REGULARIZED RANKIN-SELBERG CENTRAL L-VALUES AS SUMS OF BORCHERDS PRODUCTS

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Abstract. We study central values $L(1/2, \phi \times \theta(\chi))$ of Rankin-Selberg L-functions of cuspidal modular units ϕ for $\Gamma_0(N)$ times Hecke theta series $\theta(\chi)$ of ring class characters χ of quadratic fields k, understood in a regularized sense. In the case of k an imaginary quadratic field, we relate these values to twisted sums over CM cycles of regularized theta lifts and Borcherds products on the square $X_0(N)^2$ of the modular curve $X_0(N)$. This yields expressions in terms of finite algebraic linear combinations of logarithms of algebraic numbers (hence periods), and, more concretely, in terms of Chowla-Selberg periods, building on the work of Schofer and Kudla. In the case of k a real quadratic field, we obtain an analogous expression for $L(1/2, \phi \times \theta(\chi))$ as a twisted linear combination of sums of regularized theta lifts evaluated along real geodesic cycles related to the ideal class group of k.

Contents

1. Introduction

Fix an integer $N \geq 1$. Let $\phi \in S_0^!(\Gamma_0(N))$ be a cuspidal modular unit for $\Gamma_0(N)$, so a weakly holomorphic cusp form of weight zero and trivial nebentype character for $\Gamma_0(N)$. While there is vast classical literature on the special values of these functions at CM points, the study of their L-functions and special values appear to be somewhat more mysterious. Let k be a quadratic field of discriminant d_k prime to N and character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Let χ be a ring class character of k of conductor $c \geq 1$ prime to N, for instance a character of the ideal class group $C(\mathcal{O}_k) \cong \mathrm{Gal}(k[1]/k)$ (of conductor $c = 1$). We consider the Rankin-Selberg L-function $L(s, \phi \times \theta(\chi))$ of ϕ times the Hecke theta series $\theta(\chi)$ associated to χ, which has an analytic continuation $\Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi))L(s, \phi \times \theta(\chi))$ given in terms of a regularized Petersson inner product of ϕ

against $\theta(\chi)E^{\star}(s,\cdot)$ for $E^{\star}(s,\cdot)$ the analytic continuation of a certain Eisenstein series. This relation, which we describe below, implies that $\Lambda(s,\phi\times\theta(\chi))$ satisfies a symmetric functional equation

$$\Lambda(s,\phi\times\theta(\chi))=\eta_k(-N)|d_kc^2N|^{1-2s}\Lambda(1-s,\phi\times\theta(\chi))$$

relating values at s to $1-s$, making $s=1/2$ the central point. We derive the following expressions for the central values $\Lambda(1/2,\phi\times\theta(\chi))$ in terms of certain regularized theta lifts and Borcherds products on $X_0(N)^2$, in the spirit of the Eisenstein-Kronecker limit formula for elliptic curves with complex multiplication.

To fix ideas, let $c\geq 1$ be an integer for which there is a ring class field $k[c]$ of conductor c over k (e.g. $c=1$), and write $\mathcal{O}_c=\mathbf{Z}+c\mathcal{O}_k$ to denote the corresponding order of conductor $c=[\mathcal{O}_k:\mathcal{O}_c]$ the ring of integers $\mathcal{O}_k$ of k. Fix χ be any character of the corresponding ideal/ring class group $C(\mathcal{O}_c)\cong\mathrm{Gal}(k[c]/k)$. For each class $A\in C(\mathcal{O}_c)$, we consider the following rational quadratic spaces (V_A,Q_A) of signature $(2,2)$. Fix an integer ideal representative of the class $A=[\mathfrak{a}]$ with $\mathfrak{a}\subset\mathcal{O}_k$. Writing $\mathbf{N}_{k/\mathbf{Q}}(x)=xx^{\tau}$ for $\tau\neq\mathbf{1}\in\mathrm{Gal}(k/\mathbf{Q})$ to denote the norm homomorphism $\mathbf{N}_{k/\mathbf{Q}}:k\rightarrow\mathbf{Q}$, let $Q_{\mathfrak{a}}(x)=\mathbf{N}_{k/\mathbf{Q}}(x)/\mathbf{N}\mathfrak{a}$ denote the corresponding norm form on k, where $\mathbf{N}\mathfrak{a}=[\mathcal{O}_k:\mathfrak{a}]$ as usual denotes the absolute norm. Hence, viewed as a binary quadratic form in the natural way, $Q_{\mathfrak{a}}$ has signature $(2,0)$ (positive definite) if k is an imaginary quadratic field, and has signature $(1,1)$ if k is a real quadratic field. In either case, we obtain a rational quadratic space of signature $(2,2)$ by taking V_A to be the vector space over $\mathbf{Q}$ defined by $\mathfrak{a}_{\mathbf{Q}}\oplus\mathfrak{a}_{\mathbf{Q}}$ (for $\mathfrak{a}_{\mathbf{Q}}=\mathfrak{a}\otimes_{\mathbf{Z}}\mathbf{Q}$ the fractional ideal), and quadratic form $Q_A(z_1,z_2)=Q_{\mathfrak{a}}(z_1)-Q_{\mathfrak{a}}(z_2)$. We let $(\cdot,\cdot)_A$ to denote the corresponding inner product determined by Q_A. Writing $\mathrm{GSpin}(V_A)$ for the corresponding general spin group, this being a reductive algebraic group over $\mathbf{Q}$ which sits in a short exact sequence

$$1\longrightarrow\mathbf{G}_m\longrightarrow\mathrm{GSpin}(V_A)\longrightarrow\mathrm{SO}(V_A)\longrightarrow 1,$$

we have an exceptional isomorphism of algebraic groups $\mathrm{GSpin}(V_A)\cong\mathrm{GL}_2^2$ (Proposition 2.2 (iv)). We then take $L_A=L_A(N)$ to be the integral lattice of V_A whose adelization $L_A\otimes\mathbf{Z}$ under the of $\mathrm{GSpin}(V_A)(\mathbf{A}_f)$ by conjugation is fixed by the compact open subgroup $K_{L_A}\cong K_0(N)^2\subset\mathrm{GSpin}(V_A)(\mathbf{A}_f)\cong\mathrm{GL}_2(\mathbf{A}_f)^2$ (Proposition 2.2 (v)). We write $L_A^{\vee}$ to denote the dual lattice, with $L_A^{\vee}/L_A$ the corresponding (finite abelian) discriminant group, and $\mathbf{1}_{\mu}=\mathrm{char}(\mu+L_A\otimes\widehat{\mathbf{Z}})$ the characteristic function of a coset $\mu\in L_A^{\vee}/L_A$. Writing ω_{L_A} to denote the corresponding Weil representation, we can associate to $\phi\in S_0^!(\Gamma_0(N))$ a lift

$$f_{\phi,A}(\tau)=\sum_{\mu\in L_A^{\vee}/L_A}f_{\mu,\phi,A}(\tau)\mathbf{1}_{\mu}=\sum_{\mu\in L_A^{\vee}/L_A}\sum_{m\gg-\infty}c_{f_{\phi,A}}(\mu,m)e(m\tau)\mathbf{1}_{\mu}\in S_0^!(\omega_{L_A})$$

to a weakly holomorphic form of weight zero and type ω_{L_A} for $\Gamma=\mathrm{SL}_2(\mathbf{Z})$ (Proposition 3.2). Here, the Fourier coefficients $c_{f_{\phi,A}}(\mu,m)$ are rational integers, given in terms of those of ϕ.

We have the following regularized theta lifts $\Phi(f_{\phi,A},\cdot)$ and Borcherds products $\Psi(f_{\phi,A},\cdot)$ associated to these weakly holomorphic forms. We first describe the spaces on which these functions are defined. Let $D(V_A)$ denote the Grassmannian of oriented negative hyperplanes

$$D(V_A)=\{z\in V_A(\mathbf{R}):\dim(z)=2,Q_A|_z<0\}.$$

Hence, $D(V_A)$ has the structure of a hermitian symmetric domain, with two connected components $D^{\pm}(V_A)$, each of which can be identified as the product of two copies of the Poincaré upper-half plane $D(V_A)^{\pm}\cong\mathfrak{H}^2$. We consider the corresponding spin Shimura variety $X_{K_{L_A}}\cong Y_0(N)^2$ with complex points given by

$$\begin{aligned}X_{K_{L_A}}(\mathbf{C})&=\mathrm{GSpin}(V_A)(\mathbf{Q})\backslash D^{\pm}(V_A)\times\mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_{L_A}\\&\cong\mathrm{GL}_2(\mathbf{Q})^2\backslash\mathfrak{H}^2\times\mathrm{GL}_2(\mathbf{A}_f)^2/K_0(N)^2=Y_0(N)^2.\end{aligned}$$

Note that any rational quadratic subspace $V_{A,0}\subset V_A$ of signature $(0,2)$ determines an oriented hyperplane $V_{A,0}(\mathbf{R})=z_{A,0}^{\pm}\in D(V_A)$ and an imaginary quadratic field $k(V_{A,0})$. When the space $V_A=\mathfrak{a}_{\mathbf{Q}}\oplus\mathfrak{a}_{\mathbf{Q}}$ is constructed from an ideal class $A\in C(\mathcal{O}_c)$ of an imaginary quadratic field k, we simply take $V_{A,0}=\mathfrak{a}_{\mathbf{Q}}$ with the norm form $Q_{A,0}=Q_A|_{V_{A,0}}=Q_{\mathfrak{a}}$, so that $k=k(V_{A,0})$. In any case, a subspace $V_{A,0}\subset V_A$ of signature $(0,2)$ determines a zero cycle $Z(V_{A,0})\subset X_{K_{L_A}}\cong Y_0(N)^2$ with complex points

$$\begin{aligned}Z(V_{A,0})(\mathbf{C})&=\mathrm{GSpin}(V_A)(\mathbf{Q})\backslash\{z_{A,0}^{\pm}\}\times\mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_{L_A}\\&=\mathrm{GL}_2(\mathbf{Q})^2\backslash\{z_{A,0}^{\pm}\}\times\mathrm{GL}_2(\mathbf{A}_f)^2/K_0(N)^2.\end{aligned}$$

On the other hand, any rational quadratic ("Lorentzian") subspace $W_A \subset V_A$ of signature $(1,1)$ determines a real quadratic field $k(W_A)$. When the space $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$ is constructed from an ideal class $A \in C(\mathcal{O}_c)$ of a real quadratic field k, we simply take $W_A = \mathfrak{a}_{\mathbf{Q}}$ with the norm form $Q_A|_{W_A} = Q_{\mathfrak{a}}$, so that $k = k(W_A)$. Given such a subspace $W_A \subset V_A$, we consider the domain of oriented hyperbolic lines

$$D(W_A) = \left\{ z \subset W_A(\mathbf{R}) : \dim(z) = 1; Q_A|_z < 0 \right\}.$$

Fixing an oriented basepoint $z_{W_A}^{\pm} \in D(W_A)$, we then consider the corresponding finite set defined by

$$G(W_A) = \mathrm{GSpin}(W_A)(\mathbf{Q}) \backslash \mathrm{GSpin}(W_A)(\mathbf{A}_f)/K_{W_A}, \quad K_{W_A} := K_{L_A} \cap \mathrm{GSpin}(W_A)(\mathbf{A}_f).$$

Fixing any set of coset representatives $[h]$ for $G(W_A)$, we define for each $h \in \mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(bfA_f)$ to the corresponding locally symmetric space

$$C_{A,h} := \Gamma_{A,h} \backslash D(W_A), \quad \Gamma_{A,h} := \mathrm{GSpin}(W_A)(\mathbf{Q}) \cap h K_{W_A} h^{-1},$$

as well as the corresponding real geodesic cycle

$$\mathcal{G}(W_A) = \coprod_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} C_{A,h} = \coprod_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \Gamma_{A,h} \backslash D(W_A).$$

Although not a subvariety of $\mathrm{X}_{K_{L_A}}$, we shall use this real geodesic cycle $\mathcal{G}(W_A)$ as an evaluation locus for the regularized theta lifts we consider in the real quadratic case.

To define these theta lifts, we use the Siegel theta series $\theta_{L_A}(\tau, z, h)$ defined for $\tau = u + iv \in \mathfrak{H}$, $z \in D(V_A)$, and $h \in \mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_{L_A} \cong \mathrm{GL}_2(\mathbf{A}_f)^2/K_0(N)^2$ by

$$\theta_{L_A}(\tau, z, h) = \sum_{\mu \in L_A^{\vee}/L_A} \theta_{\mu, L_A}(\tau, z, h) \mathbf{1}_{\mu} = v \sum_{\mu \in L_A^{\vee}/L_A} \sum_{\substack{x \in V_A(\mathbf{Q}) \\ x \in \mu + L_A}} e\left(Q_A(x_{z^{\perp}}) \tau + Q_A(x_z) \overline{\tau} \right) \mathbf{1}_{\mu}.$$

Writing $\Gamma_h = \mathrm{GSpin}(V_A)(\mathbf{Q})^2 \cap h K_{L_A} h^{-1} \cong \mathrm{GL}_2(\mathbf{Q}) \cap h K_0(N)^2 h^{-1}$, this determines a nonholomorphic Γ_h-invariant function of $z \in D(V_A)$. In the variable $\tau = u + iv$, it determines a nonholomorpic modular form $\theta_{L_A}(\tau, \cdot)$ of weight 0 and type $\omega_{L_A}^{\vee}$ for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ This appears in the regularized theta lift $\Phi(f_{\phi,A}, \cdot)$ of $f_{\phi,A}(\tau) \in S_0^!(\omega_{L_A})$, defined for $\tau = u + iv \in \mathfrak{H}$, $z \in D(V_A)$, and $h \in \mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)^2$ by the regularized inner product

$$\Phi(f_{\phi,A}, z, h) = \int_{\mathcal{F}} \langle\langle f_{\phi,A}(\tau), \theta_{L_A}(\tau, z, h) \rangle\rangle d\mu(\tau) := \mathrm{CT}_{w=0} \left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f_{\phi,A}(\tau), \theta_{L_A}(\tau, z, h) \rangle\rangle v^{-w} d\mu(\tau) \right).$$

Here, we write $\mathrm{CT}_{w=0} F(w)$ for a function F of $w \in \mathbf{C}$ to denote the constant term in the Laurent series expansion around $w = 0$ of $F(w)$. We write

$$\mathcal{F} = \left\{ \tau = u + iv \in \mathfrak{H} : -1/2 \leq u \leq 1/2, \; u^2 + v^2 \geq 1 \right\}$$

to denote the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$, with

$$\mathcal{F}_T = \left\{ \tau = u + iv \in \mathcal{F} : v \leq T \right\}$$

the truncated fundamental domain of points of height $\leq T$. We also write

$$\begin{aligned} \langle\langle f_{\phi,A}(\tau), \theta_{L_A}(\tau, z, h) \rangle\rangle &= \sum_{\mu \in L_A^{\vee}/L_A} f_{\mu,\phi,A}(\tau) \theta_{\mu, L_A}(\tau, z, h) \\ &= \sum_{\mu \in L_A^{\vee}/L_A} \sum_{m \gg -\infty} c_{f_{\phi,A}}(\mu, m) e(m\tau) \sum_{\substack{x \in V_A(\mathbf{A}) \\ x \in \mu + L_A}} e\left(Q_A(x_{z^{\perp}}) \tau + Q_A(x_z) \overline{\tau} \right) \end{aligned}$$

to denote the natural pairing on vector-valued forms, and $d\mu(\tau) := \frac{dudv}{v^2}$ to denote the Poincaré measure. As we recall below, a well-known theorem of Borcherds [2] shows there is a meromorphic modular form $\Psi(f_{\phi,A}, z, h)$ of weight zero on $X_{K_{L_A}} \cong Y_0(N)^2$ for which

$$\Phi(f_{\phi,A}, z, h) = -2 \log ||\Psi(f_{\phi,A}, z, h)||^2.$$

This meromorphic modular form of weight zero $\Psi(f_{\phi,A}, z, h)$ has divisor

$$Z(f_{\phi,A}) = \sum_{\mu \in L_A^\vee / L_A} \sum_{m > 0} c_{f_{\phi,A}}(\mu, -m) Z_A(\mu, m),$$

where each $Z_A(\mu, m) \subset X_{K_{L_A}} \cong Y_0(N)^2$ denotes the special divisor with complex points given by

$$\begin{aligned} Z_A(\mu, m)(\mathbf{C}) &= \coprod_{h \in \mathrm{GSpin}(V_A)(\mathbf{Q}) \backslash \mathrm{GSpin}(V_A)(\mathbf{A}_f)/K_{L_A}} \Gamma_h \Big\backslash \Bigg(\coprod_{\substack{x \in \mu_h + L_{A\,h} \\ Q_A(x) = m}} D(V_A)_x \Bigg) \\ &\cong \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} D(V_A)_x = \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} \left\{ z \in D^{\pm}(V_A) : (z, x)_A = 0 \right\} \\ &\cong \Gamma_0(N)^2 \Big\backslash \coprod_{\substack{x \in \mu + L_A \\ Q_A(x) = m}} \left\{ z = (z_1, z_2) \in \mathfrak{H}^2 : Q_A(z + x) - Q_A(z) = m \right\} \subset Y_0(N)(\mathbf{C}) \times Y_0(N)(\mathbf{C}). \end{aligned}$$

Note that the sums over cosets $\mu \in L_A^\vee / L_A$ of these divisors $Z_A(\mu, m)$ define classical Hirzebruch-Zagier divisors on the surface $X_{K_{L_A}} \cong Y_0(N)^2$. A theorem of Howard and Madapusi Pera [10, Theorem 9.1.1] shows that the Borcherds product $\Psi(f_{\psi,A}, \cdot)$ is defined over $\mathbf{Q}$, and moreover that it takes algebraic values $\Psi(f_{\psi,A}, z, h) \in \overline{\mathbf{Q}}$ for all $(z, h) \in X_{K_{L_A}}(\mathbf{C}) \cong Y_0(N)^2$,

$$\Psi(f_{\psi,A}, \cdot) : X_{K_{L_A}}(\mathbf{C}) \cong Y_0(N)^2 \longrightarrow \overline{\mathbf{Q}}.$$

We compute the following sums of these regularized theta lifts $\Phi(f_{\phi,A}, \cdot)$ and associated logarithms of Borcherds products $\log ||\Psi(f_{\phi,A}, \cdot)||$ along CM cycles $Z(V_{A,0})$ of $X_{K_{L_A}}(\mathbf{C}) \cong Y_0(N)^2$ and real geodesic cycles $\mathcal{G}(W_A)$, namely

$$\Phi(f, Z(V_{A,0})) = \sum_{(z,h) \in Z(V_{0,A})(\mathbf{C})} \frac{\Phi(f, z, h)}{\# \operatorname{Aut}(z, h)} \tag{1}$$

and

$$\Phi(f, G(W_A)) = \sum_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \frac{1}{\# \operatorname{Aut}(h)} \int_{C_{A,h} = \Gamma_{A,h} \backslash D(W_A)} \Phi(f, z, h) d\nu(z), \tag{2}$$

where

$$\# \operatorname{Aut}(h) := \# \left(SO(W_A)(\mathbf{Q}) \cap h K_{W_A} h^{-1} \right),$$

and $d\nu(v)$ denotes the $\mathrm{O}(1,1)$-invariant length measure. We develop the calculations of [12], [13], [6], [17], and [16], averaging over the sublattices $L_{A,0} = V_{A,0} \cap L_A$ and $L_{W_A} = W_A \cap L_A$ and using the Siegel-Weil formula, to relate these finite sums to the regularized Petersson inner products. To describe the results we obtain uniformly, let (U_A, Q_{U_A}) denote either of the two-dimensional rational quadratic subspaces of (V_A, Q_A) we consider, with signature $(p(U_A), q(U_A))$ and corresponding quadratic field $k(U_A)$. Hence,

$$(p(U_A), q(U_A)) = \begin{cases} (0, 2) & \text{if } U_A = V_{0,A} \text{ is negative definite, with } k(U_A) \text{ imaginary quadraitc} \\ (1, 1) & \text{if } U_A = W_A \text{ is Lorentzian, with } k(U_A) \text{ real quadratic .} \end{cases}$$

We write $L_{U_A} = U_A \cap L_A$ to denote the corresponding lattice, with $L_{U_A}^\perp \subset L_A$ its orthogonal complement of signature $(p(U_A^\perp), q(U_A^\perp))$, and $L_{U_A} \oplus L_{U_A}^\perp \subset L_A$ the corresponding sublattice. We consider the Siegel theta series $\theta_{L_{U_A}^\perp}(\tau) = \theta_{L_{U_A}^\perp}(\tau, 1, 1)$ of weight $l(U_A^\perp) = (p(U_A^\perp) - q(U_A^\perp))/2$ of the complement. Hence,

$$\theta_{L_{U_A}^\perp}(\tau) = \theta_{L_{V_{A,0}}^\perp}(\tau) \in M_1(\omega_{L_{V_{A,0}}^\perp}^\vee)$$

is holomorphic of weight 1 when $U_A = V_{0,A}$ is negative definite, and

$$\theta_{L_{U_A}^\perp}(\tau) = \theta_{L_{W_A}^\perp}(\tau)$$

is nonholomorphic of weight 0 when $U_A = W_A$ is Lorentzian. By the Siegel-Weil formula (see Theorem 2.6), the sum over $\mathrm{SO}(U_A)(\mathbf{Q})\backslash \mathrm{SO}(U_A)(\mathbf{A})$ of the Siegel theta series $\theta_{L_{U_A}}(\tau, z, h)$ of L_{U_A} can be identified with the central value $E_{L_{U_A}}(\tau, 0) = E_{L_{U_A}}(\tau, 0; l(U))$ of the Eisenstein series of weight $l(U_A) = (p(U_A) - q(U_A))/2$ associated to the lattice L_{U_A}, defined for $\Re(s) \gg 1$ by the summation

$$E_{L_{U_A}}(\tau, s) = E_{L_{U_A}}(\tau, s; l(U_A)) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{(s+1-l(U_A))}{2}} \mathbf{1}_0 \right]_{l(U_A), \omega_{L_{U_A}}} \gamma.$$

Here, we use the same conventions as in [6, §2.2, (2.17) and §4] (for instance), writing $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbf{Z} \right\}.$$

As we explain in Theorem 2.8, each of these Eisenstein series has an analytic continuation

$$E^\star_{L_{U_A}}(\tau, s) := \Lambda(s+1, \eta_k) E_{L_{U_A}}(\tau, s)$$

to all $s \in \mathbf{C}$, and satisfies a symmetric functional equation $E^\star_{L_{U_A}}(\tau, s) = E^\star_{L_{U_A}}(\tau, -s)$. We then explain in Proposition 3.1 how we this implies that the Rankin-Selberg L-function $L(s, f_{\phi,A} \times \theta_{L^\perp_{U_A}})$, defined for $\Re(s) \gg 1$ by the regularized theta integral

$$\begin{aligned} L(s, f_{\phi,A} \times \theta_{L^\perp_{U_A}}) &= \int_{\mathcal{F}}^\star \langle\langle f_{\phi,A}(\tau), \theta_{L^\perp_{U_A}}(\tau) \otimes E_{L_{U_A}}(\tau, s) \rangle\rangle d\mu(\tau) \\ &:= \mathrm{CT}_{w=0} \left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f_{\phi,A}(\tau), \theta_{L^\perp_{U_A}}(\tau) \otimes E_{L_{U_A}}(\tau, s) \rangle\rangle v^{-w} d\mu(\tau) \right), \end{aligned}$$

has an analytic continuation

$$L^\star(s, f_{\phi,A} \times \theta_{L^\perp_{U_A}}) = \Lambda(s+1, \eta_k) L(s, f_{\phi,A} \times \theta_{L^\perp_{U_A}})$$

to all $s \in \mathbf{C}$, and satisfies the symmetric functional equation

$$L^\star(s, f_{\phi,A} \times \theta_{L^\perp_{U_A}}) = L^\star(-s, f_{\phi,A} \times \theta_{L^\perp_{U_A}}).$$

We also explain in Proposition 3.3 that we have an identification of completed L-functions

$$\Lambda(s - 1/2, \phi \times \theta(\chi)) = \sum_{A \in C(\mathcal{O}_c)} \chi(A) L^\star(2s-2, f_{\phi,A} \times \theta_{L^\perp_{U_A}}).$$

On the other hand, we show in Proposition 4.4 (see also [13], [12], [6, Lemma 4.6], [17], [16]) that this same regularized integral computes the sums of regularized theta lifts $\Phi(f_{\phi,A}, Z(V_{A,0}))$ and $\Phi(f_{\phi,A}, G(W_A))$:

$$L(0, f_{\phi,A} \times \theta_{L^\perp_{U_A}}) = \frac{1}{\mathrm{vol}(K_{L_{U_A}})} \cdot \begin{cases} \Phi(f_{\phi,A}, Z(V_{A,0})) & \text{if } U_A = V_{A,0} \text{ is negative definite} \\ \Phi(f_{\phi,A}, \mathcal{G}(W_A)) & \text{if } U_A = W_A \text{ is Lorentzian.} \end{cases}$$

Moreover (see Theorem 4.5), we obtain from the theorem of Borcherds the relations

$$\begin{aligned} &L(0, f_{\phi,A} \times \theta_{L^\perp_{U_A}}) \\ &= \begin{cases} -2\,\mathrm{vol}(K_{L_{V_{A,0}}}) \sum_{(z^\pm_{A,0}, h) \in Z(V_{A,0})} \frac{\log ||\Psi(f_{\phi,A}, z^\pm_{A,0}, h)||^2}{\#\,\mathrm{Aut}(z^\pm_{A,0}, h)} \cdot & \text{if } U_A = V_{A,0} \text{ is negative definite} \\ -2\,\mathrm{vol}(K_{L_{W_A}}) \sum_{[h] \in G(W_A)} \frac{1}{\#\,\mathrm{Aut}(h)} \int_{C_{A,h}} \log ||\Psi(f_{\phi,A}, h)||^2 d\nu(z) & \text{if } U_A = W_A \text{ is Lorentzian.} \end{cases} \end{aligned}$$

This allows us to deduce that we have the following novel expressions for the central values $\Lambda(1/2, \phi \times \theta(\chi))$.

Theorem 1.1 (Theorem 4.7, Corollary 4.8). *Let $\phi \in S^!_0(\Gamma_0(N))$ be any weakly holomorphic modular function of level $\Gamma_0(N)$ and trivial character. Let $\chi \in C(\mathcal{O}_c)^\vee$ be a ring class character of conductor $c \geq 1$ of a quadratic field k of discriminant d_k, character η_k, class number h_k, number of roots of unity w_k, and* *fundamental unit ε_k (if real quadratic). Assume that $(N, cd_k) = 1$. We have the following expressions for the central value $\Lambda(1/2, \phi \times \theta(\chi))$ of the completed Rankin-Selberg L-functions*

$$\Lambda(s - 1/2, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi)) L(s, \phi \times \theta(\chi)).$$

Given any class $A \in C(\mathcal{O}_c)$ with corresponding quadratic space (V_A, Q_A) as described in Definition 2.1 and lattice $(L_A, Q_A) = (L_A(N), Q_A)$ as described in Proposition 2.2 (v), let

$$f_{\phi,A}(\tau) = \sum_{\mu \in L_A^\vee / L_A} \sum_{m \gg -\infty} c_{f_{\phi,A}}(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_0^!(\overline{\omega}_{L_A})$$

denote the lifting of ϕ to a vector-valued weakly holomorphic form, as described in Lemma 3.2.

(i) *If k is an imaginary quadratic field, then have the central value formula*

$$\begin{aligned}\Lambda(1/2, \phi \times \theta(\chi)) &= -2\Lambda(1, \eta_k) \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{V_{A,0}}}) \sum_{(z_0,h) \in Z(V_{A,0})} \frac{\log ||\Psi(f_{\phi,A}, z_0, h)||^2}{\# \operatorname{Aut}(z_0, h)} \\ &= -\frac{4\pi h_k}{w_k} \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{V_{A,0}}}) \sum_{(z_0,h) \in Z(V_{A,0})} \frac{\log ||\Psi(f_{\phi,A}, z_0, h)||^2}{\# \operatorname{Aut}(z_0, h)}.\end{aligned}$$

(ii) *If k is a real quadratic field, then we have the central value formula*

$$\begin{aligned}\Lambda(1/2, \phi \times \theta(\chi)) &= -2\Lambda(1, \eta_K) \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{W_A}}) \sum_{[h] \in G(W_A)} \frac{1}{\# \operatorname{Aut}(h)} \int_{C_{A,h}} \log ||\Psi(f_{\phi,A}, z, h)||^2 d\nu(z) \\ &= -4 \log(\varepsilon_k) h_k \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{W_A}}) \sum_{[h] \in G(W_A)} \frac{1}{\# \operatorname{Aut}(h)} \int_{C_{A,h}} \log ||\Psi(f_{\phi,A}, z_W, h)||^2 d\nu(z).\end{aligned}$$

This identity implies the following immediate connection to "periods" in the CM case. Recall that a complex number is said to be a period if its real and imaginary parts can be expressed as integrals of rational functions over domains in $\mathbf{R}^n$ defined by polynomial inequalites with rational coefficients. Let us write $\mathcal{P}$ to denote the ring of periods. Examples include algebraic numbers and logarithms of algebraic numbers (see e.g. [11]). Folklore conjectures predict that central values $L^\star(0, M)$ and central derivative values $L^{\star(r)}(0, M)$ of motivic L-functions $L^\star(s, M)$ should lie in $\mathcal{P}$ (or even $\pi\mathcal{P}$). We refer to [11, Conjecture (Deligne-Beilinson-Scholl) §3.2 and Question 4] for more background. We obtain the following confirmation of this conjecture in the special setting we consider, along with more explicit relations to Chowla-Selberg periods. Given k be an imaginary quadratic field of discriminant d_k, we again write $h_k = \# C(\mathcal{O}_k)$ to denote the class number, with $w_k = \# \mathcal{O}_k^\times$ and the number of roots of unity, $\eta_k(\cdot) = (\frac{d_k}{\cdot})$ the quadratic Dirichlet character, and $\Lambda(s, \eta_k) = L_\infty(s, \eta_k) L(s, \eta_k)$ the completed Dirichlet series. Recall that by the formula of Chowla and Selberg, we have the logarithmic derivative formula

$$\frac{L'(0, \eta_k)}{L(0, \eta_k)} = \frac{w_k}{2h_k} \sum_{a=1}^{d_k} \eta_k(a) \log \Gamma\left(\frac{a}{d_k}\right).$$

Theorem 1.2 (Theorem 5.1, Theorem 5.2)**.** *We have the following realizations of L-values as periods.*

(a) *Let (V, Q) be any rational quadratic space of signature $(2, 2)$ with integral lattice $L \subset V$ and Weil representation ω_L. Let $f \in S_0^!(\omega_M)$ be any weakly holomorphic cusp form of weight 0 and type ω_L for Γ. Let $L_0 \subset L$ be any negative definite sublattice with orthogonal complement $L_0^\perp \subset L$, and let $\theta_{L_0^\perp}(\tau) = \theta_{L_0^\perp}(\tau, 1, 1)$ be the corresponding (holomorphic) Siegel theta series. Then, the central value $L(0, f \times \theta_{L_0^\perp})$ of the Rankin-Selberg L-function $L(s, f \times \theta_{L_0^\perp})$ described in Proposition 3.1 below is a period, $L(0, f \times \theta_{L_0^\perp}) \in \mathcal{P}$, as is the central value of the completed L-function $L^\star(0, f \times \theta_{L_0^\perp}) \in \mathcal{P}$. Moreover, if the discriminant of the imaginary quadratic field $k(L_0 \otimes \mathbf{Q})$ is odd, then we have the* *more explicit formula*

$$L(0, f \times \theta_{L_0^\perp}) = -2\operatorname{vol}(K_0)\log\left(\alpha\left((4\pi d_k)^{-h_k}\cdot\prod_{a=1}^{d_k-1}\Gamma\left(\frac{a}{d_k}\right)^{w_k\eta_k(a)}\right)^{\mathrm{CT}\langle\langle f(\tau),\theta_{L_0^\perp}(\tau)\otimes\mathbf{1}_{L_0+0}\rangle\rangle}\right)$$

$$= -2\operatorname{vol}(K_0)\log\left(\alpha\left((4\pi d_k)^{-1}\cdot e^{2\cdot\frac{L'(0,\eta_k)}{L(0,\eta_k)}}\right)^{h_k\cdot\mathrm{CT}\langle\langle f(\tau),\theta_{L_0^\perp}(\tau)\otimes\mathbf{1}_{L_0+0}\rangle\rangle}\right).$$

Here, $\alpha = \alpha(f, L_0) \in \mathbf{Q}^\times$ is a rational number related to Shimura's period invariants ([14], [8], [19]*), and $\mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau)\otimes\mathbf{1}_{L_0+0}\rangle\rangle$ denotes the constant coefficient in the Fourier series expansion of the scalar-valued modular form determined by the inner product $\langle\langle f(\tau), \theta_{L_0^\perp}(\tau)\otimes\mathbf{1}_{L_0+0}\rangle\rangle$.*

(b) *Let $\phi \in S_0^!(\Gamma_0(N))$ be a weakly holomorphic cusp form of weight zero for $\Gamma_0(N)$. Let k be any imaginary quadratic field of discriminant d_k prime to N, let $c \geq 1$ be any conductor prime to N, and let $\chi \in C(\mathcal{O}_c)^\vee$ be any ring class character of k of conductor c. Let $\theta(\chi) \in M_1(\Gamma_0(|c^2 d_k|), \eta_k)$ denote the corresponding Hecke theta series. The central value $\Lambda(1/2, \phi\times\theta(\chi))$ of the completed Rankin-Selberg L-function $\Lambda(s, \phi\times\theta(\chi)) = L_\infty(s, \phi\times\theta(\chi))L(s, \phi\times\theta(\chi))$ is a period, $\Lambda(1/2, \phi\times\theta(\chi)) \in \mathcal{P}$. Moreover, if the discriminant $d_k \equiv 1 \bmod 2$ is odd, then we have the more explicit formula*

$$\Lambda(1/2, \phi\times\theta(\chi))$$

$$= -\frac{4\pi h_k}{w_k}\sum_{A\in C(\mathcal{O}_c)}\chi(A)\operatorname{vol}(K_{L_{V_{A,0}}})\log\left(\alpha(f_{\phi,A},k)\left((4\pi d_k)^{-h_k}\cdot\prod_{a=1}^{d_k-1}\Gamma\left(\frac{a}{d_k}\right)^{w_k\eta_k(a)}\right)^{\mathrm{CT}\langle\langle f_{\phi,A}(\tau),\theta_{L_{A,0}^\perp}(\tau)\otimes\mathbf{1}_{L_{A,0}+0}\rangle\rangle}\right)$$

$$= -\frac{4\pi h_k}{w_k}\sum_{A\in C(\mathcal{O}_c)}\chi(A)\operatorname{vol}(K_{L_{V_{A,0}}})\log\left(\alpha(f_{\phi,A},k)\left((4\pi d_k)^{-1}\cdot e^{2\cdot\frac{L'(0,\eta_k)}{L(0,\eta_k)}}\right)^{h_k\cdot\mathrm{CT}\langle\langle f_{\phi,A}(\tau),\theta_{L_{A,0}^\perp}(\tau)\otimes\mathbf{1}_{L_{A,0}+0}\rangle\rangle}\right).$$

Here again, each $\alpha(f_{\phi,A}, k) \in \mathbf{Q}^\times$ is related to Shimura's period invariants ([14], [8], [19]*), and $\mathrm{CT}\langle\langle f_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau)\otimes\mathbf{1}_{L_{A,0}+0}\rangle\rangle$ denotes the constant coefficient in the Fourier series expansion of the scalar-valued for determined by the inner product $\langle\langle f_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau)\otimes\mathbf{1}_{L_{A,0}+0}\rangle\rangle$.*

We remark that the corresponding central values are conjectured to be periods for the corresponding setting of real quadratic fields. However, the deductions we use via the connection to points on the Shimura variety $X_{K_{L_A}}(\mathbf{C}) \cong Y_0(N)^2$ do not seem to be available in this setting, where we use the real geodesic cycle $\mathcal{G}(W_A)$ as an evaluation locus for the regularized theta lift. Indeed, the problem appears to be related to that of the construction of generators of ring class extensions of real quadratic fields.

2. SETUP

2.1. **Quadratic spaces.** Let (V, Q) be a rational quadratic space of signature $(n, 2)$ (for some integer $n \geq 0$) and inner product $(x, y) = Q(x+y) - Q(x) - Q(y)$. Let $L \subset V$ be an integer lattice with dual lattice $L^\vee$ and (finite abelian) discriminant group $L^\vee/L$.

2.1.1. *Quadratic spaces of signature (2,2) associated to quadratic fields.* We shall later consider the following quadratic spaces of signature $(2, 2)$ associated to ring classes of a (real or imaginary) quadratic field k. To fix ideas, let k be any quadratic field of discriminant d_k and character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. We write $\mathcal{O}_k$ to denote the ring of integers of k, with $C(\mathcal{O}_k)$ the corresponding ideal class group. More generally, we can take $c \geq 1$ any integer conductor for which there exists a ring class extension $k[c]$ of conductor c over k, noting that such an extension always exists when k is an imaginary quadratic field. Hence, writing $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_k$ to denote the $\mathbf{Z}$-order of conductor c in k, we consider the corresponding ring class group $C(\mathcal{O}_c) \cong \mathrm{Gal}(k[c]/k)$, where

$$C(\mathcal{O}_c) := \mathbf{A}_k^\times / k^\times k_\infty^\times \widehat{\mathcal{O}}_c^\times. \tag{3}$$

For each class $A \in C(\mathcal{O}_c)$, we fix an integer ideal representative $\mathfrak{a} \subset \mathcal{O}_k$ of $A = [\mathfrak{a}]$, together with a $\mathbf{Z}$-basis $[\alpha_\mathfrak{a}, z_\mathfrak{a}]$. We write $\mathfrak{a}_\mathbf{Q} = \mathfrak{a}\otimes_\mathbf{Z}\mathbf{Q}$ to denote the corresponding fractional ideal. We write $Q_\mathfrak{a}(z) = \mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$

to denote the corresponding norm form,

$$Q_{\mathfrak{a}}(z) = \frac{\mathbf{N}_{k/\mathbf{Q}}(z)}{\mathbf{N}\mathfrak{a}} = \frac{zz^{\tau}}{[\mathcal{O}_k : \mathfrak{a}]}, \quad \tau \neq \mathbf{1} \in \mathrm{Gal}(k/\mathbf{Q}).$$

Definition 2.1. *Fix k a quadratic field, and $c \geq 1$ any integer for which there exists a ring class extension $k[c]$ of conductor c over k, which is always the case if either k is imaginary quadratic or $c = 1$. For each class $A \in C(\mathcal{O}_c)$, we fix an integer ideal representative $\mathfrak{a} = [1, z_{\mathfrak{a}}]$ with corresponding fractional ideal $\mathfrak{a}_{\mathbf{Q}}$. We then consider the rational quadratic space (V_A, Q_A) of signature $(2,2)$ defined by*

$$V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}} \quad \textit{and} \quad Q_A(z_1, z_2) = Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2).$$

It is easy to see by inspection that the spaces (V_A, Q_A) have signature $(2,2)$ in either case on k. We shall also consider the subspaces $(U, Q_U) = (U_A, Q_{U_A})$ given by projection to the second component of (V_A, Q_A),

$$U_A = \mathfrak{a}_{\mathbf{Q}} \quad \text{and} \quad Q_A(z_2) = -Q_{\mathfrak{a}}(z_2).$$

Hence, (U_A, Q_{U_A}) is a negative definite subspace of signature $(0,2)$ of (V_A, Q_A) if k is an imaginary quadratic field, and (U_A, Q_{U_A}) is a Lorentzian subspace of signature $(1,1)$ of (V_A, Q_A) if k is a real quadratic field.

2.2. Spin groups.

Given (V, Q) any rational quadratic space of signature $(n, 2)$, we consider its general spin group $\mathrm{GSpin}(V)$. Hence, $\mathrm{GSpin}(V)$ is a reductive algebraic group over $\mathbf{Q}$, and sits in a short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

To be more precise, let $C(V)$ denote the Clifford algebra of V, given by the quotient $C(V) = T(V)/I(V)$ of the tensor algebra $T(V) = \bigoplus_{m \geq 0} V^{\otimes m}$ by the two-sided ideal $I(V)$ generated by elements of the form $v \otimes v - Q(v)$ for $v \in V$. There is a $\mathbf{Z}/2\mathbf{Z}$-grading

$$C(V) = C^0(V) \oplus C^1(V),$$

where $C^0(V)$ denotes the even subalgebra generated by even numbers of vectors, and $C^1(V)$ denotes the odd subalgebra generated by odd numbers of vectors. We refer to the discussion in [3, §2.2-2.3]. To summarize, multiplication by -1 defines an isometry on V, which by the universal property for $C(V)$ induces a canonical automorphism $J : C(V) \rightarrow C(V)$. The even Clifford group can be characterized equivalently as the invariant subalgebra $C^0(V) = \{v \in C(V) : J(v) = v\}$. There is also the canonical involution given by

$$^tC(V) \longrightarrow C(V), \quad (x_1 \otimes \cdots \otimes x_m)^t := x_m \otimes \cdots \otimes x_1,$$

and a (non-multiplicative) Clifford norm

$$N_{C(V)} : C(V) \longrightarrow C(V), \quad N_{C(V)}(x) := {}^t x x.$$

We then define the Clifford group

$$G_{C(V)} = \left\{ x \in C(V) : x \quad \text{invertible and} \quad xVJ(x)^{-1} = V \right\},$$

the general spin group

$$\mathrm{GSpin}(V) = G_{C(V)} \cap C^0(V),$$

and the special spin group

$$\mathrm{Spin}(V) = \left\{ x \in \mathrm{GSpin}(V) : N_{C(V)}(x) = 1 \right\}.$$

Writing $v_1 \cdots v_m$ for simplicity to denote the element of $C(V)$ represented by $v_1 \otimes \cdots \otimes v_m$ (for $v_i \in V$), we fix an orthogonal basis $v_1, \ldots v_{n+2}$ of the space V, and define from this the volume form $\delta_V := v_1 \cdots v_{n+2}$.

Proposition 2.2. *Let (V, Q) be any rational quadratic space of signature $(n, 2)$, with $C(V) = C^0(V) \oplus C^1(V)$ its corresponding Clifford algebra, and $\mathrm{GSpin}(V) = G_{C(V)} \cap C^0(V)$ its corresponding spin group.*

(i) *The centre $Z(C(V))$ of $C(V)$ is given by*

$$Z(C(V)) = \begin{cases} \mathbf{Q} & \textit{if } n+2 \equiv 0 \bmod 2 \\ \mathbf{Q} + \delta_V \mathbf{Q} & \textit{if } n+2 \equiv 1 \bmod 2, \end{cases}$$

and the centre $Z(C^0(V))$ of $C^0(V)$ is given by

$$Z(C^0(V)) = \begin{cases} \mathbf{Q} + \delta_V \mathbf{Q} & \text{if } n+2 \equiv 0 \bmod 2 \\ \mathbf{Q} & \text{if } n+2 \equiv 1 \bmod 2. \end{cases}$$

(ii) *If $n = 2$ so that $\dim V = 4$, with orthogonal basis $v_1, \ldots v_4$ and $\delta_V = v_1 \cdots v_4$, then $C^0(V)$ is isomorphic to the quaternion algebra $(-Q(v_1)Q(v_2), -Q(v_2)Q(v_3))$ over $Z(C^0(V)) = \mathbf{Q} + \delta_V \mathbf{Q}$.*

(iii) *If $n \leq 2$ so that $\dim V = n + 2 \leq 4$, then we have natural identifications*

$$\mathrm{GSpin}(V) = \left\{ x \in C^0(V) : N_{C(V)} \in \mathbf{Q}^\times \right\} \quad \text{and} \quad \mathrm{Spin}(V) = \left\{ x \in C^0(V) : N_{C(V)} = 1 \right\}.$$

(iv) *For each of the rational quadratic spaces (V_A, Q_A) of signature $(2, 2)$ described in Definition 2.1, we have exceptional isomorphisms $C^0(V_A) \cong M_2(\mathbf{Q})$, $\mathrm{GSpin}(V_A) \cong \mathrm{GL}_2^2$, and $\mathrm{Spin}(V_A) \cong \mathrm{SL}_2^2$.*

(v) *In the setup of (iv), we define for each integer $N \geq 1$ the lattice $L_A \subset V_A$ whose adelization $L_A(N) \otimes \widehat{\mathbf{Z}}$ is fixed under the action of $\mathrm{GSpin}(V_A)(\mathbf{A}_f) \cong \mathrm{GL}_2(\mathbf{A}_f)$ via conjugation by*

$$\mathrm{Stab}_{\mathrm{GSpin}(V_A)(\mathbf{A}_f)}(L_A(N) \otimes \widehat{\mathbf{Z}}) \cong K_0(N) \oplus K_0(N).$$

Here, $K_0(N) \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ denotes the congruence subgroup of level N,

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : c \equiv 0 \bmod N \right\}.$$

Proof. For (i), see [3, Theorem 2.6]. For (ii), see [3, Example 2.10]. For (iii), see [3, Lemma 2.14]. For (iv), see [16, Proposition 2.3] or [17, Proposition 3.3 (ii)]; compare this with the distinct quadratic spaces of signature $(2, 2)$ described in [3, § 2.7] and [4, § 6.1]. To sketch the proof, we take the (non-orthogonal) basis

$$w_1 = (\alpha_{\mathfrak{a}}, 0), \quad w_2 = (z_{\mathfrak{a}}, 0), \quad w_3 = (0, \alpha_{\mathfrak{a}}), \quad w_4 = (0, z_{\mathfrak{a}}).$$

and compute the determinant $d(V_A) = \det((w_i, w_j))_{i,j}$ of the Gram matrix $((w_i, w_j))_{i,j}$ to be a square

$$\begin{aligned} & d(V_A) \\ &= \frac{2\mathbf{N}_{K/\mathbf{Q}}(\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \begin{vmatrix} \frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ 0 & -\frac{2\mathbf{N}_{K/\mathbf{Q}}(\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} \\ 0 & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \end{vmatrix} \\ &- \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} \begin{vmatrix} \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} & 0 & 0 \\ 0 & -\frac{2\mathbf{N}_{K/\mathbf{Q}}(\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} \\ 0 & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)}{\mathbf{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \end{vmatrix} \\ &= \frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^2} \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right) - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right) \\ &= \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^2} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}\alpha_{\mathfrak{a}}^\tau)^2}{\mathbf{N}\mathfrak{a}^2} \right)^2 \equiv 1 \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2. \end{aligned}$$

The determinant $d(V_A) = 1 \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ does not depend on the choice of basis. It defines the discriminant of the quadratic space (V_A, Q_A). In particular, we deduce that the quadratic space (V_A, Q_A) is split. Using the relation $\delta_{V_A}^2 = (-1)^{\frac{4}{2}} 2^{-4} d(V_A) \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ (see [3, Remark 2.5]), we deduce that $Z(C^0(V_A)) = \mathbf{Q}$. Since $\dim_{\mathbf{Q}} C^0(V_A) = 8$ and $C(V_A) \otimes \mathbf{R} \cong C_{2,2} = C(\mathbf{R}^{2,2}) \cong M_4(\mathbf{R})$, we deduce that $C^0(V_A) \cong B_A^\times \oplus B_A^\times$ for some indefinite quaternion algebra B_A defined over $\mathbf{Q}$. Using that $d(V_A) = 1 \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$, we deduce that $B_A \cong M_2(\mathbf{Q})$ is the matrix quaternion algebra of discriminant $d(B_A) = 1$. For (v), see [16, Corollary 2.4] or [17, Corollary 3.4]. The lattice $L_A(N)$ is characterized by the fact that it is stabilized under the action of $R(N)^\times \oplus R(N)^\times$, where $R(N) \subset B \cong M_2(\mathbf{Q})$ denotes the Eichler order of level N. □

2.3. **Spin Shimura varieties.** Let $D(V)=\{z\subset V\otimes\mathbf{R}:\dim(z)=2, Q|_z<0\}$ denote the Grassmannian of oriented negative definite hyperplanes of $V\otimes\mathbf{R}$. Hence, $D(V)$ has two connected components $D^\pm(V)$. We fix one throughout. Note that $D(V)$ has a complex structure, and determines a hermitian symmetric domain (see e.g. [3, § 2.4]). Fix a lattice $L\subset V$, which determines a compact open subgroup $K=K_L$ of $\mathrm{GSpin}(V)(\mathbf{A}_f)$ uniquely. We write $X_K=\mathrm{Sh}_K(\mathrm{GSpin}(V),D(V))$ to denote the corresponding Shimura variety, with complex points given by

$$X_K(\mathbf{C})=\mathrm{GSpin}(V)(\mathbf{Q})\backslash D(V)\times\mathrm{GSpin}(V)(\mathbf{A}_f)/K.$$

Hence, X_K is a quasiprojective variety of dimension n over its reflex field $\mathbf{Q}$. Fix $\{h\}$ any set of representatives for the finite space $\mathrm{GSpin}(V)(\mathbf{Q})\backslash\mathrm{GSpin}(V)(\mathbf{A}_f)/K$, we have the decomposition into connected components

$$X_K(\mathbf{C})=\coprod_h\Gamma_h\backslash D(V),\quad \Gamma_h:=\mathrm{GSpin}(V)(\mathbf{Q})\cap hKh^{-1}. \tag{4}$$

Later, we take $n=2$, so that $D^\pm(V)\cong\mathfrak{H}^2$, and X_K determines a quaternionic Hilbert modular surface defined over $\mathbf{Q}$. In particular, for each of the rational quadratic spaces (V_A,Q_A) of Definition 2.1 and Proposition 2.2 (iv) with lattice $L_A\subset V_A$ described in Proposition 2.2 (v), we have $K_{L_A}\cong K_0(N)^2$ so that

$$X_{K_{L_A}}(\mathbf{C})\cong\mathrm{GL}_2(\mathbf{Q})^2\backslash D^\pm(V_A)\times\mathrm{GL}_2(\mathbf{A}_f)^2/K_0(N)^2\cong Y_0(N)^2. \tag{5}$$

2.3.1. *Zero cycles.* Let (V_0,Q_0) be any rational quadratic subspace of signature $(0,2)$, associated to an imaginary quadratic field $k=k(V_0)$ of discriminant d_k and character $\eta_k(\cdot)=(\frac{d_k}{\cdot})$. The corresponding spin group $\mathrm{GSpin}(V_0)$ sits in a short exact sequence

$$1\longrightarrow\mathbf{G}_m\longrightarrow\mathrm{GSpin}(V_0)\longrightarrow\mathrm{SO}(V_0)\longrightarrow 1,$$

and we can identify $\mathrm{GSpin}(V_0)(\mathbf{A})/\mathrm{GSpin}(V_0)(\mathbf{Q})\cong\mathbf{A}_k^\times/k^\times$. Fixing an embedding $k\hookrightarrow\mathbf{C}$, we can view $V_0\otimes\mathbf{R}$ as an oriented hyperplane in $V\otimes\mathbf{R}$, and hence as a point $z_0=z_0^\pm$ in the Grassmannian $D(V)=D^\pm(V)$. Writing $L_0=V_0\cap L$ for the corresponding lattice, with $K_0=K_{L_0}=K_L\cap\mathrm{GSpin}(V_0)(\mathbf{A}_f)$ the corresponding compact open subgroup, we have a zero-cycle $Z(V_0)\subset X_K$ with complex points given by

$$Z(V_0)(\mathbf{C})=\mathrm{GSpin}(V_0)(\mathbf{Q})\backslash\{z_0^\pm\}\times\mathrm{GSpin}(V_0)(\mathbf{A}_f)/K_0. \tag{6}$$

2.3.2. *Special divisors.* Given a vector $x\in V$ with $Q(x)>0$, let $D(V)_x=\{z\in D(V):(x,z)=0\}$ denote the complement. We have for each coset $\mu\in L^\vee/L$ and $m\in\mathbf{Q}_{>0}$ a divisor $Z(\mu,m)\subset X_K$, whose complex points $Z(\mu,m)(\mathbf{C})$ can be described in terms of the decomposition (4) as

$$Z(\mu,m)(\mathbf{C})=\coprod_h\Gamma_h\Bigg\backslash\left(\coprod_{\substack{x\in\mu_h+L_h\\ Q(x)=m}}D(V)_x\right).$$

Here, the $\mu_h\in L_h^\vee/L_h$ denote cosets of the discriminant group of the lattice L_h given by $L_h\otimes\widehat{\mathbf{Z}}=hL\otimes\widehat{\mathbf{Z}}$.

2.4. **Real geodesic cycles.** We shall also consider Lorentzian subspaces (W,Q_W) associated to real quadratic fields. That is, let $W\subset V$ be a rational quadratic space of signature $(1,1)$. Such a space determines a real quadratic field $k=k(W)$ of discriminant d_k and character $\eta_k(\cdot)=(\frac{d_k}{\cdot})$. Write $L_W=W\cap L$ for the corresponding lattice, with $K_W=K_{L_W}=K_L\cap\mathrm{GSpin}(W)(\mathbf{A}_f)$ the corresponding compact open subgroup. The corresponding spin group $\mathrm{GSpin}(W)$ sits in a short exact sequence

$$1\longrightarrow\mathbf{G}_m\longrightarrow\mathrm{GSpin}(W)\longrightarrow\mathrm{SO}(W)\longrightarrow 1,$$

and we can identify the adelic quotient $\mathrm{GSpin}(W)(\mathbf{A})/\mathrm{GSpin}(W)(\mathbf{Q})\cong\mathbf{A}_k^\times/k^\times$.

We consider the corresponding domain $D(W)=\{z\subset W\otimes\mathbf{R}:\dim(z)=1, Q_W|_z<0\}$ or oriented hyperbolic lines, and select a connected component $D^\pm(W)$. Fixing a basepoint $z_W=z_W^\pm\in D^\pm(W)$, we then consider the finite set $G(W)$ defined by the finite double quotient space

$$G(W)=\mathrm{GSpin}(W)(\mathbf{Q})\,\mathrm{GSpin}(W)(\mathbf{A}_f)/K_W. \tag{7}$$

For each class $[h] \in G(W)$ of this set represented by an element $h \in \mathrm{GSpin}(W)(\mathbf{A}_f)$, we consider the corresponding symmetric domain defined by

$$C_h = \Gamma_h \backslash D(W), \quad \Gamma_h := \mathrm{GSpin}(W)(\mathbf{Q}) \cap hK_W h^{-1}.$$

We then consider the corresponding real geodesic cycle

$$\mathcal{G}(W) := \coprod_{\substack{[h] \in G(W) \\ h \in \mathrm{GSpin}(W)(\mathbf{A}_f)}} C_h = \coprod_{\substack{[h] \in G(W) \\ h \in \mathrm{GSpin}(W)(\mathbf{A}_f)}} \Gamma_h \backslash D(W).$$

2.5. **Modular forms.** We shall work with the following vector-valued and meromorphic modular forms.

2.5.1. *Hermitian metrics, meromorphic modular forms, and Petersson norms on X_K.* Recall that $D(V)$ has the structure of a hermitian symmetric domain, and is isomorphic (see e.g. [3], [12]) to the quadric

$$\mathcal{Q}_- = \{w \in V(\mathbf{C}) : (w, w) = 0, (w, \overline{w}) < 0\} / \mathbf{C}^\times \subset \mathbf{P}(V(\mathbf{C})).$$

Let $\mathcal{L}_{D(V)}$ denote the restriction to $D(V) \cong \mathcal{Q}_-$ of the tautological line bundle on $\mathbf{P}(V(\mathbf{C}))$. Since the action of $\mathrm{O}(V)(\mathbf{R})$ on $V(\mathbf{C})$ induces an action of the connected component of the identity $\mathrm{GSpin}(V)^+(\mathbf{R})$ on $L_{D(V)}$, we have a holomorphic line bundle

$$\mathcal{L} = \mathrm{GSpin}(V)(\mathbf{Q}) \backslash \left(\mathcal{L}_{D(V)} \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K \right) \longrightarrow X_K.$$

This line bundle is known to be algebraic, with a canonical model over $\mathbf{Q}$, and to be compatible with respect to the decomposition into connected components (4). We define a hermitian metric $h_{\mathcal{L}}$ on $\mathcal{L}_{D(V)}$ by the rule

$$h_{\mathcal{L}}(w_{1,2}) = -\frac{1}{2} (w_1, \overline{w}_2).$$

This metric is invariant under the action of $\mathrm{O}(V)(\mathbf{R})$, and descends to the holomorphic line bundle $\mathcal{L}$.

Recall that we also have the following tube domain model for $D(V) \cong \mathcal{Q}_-$. Fix a Witt decomposition

$$V(\mathbf{Q}) = W \oplus \mathbf{Q}e_1 + \mathbf{Q}e_2$$

with $(e_1, e_2) = 1$ and $(e_1, e_1) = (e_2, e_2) = 0$. Here, the basis vectors e_1, e_2 span a hyperbolic plane with orthogonal complement W. The subspace $W \subset V$ has signature $(n-1, 1)$, and we consider its negative cone

$$C_W = \{y \in W(\mathbf{R}) : (y, y) < 0\}.$$

Then, $D(V) \cong \mathcal{Q}_{-1}$ is isomorphic to the tube domain

$$\mathcal{H}(V) = \{z \in W(\mathbf{C}) : y = (z) \in C_W\}.$$

To be more explicit, the isomorphism is given by the composition of the map

$$\mathcal{H}(V) \longrightarrow V(\mathbf{C}), \quad z \longmapsto w(z) := z + e_1 - Q(z)e_2$$

with the projection to $\mathcal{Q}_-$. The latter map $z \mapsto w(z)$ can be viewed as a nowhere vanishing section on $\mathcal{L}_{D(V)}$. This section has norm given by

$$||w(z)||^2 = -\frac{1}{2} (w(z), \overline{w}(z)) = -(y, y) =: |y|^2.$$

Given any $h \in \mathrm{GSpin}(V)(\mathbf{R})$, we have that

$$h \cdot w(z) = w(hz) j(h, z)$$

for a holomorphic automorphy factor

$$j : \mathrm{GSpin}(V)(\mathbf{R}) \times D(V) \longrightarrow \mathbf{C}^\times$$

The meromorphic sections of $\mathcal{L}^{\otimes l}$ for any $l \in \frac{1}{2}\mathbf{Z}$ can be viewed as functions

$$\Psi : D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f) \longrightarrow \mathbf{C}$$

satisfying the natural transformation properties

- $\Psi(z, hk) = \Psi(z, h)$ for all $k \in K$,
- $\Psi(\gamma z, \gamma h) = j(\gamma, z)^l \cdot \Psi(z, h)$ for all $\gamma \in \mathrm{GSpin}(V)(\mathbf{Q})$.

Under the decomposition (4) into geometrically connected components, such a section corresponds to a vector $(\Psi_{\Gamma_h}(\cdot,\cdot))_h$ of meromorphic functions, with each component factoring through $\Gamma_h\backslash D(V)$. We call such a function Ψ a *meromorphic modular form of weight* l *on* X_K. In this setting, we define the Petersson norm of the section $(z,h)\longmapsto \Psi(z,h)\cdot w(z)^{\otimes l}$ associated to Ψ by the formula

$$||\Psi(z,h)||^2 = |\Psi(z,h)|^2|y|^{2l}.$$

The Borcherds products we describe below give examples of such meromorphic modular forms on X_K.

2.5.2. *The Weil representation and spaces of Schwartz functions.* Let $\psi=\otimes_v\psi_v$ denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, with archimedean component $\psi_\infty(x)=e(x)=\exp(2\pi ix)$. Fix (L,Q) a quadratic module of signature $(n,2)$ as above, with $V=L\otimes\mathbf{Q}$ the corresponding space. We consider the Weil representation

$$\omega_L=\omega_{L,\psi}:\mathrm{Mp}(\mathbf{A})\longrightarrow \mathrm{Aut}\left(\mathcal{S}(V(\mathbf{A}))\right)$$

of the two-fold metaplectic cover $\mathrm{Mp}_2(\mathbf{A})$ of $\mathrm{Sp}_2(\mathbf{A})\cong \mathrm{SL}_{\mathbf{2}}(\mathbf{A})$ acting on the space of Schwartz-Bruhat functions $\mathcal{S}(V(\mathbf{A}))$. For the linear action of $\mathrm{GSpin}(V)(\mathbf{A}_f)$, we write $\omega_L(h)\Phi(x)=\Phi(h^{-1}x)$ for $\mathcal{S}(V(\mathbf{A}_f))$.

Remark 2.3. Note that if n is even, so that $\dim_{\mathbf{Q}}(V)=n+2$ is even, then ω_L factors through $\mathrm{SL}_2(\mathbf{A})$.

Let $\mathfrak{S}_L\subset\mathcal{S}(V(\mathbf{A}))$ denote the subspace of functions supported on $L^\vee\otimes\widehat{\mathbf{Z}}$ which are constant on $L\otimes\widehat{\mathbf{Z}}$. Note that this space is finite dimension, with a natural basis of characteristic functions

$$\left\{\mathbf{1}_\mu=\mathrm{char}\left(\mu+L\otimes\widehat{\mathbf{Z}}\right):\mu\in L^\vee/L\right\}$$

so that $\mathfrak{S}_L\cong\mathbf{C}[L^\vee/V]$. Note as well that $\mathcal{S}(V(\mathbf{A}_f))$ can be realized as a direct limit $\mathcal{S}(V(\mathbf{A}_f))=\varinjlim_L\mathfrak{S}_L$.

2.5.3. *Vector-valued modular forms.* Write $\Gamma=\mathrm{SL}_2(\mathbf{Z})$, with Γ' its full inverse image in $\mathrm{Mp}_2(\mathbf{R})$ We call a function $f:\mathfrak{H}\longrightarrow\mathfrak{S}_L$ a *weakly holomorphic form of weight* $l\in\frac{1}{2}\mathbf{Z}$ *and type* ω_L for Γ' if

- $f(\gamma'\tau)=(c\tau+d)^l\omega_L(\gamma')f(\tau)$ for all $\gamma'=(\gamma,\epsilon)=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\epsilon\right)\in\Gamma'$
- f is meromorphic at the cusp: It has a Fourier series expansion

  $$f(\tau)=\sum_{\mu\in L^\vee/L}\sum_{m\gg-\infty}c_f(\mu,m)e(m\tau)\mathbf{1}_\mu$$

  with only a finite number of nonzero coefficients $c_f(\mu,m)$ with $m<0$ negative.

We write $M_l^!(\omega_L)$ to denote this space of weakly holomorphic forms, with $S_l^!(\omega_L)\subset M_l^!(\omega_L)$ the subspace of cuspidal forms having vanishing constant coefficients $c_f(\mu,0)=0$.

2.6. **Siegel theta series.** For each coset $h\in\mathrm{GSpin}(V)(\mathbf{A}_f)/K$, we have a Siegel theta function

$$\theta_L(\tau,z,h):\mathfrak{H}\times D(V)\longrightarrow\mathfrak{S}_L\cong\mathbf{C}[L^\vee/L]$$

following the constructions of [2] and [6]. As a function in the Grassmannian variable $z\in D(V)$, this function is Γ_h-invariant, and hence factors through the corresponding connected component in (4). As a function in the complex variable $\tau=u+iv\in\mathfrak{H}$, it determines a non-holomorphic vector-valued modular form $\theta_L\in H_{\frac{n}{2}-1}(\omega_L^\vee)$ of weight $n/2-1$ and representation $\omega_L^\vee$.

2.6.1. *Explicit definition.* For any decomposable $\varphi=\otimes_v\varphi_v\in\mathcal{S}(V\otimes\mathbf{A})$, we define the theta kernel

$$\theta(g',h;\varphi)=\sum_{x\in V(\mathbf{Q})}\left(\omega_L(g,'h)\right)\varphi(x),\quad g'\in\mathrm{Mp}_2(\mathbf{A}),\ h\in\mathrm{GSpin}(V)(\mathbf{A}).$$

Note that the Weil representation ω_L factors through the symplectic group $\mathrm{Sp}_2(\mathbf{A})\cong\mathrm{SL}_2(\mathbf{A})$ if the dimension $\dim(V)=n+2$ is even, equivalently if the positive dimension n is even.

We make the following choice of archimedean local Schwartz function $\varphi_\infty\in\mathcal{S}(V\otimes\mathbf{R})$. Given an oriented hyperplane $z\in D(V)$, we define the corresponding majorant

$$(x,x)_z=(x_{z^\perp},x_{z^\perp})-(x_z,x_z),\quad x\in V(\mathbf{R}).$$

We then consider the function $\varphi_\infty=\varphi_\infty(\cdot,z)\in\mathcal{S}(V\otimes\mathbf{R})$ defined by the Gaussian

$$\varphi_\infty(x,z)=\exp\left(-(x,x)_z\right),\quad x\in V(\mathbf{R}).$$

Note that $\varphi_\infty(hx, hz) = \varphi_\infty(x, z)$ for all $h \in \mathrm{GSpin}(V)(\mathbf{R}) = \mathrm{GSpin}(n, 2)(\mathbf{R})$. To use this choice of Schwartz function for the theta kernel, we first fix a basepoint $z_0 \in D(V)$, then for any other point $z \in D(V)$ write $h_z \in \mathrm{GSpin}(V)(\mathbf{R})$ to denote the element for which $h_z z_0 = z$. Note that this implies the relation

$$\omega_L(h_z)\varphi_\infty(\cdot, z) = \varphi_\infty(\cdot, z).$$

We then define the corresponding theta kernel

$$\theta_L(g', h; \varphi_f) = \theta(g', h, \varphi_\infty(\cdot, z_0) \otimes \varphi_f) \quad \varphi_f = \otimes_{v<\infty}\varphi_v \in \mathcal{S}(V \otimes \mathbf{A}_f).$$

To describe this in classical terms, we first descend to the Poincaré upper-half plane $\mathfrak{H}$ using the Iwasawa decomposition for $\mathrm{Mp}_2(\mathbf{R})$. Hence, we write $g'_\tau = (g_\tau, 1) \in \mathrm{Mp}_2(\mathbf{R})$ for the image of the mirabolic matrix

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$$

associated to a point $\tau = u + iv \in \mathfrak{H}$. Given any finite adelic point $h_f \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we then define

$$\theta_L(\tau, z, h_f, \varphi_f) := \theta_L(g'_\tau, h_z h_f, \varphi_f).$$

As explained in [6, §2], we can use explicit formulae for ω_L to derive the Fourier series expansion

$$\theta_L(\tau, z, h_f, \varphi_f) = v \sum_{x \in V(\mathbf{Q})} e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right) \otimes \varphi_f(h_f^{-1}x).$$

As a function in $\tau = u + iv \in \mathfrak{H}$, this theta series can be viewed as a nonholomorphic vector-valued form of weight $(n-2)/2 = n/2 - 1$ and representation $\omega_L^\vee$.

Remark 2.4. We shall henceforth write $h = h_f$ and hence $\theta_L(\tau, h, \varphi_f) = \theta_L(\tau, h_f, \varphi_f)$ to simplify notations, as the implicit choice of basepoint $z_0 \in D(V)$ constrains the archimedean component $h_z \in \mathrm{GSpin}(V)(\mathbf{R})$.

We make the following choices of nonarchimedean local Schwartz function $\varphi_f \in \mathcal{S}(V \otimes \mathbf{A}_f)$. Given a coset μ in the finite abelian discriminant group $L^\vee/L$, we write

$$\mathbf{1}_\mu = \mathrm{char}\left(\mu + L \otimes \widehat{\mathbf{Z}}\right) \in \mathcal{S}(V \otimes \mathbf{A}_f)$$

to denote the corresponding characteristic function for the adelization of L. We then consider the theta series

$$\theta_L(\tau, z, h) = \theta_L(\tau, z, h_f) := \sum_{\mu \in L^\vee/L} \theta_L(\tau, z, h_f, \mathbf{1}_\mu)\mathbf{1}_\mu = v \sum_{\mu \in L^\vee/L} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h_f\mu}} e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right)\mathbf{1}_\mu. \tag{8}$$

2.7. **Regularized theta lifts and Borcherds products.** Fix $f \in M^!_{1-n/2}(\omega_L)$ a weakly holomorphic form of weight $1 - n/2$ and representation ω_L. We write its Fourier series expansion as

$$f(\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f(\mu, m)e(m\tau)\mathbf{1}_\mu.$$

We define the corresponding regularized theta lift $\Phi(f, z, h)$ for $z \in D(V)$ and $h = h_h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by

$$\Phi(f, z, h) = \int_{\mathcal{F}}^{\star} \langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle d\mu(\tau) := \mathrm{CT}_{s=0}\left(\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right). \tag{9}$$

This notation denotes the constant term in the Laurant series expansion around $s = 0$ of the function

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau),$$

where each $\mathcal{F}_T$ denotes the truncation to T

$$\mathcal{F}_T = \left\{\tau = u + iv \in \mathfrak{H}, -1/2 \leq u \leq 1/2, u^2 + v^2 \geq 1, v < T\right\}$$

of the standard fundamental domain $\mathcal{F}$ for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$.

Theorem 2.5 (Borcherds)**.** *There exists for each weakly holomorphic modular form* $f \in M^!_{1-n/2}(\omega_L)$ *with integral Fourier coefficients* $c_f(\mu, m) \in \mathbf{Z}$ *for each* $m < 0$ *a meromorphic modular form*

$$\Psi(f, z, h) : D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f) \longrightarrow \overline{\mathbf{Q}}$$

of weight $c_f(0, 0)/2$ *and divisor*

$$\mathrm{Div}(\Psi(f)^2) = Z(f) := \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_f(\mu, -m) Z(\mu, m)$$

which is related to the regularized theta lift $\Phi(f, z, h)$ *by the formula*

$$\Phi(f, z, h) = -2 \log ||\Psi(f, z, h)||^2 - c_f(0, 0) \left(2 \log ||y|| + \Gamma'(1)\right).$$

Proof. See [2, Theorem 13.3] or [12, Theorem 1.3]. That the Borcherds product $\Psi(f, \cdot)$ takes algebraic values is deduced from the theorem of Howard and Madapusi-Pera [10, Theorem 9.1.1]. ☐

2.8. **Langlands Eisenstein series and the Siegel-Weil formula.** We describe the Langlands Eisenstein series that occur as sums of the Siegel theta functions (8). Here, we suppose more generally that (V, Q) is any rational quadratic space of signature $(p(V), q(V))$ and even dimension $\dim(V) = p(V) + q(V) \equiv 0 \bmod 2$. We fix $L \subset V$ an integral lattice with corresponding Weil representation $\omega_L = \omega_{L,\psi}$. We then write $D(V)$ for the corresponding space of oriented negative definite $q(V)$-planes, as in [5].

2.8.1. *The Siegel-Weil formula.* Let $P \subset \mathrm{SL}_2$ denote the parabolic subgroup of upper triangular matrices, with Levi decomposition $P = MN$ for M the diagonal (maximal reductive) subgroup and N the unipotent subgroup. Here, we write the elements as

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbf{G}_m \right\} \quad \text{and} \quad N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbf{G}_a \right\}.$$

Writing $K_\infty = \mathrm{SO}_2(\mathbf{R})$ to denote maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$, we have the Iwasawa decomposition

$$\mathrm{SL}_2(\mathbf{A}) = N(\mathbf{A}) M(\mathbf{A}) K_\infty K. \tag{10}$$

Let χ_V denote the idele class character of $\mathbf{Q}$ determined by the Hilbert symbol $(\cdot, \cdot)_{\mathbf{A}}$ on $\mathbf{A}^\times$ by the rule

$$\chi_V(x) = \left(x, (-1)^{\frac{\dim(V)}{2}} \det(V) \right)_{\mathbf{A}}.$$

Here, $\det(V)$ denotes the determinant of the Gram matrix of V. Let $I(s, \chi_V)$ denote the principal series representation of $\mathrm{SL}_2(\mathbf{A})$ induced by the quasicharacter $\chi_V(\cdot)| \cdot |^s$ for $s \in \mathbf{C}$. Hence, $I(s, \chi_V)$ consists of the smooth functions $\Phi(g, s)$ on $\mathrm{SL}_2(\mathbf{A})$ which for all $a \in \mathbf{A}^\times$ and $b \in \mathbf{A}$ satisfy the transformation property

$$\Phi(n(b)m(a)g, s) = \chi_V(a)|a|^{s+1}\Phi(g, s)$$

The group $\mathrm{SL}_2(\mathbf{A})$ acts on the space $I(s, \chi_V)$ by right translations. Writing

$$s_0 = s_0(V) := \frac{\dim(V)}{2} - 1,$$

we also have an $\mathrm{SL}_2(\mathbf{A})$-intertwining map defined by

$$\lambda : \mathcal{S}(V \otimes \mathbf{A}) \longrightarrow I(s_0, \chi_V), \quad \varphi = \otimes_v \varphi_v \longmapsto \lambda(\varphi) := \left(\omega_L(g)\varphi\right)(0).$$

Recall that a section $\Phi(\cdot, s) \in I(s, \chi_V)$ is said to be *standard* if its restriction to $K_\infty K$ does not depend on the complex variable $s \in \mathbf{C}$. Given any standard section $\Phi(s) \in I(s, \chi_V)$, we consider the corresponding Eisenstein series defined (first for $\Re(s) > 1$) by

$$E(g, s; \Phi) = \sum_{\gamma \in P(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{Q})} \Phi(\gamma g, s).$$

This series converges absolutely for $\Re(s) > 1$. It also satisfies the Langlands functional equation, relating $E(g, s; \Phi)$ to $E(g, -s, M(s)\Phi)$, from which it acquires an analytic continuation to a meromorphic function of $s \in \mathbf{C}$. Via the Iwasawa decomposition (10), we see for any decomposable Schwartz function $\varphi \in \mathcal{S}(V \otimes \mathbf{A})$ that the corresponding image $\lambda(\varphi) \in I(s_0, \chi_V)$ under the $\mathrm{SL}_2(\mathbf{A})$-intertwining map λ has a unique extension $\lambda(\varphi, s)$ to $I(s, \chi_V)$ for which the specialization $\lambda(\varphi, s_0) = \lambda(\varphi)$.

Theorem 2.6 (Siegel-Weil). *Let (V, Q) be any anisotropic rational quadratic space of even dimension. Let $L \subset V$ be any integral lattice with Weil representation ω_L, and let $\varphi \in \mathcal{S}(V \otimes \mathbf{A})$ be any decomposable Schwartz function. The corresponding Eisenstein series $E(g, s; \lambda(\varphi))$ is holomorphic at the point*

$$s = s_0 = s_0(V) := \frac{\dim(V)}{2} - 1,$$

and moreover given at this point $s = s_0$ by a sum over the corresponding theta series $\theta(g, h; \varphi)$ as

$$E(g, s_0; \lambda(\varphi)) = \frac{\kappa}{2} \int\limits_{\mathrm{SO}(V)(\mathbf{Q}) \backslash \mathrm{SO}(V)(\mathbf{A})} \theta(g, h; \varphi) dh, \quad \kappa = \begin{cases} 2 & \textit{if } \dim(V) \leq 2 \\ 1 & \textit{otherwise.} \end{cases}$$

Here, dh denotes the Tamagawa measure on $\mathrm{SO}(V)(\mathbf{A})$.

Proof. The result is well-known and classical, see e.g. [12, Theorem 4.1]. □

Let $l \in \mathbf{Z}$ be any integer. Let χ_l denote the character of $K_\infty = \mathrm{SO}_2(\mathbf{R})$ defined by

$$\chi_l(k_\theta) = e^{il\theta}, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

Let $\Phi_\infty^l(s) \in I(s, \chi_{V,\infty})$ denote the standard section for which

$$\Phi_\infty^l(k_\theta, s) = \chi_l(k_\theta) = e^{il\theta},$$

equivalently by (10) for which

$$\Phi_\infty^l(n(b)m(a)k_\theta, s) = \chi_V(a)|a|^{s+1}e^{il\theta}$$

for all $b \in \mathbf{A}$, $a \in \mathbf{A}^\times$, and $k_\theta \in K_\infty$. Applying λ_∞ to $\varphi_\infty(\cdot z) \in \mathcal{S}(V \otimes \mathbf{R})$ as defined above, we obtain

Corollary 2.7. *Fix $z \in D^\pm(V)$. We have the Siegel-Weil formula*

$$E_L(\tau, s_0; n/2 - 1) = \frac{\kappa}{2} \int\limits_{\mathrm{SO}(V)(\mathbf{Q}) \backslash \mathrm{SO}(V)(\mathbf{A}_f)} \theta_L(\tau, z, h) dh.$$

2.8.2. *Subspaces associated to quadratic fields.* We now apply this to the rational quadratic subspace (U, Q_U) of dimension 2 and signature $(p(U), q(U))$. We shall be most interested in certain negative definite spaces $(U, Q_U) = (V_0, Q_0)$ associated to imaginary quadratic fields $k = k(V_0)$, as well as in certain Lorentzian spaces $(U, Q_U) = (W, Q_W)$ of signature $(1, 1)$ associated to real quadratic fields $k = k(W)$.

To fix ideas, let (V_0, Q_0) be a rational quadratic space of signature $(0, 2)$ and integral lattice $L_0 \subset V_0$. Fixing a basepoint $z_0 \in D^\pm(V_0)$, Corollary 2.7 gives us the relation

$$E_{L_0}(\tau, 0; -1) = \int\limits_{\mathrm{SO}(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \theta_{L_0}(\tau, z_0, h) dh. \tag{11}$$

Let (W, Q_W) be a Lorentzian quadratic space of signature $(1, 1)$ and integral lattice $L_W \subset W$. Fixing a basepoint $z_W \in D^\pm(W)$, Corollary 2.7 gives us the relation

$$E_{L_W}(\tau, 0; 0) = \int\limits_{\mathrm{SO}(W)(\mathbf{Q}) \backslash \mathrm{SO}(W)(\mathbf{A}_f)} \theta_{L_W}(\tau, z_W, h) dh. \tag{12}$$

We now gather some important facts about these Eisenstein series $E_{L_0}(\tau, s; -1)$ and $E_{L_W}(\tau, s; 0)$ for later. Writing U to denote either of the quadratic spaces V_0 or W, with L_U the corresponding integral lattice, let $k(U)$ the corresponding quadratic field of discriminant d_k and character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. Let

$$E_{L_U}(\tau, s) = \begin{cases} E_{L_0}(\tau, s; -1) & \text{if } (L_U, Q_U) = (L_0, Q_0) \text{ is negative definite} \\ E_{L_W}(\tau, s; 0) & \text{if } (L_U, Q_U) = (W, Q_W) \text{ is Lorentzian.} \end{cases} \tag{13}$$

Theorem 2.8. *Suppose the quadratic space* (L_U, Q_U) *corresponds to an integral ideal* $\mathfrak{a} \subset \mathcal{O}_k$ *for the corresponding quadratic field* $k = k(U)$ *of discriminant* d_k *and character* $\eta_k(\cdot) = (\frac{d_k}{\cdot})$*, with* $Q_U(z) = \pm Q_{\mathfrak{a}}(z)$ *for* $Q_{\mathfrak{a}}(z) = \mathbf{N}_{k/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a} = zz^\tau/\mathbf{N}\mathfrak{a}$ *for* $\tau \neq \mathbf{1} \in \mathrm{Gal}(k/\mathbf{Q})$ *the corresponding norm form. Writing*

$$\Lambda(s, \eta_k) = |d_k|^{\frac{s}{2}} \Gamma_{\mathbf{R}}(s+1) L(s, \eta_k), \quad \Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$$

to denote the completed Dirichlet L*-function of* η_k*, the completion*

$$E_{L_U}^\star(\tau, s) := \Lambda(s+1, \eta_k) E_{L_U}(\tau, s)$$

of the (coherent) Eisenstein series $E_{L_U}(\tau, s)$ *satisfies the symmetric functional equation*

$$E_{L_U}^\star(\tau, s) = E_{L_U}^\star(\tau, -s).$$

Proof. The result is a standard consequence of the Langlands functional equation. Cf. [6, Proposition 2.5] for the preimage $E_{L_0}(\tau, s; 1)$ under the weight-raising operator R_1. This operator switches invariants at infinity, giving an incoherent Eisenstein series $E_{L_0}(\tau, s; 1)$ which matches the natural (coherent) Eisenstein series $E_{L_0}(\tau, s; -1) = R_1 E_{L_0}(\tau, s; 1)$ at all the finite places. In particular, we can derive the functional equation by a minor adaptation of the proof given in [6, Proposition 2.5], without switching invariants at infinity. □

3. Rankin-Selberg L-functions

Let us now suppose that (V, Q) is a rational quadratic space of signature $(2, 2)$ with integral lattice $L \subset V$. We fix a subspace $U \subset V$ of dimension 2 as in the discussion above, writing $L_U \subset L$ to denote the corresponding lattice, with $L_U^\perp \subset L$ its complement with respect to the inner product determined by Q. Fix

$$f(\tau) = \sum_{\mu \in L^\vee/L} f_\mu(\tau) e(m\tau) = \sum_{\mu \in L^\vee/L} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_0^!(\omega_L)$$

a weakly holomorphic cusp form of weight zero and representation ω_L. Hence, the scalar-valued functions $f_\mu(\tau)$ are meromorphic modular forms for some congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbf{Z})$ determined by the level of the lattice L. Note that the Fourier coefficients $c_f(\mu, m)$ have exponential growth here; see e.g. [13, Lemma 2.5], which gives a detailed argument for weight 1. For this reason, the corresponding Dirichlet series

$$\sum_{m \geq 1} c_f(\mu, m) m^{-s} = \sum_{m \geq 1} c_{f_\mu}(m) m^{-s}$$

does not converge in any half plane, and we must consider the L-series of $f(\tau)$ in a regularized sense.

3.1. Rankin-Selberg L-functions of vector-valued forms.

Consider the Eisenstein series $E_{L_U}(\tau, s)$ described in Theorem 2.8 above. We also consider the corresponding Siegel theta series $\theta_{L_U^\perp}(\tau) = \theta_{L_U^\perp}(\tau, 1, 1)$. When $(U, Q_U) = (V_0, Q_0)$ is a negative definite space, the orthogonal complement $V_0^\perp \subset V_0$ (and hence $L_0^\perp$) is positive definite, and $\theta_{L_0^\perp}(\tau)$ is holomorphic of weight one. We write its Fourier series expansion as

$$\theta_{L_0^\perp}(\tau) = \sum_{\mu \in (L_0^\perp)^\vee/L_0^\perp} \sum_{x \in V_0^\perp(\mathbf{Q})} e\left(Q_0(x)\tau\right) \mathbf{1}_\mu = \sum_{\mu \in (L_0^\perp)^\vee/L_0^\perp} \sum_{m \geq 0} r_{L_0^\perp}(\mu, m) e(m\tau) \mathbf{1}_\mu.$$

When $(U, Q_U) = (W, Q_W)$ is Lorentzian, the orthogonal complement $L_W^\perp$ of the signature $(1, 1)$ sublattice $L_W \subset L$ again has signature $(1, 1)$, and the theta series $\theta_{L_W^\perp}(\tau)$ is nonholomorphic of weight zero. Fixing a basepoint $z_0 \in D(W^\perp)$, we write its Fourier series expansions as

$$\theta_{L_W^\perp}(\tau) = v \sum_{\mu \in L_W^\vee/L_W} \sum_{x \in W^\perp(\mathbf{Q})} e\left(Q_W(x_{z_0^\perp})\tau + Q_W(x_{z_0})\overline{\tau}\right) \mathbf{1}_\mu.$$

To describe either of the Siegel theta series for our later calculations, we write

$$\theta_{L_U^\vee}(\tau) = \begin{cases} \theta_{L_0^\perp}(\tau) \in M_1(\omega_{L_0^\perp}^\vee) & \text{if } (L_U, Q_U) = (L_0, Q_0) \text{ is negative definite} \\ \theta_{L_W^\perp}(\tau) \in A_0(\omega_{L_W^\perp}^\vee) & \text{if } (L_U, Q_U) = (W, Q_W) \text{ is Lorentzian.} \end{cases} \tag{14}$$

We consider the Rankin-Selberg L-function of $f(\tau) \in S_0^!(\omega_L)$ times this Siegel theta series $\theta_{L_U^\perp}(\tau)$,

$$L(s, f \times \theta_{L_U^\perp}) = \langle f, \theta_{L_U^\perp} \otimes E_{L_U}(\cdot, s)\rangle := \int_{\mathcal{F}}^{\star} \langle\langle \overline{f(\tau)}, \theta_{L_U^\perp}(\tau) \otimes E_{L_U}(\tau, s)\rangle\rangle d\mu(\tau), \quad \Re(s) > 1. \tag{15}$$

Note that this is *not* defined as a Dirichlet series, but rather as a regularized Petersson inner product. Here, $E_{L_U}(\tau, s)$ denotes the Eisenstein series defined in (13) above, and we take the regularized integral

$$\int_{\mathcal{F}}^{\star} \langle\langle \overline{f(\tau)}, \theta_{L_U^\perp}(\tau) \otimes E_{L_U}(\tau, s)\rangle\rangle d\mu(\tau) = \mathrm{CT}_{w=0}\left(\lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle\langle \overline{f(\tau)}, \theta_{L_U^\perp}(\tau) \otimes E_{L_U}(\tau, s)\rangle\rangle v^{-w} d\mu(\tau) \right).$$

To be more precise, we have the following variant of Rankin-Selberg convolution here.

Proposition 3.1. *The L-function defined first for $s \in \mathbf{C}$ with $\Re(s) \gg 1$ by the regularized integral*

$$L(s, f \times \theta_{L_W^\perp}) = \int_{\mathcal{F}}^{\star} \langle\langle \overline{f(\tau)}, \theta_{L_U^\perp}(\tau) \otimes E_{L_U}(\tau, s)\rangle\rangle d\mu(\tau).$$

has an analytic continuation

$$L^{\star}(s, f \times \theta_{L_U^\perp}) := \Lambda(s+1, \eta_k) L(s, f \times \theta_{L_U^\perp}) = \langle f, \theta_{L_U^\perp} \otimes E^{\star}_{L_U}(\cdot, s)\rangle := \int\limits_{\mathcal{F}} \langle\langle \overline{f(\tau)}, \theta_{L_U^\perp}(\tau) \otimes E^{\star}_{L_U}(\tau, s)\rangle\rangle d\mu(\tau) \tag{16}$$

to all $s \in \mathbf{C}$, and inherits from the completed Eisenstein series $E^{\star}_{L_U}(\tau, s)$ the symmetric functional equation

$$L^{\star}(s, f \times \theta_{L_U^\perp}) = L^{\star}(-s, f \times \theta_{L_U^\perp}). \tag{17}$$

In this way, $L(s, f \times \theta_{L_U^\perp})$ has an analytic continuation $L^{\star}(s, f \times \theta_{L_U^\perp})$ to an entire function of $s \in \mathbf{C}$.

Proof. Assume first that $L_U = L_0$ is a negative definite lattice, with corresponding holomorphic theta series $\theta_{L_0^\perp}(\tau)$. We first consider the formal (divergent) Dirichlet series expansion

$$L^{\mathrm{div}}(s, f \times \theta_{L_0^\perp}) := (4\pi)^{-\left(\frac{s-2}{2}\right)} \Gamma\left(\frac{s-2}{2}\right) \sum_{m \geq 1} \sum_{\mu \in (L_0^\perp)^\vee / L_0^\perp} \overline{c_f(\mu, m)} r_{L_0^\perp}(\mu, m) m^{-\left(\frac{s-2}{2}\right)}. \tag{18}$$

We have the constant coefficient formula

$$\mathrm{CT}\langle\langle \overline{f_{L_0^\perp + 0}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0+0}\rangle\rangle = \int\limits_0^1 \langle\langle \overline{f_{L_0^\perp + 0}(u+iv)}, \theta_{L_0^\perp}(u+iv) \otimes \mathbf{1}_{L_0+0}\rangle\rangle du$$

$$= \sum_{\lambda \in (L_0^\perp)^\vee/(L_0^\perp)} \sum_{m \gg -\infty} \overline{c_f(\lambda, m)} e(miv) \sum_{n \geq 0} r_{L_0^\perp}(\lambda, n) e(niv) \int\limits_0^1 e(nu - mu) du$$

$$= \sum_{\lambda \in (L_0^\perp)^\vee/(L_0^\perp)} \sum_{m \geq 0} \overline{c_f(\lambda, m)} r_{L_0^\perp}(\lambda, m) e(2miv) = \sum_{\lambda \in (L_0^\perp)^\vee/(L_0^\perp)} \sum_{m \geq 1} \overline{c_f(\lambda, m)} r_{L_0^\perp}(\lambda, m) e^{-4\pi m v}.$$

Opening up the gamma function in (18), we can then make the formal calculation

$$L^{\mathrm{div}}(s, f \times \theta_{L_0^\perp}) = \int\limits_0^\infty \sum_{\lambda \in (L_0^\perp)^\vee/(L_0^\perp)} \sum_{m \gg -\infty} \overline{c_f(\lambda, m)} r_{L_0^\perp}(\lambda, m) e^{-4\pi m v} v^{\frac{s-2}{2}} dv$$

$$= \int\limits_0^\infty \int\limits_0^1 \langle\langle \overline{f_{L_0^\perp + 0}(u+iv)}, \theta_{L_0^\perp}(u+iv) \otimes \mathbf{1}_{L_0+0}\rangle\rangle v^{\frac{s-2}{2}} du dv$$

$$= \iint\limits_{\Gamma_\infty \backslash \mathfrak{H}} \langle\langle \overline{f_{L_0^\perp + 0}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0+0}\rangle\rangle v^{\frac{s+2}{2}} d\mu(\tau).$$

Here (cf. [9, § IV.1]), we write

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbf{Z} \right\}.$$

This group acts on $\mathfrak{H}$ via fractional linear transformation. Writing $\mathcal{F} = \{\tau \in \mathfrak{H}, -1/2 \leq \Re(\tau) \leq 1/2, \tau\overline{\tau} \geq 1\}$ again to denote the standard fundamental domain for the action of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$, and γ a set of right coset representatives of $\Gamma_\infty \backslash \Gamma$, we obtain a fundamental domain $\bigcup_\gamma \mathcal{F}$ for this action $\Gamma_\infty \backslash \mathfrak{H}$. Using this in the expression above, together with the automorphy of the $f_{L_0^\perp + 0}(\tau) \in M_0^!(\omega_{L_0^\perp})$ and the $\theta_{L_0^\perp}(\tau) \in M_1(\omega_{L_0^\perp}^\vee)$, we can unfold for any $s \in \mathbf{C}$ with $\Re(s) > 1$ to find the regularized integral expression

$$\begin{aligned}
L^{\mathrm{div}}(s, f \times \theta_{L_0^\perp}) &= \iint_{\Gamma_\infty \backslash \mathfrak{H}}^\star \langle\langle \overline{f_{L_0^\perp + 0}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle v^{\frac{s+2}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma\mathcal{F}}^\star \langle\langle \overline{f_{L_0^\perp + 0}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle v^{\frac{s+2}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_0^\perp + L_0}(\gamma\tau)}, \theta_{L_0^\perp}(\gamma\tau) \otimes \omega_{L_0}^{-1}(\gamma) \mathbf{1}_{L_0 + 0} \rangle\rangle \Im(\gamma\tau)^{\frac{s+2}{2}} d\mu(\tau) \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_0^\perp + L_0}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{s+2}{2}} \mathbf{1}_0 \right] \Big|_{-1, \omega_{L_0}} \gamma \right) \rangle\rangle d\mu(\tau) \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_0 \oplus L_0^\perp}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; -1) \rangle\rangle d\mu(\tau) \\
&= \mathrm{CT}_{w=0} \left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle \overline{f_{L_0 \oplus L_0^\perp}(\tau)}, \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, s; -1) \rangle\rangle v^{-w} d\mu(\tau) \right).
\end{aligned} \tag{19}$$

In this way, we can view this regularized theta integral as giving an integral presentation for the divergent Dirichlet series (18). We can then deduce the stated analytic continuation that of the Eisenstein series $E_{L_0}(\tau, s; -1)$ in this regularized integral presentation.

Assume now that $L_U = L_W$ is a Lorentzian lattice of signature $(1, 1)$, with corresponding nonholomorphic theta series $\theta_{L_W^\perp}(\tau)$. Recall that we fix a basepoint $z_0 \in D(W^\perp)$. Note that $W_{\mathbf{R}}^\perp = z_0 \oplus z_0^\perp$. Note as well that for any vector $x \in W^\perp(\mathbf{Q})$, writing x_{z_0} and $x_{z_0^\perp}$ to denote the corresponding projections, we have $x = x_{z_0} + x_{z_0^\perp}$ and $Q(x) = Q(x_{z_0}) + Q(x_{z_0^\perp})$. Here, we write $Q = Q_W$ to simplify notations. Using these facts, we first compute the corresponding constant coefficient

$$\begin{aligned}
&\mathrm{CT} \langle\langle \overline{f_{L_W^\perp + 0}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle = \int_0^1 \langle\langle \overline{f_{L_W^\perp + 0}(u + iv)}, \theta_{L_W^\perp}(u + iv) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle du \\
&= v \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{m \gg -\infty} \overline{c_{f_{L_W^\perp + 0}}(\lambda, m)} e(miv) \sum_{\substack{x \in W^\perp(\mathbf{Q}) \\ x \in \lambda + L_W^\perp}} e\left(Q(x_{z_0^\perp}) iv - Q(x_{z_0}) iv \right) \int_0^1 e\left(-mu + Q(x_{z_0^\perp}) u + Q(x_{z_0}) u \right) du \\
&= v \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{m \gg -\infty} \overline{c_{f_{L_W^\perp + 0}}(\lambda, m)} e(miv) \sum_{\substack{x \in W^\perp(\mathbf{Q}) \\ x \in \lambda + L_W^\perp}} e\left(Q(x_{z_0^\perp}) iv - Q(x_{z_0}) iv \right) \int_0^1 e\left(-mu + Q(x) u \right) du \\
&= v \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{x \in \lambda + L_W^\perp} \overline{c_{f_{L_W^\perp + 0}}(\lambda, Q(x))} e\left(-Q(x) iv + Q(x_{z_0^\perp}) iv - Q(x_{z_0}) iv \right) \\
&= v \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{x \in \lambda + L_W^\perp} \overline{c_{f_{L_W^\perp + 0}}(\lambda, Q(x))} e\left(-2Q(x_{z_0}) iv \right) = v \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{x \in \lambda + L_W^\perp} \overline{c_{f_{L_W^\perp + 0}}(\lambda, Q(x))} e^{4\pi Q(x_{z_0}) v}.
\end{aligned}$$

Note that since $z_0 \in D(W^\perp) = \{z \subset W_{\mathbf{R}}^\perp : \dim(z) = 1, Q|_{z_0} < 0\}$, we have $Q(x_{z_0}) < 0$ for each summand. We consider the corresponding divergent Dirichlet series, unfolding formally as before to find the relation

$$\begin{aligned}
L^{\mathrm{div}}(s, f \times \theta_{L_W^\perp}) &:= (4\pi)^{-(\frac{n-3}{2})}\Gamma\left(\frac{s-3}{2}\right) \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{m \geq 1} \overline{c_f(\lambda, m)} r_{L_W^\perp}(\lambda, m) m^{-(\frac{s-3}{2})} \\
&= \int_0^\infty \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{m \geq 1} \overline{c_f(\lambda, m)} r_{L_W^\perp}(\lambda, m) e^{-4\pi m v} v^{\frac{s-3}{2}} dv \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + 0}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle v^{\frac{s+1}{2}} d\mu(\tau).
\end{aligned}$$

Unfolding the regularized integral, we then find, analogously to the CM case, that we have the presentation

$$\begin{aligned}
L^{\mathrm{div}}(s, f \times \theta_{L_W^\perp}) &:= (4\pi)^{-(\frac{n-3}{2})}\Gamma\left(\frac{s-3}{2}\right) \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{m \geq 1} \overline{c_f(\lambda, m)} r_{L_W^\perp}(\lambda, m) m^{-(\frac{s-3}{2})} \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + 0}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle v^{\frac{s+1}{2}} d\mu(\tau) \\
&= \iint_{\Gamma_\infty \backslash \mathfrak{H}}^\star \langle\langle \overline{f_{L_W^\perp + 0}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle \Im(\tau)^{\frac{s+1}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + 0}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \mathbf{1}_{L_W + 0} \rangle\rangle \Im(\tau)^{\frac{s+1}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + L_W}(\gamma\tau)}, \theta_{L_W^\perp}(\gamma\tau) \otimes \omega_{L_W}^{-1}(\gamma) \mathbf{1}_{L_W + 0} \rangle\rangle \Im(\gamma\tau)^{\frac{s+1}{2}} d\mu(\tau) \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + L_W}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{s+1}{2}} \mathbf{1}_0 \right] \Big|_{0, \omega_{L_W}} \right) \rangle\rangle d\mu(\tau) \\
&= \int_{\mathcal{F}}^\star \langle\langle \overline{f_{L_W^\perp + L_W}(\tau)}, \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, s; 0) \rangle\rangle d\mu(\tau).
\end{aligned} \tag{20}$$

Again, we deduce the claimed analytic continuation from that of the Langlands Eisenstein series $E_{L_W}(\tau, s; 0)$ in the regularized integral presentation of $L^{\mathrm{div}}(s, f \times \theta_{L_W^\perp})$. ☐

We shall relate the central value $L(0, f \times \theta_{L_0^\perp})$ in the negative definite setting $U = V_0$ to a sum over the zero cycle $Z(V_0) \subset X_K$ and the central value $L(0, f \times \theta_{L_W^\perp})$ in the Lorentzian setting $U = W$ to a sum over the geodesic set $G(W)$ of the corresponding Borcherds product $\Psi(f, \cdot)$.

3.2. **Relation to classical Rankin-Selberg L-functions.** We have the following relation to classical Rankin-Selberg L-functions of weakly holomorphic modular forms $\phi \in M_0^!(\Gamma_0(N))$ times Hecke theta series $\theta(\chi)$ associated to ring class characters χ of quadratic fields. Here, we generalize the discussion in [16, §6].

3.2.1. *Hecke theta series.* Let $k = k(U)$ be a quadratic field with integers $\mathcal{O}_k$ and class group $C(\mathcal{O}_k)$. We again write d_k to denote the discriminant, and $\eta_k(\cdot) = (\frac{d_k}{\cdot})$ the corresponding character. Fix a $\mathbf{Z}$-order $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_k$ of some conductor $c \geq 1$ (assumed to exist). We write $C(\mathcal{O}_c)$ to denote the corresponding ring class group, as described in (3) above. We can associate to each class $A = [\mathfrak{a}] \in C(\mathcal{O}_c)$ a corresponding partial theta series $\theta_A(\tau)$, then to each ring class character $\chi : C(\mathcal{O}_c) \longrightarrow \mathbf{C}^\times$ a theta series

$$\theta(\chi)(\tau) = \sum_{A \in C(\mathcal{O}_c)} \chi(A)\theta_A(\tau) \in M_{l(k)}(\Gamma_0(|c^2 d_k|), \eta_k), \quad l(k) = \begin{cases} 1 & \text{if } k \text{ is imaginary quadratic} \\ 0 & \text{if } k \text{ is real quadratic} \end{cases}. \tag{21}$$

To describe this more precisely, let $w_k = \#\mu(k)$ denote the number of roots of unity. Hence, if k is imaginary quadratic, then $w_k = \#\mathcal{O}_k^\times$ by Dirichlet's unit theorem. In this case, we define the partial theta series

$$\theta_A(\tau) = \frac{1}{w_k}\sum_{\lambda\in\mathfrak{a}} e\left(Q_{\mathfrak{a}}(\lambda)\tau\right) = \sum_{m\in\mathbf{Z}_{\geq 0}} r_A(m)e(m\tau),$$

where $r_A(m)$ denotes the counting function

$$r_A(m) = \frac{1}{w_k}\cdot\#\left\{\lambda\in\mathfrak{a} : Q_{\mathfrak{a}}(\lambda) = m\right\}.$$

Hecke showed that each partial theta series $\theta_A(\tau)\in M_1(\Gamma_0(|c^2d_k|),\eta_k)$ is a modular form of weight 1, level $\Gamma_0(|c^2d_k|)$, and character η_k. If k is a real quadratic field, then $\mathcal{O}_k^\times\cong\mathbf{Z}\times\mu(k) = \langle\varepsilon_k\rangle\times\mu(k)$ by Dirichlet's unit theorem, where ε_k denotes the fundamental unit. In this case, we first fix a fundamental domain $\mathfrak{a}^\star$ for the action of $\mathcal{O}_k^\times/\mu(k) = \langle\varepsilon_k\rangle$ on $\mathfrak{a}\subset k_\infty\cong\mathbf{R}^2$. In this case, we define the partial theta series

$$\theta_A(\tau) = \frac{1}{w_k}\sum_{\lambda\in\mathfrak{a}^\star} e\left(Q_{\mathfrak{a}}(\lambda)\tau\right) = \sum_{m\in\mathbf{Z}_{\geq 0}} r_A(m)e(m\tau),$$

where $r_A(m)$ denotes the counting function

$$r_A(m) = \frac{1}{w_k}\cdot\#\left\{\lambda\in\mathfrak{a}^\star : Q_{\mathfrak{a}}(\lambda) = m\right\}.$$

Hecke showed that each of the partial theta series $\theta_A\in A_0(\Gamma_0(c^2d_k),\eta_k)$ is a (nonholomorphic) modular form of weight 0, level $\Gamma_0(c^2d_k)$, and character η_k.

3.2.2. *Classical Rankin-Selberg L-functions.* Let us now consider the corresponding Rankin-Selberg L-function $L(s,\phi\times\theta(\chi))$ of $\phi\in S_0^!(\Gamma_0(N))$ of a weakly holomorphic modular function for $\Gamma_0(N)$ against the Hecke theta series $\theta(\chi)$ described in (21) above. Again, we must interpret these in a regularized sense.

Suppose more generally that $S_{l(\phi)}^!(\Gamma_0(N))$ is any weakly holomorphic cusp form of weight $l(\phi)\geq 0$ on $\Gamma_0(N)$. We write its Fourier series expansion as

$$\phi(\tau) = \sum_{m\gg-\infty} c_\phi(m)e(m\tau) = \sum_{m\gg-\infty} a_\phi(m)m^{\frac{l(\phi)-1}{2}}e(m\tau)\in S_{l(\phi)}^!(\Gamma_0(N))$$

so that its standard L-function $\Lambda(s,\phi) = L_\infty(s,h)L(s,h)$ corresponds to the formal Dirichlet series expansion

$$L^{\mathrm{div}}(s,\phi) := \sum_{m\geq 1} a_\phi(m)m^{-s} = \sum_{m\geq 1} c_\phi(m)m^{-\left(s+\frac{l(\phi)-1}{2}\right)}.$$

Consider the theta series $\theta(\chi)(\tau)\in M_{l(k)}(\Gamma_0(|c^2d_k|),\eta_k)$ from (21), writing its Fourier series abstractly as

$$\theta(\chi)(\tau) = \sum_{m\gg-\infty} c_{\theta(\chi)}(m)e(m\tau) = \sum_{m\gg-\infty} a_{\theta(\chi)}(m)m^{\frac{l(k)-1}{2}}e(m\tau).$$

We consider the standard L-function

$$\Lambda(s,\phi\times\theta(\chi)) = L_\infty(s,\phi\times\theta(\chi))L(s,\phi\times\theta(\chi))$$

associated to such a ϕ and $\theta(\chi)$, which can be defined as an Euler product in terms of the corresponding $\mathrm{GL}_2(\mathbf{A})$-automorphic representations, and again requires a suitable regularization. That is, the underlying Euler product over finite primes $L(s,\phi\times\theta(\chi))$ corresponding to the divergent Dirichlet series

$$L^{\mathrm{div}}(s,\phi\times\theta(\chi)) := (4\pi)^{-(s+l(k)-3)}\Gamma\left(s+l(k)-3\right)\sum_{m\geq 1}\overline{c_\phi(m)}\left(\sum_{A\in C(\mathcal{O}_c)}\chi(A)r_A(m)\right)m^{-(s+l(k)-3)}$$

has an analytic continuation $\Lambda(s,\phi\times\theta(\chi))$ to all $s\in\mathbf{C}$, and satisfies a symmetric functional equation

$$\Lambda(s,\phi\times\theta(\chi)) = \epsilon(s,\phi\times\theta(\chi))\Lambda(1-s,\phi\times\theta(\chi)).$$

For each class $A\in C(\mathcal{O}_c)$, we let (V_A,Q_A) denote the quadratic space of Definition 2.1 above. Fixing an integer $N\geq 1$, we let $L_A = L_A(N)\subset V_A$ denote the lattice corresponding to $K_0(N)^2$ under the exceptional isomorphism $\mathrm{GSpin}(V_A)\cong\mathrm{GL}_2^2$, as described in Proposition 2.2 (v) above.

Proposition 3.2. *Let $\phi \in M_0^!(\Gamma_0(N))$ be any weakly holomorphic form of weight zero for $\Gamma_0(N)$. Given any class $A \in C(\mathcal{O}_c)$ with corresponding quadratic lattice $(L_A, Q_A) = (L_A(N), Q_A)$ described by Definition 2.1 and Proposition 2.2 (v), there exists a unique lifting $f(\tau) = f_{\phi,A}(\tau) \in M_0^!(\omega_{L_A})$ of ϕ to a vector-valued weakly holomorphic form of weight zero and representation ω_{L_A}. Explicitly, it is given by the expansion*

$$f(\tau) = f_{\phi,A}(\tau) = \sum_{\gamma \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{Z})} c_\gamma \phi|_0(\gamma\tau), \quad c_\gamma := \omega_{L_A}(\gamma)^{-1} \sum_{\mu \in L_A(N)^\vee / L_A(N)} \mathbf{1}_\mu.$$

Proof. This follows from [15, Theorem 5.2], applied to the quadratic lattice $(L_A(N), Q_A)$ of level N, square discriminant $d(V_A)$, and trivial character $\chi_{d(V_A)} = (\frac{d(V_A)}{\cdot})$, taking the isotropic group $S_0 = L_A^\vee(N)/L_A(N)$. $\square$

Conversely, to any weakly holomorphic modular form $f(\tau) \in M_0^!(\omega_L)$ as considered above, we have a canonical lift to a meromorphic modular function $\phi(\tau) = \phi_f(\tau) \in M_0^!(\Gamma_0(N))$ of level N determined by the level of the lattice $L \subset V$. According to [15], this lifting is given explicitly by the expansion

$$\phi(\tau) = \phi_f(\tau) = \sum_{\gamma \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbf{Z})} \left[f(\tau)\mathbf{1}_0\right]\Big|_{0,\omega_L} \gamma \in M_0^!(\Gamma_0(N)).$$

Proposition 3.3. *Let $\phi \in S_0^!(\Gamma_0(N))$ be any weakly holomorphic modular function of level $\Gamma_0(N)$ and trivial character. Fix $\chi \in C(\mathcal{O}_c)^\vee$ a ring class character of conductor $c \geq 1$ of a quadratic field k. Given any class $A \in C(\mathcal{O}_c)$ with corresponding quadratic lattice $(L_A, Q_A) = (L_A(N), Q_A)$ described above, let*

$$f_{\phi,A}(\tau) = \sum_{\mu \in L_A^\vee / L_A} \sum_{m \gg -\infty} c_{f_{\phi,A}}(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_0^!(\omega_{L_A}^\vee)$$

denote the lifting of ϕ to a vector-valued weakly holomorphic form of weight zero and representation $\omega_{L_A}^\vee$, as described in Proposition 3.2. We have the following equivalences of Rankin-Selberg L-functions.

(i) *If $k = k(U) = k(V_{A,0})$ is the imaginary quadratic field associated to a negative definite subspace $V_{A,0} \subset V_A$ with lattice $L_{A,0} = L_A(N) \cap V_{A,0}$, then we have an identification of completed L-functions*

$$L^\star(2s-2, f_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

for each class $A \in C(\mathcal{O}_k)$, and for each ring class character $\chi \in C(\mathcal{O}_c)^\vee$ the identification

$$\sum_{A \in C(\mathcal{O}_c)} \chi(A) L^\star(2s-2, f_{\phi,A} \times \theta_{L_{A,0}^\perp}) = \Lambda(s - 1/2, \phi \times \theta(\chi)).$$

(ii) *If $k = k(U) = k(W_A)$ is the real quadratic field associated to a Lorentzian subspace $W_A \subset V_A$ with lattice $L_{A,W} = L_A(N) \cap W_A$, then we have the identifications of completed L-functions*

$$L^\star(2s-2, f_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

for each class $A \in C(\mathcal{O}_c)$, and for each ring class character $\chi \in C(\mathcal{O}_c)^\vee$ that

$$\sum_{A \in C(\mathcal{O}_c)} \chi(A) L^\star(2s-2, f_{\phi,A} \times \theta_{L_{A,W}^\perp}) = \Lambda(s - 1/2, \phi \times \theta(\chi)).$$

Proof. Cf. [16, Proposition 6.6], [17, Corollary 4.18]. Fix a class $A \in C(\mathcal{O}_c)$. In each of the respective cases, a direct comparison of the Fourier series expansions reveals that

$$\langle\langle \overline{f_{A,\phi}(\tau)}, \theta_{L_{A,0}^\perp}(\tau) \otimes \mathbf{1}_{L_{A,0}+0} \rangle\rangle = \overline{\phi_A(\tau)} \theta_A(\tau)$$

and

$$\langle\langle \overline{f_{A,\phi}(\tau)}, \theta_{L_{A,W}^\perp}(\tau) \otimes \mathbf{1}_{L_{A,W}+0} \rangle\rangle = \overline{\phi_A(\tau)} \theta_A(\tau).$$

Here, we drop the subscripts denoting restriction to the sublattice for the vector-values form $f_{A,\phi}$. The deduction is then a direct consequence of the discussion above with regularized integral presentations.

Let us suppose first that $k = k(U) = k(V_{A,0})$ is an imaginary quadratic field associated to a negative definite subspace $V_{A,0} \subset V_A$. We have for each class $A \in C(\mathcal{O}_c)$ the corresponding partial theta series

$$\theta_{L_{A,0}^{\perp}}(\tau) = \sum_{\mu \in (L_{A,0}^{\perp})^{\vee}/L_{A,0}^{\perp}} \sum_{m \geq 0} r_{L_{A,0}^{\perp}}(\mu, m) e(m\tau) \mathbf{1}_{\mu} \in M_1(\omega_{L_{A,0}^{\perp}}).$$

we consider the corresponding Rankin-Selberg L-function, defined first by the formal (divergent) series

$$L^{\mathrm{div}}(s, f_{\phi,A} \times \theta_{L_{A,0}^{\perp}}) := (4\pi)^{-\left(\frac{s-2}{2}\right)} \Gamma\left(\frac{s-2}{2}\right) \sum_{\mu \in (L_{A,0}^{\perp})^{\vee}/L_{A,0}^{\perp}} \sum_{m \geq 1} \overline{c_{f_{\phi,A}}(\mu, m)} r_{L_{A,0}^{\perp}}(\mu, m) m^{-\left(\frac{s-2}{2}\right)}.$$

Note that

$$(L_{A,0}^{\perp})^{\vee}/L_{A,0}^{\perp} \cong \mathfrak{d}_k^{-1} N^{-1}\mathfrak{a}/N^{-1}\mathfrak{a} \cong \mathfrak{d}_k^{-1}\mathcal{O}_k/\mathcal{O}_k,$$

and

$$r_{L_{A,0}^{\perp}}(\mu, m) = \frac{1}{w_k} \cdot \# \left\{ \lambda \in \mu + L_{A,0}^{\perp} : Q_A|_{L_{A,0}^{\perp}}(\lambda) = m \right\} = \frac{1}{w_k} \cdot \# \left\{ \lambda \in \mu + N^{-1}\mathfrak{a} : Q_{\mathfrak{a}}(\lambda) = m \right\}.$$

We have the identification[1] of counting functions

$$\sum_{\mu \in (L_{A,0}^{\perp})^{\vee}/L_{A,0}^{\perp}} r_{L_{A,0}^{\perp}}(\mu, m) = \frac{1}{w_k} \cdot \# \left\{ \lambda \in N^{-1}\mathfrak{a} : Q_{\mathfrak{a}}(\lambda) = m \right\} = r_A(m).$$

Similarly, we deduce that we have the relation of Fourier coefficients

$$\sum_{\mu \in (L_{A,0}^{\perp})^{\vee}/L_{A,0}^{\perp}} c_{f_{\phi,A}}(\mu, m) = c_{\phi}(m),$$

and more generally, that we have an identification of scalar-valued forms

$$\langle\langle \overline{f_{\phi,A}(\tau)}, \theta_{L_{A,0}^{\perp}}(\tau) \otimes E_{L_{A,0}}(\tau, s; -1) \rangle\rangle = \overline{\phi_A(\tau)} \theta_A(\tau) E_A(\tau, s; -1),$$

where $E_A(\tau, s; -1)$ denotes the scalar-valued Eisenstein series determined by lifting $E_{L_{A,0}}(\tau, s; -1) \in H_{-1}(\omega_{L_{A,0}})$. By Proposition 3.1, we have the regularized integral presentation

$$L^{\mathrm{div}}(s, f_{\phi,A} \times \theta_{L_{A,0}^{\perp}}) = \int_{\mathcal{F}}^{\star} \langle\langle \overline{f_{\phi,A}(\tau)}, \theta_{L_{A,0}^{\perp}}(\tau) \otimes E_{L_{A,0}}(\tau, s; -1) \rangle\rangle d\mu(\tau).$$

We deduce that the completed L-function $\Lambda(s, \phi_A \times \theta_A)$ defined by the relation

$$\Lambda(s/2 + 3/4, \phi_A \times \theta_A) := L^{\star}(s, f_{A,\phi} \times \theta_{L_{A,0}^{\perp}}) = \int_{\mathcal{F}}^{\star} \langle\langle \overline{f_{\phi,A}(\tau)}, \theta_{L_{A,0}^{\perp}}(\tau) \otimes E_{L_{A,0}}^{\star}(\tau, s; -1) \rangle\rangle d\mu(\tau)$$

has an analytic continuation to all of $s \in \mathbf{C}$, and satisfies the symmetric functional equation

$$\Lambda(s, \phi_A \times \theta_A) = \epsilon(s, \phi_A \times \theta_A) \Lambda(1 - s, \phi_A \times \theta_A).$$

That is, we deduce that we have the identification of completed L-functions

$$L^{\star}(2s - 2, f_{\phi,A} \times \theta_{L_{A,0}^{\perp}}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

with

$$\sum_{A \in C(\mathcal{O}_k)} \chi(A) L^{\star}(2s - 2, g_{\phi,A} \times \theta_{L_{A,0}^{\perp}}) = \sum_{A \in C(\mathcal{O}_k)} \chi(A) \Lambda(s - 1/2, \phi \times \theta_A) = \Lambda(s - 1/2, \phi \times \theta(\chi)).$$

Suppose now that $k = k(W) = k(W_A)$ is a real quadratic field associated to a Lorentzian subspace $W_A \subset V_A$. We argue in the same way. That is, we first open the formal Dirichlet series expansion

$$L^{\mathrm{div}}(s, f_{\phi,A} \times \theta_{L_{A,W}^{\perp}}) := \frac{\Gamma\left(\frac{s-3}{2}\right)}{(4\pi)^{\frac{s-3}{2}}} \sum_{\mu \in (L_{A,W}^{\perp})^{\vee}/L_{A,W}^{\perp}} \sum_{m \geq 1} \frac{c_{f_{\phi,A}}(\mu, m) r_{L_{A,W}^{\perp}}(\mu, m)}{m^{\frac{s-3}{2}}}.$$

---

[1] Here, we have $[N^{-1}\mathfrak{a}] = [(N^{-1})\mathfrak{a}] = [\mathfrak{a}] \in C(\mathcal{O}_c)$.

In this case, we have identifications

$$(L_{A,W}^{\perp})^{\vee}/L_{A,W}^{\perp} \cong \mathfrak{d}_k^{-1}N^{-1}\mathfrak{a}/N^{-1}\mathfrak{a} \cong \mathfrak{d}_k^{-1}\mathcal{O}_k/\mathcal{O}_k,$$

and

$$r_{L_{A,W}^{\perp}}(\mu, m) = \frac{1}{w_k} \cdot \# \left\{ \lambda \in \mu + L_{A,W}^{\perp}/\langle \varepsilon_k \rangle : Q_A|_{L_{A,W}^{\perp}}(\lambda) = m \right\} = \frac{1}{w_k} \cdot \# \left\{ \lambda \in \mu + N^{-1}\mathfrak{a}^{\star} : Q_{\mathfrak{a}}(\lambda) = m \right\}$$

so that

$$\sum_{\mu \in (L_{A,W}^{\perp})^{\vee}/L_{A,W}^{\perp}} r_{L_{A,W}^{\perp}}(\mu, m) = \frac{1}{w_k} \cdot \# \left\{ \lambda \in N^{-1}\mathfrak{a}^{\star} : Q_{\mathfrak{a}}(\lambda) = m \right\} = r_A(m).$$

We also have

$$\sum_{\mu \in (L_{A,W}^{\perp})^{\vee}/L_{A,W}^{\perp}} c_{f_{\phi,A}}(\mu, m) = c_{\phi}(m)$$

and

$$\langle\langle f_{\phi,A}(\tau), \theta_{L_{A,W}^{\perp}}(\tau) \otimes E_{L_{A,W}}(\tau, s; 0) \rangle\rangle = \phi(\tau)\theta_A(\tau)E_A(\tau, s; 0).$$

By Proposition 3.1, we have the regularized integral presentation

$$L^{\mathrm{div}}(s, f_{\phi,A} \times \theta_{L_{A,W}^{\perp}}) = \int_{\mathcal{F}} \langle\langle f_{\phi,A}(\tau), \theta_{L_{A,W}^{\perp}}(\tau) \otimes E_{L_{A,W}}(\tau, s; 0) \rangle\rangle.$$

Again, we use this to deduce that the corresponding completed L-function $\Lambda(s, \phi_A \times \theta_A)$ defined by

$$\Lambda(s/2 + 3/4, \phi_A \times \theta_A) := L^{\star}(s - 1/2, f_{\phi,A} \times \theta_{L_{A,W}^{\perp}}) = \int_{\mathcal{F}} \langle\langle f_{\phi,A}(\tau), \theta_{L_{A,W}^{\perp}}(\tau) \otimes E_{L_{A,W}}^{\star}(\tau, s; 0) \rangle\rangle$$

has an analytic continuation to all $s \in \mathbf{C}$, and satisfies the symmetric functional equation

$$\Lambda(s, \phi_A \times \theta_A) = \epsilon(s, \phi_A \times \theta_A)\Lambda(1 - s, \phi_A \times \theta_A).$$

That is, we deduce that we have the identification of completed L-functions

$$L^{\star}(2s - 2, f_{\phi,A} \times \theta_{L_{A,W}^{\perp}}) = \Lambda(s - 1/2, \phi \times \theta_A)$$

with

$$\sum_{A \in C(\mathcal{O}_c)} \chi(A)L^{\star}(2s - 2, f_{\phi,A} \times \theta_{L_{A,W}^{\perp}}) = \sum_{A \in C(\mathcal{O}_c)} \chi(A)\Lambda(s, \phi \times \theta_A) = \Lambda(s, \phi \times \theta(\chi)).$$

□

4. SUMS OVER ZERO CYCLES AND GEODESIC SETS AS CENTRAL VALUES OF L-FUNCTIONS

We now describe the central values of the Rankin-Selberg L-functions of Proposition 3.3 as finite sums of Borcherds products of the weakly holomorphic functions $f_{\phi,A}$.

4.0.3. *Conventions.* We retain the setup described above, with (V, Q) a rational quadratic space of signature $(n, 2)$, and (U, Q_U) a rational quadratic subspace of dimension 2 corresponding to a quadratic field $k = k(U)$ of discriminant d_k and character $\eta_k(\cdot) = (\frac{d_k}{\cdot})$. That is, U denotes either a negative definite subspace (V_0, Q_0) associated to an imaginary quadratic field $k = k(V_0)$, or a Lorentzian subspace (W, Q_W) associated to a real quadratic field $k = k(W)$.

4.1. **General summation formulae.** We now show the main summation formula (Proposition 4.4, Corollary 4.5), a variation of the calculations of [13, Theorem 3.4], [6, Theorem 4.7], and [12], describing the integral of $\Phi(f, z, h)$ along a zero cycle $Z(V_0)$ of X_K when $U = V_0$ is negative definite, or a variation of the calculations in [16] and [17] describing the integral of $\Phi(f, z, h)$ along the geodesic set $G(W)$ when $U = W$ is Lorentzian. Hence, for $f \in M_0^!(\omega_L)$ any weakly holomorphic form of weight zero and representation ω_L, we compute the sums (1) and (2) defined above. Note that these sums run over the finite sets (6) and (7) respectively. In each case, we fix a basepoint $z = z_U^{\pm} \in D^{\pm}(U)$. Although this section is largely expository, we clarify the setup for the special case of $n = 2$ to relate the central values $L(0, f \times \theta_{L_U^{\perp}})$ of the Rankin-Selberg L-functions $L(s, f \times \theta_{L_U^{\perp}})$ to the corresponding sums of the Borcherds products $\Psi(f, \cdot)$. This relation does not seem to appear elsewhere.

4.1.1. *Rational splittings and decompositions of theta series.* Recall that we fix an integral lattice $L \subset V$, and consider the corresponding Siegel theta series $\theta_L(\tau, z, g)$ as a function of $\tau = u + iv \in \mathfrak{H}$, $z \in D(V)$, and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)/K$, for $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ the corresponding compact open subgroup. Writing $L_U = U \cap L$ for the corresponding sublattice, we again write $L_U^{\perp} \subset L$ to denote the orthogonal complement.

We consider the following restrictions of this theta series to the sublattice $L_U \oplus L_U^{\perp} \subset L$ in our calculations. Note that for this sublattice, we have the following natural decomposition of the Siegel theta series

$$\theta_{L_U \oplus L_U^{\perp}}(\tau, z_U, h_U) = \theta_{L_U}(\tau, z_U, h_U) \otimes \theta_{L_U^{\perp}}(\tau), \quad \theta_{L_U^{\perp}}(\tau) := \theta_{L_U^{\perp}}(\tau, 1, 1) \tag{22}$$

for any $z_U \in D(V_U)$ and $h_U \in \mathrm{GSpin}(V_U)(\mathbf{A}_f)$. These determine Maass forms in the variable $\tau \in \mathfrak{H}$. In the negative definite case with $L_U = L_0$, the theta series $\theta_{L_0}(\tau)$ nonholomorphic, the theta series of the complement $\theta_{L_0^{\perp}}(\tau)$ is holomorphic, and the tensor product is given more explicitly by

$$\theta_{L_0}(\tau, z_0, h_0) \otimes \theta_{L_0^{\perp}}(\tau) = \sum_{\substack{\mu \in L_0^{\vee}/L_0 \\ \nu \in (L_0^{\perp})^{\vee}/L_0^{\perp}}} \theta_{L_0, \mu}(\tau) \theta_{L_0^{\perp}, \nu}(\tau) \mathbf{1}_{\mu + \nu}.$$

In the Lorentzian case with $L_U = L_W$, the theta series $\theta_{L_W}(\tau)$ is nonholomorphic, the theta series of the complement $\theta_{L_U^{\perp}}(\tau)$ is nonholomorphic, and the tensor product is given more explicitly by the expansion

$$\theta_{L_W}(\tau, z_W, h_W) \otimes \theta_{L_U^{\perp}}(\tau) = \sum_{\substack{\mu \in L_W^{\vee}/L_W \\ \nu \in (L_W^{\perp})^{\vee}/L_W^{\perp}}} \theta_{L_W, \mu}(\tau) \theta_{L_W^{\perp}, \nu}(\tau) \mathbf{1}_{\mu + \nu}.$$

We use the following standard result to argue that we can replace the Siegel theta series $\theta_L(\tau, z, h)$ appearing in the definition of the regularized theta lift $\Phi(f, \cdot)$ with that of the sublattice $L_U \oplus L_U^{\perp} \subset L$ described in (22). Suppose more generally that $M \subset L$ is any sublattice of finite index. Observe that we have the natural inclusions $M \subset L \subset L^{\vee} \subset M^{\vee}$ and hence $L/M \subset L^{\vee}/M \subset M^{\vee}/M$. It follows that we have natural inclusions $H_l(\omega_L) \subset H_l(\omega_M)$ for each weight $l \in \frac{1}{2}\mathbf{Z}$. To describe this more precisely, consider the natural map

$$L^{\vee}/M \longrightarrow L^{\vee}/L, \quad \mu \longmapsto \overline{\mu}.$$

Lemma 4.1. *Fix a sublattice $M \subset L$ of finite index. There is a natural restriction map*

$$\mathrm{res}_{L/M} : H_l(\omega_L) \longrightarrow H_l(\omega_M), \quad f \longmapsto f_M$$

and a natural trace map

$$\mathrm{tr}_{L/M} : H_l(\omega_M) \longrightarrow H_l(\omega_L), \quad g \longmapsto g^L$$

such that for all $f \in H_l(\omega_L)$ and $g \in H_l(\omega_M)$ of any weight $l \in \frac{1}{2}\mathbf{Z}$, we have the identification

$$\langle\langle f(\tau), g^L(\tau) \rangle\rangle = \langle\langle f_M(\tau), g(\tau) \rangle\rangle.$$

These maps are given respectively as follows. For $\mu \in M^{\vee}/M$ and $f \in H_l(\omega_L)$, we have

$$f_{M,\mu}(\tau) = \begin{cases} f_{\overline{\mu}}(\tau) & \text{if } \mu \in L^{\vee}/M \\ 0 & \text{if } \mu \notin L^{\vee}/M. \end{cases}$$

For $\overline{\mu} \in L^\vee/L$ and $g \in H_L(\omega_M)$ with μ a fixed preimage of $\overline{\mu}$ in $L^\vee/M$, we have

$$g^L_{\overline{\mu}}(\tau) = \sum_{\nu \in L/M} g_{\nu+\mu}(\tau).$$

Proof. See [6, Lemma 3.1]. $\square$

As a consequence of this result, we see that $\theta_L = (\theta_M)^L$, and in particular that we have for any harmonic weak Maass form $f \in H_{1-n/2}(\omega_L)$ the identification of inner products

$$\langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle = \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle = \langle f_{L_U \oplus L_U^\perp}(\tau), \theta_{L_U \oplus L_U^\perp}(\tau, z, h)\rangle\rangle. \tag{23}$$

4.1.2. *Normalizations of measures.* Recall that we have the short exact sequences of algebraic groups

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(U) \longrightarrow \mathrm{SO}(U) \longrightarrow 1.$$

Again, we can identify the spin group $\mathrm{GSpin}(U)$ with the torus $k^\times = k(U)^\times$ determined by the quadratic field $k(U)$ associated to U, and $\mathrm{SO}(U)$ with the norm one elements $k^{(1)}$. Taking the finite adelic points modulo the rational points of each term, we recover the Hilbert exact sequence

$$1 \longrightarrow \mathbf{Q}^\times \backslash \mathbf{A}_f^\times \longrightarrow k^\times \backslash \mathbf{A}_{k,f}^\times \longrightarrow k^{(1)} \backslash \mathbf{A}_k^{(1)} \longrightarrow 1. \tag{24}$$

Let us now fix the Haar measure $dh = dh_\infty \times dh_f$ on $\mathrm{SO}(U)$ so that $\mathrm{vol}(\mathrm{SO}(U)(\mathbf{Q}) \backslash \mathrm{SO}(U)(\mathbf{A})) = 2$ with $\mathrm{vol}(\mathrm{SO}(U)(\mathbf{R}) = 1$. Fixing the Haar measure on $\mathbf{A}_f^\times$ so that $\mathrm{vol}(\widehat{\mathbf{Z}}^\times) = 1$ and $\mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}_f^\times) = 1/2$, we obtain from the exact sequence (24) a choice a of Haar measure dh_f on $\mathrm{GSpin}(U)(\mathbf{A}_f)$ for which

$$\mathrm{vol}(\mathrm{GSpin}(U)(\mathbf{Q}) \backslash \mathrm{GSpin}(U)(\mathbf{A}_f)) = \mathrm{vol}(k^\times \backslash \mathbf{A}_{k,f}^\times) = \mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}_f^\times) \cdot \mathrm{vol}(k^{(1)} \backslash \mathbf{A}_k^{(1)}) = 1.$$

Lemma 4.2. *We have the following identifications of finite sums over the CM cycle $Z(V_0)$ and $G(W)$.*

(i) *If $U = V_0$ is negative definite, then*

$$\int_{\mathrm{SO}(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} B(h) dh = \mathrm{vol}(K_T) \sum_{\mathrm{GSpin}(V_0)(\mathbf{Q}) \backslash \mathrm{GSpin}(V_0)(\mathbf{A}_f)/K_T} B(h).$$

In particular, we have the identification

$$\Phi(f, Z(V_0)) = \sum_{(z,h) \in Z_K(V_0)(\mathbf{C})} \frac{\Phi(f, z, h)}{\#\,\mathrm{Aut}(z, h)} = \frac{1}{\mathrm{vol}(K_0)} \int_{\mathrm{SO}(V_0)(\mathbf{Q}) \backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \Phi(f, z, h) dh.$$

(ii) *If $U = W$ is Lorentzian, then*

$$\begin{aligned}\Phi(f, \mathcal{G}(W)) &= \sum_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \frac{1}{\#\,\mathrm{Aut}(h)} \int_{C_{A,h} = \Gamma_{A,h} \backslash D(W_A)} \Phi(f, z, h) d\nu(z) \\ &= \frac{1}{\mathrm{vol}(K_W)} \int_{\mathrm{SO}(W)(\mathbf{Q}) \backslash \mathrm{SO}(W)(\mathbf{A}_f)} \Phi(f, z, h) dh.\end{aligned}$$

Proof. See [13, Lemma 2.13] and [6, Lemma 4.4] for (i), as well as [16, Lemma 5.9] and [17] for (ii). $\square$

4.1.3. *A simplified expression for the regularized theta lift.* Fix $f(\tau) \in M^!_{1-\frac{n}{2}}(\omega_L)$ a weakly holomorphic form of weight $1-n/2$ and representation ω_L. We consider its restriction $f_M(\tau)$ to the sublattice $M = L_0 \oplus L_0^\perp \subset L$.

Recall that we consider the regularized theta integral defined on $z \in D(V)$ and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by

$$\Phi(f, z, h) = \mathrm{CT}_{s=0}\left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right).$$

We have the following more convenient limiting expression at points $z_0 \in D(V_0)$.

Proposition 4.3. *Fix $z = z_U \in D(U)$. We have for any $f \in H_{1-n/2}(\omega_L)$ and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ that*

$$\Phi(f, z, h) = \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f_{L_U \oplus L_U^\perp}(\tau), \theta_{L_U \oplus L_U^\perp}(\tau, z, h)\rangle\rangle \; d\mu(\tau) - A_0 \log T \right].$$

Here, we write

$$A_0 := \sum_{\lambda \in (L_U^\perp)^\vee / L_U^\perp} \sum_{\substack{x \in U^\perp(\mathbf{Q}) \\ x \in \lambda + L_U^\perp}} c_{f_M}(\lambda, -Q(x)).$$

Proof. See [12, Proposition 2.5]. We give a proof for the convenience of the reader. Starting with the definition followed by (23), we have for the sublattice $M = L_U \oplus L_U^\perp \subset L$ that

$$\begin{aligned}\Phi(f, z, h) &= \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right) \\ &= \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right).\end{aligned}$$

Observe that the function of $s \in \mathbf{C}$ defined by the integral

$$\int\limits_{\mathcal{F}_1} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)$$

is holomorphic, and hence that

$$\Phi(f, z, h) = \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int\limits_1^T \int\limits_{-1/2}^{1/2} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right) + \int\limits_{\mathcal{F}_1} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle d\mu(\tau).$$

To describe the first term in this latter expression, let us write it as a limit of Mellin transforms

$$\mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int\limits_1^T \int\limits_{-1/2}^{1/2} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right) = \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int\limits_1^T C(v, h) v^{-s} \frac{dv}{v} \right),$$

where $C(v, h)$ denotes the constant term in the Fourier expansion of $v^{-1}\langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle$. Hence,

$$\begin{aligned} C(v, h) &= v^{-1} \int_{-1/2}^{1/2} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle du = v^{-1} \int_{-1/2}^{1/2} \sum_{\mu \in M^\vee/M} f_{M,\mu}(\tau)\theta_{M,\mu}(\tau, z, h) du \\ &= \int_{-1/2}^{1/2} \sum_{\mu \in M^\vee/M} \sum_{m \in \mathbf{Q}} c_{f_M}(\mu, m) e(m\tau) \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right) du \\ &= \sum_{\mu \in M^\vee/M} \sum_{m \in \mathbf{Q}} c_{f_M}(\mu, m) e(miv) \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} e\left(Q(x_{z^\perp})iv - Q(x_z)iv\right) \int_{-1/2}^{1/2} e\left(mu + Q(x_{z^\perp})u + Q(x_z)u\right) du \\ &= \sum_{\mu \in M^\vee/M} \sum_{m \in \mathbf{Q}} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu \\ m = -Q(x_{z^\perp}) - Q(x_z)}} c_{f_M}(\mu, m) e\left(miv + Q(x_{z^\perp})iv - Q(x_z)iv\right) \\ &= \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} c_{f_M}(\mu, -Q(x_{z^\perp}) - Q(x_z)) e\left(-Q(x_{z^\perp})iv - Q(x_z)iv + Q(x_{z^\perp})iv - Q(x_z)iv\right) \\ &= \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} c_{f_M}(\mu, -Q(x_{z^\perp}) - Q(x_z)) e^{4\pi Q(x_z)v}. \end{aligned}$$

We now make some observations. First, since $z \in D(U)$ implies $Q|_z < 0$, we see that the integral

$$\lim_{T \to \infty} \int_1^T C(v,h) v^{-s} \frac{dv}{v} = \int_1^\infty C(v,h) v^{-s} \frac{dv}{v}$$

converges absolutely. Here, we use the well-known bound

$$c_{f_M}(\mu, m) = O\left(e^{C\sqrt{|m|}} \right) \quad \forall\ C > 0$$

for the Fourier coefficients $c_{f_M}(\mu, m)$ (see e.g. [13, Lemma 2.5]). Let us first consider the contributions from the orthogonal terms $x \in h\mu$ with $(x,z) = 0$, equivalently $x_z = 0$. These contribute the quantity

$$C_{z^\perp}(v,h) := \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu \\ (x,z)=0}} c_{f_M}(\mu, -Q(x_{z^\perp}) - Q(x_z)) e^{4\pi Q(x_z) v} = \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} c_{f_M}(\mu, -Q(x_{z^\perp})),$$

which we see does not depend on the variable v. Hence, this quantity factors out of the Mellin transform as

$$\lim_{T \to \infty} \int_1^T C_{z^\perp}(v,h) v^{-s} \frac{dv}{v} = \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} c_{f_M}(\mu, -Q(x_{z^\perp})) \cdot \lim_{T \to \infty} \int_1^T v^{-s} \frac{dv}{v},$$

with

$$\lim_{s \to 0} \left(\lim_{T \to \infty} \int_1^T C_{z^\perp}(v,h) v^{-s} \frac{dv}{v} \right) = \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu}} c_{f_M}(\mu, -Q(x_{z^\perp})) \cdot \lim_{T \to \infty} \log(T)$$

$$= \mathrm{CT}\langle\langle f_{L_U \oplus L_U^\perp}(\tau), \theta^{\mathrm{dg}}_{L_U^\perp}(\tau, 1, 1) \otimes \mathbf{1}_{0 + L_U} \rangle\rangle \cdot \lim_{T \to \infty} \log(T) = A_0 \cdot \lim_{T \to \infty} \log(T).$$

For the remaining non-orthogonal terms

$$C_z(v,h) := \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\mu \\ (x,z) \neq 0}} c_{f_M}(\mu, -Q(x_{z^\perp}) - Q(x_z)) e^{4\pi Q(x_z) v}$$

with $x_z \neq 0$, we argue as in [12, Proposition 2.5] that the integrals

$$\beta_{s+1}(t) := \int_1^\infty e^{-tv} v^{-s} \frac{dv}{v}, \quad t \in \mathbf{R}_{>1}$$

converge for all values of $s \in \mathbf{C}$ and determine holomorphic functions at $s = 0$. That is, we now consider

$$\int_1^T C(v,h) v^{-s-1}\, dv = \int_1^T \left(C_{z^\perp}(v,h) + C_z(v,h) \right) v^{-s} \frac{dv}{v}$$

$$= \sum_{\mu \in M^\vee/M} \int_1^T \sum_{\substack{x \in h\mu \\ x \not\perp z}} c_{f_M}(-Q(x), \mu) e^{4\pi v Q(x_z)} v^{-s-1}\, dv + \sum_{\mu \in M^\vee/M} \int_1^T \sum_{\substack{x \in h\mu \\ x \perp z}} c_{f_M}(-Q(x), \mu) v^{-s-1}\, dv.$$

Here, we can interchange $\int_1^\infty$ and $\sum_{x \not\perp z_0}$ in the first term because $\beta_{s+1}(t) = O(e^{-t})$ as $t \to \infty$. The last term is a finite sum since $c_{f_M}(m, \mu) = 0$ for all but finitely many $m < 0$, so we need only consider $x \perp z$ with bounded $Q(x)$. It is equal to

$$\sum_{\lambda \in (L_U^\perp)^\vee / L_U^\perp} \sum_{\substack{x \in V(\mathbf{Q}) \\ x \in h\lambda}} c_{f_M}(-Q(x), \lambda)(1 - T^{-s}) s^{-1}$$

For $\Re s > 0$, we have $s^{-1}(1 - T^{-s}) \to s^{-1}$ as $T \to \infty$, which has no constant term as a Laurent series. Hence, the last term does not contribute to $\Phi(f, z, h)$.

Summing up, the previous paragraph shows that

$$\lim_{T \to \infty} \int_1^T (C(v,h) - A_0) v^{-s-1}\, dv = \sum_{\mu \in M^\vee/M} \sum_{\substack{x \in h\mu \\ x \not\perp z}} c_{f_M}(-Q(x), \mu) \beta_{s+1}(4\pi v Q(x_z))$$

is convergent and holomorphic at $s = 0$. Since

$$\Phi(f, z, h) = \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_1^T C(v, h) v^{-s-1}\, dv + \int_{\mathcal{F}_1} \langle\langle f_M(\tau), \theta_m(\tau, z, h)\rangle\rangle d\mu(\tau) \right)$$

and

$$\begin{aligned}
&\mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_1^T C(v, h) v^{-s-1} dv \right) \\
&= \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_1^T (C(v, h) - A_0) v^{-s-1}\, dv \right) \\
&= \int_1^\infty (C(v, h) - A_0) v^{-1}\, dv \quad \text{(the integral is holomorphic)} \\
&= \lim_{T\to\infty} \left[\int_1^T C(v, h) v^{-s-1}\, dv - A_0 \log T \right] \\
&= \lim_{T\to\infty} \left[\int_{\mathcal{F}_T - \mathcal{F}_1} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle\ d\mu(\tau) - A_0 \log T \right],
\end{aligned}$$

we conclude that

$$\Phi(f, z, h) = \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f_M(\tau), \theta_M(\tau, z, h)\rangle\rangle\ d\mu(\tau) - A_0 \log T \right].$$

□

Putting the pieces together, we derive the following

Proposition 4.4. *Let (V, Q) be any rational quadratic space of signature $(n, 2)$. Let $L \subset V$ any integral lattice with corresponding Weil representation ω_L. Let $f \in M^!_{1-n/2}(\omega_L)$ be any weakly holomorphic form of weight $1 - n/2$ and representation ω_L.*

(i) *Given $V_0 \subset V$ any negative definite subspace with corresponding lattice $L_0 = V_0 \cap L$ and zero-cycle $Z(V_0) \subset X_K = X_{K_L}$, we have the summation formula*

$$\Phi(f, Z(V_0)) = \frac{1}{\mathrm{vol}(K_0)} \cdot \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1)\rangle\rangle d\mu(\tau) - A_0 \log(T) \right].$$

Here,

$$A_0 = \mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0}\rangle\rangle = \sum_{\lambda\in(L_0^\perp)^\vee/L_0^\perp} \sum_{\substack{x\in V_0^\perp(\mathbf{Q}) \\ x\in\mu}} c_{f_{L_0\oplus L_0^\perp}}(-Q(x), \lambda).$$

(ii) *Given $W \subset V$ any Lorentzian subspace with corresponding lattice $L_W = W \cap L$ and geodesic set $G(W)$, we have the summation formula*

$$\Phi(f, \mathcal{G}(W)) = \frac{1}{\mathrm{vol}(K_W)} \cdot \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0)\rangle\rangle d\mu(\tau) - A_0 \log(T) \right].$$

Here,

$$A_0 = \sum_{\lambda\in(L_W^\perp)^\vee/L_W^\perp} \sum_{\substack{x\in W^\perp(\mathbf{Q}) \\ x\in\mu}} c_{f_{L_W\oplus L_W^\perp}}(-Q(x), \lambda).$$

Proof. Cf. [13, Theorem 3.4], [6, Lemma 4.6, Theorem 4.7], and [Corollary 5.11][16]. Starting with Lemma 4.2 (i), we have for that

$$\Phi(f, Z(V_0)) = \frac{1}{\mathrm{vol}(K_0)} \int\limits_{\mathrm{SO}(V_0)(\mathbf{Q})\backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \Phi(f, z_0, h) dh.$$

Evaluating the regularized theta lift using Proposition 4.3, then switching the order of summation of the absolutely convergent integrals and applying the Siegel-Weil formula (11), we find that

$$\begin{aligned}
\Phi(f, Z(V_0)) &= \frac{1}{\mathrm{vol}(K_0)} \int\limits_{\mathrm{SO}(V_0)(\mathbf{Q})\backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \lim_{T\to\infty} \left[\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \theta_{L_0}(\tau, z_0, h) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right] dh \\
&= \frac{1}{\mathrm{vol}(K_0)} \cdot \lim_{T\to\infty} \left[\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \left(\int\limits_{\mathrm{SO}(V_0)(\mathbf{Q})\backslash \mathrm{SO}(V_0)(\mathbf{A}_f)} \theta_{L_0}(\tau, z_0, h) dh \right) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right] \\
&= \frac{1}{\mathrm{vol}(K_0)} \cdot \lim_{T\to\infty} \left[\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1) \rangle\rangle d\mu(\tau) - A_0 \log(T) \right].
\end{aligned}$$

This proves (i). Starting with Lemma 4.2 (ii), then applying the regularized integral formula of Proposition 4.3 and the corresponding Siegel-Weil formula (12), we deduce (ii) in the same way. ☐

We obtain the following equivalent formulation, after expressing the regularized theta lifts on the left-hand sides in terms of the corresponding Borcherds products, and the limiting expressions on the right-hand sides as regularized theta integrals with the corresponding Eisenstein series.

Corollary 4.5. *Evaluating $\Phi(f, \cdot)$ in terms of its Borcherds product $\Psi(\cdot)$ via Theorem 2.5, we have*

(i)

$$\begin{aligned}
&- \sum_{(z,h)\in Z(V_0)} \left(2\log ||\Psi(f, z, h)||^2 + c_f(0,0) \left(2\log ||y|| + \Gamma'(1) \right) \right) \\
&= \frac{1}{\mathrm{vol}(K_0)} \cdot \lim_{T\to\infty} \left[\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1) \rangle\rangle d\mu(\tau) - \sum_{\lambda \in (L_0^\perp)^\vee / L_0^\perp} \sum_{\substack{x \in V_0^\perp(\mathbf{Q}) \\ x\in\mu}} c_f(-Q(x), \lambda) \log(T) \right] \\
&= \frac{1}{\mathrm{vol}(K_0)} \cdot \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1) \rangle\rangle d\mu(\tau).
\end{aligned}$$

(ii)

$$\begin{aligned}
&- \sum_{\substack{[h]\in G(W_A) \\ h\in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \frac{1}{\#\,\mathrm{Aut}(h)} \int\limits_{C_{A,h} = \Gamma_{A,h}\backslash D(W_A)} \left(2\log ||\Psi(f, z, h)||^2 + c_f(0,0) \left(2\log ||y|| + \Gamma'(1) \right) \right) d\nu(z) \\
&= \frac{1}{\mathrm{vol}(K_W)} \cdot \lim_{T\to\infty} \left[\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0) \rangle\rangle d\mu(\tau) - \sum_{\lambda \in (L_W^\perp)^\vee / L_W^\perp} \sum_{\substack{x \in W^\perp(\mathbf{Q}) \\ x\in\mu}} c_f(-Q(x), \lambda) \log(T) \right] \\
&= \frac{1}{\mathrm{vol}(K_W)} \cdot \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0) \rangle\rangle d\mu(\tau).
\end{aligned}$$

Proof. By Theorem 2.5, we obtain from (i) the formula

$$\Phi(f, Z(V_0)) = \sum_{(z,h)\in Z(V_0)} \Phi(f, z, h) = \sum_{(z,h)\in Z(V_0)} \left(-2\log ||\Psi(f, z, h)||^2 - c_f(0,0) \left(2\log ||y|| + \Gamma'(1) \right) \right).$$

We then restate Proposition 4.4 in terms of this latter identity. We derive (ii) in the same way, after interpreting the real geodesic cycle $\mathcal{G}(W)$ as an evaluation locus for the Borcherds product. $\square$

4.2. **Links to Rankin-Selberg central values for the case of $n = 2$.** We now interpret Theorem 4.4 and Corollary 4.5 in the case of $n = 2$, where the spin Shimura variety X_K is a quaternionic Hilbert modular surface defined over $\mathbf{Q}$. Here, we first obtain the following direct consequence of Proposition 3.1.

Theorem 4.6. *Let (V, Q) be any rational quadratic space of signature $(2, 2)$. Let $L \subset V$ be any integral lattice with corresponding Weil representation ω_L. Let $f \in S_0^!(\omega_L)$ be any weakly holomorphic cusp form of weight zero and representation ω_L.*

(i) *Given any negative definite subspace $V_0 \subset V$ with corresponding sublattice $L_0 = V_0 \cap L$ and positive definite complement $L_0^\perp \subset L$, we have the following formula for the central value $L(0, f \times \theta_{L_0^\perp})$ of the Rankin-Selberg L-function $L(s, f \times \theta_{L_0^\perp})$ described in Proposition 3.1 (i):*
$$L(0, f \times \theta_{L_0^\perp}) = -2\operatorname{vol}(K_0) \sum_{(z_0,h)\in Z(V_0)} \log||\Psi(f, z_0, h)||^2.$$

(ii) *Given any Lorentzian subspace $W \subset V$ with corresponding sublattice $L_W = W \cap L$ and complement $L_W^\perp \subset L$, we have the following formula for the central value $L(0, f \times \theta_{L_W^\perp})$ of the Rankin-Selberg L-function $L(s, f \times \theta_{L_W^\perp})$ described in Proposition 3.1 (ii):*
$$L(0, f \times \theta_{L_W^\perp}) = -2\operatorname{vol}(K_W) \sum_{\substack{[h]\in G(W)\\ h\in\mathrm{GSpin}(W)(\mathbf{A}_f)}} \frac{1}{\#\operatorname{Aut}(h)} \int_{C_{A,h}=\Gamma_{A,h}\backslash D(W_A)} \log||\Psi(f, z, h)||^2 d\nu(z).$$

Proof. By Corollary 4.5 (i) with Proposition 3.1, we have the relation
$$\begin{aligned}
&-2\sum_{(z_0,h)\in Z(V_0)} \log||\Psi(f, z_0, h)||^2\\
&= \frac{1}{\operatorname{vol}(K_0)} \cdot \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1)\rangle\rangle d\mu(\tau) - \sum_{\lambda\in(L_0^\perp)^\vee/L_0^\perp} \sum_{\substack{x\in V_0^\perp(\mathbf{Q})\\ x\in\mu}} c_f(-Q(x), \lambda)\log(T) \right]\\
&= \frac{1}{\operatorname{vol}(K_0)} \cdot \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes E_{L_0}(\tau, 0; -1)\rangle\rangle d\mu(\tau) = L(0, f \times \theta_{L_0^\perp}).
\end{aligned}$$
By Corollary 4.5 (ii) with Proposition 3.1, we have the relation
$$\begin{aligned}
&-2\sum_{\substack{[h]\in G(W)\\ h\in\mathrm{GSpin}(W)(\mathbf{A}_f)}} \frac{1}{\#\operatorname{Aut}(h)} \int_{C_{A,h}=\Gamma_{A,h}\backslash D(W_A)} \log||\Psi(f, z, h)||^2 d\nu(z)\\
&= \frac{1}{\operatorname{vol}(K_W)} \cdot \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0)\rangle\rangle d\mu(\tau) - \sum_{\lambda\in(L_W^\perp)^\vee/L_W^\perp} \sum_{\substack{x\in W^\perp(\mathbf{Q})\\ x\in\mu}} c_f(-Q(x), \lambda)\log(T) \right]\\
&= \frac{1}{\operatorname{vol}(K_0)} \cdot \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_W^\perp}(\tau) \otimes E_{L_W}(\tau, 0; 0)\rangle\rangle d\mu(\tau) = L(0, f \times \theta_{L_W^\perp}).
\end{aligned}$$
$\square$

We obtain the following relation to central values of Rankin-Selberg L-functions via Proposition 3.3. Here, we use the spaces (V_A, Q_A) defined above, with negative definite subspaces $V_{A,0} \subset V_A$ when $k = k(V_A)$ is imaginary quadratic, and Lorentzian subspaces $W_A \subset V_A$ when $k = k(V_A)$ is real quadratic.

Theorem 4.7. *Let $\phi \in S_0^!(\Gamma_0(N))$ be any weakly holomorphic modular function of level $\Gamma_0(N)$ and trivial character. Let $\chi \in C(\mathcal{O}_c)^\vee$ be a ring class character of conductor $c \geq 1$ of a quadratic field k. Assume that $(N, cd_k) = 1$. We have the following expressions for the central value $\Lambda(1/2, \phi \times \theta(\chi))$ of the completed*

Rankin-Selberg L-functions $\Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi))L(s, \phi \times \theta(\chi))$*. Given any class* $A \in C(\mathcal{O}_c)$ *with corresponding quadratic space* (V_A, Q_A) *as described in Definition 2.1 and lattice* $(L_A, Q_A) = (L_A(N), Q_A)$ *as described in Proposition 2.2 (v), let*

$$f_{\phi,A}(\tau) = \sum_{\mu \in L_A^\vee/L_A} \sum_{m \gg -\infty} c_{f_{\phi,A}}(\mu, m) e(m\tau) \mathbf{1}_\mu \in S_0^!(\overline{\omega}_{L_A})$$

denote the lifting of ϕ *to a vector-valued weakly holomorphic form, as described in Lemma 3.2.*

(i) *If* $k = k(U)$ *is an imaginary quadratic field, then have the central value formula*

$$\Lambda(1/2, \phi \times \theta(\chi)) = -2\Lambda(1, \eta_k) \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{A,0}}) \sum_{(z_0,h) \in Z(V_{A,0})} \log ||\Psi(f_{\phi,A}, z_0, h)||^2.$$

(ii) *If* $k = k(U)$ *is a real quadratic field, then we have the central value formula*

$$\begin{aligned}&\Lambda(1/2, \phi \times \theta(\chi)) \\ &= -2\Lambda(1, \eta_K) \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{W_A}}) \sum_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \frac{1}{\# \operatorname{Aut}(h)} \int_{C_{A,h} = \Gamma_{A,h} \backslash D(W_A)} \log ||\Psi(f_{\phi,A}, z, h)||^2 d\nu(z).\end{aligned}$$

Proof. We use Theorem 4.6 to evaluate the expression for the central value $\Lambda(1/2, \phi \times \theta(\chi))$ implied by Proposition 3.3 in each case. ☐

Using the Dirichlet analytic class number formula

$$L(1, \eta_k) = \begin{cases} \frac{2\pi h_k}{w_k \sqrt{|d_k|}} & \text{if } k \text{ is imaginary quadratic} \\ \frac{2 \ln(\varepsilon_k) h_k}{\sqrt{d_k}} & \text{if } k \text{ is real quadratic} \end{cases}$$

to evaluate

$$\Lambda(1, \eta_k) = |d_k|^{\frac{1}{2}} \Gamma_{\mathbf{R}}(2) L(1, \eta_k) = \begin{cases} \frac{2\pi h_k}{w_k} & \text{if } k \text{ is imaginary quadratic} \\ 2 \log(\varepsilon_k) h_k & \text{if } k \text{ is real quadratic} \end{cases},$$

we obtain the following

Corollary 4.8. *We have the following expressions for the central value* $\Lambda(1/2, \phi \times \theta(\chi))$ *of the completed Rankin-Selberg L-functions* $\Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi))L(s, \phi \times \theta(\chi))$*.*

(i) *If* $k = k(U)$ *is an imaginary quadratic field, then have the central value formula*

$$\Lambda(1/2, \phi \times \theta(\chi)) = -\frac{4\pi h_k}{w_k} \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{A,0}}) \sum_{(z_0,h) \in Z(V_{A,0})} \log ||\Psi(f_{\phi,A}, z_0, h)||^2.$$

(ii) *If* $k = k(U)$ *is a real quadratic field, then we have the central value formula*

$$\begin{aligned}&\Lambda(1/2, \phi \times \theta(\chi)) \\ &= -4 \log(\varepsilon_k) h_k \sum_{A \in C(\mathcal{O}_c)} \chi(A) \operatorname{vol}(K_{L_{W_A}}) \sum_{\substack{[h] \in G(W_A) \\ h \in \mathrm{GSpin}(W_A)(\mathbf{A}_f)}} \frac{1}{\# \operatorname{Aut}(h)} \int_{C_{A,h} = \Gamma_{A,h} \backslash D(W_A)} \log ||\Psi(f_{\phi,A}, z, h)||^2 d\nu(z)\end{aligned}$$

5. Applications to periods

We obtain the following arithmetic applications of Theorem 4.7 (i) and Corollary 4.8 (i) for the CM case.

5.1. **Central values as periods.** We derive the following consequence of Theorem 4.7 and Corollary 4.8, using that the Borcherds products $\Psi(f, z, h)$ are known to take algebraic values by [10, Theorem 9.1.1].

Theorem 5.1. *The following central values of Rankin-Selberg L-functions lie in the ring of periods.*

(a) *Let* (V, Q) *be any rational quadratic space of signature* $(2, 2)$ *with integral lattice* $L \subset V$ *and Weil representation* ω_L. *Let* $f \in S_0^!(\omega_M)$ *be any weakly holomorphic cusp form of weight* 0 *and type* ω_L *for* Γ. *Let* $L_0 \subset L$ *be any negative definite sublattice with orthogonal complement* $L_0^\perp \subset L$, *and let* $\theta_{L_0^\perp}(\tau) = \theta_{L_0^\perp}(\tau, 1, 1)$ *be the corresponding (holomorphic) Siegel theta series. Then, the central value* $L(0, f \times \theta_{L_0^\perp})$ *of the Rankin-Selberg L-function* $L(s, f \times \theta_{L_0^\perp})$ *described in Proposition 3.1 above is a period,* $L(0, f \times \theta_{L_0^\perp}) \in \mathcal{P}$, *as is the central value of the completed L-function* $L^\star(0, f \times \theta_{L_0^\perp}) \in \mathcal{P}$.

(b) *Let* $\phi \in S_0^!(\Gamma_0(N))$ *be a weakly holomorphic cusp form of weight zero for* $\Gamma_0(N)$. *Let* k *be any imaginary quadratic field of discriminant* d_k *prime to* N, *let* $c \geq 1$ *be any conductor prime to* N, *and let* $\chi \in C(\mathcal{O}_c)^\vee$ *be any ring class character of* k *of conductor* c. *Let* $\theta(\chi) \in M_1(\Gamma_0(|c^2 d_k|), \eta_k)$ *denote the corresponding Hecke theta series. The central value* $\Lambda(1/2, \phi \times \theta(\chi))$ *of the completed Rankin-Selberg L-function* $\Lambda(s, \phi \times \theta(\chi)) = L_\infty(s, \phi \times \theta(\chi)) L(s, \phi \times \theta(\chi))$ *is a period,* $\Lambda(1/2, \phi \times \theta(\chi)) \in \mathcal{P}$.

Proof. The proofs of (a) and (b) follow from Theorem 4.7 (i) and Corollary 4.8 (i) respectively, using that the Borcherds products appearing on the right-hand sides take algebraic values by [10, Theorem 9.1.1]. Here, we use the fact that the sums defining the CM cycles $Z(V_0)$ and $Z(V_{A,0})$ have finite cardinality, equal to $2h_k$. From this, we can deduce that the central values $L(0, f \times \theta_{L_0^\perp})$, $L^\star(0, f \times \theta_{L_0^\perp})$ and $\Lambda(1/2, \phi \times \theta(\chi))$ are logarithms of algebraic numbers, and hence periods. □

5.2. **Relation to Chowla-Selberg periods.** Finally, we can also deduce the following refinement of Schofer [13, Corollary 3.5 (ii)].

Theorem 5.2. *Let* k *be an imaginary quadratic field of odd discriminant* d_k, *character* η_k, *class number* h_k, *and number of roots of unity* w_k.

(a) *Let* (V, Q) *be any rational quadratic space of signature* $(2, 2)$ *with negative definite subspace* $V_0 \subset V$ *corresponding to this* $k = k(V_0)$. *We fix an integral lattice* $L \subset V$, *writing* $L_0 = V_0 \cap L$ *for its image in* V_0, *and* $L_0^\perp$ *its orthogonal complement. Let* $\theta_{L_0^\perp}(\tau)$ *denote the corresponding holomorphic Siegel theta series. Let* $f \in S_0^!(\omega_L)$ *be any weakly holomorphic cusp form of weight zero and type* ω_L *for* Γ. *The product over the CM cycle* $Z(V_0)$ *of the corresponding Borcherds product* $\Psi(f, \cdot)$ *factorizes as*

$$\prod_{(z,h) \in Z(V_0)} ||\Psi(f, z, h)||^2 = \alpha \left((4\pi d_k)^{-h_k} \cdot \prod_{a=1}^{d_k - 1} \Gamma\left(\frac{a}{d_k}\right)^{w_k \eta_k(a)} \right)^{\mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle}$$

$$= \alpha \left((4\pi d_k)^{-1} \cdot e^{2 \cdot \frac{L'(0, \eta_k)}{L(0, \eta_k)}} \right)^{h_k \cdot \mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle}$$

for some rational number $\alpha = \alpha(f, k) \in \mathbf{Q}$, *where the transcendental factors on the right are related to Shimura's period invariants* [14], [8], [19]. *Taking logarithms of both sides, we deduce that*

$$L(0, f \times \theta_{L_0^\perp}) = -2 \operatorname{vol}(K_{L_{A,0}}) \sum_{(z,h) \in Z(V_0)} \log ||\Psi(f, z, h)||^2$$

$$= -2 \operatorname{vol}(K_{L_{A,0}}) \log \left(\alpha \left((4\pi d_k)^{-h_k} \cdot \prod_{a=1}^{d_k - 1} \Gamma\left(\frac{a}{d_k}\right)^{w_k \eta_k(a)} \right)^{\mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle} \right)$$

$$= -2 \operatorname{vol}(K_{L_{A,0}}) \log \left(\alpha \left((4\pi d_k)^{-1} \cdot e^{2 \cdot \frac{L'(0, \eta_k)}{L(0, \eta_k)}} \right)^{h_k \cdot \mathrm{CT}\langle\langle f(\tau), \theta_{L_0^\perp}(\tau) \otimes \mathbf{1}_{L_0 + 0} \rangle\rangle} \right).$$

(b) *Let* $\phi \in S_0^!(\Gamma_0(N))$ *be a weakly holomorphic modular form of weight zero for* $\Gamma_0(N)$. *Assume that* $(N, d_k) = 1$, *and let* $c \geq 1$ *be a conductor prime to* N. *For each ring class* $A \in C(\mathcal{O}_c)$, *let* (V_A, Q_A)

be the corresponding quadratic space defined in Definition 2.1, with $L_A = L_A(N) \subset V_A$ *the integral lattice defined in Proposition 2.2 (v). We have a canonical negative definite subspace* $V_{A,0} \subset V_A$ *with lattice* $L_{A,0} = L_A \cap V_{A,0}$*. Let* $f_{\phi,A} \in S_0^!(\omega_{L_A})$ *the lifting of* ϕ *to a vector-valued form by Proposition 3.2, with* $\Psi(f_{\phi,A}, \cdot)$ *its corresponding Borcherds product. Let* $\chi \in C(\mathcal{O}_c)^\vee$ *be any ring class character of* k *of conductor* c*, with corresponding Hecke theta series* $\theta(\chi) \in M_1(\Gamma_0(|c^2 d_k|), \eta_k)$*. We have the central value formula for the Rankin-Selberg L-function* $\Lambda(s, \phi\times\theta(\chi)) = L_\infty(s, \phi\times\theta(\chi))L(s, \phi\times\theta(\chi))$*:*

$$\Lambda(1/2, \phi \times \theta(\chi)) = -\frac{4\pi h_k}{w_k} \sum_{A\in C(\mathcal{O}_c)} \chi(A)\,\mathrm{vol}(K_{L_{A,0}}) \sum_{(z_0,h)\in Z(V_{A,0})} \log ||\Psi(f_{\phi,A}, z_0, h)||^2.$$

$$= -\frac{4\pi h_k}{w_k} \sum_{A\in C(\mathcal{O}_c)} \chi(A)\,\mathrm{vol}(K_{L_{A,0}}) \log \left(\alpha(f_{\phi,A}, k) \left((4\pi d_k)^{-h_k} \cdot \prod_{a=1}^{d_k-1} \Gamma\left(\frac{a}{d_k}\right)^{w_k \eta_k(a)} \right)^{\mathrm{CT}\langle\langle f_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau)\otimes \mathbf{1}_{L_{A,0}+0}\rangle\rangle} \right)$$

$$= -\frac{4\pi h_k}{w_k} \sum_{A\in C(\mathcal{O}_c)} \chi(A)\,\mathrm{vol}(K_{L_{A,0}}) \log \left(\alpha(f_{\phi,A}, k) \left((4\pi d_k)^{-1} \cdot e^{2\cdot \frac{L'(0,\eta_k)}{L(0,\eta_k)}} \right)^{h_k \cdot \mathrm{CT}\langle\langle f_{\phi,A}(\tau), \theta_{L_{A,0}^\perp}(\tau)\otimes \mathbf{1}_{L_{A,0}+0}\rangle\rangle} \right).$$

Proof. We deduce these results from [13, Corollary 3.5] via Theorem 4.7 for (a) and Corollary 4.8 for (b). ☐

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