathbf{R})$ for each $z \in D(V)$. In particular, we obtain from this complex structure for each $z \in D(V)$ a corresponding abelian variety

$$A_z = C(V_{\mathbf{R}})/C(L)$$

of dimension 2^{n+1} known as the *Kuga-Satake abelian variety associated to* $X = X_{K_L}$ *at a point* $z \in D(V)$. As explained in [1], this construction extends[10] to give an abelian scheme $\mathcal{A} \to \mathcal{X}$.

Example 7.1. Suppose we consider a rational quadratic space (V_0, Q_0) of signature $(0,2)$ with maximal lattice $L_0 \subset V_0$. In this case, the submodule $C^0(L_0) \subset C^0(V_0)$ corresponds to an order $\mathcal{O} \subset \mathcal{O}_{k(V_0)}$ of the imaginary quadratic field $k(V_0)$ associated to V_0, as described above. Each point $z \in D(V_0)$ determines a Kuga-Satake abelian surface $A_z = A_z^+ \times A_z^- = C(V_0(\mathbf{R}))/C(L_0)$, where A_z^+ is an elliptic curve with complex multiplication by the order $\mathcal{O} \cong C^0(V_0)$, and A_z is the elliptic curve with CM by $\mathcal{O}$ given by $A_z^- = A_z^+ \otimes_{\mathcal{O}} L_0 = A_z^+ \otimes_{C^0(L_0)} L_0$.

Example 7.2. Consider the rational quadratic space (V, Q) of signature $(1,2)$ given by $V = M_2^{\mathrm{tr}=0}(\mathbf{Q})$ and $Q(\cdot) = N \det(\cdot)$. As explained in [13, §7.3] and Appendix A below, we have an accidental isomorphism $\mathrm{GSpin}(V) \cong \mathrm{GL}_2$ of algebraic groups over $\mathbf{Q}$. Let $L \subset V$ denote the lattice

$$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}$$

with dual lattice

$$L^\vee = \left\{ \begin{pmatrix} b/2n & -a/N \\ c & -b/2N \end{pmatrix} : a, b, c \in \mathbf{Z} \right\},$$

so that the discriminant group $L^\vee/L$ can be identified as

$$\mathbf{Z}/2N\mathbf{Z} \longrightarrow L^\vee/L, \quad r \longmapsto \mu_r := \begin{pmatrix} r/2N & \\ & -r/2N \end{pmatrix}.$$

Hence, L has level $4N$, and the corresponding quadric

$$\Omega_{\mu,m}(\mathbf{Q}) := \{x \in \mu + L : Q(x) = m\}$$

is nonempty unless $Q(\mu) \equiv m \bmod 1$. The corresponding compact open subgroup $K = K_L$ is given by

$$K = \prod_{p<\infty} K_p \subset \mathrm{GSpin}(V)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{Q}_p), \quad K_p := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) : c \in N\mathbf{Z}_p \right\}.$$

In this setting, we have an isomorphism of Shimura varieties

$$Y_0(N) := \Gamma_0(N)\backslash\mathfrak{H} \cong X_K(\mathbf{C}), \quad \Gamma_0(N)z \longmapsto \mathrm{GSpin}(V)(\mathbf{Q})(z,1)K.$$

In the other direction, using the moduli description of the noncompactified modular curve $Y_0(N)$, we have

$$X_K(\mathbf{C}) := \mathrm{GSpin}(V)(\mathbf{Q})\backslash D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K \cong Y_0(N), \quad z \longmapsto (E_z \to E_z')$$

for (E_z, E_z') a pair of elliptic curves with CM by some order $\mathcal{O} \subset \mathcal{O}_{k(V_0)}$ in the imaginary quadrtic field $k(V_0)$ determined by a negative definite subspace $V_0 \subset V$ (given by a Heegner embedding). In this case, each point $z \in D(V) = D^\pm(V) \cong \mathfrak{H}$ has the corresponding Kuga-Satake abelian fourfold $A_z = C(V(\mathbf{R}))/C(L)$ given by

$$A_z = A_z^+ \times A_z^-, \qquad A_z^+ = A_z^- = E_z \times E_z'.$$

We shall return to this example in Appendix A below to explain how to recover the formula of Gross-Zagier [25, Theorem I (6.3)] from [13, Theorem 4.7, Theorem 7.7] and [1, Theorem A].

[10] They also show that the associated vector bundle with connection $(C(V)_{\mathrm{dR}}, \nabla)$ extends to the integral model $\mathcal{X}$, with vector bundle $C(V)_{\mathrm{dR}}$ given by the relative de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{A})$, connection ∇ given by the Gauss-Manin connection, and filtration $\mathrm{Fil}^J(C(V)_{\mathrm{dR},z})$ given by the Hodge filtration $0 \longrightarrow R_!\pi_*(\mathcal{O}_{\mathcal{A}}) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{A}) \longrightarrow \pi_*(\Omega^1_{\mathcal{A}}) \longrightarrow 0$.

Now, the Kuga-Satake abelian scheme $\mathcal{A} \longrightarrow \mathcal{X}$ comes equipped with an action of the Clifford algebra $C(L) = C^0(L) \oplus C^1(L)$, and acquires from this a $\mathbf{Z}/2\mathbf{Z}$-grading $\mathcal{A} = \mathcal{A}^+ \times \mathcal{A}^-$. For each scheme $S \longrightarrow \mathcal{X}$, we can associate to the pullback $\mathcal{A}_S$ a distinguished $\mathbf{Z}$-module of *special endomorphisms*

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S),$$

together with an associated quadratic form $q : V(\mathcal{A}_S) \longrightarrow \mathbf{Z}$ defined by $x \circ x = q(x) \cdot \mathrm{id}$. More generally, for each coset $\mu \in L^\vee/L$, we can associate to the pullback $\mathcal{A}_S$ a distinguished subset $V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S) \otimes \mathbf{Q}$ with the property that $V_0(\mathcal{A}_S) = V(\mathcal{A}_S)$.

As explained in [1], we can define from each of these subsets $V_\mu(\mathcal{A}_S)$ a divisor on $\mathcal{X}$ as follows: For each coset $\mu \in L^\vee/L$ and rational number $m \in \mathbf{Q}$, let

$$\mathcal{Z}(\mu, m) \longrightarrow \mathcal{X}$$

denote the moduli stack that assigns to each $\mathcal{X}$-scheme $S \longrightarrow \mathcal{X}$ the set

$$\mathcal{Z}(\mu, m)(S) := \{x \in V_\mu(\mathcal{A}_S) : q(x) = m\}.$$

This morphism $\mathcal{Z}(\mu, m) \longrightarrow \mathcal{X}$ turns out to be finite and relatively representable, and to determine a Cartier divisor on $\mathcal{X}$. It agrees on the generic fibre with the special cycles (5) defined above, $\mathcal{Z}(\mu, m)(\mathbf{C}) = Z(\mu, m)$.

Recall that we write $T_0 = \mathrm{GSpin}(V_0) = \mathrm{Res}_{k(V_0)/\mathbf{Q}}\, \mathbf{G}_m$ to denote the torus corresponding to a rational quadratic subspace (V_0, Q_0) of signature $(0,2)$ and associated imaginary quadratic field $k(V_0)$. We consider the corresponding zero-dimensional Shimura variety $Z(V_0) \longrightarrow \mathrm{Spec}(k(V_0))$ with complex points given by (6). Note that we can identify the corresponding compact open subgroup $K_0 := K \cap T_0(\mathbf{A}_f)$ with $K_0 = \widehat{O}_{k(V_0)}^\times$, and that this acts trivially on the discriminant group $L_0^\vee/L_0$. Hence, we can identify the complex points

$$Z(V_0)(\mathbf{C}) = k(V_0)^\times \backslash \{z_{V_0}^\pm\} \times \mathbf{A}_{k(V_0),f}^\times / \widehat{O}_{k(V_0)}^\times$$

with two copies of the ideal class group $C(\mathcal{O}_{k(V_0)})$. Observe that by Lemma 5.8 (cf. [13, Lemma 6.3]) with the Dirichlet analytic class number formula (67), we have that the degree $\deg Z(V_0) = 4/\mathrm{vol}(K_0)$ of $Z(V_0)$ as defined in Lemma 5.9 (i) is given by the relation

$$\frac{\deg Z(V_0)}{4} = \frac{1}{\mathrm{vol}(K_0)} = \frac{h_{k(V_0)}}{w_{k(V_0)}} = \frac{|d_{k(V_0)}|^{\frac{1}{2}}}{2\pi} \cdot L(1, \eta_{k(V_0)}).$$

Viewing $Z(V_0) \longrightarrow \mathrm{Spec}(k(V_0))$ as the moduli space of elliptic curves with complex multiplication by $\mathcal{O}_{k(V_0)}$, we obtain a smooth integral model $\mathcal{Z}(V_0) \longrightarrow \mathrm{Spec}(\mathcal{O}_{k(V_0)})$. As explained in [1], if the imaginary quadratic field $k = k(V_0)$ has odd discriminant $d_{k(V_0)}$, then the embedding of reductive groups $T_0 \subset \mathrm{GSpin}(V)$ induced by the embedding of quadratic spaces $V_0 \subset V$ gives a finite, relatively representable, umramified morphism

$$\mathcal{Z}(V_0) \longrightarrow \mathcal{X}.$$

This algebraic stack has its own Kuga-Satake abelian scheme $\mathcal{A}_0 \longrightarrow \mathcal{Z}(V_0)$ equipped with an action of the Clifford algebra $C(L_0) = C^0(L_0) \oplus C^1(L_0)$ and hence a $\mathbf{Z}/2\mathbf{Z}$-grading $\mathcal{A}_0 \cong \mathcal{A}_0^+ \times \mathcal{A}_0^-$. Here, $\mathcal{A}_0^+$ can be identified with the universal elliptic curve with CM by $\mathcal{O}_{k(V_0)}$, with $\mathcal{A}_0 \cong \mathcal{A}_0^+ \otimes_{\mathcal{O}_{k(V_0)}} C(L_0)$ and $\mathcal{A}_0^- \cong \mathcal{A}_0^+ \otimes_{\mathcal{O}(k(V_0))} L_0$. Moreover, this Kuga-Satake abelian scheme $\mathcal{A}_0 \longrightarrow \mathcal{Z}(V_0)$ is related to the Kuga-Satake abelian scheme $\mathcal{A} \longrightarrow \mathcal{X}$ by a $C(L)$-linear isomorphism

$$\mathcal{A}|_{\mathcal{Z}(V_0)} \cong \mathcal{A}_0|_{\mathcal{Z}(L_0)} \otimes_{C(L_0)} C(L).$$

7.1.2. *Arithmetic degrees along CM cycles and central derivative Rankin-Selberg L-values.* Recall we saw in Theorem 5.12 above that we have for any harmonic weak Maass form $f \in H_{1-\frac{n}{2}}(\omega_L)$ the formula

$$\Phi(f, Z(V_0)) = -\deg(Z(V_0)) \cdot \left(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, \xi_{1-n/2}(f) \times \theta_{L_0^\perp} \right). \tag{70}$$

We can now describe this formula in terms of arithmetic heights, according to the calculations of [13, §5-6] and more generally [1, Theorem A], which we now summarize. Recall that an arithmetic divisor $\widehat{x} = (x, G_x)$ on the integral model $\mathcal{X} \rightarrow \mathrm{Spec}(\mathbf{Z})$ consists of a divisor x on $\mathcal{X}$ and a corresponding Green's function G_x for the the divisor $x(\mathbf{C})$ induced by x on the complex variety $\mathcal{X}(\mathbf{C}) = X(\mathbf{C})$. That is, the G_x is a smooth function on $\mathcal{X}(\mathbf{C}) \backslash x(\mathbf{C})$ with a logarithmic singularity along $x(\mathbf{C})$ which satisfies the Green's current equation

$$dd^c[G_x] + \delta_{x(\mathbf{C})} = [\Omega_x]$$

for some smooth $(1,1)$-form Ω_x on $\mathcal{X}(\mathbf{C})$. Let $\widehat{\mathrm{Ch}}^1(\mathcal{X})$ denote the first arithmetic Chow group of $\mathcal{X}$, so the free abelian group generated by arithmetic divisors on $\mathcal{X}$ modulo rational equivalence. Let

$$[\cdot,\cdot] : \widehat{\mathrm{Ch}}^1(\mathcal{X}) \times Z^n(\mathcal{X}) \longrightarrow \mathbf{R}$$

denote the height pairing defined in Bost-Gillet-Soulé [4, § 2.3]. Given an arithmetic divisor $\widehat{x} \in \widehat{\mathrm{Ch}}^1(\mathcal{X})$ and an n-cycle $y \in Z^n(\mathcal{X})$ intersecting properly, we know that this pairing is given by the Faltings height

$$[\widehat{x}, y] = [\widehat{x}, y]_{\mathrm{Fal}} = [x, y]_{\mathrm{fin}} + [\widehat{x}, y]_\infty .$$

Here, the archimedean component is given by half the value of the Green function G_x along $y(\mathbf{C})$,

$$[\widehat{x}, y]_\infty = \frac{1}{2} \cdot G_x\left(y(\mathbf{C})\right).$$

The setup we consider above with regularized theta lifts $\Phi(f) = \Phi(f,\cdot)$ is relevant here as it provides us with a supply of such arithmetic divisors $\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f))$ on the integral model $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathbf{Z})$. To be more precise, each CM-cycle $\mathcal{Z}(V_0) \longrightarrow \mathcal{X}_{\mathcal{O}_{k(V_0)}}$ associated to a rational quadratic subspace $V_0 \subset V$ of signature $(0,2)$ provides us with a zero-cycle $y \in Z^0(\mathcal{X})$ which intersects $\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f))$ properly. In particular, this allows us to reinterpret (70) in terms of the archimedean local height as

(71)
$$\left[\widehat{\mathcal{Z}}(f), \mathcal{Z}(V_0)\right]_\infty = \frac{1}{2}\cdot\Phi(f,Y) \quad = -\frac{\deg(Z(V_0))}{2}\cdot\Big(\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_0^\perp}(\tau)\otimes\mathcal{E}_{L_0}(\tau)\rangle\rangle + L'(0, \xi_{1-n/2}(f), \theta_\Lambda)\Big).$$

Remark 7.3. When the Shimura variety $X = X_K$ is not compact, we can add suitable boundary components $\mathcal{C}(f)$ to the divisor $\mathcal{Z}(f)$ as in (30) to get an arithmetic divisor

$$\widehat{\mathcal{Z}}^c(f) = (\mathcal{Z}^c(f), \Phi) \in \widehat{\mathrm{Ch}}^1(\mathcal{X}^\star)$$

on the integral model $\mathcal{X}^\star$ of the compactification $X^\star$. See [13, §5-7] for more details. When $f \in H_{1-n/2}(\omega_L)$ is not cuspidal, we also have to work with generalized arithmetic Chow groups in the sense of [17].

Suppose now that $f = f^+ + f^- \in H_{1-n/2}(\omega_L)$ has integral holomorphic part f^+, so that the Fourier coefficients $c_f^+(\mu, m)$ are integers for all $m \in \mathbf{Q}$ and $\mu \in L^\vee/L$. Then,

$$Z(f) = \sum_{\mu\in L^\vee/L}\sum_{\substack{m\in\mathbf{Q}\\ m>0}} c_f^+(\mu,-m) Z(\mu,m)$$

determines a divisor on X, with extension

$$\mathcal{Z}(f) = \sum_{\mu\in L^\vee/L}\sum_{\substack{m\in\mathbf{Q}\\ m>0}} c_f^+(\mu,-m) \mathcal{Z}(\mu,m)$$

to a Cartier divisor on the integral model $\mathcal{X}$, and with corresponding arithmetic divisor

$$\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), G_{\mathcal{Z}(f)}) \in \widehat{\mathrm{Ch}}^1(\mathcal{X}).$$

If $f \in \ker(\xi_{1-n/2}) \cong M^!_{1-n/2}(\omega_L)$ is weakly holomorphic, then we expect $\widehat{Z}(f) = (\mathcal{Z}(f), \Phi(f,\cdot))$ to be rationally equivalent to a torsion element, by the relation given by the Borcherds lift $\Psi(f,\cdot)$ described in (20) above (Theorem 4.1). If this rational equivalence to zero were known to be true, we would derive the corresponding vanishing of the Faltings height the relation

$$\left[\widehat{Z}(f), \mathcal{Z}(V_0)\right] = [\mathcal{Z}(f), \mathcal{Z}(V_0)]_{\mathrm{fin}} + \frac{1}{2}\cdot\Phi(f, Z(V_0)) = 0, \tag{72}$$

from which it would follow that the nonarchimedean height pairing is given by the constant coefficient term

$$[\mathcal{Z}(f), \mathcal{Z}(V_0)]_{\mathrm{fin}} = -\frac{\deg(V_0)}{2}\cdot\mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau)\otimes\mathcal{E}_{L_0}(\tau)\rangle\rangle. \tag{73}$$

Expanding out both sides of this relation (73) leads to the following expectation for the general case.

Conjecture 7.4 (Bruinier-Yang). *Let $f = f^+ + f^- \in H_{1-n/2}(\omega_L)$ be any weakly harmonic Maass form whose holomorphic part has integral Fourier coefficients,*

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(\mu, m) e(m\tau) \mathbf{1}_\mu, \quad c_f^+(\mu, m) \in \mathbf{Z}.$$

We have for each coset $\mu \in L^\vee/L$ and each positive rational $m \in Q(\mu) + \mathbf{Z}$ that the nonarchimedean local height $[\mathcal{Z}(\mu, m), \mathcal{Y}]_{\mathrm{fin}}$ is given by the (μ, m)-th Fourier coefficient of the modular form $\theta_{L_0^\perp}(\tau) \otimes \mathcal{E}_{L_0}(\tau)$,

$$[\mathcal{Z}(\mu, m), \mathcal{Z}(V_0)]_{\mathrm{fin}} = - \sum_{\substack{\mu_1 \in (L_0^\perp)^\vee/L_0^\perp, \mu_2 \in L_0^\vee/L_0 \\ \mu_1 + \mu_2 \equiv \mu \bmod L}} \sum_{\substack{m_1, m_2 \in \mathbf{Q}_{\geq 0} \\ m_1 + m_2 = m}} r_{L_0^\perp}(\mu_1, m_1) \kappa_{L_0}(\mu_2, m_2).$$

Putting this into (70) *(Theorem 5.12), taking the sum over $m > 0$, we obtain the arithmetic height formula*

$$(74) \qquad \left[\widehat{\mathcal{Z}}(f), \mathcal{Z}(V_0)\right] = \left[\widehat{\mathcal{Z}}(f), \mathcal{Z}(V_0)\right]_{\mathrm{Fal}} = -\frac{\deg(Z(V_0))}{2} \cdot \left(c_f^+(0,0) \cdot \kappa_{L_0}(0,0) + L'(0, \xi_{1-n/2}(f), \theta_{L_0^\perp}) \right).$$

Theorem 7.5 (Bruinier-Yang, Andreatta-Goren-Howard-Madapusi Pera). *Let $(V_0, Q_0) = (\mathfrak{a}_{\mathbf{Q}}, -Q_{\mathfrak{a}}(\cdot))$ be a rational quadratic subspace of signature $(0,2)$ given by a fractional ideal $\mathfrak{a}_{\mathbf{Q}}$ in an imaginary quadratic field $k(V_0)$ of odd discrimant $d_{k(V_0)}$ determined by a nonzero integral ideal $\mathfrak{a} \subset \mathcal{O}_{k(V_0)}$. Let $L_0 = \mathfrak{a}$ denote the corresponding lattice. Assume that the even part $C^0(L_0)$ of the Clifford algebra $C(L_0)$ is identified with the maximal order $\mathcal{O}_{k(V_0)} \cong C^0(V_0)$. Then, Conjecture 7.4 is true. In particular, the arithmetic height formula* (74) *is true. Equivalently, writing $\widehat{\mathbf{T}} \in \widehat{\mathrm{Ch}}^1(\mathcal{X})$ to denote the metrized cotautological defined in* [1, §5.3],

$$\left[\widehat{\mathcal{Z}}(f) : \mathcal{Z}(V_0)\right] + c_f^+(0,0) \cdot \left[\widehat{\mathbf{T}} : \mathcal{Z}(f)\right] = -\frac{h_k}{w_k} \cdot L'(0, \xi_{1-n/2}(f) \times \theta_{L_0^\perp}).$$

Proof. This follows from the combined results of [13, Theorem 1.2] and [1, Theorem A, Theorem 5.7.3]. □

Remark 7.6. Note that Conjecture 7.4 is not yet established in general; see [13, Conjectures 5.1 and 5.2]. That is, the conjecture is posed more generally for (V_0, Q_0) any negative definite quadratic subspace of signature $(0,2)$. In particular, it should be possible to take $(V_0, Q_0) = (\mathfrak{a}, -Q_{\mathfrak{a}}(\cdot))$ with $C^+(L_0) \cong \mathcal{O}$ any (non-maximal) order $\mathcal{O} \subset \mathcal{O}_{k(V_0)}$, and without any condition on the partity of the discriminant $d_{k(V_0)}$.

7.2. **Gross-Zagier via special (Hirzebruch-Zagier) divisors on $X_0(N) \times X_0(N)$.** We now return to the quadratic spaces (V_A, Q_A) of signature $(2,2)$ parametrizing $X_{K_A} = Y_0(N) \times Y_0(N)$. Here, we give a geometric interpretation of the formulae of Theorem 6.8 and Corollary 6.9 above. In case (i) where k is imaginary quadratic, this will give a new proof of the formula of Gross and Zagier [25, I Theorem (6.3)], including a comparison of the arithmetic heights of Heegner divisorss on $X_0(N)$ and the corresponding arithmetic heights of Hirzebruch-Zagier divisors on $X_0(N) \times X_0(N)$. For the convenience of the reader, we explain in Appendix A how the Gross-Zagier formula [25, Theorem I (6.3)] can be derived by a variation of the proof of Bruinier-Yang [13, Theorem 7.7], developing Theorem 7.5 for the special case of signature $(1,2)$ described in Example 7.2 above. Here, we give a distinct deduction of the formula via the Hilbert modular surfaces $Y_0(N) \times Y_0(N)$ and $X_0(N) \times X_0(N)$.

For each class $A \in C(\mathcal{O}_k)$, recall that we fix a representative $\mathfrak{a} \subset \mathcal{O}_k$, and consider the quadratic space (V_A, Q_A) of signature (2,2) defined by $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$ and quadratic form $Q_A(z_1, z_2) = Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2)$ (for $Q_{\mathfrak{a}}(z) := \mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ the norm form), so that $(V_{A,0}, Q_{A,0}) = (\mathfrak{a}, -Q_{\mathfrak{a}})$ determines a rational quadratic space of signature $(0,2)$. Recall that we have an accidental isomorphism $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$ of algebraic groups over $\mathbf{Q}$ by Proposition 2.3, and that we take $L_A \subset V_A$ to be the maximal lattice corresponding to the compact open subgroup $K_A = K_{L_A} \cong K_0(N)^2 \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})^2$ so that $X_A \cong Y_0(N) \times Y_0(N)$ as in Corollary 2.4. Hence, we consider the corresponding integral model $\mathcal{X}_A \cong \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$. Let us fix a compactification $X_A^\star \cong X_0(N) \times X_0(N)$ (see e.g. [7, §§1.2 and 2.4]), so that we can identify the corresponding integral model $\mathcal{X}_A^\star$ with $\mathcal{X}_0(N) \times \mathcal{X}_0(N)$.

Remark 7.7. Note that we have the following moduli descriptions of these Hilbert modular surfaces and their special divisors. Recall that the noncompactified modular curve $Y_0(N)$ has the following moduli description (see e.g. [25]). For any scheme S over $\mathbf{Q}$, $Y_0(N)(S)$ represents the isomorphism class of triples (E, E', φ) consisting of a pair of elliptic curves E/S, E'/S and an isogeny $\varphi : E \to E'$ of degree N. Hence, φ is finite flat of degree N, and its kernel $\ker(\varphi) \cong \mathbf{Z}/N\mathbf{Z}$ is a finite locally free group scheme over S. The compactified modular curve $X_0(N)(S)$ represents triples (E, E', φ) of generalized elliptic curves E/S, E'/S and a cyclic isogeny $\varphi : E \rightarrow E'$ of degree N. We use this to deduce that $X_A(S) = Y_0(N)(S) \times Y_0(N)(S)$ represents pairs of triples (E_1, E_1', φ_1), (E_2, E_2', φ_2). More precisely, $Y_0(N)(S) \times Y_0(N)(S)$ represents the moduli space of triples $\mathbf{A} = (A, \kappa_A(\varphi), \lambda_A)$ made up of the abelian surface $A = E_1 \times E_2$, the endomorphism $\kappa_A(\varphi)$ of A determined by the isogeny $\varphi = \varphi_1 \times \varphi_2 : A \rightarrow A'$ for $A' = E_1' \times E_2'$ and its dual $\varphi^\vee : A' \rightarrow A$, and the product principal polarization $\lambda_A = \lambda_{E_1} \times \lambda_{E_2}$. Similarly, $X_0(N)(S) \times X_0(N)(S)$ represents the moduli space of triples $\mathbf{A} = (A, \kappa_A(\varphi), \lambda_A)$ made up of the abelian surface $A = E_1 \times E_2$ with E_1/S, E_2/S generalized elliptic curves, special endomorphism $\kappa_A(\varphi) \in \mathrm{End}(A)$ determined by the isogeny $\varphi : A \rightarrow A'$ and its dual $\varphi^\vee : A' \rightarrow A$, and the product principal polarization λ_A. We can then describe the special arithmetic (Hirzebruch-Zagier) divisor $Z_A(\mu, m)$ in either of these spaces as the moduli of triples $(A, \kappa_A(x), \lambda_A) = (E_1 \times E_2, \kappa_{E_1 \times E_2}(x), \lambda_{E_1} \times \lambda_{E_2})$ with endomorphism $\kappa_A(x)$ of degree $\deg(\kappa_A(\varphi)) = m$ supported on $\mu + L_A$. We can also describe the CM cycles $\mathcal{Z}(V_{A,0})$ – cf. [13, Proposition 7.2], and the descriptions of Heegner divisors in [25] and [26]. We can identify the CM cycles $\mathcal{Z}(V_{A,0})$ with Heegner divisors corresponding in the moduli descriptions of $Y_0(N)$ and $X_0(N)$ to a triple (E, E', φ) consisting of elliptic curves E and E' with complex multiplication by $\mathcal{O}_k$ and a cyclic isogeny $\varphi : E \rightarrow E'$ of degree N annihilated by a primitive ideal of the form $\mathfrak{n} = [N, (r + \sqrt{d_k})/2]$. We then deduce that the CM cycles $\mathcal{Z}(V_{A,0})$ we consider will correspond to triples $\mathbf{A} = (A, \kappa_A(\varphi), \lambda_A)$ with $A = E \times E$ the self-product of the elliptic curve E with CM by $\mathcal{O}_k$, and $\kappa_A(\varphi)$ the endomorphism corresponding to the cyclic isogeny $\varphi : E \rightarrow E'$ and its dual $\varphi^\vee : E' \rightarrow E$. We refer to the discussions in [53] and [29] for a more general description of these moduli spaces of abelian surfaces with special endomorphisms.

As in Remark 7.3, we extend each arithmetic divisor

$$\widehat{\mathcal{Z}}_A(f) = (\mathcal{Z}_A(f), \Phi(f, \cdot)), \quad \widehat{\mathcal{Z}}_A(\mu, m) = (\mathcal{Z}_A(\mu, m), \Phi_{\mu,m}(\cdot)) \in \widehat{\mathrm{Ch}}^1 (\mathcal{Y}_0(N) \times \mathcal{Y}_0(N))$$

to the compatification

$$\widehat{\mathcal{Z}}_A^c(f) := (\mathcal{Z}_A^c(f), \Phi(f, \cdot)), \quad \widehat{\mathcal{Z}}_A^c(\mu, m) := (\mathcal{Z}_A^c(\mu, m), \Phi_{\mu,m}(\cdot)) \in \widehat{\mathrm{Ch}}^1 (\mathcal{X}_0(N) \times \mathcal{X}_0(N)) . \tag{75}$$

We derive the following consequence of Theorem 7.5 in this setting, using Proposition 6.7 and Theorem 6.8.

Theorem 7.8. *Let $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$ be a cuspidal newform of level N and trivial character. Let k be an imaginary quadratic field of odd discriminant d_k and (odd) quadratic Dirichlet character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Assume that $(N, d_k) = 1$, and that $\eta_k(-N) = -\eta_k(N) = -1$. Let $g_{\phi,A} \in S_2(\overline{\omega}_{L_A})$ denote the vector-valued lift of ϕ described lift in Corollary 6.5. Let $f_{0,A} \in H_0(\omega_{L_A})$ be a harmonic weak Maass form of weight zero and representation for which*

$$\xi_0(f_{0,A})(\tau) = g_{\phi,A}(\tau) \in S_2(\overline{\omega}_{L_A}),$$

where $\xi_0 : H_0(\omega_{L_A}) \rightarrow S_2(\overline{\omega}_{L_A})$ denotes the antilinear differential operator defined in (17). Then, for any class group character $\chi \in C(\mathcal{O}_k)^\vee$, we have the central derivative value formula

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A(f_{0,A}) : \mathcal{Z}(V_{A,0}) \right]$$

for the completed Rankin-Selberg L-function $\Lambda(s, \phi \times \theta(\chi))$ of ϕ times the Hecke theta series $\theta(\chi)$, where each term on the right-hand side denotes the arithmetic height of the arithmetic special divisor

$$\widehat{\mathcal{Z}}_A(f_{0,A}) = \sum_{\mu \in L_A^\vee / L_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_{0,A}}^+(\mu, -m) \mathcal{Z}_A(\mu, m)$$

on the integral model $\mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$ of the Hilbert modular surface $X = Y_0(N) \times Y_0(N)$ evaluated along the corresponding CM cycle $\mathcal{Z}(V_{A,0}) \subset \mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$. Here, each $\mathcal{Z}_A(\mu, m)$ is the Hirzebruch-Zagier divisor $\widehat{\mathcal{Z}}_A(\mu, m) = (\mathcal{Z}_A(\mu, m), \Phi_{\mu,m}^{L_A})$ on $\mathcal{X} = \mathcal{Y}_0(N) \times \mathcal{Y}_0(N)$ described above. We can also extend arithmetic divisors to the compactification $\mathcal{X}^\star \cong \mathcal{X}_0(N) \times \mathcal{X}_0(N)$ as described in (75) to get the corresponding formula

$$\Lambda'(1/2, \phi \times \theta(\chi)) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A^c(f_{0,A}) : \mathcal{Z}(V_{A,0}) \right]$$

Proof. For each class $A \in C(\mathcal{O}_k)$, we have by Theorem 7.5 the arithmetic height formula

$$\left[\widehat{Z}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right] = -\frac{h_k}{w_k} \cdot L'(0, \xi_{1-n/2}(f_{0,A}) \times \theta_{L_{A,0}^\perp}) = -\frac{h_k}{w_k} \cdot L'(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}).$$

Using the relation (61) from the proof of Theorem 6.8, we can extend this to the completed Rankin-Selberg L-function $L^\star(s, g_{\phi,A} \times \theta_{L_{A,0}^\vee}) = \Lambda(1+s, \eta_k) L(s, g_{\phi,A} \times \theta_{L_{A,0}^\vee})$ to get the corresponding formula

$$\Lambda(1, \eta_k) \left[\widehat{Z}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right] = -\frac{h_k}{w_k} \cdot \Lambda(1, \eta_k) L'(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}) = -\frac{h_k}{w_k} \cdot L^{\star\prime}(0, g_{\phi,A} \times \theta_{L_{A,0}^\perp}),$$

which by Proposition 6.7 (i) is the same as

$$\Lambda(1, \eta_k) \left[\widehat{Z}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right] = -\frac{h_k}{w_k} \cdot \Lambda'(1/2, \phi \times \theta_A),$$

and which after evaluating $\Lambda(1, \eta_k)$ via (68) then dividing each side by $-h_k/w_k$ is the same as

$$-2\pi \left[\widehat{Z}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right] = \Lambda'(1/2, \phi \times \theta_A).$$

Taking the twisted sum $\Lambda'(1/2, \phi \times \theta(\chi)) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) \Lambda'(1/2, \phi \times \theta_A)$ then gives the stated formula. □

Corollary 7.9. *Let E be an elliptic curve of level N defined over $\mathbf{Q}$, parametized via modularity by a cuspidal newform $\phi = \phi_E \in S_2(\Gamma_0(N))$, so that the Hasse-Weil L-function $L(E, s)$ has an analytic continuation $\Lambda(E, s) = \Lambda(s - 1/2, \phi)$ given by a shift of the standard L-function $\Lambda(s, \phi) = L_\infty(s, \phi) L(s, \phi)$ of ϕ. Let k be an imaginary quadratic field of odd discriminant d_k and (odd) quadratic Dirichlet character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Assume that $(N, d_k) = 1$, and that the "Heegner hypothesis" $\eta_k(-N) = -\eta_k(N) = -1$ holds. Then, for any class group character $\chi \in C(\mathcal{O}_k)^\times$, we have the following central derivative value formula*

$$\Lambda'(E/K, \chi, 1) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A(f_{0,A}) : \mathcal{Z}(V_{A,0})\right]$$

for the Hasse-Weil L-function $\Lambda(E/K, \chi, s) = \Lambda(s - 1/2, \phi \times \theta(\chi))$ of E over K twisted by χ in terms of arithmetic divisors on the Hilbert modular surface $\mathcal{Y}_0(N) \times \mathcal{Y}_0(N) \longrightarrow \mathrm{Spec}(\mathbf{Z})$. Extending to the compactification $\mathcal{X}_0(N) \times \mathcal{X}_0(N) \longrightarrow \mathrm{Spec}(\mathbf{Z})$, we also have the central derivative value formula

$$\Lambda'(E/K, \chi, 1) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \chi(A) \left[\widehat{\mathcal{Z}}_A^c(f_{0,A}) : \mathcal{Z}(V_{A,0})\right].$$

Note that by comparing with the formula of Gross-Zagier [25, Theorem I (6.3)], we also obtain a relation of arithmetic heights on $\mathcal{X}_0(N)$ and $\mathcal{X}_0(N) \times \mathcal{X}_0(N)$. We explain this in more detail in Appendix A below, where we develop the discussion of [13, §7.3, Theorem 7.7] to prove the full version of [25, Theorem I (6.3)] this way, i.e. via Theorem 7.5 applied to the setup described in Example 7.2.

7.3. **Relations to Birch-Swinnerton-Dyer constants and periods.** As explained in the introduction, we have the following application to the refined conjecture of Birch and Swinnerton-Dyer.

Theorem 7.10. *Let $E/\mathbf{Q}$ be an elliptic curve parametrized by a cuspidal newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$. Let k be a quadratic field of discriminant d_k prime to N. Assume that E has semistable reduction, hence N squarefree. Assume that the completed L-function $\Lambda(E/K, s) = \Lambda(E, s)\Lambda(E^{(d_k)}, s) = \Lambda(s - 1/2, \phi)\Lambda(s - 1/2, \phi \otimes \eta_k)$ has order of vanishing $\mathrm{ord}_{s=1} \Lambda(E/K, s) = 1$, so that exactly one of the central values $\Lambda(E, 1) = \Lambda(1/2, \phi)$ or $\Lambda(E^{(d_k)}, 1) = \Lambda(1/2, \phi \otimes \eta_k)$ vanishes. Write $[e, e]$ to denote either the regulator $R(E/\mathbf{Q})$ or the regulator $R(E^{(d_k)}/\mathbf{Q})$ according to which factor vanishes. Let us also assume for each prime $p \geq 5$ that*

- *The residual Galois representations $E[p]$ and $E^{(d_k)}[p]$ are irreducible.*
- *There exists a prime $l \mid N$ distinct from p where $E[p]$ is ramified, and a prime $q \mid N$ distinct from p where $E^{(d_k)}[p]$ is ramified.*

Then, up to powers of 2 *and* 3*, we have the following unconditional identifications for the constant(s)*

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) = \frac{\#\text{Ш}(E/\mathbf{Q}) \cdot \#\text{Ш}(E^{(d_k)}/\mathbf{Q}) \cdot [e,e] \cdot T(E/\mathbf{Q}) \cdot T(E^{(d_k)}\mathbf{Q}) \cdot \Omega_\infty(E/\mathbf{Q}) \cdot \Omega_\infty(E^{(d_k)}/\mathbf{Q})}{\#E(\mathbf{Q})^2_{\text{tors}} \cdot \#E^{(d_k)}(\mathbf{Q})^2_{\text{tors}}}.$$

(i) *If k is imaginary quadratic with $\eta_k(-N) = -\eta_k(N) = -1$, then*

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) \approx \Lambda'(E/k,1) = -\frac{\pi}{2} \sum_{A \in C(\mathcal{O}_k)} \Phi(f_{0,A}, Z(V_{A,0})) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \left[\widehat{\mathcal{Z}}^c_A(f_{0,A}) : \mathcal{Z}(V_{A,0}) \right].$$

(ii) *If k is real quadratic with $\eta_k(-N) = \eta_k(N) = -1$, then*

$$\kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q}) \approx \Lambda'(E/k,1)$$
$$= -2\ln(\varepsilon_k) h_k \sum_{A \in C(\mathcal{O}_k)} \left[\left(\frac{w_k \ln(\varepsilon_k)}{4h_k} \right) \Phi(f_{0,A}, G(W_A)) + \text{CT}\langle\langle f^+_{0,A}(\tau), \theta_{L^\perp_{A,W}}(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle \right].$$

Here, in either case, we write $\approx$ to denote equality up to powers of 2 *and* 3*.*

Proof. Cf. [54, Theorem 5.1] for the case of k real quadratic. In either case, we use the product rule with the Artin decomposition $\Lambda(E/k,s) = \Lambda(E,s)\Lambda(E^{(d_k)},s) = \Lambda(E/\mathbf{Q},s)\Lambda(E^{(d_k)}/\mathbf{Q},s)$ to compute

$$\Lambda'(E/k,1) = \Lambda'(E,1)\Lambda(E^{(d_k)},1) + \Lambda'(E^{(d_k)},1)\Lambda(E,1),$$

equivalently

$$\Lambda'(1/2,\Pi(\pi)) = \Lambda'(1/2,\phi)\Lambda(1/2,\phi \otimes \eta_k) + \Lambda'(1/2,\phi \otimes \eta_k)\Lambda(1/2,\phi),$$

where exactly one of the summands on the right-hand side vanishes. For the nonvanishing summand, we can take for granted the refined conjecture of Birch and Swinnerton-Dyer up to powers of 2 and 3 using the combinations of various theorems on the Iwasawa main conjectures and subsequent Euler characteristic calculations; see [15] and [16] for details. In brief, we use the combined works of Kato [33], Kolyvagin [37], Rohrlich [45], and Skinner-Urban [48] to establish the cyclotomic main conjectures[11], followed by the relevant Euler characteristic calculations of Burungale-Skinner-Tian [15], [16], and Castella [18] for the rank zero factor, and those of Jetchev-Skinner-Wan [32], Skinner-Zhang [50], and Zhang [66] for the rank one factor. This allows us to deduce that $\Lambda'(E/k,1) \approx \kappa_E(\mathbf{Q}) \cdot \kappa_{E^{(d_k)}}(\mathbf{Q})$. We then identify the central derivative values according to Theorem 6.8 and Corollary 6.9, as well as Theorem 7.8 and Corollary 7.9 when k is imaginary quadratic. ☐

Recall that a complex number $\alpha = \sigma + it$ is said to be a *period* if its real and imaginary parts σ and it can be expressed as integrals of rational functions, over domains in $\mathbf{R}^n$ given by polynomial inequalities with rational coefficients. We write $\mathcal{P} \subset \mathbf{C}$ to denote the set of all such numbers. We refer to the paper of Kontsevich-Zagier [38] for a definitive expository account of this countable subring $\mathcal{P}$ of $\mathbf{C}$, which contains the algebraic numbers $\overline{\mathbf{Q}}$ and their logarithms $\log \overline{\mathbf{Q}}$ (for instance). This paper also describes the conjecture of Birch-Swinnerton-Dyer from this perpective, including the conjecture [38, Question 4] that the central derivative value $\Lambda^{(r_E(k))}(E/k,1)$ should lie in the ring of periods $\mathcal{P}$. Assuming the finiteness of the Tate-Shafarevich group, the argument of [38, §3.5] shows that the Birch-Swinnerton-Dyer constant $\kappa_E(\mathbf{Q}) \in \mathcal{P}$ is a period. The same argument works for the more general setting of number fields, to show that $\kappa_E(k) \in \mathcal{P}$.

Corollary 7.11. *Let us retain the setup of Theorem 7.10.*

(i) *If k is imaginary quadratic with $\eta_k(-N) = -\eta_k(N) = -1$, then the central derivative value*

$$\Lambda'(E/k,1) = -\frac{\pi}{2} \sum_{A \in C(\mathcal{O}_k)} \Phi(f_{0,A}, Z(V_{A,0})) = -2\pi \sum_{A \in C(\mathcal{O}_k)} \left[\widehat{\mathcal{Z}}^c_A(f_{0,A}) : \mathcal{Z}(V_{A,0}) \right]$$

[11] We also consider the anticyclotomic main conjecture for the rank one factor, after passing to an imaginary quadratic field, then descend back down to $\mathbf{Q}$ using various converse theorems and Euler characteristic calculations – see [15] and [16] for a desciption of the state of the art.

lies in the ring of periods $\mathcal{P}$.

(ii) *If k is real quadratic with $\eta_k(-N) = \eta_k(N) = -1$, then the central derivative value*

$$\Lambda'(E/k,1) = -2\ln(\varepsilon_k)h_k \sum_{A \in C(\mathcal{O}_k)} \left[\left(\frac{w_k \ln(\varepsilon_k)}{4h_k} \right) \Phi(f_{0,A}, G(W_A)) + \mathrm{CT}\langle\langle f_{0,A}^+(\tau), \theta_{L_{A,W}^\perp}(\tau) \otimes \mathcal{E}_{L_{A,W}}(\tau) \rangle\rangle \right]$$

lies in the ring of periods $\mathcal{P}$.

Proof. We use the argument of [38, §3.5] to deduce that $\kappa_E(k) \in \mathcal{P}$, up to powers of 2 and 3. That is, we take for granted the conditions of Theorem 7.10 so that various theorems on Iwasawa main conjectures described above allow us to bound the p-primary Tate-Shafarevich groups $Ш(E/\mathbf{Q})[p^\infty]$ and $Ш(E^{(d_k)}/\mathbf{Q})[p^\infty]$ for primes $p \geq 5$ through the corresponding bounds for the p-primary Selmer groups. The argument [38, § 3.5] then shows that each of the corresponding Birch-Swinnerton-Dyer constants $\kappa_E(\mathbf{Q})$ and $\kappa_{E^{(d_k)}}(\mathbf{Q})$, up to powers of 2 and 3, lies in the ring of periods $\mathcal{P}$. □

Appendix A. Gross-Zagier via the signature (1,2) setting

We now explain how a variation of the argument of Bruinier-Yang [13, Theorem 7.7 and Corollary 7.8] can be used to deduce the full Gross-Zagier formula [25, Theorem I (6.3)], for twists by class group characters. This generalizes [25, Theorem 7.7], which recovers the formula of [25, Theorem I(6.3)] for the case of trivial/principal ring class character $\chi_0 = \mathbf{1} \in C(\mathcal{O}_k)^\vee$. Although perhaps well-known to experts, we include details for lack of reference. We also do this to compare with the arithmetic Hirzebruch-Zagier divisors in Theorem 7.8 and Corollary 7.9.

A.1. $X_0(N)$ **as spin Shimura variety.** See [12, §2.4] and [13, §7.3]. Fix an integer $N \geq 1$. Let (V, Q) be the rational quadratic space with underlying vector space

$$V = \mathrm{Mat}_{2\times 2}^{\mathrm{tr}=0}(\mathbf{Q})$$

given by 2×2 matrices with rational coordinates and trace zero, and quadratic form given by $Q(x) = N\det(x)$. The corresponding bilinear form is then given by $(x,y) = -N\,\mathrm{tr}(xy)$ for $x, y \in V$. This rational quadratic space (V,Q) has signature $(1,2)$. The group $\mathrm{GL}_2(\mathbf{Q})$ acts on the trace zero matrices V by conjugation $\gamma \cdot x = \gamma x \gamma^{-1}$ for $x \in V$ and $\gamma \in \mathrm{GL}_2(\mathbf{Q})$. This action leaves the form Q invariant, and induces isomorphisms

$$\mathrm{GSpin}(V) \cong \mathrm{GL}_2, \qquad \mathrm{Spin}(V) \cong \mathrm{SL}_2$$

of algebraic groups over $\mathbf{Q}$. The Grassmannian $D(V) = D^\pm(V)$ can be identified with $\mathfrak{H} \cup \overline{\mathfrak{H}}$ via the map

$$z = x + iy \in \mathfrak{H} \longmapsto \mathbf{R}\Re \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} + \mathbf{R}\Im \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \in D(V).$$

Note that $\mathrm{GSpin}(V)(\mathbf{R})$ acts on $D(V) \cong \mathfrak{H} \cup \overline{\mathfrak{H}}$ by fractional linear transformation. The congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$ determines both a lattice $L \subset V$ and a compact open subgroup $K = K_L = \prod_p K_p$ of $\mathrm{GSpin}(V)(\mathbf{A}_f)$. To be more concrete, we take the lattice

$$L = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}, \tag{76}$$

with dual lattice

$$L^\vee = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}.$$

We have a natural identification of the corresponding discriminant group

$$(\mathbf{Z}/N\mathbf{Z}) \cong L^\vee/L, \quad r \longmapsto \mu_r := \begin{pmatrix} r/2N & \\ & -r/2N \end{pmatrix}. \tag{77}$$

The lattice $L \subset V$ has level $4N$, and the quadratic form on $L^\vee/L$ can be identified with $x \mapsto -x^2$ on $\mathbf{Z}/4N\mathbf{Z}$. The compact open subgroup $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)$ is given by $K = \prod_p K_p$, with each $K_p \subset \mathrm{GSpin}(V)(\mathbf{Z}_p) \cong \mathrm{GL}_2(\mathbf{Z}_p)$ defined by

$$K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p) : c \in N\mathbf{Z}_p \right\}.$$

In this way, we obtain the identification of Shimura curves

(78)

$$Y_0(N) = \Gamma_0(n)\backslash\mathfrak{H} \longrightarrow X_K(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q})\backslash D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K, \quad \Gamma_0(N)z \longmapsto \mathrm{GSpin}(V)(\mathbf{Q})[z,1]K.$$

A.2. **Heegner divisors as special divisors.**

A.2.1. *Special divisors and CM cycles associated to the lattice L.* We have the following correspondence between the special divisors $Z(\mu, m)$ and Heegner divisors $P_{D,r}$ described[12] in Gross-Kohnen-Zagier [26]. Given $m \in \mathbf{Q}_{>0}$ and a coset $\mu \in L^\vee/L$ such that $Q(\mu) \equiv m \bmod L$, we again consider the quadric

$$\Omega_{\mu,m}(\mathbf{Q}) = \{x \in \mu + L : Q(x) = m\}.$$

Note that $\Omega_{\mu,m}(\mathbf{Q}) = \emptyset$ unless $Q(\mu) \equiv m \bmod L$.

Let us for each $m \in \mathbf{Q}_{>0}$ and $\mu \in L^\vee/L$ with $Q(\mu) \equiv m \bmod L$ consider the fundamental discriminant

$$D = -4Nm \in \mathbf{Z}.$$

Given an integer $r \in \mathbf{Z}$ with coset representative $\mu = \mu_r$ under the natural bijection (77),

$$\mu = \mu_r = \begin{pmatrix} r/2N & \\ & -r/2N \end{pmatrix} \in \Omega_{\mu,r}(\mathbf{Q}),$$

we have that $D \equiv r^2 \bmod 4N$. In this way, we produce a positive norm vector in the quadric

$$x = x(\mu, m) = x(\mu_r, -D/4N) = \begin{pmatrix} r/2N & 1/N \\ (D-r^2)/4N & -r/2N \end{pmatrix} \in \Omega_{\mu,m}(\mathbf{Q}). \tag{79}$$

Conversely, given integers $D < 0$ and r such that $D \equiv r^2 \bmod 4N$, let $m = -D/4N$ and $\mu = \mu_r$. Observe that $m \in Q(\mu) + \mathbf{Z}$ is positive. As in [13, §7.1], we take this identification for granted, and note that the corresponding special divisor $Z(\mu, m) = Z(\mu_r, -D/4N)$ defined in (5) above can be identified with a sum of Heegner divisors $P_{D,r} + P_{D,-r}$ defined in Gross-Kohnen-Zagier [26, IV.1(1)]. We remark that each of these Heegner divisors $P_{D,\pm r}$ has degree equal to the Hurwitz class number $H(D)$,

$$\deg(P_{D,\pm r}) = H(D) = \frac{h(D)}{2w(D)}.$$

Here, $h(D)$ denotes the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{D})$, equivalently the cardinality of the class group of positive definite binary quadratic forms of discriminant D. We also write $w(D)$ to denote the number of roots of unity in $\mathbf{Q}(\sqrt{D})$, equivalently the cardinality of the unit group of $\mathbf{Q}(\sqrt{D})$. Hence, we deduce that for a pair $(\mu, m) = (\mu_r, -D/4N)$ corresponding to (D, r) in this way, we have the relation

$$\deg Z(\mu, m) = \deg Z(\mu_r, -D/4N) = \deg(P_{D,\pm r}) = H(D) = \frac{h(D)}{2w(D)}.$$

Fixing a positive norm vector $x = x(\mu, m) \in \Omega_{\mu,m}(\mathbf{Q})$ as in (79) above, we consider the positive and negative definite subspaces defined by

$$V_+ := \mathbf{Q}x, \quad U := V \cap x^\perp$$

[12]To be more precise, let $\tau \in \mathfrak{H}$ be a root of the quadratic equation $a\tau^2 + b\tau + c = 0$ for $a, b, c \in \mathbf{Z}$, $a > 0$, $a \equiv 0 \bmod N$, $b \equiv r \bmod 2N$ and $D = b^2 - 4ac$. The image $\tau_{a,b,c}$ of such a root in $X_0(N)$ is rational over the Hilbert class field $k[1]$ of the imaginary quadratic field k of discriminant D, and the Galois group $\mathrm{Gal}(k[1]/k) \cong C(\mathcal{O}_k)$ permutes these images simply transitively. We then define $P_{D,r} = \frac{w_k}{2}\sum_{[a,b,c]\in\mathcal{Q}_D\cong C(\mathcal{O}_k)} \tau_{a,b,c}$ as $\frac{w_k}{2}$ times the sum of these h_k points. In the moduli description of $X_0(N)$, this point $P_{D,r}$ corresponds to a triple (E, E', φ) of elliptic curves E and E' with complex multiplication by $\mathcal{O}_k$ and $\varphi : E \to E'$ is an isogeny of kernel annihilated by the primitive ideal $\mathfrak{n} = [N, (r + \sqrt{D})/2]$ of norm N.

of $(V,Q) = (\mathrm{Mat}_{2\times 2}^{\mathrm{tr}=0}(\mathbf{Q}), N\cdot \det(\cdot))$, as well as the respective positive definite and negative definite lattices

$$\mathcal{P} := L \cap V_+, \quad \mathcal{N} := L \cap x^{\perp}.$$

Notice that we can present the negative definite lattice $\mathcal{N} \subset L$ more explicitly as

$$\mathcal{N} = \mathbf{Z}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix} \oplus \mathbf{Z}\begin{pmatrix} 0 & 1/N \\ (r^2-D)/4N & 0 \end{pmatrix},$$

and also that $\mathcal{N}$ has determinant $-D$. Writing $t = \gcd(r, 2N)$, the positive definite lattice $\mathcal{P} \subset L$ and its dual lattice $\mathcal{P}^{\vee}$ can be presented more explicitly as

$$\mathcal{P} = \mathbf{Z}\begin{pmatrix} r & 2 \\ (D-r^2)/2 & -r \end{pmatrix} = \mathbf{Z}\frac{2N}{t}x, \quad (L_0^{\perp})^{\vee} = P^{\vee} = \mathbf{Z}\frac{t}{D}x.$$

Let us now consider the ideal $\mathfrak{n} = [N, (r+\sqrt{D})/2]$ of $\mathbf{Z}[(D+\sqrt{D})/2]$. This ideal has norm N, and we can associate with it the quadratic form given by the corresponding norm form

$$Q_{\mathfrak{n}}(z) := \frac{z\overline{z}}{N} = \frac{\mathbf{N}(z)}{\mathbf{N}\mathfrak{n}}.$$

As shown in [13, Lemma 7.1], if D is a fundamental discriminant of the imaginary quadratic field $k = \mathbf{Q}(\sqrt{D})$, we have an isomorphism of quadratic lattices

$$(\mathfrak{n}, -Q_{\mathfrak{n}}) \longrightarrow (\mathcal{N}, -Q_{\mathfrak{n}}), \quad xN + y\left(\frac{r+\sqrt{D}}{2}\right) \longmapsto \begin{pmatrix} x & -y/N \\ -rx - y(r^2-D)/4N & \end{pmatrix}.$$

Both lattices are equivalent to the integral quadratic form defined by

$$[-N, -r, -(r^2-D)/4N] = -Nx^2 - rxy - (r^2-D)/4Ny^2.$$

Now, recall that the spin group $\mathrm{GSpin}(U) = \mathrm{GSpin}(U(x))$ can be identified as the multiplicative group $T_U = \mathrm{GSpin}(U) \cong k^{\times}$, with $K_T = K \cap T \cong \widehat{\mathcal{O}}_k^{\times}$ maximal. According to [13, Proposition 7.2], if the fundamental discriminant D is coprime to N, then we have an identification of zero cycles $Z(U) = Z(m,\mu)$. More precisely, writing $F_{\mu,m}(\tau) = F_{\mu,m}^{L}(\tau, 3/4)$ for the Hejhal Poincaré series of Definition 4.4 above, we have the identifications of CM cycles

$$Z(U) = Z(U(x)) = Z(\mu, m) = Z(F_{\mu,m}) = Z(F_{\mu,m}^{L})$$

on $Y_0(N)(\mathbf{C}) = X_K(\mathbf{C}) = \mathrm{Sh}_K(\mathrm{GSpin}(V), D(V))(\mathbf{C})$.

A.2.2. *Ideal class representatives.* Let k be any quadratic field (real or imaginary) of discriminant d_k and class group $C(\mathcal{O}_k)$. In the subsections of this appendix, we shall take k to be an imaginary quadratic field of discriminant $d_k = D$, though we consider the more general situation for future reference. Let $\mathcal{Q}_{d_k}$ denote the class group of binary quadratic forms $q_{a,b,c}(x,y) = ax^2 + bx + c$ of discriminant $d_k = b^2 - 4a$. Write $[a,b,c] = [q_{a,b,c}] \in \mathcal{Q}_{d_k}$ to denote the class represented by a binary quadratic form $q_{a,b,c}(x,y)$ of discriminant $d_k = b^2 - 4ac$. A classical theorem shows that we have an isomorphism of class groups $\mathcal{Q}_{d_k} \cong C(\mathcal{O}_k)$. For instance (see e.g. [19, Theorem 7.7]), we have the explicit isomorphism

$$\varphi : \mathcal{Q}_{d_k} \cong C(\mathcal{O}_k), \qquad [a,b,c] \longmapsto [a, (-b+\sqrt{d_k})/2].$$

Recall that for each class $A \in C(\mathcal{O}_k)$ we fix an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_k$ of A. Let us then consider the sublattice $L_A \subset L$ defined by

$$(80) \qquad L_A = \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} : a,b,c \in \mathbf{Z}, \quad N\det\begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix}\Bigg|_{L_{A,U}} \equiv -q_{a,b,c} \text{ for } \varphi([a,b,c]) = A \right\},$$

with dual lattice

$$L_A^{\vee} = \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} : a,b,c \in \mathbf{Z}, \quad N\det\begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix}\Bigg|_{L_{A,U}} \equiv -q_{a,b,c} \text{ for } \varphi([a,b,c]) = A \right\}.$$

Again, we have a natural identification of the corresponding discriminant group

$$(\mathbf{Z}/N\mathbf{Z}) \cong L_A^\vee/L_A, \quad r \longmapsto \mu_r := \begin{pmatrix} r/2N & \\ & -r/2N \end{pmatrix},$$

and the quadratic form on $L_A^\vee/L_A$ can again be identified with $x \mapsto -x^2$ on $\mathbf{Z}/4N\mathbf{Z}$. The corresponding compact open subgroup $K_A = K_{L_A} \subset \mathrm{GSpin}(V)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)$ is given by $K_A = \prod_p K_{A,p}$, with each $K_{A,p} = K_p$ for $K_p \subset \mathrm{GSpin}(V)(\mathbf{Z}_p) \cong \mathrm{GL}_2(\mathbf{Z}_p)$ as defined above. That is, we have identified the corresponding compact open subgroups $K = K_A$ for each $A \in C(\mathcal{O}_k)$.

Let us now assume k is imaginary quadratic with discriminant $d_k = D$. Adapting the discussion above, we consider for each $m \in \mathbf{Q}_{>0}$ and $\mu \in L_A^\vee/L_A$ with $Q(\mu) \equiv m \bmod L_A$ the corresponding quadric

$$\Omega_{A,\mu,m}(\mathbf{Q}) = \{x \in \mu + L_A : Q(x) = m\}.$$

Here, we see that $\Omega_{A,\mu,m}(\mathbf{Q})$ is empty unless $Q(\mu) \equiv m \bmod L_A$. Given $m \in \mathbf{Q}_{>\mathbf{0}}$, let $\mu \in L_A^\vee/L_A$ be such that $Q(\mu) \equiv m \bmod 1$. Hence, $D = -4Nm \in \mathbf{Z}$ is again a negative discriminant. If $r \in \mathbf{Z}$ with $\mu =_r \bmod L_A$, then again $D \equiv r^2 \bmod N$, and we have a positive norm vector

$$x_A = x_A(\mu, m) = x_A(\mu_r, -D/4N) = \begin{pmatrix} r/2N & 1/N \\ (D-r^2)/4N & -r/2N \end{pmatrix} \in \Omega_{A,\mu,m}(\mathbf{Q}).$$

Conversely, given $D < 0$ and r with $D \equiv r^2 \bmod 4N$, put $m = -D/4N$ and $\mu = \mu_r$. Then, $m \in Q(\mu) + \mathbf{Z}$ is positive. The corresponding special divisor

$$Z_A(\mu, m) = Z_A(\mu_r, -D/4N) = \Gamma_0(N) \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q(x) = m}} D(V)_x = P_{D,r}^A + P_{D,-r}^A$$

corresponds to the Heegner divisor $P_{D,r}^A + P_{D,-r}^A$, where each point $P_{D,\pm r} \in X_0(N)(k[1])$ in the moduli description is represented by a triple (E, E', φ) with $E(\mathbf{C}) \cong \mathbf{C}/\mathfrak{a}$ and $E'(\mathbf{C}) \cong \mathbf{C}/\mathfrak{n}^{-1}\mathfrak{a}$ and kernel $\ker(\varphi)$ of the isogeny $\varphi : E \to E'$ annihilated by the primitive ideal $\mathfrak{n}$ (see e.g. [25, §II.1]). In this way, we see that each class $A \in C(\mathcal{O}_k)$ has a representative special (Heegner) divisor $Z_A(\mu, m) = P_{D,r}^A + P_{D,-r}^A$, as well as a representative positive norm vector $x_A = x_A(\mu, m) = x_A(\mu_r, -D/4N) \in \Omega_{A,\mu,m}(\mathbf{Q})$. Let us henceforth fix this set of representative special (Heegner) divisors and positive norm vectors

$$\{Z_A(\mu, m) = Z_A(\mu_r, -D/4N)\}_{A \in C(\mathcal{O}_k)}, \quad \{x_A(\mu, m) = x_A(\mu_r, -D/4N)\}_{A \in C(\mathcal{O}_k)} \longleftrightarrow A \in C(\mathcal{O}_k). \tag{81}$$

Fixing such a set of representatives (81), we consider the positive and negative definite subspaces

$$V_{A,+} = V_{A,+}(x_A) := \mathbf{Q}x_A, \quad U_A = U_A(x_A) := V \cap x_A^\perp$$

of $(V, Q) = (\mathrm{Mat}_{2\times 2}^{\mathrm{tr}=0}(\mathbf{Q}), N \cdot \det(\cdot))$, with corresponding positive and negative definite lattices

$$\mathcal{P}_A = \mathcal{P}_A(x_A) := L_A \cap V_{A,+}, \quad \mathcal{N}_A = \mathcal{N}_A(x_A) = L_A \cap U_A = L_A \cap x_A^\perp.$$

In what follows, we shall apply the results of Theorems 5.12 and 7.5 to these negative definite subspaces $V_{A,+} = \mathcal{N}_A \otimes_{\mathbf{Z}} \mathbf{Q} \subset V$ for each class $A \in C(\mathcal{O}_k)$.

A.3. **Cuspidal eigenforms from vector-valued Shimura lifts.** We now describe the harmonic weak Maass forms $f \in H_{1-n/2}(\omega_L) = H_{1/2}(\omega_L)$ that appear. Let $L \subset V$ be the lattice described in (76) corresponding to the compact open subgroup $K_0(N) \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$. Let $S_{3/2}(\omega_L)$ denote the space of holomorphic cuspidal modular forms of weight 3/2 and representation ω_L. A theorem of Eichler-Zagier [23, Theorem 5.1] shows that we have the identification

$$S_{3/2}(\omega_L) \cong J_{2,N}^{\mathrm{cusp}} \tag{82}$$

with the space $J_{2,N}^{\mathrm{cusp}}$ of Jacobi cusp forms of weight 2 and index N. There is a theory of Hecke operators and newforms for the space of Jacobi cusp forms $J_{2,N}^{\mathrm{cusp}}$. We write $J_{2,N}^{\mathrm{new,cusp}} \subset J_{2,N}^{\mathrm{cusp}}$ to denote its subspace of newforms, with $S_{3/2}^{\mathrm{new}}(\omega_L) \subset S_{3/2}(\omega_L)$ the induced subspace of vector-valued newforms.

To describe this in a more exact way, we again consider the space $S_2(\Gamma_0(N))$ of scalar-valued holomorphic cusp forms of weight 2 on $X_0(N)$ with trivial nebentype character. Let $S_2^\star(\Gamma_0(N)) \subset S_2(\Gamma_0(N))$ denote the subspace of holomorphic cusp forms which are invariant under the Fricke involution W_N. Note that a cusp form $\phi \in S_2(\Gamma_0(N))$ is invariant under the Fricke involution if and only if its corresponding standard

L-function[13] $\Lambda(s,\phi) = L_\infty(s,\phi)L(s,\phi)$ has an odd, symmetric functional equation $\Lambda(s,\phi) = -\Lambda(1-s,\phi)$. Let $S_2^{\mathrm{new}}(\Gamma_0(N)) \subset S_2(\Gamma_0(N))$ denote the subspace of newforms, and

$$S_2^{\mathrm{new},\star}(\Gamma_0(N)) = S_2^\star(\Gamma_0(N)) \cap S_2^{\mathrm{new}}(\Gamma_0(N)) \subset S_2(\Gamma_0(N))$$

the subspace of Fricke-invariant newforms. The theorem of Skoruppa-Zagier [49] shows that the Shimura correspondence can be realized explicitly as an isomorphism of $J_{2,N}^{\mathrm{new,cusp}}$-Hecke modules

$$S_2^{\mathrm{new},\star}(\Gamma_0(N)) \cong J_{2,N}^{\mathrm{new,cusp}}. \tag{83}$$

Explicitly, fix a positive rational $m_0 \in \mathbf{Q}_{>0}$ and a coset $\mu_0 \in L^\vee/L$ such that $m_0 \equiv Q(\mu_0) \bmod 1$. Suppose that $D_0 := -4Nm_0 \in \mathbf{Z}$ determines a fundamental discriminant, with $x \in \Omega_{\mu_0,m_0}(\mathbf{Q})$ the corresponding positive norm vector defined in (79), and $U = U(x) = V \cap x^\perp$ the corresponding negative definite space. Consider the space $S_{3/2}(\omega_L) \subset M_{3/2}(\omega_L) \subset M_{3/2}^!(\omega_L) \subset H_{3/2}(\omega_L)$ of holomorphic cuspidal forms of weight $3/2$ and representation ω_L. We have for each such pair (μ_0, m_0) a linear map

$$\mathcal{S}_{\mu_0,m_0} : S_{3/2}(\omega_L) \longrightarrow S_2(\Gamma_0(N)), \quad g \longmapsto \mathcal{S}_{m_0,\mu_0}(g)$$

defined on Fourier series expansions

$$g(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \in \mathbf{Q}_{>0}} c_g(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_{3/2}(\omega_L)$$

by the rule

$$\mathcal{S}_{\mu_0,m_0}(g)(\tau) := \sum_{n\geq 1} \left(\sum_{d \mid n} \left(\frac{D_0}{d}\right) c_g\left(\mu_0 \cdot \frac{n}{d}, m_0 \cdot \frac{n^2}{d^2}\right)\right) e(n\tau).$$

Here, we shall also write the Fourier series expansion of $\mathcal{S}_{\mu_0,m_0}(g)(\tau) \in S_2(\Gamma_0(N))$ with the simpler notations

$$\mathcal{S}_{\mu_0,m_0}(g)(\tau) = \sum_{n\geq 1} c_{\mathcal{S}_{\mu_0,m_0}(g)}(n) e(n\tau), \quad c_{\mathcal{S}_{\mu_0,m_0}(g)}(n) := \sum_{d \mid n} \left(\frac{D_0}{d}\right) c_g\left(\mu_0 \cdot \frac{n}{d}, m_0 \cdot \frac{n^2}{d^2}\right),$$

as well as the normalized Fourier series expansion

$$\mathcal{S}_{\mu_0,m_0}(g)(\tau) = \sum_{n\geq 1} n^{\frac{1}{2}} a_{\mathcal{S}_{\mu_0,m_0}(g)}(n) e(n\tau), \quad a_{\mathcal{S}_{\mu_0,m_0}(g)}(n) = c_{\mathcal{S}_{\mu_0,m_0}(g)}(n) n^{-\frac{1}{2}}.$$

Hence, the standard L-function $\Lambda(s, \mathcal{S}_{\mu_0,m_0}(g), s) = L_\infty(s, \mathcal{S}_{\mu_0,m_0}(g)) L(s, \mathcal{S}_{\mu_0,m_0}(g))$ has Dirichlet series expansion for $\Re(s) > 1$ given by

$$L(s, \mathcal{S}_{\mu_0,m_0}(g)) = \sum_{n\geq 1} a_{\mathcal{S}_{\mu_0,m_0}(g)}(n) n^{-s} = \sum_{n\geq 1} c_{\mathcal{S}_{\mu_0,m_0}(g)}(n) n^{-(s+\frac{1}{2})}.$$

Writing $\eta_{D_0}(\cdot) = \left(\frac{D_0}{\cdot}\right)$ for the quadratic Dirichlet character of discriminant D_0, this can also be written as

$$L(s-1/2, \mathcal{S}_{\mu_0,m_0}(g)) = \sum_{n\geq 1} c_{\mathcal{S}_{\mu_0,m_0}(g)}(n) n^{-s} = L(s, \eta_{D_0}) \sum_{n\geq 1} c_g(\mu_0 n, m_0 n^2) n^{-s}. \tag{84}$$

Each of the linear maps $\mathcal{S}_{\mu_0,m_0} : S_{3/2}(\omega_L) \longrightarrow S_2(\Gamma_0(N))$ is Hecke-equivariant, and some linear combination of them supplies the isomorphism $S_{3/2}^{\mathrm{new}}(\omega_L) \cong S_2^{\mathrm{new},\star}(\Gamma_0(N))$ implicit in the combination of (83) and (82).

Observe from the Dirichlet series expansion of (84) that if $g \in S_{3/2}^{\mathrm{new}}(\omega_L)$ is related via Shimura correspondence to a scalar-valued cusp form $\phi = \phi_g \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$, then we have the relation of L-series

$$L(s, \mathcal{S}_{\mu_0,m_0}(g)) = c_g(\mu_0, m_0) \cdot L(s, \phi), \tag{85}$$

and hence the relation of central derivative values

$$L'(1/2, \mathcal{S}_{\mu_0,m_0}(g)) = c_g(\mu_0, m_0) \cdot L'(1/2, \phi).$$

[13] Which we normalize here to have central value at $s = 1/2$, as in the discussion above, but distinct from the classical normalizations used by [13, §7.3], [26], and [25].

Lemma A.1. *Fix $m_0 \in \mathbf{Q}_{>0}$ and $\mu_0 \in L^\vee/L$ with $m_0 \equiv Q(\mu_0) \bmod 1$. Consider a fundamental discriminant $D_0 = -4Nm_0 \in \mathbf{Z}$, with corresponding positive norm vector $x_0 \in \Omega_{\mu_0,m_0}(\mathbf{Q})$ and negative definite space $U = U(x_0) = V \cap x_0^\perp$ as described above. Given any vector-valued cuspidal form $g \in S_{3/2}(\overline{\omega}_L)$ whose (μ_0, m_0) Fourier coefficient $c_g(\mu_0, m_0)$ does not vanish, we have the identification of L-series*

$$L(s, g, U) = (4\pi m_0)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\frac{L(s+1/2, \mathcal{S}_{\mu_0,m_0}(g))}{L(s+1, \eta_{D_0})}.$$

In particular, if $g \in S_{3/2}(\overline{\omega}_L)$ and $\phi \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$ are linked by the Shimura correspondence (83) via (85), then we have the relation of L-series

$$L(s, g, U) = (4\pi m_0)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\frac{c_g(\mu_0, m_0) \cdot L(s+1/2, \phi)}{L(s+1, \eta_{D_0})},$$

from which we can derive the identification of central derivative values

$$L'(0, g, U) = \frac{\sqrt{N}}{4\pi}\left(\frac{c_g(\mu_0, m_0)}{\deg Z(\mu_0, m_0)}\right) \cdot L'(1/2, \phi).$$

Proof. See [13, Lemma 7.3] with [13, (4.24)], which we state here in the unitary normalization for the standard L-function of $\phi \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$. Since we obtain a slightly distinct identification for the central derivative value (by a factor of 2^{-4}), we provide details. To be clear, we have from definitions (first for $\Re(s) > 1$) that

$$L(s, g, U) = (4\pi)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\sum_{m\geq 1}\sum_{\mu \in \mathcal{P}^\vee/\mathcal{P}} r_{\mathcal{P}}(\mu, m)c_g(\mu, m)m^{-\left(\frac{s+1}{2}\right)}.$$

Viewing $g \in S_{3/2}(\overline{\omega}_L)$ as a form of weight 3/2 and representation $\omega_{\mathcal{P}\oplus\mathcal{N}}$ via [13, Lemma 3.1], we argue as in [13, Lemma 7.3] that $c_g((\lambda), \lambda) = 0$ for all $\lambda \in \mathcal{P}^\vee$ unless $\lambda \in \mathcal{P}^\vee \cap L^\vee = \mathbf{Z}x$ to deduce that

$$L(s, g, U) = (4\pi)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\sum_{\lambda \in \mathcal{P}^\vee} c_g(\lambda, Q(\lambda))Q(\lambda)^{-\left(\frac{s+1}{2}\right)}.$$

On the other hand, we deduce from (84) that we have the relation

$$\begin{aligned} L(s+1/2, \mathcal{S}_{\mu_0,m_0}(g)) &= L(s+1, \eta_{D_0})\sum_{m\geq 1} c_g(\mu_0 m, m_0 m^2)m^{-(s+1)} \\ &= L(s+1, \eta_{D_0})\sum_{\lambda \in \mathcal{P}^\vee} c_g(\mu_0\lambda, m_0 Q(\lambda)) \cdot (m_0 Q(\lambda))^{-\left(\frac{s+1}{2}\right)} \\ &= L(s+1, \eta_{D_0}) \cdot m_0^{-\left(\frac{s+1}{2}\right)}\sum_{\lambda \in \mathcal{P}^\vee} c_g(\lambda, Q(\lambda))Q(\lambda)^{-\left(\frac{s+1}{2}\right)} \end{aligned}$$

and hence

$$\sum_{\lambda \in \mathcal{P}^\vee} c_g(\lambda, Q(\lambda))Q(\lambda)^{-\left(\frac{s+1}{2}\right)} = m_0^{-\left(\frac{s+1}{2}\right)} \cdot \frac{L(s+1/2, \mathcal{S}_{\mu_0,m_0}(g))}{L(s+1, \eta_{D_0})},$$

so that

$$L(s, g, U) = (4\pi m_0)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\frac{L(s+1/2, \mathcal{S}_{\mu_0,m_0}(g))}{L(s+1, \eta_{D_0})}.$$

which by (85) gives the desired relation of L-series

$$L(s, g, U) = (4\pi m_0)^{-\left(\frac{s+1}{2}\right)}\Gamma\left(\frac{s+1}{2}\right)\frac{c_g(\mu_0, m_0) \cdot L(s+1/2, \phi)}{L(s+1, \eta_{D_0})}.$$

Using that $L(1, \phi) = 0$ as ϕ is invariant under the Fricke involution, we deduce via the product rule that

$$L'(0, g, U) = (4\pi m_0)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\frac{c_g(\mu_0, m_0) \cdot L'(1/2, \phi)}{L(1, \eta_{D_0})} = \frac{c_g(\mu_0, m_0) \cdot L'(1/2, \phi)}{2\sqrt{m_0} \cdot L(1, \eta_{D_0})}. \tag{86}$$

Using the Dirichlet class number formula (67) for the imaginary quadratic field $k = \mathbf{Q}(\sqrt{D_0})$, we evaluate

$$L(1, \eta_{D_0}) = \frac{2\pi h_k}{w_k \sqrt{|D_0|}} = \frac{4\pi}{\sqrt{|D_0|}} \cdot \frac{h_k}{2w_k} = \frac{4\pi}{\sqrt{|D_0|}} \cdot H(D_0) = \frac{2\pi}{\sqrt{Nm_0}} \cdot \deg Z(\mu_0, m_0).$$

Again, $H(D_0) = h_k/2w_k$ denotes the Hurwitz class number, and we have $H(D_0) = \deg Z(\mu_0, m_0)$. Hence,

(87)

$$L'(0, g, U) = \frac{c_g(\mu_0, m_0) \cdot L'(1/2, \phi)}{2\sqrt{m_0} \cdot L(1, \eta_{D_0})} = \frac{\sqrt{Nm_0} \cdot c_g(\mu_0, m_0) \cdot L'(1/2, \phi)}{2\sqrt{m_0} \cdot 2\pi \cdot \deg Z(\mu_0, m_0)} = \frac{N^{\frac{1}{2}}}{4\pi} \cdot \frac{c_g(\mu_0, m_0) \cdot L'(1/2, \phi)}{\deg Z(\mu_0, m_0)}.$$

□

To relate this to the discussion of Theorems 5.12 and 7.5 above, we choose the harmonic weak Maass form $f = f^+ + f^- \in H_{1/2}(\overline{\omega}_L)$ according to the following result.

Lemma A.2. *Fix any cuspidal form $g \in S_{3/2}^{\mathrm{new}}(\overline{\omega}_L)$, and let $\phi \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$ denotes its image under the Shimura correspondence via (83). There exists a harmonic weak Maass form $f \in H_{1/2}(\overline{\omega}_L)$ with Fourier coefficients $c_f^{\pm}(m, \mu)$ as above such that:*

(i) *We have the relation $\xi_{1/2}(f) = g/||g||^2$.*

(ii) *The Fourier coefficients $c_f^+(\mu, m)$ of the principal part P_f of f lie in the Hecke field $\mathbf{Q}(\phi)$ obtained by adjoining to $\mathbf{Q}$ the Fourier coefficients of the cuspidal newform $\phi \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$.*

(iii) *The constant Fourier coefficient $c_f^+(0, 0)$ of f vanishes.*

Proof. See [13, Lemma 7.4] or [12, Lemma 7.3]. □

We also have the following result, to ensure the nonvanishing of coefficients $c_g(\mu_0, m_0)$ in Lemma A.1.

Lemma A.3. *Fix a newform*

$$g(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m>0} c_g(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_{3/2}^{\mathrm{new}}(\omega_L).$$

There exist infinitely many fundamental discriminants $D < 0$ such that

(i) *Each prime divisor $q \mid N$ splits in the imaginary quadratic extension $\mathbf{Q}(\sqrt{D})$.*

(ii) *The coefficient $c_g(\mu, m)$ does not vanish for $m = -\frac{D}{4N}$ and any $\mu \in L^\vee/L$ for which $m \equiv Q(\mu) \bmod 1$.*

Proof. See [13, Lemma 7.5]. This is deduced from the nonvanishing theorem of Bump-Friedberg-Hoffstein [14] together with the Waldspurger formula shown in [26, §II.4 Corollary 1] and [49]. □

A.4. **Relation to heights.** We now consider the moduli stack $\mathcal{Y}_0(N)$ over $\mathbf{Z}$ of cyclic isogenies of degree N of elliptic curves $\pi : E \longrightarrow E'$ for which $\ker(\pi)$ meets each irreducible component of each geometric fibre. We also consider the moduli stack $\mathcal{X}_0(N)$ over $\mathbf{Z}$ of cyclic isogenies of degree N of generalized elliptic curves $\pi : E \longrightarrow E'$ for which $\ker(\pi)$ meets each irreducible component of each geometric fibre. Hence, we have the relation $\mathcal{X}_0(N)(\mathbf{C}) = X_0(N) = X_K^\star(\mathbf{C})$, and $\mathcal{X}_0(N)$ is smooth over $\mathbf{Z}[1/N]$, regular away from supersingular points $\underline{x}$ in characteristic p for $p \mid N$ any prime divisor.

Recall that each of the special divisors $Z(\mu, m)$ has an extension $\mathcal{Z}(\mu, m)$ to the integral model $\mathcal{X} = \mathcal{Y}_0(N)$. More precisely, we can view each $\mathcal{Z}(\mu, m)$ as a Deligne-Mumford stack which assigns to a base scheme S over $\mathbf{Z}$ a set of pairs $(\pi : E \longrightarrow E', \iota)$ consisting of

- A cyclic isogeny $\pi : E \longrightarrow E'$ of elliptic curves E, E' over S of degree N

- An action $\iota : \mathcal{O}_{\mathbf{Q}(\sqrt{D})} \hookrightarrow \mathrm{End}(\pi) = \left\{\alpha \in \mathrm{End}(E) : \pi\alpha\pi^{-1} \in \mathrm{End}(E')\right\}$ of $\mathcal{O}_{\mathbf{Q}(\sqrt{D})}$ on π for which $\iota(\mathfrak{n})\ker(\pi) = 0$.

Again, we take $\mathfrak{n}$ to be the ideal $\mathfrak{n} = [N, (r+\sqrt{d})/2]$ in $k = \mathbf{Q}(\sqrt{D})$, with

$$D = -4Nm \quad \text{and} \quad \mu = \mu_r = \begin{pmatrix} \frac{r}{2N} & \\ & -\frac{r}{2N} \end{pmatrix}.$$

Remark A.4. Although $\mathcal{X}_0(N)$ is not regular, we may use intersection theory for the special divisors $\mathcal{Z}(\mu, m)$ and for cuspidal divisors on $\mathcal{X}_0(N)$. To justify this, we consider the corresponding forgetful maps

$$\mathcal{Z}(\mu, m) \longrightarrow \mathcal{Y}_0(N), \quad (\pi : E \to E', \iota) \longmapsto (\pi : E \to E'),$$

each of which is finite étale, and generally 2 to 1. The image of each of these forgetful maps consists of the flat closure of $\mathcal{Z}(\mu, m)$ in $\mathcal{X}_0(N)$, which does not intersect the boundary $\mathcal{X}_0(N)\backslash\mathcal{Y}_0(N)$, and which moreover lies in the regular locus of $\mathcal{X}_0(N)$.

Let us now fix a harmonic weak Maass form $f(\tau) = f^+(\tau) + f^-(\tau) \in H_{1/2}(\overline{\omega}_L)$ as in Lemma A.2 above. Hence, the principal part $P_f(\tau)$ has Fourier coefficients contained in the Hecke field $\mathbf{Q}(\phi)$, and $c_f^+(0,0) = 0$. Note that if $\phi(\tau) \in S_2^{\mathrm{new},\star}(\Gamma_0(N))$ is the eigenform parametrizing a modular elliptic curve E over $\mathbf{Q}$ of conductor N, then we can deduce that the Fourier coefficients of $P_f(\tau)$ are in fact rational integers. Now, recall that we have associated to this harmonic weak Maass form f a divisor

$$Z(f) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f^+(\mu, -m) Z(\mu, m) \in \mathrm{Div}(Y_0(N)),$$

and that the corresponding reguarized theta lift $\Phi(f; \cdot) = \Phi(f; z, h)$ can be identified as the automorphic Green's function $G_{Z(f)}(\cdot) = G_{Z(f)}(z, h)$ with logarithmic singularity along this divisor $Z(f)$. As explained in [13, § 7.3], there exists a divisor $C(f)$ on $X_0(N)$ supported on the cusps for which the divisor

$$Z^c(f) := Z(f) + C(f)$$

has degree zero on $X_0(N)$. Moreover, the regularized theta lift $\Phi(f, \cdot)$ can be viewed as the automorphic Green's function $G_{Z^c(f)}(\cdot)$ for this divisor $Z^c(f)$ on the compactification $X_0(N) = X_K$. We write

$$\mathcal{Z}^c(f) = \mathcal{Z}(f) + \mathcal{C}(f)$$

to denote its flat closure in $\mathcal{X}_0(N)$, and consider the corresponding arithmetic divisor

$$\widehat{\mathcal{Z}}^c(f) = (\mathcal{Z}^c(f), \Phi(f, \cdot)) = (\mathcal{Z}^c(f), G_{Z^c(f)}(\cdot)) \in \widehat{\mathrm{Ch}}^1(\mathcal{X}_0(N))_{\mathbf{Q}(\phi)}.$$

Given a rational number $m \in \mathbf{Q}_{>0}$ and a coset $\mu \in L^\vee/L$, we consider this divisor on $X_0(N)$ given by

$$y(\mu, m) := Z(\mu, m) - \frac{\deg(Z(\mu, m))}{2}\left((\infty) + (0)\right).$$

Note that this divisor $y(\mu, m)$ has degree zero, and is invariant under the Fricke involution. Let $\mathcal{Y}(\mu, m)$ denote its flat closure in $\mathcal{X}_0(N)$. As explained in [13, § 7.3], for each prime p not dividing the discriminant $D = -4Nm$, this latter divisor $\mathcal{Y}(\mu, m)$ has zero intersection with each fibre component of $\mathcal{X}_0(N)$ over $\mathbf{F}_p$. We also consider the divisor defined by

$$y(f) := \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f^+(\mu, -m) y(\mu, m) \in \mathrm{Div}(X_0(N)) = \mathrm{Div}(X_K^\star),$$

and write $\mathcal{Y}(f)$ to denote its flat closure in $\mathcal{X}_0(N)$.

Let $J_0(N)$ denote the Jacobian of $X_0(N)$, with $J_0(N)(F)$ the F-rational points for some number field F. Hence, elements of $J_0(N)(F)$ correspond to divisor classes of degree zero on $X_0(N)$ which are rational over F. Now, observe that $y(f)$ is a divisor of degree zero on $X_0(N)$ which differs from the $Z^c(f)$ by a divisor of degree zero supported at the cusps. We deduce from the Manin-Drinfeld theorem that $y(f)$ and $Z^c(f)$ represent the same point in $J_0(N) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$. Keeping with the setup of Lemmas A.1, A.2, and A.3 above, let us now consider the generating series

$$\text{Ӿ}(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} y(\mu, m) e(m\tau) \mathbf{1}_\mu.$$

By the theorem[14] of Gross-Kohnen-Zagier [26], this generating series $\mathcal{H}(\tau)$ can be viewed as a modular form taking values $J_0(N)(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}$. Given a normalized newform $\phi \in S^{\mathrm{new},\star}(\Gamma_0(N))$ as above, we can consider the corresponding projection $\mathcal{H}^\phi(\tau)$ of $\mathcal{H}(\tau)$ to the ϕ-isotypical component. The coefficients of this projection $\mathcal{H}^\phi(\tau)$ consist of the projections $y^\phi(\mu,m)$ of each of the divisors $y(\mu,m)$ to the ϕ-isotypical component,

$$\mathcal{H}^\phi(\tau) = \sum_{\mu\in L^\vee/L}\sum_{\substack{m\in\mathbf{Q}\\ m>0}} y^\phi(\mu,m)e(m\tau)\mathbf{1}_\mu.$$

Theorem A.5. *Let us retain the setups of Lemmas A.1, A.2, and A.3 above, so that $g=\xi_{1/2}(f)\in S_{3/2}(\omega_L)$ is the vector-valued Shimura lift of the Fricke-invariant newform $\phi\in S_2^{\mathrm{new},\star}(\Gamma_0(N))$. We have the identity*

$$\mathcal{H}^\phi(\tau) = g(\tau)\otimes y(f) \in S_{3/2}(\overline{\omega}_L)\otimes J_0(N)(\mathbf{Q}).$$

In particular, the divisor $y(f)$ factors through the ϕ-isotypical component of the Jacobian $J_0(N)(\mathbf{Q})\otimes_{\mathbf{Z}}\overline{\mathbf{Q}}$.

Proof. See [13, Theorem 7.6], which explains how to deduce this from [26] and [12, Theorem 7.7]. □

Theorem A.6. *Let us retain the setups of Lemmas A.1, A.2, and A.3 above, so that $g=\xi_{1/2}(f)\in S_{3/2}(\omega_L)$ is the vector-valued Shimura lift of the Fricke-invariant newform $\phi\in S_2^{\mathrm{new},\star}(\Gamma_0(N))$. The Néron-Tate height $[y(f),y(f)]_{\mathrm{NT}}$ of the divisor $y(f)$ is given by the preliminary Gross-Zagier formula*

$$[y(f),y(f)]_{\mathrm{NT}} = \frac{\sqrt{N}}{8\pi||g||^2}\cdot L'(1/2,\phi).$$

Proof. See [13, Theorem 7.7]; we modify the proof via Lemma A.1 and Theorem 5.12 above as follows. Observe that Theorem A.5 implies the identification of Fourier coefficient divisors $c_g(\mu,m)y(f)=y^\phi(\mu,m)$ for each of the pairs (μ,m) we consider. Using this identification together with the Manin-Drinfeld theorem, we deduce that

$$[y(f),y(f)]_{\mathrm{NT}}\cdot c_g(\mu,m) = \left[y(f),y^\phi(\mu,m)\right]_{\mathrm{NT}} = [y(f),y(\mu,m)]_{\mathrm{NT}} = [Z^c(f),y(\mu,m)]_{\mathrm{NT}} \tag{88}$$

for each pair (μ,m) contributing to the principal part $P_f(\tau)$ of $f\in H_{1/2}(\overline{\omega}_L)$.

Let us now fix two distinct pairs (μ_0,m_0) and (μ_1,m_1), and for simplicity write

$$d(\mu_j,m_j)=\deg Z(\mu_j,m_j) \qquad \text{for } j=0,1.$$

Define the constant

$$c=c(\mu_0,\mu_1,m_0,m_1) := d(\mu_1,m_1)c_g(\mu_0,m_0)-d(\mu_0,m_0)c_g(\mu_1,m_1).$$

Consider the divisor of degree zero on $X_0(N)$ defined by

$$Z = d(\mu_1,m_1)y(\mu_0,m_0)-d(\mu_0,m_0)y(\mu_1,m_1) = d(\mu_1,m_1)Z(\mu_0,m_0)-d(\mu_0,m_0)Z(\mu_1,m_1).$$

We also write $\mathcal{Z}$ to denote its flat closure in $\mathcal{X}_0(N)$. Observe that Z is supported outside of the cusps of $X_0(N)$. Let M denote the least common multiple of all the discriminants of the special divisors $Z(\mu,m)$ in the support of $Z(f)$. Assume that for each $j=0,1$ the discriminant $D_j=-4Nm_j$ is coprime to MN. This ensures that the divisors Z and $Z^c(f)$ are coprime. It also allows ensures for each prime p that $\mathcal{Z}$ and $\mathcal{Z}^c(f)$ have zero intersection with each fibral component of $\mathcal{X}_0(N)$ over $\mathbf{F}_p$. Via (88), we compute

$$\begin{aligned} c\cdot[y(f),y(f)]_{\mathrm{NT}} &= [Z^c(f), d(\mu_1,m_1)Z(\mu_0,m_0)-d(\mu_0,m_0)Z(\mu_1,m_1)]_{\mathrm{NT}} \\ &= -d(\mu_1,m_1)\left[\widehat{\mathcal{Z}}^c(f),\mathcal{Z}(\mu_0,m_0)\right]_{\mathrm{Fal}} + d(\mu_0,m_0)\left[\widehat{\mathcal{Z}}^c(f),\mathcal{Z}(\mu_1,m_1)\right]_{\mathrm{Fal}}. \end{aligned}$$

Note that the cuspidal divisor $\mathcal{C}(f)$ does not intersect with any of the special divisors $\mathcal{Z}(\mu,m)$. We now apply the arithmetic height formula (74) shown[15] in Theorem 7.5 for each of the negative definite spaces $U_j=V\cap x(\mu_j,m_j)^\perp$ and lattices $\mathcal{N}_j=U_j\cap L$ and $\mathcal{P}_j=\mathcal{N}_j^\perp\subset L$ with Lemma A.1 (cf. [13, Lemma 7.3])

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[14] This was reproven later by Borcherds using Borcherds products for weakly holomorphic forms in the space $M^!_{1/2}(\omega_L)$.

[15] Here, we could also use the original argument of Bruinier-Yang [13, Theorem 7.7], replacing their substitution of the formula [13, Theorem 4.7] with the slightly modified version we derive in Theorem 5.12 above to obtain the same result, i.e. without any condition on the parity of the discriminant d_k.

and the identification of central derivative L-values (87) and Lemma A.2 (i) and (iii) for the cuspidal form $f \in H_{1/2}(\omega_L)$ to each index $j = 0, 1$ to obtain the arithmetic height formulae

$$\begin{aligned}\left[\widehat{\mathcal{Z}}^c(f), \mathcal{Z}(\mu_j, m_j)\right] &= \frac{1}{2} \cdot \Phi(f, Z(\mu_j, m_j)) + [\mathcal{Z}(f), \mathcal{Z}(\mu_j, m_j)]_{\mathrm{fin}} + [\mathcal{C}(f), \mathcal{Z}(\mu_j, m_j)]_{\mathrm{fin}} \\ &= -\frac{d(\mu_j, m_j)}{2} \left(L'(0, \xi_{1/2}(f), U_j) + c_f^+(0,0) \cdot \kappa_{\mathcal{N}_j}(0,0) \right) + [\mathcal{C}(f), \mathcal{Z}(\mu_j, m_j)]_{\mathrm{fin}} \\ &= -\frac{d(\mu_j, m_j)}{2} \cdot L'(0, \xi_{1/2}(f), U_j) \\ &= -\frac{d(\mu_j, m_j)}{2||g||^2} \cdot \frac{N^{\frac{1}{2}}}{4\pi} \cdot \frac{c_g(\mu_j, m_j) L'(1/2, \phi)}{\deg Z(\mu_j, m_j)} \\ &= -\frac{N^{\frac{1}{2}}}{8\pi ||g||^2} \cdot c_g(\mu_j, m_j) \cdot L'(1/2, \phi)\end{aligned}$$

so that

$$\begin{aligned}c \cdot [y(f), y(f)]_{\mathrm{NT}} &= -d(\mu_1, m_1) \left[\widehat{\mathcal{Z}}^c(f), \mathcal{Z}(\mu_0, m_0)\right]_{\mathrm{Fal}} + d(\mu_0, m_0) \left[\widehat{\mathcal{Z}}^c(f), \mathcal{Z}(\mu_1, m_1)\right]_{\mathrm{Fal}} \\ &= d(\mu_1, m_1) \cdot \frac{N^{\frac{1}{2}}}{8\pi |g||^2} \cdot c_g(\mu_0, m_0) \cdot L'(1/2, \phi) - d(\mu_0, m_0) \cdot \frac{N^{\frac{1}{2}}}{8\pi ||8||^2} \cdot c_g(\mu_1, m_1) \cdot L'(1/2, \phi).\end{aligned} \tag{89}$$

Now, it is not hard to show that we can choose the pairs (μ_0, m_0) and (μ_1, m_1) in such a way that the constant $c = c(\mu_0, \mu_1, m_0, m_1)$ does not vanish. We can then deduce from the calculation (89) that

$$c \cdot [y(f), y(f)]_{\mathrm{NT}} = c \cdot \frac{\sqrt{N}}{8\pi ||g||^2} \cdot L'(1/2, \phi),$$

so that the claimed formula follows after dividing out by the nonzero constant c. □

Corollary A.7. *For any coset $\mu \in L^\vee / L$ and positive integer $m \in Q(\mu) + \mathbf{Z}$, we have for $D := -4Nm$ that*

$$\left[y^\phi(\mu, m), y^\phi(\mu, m)\right]_{\mathrm{NT}} = \frac{\sqrt{|D|}}{8\pi^2 ||\phi||^2} \cdot L(1/2, \phi \otimes \eta_D) \cdot L'(1/2, \phi).$$

Proof. Cf. [13, Corollary 7.8]. We deduce this from Theorem A.6 using the relation $y^\phi(\mu, m) = c_g(\mu, m) \cdot y(f)$ with the Waldspurger-like formula theorem shown shown in Gross-Kohnen-Zagier [26, II, §4 Corollary 1]:

$$\frac{c_{j_g}(\mu, m)^2}{\langle j_g, j_g \rangle} = \frac{\sqrt{|D|}}{2\pi} \cdot \frac{L(1/2, \phi \otimes \eta_D)}{\langle \phi, \phi \rangle},$$

where $j_g \in J_{2,N}^{\mathrm{new,cusp}}$ denotes the Jacobi form corresponding to g. Using that the Petersson norm $||g||$ is equal to $N^{\frac{1}{4}} ||j_g||$, by Eichler-Zagier [23, Theorem 5.3], we derive from this the coefficient formula

$$c_g(\mu, m)^2 = c_{j_g}(\mu, m)^2 = \frac{||j_g||^2}{2\pi ||\phi||^2} \cdot |D|^{\frac{1}{2}} \cdot L(1/2, \phi \otimes \eta_D) = \frac{||g||^2}{2\pi N^{\frac{1}{2}} ||\phi||^2} \cdot |D|^{\frac{1}{2}} \cdot L(1/2, \phi \otimes \eta_D) \tag{90}$$

to get

$$\begin{aligned}\left[y^\phi(\mu, m), y^\phi(\mu, m)\right]_{\mathrm{NT}} &= [c_g(\mu, m) y(f), c_g(\mu, m) y(f)]_{\mathrm{NT}} = c_g(\mu, m)^2 \cdot [y(f), y(f)]_{\mathrm{NT}} \\ &= \frac{|D|^{\frac{1}{2}} ||g||^2}{2\pi N^{\frac{1}{2}} ||\phi||^2} \cdot L(1/2, \phi \otimes \eta_D) \cdot \frac{N^{\frac{1}{2}}}{4\pi ||g||^2} \cdot L'(1/2, \phi) = \frac{|D|^{\frac{1}{2}}}{8\pi^2 ||\phi||^2} \cdot L(1/2, \phi \otimes \eta_D) \cdot L'(1/2, \phi).\end{aligned}$$

□

A.5. **Class group twists.** We now explain how to adapt Theorem A.6 and Corollary A.9 to derive the full Gross-Zagier formula [25, I Theorem (6.3)], which applies to twists by any character χ of the ideal class group $C(\mathcal{O}_k)$ for $k = \mathbf{Q}(\sqrt{D})$ as we consider above. Note that this general form of the Gross-Zagier formula is not derived in [13]. We take for granted all of the discussion above leading to Corollary A.9, and fix a set of representatives as in (81) above. Hence, we choose for each class A a point $x_A \in \Omega_{A,\mu,m}(\mathbf{Q})$ which gives rise to a negative definite space $U_A = V \cap x_A^\perp$. Note that each space U_A corresponds to a fractional ideal representative of the class A of $C(\mathcal{O}_k)$. We also obtain a negative definite lattice $\mathcal{N}_A = U_A \cap L_A$ and a positive definite lattice $\mathcal{P}_A = \mathcal{N}_A^\perp \subset L_A$. Recall that in (81), we also fix for each class $A \in C(\mathcal{O}_k)$ a divisor $Z_A(\mu, m) = Z_A(\mu_r, -D/4N)$ on $X_{K_A} \cong Y_0(N)$. Fixing $f \in H_{1/2}(\omega_L)$ as in Lemma A.2, and taking the restriction f_A to the sublattice $L_A \subset L$ as in Lemma 5.7 and (48) we then define the corresponding divisor

$$Z_A(f_A) = \sum_{\mu \in L_A^\vee / L_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_A}^+(\mu, -m) Z_A(\mu, m) \in \mathrm{Div}(Y_0(N)).$$

There exists for each $A \in C(\mathcal{O}_k)$ a divisor $C_A(f)$ supported on the cusps $X_0(N) \backslash Y_0(N)$ for which the divisor $Z_A^c(f) = Z_A(f) + C_A(f)$ on $X_0(N)$ has degree zero. Again, the regularized theta lift $\Phi(f_A, \cdot)$ determines the automorphic Green's function for this divisor $Z_A^c(f_A) \in \mathrm{Div}(X_0(N))$. Writing

$$\mathcal{Z}_A^c(f_A) = \mathcal{Z}_A(f_A) + \mathcal{C}_A(f_A)$$

for the extension to the flat closure in $\mathcal{X}_0(N)$ of this divisor $Z_A^c(f_A)$, we obtain an arithmetic divisor

$$\widehat{\mathcal{Z}}_A^c(f_A) := (\mathcal{Z}_A^c(f_A), \Phi(f_A, \cdot)) = (\mathcal{Z}_A^c(f_A), G_{Z_A^c(f_A)}(\cdot)) \in \widehat{\mathrm{Ch}}^1(\mathcal{X}_0(N)).$$

We also consider for each class $A \in C(\mathcal{O}_k)$ the Fricke-invariant divisor of degree zero on $X_0(N)$ defined by

$$y_A(\mu, m) = Z_A(\mu, m) - \frac{\deg Z_A(\mu, m))}{2} \left((\infty) + (0) \right),$$

with $\mathcal{Y}_A(\mu, m)$ its flat closure in $\mathcal{X}_0(N)$. We also consider

$$y_A(f_A) = \sum_{\mu \in L_A^\vee / L_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_A}^+(\mu, -m) y_A(\mu, m) \in \mathrm{Div}(X_0(N)),$$

with $\mathcal{Y}_A(f)$ its flat closure in $\mathcal{X}_0(N)$.

A.5.1. *Decompositions of basechange L-functions.* For each class, a minor variation of the argument of Lemma A.1 allows us to make the identification

$$\begin{aligned} L(s, g, U_A) &= (4\pi m)^{-\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) \sum_{m \geq 1} \sum_{\mu \in \mathcal{P}_A^\vee / \mathcal{P}_A} r_{\mathcal{P}_A}(\mu, m) c_{g_A}(\mu, m) m^{-\left(\frac{s+1}{2}\right)} \\ &= (4\pi)^{-\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) \sum_{\lambda \in \mathcal{P}_A^\vee} c_{g_A}(\lambda, Q(\lambda)) Q(\lambda)^{-\left(\frac{s+1}{2}\right)} \\ &= (4\pi m)^{-\left(\frac{s+1}{2}\right)} \Gamma\left(\frac{s+1}{2}\right) \frac{L_A(s+1/2, \phi)}{L(s+1, \eta_D)}. \end{aligned}$$

Here, we view $g = g_A \in S_{3/2}(\overline{\omega}_{L_A})$ as a form of weight 3/2 and representation $\omega_{\mathcal{P}_A \oplus \mathcal{N}_A}$ via Lemma 5.7 and (48) (cf. [13, Lemma 3.1]), and we argue as in [13, Lemma 7.3] that $c_{g_A}((\lambda), \lambda) = 0$ for all $\lambda \in \mathcal{P}_A^\vee$ unless $\lambda \in \mathcal{P}_A^\vee \cap L_A^\vee = \mathbf{Z} x_A$. We then define $L_A(s, \phi)$ by the corresponding relation

$$L(s, \mathcal{S}_{\mu, m}(g_A)) = c_{g_A}(\mu, m) \cdot L_A(s, \phi).$$

Using the same argument as for (87), we compute

$$L'(0, g_A, U_A) = (4\pi m)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \frac{L_A'(1/2, \phi)}{L(1, \eta_D)} = \frac{\sqrt{Nm}}{2\sqrt{m}} \frac{c_{g_A}(\mu, m) L_A'(1/2, \phi)}{2\pi H(D)} = \frac{N^{\frac{1}{2}}}{4\pi} \cdot \frac{c_{g_A}(\mu, m) L_A'(1/2, \phi)}{\deg Z_A(\mu, m)},$$

which via Lemma A.2 (i) and (iii) is the same as

$$L'(0, g_A, U_A) = \frac{N^{\frac{1}{2}}}{4\pi ||g_A||^2} \cdot \frac{c_{g_A}(\mu, m) L_A'(1/2, \phi)}{\deg Z_A(\mu, m)}. \tag{91}$$

We can intepret $L_A(s, \phi)$ as the partial/class basechange L-function of ϕ to k with our unitary normalizations for the standard L-function (cf. [25]), so that in this setup we have the identifications of L-functions

$$L(s, \phi \times \theta(\chi)) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) L_A(s, \phi) L_A(s, \phi \otimes \eta_k) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) L(s, \phi \times \theta_A).$$

To be clear, we define each $L_A(s,\phi)$ according to the partition of the lattice $L = \bigoplus_{A \in C(O_k) \cong \mathcal{Q}_D} L_A$ so that

$$\sum_{A \in C(\mathcal{O}_k)} L_A(s,\phi) = L(s,\phi)$$

is the finite part $L(s,\phi)$ of the standard L-function $\Lambda(s,\phi) = L_\infty(s,\phi)L(s,\phi)$. We can then define $L_A(s,\phi\otimes\eta)$ simply as the quadratic twist. Writing $\Pi = \mathrm{BC}_{k/\mathbf{Q}}(\pi(\phi))$ to denote the quadratic basechange lifting of the cuspidal automorphic representation $\pi(\phi)$ of $\mathrm{GL}_2(\mathbf{A})$ associated to ϕ to $\mathrm{GL}_2(\mathbf{A}_k)$,with standard L-function

$$\Lambda(s,\Pi) = L_\infty(s,\Pi)L(s,\Pi) = \Lambda(s,\phi)\Lambda(s,\phi\otimes\eta_k) = L_\infty(s,\phi)L(s,\pi)L_\infty(s,\phi\otimes\eta_k)L(s,\phi\otimes\eta_k),$$

we have an identification of L-functions

$$\sum_{A \in C(\mathcal{O}_k)} L_A(s,\phi)L_A(s,\phi\otimes\eta_k) = L(s,\phi)L(s,\phi\otimes\eta_k) = L(s,\Pi).$$

Moreover, we have for any character $\chi \in C(\mathcal{O}_k)^\vee$ the equivalence of L-functions

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A)L_A(s,\phi)L_A(s,\phi\otimes\eta_k) = L(s,\Pi\otimes\chi).$$

Note that we can justify this latter identification after opening up Dirichlet series expansions for $\Re(s) > 1$ to see that both sides describe the Dirichlet series expansion of the basechange L-function $L(s,\Pi\otimes\chi)$ over nonzero ideals $\mathfrak{a} \subset \mathcal{O}_k$ corresponding to sums over partial basechange L-functions (first for $\Re(s) > 1$)

$$L_A(s,\Pi\otimes\chi) = \chi(A)L_A(s,\Pi) = \chi(A) \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_k\setminus\{0\} \\ [\mathfrak{a}]=A\in C(\mathcal{O}_k)}} \frac{a_\phi(\mathbf{N}\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s} = \chi(A) \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_k\setminus\{0\} \\ [\mathfrak{a}]=A\in C(\mathcal{O}_k)}} \frac{a_\Pi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s}$$

determined by sums over ideals $\mathfrak{a} \in A$ in each ideal class $A \in C(\mathcal{O}_k)$. That is, we have for each class $A \in C(\mathcal{O}_k)$ the identification of partial/class basechange L-functions (first for $\Re(s) > 1$)

$$L_A(s,\phi)L_A(s,\phi\otimes\eta) = L_A(s,\Pi) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_k\setminus\{0\} \\ [\mathfrak{a}]=A\in C(\mathcal{O}_k)}} \frac{a_\phi(\mathbf{N}\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s} = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_k\setminus\{0\} \\ [\mathfrak{a}]=A\in C(\mathcal{O}_k)}} \frac{a_\Pi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s}.$$

Hence, we have the identifications of quadratic basechange and Rankin-Selberg L-functions

$$\begin{aligned}&\sum_{A \in C(\mathcal{O}_k)} \chi(A)L_A(s,\phi)L_A(s,\phi\otimes\eta_k) \\ &= L(s,\Pi\otimes\chi) := \sum_{A \in C(\mathcal{O}_k)} L_A(s,\Pi\otimes\chi) = \sum_{A \in C(\mathcal{O}_k)} \chi(A)L_A(s,\Pi) \\ &= L(s,\phi\times\theta(\chi)) := \sum_{A \in C(\mathcal{O}_k)} \chi(A)L(s,\phi\times\theta_A).\end{aligned}$$

Remark A.8. Note that this realization of the Rankin-Selberg L-function $L(s,\phi\times\theta(\chi))$ is distinct from that considered in Gross-Zagier [25], where the corresponding $L_{\mathcal{A}}(s,f)$ for $\mathcal{A} \in C(\mathcal{O}_k)$ denotes to the partial Rankin-Selberg L-function $L(s-1/2,\phi\times\theta_A)$ in our description above. In particular, the $L_A(s,\phi)$ here forms a summand of the $\mathrm{GL}_2(\mathbf{A})$-automorphic L-function $L(s,\phi)$ as described in Lemma A.1. In other words, we are working with some explicit form of the basechange equivalence $L(s,\Pi\otimes\chi) = L(s,\phi\times\theta(\chi))$ in this setup.

A.5.2. *Relation to arithmetic heights of Heegner divisors.* We argue as in the proof of Theorem A.6 that when D is prime to $2N$, we have for each class $A \in C(\mathcal{O}_k)$ the corresponding arithmetic height formulae

$$\left[\widehat{\mathcal{Z}}_A^c(f_A), \mathcal{Z}_A(\mu,m)\right]_{\mathrm{Fal}} = -\frac{N^{\frac{1}{2}}}{8\pi||g_A||^2} \cdot c_A(\mu,m) \cdot L_A'(1/2,\phi)$$

and

$$[y_A(f_A), y_A(f_A)]_{\mathrm{NT}} = \frac{N^{\frac{1}{2}}}{8\pi||g_A||} \cdot L_A'(1/2,\phi).$$

We then argue as in Corollary A.6 that we can use the Waldspurger-like formula (90) to derive the formula

$$\begin{aligned}\left[y_A^{\phi}(\mu,m), y_A^{\phi}(\mu,m)\right]_{\mathrm{NT}} &= [c_{g_A}(\mu,m)y_A(f_A), c_A(\mu,m)y_A(f_A)]_{\mathrm{NT}} = c_{g_A}(\mu,m)^2\cdot[y_A(f_A),y_A(f_A)]_{\mathrm{NT}} \\ &= \frac{|D|^{\frac{1}{2}}||g_A||^2}{2\pi N^{\frac{1}{2}}||\phi||^2}\cdot L(1/2,\phi\otimes\eta_D)\cdot\frac{N^{\frac{1}{2}}}{4\pi||g_A||^2}\cdot L_A'(1/2,\phi) = \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot L(1/2,\phi\otimes\eta_D)\cdot L_A'(1/2,\phi)\end{aligned} \tag{92}$$

for the ϕ-isotypical components. Taking the χ-twisted linear combination for any $\chi\in C(\mathcal{O}_k)^\vee$, we obtain

$$\begin{aligned}\frac{1}{h_k}\cdot\widehat{h}_k(y_\chi^\phi) = [y_\chi^\phi,y_\chi^\phi]_{\mathrm{NT}} &:= \sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[y_A^{\phi}(\mu,m), y_A^{\phi}(\mu,m)\right]_{\mathrm{NT}} \\ &= \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot L(1/2,\phi\otimes\eta_D)\sum_{A\in C(\mathcal{O}_k)}\chi(A)L_A'(1/2,\phi) \\ &= \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot L'(1/2,\Pi\otimes\chi) = \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot L'(1/2,\phi\times\theta(\chi)).\end{aligned} \tag{93}$$

This is the same as the formula of Gross-Zagier [25, Theorem I (6.3)] when $|D|>4$:

Theorem A.9 (Gross-Zagier)**.** *Assume the fundamental discriminant D is coprime to $2N$, and that $|D|>4$. Let $f\in H_{1/2}(\overline{\omega}_L)$ as described in Lemma A.2 with $g/||g||^2=\xi_{1/2}(f)\in S_{3/2}(\overline{\omega}_L)$ and $\phi\in S_2^{\mathrm{new},\star}(\Gamma_0(N))$ the corresponding Shimura lift. For χ any character of the ideal class group $C(\mathcal{O}_k)$ of $k=\mathbf{Q}(\sqrt{D})$, we have that*

$$\frac{1}{h_k}\cdot\widehat{h}_k(y_\chi^\phi) = [y_\chi^\phi,y_\chi^\phi]_{\mathrm{NT}} := \sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[y_A^{\phi}(\mu,m), y_A^{\phi}(\mu,m)\right]_{\mathrm{NT}} = \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot L'(1/2,\phi\times\theta(\chi)).$$

Proof. See [25, Theorem I (6.3)], as presented in Theorem 1.1, which gives a *distinct* proof, e.g. without using the Shimura correspondence. This approach is also distinct from the re-proofs and generalizations established by Zhang [64], [65], [62], [63] and Yuan-Zhang-Zhang [59] for Shimura curves over totally real fields. ☐

Phrasing the result in terms of completed L-functions

$$\Lambda'(1/2,\phi\times\theta(\chi)) = \Lambda(1,\eta_k)L'(1/2,\phi\times\theta(\chi)) = \frac{2\pi h_k}{w_k}\cdot L'(1/2,\phi\times\theta(\chi))$$

and comparing with Theorem 7.8 and Corollary 7.9, we obtain the following relations.

Corollary A.10. *We have the arithmetic height formula*

$$2\pi\cdot\widehat{h}_k(y_\chi^\phi) = \frac{|D|^{\frac{1}{2}}}{8\pi^2||\phi||^2}\cdot\Lambda'(1/2,\phi\times\theta(\chi)) = -2\pi\sum_{A\in C(\mathcal{O}_k)}\chi(A)\left[\widehat{Z}^c(f_{0,A}):\mathcal{Z}(V_{A,0})\right],$$

where the twisted sum of special arithmetic divisors on the right-hand side represents the arithmetic Hirzebruch-Zagier divisors on $X_0(N)\times X_0(N)$ described in Remark 7.7, Theorem 7.8, and Corollary 7.9 above.

Appendix B. Relation to metaplectic Fourier coefficients

We now use the connection to the regularized theta lifts $\Phi(f_{1/2},z)=G_{Z(f_{1/2})}(z)\in L^2(X_0(N))$ of Theorem A.9 and $\Phi(f_0,z)=G_{Z(f_0)}(z)\in L^2(X_0(N)\times X_0(N))$ of Theorem 7.8 to relate the central derivative values $\Lambda'(1/2,\phi\times\theta(\chi))$ to Fourier coefficients of half-integral weight forms; cf. the works [12], [34], and [11].

B.1. **The setting of signature** $(1,2)$ **with** $\Phi(f_{1/2},z)\in L^{1+\varepsilon}(X_0(N))$**.** Let us first recall the harmonic weak Maass form described in Lemma A.2, adapted to class group representatives as in Theorem A.9 above. Hence, we retain all of the setup of the previous section, with $(V,Q)=(\mathrm{Mat}_{2\times 2}^{\mathrm{tr}=0}(\mathbf{Q}),N\det(\cdot))$, the lattice $L\subset V$ giving rise to the congruence subgroup $\Gamma_0(N)\subseteq\mathrm{SL}_2(\mathbf{Z})$, and the sublattices $L_A\subset V$ corresponding to ideal classes $A\in C(\mathcal{O}_k)$ described in (80). To be clear, we fix a cuspidal newform $\phi\in S_2^{\mathrm{new},\star}(\Gamma_0(N))$ which is invariant under the Fricke involution w_N.

B.1.1. *Quadratic sublattices.* If k is an imaginary quadratic field of discriminant $d_k < 0$, we fix a set of lattice representatives as in (80) above, together with positive-norm vectors $x \in \Omega_{A,\mu,m}(\mathbf{Q})$ and Heegner divisors $Z_A(\mu, m) \in Y_0(N)$ as in (81). If k is a real quadratic field of discriminant $d_k > 0$, we fix for each class $A \in C(\mathcal{O}_k)$ an integer ideal representative $\mathfrak{a} \subset \mathcal{O}_k$. We then fix a Witt decomposition

$$V_A = \mathcal{K}_A \oplus \mathbf{Q}e_{A,1} \oplus \mathbf{Q}e_{A,2},$$

so that the orthogonal complement $W_A = \mathcal{K}_A^\perp \subset V$ of the Lorentzian subspace $\mathcal{K}_A$ of signature $(1,0)$ is isomorphic to the signature $(1,1)$ subspace determined by $(\mathfrak{a}_\mathbf{Q}, Q_\mathfrak{a}) = (\mathfrak{a}_\mathbf{Q}, \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a})$. Here, we write $e_{A,j} \in V$ to denote the corresponding nonzero isotropic basis vectors with $(e_{A,j}, e_{A,j}) = 0$ for $j = 1, 2$ and $(e_{A,1}, e_{A,2}) = 1$. We then write $L_{A,W} = L \cap W_A$ for the corresponding lattice as in the discussion above, leading to Theorem 5.14 (for the case of $n = 1$). Note that $(L_{A,W}, Q|_{W_A})$ corresponds to the quadratic lattice $(\mathfrak{a}, Q_\mathfrak{a}(\cdot))$. Note as well that this quadratic lattice $L_{A,W}$ is isomorphic to the sublattice $L_A \subset V$ defined in (80) above. Hence, for either case on of the quadratic field k, we have for each class $A \in C(\mathcal{O}_k)$ a corresponding sublattice $L_A \subset L$ corresponding to an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_k$.

B.1.2. *Vector-valued Shimura lifts.* Fix a quadratic field k, real or imaginary, with discriminant d_k, and character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. For each ideal class $A \in C(\mathcal{O}_k)$, we let $g_A \in S_{3/2}^{\mathrm{new}}(\omega_{L_A})$ denote the holomorphic vector-valued cusp form of weight $3/2$ and (Weil) representation ω_{L_A} associated to ϕ by the Shimura correspondence via the isomorphism (83). By Lemma A.2 (cf. [13, Lemma 7.4], [12, Lemma 7.3]), there exists for each class $A \in C(\mathcal{O}_k)$ a harmonic weak Maass form $f_{1/2,A} \in H_{1/2}(\omega_{L_A})$ such that

(i) We have the relation $\xi_{1/2}(f_{1/2,A}) = g_A/||g_A||^2$.

(ii) The Fourier coefficients $c^+_{f_{1/2,A}}(\mu, m)$ of the principal/holomorphic part $f^+_{1/2,A}$ lie in $\mathbf{Q}(\phi)$.

(iii) The constant coefficient $c^+_{f_{1/2,A}}(0, 0)$ vanishes.

Proposition B.1. *For each class $A \in C(\mathcal{O}_K)$, consider the corresponding regularized theta lift*

$$\Phi(f_{1/2,A}, z) = \Phi(f_{1/2,A}, z, 1) = \int_{\mathcal{F}}^{\star} \langle\langle f_{1/2,A}(\tau), \theta_{L_A}(\tau, z, 1)\rangle\rangle d\mu(\tau)$$
$$= \mathrm{CT}_{s=0}\left\{\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_{1/2,A}(\tau), \theta_{L_A}(\tau, z, 1)\rangle\rangle v^{-s} d\mu(\tau)\right\}$$

as a function of the variable $z \in D(V) = D^\pm(V) \cong \mathfrak{H}$. The following assertions are true.

(i) *$\Phi(f_{1/2,A}, z)$ determines a (weight zero) modular function on $X_0(N)$ with Laplacian eigenvalue* 0.

(ii) *$\Phi(f_{1/2,A}, z)$ is the automorphic Green's function for the special divisor $\mathcal{Z}^c(f_{1/2,A})$ on $\mathcal{X}_0(N)$.*

(iii) *$\Phi(f_{1/2,A}, z) \in L^{1+\varepsilon}(X_0(N))$.*

Proof. All three assertions follow from Theorem 4.2, using the identification $X_{K_A} \cong Y_0(N)$ and extending to the cusps, as well as that $c^+_{f_{1/2,A}}(0, 0) = 0$. ☐

Remark B.2. Note that for $A \in C(\mathcal{O}_k)$ with k real quadratic, the Fourier series expansions of these modular functions $\Phi(f_{1/2,A}, z)$ can be calculated according to [12, Theorem 5.3, cf. (5.10)]. More precisely, let $r \in \mathbf{Z}$ be any integer such that $d_k \equiv r^2 \bmod 4N$. Let us for each lattice L_A as defined in (80) above fix a primitive isotropic vector $l_A \in L_A$ and $l'_A \in L_A^\vee$ so that $(l_A, l'_A) = 1$. We then have the corresponding positive definite subspace defined by $\mathcal{K}_A = L_A \cap l_A^\perp \cap l'^\perp_A$. We choose these vectors so that

$$\mathcal{K}_A = \mathbf{Z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given a vector $\lambda \in \mathcal{K}_A \otimes \mathbf{R}$, we write $\lambda > 0$ if it is a positive multiple of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We then have

$$\Phi(f_{1/2,A}, z) = -4 \sum_{\substack{\lambda \in \mathcal{K}_A \\ \lambda > 0}} \sum_{b \bmod d_k} \eta_k(b) c^+_{f_{1/2,A}} \left(\frac{d_k \lambda^2}{2}, r\lambda \right) \log \left| 1 - e\left((\lambda, z) + \frac{b}{d_k} \right) \right|.$$

B.1.3. *Relation to metaplectic Fourier coefficients.* If we knew the functions $\Phi(f_{1/2,A}, z) \in L^{1+\varepsilon}(X_0(N))$ were square integrable, then a straightforward generalization of the theorem of Katok-Sarnak [34] would relate the sums over CM cycles and geodesic sets we compute in Theorems 5.12 and 5.14 for the case of $n = 1$ to twisted sums over the half-integal weight forms F_j related to $\Phi(f_{1/2,A}, z)$ by the Shimura correspondence $\mathrm{Shim}(F_j) = \Phi(f_{1/2,A}, \cdot)$ of the Fourier coefficients $c_{F_j}(d_k)\overline{c_{F_j}(1)}$. On the other hand, we have the following more precise result in this direction. To describe it, we first need to describe the following trace coefficients $\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A}))$ for each case on the quadratic field k.

When $d_k < 0$ so that k is imaginary quadratic, define for each $m \in \mathbf{Q}_{>0}$ and $\mu \in L_A^\vee / L_A$ the trace function

$$\mathrm{tr}_{\mu,m}\left(\Phi(f_{1/2,A})\right) = \sum_{x \in \Gamma_0(N) \backslash \Omega_{A,\mu,m}(\mathbf{Q})} \frac{1}{\# \mathrm{Stab}_{\Gamma_0(N)}(x)} \cdot \Phi\left(f_{1/2,A}, \iota(D(V)_x)\right).$$

Here, we write ι to denote the identification $\iota : D(V) \cong \mathfrak{H}$. We also write $\iota(D(V)_x)$ to denote the image of $D(V)_x$ in the modular curve $X_0(N)$, so that the special (Heegner) divisors $Z_A(\mu, m)$ we consider above is

$$Z_A(\mu, m) = \sum_{x \in \Gamma_0(N) \backslash \Omega_{A,\mu,m}(\mathbf{Q})} \iota(D(V)_x).$$

When $d_k > 0$ so that k is real quadratic, we fix a vector $x \in V(\mathbf{Q})$ of positive length $m \in \mathbf{Q}$, and consider the corresponding geodesic in $D(V) \cong \mathfrak{H}$ defined by $\gamma_x = D(V)_x$. Here, we fix the following orientation:

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \gamma_x = \pm(0, i\infty) \quad \text{is the imaginary axis with the orientation } \pm.$$

The orientation-preserving action of $\mathrm{SL}_2(\mathbf{R})$ then induces an orientation on each geodesic γ_x. We also define

$$dz_x = \pm dz/\sqrt{m} z \quad \text{for} \quad x = \pm\sqrt{\frac{m}{N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\alpha(x) = \mathrm{Stab}_{\Gamma_0(N)}(x) \backslash \gamma_x$, as well as its image in $X_0(N)$. Note that when the stabilizer $\mathrm{Stab}_{\Gamma_0(N)}(x)$ is infinite, $\alpha(x)$ determines a closed geodesic in $X_0(N)$. We then define for each $m \in \mathbf{Q}_{>0}$ and $\mu \in L_A^\vee / L_A$ the corresponding trace function

$$\mathrm{tr}_{\mu,m}\left(\Phi(f_{1/2,A})\right) = \frac{1}{2\pi} \sum_{x \in \Gamma_0(N) \backslash \Omega_{A,\mu,m}(\mathbf{Q})} \int_{\alpha(x)} \Phi(f_{1/2,A}, z) dz_x.$$

Here, each $\alpha(x)$ is a closed geodesic, equivalently, $x^\perp$ is nonsplit over $\mathbf{Q}$, and we again write

$$\Omega_{A,\mu,m}(\mathbf{Q}) = \{ x \in \mu + L_A : Q(x) = m \}$$

for the corresponding quadric[16].

Theorem B.3 (Bruinier-Funke-Imamoglu). *Fix any class $A \in C(\mathcal{O}_k)$. The generating series defined by*

$$\begin{aligned} &I_{1/2,\mu}(\Phi(f_{1/2,A}, \cdot), \tau) \\ &= -2\sqrt{v}\, \mathrm{tr}_{\mu,0}(\Phi(f_{1/2,A})) + \sum_{m<0} \mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) \frac{\mathrm{erf}(2\sqrt{\pi |m| v})}{2\sqrt{|m|}} e(m\tau) + \sum_{m>0} \mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) e(m\tau) \end{aligned}$$

[16] In the remaining case where $\alpha(x)$ is an infinite geodesic, equivalently when $\mathrm{Stab}_{\Gamma_0(N)}(x)$ is trivial and the complement $x^\perp \subset V$ determines an isotropic quadratic space – which only happens if $Q(x) \in \mathbf{N}(\mathbf{Q}^\times)^2$ – then the trace is defined according to the regularization procedure described in [11, §3.3], and the corresponding complementary trace in [11, §3.3.2]. However, as the orthogonal complement $x^\perp \subset V$ always determines an anisotropic subspace $(W_A, Q|_{W_A}) \cong (a_{\mathbf{Q}}, Q_{\mathfrak{a}})$ in the setup we consider above, we do not need to consider these variations of the trace here.

determines the μ part of a harmonic weak Maass form in $H_{1/2}(\omega_{L_A})$,

$$I_{1/2}(\Phi(f_{1/2,A}),\tau) = \sum_{\mu \in L_A^\vee/L_A} I_{1/2,\mu}(\Phi(f_{1/2,A}),\tau) \in H_{1/2}(\omega_{L_A}).$$

Proof. This is special case of [11, Theorem 4.1] detailed for the setup we consider above. □

Now, we identify these traces with the CM cycles $Z(U_A) = Z(L_{A,0})$ and geodesic sets $G(W_A)$ described for Theorems 5.12 and 5.14 above (for the special case of $n = 1$).

Proposition B.4. *Let k be any quadratic field of discriminant d_k prime to the level N. We have the following identifications of the traces defined above in terms of the sums* (50) *of the regularized theta lift $\Phi(f_{1/2,A}, z)$ along the CM cycles $Z(U_A)$ defined in* (6) *and the geodesic sets $G(W_A)$ defined in* (7).

(i) *If $A \in C(\mathcal{O}_k)$ for k an imaginary quadratic field of discriminant $d_k < 0$, then for each positive norm vector $x \in \Omega_{A,\mu,m}(\mathbf{Q})$ with orthogonal complement $U_A = U_A(x) := x^\perp \subset V$, we have that*

$$\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) = \Phi(f_{1/2,A}, Z(U_A)).$$

(ii) *If $A \in C(\mathcal{O}_k)$ for k a real quadratic field of discriminant $d_k > 0$, then for each positive norm vector $x \in \Omega_{A,\mu,m}(\mathbf{Q}) \subset \mathcal{K}_A$ with orthogonal complement $W_A = W_A(x) := x^\perp \subset V$, we have that*

$$\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) = \Phi(f_{1/2,A}, G(W_A)).$$

Proof. Cf. [13, Proposition 7.2]. We use that $\Omega_{A,\mu,m}(\mathbf{A}_f) = K_A x = K_{L_A} x$ in either case. Hence for (i), we see from the corresponding definition (5) of $Z_A(\mu,m)$ that $Z_A(\mu,m) = Z(U_A)$. The result then follows from the definitions. Similarly, for (ii), we see from the corresponding definition (7) of the geodesic $G(W_A)$ with the natural identification $D(V)_x \cong D(W_A)$ that the identification follows from the definitions. □

Putting these results together with Theorem 5.12, Theorem 5.14, and Lemma A.2 (cf. [13, Lemma 7.4]), we obtain the following.

Theorem B.5. *We have via Theorem 5.12 and Theorem 5.14 for the quadratic space (V,Q) of signature $(1,2)$ described above the following identification of central derivative values of L-functions as Fourier coefficients of half-integral weight forms.*

(i) *Let k be an imaginary quadratic field of discriminant $D = d_k < 0$. Assume as in Lemma A.1 that $m = -D/4N$ for $D = -4Nm$ with $D \equiv r^2 \bmod 4N$, and take $\mu = \mu_r$. Then, for χ and character of the ideal class group $C(\mathcal{O}_k)$, we have the relation*

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) \cdot c_{g_A}(\mu,m) \cdot \mathrm{tr}_{\mu,m}\left(\Phi(f_{A,1/2})\right) = -\frac{|D|^{\frac{1}{2}}}{16\pi^2 ||\phi||^2} \cdot L'(1/2, \phi \times \theta(\chi)).$$

(ii) *Let k be a real quadratic field of discriminant $d_k > 0$, and $x \in \Omega_{A,\mu,m}(\mathbf{Q}) \subset \mathcal{K}_A$ a positive norm vector with orthogonal complement $W_A = W_A(x) := x^\perp \subset V$ as in Proposition B.4 (ii). We have for each class $A \in C(\mathcal{O}_k)$ the relation*

$$\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) = -\frac{4h_k}{w_k \ln(\varepsilon_k)} \cdot L'(0, g_A \times \theta_{L_{A,W}^\perp}).$$

Proof. For (i), we combine Proposition B.4, Theorem 5.12, Lemma A.1 (cf. (91)) and Lemma A.2 to find

$$\begin{aligned}\mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) &= \Phi(f_{1/2,A}, Z(U_A)) = -\frac{\deg(Z(U_A))}{2} \cdot L'(0, \xi_{1/2}(f_{1/2,A}) \times \theta_{L_A^\perp}) \\ &= -\frac{\deg(Z(U_A))}{2} \cdot \frac{N^{\frac{1}{2}}}{4\pi ||g_A||^2} \cdot \frac{c_{g_A}(\mu,m) L_A'(1/2,\phi)}{\deg Z_A(\mu,m)} \qquad (94) \\ &= -\frac{N^{\frac{1}{2}} \cdot c_{g_A}(\mu,m)}{8\pi ||g_A||^2} \cdot L_A'(1/2,\phi).\end{aligned}$$

A variation of the formula (90) implied by [26, §II.4, Corollary 1] gives us the corresponding relation

$$c_{g_A}(\mu, m)^2 = \frac{||g_A||^2 |D|^{\frac{1}{2}}}{2\pi N^{\frac{1}{2}} ||\phi||^2} \cdot L_A(1/2, \phi \otimes \eta_D). \tag{95}$$

Multiplying both sides of (94) by the Fourier coefficient $c_{g_A}(\mu, m)$ and applying (95), we find that

$$\begin{aligned} c_{g_A}(\mu, m) \cdot \mathrm{tr}_{\mu,m}\left(\Phi(f_{A,1/2})\right) &= -\frac{N^{\frac{1}{2}} \cdot c_{g_A}(\mu, m)}{8\pi ||g_A||^2} \cdot L_A'(1/2, \phi) \\ &= -\frac{|D|^{\frac{1}{2}}}{16\pi^2 ||\phi||^2} \cdot L_A(1/2, \phi \otimes \eta_D) \cdot L_A'(1/2, \phi). \end{aligned}$$

Taking the twisted linear combination for any class group character $\chi \in C(\mathcal{O}_k)^\vee$, we then find that

$$\begin{aligned} \sum_{A \in C(\mathcal{O}_k)} \chi(A) \cdot c_{g_A}(\mu, m) \cdot \mathrm{tr}_{\mu,m}\left(\Phi(f_{A,1/2})\right) &= -\frac{|D|^{\frac{1}{2}}}{16\pi^2 ||\phi||^2} \sum_{A \in C(\mathcal{O}_k)} \chi(A) L_A(1/2, \phi \otimes \eta_D) L_A'(1/2, \phi) \\ &= -\frac{|D|^{\frac{1}{2}}}{16\pi^2 ||\phi||^2} \cdot L'(1/2, \phi \times \theta(\chi)). \end{aligned}$$

For (ii), we put together Proposition B.4, Theorem 5.14, and Lemma 5.8 to get the relation

$$\begin{aligned} \mathrm{tr}_{\mu,m}(\Phi(f_{1/2,A})) = \Phi(f_{1/2,A}, G(W_A)) &= -\frac{4}{\mathrm{vol}(K_{W_A})} \cdot L'(0, \xi_{1/2}(f_{1/2,A}) \times \theta_{L_A^\perp}) \\ &= -\frac{4h_k}{w_k \ln(\varepsilon_k)} \cdot L'(0, g_A \times \theta_{L_{A,W}^\perp}). \end{aligned}$$

□

Remark B.6. In the situation of Theorem B.5 (ii), we expect there to be an analogue of the adjoint map of [26, §II.4] to give a precise relation between the L-function of the vector-valued Shimura lift $L(s, g_A)$ and the standard L-function $L(s, \phi)$, and from this an analogue of [26, §II.4, Corollary 1] to derive a precise relation between the squares of the Fourier coefficients $c_{g_A}(\mu, m)^2$ and the twisted central values $L(1/2, \phi \otimes \eta_D)$. Of course, some relation between these values is known fairly generally by the works of Waldspurger [56], [55] and Kohnen-Zagier [38]. We hope to make this more explicit this in a subsequent work, and in this way perhaps to resolve the conjecture implied by Bruinier-Ono [12, Theorem 1.1 (2)] in this way.

B.2. **The setting of signature** $(2, 2)$ **with** $\Phi(f_0, z) \in L^{1+\varepsilon}(X_0(N) \times X_0(N))$**.** We now return to the setup of Theorem 7.8 and Corollary 7.9. Hence, for k a real or imaginary quadratic field of discriminant d_k prime to N and quadratic Dirichlet character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$, we fix for each ideal class $A \in C(\mathcal{O}_k)$ an integer ideal representative $\mathfrak{a} \subset \mathcal{O}_k$ with norm form $Q_{\mathfrak{a}}(\cdot) = \mathbf{N}_{k/\mathbf{Q}}(\cdot)/\mathbf{N}\mathfrak{a}$. We then consider the quadratic space (V_A, Q_A) of signature $(2, 2)$ defined by $(V_A, Q_A) = (\mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}, Q_{\mathfrak{a}} - Q_{\mathfrak{a}})$. We fix $L_A = N^{-1}\mathfrak{a} + N^{-1}\mathfrak{a} \subset V_A$ to be the lattice whose adelization corresponds to the compact open subgroup $K_A = K_{L_A} = K_0(N) \oplus K_0(N)$, as described in Corollary 2.4. We fix a cuspidal newform $\phi \in S_2^{\mathrm{new}}(\Gamma_0(N))$. We then argue following Corollary 6.5 that there exists a vector-valued form $g_{\phi,A} \in S_2(\overline{\omega}_{L_A})$ lifting ϕ. For each ideal class $A \in C(\mathcal{O}_k)$, we then take $f_{0,A} \in H_0(\omega_{L_A})$ to be any cuspidal harmonic weak Maass form of weight zero and representation ω_{L_A} whose image under the differential operator $\xi_{1/2}$ equals $g_{\phi,A}$.

Proposition B.7. *For each class $A \in C(\mathcal{O}_K)$, consider the corresponding regularized theta lift*

$$\begin{aligned} \Phi(f_{0,A}, z) = \Phi(f_{0,A}, z, 1) &= \int_{\mathcal{F}}^{\star} \langle\langle f_{0,A}(\tau), \theta_{L_A}(\tau, z, 1)\rangle\rangle d\mu(\tau) \\ &= \mathrm{CT}_{s=0}\left\{ \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f_{0,A}(\tau), \theta_{L_A}(\tau, z, 1)\rangle\rangle v^{-s} d\mu(\tau) \right\} \end{aligned}$$

as a function of the variable $z \in D(V_A) = D^\pm(V_A) \cong \mathfrak{H}^2$. The following assertions are true.

(i) $\Phi(f_{0,A}, z)$ *determines a (parallel weight* 0*) modular function on* $X_0(N)^2$ *with Laplacian eigenvalue* 0.

(ii) $\Phi(f_{0,A}, z)$ *is the automorphic Green's function for the special divisor* $\mathcal{Z}^c(f_{0,A})$ *on* $\mathcal{X}_0(N)^2$.

(iii) $\Phi(f_{0,A}, z) \in L^{1+\varepsilon}(X_0(N)^2)$.

Proof. Again, all three assertions follow from Theorem 4.2, using the identification $X_{K_A} \cong Y_0(N)^2$ and extending to the cusps, as well as that $c^+_{f_{0,A}}(0,0) = 0$. ☐

Again, we note that if these regularized theta lifts $\Phi(f_{0,A}, z) \in L^{1+\varepsilon}(X_0(N)^2)$ were known to be square integrable, then a straightforward generalization of the main theorem of Katok-Sarnak [34] would allow us to relate the twisted linear combinations of Theorem 6.8 and Corollary 6.9 to twisted linear combinations of metaplectic Hilbert modular forms (on $X_0(4N)^2$) of parallel weight $1/2$. Here, we expect there to be some version of Theorem B.3 that generalizes the theorems of Waldspurger [56] and Gross-Kohnen-Zagier [26, §II.4, Corollary 1] relating central values of quadratic twists of $\mathrm{GL}_2(\mathbf{A})$-automorphic L-functions to Fourier coefficients of half-integral weight forms to central derivative values.

Conjecture B.8. *Retain the setups of Theorem 5.12 and 5.14, with* k *the corresponding quadratic field. Through the connection to the sums* $\Phi(f_{0,A}, V_{A,0})$ *and* $\Phi(f_{0,A}, G(W_A))$ *of the automorphic Green's functions* $\Phi(f_{0,A})$ *along CM cycles* $V_{A,0} \subset V_A$ *or geodesic sets respectively via Theorem 6.8 and Corollary 6.9, we have for any class group character* $\chi \in C(\mathcal{O}_k)$ *a relation between the central derivative value* $L'(1/2, \phi \times \theta(\chi))$ *and a twisted linear combination of Fourier coefficients of some Hilbert modular form of parallel weight* $3/2$ *on the Hilbert modular surface* $X_0(4N) \times X_0(4N)$*, as well as a relation to a twisted linear combintation of Fourier coefficient of a harmonic weak Maass form of parallel weight* $1/2$ *and representations* ω_{L_A}.

That is, we expect to have an analogue of the main theorem of Bruinier-Funke-Imamoglu [11] for this setup of type with rational quadratic spaces (V_A, Q_A) of signature $(2,2)$, linking to Fourier coefficients of harmonic weak Hilbert Maass forms of parallel weight $1/2$. We also expect to have an analogue of the theorems of Waldspurger [56], [55], Kohnen-Zagier [38], and Gross-Kohnen-Zagier [26, § II.4, Corollary 1] to express the central derivative values $L'(1/2, \phi \times \theta(\chi))$ – at least for the principal class group character $\chi = \chi_0$ where $L'(1/2, \phi \times \theta(\chi_0)) = L'(1/2, \phi) L(1/2, \phi \otimes \eta_k)$ – relating to squares of Fourier coefficients of Hilbert modular forms of parallel weight $3/2$. Although we do not pursue the idea here, we use this method of proof of the Gross-Zagier formula via regularized theta lifts to illustrate the natural connection for some future work.

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