ON THE DIHEDRAL MAIN CONJECTURES OF IWASAWA THEORY FOR HILBERT MODULAR EIGENFORMS

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ABSTRACT. We construct a bipartite Euler system in the sense of Howard for Hilbert modular eigenforms of parallel weight two over totally real fields, generalizing works of Bertolini-Darmon, Longo, Nekovar, Pollack-Weston and others. The construction has direct applications to Iwasawa main conjectures. For instance, it implies in many cases one divisibility of the associated dihedral or anticyclotomic main conjecture, at the same time reducing the other divisibility to a certain nonvanishing criterion for the associated p-adic L-functions. It also has applications to cyclotomic main conjectures for Hilbert modular forms over CM fields via the technique of Skinner and Urban.

CONTENTS

1. INTRODUCTION

Let F be a totally real field of degree d, and fix a prime $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. Let $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ be a cuspidal Hilbert modular eigenform of parallel weight 2, level $\mathfrak{N} \subset \mathcal{O}_F$, and trival character. Assume that $\mathbf{f}$ is $\mathfrak{p}$-ordinary, in the sense that its $T_{\mathfrak{p}}$-eigenvalue is a p-adic unit with respect to any fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. Assume as well that $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}) = 1$, with $\mathbf{f}$ being either new of level $\mathfrak{N}$, or

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else arising from a newform of level $\mathfrak{N}/\mathfrak{p}$. Let us always view $\mathbf{f}$ is a p-adic modular form via a fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$, writing $\mathcal{O}_0$ to denote the $\mathbf{Z}_p$-subalgebra of $\overline{\mathbf{Q}}_p$ generated by the Fourier coefficients of $\mathbf{f}$, $\mathcal{O}$ the integral closure of $\mathcal{O}_0$ in its field of fractions L, and $\mathfrak{P}$ the maximal ideal of $\mathcal{O}$. We assume for simplicity that $\mathfrak{P}$ is contained in $\mathcal{O}_0$. Fix a totally imaginary quadratic extension K of F, with relative discriminant prime to $\mathfrak{N}$. The choice of K then determines the following factorization of $\mathfrak{N}$ in $\mathcal{O}_F$:

$$\mathfrak{N} = \mathfrak{p}\mathfrak{N}^+\mathfrak{N}^-, \tag{1}$$

where $\mathfrak{N}^+$ is divisible only by primes that split in K, and $\mathfrak{N}^-$ is divisible only by primes that remain inert in K. Assume that $\mathfrak{N}^-$ is the squarefree product of a number of primes congruent to d mod 2. In this setting, the root number of the Rankin-Selberg L-function $L(\mathbf{f}, K, s)$ at its central value $s = 1$ is equal to 1. Moreover, the central value (as well as those of the associated twists by ring class characters) can be described by the toric integral formula of Waldspurger [67], as generalized for instance by Yuan-Zhang-Zhang [70]. Ultimately, this formula can be used to study the arithmetic behaviour of $\mathbf{f}$ in the dihedral or anticyclotomic $\mathbf{Z}_p^{\delta}$-extension $K_{\mathfrak{p}^\infty}$ of K, where δ denotes the index $[F_{\mathfrak{p}} : \mathbf{Q}_p]$. That is, let $G_{\mathfrak{p}^\infty}$ denote the Galois group $\mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)$, with $\Lambda = \mathcal{O}[[G_{\mathfrak{p}^\infty}]]$ the associated $\mathcal{O}$-Iwasawa algebra. Using these toric integral formulae, as well as the class field theoretic description of $G_{\mathfrak{p}^\infty}$, there is a natural construction of the associated p-adic L-function $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty}) \in \Lambda$, as shown in the prequel paper [62] (following the constructions of Bertolini-Darmon [3], [2]). In particular, in addition to satisfying the usual interpolation property, this p-adic L-function is nontrivial thanks to the nonvanishing theorem of Cornut and Vatsal [16, Theorem 1.4]. The main purpose of the present work is to use this construction to prove the one divisibility of the associated dihedral or anticyclotomic main conjecture, as well as to outline some applications beyond this. To be more precise, let $\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ denote the $\mathfrak{P}^\infty$-Selmer group of $\mathbf{f}$ in $K_{\mathfrak{p}^\infty}/K$, with $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$ its Pontryagin dual. The Iwasawa main conjecture in this setting predicts that $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$ is a torsion Λ module, and moreover that there is an equality of principal ideals $(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) = (\mathrm{char}_\Lambda(X(\mathbf{f}, K_{\mathfrak{p}^\infty})))$ in Λ. Here, $\mathrm{char}_\Lambda(X(\mathbf{f}, K_{\mathfrak{p}^\infty}))$ is the Λ-characteristic power series of $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$, which exists (by the structure theorem of [6]) as $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$ is Λ-torsion. We show the following results towards this conjecture. Let us first impose the following hypotheses, writing $\rho_{\mathbf{f}} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O})$ to denote the $\mathfrak{P}$-adic Galois representation associated to $\mathbf{f}$ by the construction of Carayol [10], Taylor [60] and Wiles [69] (see Theorem 4.1 below). Here, G_F denotes the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$.

Hypothesis 1.1.

(i) *The prime p is odd.*
(ii) *The prime $\mathfrak{p} \subset \mathcal{O}_F$ is the unique prime above p in $K_{\mathfrak{p}^\infty}$.*
(iii) *The eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ is $\mathfrak{p}$-ordinary.*
(iv) *The Galois representation $\rho_{\mathbf{f}}$ is residually irreducible.*
(v) *The image of the residual Galois representation $\overline{\rho}_{\mathbf{f}}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$.*
(vi) *The degree d is either odd, or else even with the condition that $\mathfrak{N}^- \neq \mathcal{O}_F$.*

Remark Thanks to Dimitrov [19, Proposition 0.1], we have the following generalizations of relevant results of Serre [55] and Ribet [54] here: (i) for all but finitely many rational primes p, the Galois representation $\rho_{\mathbf{f}}$ is residually irreducible ([19, Proposition 3.1]), and (ii) for all but finitely many rational primes p, there exists some power $q = p^a$ of p such that the image of the residual Galois representation $\overline{\rho}_{\mathbf{f}}$ contains $\mathrm{SL}_2(\mathbf{F}_q)$ ([18, Proposition 3.8]). Thus, Hypotheses 1.1 (iv) and (v) are not prohibitively strong.

Theorem 1.2 (Proposition 7.5, Corollary 7.6)**.** *Let $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ be a cupsidal Hilbert eigenform as above, with $\mathfrak{N} \subset \mathcal{O}_F$ having the factorization* (1)*, and with the conditions of Hypothesis 1.1. Assume also that the following standard hypotheses hold:*

(A) *The totally real field F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$,*
(B) *The Galois representation $\rho_{\mathbf{f}}$ satisfies a certain multiplicity one condition: Hypothesis 11.5.*
(C) *A variant of Ihara's lemma for Shimura curves holds: Hypothesis 11.12.*

Then, $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$ is Λ-torsion, and there is an inclusion of ideals

$$(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) \subseteq (\mathrm{char}_\Lambda(X(\mathbf{f}, K_{\mathfrak{p}^\infty}))) \quad \text{in } \Lambda. \tag{2}$$

Remark on hypotheses. We refer the reader to the statements of Hypotheses 11.5 and 11.12 below for more details. We state them here in this form not only for simplicity of exposition, but also because they are at present works in progress by others (see for instance [12] and [13]). Condition (A) is used to prove a level raising at two primes result, see Proposition 11.9 below. Condition (B) is a standard hypothesis (cf. [50, Proposition 6.3]) that is crucial to our arguments. It is proved with our hypotheses on $\rho_{\mathbf{f}}$ given above granted that p is unramified in F by Cheng in [12]. Condition (C) is also treated by Cheng in [13], assuming that the level $\mathfrak{N}$ is sufficiently large, and that $p > d$. It is likely that these latter two technical hypotheses can be loosened.

Remark on μ-invariants. We can also deduce from (2) one divisibility in the μ-part of the main conjecture, following the characterization of the μ-invariant associated to $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ in the author's previous work [62] (see [62, Theorem 4.10]). Roughly, following the approach of Vatsal [63], we find that $\mu(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) = 2\nu$, where $\nu = \nu_{\mathbf{f}}$ is the largest integer such that $\mathbf{f}$ is congruent to a constant mod $\mathfrak{P}^\nu$. Following the line of argument of Pollack-Weston [50, §2.3], the hypothesis that $\rho_{\mathbf{f}}$ be residually irreducible should indicate that $\mu(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) = 0$, and hence via (2) that $\mu(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) = \mu(\mathrm{char}_\Lambda(X(\mathbf{f}, K_{\mathfrak{p}^\infty}))) = 0$. We hope to take up a more detailed study of this interesting and subtle issue in a later work, perhaps in the context of Euler characteristic computations (cf. [61]).

The strategy of proof is to generalize the refined Euler system method of Pollack-Weston [50] (following Bertolini-Darmon [3]) to the setting of totally real fields. In doing so, we construct a bipartite Euler system in the sense of Howard [29, Definition 2.3.2]. In particular, we obtain from [29, Theorem 3.2.3] the following criterion for equality in (2), as explained in [62, §5]. Let us now assume for simplicity that $\mathfrak{N}$ is prime to the relative discriminant of K over F. Fix a positive integer k. Define a set of admissible primes $\mathfrak{L}_k$ of F, each being inert in K, by the condition that for any ideal $\mathfrak{n} \subset \mathcal{O}_F$ in the set $\mathfrak{S}_k$ of squarefree products of primes in $\mathfrak{L}_k$, there exists a nontrivial eigenform $\mathbf{f}^{(\mathfrak{n})}$ of level $\mathfrak{n}\mathfrak{N}$ such that the following congruence on Hecke eigenvalues holds:

$$\mathbf{f}^{(\mathfrak{n})} \equiv \mathbf{f} \mod \mathfrak{P}^k.$$

Let $\mathfrak{S}_k^+ \subset \mathfrak{S}_k$ denote the subset of ideals $\mathfrak{n} \in \mathfrak{S}_k$ for which $\omega_{K/F}(\mathfrak{n}\mathfrak{N}) = -1$, where $\omega_{K/F}$ denotes the quadratic Hecke character associated to K/F. Equivalently, $\mathfrak{S}_k^+ \subset \mathfrak{S}_k$ denotes the subset of ideals $\mathfrak{n} \in \mathfrak{S}_k$ for which the root number of the $L(\mathbf{f}, K, s)$ is equal to $+1$. Note that by our hypotheses of $\mathfrak{N}$, this set $\mathfrak{S}_k^+$ includes the trivial ideal $\mathfrak{n} = \mathcal{O}_F$. Let $\mathfrak{S}_k^- \subset \mathfrak{S}_k$ denote the subset of ideals $\mathfrak{n} \in \mathfrak{S}_k$ for which $\omega_{K/F}(\mathfrak{n}\mathfrak{N}) = +1$, equivalently for which the root number of $L(\mathbf{f}, K, s)$ is equal to -1. Given an ideal $\mathfrak{n} \in \mathfrak{S}_k^+$, there is an associated p-adic L-function $\mathcal{L}_\mathfrak{p}(\mathbf{f}^{(\mathfrak{n})}, K_{\mathfrak{p}^\infty})$ in Λ. As explained below, $\mathcal{L}_\mathfrak{p}(\mathbf{f}^{(\mathfrak{n})}, K_{\mathfrak{p}^\infty}) = \mathcal{L}_{\mathbf{f}^{(\mathfrak{n})}} \mathcal{L}^*_{\mathbf{f}^{(\mathfrak{n})}}$, where $\mathcal{L}_{\mathbf{f}^{(\mathfrak{n})}} \in \Lambda$ is a completed group ring element constructed in a natural way from $\mathbf{f}^{(\mathfrak{n})}$, and $\mathcal{L}^*_{\mathbf{f}^{(\mathfrak{n})}}$ is the image of $\mathcal{L}_{\mathbf{f}^{(\mathfrak{n})}}$ under the involution $\Lambda \rightarrow \Lambda$ sending σ to σ^{-1} in $G_{\mathfrak{p}^\infty}$. Let us write $\lambda_\mathfrak{n}$ to denote this completed group ring element $\mathcal{L}_{\mathbf{f}^{(\mathfrak{n})}}$, which is only well defined up to multiplication by elements of $G_{\mathfrak{p}^\infty}$. Given an ideal $\mathfrak{n} \in \mathfrak{S}_k^-$, there is an associated collection of CM points of $\mathfrak{p}$-power conductor on the quaternionic Shimura curve $\mathfrak{M}(\mathfrak{N}^+, v\mathfrak{n}\mathfrak{N}^-)$, where v is a k-admissible prime with respect to $\mathbf{f}$, as we explain in §§7 − 11 below. As we also explain below, these points can be used to construct classes in the cohomology group $H^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},k})$, which we denote here by $\kappa_\mathfrak{n}$ (i.e. so that $\kappa_{v\mathfrak{n}} = \zeta(\mathfrak{n})$ in our notations below). We refer the reader to the discussion below for more explanation, as well as to §4 for a definition of the mod $\mathfrak{P}^k$ Galois representation $T_{\mathbf{f},k}$ associated to $\mathbf{f}$. Anyhow, we construct for each integer $k \geq 1$ a pair of families

$$\{\lambda_\mathfrak{n} \in \Lambda/\mathfrak{P}^k\Lambda : \mathfrak{n} \in \mathfrak{S}_k^+\} \quad \text{and} \quad \{\kappa_\mathfrak{n} \in \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},k}) : \mathfrak{n} \in \mathfrak{S}_k^-\} \tag{3}$$

which, as k varies, are compatible with respect to the inclusion $\mathfrak{S}_{k+1} \subset \mathfrak{S}_k$, as well as with respect to the natural maps $T_{\mathbf{f},k+1} \rightarrow T_{\mathbf{f},k}$ and $\Lambda/\mathfrak{P}^{k+1} \rightarrow \Lambda/\mathfrak{P}^k$. We show here that these classes satisfy the following *first and second explicit reciprocity laws*:

The first explicit reciprocity law (Theorem 13.1). For any $v\mathfrak{n} \in \mathfrak{S}_k^-$ with v a prime, there is an isomorphism of Λ-modules

$$\widehat{H}^1_{\text{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},k}) \cong \Lambda/\mathfrak{P}^k\Lambda$$

sending $\text{loc}_v(\kappa_{v\mathfrak{n}})$ to $\lambda_\mathfrak{n}$, where loc_v denotes the localization map at v.

The second explicit reciprocity law (Theorem 13.2). For any $v\mathfrak{n} \in \mathfrak{S}_k^+$ with v a prime, there is an isomorphism of Λ-modules

$$\widehat{H}^1_{\text{unr}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},k}) \cong \Lambda/\mathfrak{P}^k\Lambda$$

sending $\text{loc}_v(\kappa_\mathfrak{n})$ to $\lambda_{v\mathfrak{n}}$, where loc_v denotes the localization map at v

Now, since the empty product lies in $\mathfrak{S}_k$ for each integer $k \geq 1$, we can construct a distinguished element

$$\begin{cases} \lambda^\infty \in \Lambda & \text{if } \omega_{K/F}(\mathfrak{N}^-) = -1 \\ \kappa^\infty \in \mathfrak{S}(\mathbf{f}/K_{\mathfrak{p}^\infty}) & \text{if } \omega_{K/F}(\mathfrak{N}^-) = +1 \end{cases}$$

by taking the inverse limit of λ_1 or κ_1 as k varies. Here, $\mathfrak{S}(\mathbf{f}/K_{\mathfrak{p}^\infty})$ denotes the compactified Selmer group of $\mathbf{f}$ over $K_{\mathfrak{p}^\infty}$. Note that while the element κ^∞ has been studied independently by Howard in [30], it can also be recovered directly from the construction given below. Note as well that by the nonvanishing theorems of Cornut and Vastal [16], neither of these distinguished elements vanishes. Hence, we deduce that the pair of families (3) defines a nontrivial bipartite Euler system in the sense of Howard [29, Definition 2.3.2]. In particular, via Howard's theory of bipartite Euler systems, we obtain the following result. Here, given any eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ with $\mathfrak{N} \subset \mathcal{O}_F$ having the factorization (1), we assume Hypothesis 1.1 along with the hypotheses (A), (B) and (C) of Theorem 1.2.

Theorem 1.3. *Let $X(\mathbf{f}, K_{\mathfrak{p}^\infty})_{\mathrm{tors}}$ denote the Λ-torsion submodule of $X(\mathbf{f}, K_{\mathfrak{p}^\infty})$.*

(i) *We have the following rank formula:*

$$\mathrm{rank}_\Lambda \mathfrak{S}(\mathbf{f}/K_{\mathfrak{p}^\infty}) = \mathrm{rank}_\Lambda X(\mathbf{f}, K_{\mathfrak{p}^\infty}) = \begin{cases} 0 & \text{if } \omega_{K/F}(\mathfrak{N}^-) = -1 \\ 1 & \text{if } \omega_{K/F}(\mathfrak{N}^-) = +1. \end{cases}$$

(ii) *For each height one prime $\mathfrak{Q}$ of Λ,*

$$\mathrm{ord}_{\mathfrak{Q}}\left(\mathrm{char}_\Lambda\left(X(\mathbf{f}, K_{\mathfrak{p}^\infty})_{\mathrm{tors}}\right)\right) \leq$$
$$2 \times \begin{cases} \mathrm{ord}_{\mathfrak{Q}}(\lambda^\infty) & \text{if } \omega_{K/F}(\mathfrak{N}^-) = -1 \\ \mathrm{ord}_{\mathfrak{Q}}\left(\mathrm{char}_\Lambda\left(\mathfrak{S}(\mathbf{f}/K_{\mathfrak{p}^\infty})/\Lambda\kappa^\infty\right)\right) & \text{if } \omega_{K/F}(\mathfrak{N}^-) = +1. \end{cases}$$

(iii) *Equality holds in (ii) if the following condition is satisfied: there exists an integer k_0 such that for all integers $k \geq k_0$, the set*

$$\{\lambda_{\mathfrak{n}} \in \Lambda/\mathfrak{P}^k\Lambda : \mathfrak{n} \in \mathfrak{S}_k^+\}$$

contains at least one element with nontrivial image in $\Lambda/(\mathfrak{Q}, \mathfrak{P}^{k_0})$. In particular, equality in (ii) holds if one of the elements $\lambda_{\mathfrak{n}}$ is a unit in Λ.

Proof. The result follows from the proof of Howard [29, Theorem 3.2.3], which carries over to this setting with minor changes. See also the discussion in [30]. We sketch the deduction for lack of better reference. First, note that the general theory of Euler systems over Artinian ring developed in Howard [29, §2] applies to this setting. In particular, [29, Proposition 3.3.1] (cf. [44, Lemma 5.3.13]) and [29, Proposition 3.3.3] carry over to this setting. The result of [29, Lemma 3.3.2] is also standard here, see for instance [28, Theorem 7.1] , using the basic fact that $A(K_{\mathfrak{p}^\infty})_{p^\infty} \subset A(K_{\mathfrak{p}^\infty})_{\mathrm{tors}}$ is finite for any abelian variety A defined over $K_{\mathfrak{p}^\infty}$, in particular for the abelian variety $A_{\mathbf{f}}$ associated to $\mathbf{f}$ in Proposition 4.2 below. The proof of Howard [29, Theorem 3.2.3 (c)] can then be given by the argument of [29, § 3.4], with minor modifications, following [30, §3.3]. That is, fix a height one prime ideal $\mathfrak{Q}$ of Λ. Fix a sequence of specializations $\phi_i : \Lambda \longrightarrow S$, in the sense of [30, Definition 3.2.5]. Suppose that this sequence converges to $\mathfrak{Q}$, following [30, Definition 3.3.3]. Note that such a sequence always exists by [30, Proposition 3.3.3.]. The argument of [29, §3.4 p. 21] can then be modified by taking tensors $\otimes_\Lambda S$ as done in [30, §3.4] to obtain the analogous result of [29, Theorem 3.2.3] in this setting. □

Combined, Theorems 1.2 and 1.3 imply following criterion for equality in (2).

Corollary 1.4. *Suppose that for each height one prime ideal $\mathfrak{Q}$ of Λ, there exists a positive integer k_0 such that for each integer $k \geq k_0$, the set $\mathfrak{S}_k^+$ contains an ideal $\mathfrak{n}$ for which the image of the associated completed group ring element $\lambda_{\mathfrak{n}}$ in the quotient $\Lambda/(\mathfrak{Q}, \mathfrak{P}^{k_0})$ is not trivial. Then, there is an equality of ideals in (2),* *i.e. the full dihedral (or anticyclotomic) main conjecture of Iwasawa theory holds:*

$$(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) = (\mathrm{char}_{\Lambda}(X(\mathbf{f}, K_{\mathfrak{p}^\infty}))) \text{ in } \Lambda. \tag{4}$$

In particular, if one of the completed group ring elements $\lambda_{\mathfrak{n}}$ *is a unit in* Λ*, then the full main conjecture equality* (4) *holds.*

Some further remarks are in order at this point. The result of Theorem 1.2 has many antecendents in the literature, among them the original work of Bertolini-Darmon [3], as well as subsequent generalizations to totally real fields by Longo ([43], [42] and [41]), Fouquet [23] and Nekovar [49]. The main novelty here is that the remove the restrictive p-isolatedness hypotheses found in these works of Bertolini-Darmon and Longo, following the approach of Pollack-Weston [50]. This innovation is not merely technical, as it allows is to invoke the theory of bipartite Euler systems due to Howard [29] to both reduce the other divisibility of the main conjecture to a nonvanishing criterion for p-adic L-functions, as well as to treat both definite and indefinite cases on the root number simultaneously. Perhaps more intriguingly, Theorem 1.2 above can also be combined with techniques of Skinner-Urban [57] to give a new proof of the associated cyclotomic main conjecture, which previously had only been accessible by the Euler system method of Kato [36]. Moreover, it seems that the techniques of Skinner-Urban [57] extend to the more general setting of totally real fields (by work in progress of [68]), in which case the result of Theorem 1.2 would allow one to deduce the associated cyclotomic main conjecture for totally real fields, which at present is not accessible even by the method of Kato's Euler system.

Application to modular abelian varieties. We obtain the following consequence for modular abelian varieties. Let A be an abelian variety over F of arithmetic conductor $\mathfrak{N} \subset \mathcal{O}_F$. Given any integer $n \geq 1$ and any Galois extension L over F with $\mathfrak{P} \mid \mathfrak{p}$ a prime above $\mathfrak{p}$ in L, we can associate to A a residual Selmer group $\mathrm{Sel}_{\mathfrak{P}^n}(A/L)$, defined by its inclusion in the exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathfrak{P}^n}(A/L) \longrightarrow H^1(L, A[\mathfrak{P}^n]) \longrightarrow \bigoplus_v H^1(L_v, A[\mathfrak{P}^n])/\mathrm{im}(\mathfrak{K}_v).$$

Here, the sum runs over all primes $v \subset \mathcal{O}_L$, and

$$\mathfrak{K}_v : A(L_v)/\mathfrak{P}^n A(L_v) \longrightarrow H^1(L_v, A[\mathfrak{P}^n])$$

denotes the local Kummer map at v. Now, an abelian variety A/F is said to be of GL_2-*type* if the endomorphism algebra $\mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ contains a number field L of degree equal to $\dim(A)$. An abelian variety A defined over F of GL_2-type is said to be *modular* if there exists a Hilbert modular eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ such that the Galois representation

$$\rho_{A,\lambda} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O}_L) \cong \mathrm{Aut}\left(\mathrm{Ta}_{\lambda}\left(A\right)\right)$$

associated to the λ-adic Tate module $\mathrm{Ta}_{\lambda}(A)$ of A is equivalent to the Galois representation

$$\rho_{\mathbf{f},\lambda} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O}_L)$$

associated to $\mathbf{f}$ by the construction of Carayol [10], Taylor [60] and Wiles [69] (Theorem 4.1 below) for any prime $\lambda \subset \mathcal{O}_L$, where $\mathcal{O}_L$ contains all of the Fourier coefficients of $\mathbf{f}$. One can make analogous definitions for the unramified and ordinary local cohomology groups $H^1_{\mathrm{unr}}(K_v, A[\mathfrak{P}^n]) \subset H^1(K_v, A[\mathfrak{P}^n])$ and $H^1_{\mathrm{ord}}(K_v, A[\mathfrak{P}^n]) \subset$

$H^1(K_v, A[\mathfrak{P}^n])$ as given below for Hilbert modular eigenforms. See for instance the discussion in [42, §4]. Rather than give them here, let us just state the following characterization.

Proposition 1.5. *If A/F is a modular abelian variety associated to an eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ as above, then we have the following description of $\operatorname{im}(\mathfrak{K}_v)$:*

(i) $\operatorname{im}(\mathfrak{K}_v) = H^1_{\mathrm{unr}}(L_v, A_{\mathbf{f},n})$ *if* $v \nmid \mathfrak{N} \subset \mathcal{O}_F$.
(ii) $\operatorname{im}(\mathfrak{K}_v) = H^1_{\mathrm{ord}}(L_v, A_{\mathbf{f},n})$ *if* $v = \mathfrak{p} \subset \mathcal{O}_F$.

Here, $A_{\mathbf{f},n}$ is the mod $\mathfrak{P}^n$ Galois representation arising from the abelian variety $A_{\mathbf{f}}$ associated to $\mathbf{f}$, as defined in §4 below.

Proof. The result is well known, see for instance [14] or [42, §4.1]. □

Let
$$\mathrm{Sel}_{\mathfrak{P}^\infty}(A/K_{\mathfrak{p}^\infty}) = \varinjlim_n \mathrm{Sel}_{\mathfrak{P}^n}(A/K_{\mathfrak{p}^\infty}),$$
where the limit is taken with respect to the natural maps $A[\mathfrak{P}^n] \to A[\mathfrak{P}^{n+1}]$. Let $X(A/K_{\mathfrak{p}^\infty}) = \mathrm{Hom}\,(\mathrm{Sel}_{\mathfrak{P}^\infty}(A/K_{\mathfrak{p}^\infty}), \mathbf{Q}_p/\mathbf{Z}_p)$. By Proposition 1.5, we can identify $\mathrm{Sel}_{\mathfrak{P}^\infty}(A/K_{\mathfrak{p}^\infty}) = \mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ to obtain the following result.

Corollary 1.6. *Let A/F be a modular abelian variety. If the eigenform $\mathbf{f}$ associated to A satisfies all of the conditions of Theorem 1.2 above, then the dual Selmer group $X(A/K_{\mathfrak{p}^\infty})$ is Λ-torsion, and there is an inclusion of ideals*
$$(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) \subseteq (\mathrm{char}_\Lambda\, X(A/K_{\mathfrak{p}^\infty})) \ \text{ in } \Lambda.$$

Note as well that the analogous formulations of Theorem 1.3 and Corollary 1.4 carry over to the setting of modular abelian varieties. Now, consider the short exact descent sequence

$$0 \longrightarrow A(K_{\mathfrak{p}^\infty}) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathrm{Sel}_{\mathfrak{P}^\infty}(A/K_{\mathfrak{p}^\infty}) \longrightarrow Ш(A/K_{\mathfrak{p}^\infty})[\mathfrak{P}^\infty] \longrightarrow 0. \tag{5}$$

Here, $Ш(A/K_{\mathfrak{p}^\infty})[\mathfrak{P}^\infty]$ denotes the $\mathfrak{P}$-primary part of the Tate-Shafarevich group $Ш(A/K_{\mathfrak{p}^\infty})$ of A over $K_{\mathfrak{p}^\infty}$.

Corollary 1.7. *Let A/F be a modular abelian variety. If the eigenform $\mathbf{f}$ associated to A satisfies all of the conditions of Theorem 1.2 above, then for ρ any finite order character of $G_{\mathfrak{p}^\infty}$ for which the specialization $\rho^{-1}(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty}))$ does not vanish, the components $A(K_{\mathfrak{p}^\infty})^\rho$, $\mathrm{Sel}_{\mathfrak{P}^\infty}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\rho$, and $Ш(A/K_{\mathfrak{p}^\infty})[\mathfrak{P}^\infty]^\rho$ are finite.*

Proof. This is a direct consequence of Corollary 1.6 applied to (5). □

Note that by the nonvanishing theorem of Cornut-Vatsal [16, Theorem 1.4], the nonvanishing hypothesis of Corollary 1.7 is satisfied for for all but finitely many finite order characters ρ of $G_{\mathfrak{p}^\infty}$, as can be deduced as can be deduced from the algebraicity theorem of Shimura [56].

Notations. We write $\mathbf{A}_F$ to denote the adeles of F, with $\mathbf{A} = \mathbf{A}_{\mathbf{Q}}$, and $\mathbf{A}_f$ the finite adeles of $\mathbf{Q}$. We shall sometimes write $\widehat{F}^\times$ to denote the finite adeles of F. Given a finite prime v of F, we fix a uniformizer ϖ_v of F_v. We let κ_v denote the residue fields of F_v at v, with $q = q_v$ its cardinality, which is not to be confused with the cohomology class constructed in (64) below. Throughout, we write $\mathcal{O}_0$ to denote the $\mathbf{Z}_p$-algebra generated by the images of the Fourier coefficients of $\mathbf{f}$, and $\mathcal{O}$ the integral closure of $\mathcal{O}_0$ in its fraction field L. We let $\mathfrak{P}$ denote the maximal ideal of $\mathcal{O}$, and for each integer $n \geq 1$ put $\mathfrak{P}_n = \mathfrak{P}^n \cap \mathcal{O}_0$.

2. Automorphic forms

Hilbert modular forms. Given an ideal $\mathfrak{N} \subset \mathcal{O}_F$, let $\mathcal{S}_2(\mathfrak{N})$ denote the space of cuspidal Hilbert modular forms of parallel weight 2, level $\mathfrak{N}$, and trivial character. The space $\mathcal{S}_2(\mathfrak{N})$ comes equipped with the action of standard (classically or adelically defined) operators T_v for each prime $v \nmid \mathfrak{N} \subset \mathcal{O}_F$ and U_v for each prime $v \mid \mathfrak{N} \subset \mathcal{O}_F$. Let $\mathbf{T}(\mathfrak{N})$ denote the $\mathbf{Z}$-algebra generated by these operators. Given a Hilbert modular form $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$, let $a_{\mathfrak{m}}(\mathbf{f})$ denote the normalized Fourier coefficient of $\mathbf{f}$ at an ideal $\mathfrak{m}$ of F. We refer to [24] [25], or [26] for precise definitions and further background.

Definition A Hilbert modular form $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ is said to be a *normalized eigenform* if it is a simultaneous eigenvector for all of the Hecke operators T_v and U_v, with $T_v\mathbf{f} = a_v(\mathbf{f}) \cdot \mathbf{f}$ if $v \nmid \mathfrak{N}$, $U_v\mathbf{f} = a_v(\mathbf{f}) \cdot \mathbf{f}$ if $v \mid \mathfrak{N}$, and $a_{\mathcal{O}_F}(\mathbf{f}) = 1$.

Definition A normalized eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ is said to be a *newform* if there does not exist a form $\mathbf{g} \in \mathcal{S}_2(\mathfrak{M})$ for $\mathfrak{M} \mid \mathfrak{N} \subset \mathcal{O}_F$ an ideal not equal to $\mathfrak{N}$ such that $a_{\mathfrak{n}}(\mathbf{f}) = a_{\mathfrak{n}}(\mathbf{g})$ for all ideals $\mathfrak{n} \subset \mathcal{O}_F$ prime to $\mathfrak{N}$. Given a prime $\mathfrak{q} \mid \mathfrak{N}$, we say that $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ is *new at* $\mathfrak{q}$ if it does not arise from another form $\mathbf{g} \in \mathcal{S}_2(\mathfrak{N}/\mathfrak{q})$ in this way.

In general, given a normalized eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$, we fix a factorization $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$ such that $\mathbf{f}$ is new at all primes dividing $\mathfrak{N}^-$. We then write $\mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ to denote the space of Hilbert modular cusp forms of parallel weight 2 and level $\mathfrak{N}$ that are new at all primes dividing $\mathfrak{N}^-$, with $\mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$ the corresponding algebra of Hecke operators. Finally, let us also make the following

Definition An eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ is $\mathfrak{p}$-*ordinary* if its $T_{\mathfrak{p}}$-eigenvalue $a_{\mathfrak{p}}(\mathbf{f})$ is a p-adic unit with respect to any fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. In this case, if $\mathfrak{p} \nmid \mathfrak{N}$, then there exists a p-adic unit root $\alpha_{\mathfrak{p}}(\mathbf{f})$ to the polynomial

$$x^2 - a_{\mathfrak{p}}(\mathbf{f})x + q, \tag{6}$$

where q denotes the cardinality of the residue field at $\mathfrak{p}$.

In what follows, let us always view $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ as a p-adic modular form via a fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. Let $\mathcal{O}_0$ denote the $\mathbf{Z}_p$-subalgebra of $\overline{\mathbf{Q}}_p$ generated by the Fourier coefficients of $\mathbf{f}$, let $\mathcal{O}$ denote the integral closure of $\mathcal{O}_0$ in its field of fractions L, and let $\mathfrak{P}$ denote the maximal ideal of $\mathcal{O}$.

The Jacquet-Langlands correspondence. Let B denote a quaternion algebra defined over F, with $\mathrm{Ram}(B)$ the set of places of F where B is ramified. The theorem of Jacquet and Langlands [32] establishes a bijection from the space of automorphic representations of $(B \otimes \mathbf{A}_F)^\times$ of dimension greater than 1 to the space of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ that are discrete series (i.e. square integrable) at each place $v \in \mathrm{Ram}(B)$. This injection being a bijection on its image, we obtain the following more concrete result. Fix a compact open subgroup $H \subset \widehat{B}^\times$. Let $\mathbb{S}_2(H, B)$ denote the space of automorphic forms of weight 2 and level H on B (to be defined precisely later), with the additional assumption that the forms are cuspidal if B is an indefinite quaternion algebra. The space $\mathbb{S}_2(H, B)$ comes equipped with actions of standard Hecke operators at each prime $v \subset \mathcal{O}_F$ (to be defined precisely later). Let $\mathbb{T}(H, B)$ denote the $\mathbf{Z}$-algebra generated by these operators. Suppose that the discriminant $\mathrm{disc}(B)$ of B is equal to some ideal $\mathfrak{N}^- \subset \mathcal{O}_F$, and that the level of H is equal to some ideal $\mathfrak{N}^+ \subset \mathcal{O}_F$. Let us then write $\mathbb{S}_2(\mathfrak{N}^+, \mathfrak{N}^-) = \mathbb{S}_2(H, B)$ with $\mathbb{T}(\mathfrak{N}^+, \mathfrak{N}^-) = \mathbb{T}(H, B)$. The correspondence of Jacquet and Langlands then establishes an isomorphism of Hecke modules $\mathbb{S}_2(\mathfrak{N}^+, \mathfrak{N}^-) \cong \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$.

Automorphic forms on totally definite quaternion algebras. Let $\mathcal{O}$ be any ring. Let D be any totally definite quaternion algebra defined over F. Fix a compact open subgroup $U \subset \widehat{D}^\times$. Let $\mathbb{S}_2(U;\mathcal{O}) = \mathbb{S}_2(U, D;\mathcal{O})$ denote the space of $\mathcal{O}$-valued automorphic forms of weight 2 and level U on D, i.e. the space of functions

$$\Phi : D^\times \backslash \widehat{D}^\times / U \longrightarrow \mathcal{O}$$

such that $\Phi(dgu) = \Phi(g)$ for all $d \in D^\times$, $g \in \widehat{D}^\times$, and $u \in U$. Let $\mathbb{S}_2(U;\mathcal{O})_{\mathrm{triv}} \subset \mathbb{S}(U;\mathcal{O})$ denote the subspace of functions that factor through the adelization of the reducted norm homomorphism, $\mathrm{nrd} : \widehat{D}^\times \longrightarrow F^\times$.

Definition Let $\mathcal{S}_2(U;\mathcal{O}) = \mathbb{S}_2(U;\mathcal{O})/\mathbb{S}_2(U;\mathcal{O})_{\mathrm{triv}}$. Functions in this space are called *$\mathcal{O}$-valued modular forms of weight 2 and level U on D.*

The space $\mathbb{S}(U;\mathcal{O})$ comes equipped with actions of Hecke operators, defined via double coset operators as follows. Given two compact open subgroups $U, U' \subset \widehat{D}^\times$ and an element $g \in \widehat{D}^\times$, the group gUg^{-1} is commensurable with U'. Fixing a decomposition of U' into a disjoint union of cosets $\coprod_i \alpha_i(U' \cap gUg^{-1})$ gives an identification $U'gU = \coprod_i g_i U$, where $g_i = \alpha_i g$. The associated double coset operator $[U'gU]$ is then given by the linear map

$$[U'gU] : \mathbb{S}_2(U;\mathcal{O}) \longrightarrow \mathbb{S}_2(U';\mathcal{O}),\ \left([U'gU]\Phi\right)(x) = \sum_i \Phi(xg_i).$$

Definition Fix a compact open subgroup $U \subset \widehat{D}^\times$. Fix a finite set of places $S \supset \mathrm{Ram}(D)$ of F such that U admits a decomposition $U_S \times U^S$. The Hecke algebra $\mathbb{T}^S(U) = \mathbb{T}^S(U, B)$ is the (commutative) subring of $\mathbf{Z}[U \backslash \widehat{D}^\times / U]$ generated by double coset operators $[UgU]$ with $g \in (\widehat{D}^S)^\times$. It is isomorphic as a ring to $\mathbf{Z}[T_w, S_w, S_w^{-1} : w \notin S]$, where T_w and S_w are the standard Hecke operators $T_w = [U\eta_w U]$ and $S_w = [U\varpi_w U]$.

Fix a prime $v \notin S$ of F. Hence, D is split at v, and we may fix an isomorphism $D_v \cong \mathrm{M}_2(F_v)$. Let us assume additionally that this isomorphism sends the component U_v to $\mathrm{GL}_2(\mathcal{O}_{F_v})$. In what follows, we shall make the identification $U_v \cong \mathrm{GL}_2(\mathcal{O}_{F_v})$ implicitly. Let $U(v) \subset U$ be the subgroup defined by

$$U(v) = \{u \in U : u_v \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \varpi_v\}.$$

We have a pair of natural degeneracy maps

$$\alpha^* = [U(v)\mathbf{1}U] : \mathbb{S}_2(U;\mathcal{O}) \longrightarrow \mathbb{S}_2(U(v);\mathcal{O}),\ (\alpha^*\Phi)(g) = \Phi(g)$$
$$\beta^* = [U(v)\eta_v U] : \mathbb{S}_2(U;\mathcal{O}) \longrightarrow \mathbb{S}_2(U(v);\mathcal{O}),\ (\beta^*\Phi)(g) = \Phi(g\eta_v),$$

as well as a pair of associated trace maps

$$\alpha_* = [U\mathbf{1}U(v)] : \mathbb{S}_2(U(v);\mathcal{O}) \longrightarrow \mathbb{S}_2(U;\mathcal{O})$$
$$\beta_* = [U\eta_v U(v)] : \mathbb{S}_2(U(v);\mathcal{O}) \longrightarrow \mathbb{S}_2(U;\mathcal{O}).$$

These degeneracy and trace maps commute with the actions of Hecke operators T_w (for $w \notin \mathrm{Ram}(D) \cup \{v\}$) on $\mathbb{S}_2(U;\mathcal{O})$ and $\mathbb{S}_2(U(v);\mathcal{O})$, as well as with the action by the centre of $\overline{D}^\times$.

Definition The *v-new susbspace of* $\mathbb{S}_2(U(v);\mathcal{O})$ is the subspace defined by

$$\mathbb{S}_2(U(v);\mathcal{O})^{\text{v-new}} = \ker(\mathbb{S}_2(U(v);\mathcal{O}) \xrightarrow{\alpha_*,\beta_*} \mathbb{S}_2(U;\mathcal{O})^{\oplus 2}).$$

This subspace is stable under the actions of Hecke operators T_w (for $w \notin \mathrm{Ram}(D) \cup \{v\}$), as well as under the action of the centre of $\widehat{D}^\times$.

Shimura curves. Fix a place τ_1 in the set of archimedean places $\{\tau_1, \ldots, \tau_d\}$ of F. Let B be any quaternion algebra over F that is split at τ_1 and ramified at the remaining set of real places $\{\tau_2, \ldots, \tau_d\}$. Let $X = \mathbf{C} - \mathbf{R}$, which is two copies of the Poincaré upper-half plane. The group $B^\times \subset B^\times_{\tau_1} \cong \mathrm{GL}_2(\mathbf{R})$ acts naturally on X, via fractional linear transformation. Let $H \subset \widehat{B}^\times$ be any compact open subgroup. The diagonal left action of $B^\times$ on $\widehat{B}^\times/H \times X$ defines a Riemann surface

$$M_H(\mathbf{C}) = M_H(B,X)(\mathbf{C}) = B^\times \backslash \widehat{B}^\times \times X/H. \tag{7}$$

Shimura proved that the curve $M_H(\mathbf{C})$ has a canonical model defined over the totally real field F. We adopt the standard convention of writing M_H to denote this model, whose complex points are identified with the Riemann surface $M_H(\mathbf{C})$. The curve M_H is irreducible, but not necessarily geometrically irreducible. Indeed, by strong approximation and the theorem of the norm, the reduced norm homomorphism $\mathrm{nrd} : B \longrightarrow F$ is seen to induce a bijection of finite sets

$$\pi_0\left(M_H(\mathbf{C})\right) \cong F_+^\times \backslash \widehat{F}^\times/\mathrm{nrd}(H), \tag{8}$$

as explained for instance in [9, § 1.2].

We have the following description of Hecke operators acting on M_H. Given $g \in \widehat{B}^\times$, fix a pair of compact open subgroups $H, H' \subset \widehat{B}^\times$ such that $g^{-1}Hg \subset H'$. Multiplication on the right by g induces natural maps $M_H(\mathbf{C}) \rightarrow M_{H'}(\mathbf{C})$ which descend to finite flat morphisms $M_H \rightarrow M_{H'}$. We may then define from these maps Hecke operators via double coset operators. To be more precise, given a complex point $x = [g,h] \in M_H(\mathbf{C})$, the double coset operator $[HgH]$ acts as

$$[HgH]\,(x) = \sum_i [xg_i, h] \in \mathrm{Div}(M_H(\mathbf{C})),$$

where $HgH = \coprod_i g_i H$. The algebra of Hecke operators

$$\mathbb{T}_H = \mathrm{End}_{\mathbf{Z}[\widehat{B}^\times]}\left(\mathbf{Z}[\widehat{B}^\times/H]\right) \cong \mathbf{Z}[H\backslash \widehat{B}^\times/H]$$

has a natural left action on the jacobian J_H of M_H. Hence, we also write $\mathbb{T}_H$ to denote the subring of $\mathrm{End}(J_H)$ generated by these Hecke operators.

Automorphic forms on indefinite quaternion algebras. Let us now fix an indefinite quaternion algebra B as above, ramified at all but but one real place τ_1 of F. Let us also fix a compact open subgroup $H \subset \widehat{B}^\times$. Recall that we let $M_H(\mathbf{C})$ denote the associated complex Shimura curve (7), with M_H its canonical model defined over F.

Definition Let $\mathbb{S}_2(H,B)$ denote the space of functions $\Phi : (B \otimes \mathbf{A}_F)^\times \longrightarrow \mathbf{C}$ such that

(i) Φ is left $B^\times$-invariant.
(ii) Φ is right invariant under $\mathbf{R}^\times \times \prod_{i=2}^d B^\times_{\tau_i} \subset (B \otimes \mathbf{R})^\times$.
(iii) Φ is right invariant under H.

(iv) For each $g \in (B \otimes \mathbf{A}_F)^\times$ and $\theta \in \mathbf{R}$,
$$\Phi\left(g\left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, 1, \ldots, 1\right]\right) = \exp(2i\theta)\cdot\Phi(g).$$

(v) For each $g \in (B \otimes \mathbf{A}_F)^\times$, the function defined by
$$z = x + iy \longmapsto \Phi(g, z) := \frac{1}{y}\cdot\Phi\left(g\left[\begin{pmatrix} -y & x \\ 0 & 1 \end{pmatrix}, 1, \ldots, 1\right]\right)$$
is holomorphic on the lower half plane $\mathfrak{H}^-$.

Note that there is a left action of $g \in \widehat{B}^\times$ on $\mathbb{S}_2(H, B)$ via the rule $(g\cdot\Phi)(x) = \Phi(xg)$. We refer the reader to [16, §3.6] for more details.

Let Ω_H denote the sheaf of differentials on the Shimura curve M_H, with $\Omega_H(\mathbf{C})$ its pullback to $M_H(\mathbf{C})$. Hence, $\Omega_H(\mathbf{C})$ is the sheaf of holomorphic 1-forms on $M_H(\mathbf{C})$. Let $\Gamma(\Omega_H(\mathbf{C}))$ denote the global sections of $\Omega_H(\mathbf{C})$.

Proposition 2.1. *There is a $\widehat{B}^\times$-equivariant bijection of $\mathbf{C}$ -vector spaces $\Gamma(\Omega_H(\mathbf{C})) \cong \mathbb{S}_2(H, B)$.*

Proof. The identification is standard, see for instance [16, Proposition 3.10]. □

Let J_H denote the Jacobian of M_H, with $J_H^*(0)$ the complex cotangent space of J_H at 0.

Corollary 2.2. *There is a $\widehat{B}^\times$-equivariant bijection of $\mathbf{C}$-vector spaces $J_H^*(0) \cong \mathbb{S}_2(H, B)$. In particular, there is an identification of the associated Hecke algebras: $\mathbb{T}_H = \mathbb{T}(H, B)$.*

Proof. This follows from the (canonical) identification of $J_H^*(0)$ with $\Gamma(\Omega_H(\mathbf{C}))$. □

3. p-adic L-functions

We sketch here the construction of p-adic L-functions given in [62].

The integer factorization. Recall that we fix a prime $\mathfrak{p} \subset \mathcal{O}_F$, with p the underlying rational prime. Fix an integral ideal $\mathfrak{N}_0$ with $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}_0) \leq 1$. Let

$$\mathfrak{N} = \begin{cases} \mathfrak{N}_0 & \text{if } \mathfrak{p} \mid \mathfrak{N}_0 \\ \mathfrak{p}\mathfrak{N}_0 & \text{if } \mathfrak{p} \nmid \mathfrak{N}_0. \end{cases} \tag{9}$$

Hence, $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{N}) = 1$. Fix a totally imaginary quadratic extension K of F, with relative discriminant prime to $\mathfrak{N}/\mathfrak{p}$. The choice of K then determines a unique factorization

$$\mathfrak{N} = \mathfrak{p}\mathfrak{N}^+\mathfrak{N}^-$$

of $\mathfrak{N}$ in $\mathcal{O}_F$, with $v \mid \mathfrak{N}^+$ if and only if v is split in K, and $v \mid \mathfrak{N}^-$ if and only if v is inert in K. Let us assume additionally that $\mathfrak{N}^-$ is the squarefree product of a number of primes congruent to $d \mod 2$. Hence, there exists a totally definite quaternion algebra D say of discriminant $\mathfrak{N}^-$ defined over F. Observe that D is split at $\mathfrak{p}$ by hypothesis. Hence, we can and do fix an isomorphism $\iota_{\mathfrak{p}} : D_{\mathfrak{p}} \cong \mathrm{M}_2(F_{\mathfrak{p}})$. Fix a cuspidal Hilbert modular eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$. Let us assume that $\mathbf{f}$ is either a newform, or else that it arises from a newform of level $\mathfrak{N}/\mathfrak{p}$ via the process of $\mathfrak{p}$-stablization. Fix a compact open subgroup $U \subset \widehat{D}^\times$ of level $\mathfrak{M} \subset \mathcal{O}_F$ prime to $\mathfrak{N}^-$ (we shall often just take $\mathfrak{M} = \mathfrak{p}\mathfrak{N}^+$), assumed to be maximal at $\mathfrak{p}$.

Ring class towers. Given an ideal $\mathfrak{c} \subset \mathcal{O}_F$, let $\mathcal{O}_{\mathfrak{c}} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ denote the $\mathcal{O}_F$-order of conductor $\mathfrak{c}$ of K. Let $K[\mathfrak{c}]$ denote the abelian extension of K characterized by class field theory via the isomorphism:

$$\widehat{K}^\times / \widehat{\mathcal{O}}_{\mathfrak{c}}^\times K^\times \xrightarrow{\mathrm{rec}_K} \mathrm{Gal}(K[\mathfrak{c}]/K).$$

Here, rec_K denotes the Artin reciprocity map, normalized to send uniformizers to geometric Frobenius elements. Write $G[\mathfrak{c}]$ to denote the Galois group $\mathrm{Gal}(K[\mathfrak{c}]/K)$. Let $K[\mathfrak{p}^\infty] = \bigcup_{n \geq 0} K[\mathfrak{p}^n]$ with Galois group $G[\mathfrak{p}^\infty] = \mathrm{Gal}(K[\mathfrak{p}^\infty])$. Hence, $G[\mathfrak{p}^\infty]$ has the structure of a profinite group, $G[\mathfrak{p}^\infty] = \varprojlim_n G[\mathfrak{p}^n]$. It is well known that the torsion subgroup $G[\mathfrak{p}^\infty]_{\mathrm{tors}} \subset G[\mathfrak{p}^\infty]$ is finite, and moreover that the quotient $G[\mathfrak{p}^\infty]/G[\mathfrak{p}^\infty]_{\mathrm{tors}}$ is topologically isomorphic to $\mathbf{Z}_p^\delta$, where $\delta = [F_{\mathfrak{p}} : \mathbf{Q}_p]$ (see for instance [16, Corollary 2.2]). Let $G_{\mathfrak{p}^\infty} = G[\mathfrak{p}^\infty]/G[\mathfrak{p}^\infty]_{\mathrm{tors}}$ denote the $\mathbf{Z}_p^\delta$ quotient of $G[\mathfrak{p}^\infty]$. Let $K_{\mathfrak{p}^\infty}$ denote the dihedral or anticyclotomic $\mathbf{Z}_p^\delta$ of K, so that $G_{\mathfrak{p}^\infty} = \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \cong \mathbf{Z}_p^\delta$. Given a positive integer n, let $K_{\mathfrak{p}^n}$ denote the extension of K for which $G_{\mathfrak{p}^n} = \mathrm{Gal}(K_{\mathfrak{p}^n}/K) \cong (\mathbf{Z}/p^n\mathbf{Z})^\delta$, so that $G_{\mathfrak{p}^\infty} = \varprojlim_n G_{\mathfrak{p}^n}$.

Strong approximation. Fix a set of representatives $\{x_i\}_{i=1}^h$ for the modified class group $\mathfrak{Cl}_F/F_{\mathfrak{p}}^\times$, where $\mathfrak{Cl}_F$ denotes the narrow class group

$$\mathfrak{Cl}_F = F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times$$

of F, with the condition that $(x_i)_{\mathfrak{p}} = 1$ for each $i = 1, \ldots h$. A standard consequence of the strong approximation theorem ([66, § III.4, Thm. 4.3]) with the theorem of the norm ([66, § III.4, Thm. 4.1]) shows that there is a canonical bijection

$$\coprod_{i=1}^h D^\times \xi_i D_{\mathfrak{p}}^\times U \xrightarrow{\eta_{\mathfrak{p}}} \widehat{D}^\times.$$

Here, each ξ_i is an element of $\widehat{D}^\times$ such that $(\xi_i)_{\mathfrak{p}} = 1$ and $\mathrm{nrd}(\xi_i) = x_i$. For each $i = 1, \ldots h$, let us then define a subgroup

$$\Gamma_i = \{d \in D^\times : d_v \in (\xi_{i,v}) U_v (\xi_{i,v})^{-1} \text{ for all } v \nmid \mathfrak{p}\} \subset D^\times. \tag{10}$$

A standard argument shows that these subgroups $\Gamma_i \subset D^\times$ embed discretely into $D_{\mathfrak{p}}^\times$. Hence, via our fixed isomorphism $\iota_{\mathfrak{p}} : D_{\mathfrak{p}}^\times \cong \mathrm{GL}_2(F_{\mathfrak{p}})$, we can and do view these subgroups as discrete subgroups of $\mathrm{GL}_2(F_{\mathfrak{p}})$. Now, $\eta_{\mathfrak{p}}$ induces a canonical bijection (which we also denote by $\eta_{\mathfrak{p}}$)

$$\coprod_{i=1}^h \Gamma_i \backslash D_{\mathfrak{p}}^\times / U_{\mathfrak{p}} \xrightarrow{\eta_{\mathfrak{p}}} D^\times \backslash \widehat{D}^\times / U$$

via the map given on each component by $[d] \mapsto [\xi_i \cdot d]$. Hence, we can view each modular form $\Phi \in \mathcal{S}_2(H; \mathcal{O}) = \mathcal{S}_2^D(H; \mathcal{O})$ as an h-tuple of functions $\left(\phi^i\right)_{i=1}^h$ on $\mathrm{GL}_2(F_{\mathfrak{p}})$ such that

$$\phi^i(\gamma d u z) = \phi^i(d)$$

for each $i = 1, \ldots, h$, with $\gamma \in \Gamma_i$, $d \in D_{\mathfrak{p}}^\times$, $u \in U_{\mathfrak{p}}$, and $z \in \widehat{F}_{\mathfrak{p}}^\times$. A simple argument shows that these functions ϕ^i factor through homothety classes of full rank lattices of $F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$, and hence can be viewed as functions on the edgeset of the Bruhat-Tits tree of $D_{\mathfrak{p}}^\times / F_{\mathfrak{p}}^\times \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$. To be more precise, let $\mathcal{T}_{\mathfrak{p}} = (\mathcal{V}_{\mathfrak{p}}, \mathcal{E}_{\mathfrak{p}})$ denote the Bruhat-Tits tree of $B_{\mathfrak{p}}^\times / F_{\mathfrak{p}}^\times \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$, which is the tree of maximal orders of $\mathrm{M}_2(F_{\mathfrak{p}}) \cong B_{\mathfrak{p}}$ such that

(i) The vertex set $\mathcal{V}_{\mathfrak{p}}$ is indexed by maximal orders of $\mathrm{M}_2(F_{\mathfrak{p}})$.
(ii) The edgeset $\mathcal{E}_{\mathfrak{p}}$ is indexed by Eichler orders of level $\mathfrak{p}$ of $\mathrm{M}_2(F_{\mathfrak{p}})$.

(iii) The edgeset $\mathcal{E}_{\mathfrak{p}}$ has an orientation, i.e. a pair of maps

$$s, t : \mathcal{E}_{\mathfrak{p}} \longrightarrow \mathcal{V}_{\mathfrak{p}}, \ \mathfrak{e} \mapsto (s(\mathfrak{e}), t(\mathfrak{e}))$$

that assigns to each edge $\mathfrak{e} \in \mathcal{E}_{\mathfrak{p}}$ a *source* $s(\mathfrak{e})$ and a *target* $t(\mathfrak{e})$. Once such a choice of orientation is fixed, let us write $\mathcal{E}_{\mathfrak{p}}^*$ to denote the so-called "directed" edgeset of $\mathcal{T}_{\mathfrak{p}}$.

The group $D_{\mathfrak{p}}^\times / F_{\mathfrak{p}}^\times$ acts naturally by conjugation on $\mathcal{T}_{\mathfrak{p}}$. It is a standard result that this action is transitive, and moreover that there is an identification $\mathcal{V}_{\mathfrak{p}} \cong \mathrm{PGL}_2(F_{\mathfrak{p}})/\mathrm{PGL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$. In particular, we see that the discrete subgroups $\Gamma_i \subset D_{\mathfrak{p}}^\times \cong \mathrm{GL}_2(F_{\mathfrak{p}})$ modulo $F_{\mathfrak{p}}^\times$ act transitively by conjugation on $\mathcal{T}_{\mathfrak{p}}$. Now, each quotient graph $\Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}$ is a finite graph. Hence, we may consider the disjoint union of finite quotient graphs

$$\coprod_{i=1}^h \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}} = \left(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{V}_{\mathfrak{p}}, \coprod_{i=1}^h \Gamma_i \backslash \mathcal{E}_{\mathfrak{p}}^* \right).$$

Definition Let $\mathcal{S}_2 \left(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathcal{O} \right)$ denote the space of vectors $\left(\phi^i\right)_{i=1}^h$ of $\mathcal{O}$-valued, $(\Gamma_i)_{i=1}^h$-invariant functions on $\mathcal{T}_{\mathfrak{p}}$.

Remark Here, it is understood that $\Phi \in \mathcal{S}_2 \left(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathcal{O} \right)$ is a function on $\coprod_{i=1}^h \Gamma_i \backslash \mathcal{V}_{\mathfrak{p}}$ if $\mathfrak{p} \nmid \mathfrak{M}$, and a function on $\coprod_{i=1}^h \Gamma_i \backslash \mathcal{E}_{\mathfrak{p}}^*$ if $\mathfrak{p} \mid \mathfrak{M}$. We refer the reader to the discussion in [62, § 3] for more explanation.

A simple argument ([62, Proposition 3.6]) shows that the canonical bijection $\eta_{\mathfrak{p}}$ induces a bijection of $\mathcal{O}$-modules

$$\mathcal{S}_2 \left(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathcal{O} \right) \longrightarrow \mathcal{S}_2(U; \mathcal{O}). \tag{11}$$

Let us for simplicity of notation write Φ to denote both a modular form in $\mathcal{S}_2(U;\mathcal{O})$, as well as its correponding vector of functions $(\phi^i)_{i=1}^h$ in the space $\mathcal{S}_2 \left(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathcal{O} \right)$. We obtain from (11) the following combinatorial description of the standard Hecke operators $T_{\mathfrak{p}}$ and $U_{\mathfrak{p}}$ acting on $\mathcal{S}_2(H;\mathcal{O})$. Here, $U_{\mathfrak{p}}$ denotes the standard Hecke operator defined in [62, § 3], and *not* the compact open subgroup $U \subset \widehat{D}^\times$ defined above.

Case I: $\mathfrak{p} \nmid \mathfrak{M}$. Let $\Phi(\mathfrak{v})$ denote evaluation of the corresponding h-tuple of functions $\left(\phi^i\right)_{i=1}^h$ at a vertex $\mathfrak{v} \in \mathcal{V}_{\mathfrak{p}}$. Then,

$$\left(T_{\mathfrak{p}} \Phi\right)(\mathfrak{v}) = \sum_{\mathfrak{w} \rightarrow \mathfrak{v}} c_\Phi(\mathfrak{w}). \tag{12}$$

Here, the sum ranges over all $q+1$ vertices $\mathfrak{w}$ adjacent to $\mathfrak{v}$.

Case II: $\mathfrak{p} \mid \mathfrak{M}$. Let $\Phi(\mathfrak{e})$ denote evaluation of the corresponding h-tuple of functions $\left(\phi^i\right)_{i=1}^h$ at a directed edge $\mathfrak{e} \in \mathcal{E}_{\mathfrak{p}}^*$. Then,

$$\left(U_{\mathfrak{p}} \Phi\right)(\mathfrak{e}) = \sum_{s(\mathfrak{e}')=t(\mathfrak{e})} c_\Phi(\mathfrak{e}'). \tag{13}$$

Here, the sum runs over the $q+1$ edges $\mathfrak{e}' \in \mathcal{E}_{\mathfrak{p}}^*$ such that $s(\mathfrak{e}') = t(\mathfrak{e})$, minus the edge obtained by reversing orientation.

We also obtain the following explicit version of the Jacquet-Langlands correspondence induced by the bijection $\eta_{\mathfrak{p}}$. That is, fix an integral ideal $\mathfrak{N} \subset \mathcal{O}_F$ as defined in (9) above, with underlying integral ideal $\mathfrak{N}_0 \subset \mathcal{O}_F$. Fix a Hilbert modular eigenform $\mathbf{f}_0 \in \mathcal{S}_2(\mathfrak{N}_0)$ that is new at all primes dividing the ideal $\mathfrak{N}^-$.

Definition Let $\mathfrak{N}_0 \subset \mathcal{O}_F$ be an integral ideal that is not divisible by $\mathfrak{p}$. The $\mathfrak{p}$-*stabilization* $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ of $\mathbf{f}_0 \in \mathcal{S}_2(\mathfrak{N}_0)$ is the eigenform given by

$$\mathbf{f} = \mathbf{f}_0 - \beta_{\mathfrak{p}}(\mathbf{f}_0) \cdot (T_{\mathfrak{p}}\mathbf{f}_0)\,, \tag{14}$$

where $\beta_{\mathfrak{p}}(\mathbf{f}_0)$ denotes the non-unit root to (6). This is a $\mathfrak{p}$-ordinary eigenform in $\mathcal{S}_2(\mathfrak{N})$ with $U_{\mathfrak{p}}$-eigenvalue $\alpha_{\mathfrak{p}} = \alpha_{\mathfrak{p}}(\mathbf{f}_0)$.

We now consider an eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ that is given by $\mathbf{f}_0$ if $\mathfrak{p}$ divides $\mathfrak{N}_0$, or given by the $\mathfrak{p}$-stabilization of $\mathbf{f}_0$ if $\mathfrak{p}$ does not divide $\mathfrak{N}_0$. We have the following quaternionic description of $\mathbf{f}$ in either case. To be consistent with the notations above, let us write U_v to denote the Hecke operator T_v acting on $S_2^B(H;\mathcal{O})$ at a prime $v \mid \mathfrak{N}^+$ (again not to be confused with the component at v of the level structure $U \subset \widehat{D}^\times$).

Proposition 3.1 (Jacquet-Langlands)**.** *Given an eigenform* $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ *as defined above, there exists a function* $\Phi \in \mathcal{S}_2\left(\coprod_{i=1}^{\mathfrak{h}} \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathbf{C}\right)$ *such that*

- $T_v \Phi = a_v(\mathbf{f}) \cdot \Phi$ *for all* $v \nmid \mathfrak{N}$.
- $U_v \Phi = \alpha_v(\mathbf{f}) \cdot \Phi$ *for all* $v \mid \mathfrak{N}^+$.
- $U_{\mathfrak{p}} \Phi = \alpha_{\mathfrak{p}} \cdot \Phi$.

This function is unique up to multiplication by non-zero complex numbers. Conversely, given an eigenform $\Phi \in \mathcal{S}_2\left(\coprod_{i=1}^{\mathfrak{h}} \Gamma_i \backslash \mathcal{T}_{\mathfrak{p}}; \mathbf{C}\right)$*, there exists an eigenform* $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ *such that*

- $T_v \mathbf{f} = a_v(\Phi) \cdot \mathbf{f}$ *for all* $v \nmid \mathfrak{N}$.
- $U_v \mathbf{f} = \alpha_v(\Phi) \cdot \mathbf{f}$ *for all* $v \mid \mathfrak{N}^+$.
- $U_{\mathfrak{p}} \mathbf{f} = \alpha_{\mathfrak{p}}(\Phi) \cdot \mathbf{f}$.

Here, $a_v(\Phi)$ *denotes the eigenvalue for* T_v *of* Φ *if* $v \nmid \mathfrak{N}$*, and* $\alpha_v(\Phi)$ *the eigenvalue for* U_v *of* Φ *if* $v \mid \mathfrak{N}$*.*

Proof. See [62, Proposition 3.7], which is a direct generalization of [3, Proposition 1.3] to totally real fields. □

Construction of measures. We now sketch the construction of p-adic measures given in [62], which generalizes that of [3, § 1.2]. Fix an integral ideal $\mathfrak{N} \subset \mathcal{O}_F$ having the factorization (9). Recall that we write D to denote the totally definite quaternion algebra of discriminant $\mathfrak{N}^-$ defined over F. Let Z denote the maximal $\mathcal{O}_F[\frac{1}{\mathfrak{p}}]$-order of K. Let $R \subset D$ denote an Eichler $\mathcal{O}_F[\frac{1}{\mathfrak{p}}]$-order of level $\mathfrak{N}^+$. Let us fix an *optimal embedding* Ψ of Z into R, i.e. an injective F-algebra homomorphism $\Psi: K \longrightarrow D$ such that $\Psi(K) \cap R = \Psi(Z)$. Such an embedding exists if and only if K is split at all primes dividing the level of R ([66, § II.3]). Hence, such an embedding exists by our choice of integer factorization (9).

The Galois group $G[\mathfrak{p}^\infty]$ has a natural action on the directed edgeset $\mathcal{E}_\mathfrak{p}^*$ of $\mathcal{T}_\mathfrak{p}$. That is, the reciprocity map rec_K induces a bijection

$$\widehat{K}^\times / \left(K^\times \prod_{v \nmid \mathfrak{p}} Z_v^\times \right) \xrightarrow{\ r_K\ } G[\mathfrak{p}^\infty].$$

Passing to the adelization, the optimal embedding Ψ induces an embedding

$$\widehat{K}^\times / \left(K^\times \prod_{v \nmid \mathfrak{p}} Z_v^\times \right) \xrightarrow{\ \widehat{\Psi}\ } D^\times \backslash \widehat{D}^\times / \prod_{v \nmid \mathfrak{p}} R_v^\times .$$

Consider the compact open subgroup of $\widehat{D}^\times$ defined by $\prod_{v \nmid \mathfrak{p}} R_v^\times$, with associated subgroups Γ_i as defined in (10). As explained above, strong approximation induces a canonical bijection

$$\coprod_{i=1}^{\mathfrak{h}} \Gamma_i \backslash D_\mathfrak{p}^\times / F_\mathfrak{p}^\times \xrightarrow{\ \eta_\mathfrak{p}\ } D^\times \backslash \widehat{D}^\times / \prod_{v \nmid \mathfrak{p}} R_v^\times .$$

The composition $\eta_\mathfrak{p}^{-1} \circ \widehat{\Psi} \circ r_K^{-1}$ then gives rise to a natural action $\star$ of the Galois group $G[\mathfrak{p}^\infty]$ on the Bruhat-Tits tree $\mathcal{T}_\mathfrak{p} = (\mathcal{V}_\mathfrak{p}, \mathcal{E}_\mathfrak{p}^*)$: it is the induced conjugation action on maximal orders of $D_\mathfrak{p} \cong \mathrm{M}_2(F_\mathfrak{p})$. This action factors through that of $K_\mathfrak{p}^\times / F_\mathfrak{p}^\times$ on $\mathcal{T}_\mathfrak{p}$ via the local optimal embedding $\Psi_\mathfrak{p} : K_\mathfrak{p} \longrightarrow D_\mathfrak{p}$. Moreover, it fixes a single vertex if $\mathfrak{p}$ is inert in K, or no vertex if $\mathfrak{p}$ is split in K. Given an integer $n \geq 1$, let us write $\mathcal{U}_n$ to denote the standard compact open subgroup of level n of $K_\mathfrak{p}^\times / F_\mathfrak{p}^\times$,

$$\mathcal{U}_n = \left(1 + \mathfrak{p}^n \mathcal{O}_K \otimes \mathcal{O}_{F_p}\right)^\times / \left(1 + \mathfrak{p}^n \mathcal{O}_{F_p}\right)^\times .$$

We can then fix a sequence of consecutive edges $\{\mathfrak{e}_j\}_{j \geq 1}$ in $\mathcal{E}_\mathfrak{p}^*$ such that

$$\mathrm{Stab}_{K_\mathfrak{p}^\times / F_\mathfrak{p}^\times}(\mathfrak{e}_j) = \mathcal{U}_j .$$

Now, the choice of an eigenform $\Phi \in \mathcal{S}_2(\coprod_{i=1}^h \Gamma_i \backslash \mathcal{E}_\mathfrak{p}^*; \mathcal{O})$ determines a pairing

$$\begin{aligned} [\ ,\ ]_\Phi : G[\mathfrak{p}^\infty] \times \mathcal{E}_\mathfrak{p}^* &\longrightarrow \mathcal{O} \\ (\sigma, \mathfrak{e}) &\longmapsto \Phi\left(\eta_\mathfrak{p}^{-1} \circ \widehat{\Phi} \circ r_K^{-1}(\sigma) \star \mathfrak{e} \right). \end{aligned}$$

Let $\mathcal{H}_\infty$ denote the group $\mathrm{rec}_K^{-1}(G[\mathfrak{p}^\infty])$, with profinite structure $\mathcal{H}_\infty = \varprojlim_n \mathcal{H}_n$, where $\mathcal{H}_n = \mathcal{H}_\infty / \mathcal{U}_n$. Since the $U_\mathfrak{p}$-eigenvalue $\alpha_\mathfrak{p}$ is invertible in the ring of values $\mathcal{O}$, the pairing $[\ ,\ ]_\Phi$ can be shown to give rise to a natural $\mathcal{O}$-valued measure ϑ_Φ on $\mathcal{H}_\infty$ via the rule

$$\vartheta_\Phi(\sigma \mathcal{U}_j) = \alpha_\mathfrak{p}^{-j} \cdot [\sigma, \mathfrak{e}_j]_\Phi$$

for all compact open subgroups of $\mathcal{H}_\infty$ of the form $\sigma \mathcal{U}_j$, with $\sigma \in \mathcal{H}_\infty$. This distribution gives rise to an element $\mathcal{L}_\Phi$ in the completed group ring $\mathcal{O}[[G[\mathfrak{p}^\infty]]]$ via the rule

$$(\mathcal{L}_\Phi)_n = \sum_{h \in \mathcal{H}_n} \vartheta_\Phi(h \mathcal{U}_n) \cdot h.$$

Let us commit an abuse of notation in also writing $\mathcal{L}_\Phi$ to denote the image of this element in the Iwasawa algebra $\Lambda = \mathcal{O}[[G_{\mathfrak{p}^\infty}]]$. This image element is not well defined, since a different choice of sequence of consecutive edges $\{\mathfrak{e}_j\}_{j \geq 1}$ has the effect of multiplying $\mathcal{L}_\Phi$ by an element of $G_{\mathfrak{p}^\infty}$. Hence, let $\mathcal{L}_\Phi^*$ denote the image of $\mathcal{L}_\Phi$ under the involution $\Lambda \rightarrow \Lambda$ induced by inversion $\sigma \mapsto \sigma^{-1} \in G_{\mathfrak{p}^\infty}$. We then let

$$\mathcal{L}_\mathfrak{p}(\Phi, K_{\mathfrak{p}^\infty}) = \mathcal{L}_\Phi \mathcal{L}_\Phi^* . \tag{15}$$

Observe that $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ is then a well-defined element of Λ. We shall refer to this element $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty}) \in \Lambda$ as the *p-adic L-function associated to* Φ. Moreover, if Φ is associated to a Hilbert modular eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ via the Jacquet-Langlands correspondence (as described above), then we shall also write $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty}) = \mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$, with $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty}) = \mathcal{L}_{\mathbf{f}}\mathcal{L}_{\mathbf{f}}{}^*$.

Interpolation properties. The p-adic L-function $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ satisfies the following rough interpolation property; we refer the reader to [62, Theorem 4.7] for a more precise version. Given a finite order character ρ of $G_{\mathfrak{p}^\infty}$, let

$$\rho\left(\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})\right) = \int_{G_{\mathfrak{p}^\infty}} \rho(\sigma) d\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})(\sigma)$$

denote the specialization of $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ to ρ. Here, $d\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ denotes the measure of Λ defined by $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$. Let $\langle \Phi, \Phi \rangle$ denote the Petersson inner product of Φ. Fix embeddings $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ and $\overline{\mathbf{Q}}_p \rightarrow \mathbf{C}$. Fix a finite order character ρ of $G_{\mathfrak{p}^\infty}$. Suppose that ρ factors through $G_{\mathfrak{p}^m}$ for some integer $m \geq 1$. Let us view the values of ρ and $d\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ as complex values via the embedding $\overline{\mathbf{Q}}_p \rightarrow \mathbf{C}$, in which case we write $|\rho\left(\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})\right)|$ to denote the complex absolute value of the specialization $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$. We then have the following interpolation formula:

$$|\rho\left(\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})\right)|^2 = \alpha_{\mathfrak{p}}^{-4m} \cdot \kappa(\Phi, F) \cdot L(\Phi \times \rho, 1). \tag{16}$$

Here, $\kappa(\Phi, F)$ is a nonvanishing product of algebraic constants (which can be given precisely in terms of certain special values of some related L-functions), and $L(\Phi \times \rho, 1)$ is the central value of the Rankin-Selberg L-function of Φ times the twisted theta series associated to ρ. Moreover, both sides of (16) belong to $\overline{\mathbf{Q}}_p$. This result is deduced from the generalization of Waldspurger's formula shown in Yuan-Zhang-Zhang [70]. In particular, we see from this that the specialization $\rho\left(\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})\right)$ vanishes if and only if the central value $L(\Phi \times \rho, 1)$ vanishes. Hence, we deduce from the nonvanishing theorem of Cornut-Vatsal [16, Theorem 1.4] that the p-adic L-function $\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})$ does not vanish identically.

The dihedral μ-invariant. Recall that we let $\mathfrak{P}$ denote the maximal ideal of the local ring $\mathcal{O}$. Given an element $\lambda \in \Lambda$, we define the *dihedral mu-invariant* $\mu(\lambda)$ *of* λ to be the largest exponent c such that $\lambda \in \mathfrak{P}^c\Lambda$. Following the method of Vatsal [63] it can be shown ([62, Theorem 4.10]) that $\mu(\mathcal{L}_{\mathfrak{p}}(\Phi, K_{\mathfrak{p}^\infty})) = 2\nu$, where $\nu = \nu_\Phi$ is defined to be the largest positive integer such that Φ is congruent to a constant modulo $\mathfrak{P}^\nu$.

4. Galois representations

Galois representations associated to Hilbert modular forms. Recall that we write G_F denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$.

Theorem 4.1 (Carayol-Taylor-Wiles)**.** *Fix an eigenform* $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$*, with* $\pi_{\mathbf{f}}$ *its associated automorphic representation of* $\mathrm{GL}_2(F)$*. Let* $\mathcal{O}_{\mathbf{f}}$ *be the ring of integers of any number field such that there exists a morphism*

$$\theta_{\mathbf{f}} : \mathbf{T}(\mathfrak{N}) \longrightarrow \mathcal{O}_{\mathbf{f}}$$

with $\mathbf{f}|T = \theta_{\mathbf{f}}(T)\mathbf{f}$ *for any Hecke operator* $T \in \mathbf{T}(\mathfrak{N})$*. Then, for each prime* $\lambda \subset \mathcal{O}_{\mathbf{f}}$*, there exists a continuous representation*

$$\rho_{\mathbf{f},\lambda} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O}_{\mathbf{f},\lambda})$$

such that the following property holds: for any prime $v \subset \mathcal{O}_F$ of residue characteristic not equal to that of λ, the restriction of the representation $\rho_{\mathbf{f},\lambda}$ to the decomposition subgroup at v is conjugate to the λ-adic representation of associated by the local Langlands correspondence to the local component of $\pi_{\mathbf{f}}$ at v. Here, $\mathcal{O}_{\mathbf{f},\lambda}$ denotes the localization at λ of $\mathcal{O}_{\mathbf{f}}$.

Proof. This results follows from the specializations of works of Carayol [10], Taylor [60] and Wiles [69] to parallel weight 2. □

Recall that we view $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ as a p-adic modular form via a fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$, writing $\mathcal{O}_0$ to denote the $\mathbf{Z}_p$-subalgebra of $\overline{\mathbf{Q}}_p$ generated by the Fourier coefficients of $\mathbf{f}$, $\mathcal{O}$ the integral closure of $\mathcal{O}_0$ in its field of fractions L, and $\mathfrak{P}$ to denote the maximal ideal of $\mathcal{O}$. We then write

$$\rho_{\mathbf{f}} = \rho_{\mathbf{f},\mathfrak{P}} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O})$$

to denote the $\mathfrak{P}$-adic Galois representation associated to $\mathbf{f}$ by Theorem 4.1. Let $T_{\mathbf{f}}$ be the lattice $\mathcal{O}^2$, together with the action of G_F given by $\rho_{\mathbf{f}}$. If (as we shall always assume) the residual representation $T_{\mathbf{f}}/\mathfrak{P}$ is irreducible, then $T_{\mathbf{f}}$ is the unique G_F-stable sublattice of L^2 up to homothety. We then define $A_{\mathbf{f}} = (T_{\mathbf{f}} \otimes L)/T_{\mathbf{f}} \cong (L/\mathcal{O})^2$. We also define G_F-modules

$$T_{\mathbf{f},n} = T_{\mathbf{f}}/\mathfrak{P}T_{\mathbf{f}} \text{ and } A_{\mathbf{f},n} = A_{\mathbf{f}}[\mathfrak{P}^n].$$

These modules are of course isomorphic. However, we maintain a notational distinction as these modules form respective projective and injective systems

$$T_{\mathbf{f}} = \varprojlim_n T_{\mathbf{f},n} \text{ and } A_{\mathbf{f}} = \varinjlim_n A_{\mathbf{f},n}. \tag{17}$$

Abelian varieties associated to Hilbert modular forms. We now explain how to associate to $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ an abelian variety over F, following the construction of Carayol [10]. Recall we assume for simplicity that $\mathfrak{P}$ is contained in $\mathcal{O}_0$.

Proposition 4.2. *Fix an eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$, with $\pi_{\mathbf{f}}$ the associated automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Assume that either d is odd, or else d is even with the condition that there exists a finite place $v \subset \mathcal{O}_F$ at which $\pi_{\mathbf{f}}$ is either special or supercuspidal. Then, we can associate to $\mathbf{f}$ an abelian variety A defined over F. Moreover, there is a G_F-module isomorphism $\mathrm{Ta}_{\mathfrak{P}}A \otimes_{\mathcal{O}_0} L \cong A_{\mathbf{f}} \otimes_{\mathcal{O}} L$.*

Proof. The result is presumably well known, see for instance [16, § 3]. We sketch the construction for lack of a better reference.

Fix positive integers $k \geq 2$ and w having the same parity. Let $D_{k,w}$ denote the representation of $\mathrm{GL}_2(\mathbf{R})$ that occurs via unitary induction as $\mathrm{Ind}(\mu, \nu)$, where μ and ν are the characters on $\mathbf{R}^\times$ given by

$$\mu(t) = |t|^{\frac{1}{2}(k-1-w)} \,\mathrm{sgn}(t)^k$$
$$\nu(t) = |t|^{\frac{1}{2}(-k+1-w)}.$$

Fix integers $k_1, \ldots k_d$ all having the same parity. Let $\pi \cong \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ such that for each real place τ_i of F, there is an isomorphism $\pi_{\tau_i} \cong D_{k_i,w}$. It is well know that such representations correspond to holomorphic Hilbert modular forms of weight $\mathbf{k} = (k_1, \ldots, k_d)$. If d is even, then we assume that there exists a finite prime $v \subset \mathcal{O}_F$ where the local component π_v is either a special or supercuspidal representation of $\mathrm{GL}_2(F_v)$.

Let B be a quaternion algebra over F that is ramified at $\{\tau_2, \ldots, \tau_d\}$ if d is odd, and ramified at $\{\tau_2, \ldots, \tau_d, v\}$ if d is even. Let $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$ denote the associated algebraic group over $\mathbf{Q}$. Hence, we have an isomorphism $G(\mathbf{R}) \cong \mathrm{GL}_2(\mathbf{R}) \times (\mathbb{H}^\times)^{d-1}$, where $\mathbb{H}$ denotes the Hamiltonian quaternions. Let $\overline{D}_{k,w}$ denote the representation of $\mathbb{H}^\times$ corresponding to $D_{k,w}$ via the Jacquet-Langlands correspondence . We then consider cuspidal automorphic representations $\pi' = \bigotimes_v \pi'_v$ of $G(\mathbf{A}_F)$ such that $\pi'_{\tau_1} \cong D_{k_1,w}$ and $\pi_{\tau_i} \cong \overline{D}_{k_i,w}$ for $i = 2, \ldots, d$. Let us now fix such a representation π' associated to $\pi = \pi_{\mathbf{f}}$. Hence, $\mathbf{k} = (2, \ldots, 2)$. Fix a vector $\Phi \in \pi'$. Hence, Φ is seen to be a function in the space $\mathbb{S}_2(H, B)$ for some compact open subgroup $H \subset G(\mathbf{A}_f)$. By Proposition 2.1, we can identify Φ with a section of the sheaf of holomorphic 1-forms $\Omega_H(\mathbf{C})$ on the complex Shimura curve $M_H(\mathbf{C}) = M_H(B, X)(\mathbf{C})$. Recall that we let J_H denote the Jacobian of the canonical model M_H. Let $\mathbb{T}$ denote the subalgebra of $\mathrm{End}(J_H)$ generated by Hecke correspondences acting on J_H. By Proposition 2.1, we deduce the identification $\mathbb{T} = \mathbb{T}(H, B)$. Consider the homorphism

$$\theta_\Phi : \mathbb{T} \longrightarrow \mathcal{O}$$

that sends each operator in $\mathbb{T}$ the the eigenvalue for its action on Φ. Let $I_\Phi = \ker(\theta_\Phi)$. Consider the quotient abelian variety defined by

$$A_H = J_H / I_\Phi J_H.$$

Hence, we have constructed from $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N})$ an abelian variety $A = A_H$ defined over F. Now, by the construction of Carayol [10], we claim that the Galois representation $\rho_{\mathbf{f}}$ is equivalent to the Galois representation arising from the $\mathfrak{P}$-adic Tate module of A. Hence, we deduce that there is an identification of G_F-modules $\mathrm{Ta}_{\mathfrak{P}} A \otimes_{\mathcal{O}_0} L \cong A_{\mathbf{f}} \otimes_{\mathcal{O}} L$. □

Corollary 4.3. *For each integer $n \geq 1$, there is a canonical, nondegenerate G_F-equivariant pairing*

$$T_{\mathbf{f},n} \times A_{\mathbf{f},n} \longrightarrow \mu_{p^n}.$$

Proof. The pairing is induced from the Weil pairing, after composition with a suitable choice of polarization map. See for instance the discussion in [30, § 2.3]. □

Given a prime $w \subset \mathcal{O}_F$, let us choose a decomposition subgroup of G_F above w, which we can and will identify with the Galois group $G_{F_w} = \mathrm{Gal}(\overline{F}_w/F_w)$ for some choice of algebraic closure $\overline{F}_w$. Let $I_w = I_{F_w}$ denote the inertia subgroup at w.

Lemma 4.4. *For each prime $w \mid \mathfrak{N}^- \subset \mathcal{O}_F$, the maximal I_w-invariant submodule of $A_{\mathbf{f}}$ is divisible of $\mathcal{O}$-corank one.*

Proof. If $\mathbf{f}$ is associated to a modular abelian variety defined over F of arithmetic conductor $\mathfrak{N}$ having good reduction outside of $\mathfrak{N}$, ordinary reduction at $\mathfrak{p}$, and purely toric reduction at each prime $w \mid \mathfrak{N}^-$, then this condition is satisfied (cf. [3, Remark 1, p. 14] with the relevant sections of [42] or [41]). Granted that $\mathbf{f}$ is $\mathfrak{p}$-ordinary, we can deduce from Proposition 4.2 that the associated abelian variety A always has these properties. In particular, the toric reduction of A at primes $w \mid \mathfrak{N}^-$ can be deduced from the general theory of Néron models given in the standard reference [5, Ch. 9], using the fact that the associated Shimura curve has a semistable reduction at w. See for instance the discussion in [49, 1.6.4] along with the description of integral models given below. □

5. Selmer groups

Global cohomology. Given integers $m, n \geq 1$, we define continuous cohomology groups of $G_{K_{\mathfrak{p}^m}} = \mathrm{Gal}(\overline{\mathbf{Q}}/K_{\mathfrak{p}^m})$ with coefficients in the modules $T_{\mathbf{f}} = \varprojlim_n T_{\mathbf{f},n}$ and $A_{\mathbf{f}} = \varinjlim_n A_{\mathbf{f},n}$,

$$H^1(K_{\mathfrak{p}^m}, T_{\mathbf{f}}) = \varprojlim_n H^1(K_{\mathfrak{p}^m}, T_{\mathbf{f},n})$$
$$H^1(K_{\mathfrak{p}^m}, A_{\mathbf{f}}) = \varinjlim_n H^1(K_{\mathfrak{p}^m}, A_{\mathbf{f},n}).$$

Note that these identifications can be justified (see [59, Proposition 2.2]). We also define cohomology groups of $G_{K_{\mathfrak{p}^\infty}} = \mathrm{Gal}(\overline{\mathbf{Q}}/K_{\mathfrak{p}^\infty})$,

$$\widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}}) = \varprojlim_m H^1(K_{\mathfrak{p}^m}, T_{\mathbf{f}}),$$
$$H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f}}) = \varinjlim_m H^1(K_{\mathfrak{p}^m}, A_{\mathbf{f}}).$$

Here, the direct limit is taken with respect to natural restriction maps, and the inverse limit with respect to natural corestriction maps. Note that the compatible actions of the group rings $\mathcal{O}[G_{\mathfrak{p}^m}]$ on the cohomology groups $H^1(K_{\mathfrak{p}^m}, T_{\mathbf{f}})$ and $H^1(K_{\mathfrak{p}^m}, A_{\mathbf{f}})$ for each integer $m \geq 1$ induce an action of the Iwasawa algebra $\Lambda = \mathcal{O}[[G_{\mathfrak{p}^\infty}]]$ on the cohomology groups $\widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}})$ and $H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f}})$.

Local cohomology. Fix an integer $m \geq 1$. Given a finite prime v of F, let us for notational simplicity write

$$K_{\mathfrak{p}^m, v} = K_{\mathfrak{p}^m} \otimes F_v = \bigoplus_{\mathfrak{v}|v} K_{\mathfrak{p}^m, \mathfrak{v}}$$

to denote the direct sum over completions of $K_{\mathfrak{p}^m}$ at each prime $\mathfrak{v}$ above v in $K_{\mathfrak{p}^m}$. Hence, we can define local cohomology groups

$$\widehat{H}^1(K_{\mathfrak{p}^\infty, v} T_{\mathbf{f},n}) = \varprojlim_m \bigoplus_{\mathfrak{v}|v} H^1(K_{\mathfrak{p}^m, \mathfrak{v}}, T_{\mathbf{f},n}),$$
$$\widehat{H}^1(K_{\mathfrak{p}^\infty, v} T_{\mathbf{f}}) = \varprojlim_m \bigoplus_{\mathfrak{v}|v} H^1(K_{\mathfrak{p}^m, \mathfrak{v}}, T_{\mathbf{f}}),$$
$$H^1(K_{\mathfrak{p}^\infty, v}, A_{\mathbf{f},n}) = \varinjlim_m \bigoplus_{\mathfrak{v}|v} H^1(K_{\mathfrak{p}^m, \mathfrak{v}}, A_{\mathbf{f},n}),$$
$$H^1(K_{\mathfrak{p}^\infty, v}, A_{\mathbf{f}}) = \varinjlim_m \bigoplus_{\mathfrak{v}|v} H^1(K_{\mathfrak{p}^m, \mathfrak{v}}, A_{\mathbf{f}}).$$

Here (as in the global case), the direct limits are taken with respect to natural restriction maps, and the inverse limits with respect to natural corestriction maps. Taking appropriate limits from (17) again, we then define

$$\widehat{H}^1(K_{\mathfrak{p}^\infty, v}, T_{\mathbf{f}}) = \varprojlim_n H^1(K_{\mathfrak{p}^\infty, v}, T_{\mathbf{f},n})$$
$$H^1(K_{\mathfrak{p}^\infty, v}, A_{\mathbf{f}}) = \varinjlim_n H^1(K_{\mathfrak{p}^\infty, v}, A_{\mathbf{f},n}).$$

These identifications as before can be justified (see [58, Proposition 2.2]). As in the global case, the Iwasawa algebra Λ acts on the local cohomology groups

$\widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f}})$ and $H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f}})$ in such a way that is compatible with respect to the respective corestriction and restriction maps.

Local Tate pairings. Recall that for each integer $n \geq 1$, we have an isomorphism of G_F-modules $T_{\mathbf{f},n} \cong A_{\mathbf{f},n}$. Recall as well that by Corollary 4.3 (cf. [30, §2.3]), there exists a canonical, G_F-equivariant pairing

$$T_{\mathbf{f},n} \times A_{\mathbf{f},n} \longrightarrow \mathbf{Z}/p^n\mathbf{Z}(1) = \mu_{p^n}.$$

Composition with the cup product of local cohomology then gives a collection of local Tate pairings

$$\langle\ ,\ \rangle_{m,v} : \widehat{H}^1(K_{\mathfrak{p}^m,v}, T_{\mathbf{f},n}) \times H^1(K_{\mathfrak{p}^m,v}, A_{\mathbf{f},n}) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

Passage to the limit(s) then induces a perfect pairing

$$\langle\ ,\ \rangle_{v} : \widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \times H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p. \tag{18}$$

We refer the reader to [46, § 1], [1] or [30] for relevant background on local Tate duality. The main fact we shall use is that (18) induces an isomorphism of Λ-modules

$$\widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \cong H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})^\vee.$$

Here, $H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})^\vee$ denotes the Pontryagin dual of $H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$, endowed with the usual Λ-module structure. (cf. [3, §2]).

Singular/unramified structures. Given a prime $v \subset \mathcal{O}_F$, let $I_{m,v} = \bigoplus_{\mathfrak{v}|v} I_{m,\mathfrak{v}}$ denote the direct sum over all primes $\mathfrak{v}$ above v in $K_{\mathfrak{p}^m}$ of the inertia subgroups $I_{m,\mathfrak{v}} = I_{K_{\mathfrak{p}^m},\mathfrak{v}}$ in $G_{K_{\mathfrak{p}^m}}$.

Definition Let $v \nmid \mathfrak{N} \subset \mathcal{O}_F$ be a prime that does not divide the residue characteristic of $\mathfrak{p}$. Let $M_{\mathbf{f},n}$ denote either $A_{\mathbf{f},n}$ or $T_{\mathbf{f},n}$.

(i) The *singular structure* $H^1_{\text{sing}}(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n}) \subset H^1(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n})$ is

$$H^1_{\text{sing}}(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n}) = H^1(I_{m,v}, M_{\mathbf{f},n})^{G_{K_v}}.$$

(ii) The *residue map* ∂_v is the natural restriction map

$$\partial_v : H^1(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n}) \longrightarrow H^1_{\text{sing}}(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n}).$$

(iii) The *unramified structure* $H^1_{\text{unr}}(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n}) \subset H^1(K_{\mathfrak{p}^m,v}, M_{\mathbf{f},n})$ is the kernel of the residue map ∂_v.

Analogous definitions hold under passage to projective limits in the case that $M_{\mathbf{f},n} = T_{\mathbf{f},n}$, and under inductive limits in the case that $M_{\mathbf{f},n} = A_{\mathbf{f},n}$. Let us also write ∂_v to denote the induced residue maps on $G_{K_{\mathfrak{p}^\infty}}$-cohomology.

Ordinary structures. Recall that for each prime divisor $w \mid \mathfrak{N}^-$, Lemma 4.4 shows that the maximal I_w-invariant submodule of $A_{\mathbf{f}}$ is divisible of $\mathcal{O}$-corank one. Hence, we have an exact sequence of I_w-modules

$$0 \longrightarrow A_{\mathbf{f}}^{(w)} \longrightarrow A_{\mathbf{f}} \longrightarrow A_{\mathbf{f}}^{(1)} \longrightarrow 0, \tag{19}$$

where $A_{\mathbf{f}}^{(1)}$ is the maximal submodule of $A_{\mathbf{f}}$ on which I_w acts trivially, giving trivial isomorphisms of I_w-modules

$$A_{\mathbf{f}}^{(w)} \cong A_{\mathbf{f}}^{(1)} \cong L/\mathcal{O}. \tag{20}$$

Suppose now that we consider the prime $\mathfrak{p} \subset \mathcal{O}_F$. Recall we assume that $\mathbf{f}$ is $\mathfrak{p}$-ordinary, in which case it is known that there is an exact sequence of $I_{\mathfrak{p}}$-modules

$$0 \longrightarrow A_{\mathbf{f}}^{(\mathfrak{p})} \longrightarrow A_{\mathbf{f}} \longrightarrow A_{\mathbf{f}}^{(1)} \longrightarrow 0, \tag{21}$$

where $I_{\mathfrak{p}}$ acts on $A_{\mathbf{f}}^{(\mathfrak{p})}$ by the cyclotomic character $\varepsilon_p : G_F \longrightarrow \mathrm{Aut}(\mu_{p^\infty})$ times a multiplicative factor of ± 1.

Definition Given a prime $w \mid \mathfrak{p}\mathfrak{N}^- \subset \mathcal{O}_F$, we define the *ordinary structure* $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{\mathbf{f},n}) \subset H^1(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{\mathbf{f},n})$ as follows.

(i) If $w \mid \mathfrak{N}^- \subset \mathcal{O}_F$, then it is the unramified cohomology

$$H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{\mathbf{f},n}) = H^1(K_{\mathfrak{p}^\infty,\mathfrak{q}}, A_{\mathbf{f},n}^{(w)}).$$

(ii) At the prime $\mathfrak{p} \subset \mathcal{O}_F$,

$$H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,\mathfrak{p}}, A_{\mathbf{f},n}) = \mathrm{res}_{\mathfrak{p}}^{-1} H^1\left(I_{K_{\mathfrak{p}^\infty,\mathfrak{p}}}, A_{\mathbf{f},n}^{(\mathfrak{p})}\right).$$

Here, $\mathrm{res}_{\mathfrak{p}} : H^1(K_{\mathfrak{p}^\infty,\mathfrak{p}}, A_{\mathbf{f},n}) \longrightarrow H^1(I_{K_{\mathfrak{p}^\infty,\mathfrak{p}}}, A_{\mathbf{f},n})$ denotes the map induced from the restriction at the prime above $\mathfrak{p}$ in $K_{\mathfrak{p}^\infty}$.

Note that we do not define ordinary parts at primes $w \mid \mathfrak{N}^+ \subset \mathcal{O}_F$, as these groups are seen easily to vanish by variant of the argument given in Corollary 5.3 below (cf. [41, 5.2.2]). Note as well that we may also define ordinary cohomology groups for the G_F-modules $A_{\mathbf{f}}$ and $T_{\mathbf{f}}$ by taking the limits (17).

Admissible primes. Here, we define the notation of an n-admissible prime with respect to $\mathbf{f}$, for $n \geq 1$ an integer. As we shall see below in Proposition 6.2, the set of n-admissible primes controls the Selmer group $\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})$.

Definition A prime $v \subset \mathcal{O}_F$ is siad to be *n-admissible with respect to* $\mathbf{f}$ *in* K for some integer $n \geq 1$ if

(i) $v \nmid \mathfrak{p}\mathfrak{N}$.
(ii) v is inert in K.
(iii) $\mathfrak{P}$ does not divide $\mathbf{N}(v)^2 - 1$.
(iv) $\mathfrak{P}_n$ divides one of $\mathbf{N}(v) + 1 - a_v(\mathbf{f})$ or $\mathbf{N}(v) + 1 + a_v(\mathbf{f})$.

We shall use the following two facts repeatedly throughout.

(1) If $v \subset \mathcal{O}_F$ is an n-admissible prime with respect to $\mathbf{f}$, then the associated mod $\mathfrak{P}^n$ Galois representation $T_{\mathbf{f},n}$ is unramified at v. Moreover, the arithmetic Frobenius at v acts semisimply on $T_{\mathbf{f},n}$ with eigenvalues 1 and $\mathbf{N}(v)^2$, both of which are distinct mod $\mathfrak{P}^n$.
(2) If $v \subset \mathcal{O}_F$ is an n-admissible prime with respect to $\mathbf{f}$, then by condition (ii) it is inert in K. We commit an abuse of notation in writing v to also denote the prime above v in K. Hence, K_v denotes the localization at the prime above v in F, which is isomorphic to the quadratic unramified extension of F_v. Writing F_{v^2} to denote the quadratic unramified extension of F_v, we shall then always make the implicit identification $K_v \cong F_{v^2}$.

Some identifications. Here, we give some identifications for the finite, singular and ordinary structures of the local Galois cohomology groups defined above. The results here are analogous to the case of $F = \mathbf{Q}$ (cf. [3, §2]).

Proposition 5.1. *Let $v \subset \mathcal{O}_F$ be a finite prime.*

(i) *If $v \nmid \mathfrak{N}$, then the cohomology groups $\widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$ and $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$ annihilate each other under the local Tate pairing $\langle\ ,\ \rangle_v$.*

(ii) *If $v \mid \mathfrak{N}$ with $\mathrm{ord}_v(\mathfrak{N}) = 1$, then the cohomology groups $\widehat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$ and $H^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$ annihilate each other under the local Tate pairing $\langle\ ,\ \rangle_v$.*

In particular, the local cohomology groups $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ and $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty}, A_{\mathbf{f},n})$ are Pontryagin dual to each other.

Proof. The result over finite layers $K_{\mathfrak{p}^m,v}$ is a standard consequence of local Tate duality, see [3, Proposition 2.3] or [4, §2.1]. Passage to limits then proves the claim. □

Let us from now on write

$$\langle\ ,\ \rangle_v : \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \times H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p \tag{22}$$

to denote the perfect pairing induced by the local Tate duality of Proposition 5.1.

Lemma 5.2. *Let $v \nmid p$ be any finite prime of $\mathcal{O}_F$. If v is inert in K, then v splits completely in $K_{\mathfrak{p}^m}$ for any integer $m \geq 1$.*

Proof. This is a standard consequence of global class field theory. See for instance [58, Proposition 2.3], using that $K_{\mathfrak{p}^m}$ is of generalized dihedral type over F. □

Corollary 5.3. *Let $v \nmid \mathfrak{N}$ be a finite prime of $\mathcal{O}_F$.*

(i) *If v splits in K, then we have identifications*

$$\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) = H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) = 0.$$

(ii) *If v is inert in K with $v \nmid \mathfrak{N}$, then we have identifications*

$$\begin{aligned}\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) &\cong H^1_{\mathrm{sing}}(K_v, T_{\mathbf{f},n}) \otimes \Lambda,\\ H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) &\cong \mathrm{Hom}\left(H^1_{\mathrm{sing}}(K_v, T_{\mathbf{f},n}) \otimes \Lambda, \mathbf{Q}_p/\mathbf{Z}_p\right).\end{aligned}$$

Proof. The first assertion is shown in [3, Lemma 2.4]. That is, it suffices by nondegeneracy of the local Tate pairing (22) to show that $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$ vanishes. If $v\mathcal{O}_K = v_1 v_2$, then a similar application of global class field theory as used in Lemma 5.2 shows that that the Frobenius at each v_i is the topological generator of a finite index subgroup of $G_{\mathfrak{p}^\infty} \cong \mathbf{Z}_p^\delta$. We can then view $K_{\mathfrak{p}^\infty,v}$ as the direct sum of copies of the maximal unramified p-extension of F_v. Since $A_{\mathbf{f},n} = A_{\mathbf{f}}[\mathfrak{P}^n]$ has exponent $\mathfrak{P}^n$, we deduce that any unramified class in $H^1(K_{\mathfrak{p}^m,v}, A_{\mathbf{f},n})$ must have trivial restriction to $H^1(K_{\mathfrak{p}^{m'},v}, A_{\mathbf{f},n})$ for m' sufficiently large. Hence, $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) = 0$. The second assertion is shown in [3, Lemma 2.5]. That is, since v splits completely in $K_{\mathfrak{p}^\infty}$ by Lemma 5.2, any choice of prime $\mathfrak{v}_m$ above v in $K_{\mathfrak{p}^m}$ determines an isomorphism

$$H^1(K_{\mathfrak{p}^m,v}, T_{\mathbf{f},n}) \longrightarrow H^1(K_v, T_{\mathbf{f},n}) \otimes \mathcal{O}[G_{\mathfrak{p}^m}].$$

A compatible system of choices of primes $\mathfrak{v}_m$ above v in $K_{\mathfrak{p}^\infty}$ then determines an isomorphism

$$\widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \cong H^1(K_v, T_{\mathbf{f},n}) \otimes \Lambda.$$

Passage to the singular cohomology then proves the claim, with the latter isomorphism being a consequence of Proposition 5.1. □

Lemma 5.4. *If $v \subset \mathcal{O}_F$ is an n-admissible prime with respect to $\mathbf{f}$, then the local cohomology groups $H^1_{\mathrm{sing}}(K_v, T_{\mathbf{f},n})$ and $H^1_{\mathrm{unr}}(K_v, A_{\mathbf{f},n})$ are both isomorphic to $\mathcal{O}/\mathfrak{P}^n$. Moreover, the local cohomology groups $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$ and $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$ are both free of rank one over $\Lambda/\mathfrak{P}^n$.*

Proof. The first assertion follows from the same proof as given in [3, Lemma 2.6], using the identification $H^1_{\mathrm{sing}}(K_v, T_{\mathbf{f},n}) = H^1(I_{K_v}, T_{\mathbf{f},n})^{G_{K_v}}$ along with the fact that $T_{\mathbf{f},n}$ is unramified at v. The second assertion then follows directly from the second part of Lemma 5.3 above (cf. [3, Lemma 2.7]). $\square$

Residual Selmer groups. Recall that we defined the residue maps ∂_v on local cohomology to be the natural restriction maps

$$\partial_v : H^1(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n}) \longrightarrow H^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$$
$$\partial_v : \widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \longrightarrow \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}).$$

Let us commit an abuse of notation in also writing ∂_v to denote the composition of these maps with the restriction from $K_{\mathfrak{p}^\infty}$ to $K_{\mathfrak{p}^\infty,v}$, which gives residue maps on global cohomology

$$\partial_v : H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f},n}) \longrightarrow H^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$$
$$\partial_v : \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n}) \longrightarrow \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}).$$

Let us establish for future reference the following notations:

- If $\partial_v(c) = 0$ for a class $c \in \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$, then $\vartheta_v(c)$ denotes the image of c in $\widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$.
- If $\partial_v(c) = 0$ for a class $c \in H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f},n})$, then $\vartheta_v(c)$ denotes the image of c in $H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, A_{\mathbf{f},n})$.

Recall that the integer factorization $\mathfrak{N} = \mathfrak{p}\mathfrak{N}^+\mathfrak{N}^- \subset \mathcal{O}_F$ of (9) is assumed. Let us write s_v denote the image of a class s under the restriction from $K_{\mathfrak{p}^\infty}$ to $K_{\mathfrak{p}^\infty,v}$.

Definition The *residual Selmer group* $\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$ associated to $(\mathbf{f}, n, K_{\mathfrak{p}^\infty})$ is defined to be the group of classes $s \in H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f},n})$ such that

(i) The residue $\partial_v(s)$ vanishes at all primes $v \nmid \mathfrak{N}$.
(ii) The restriction s_v is ordinary at all primes $v \mid \mathfrak{p}\mathfrak{N}^-$.
(iii) The restriction s_v is trivial at all primes $v \mid \mathfrak{N}^+$.

Observe that the residual Selmer group $\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$ depends only on the mod $\mathfrak{P}^n$ Galois representation $T_{\mathbf{f},n}$ associated to $\mathbf{f}$, and not on $T_{\mathbf{f}}$ itself!

Compactified Selmer groups. We now define compactified Selmer groups.

Definition Let $\mathfrak{S} \subset \mathcal{O}_F$ be any integral ideal prime to $\mathfrak{N}$. The *compactified Selmer group* $\widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ associated to $(\mathbf{f}, n, K_{\mathfrak{p}^\infty})$ is defined to be the group of classes $\mathfrak{s} \in \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ such that

(i) The residue $\partial_v(\mathfrak{s})$ vanishes at all primes $v \nmid \mathfrak{S}\mathfrak{N}$.
(ii) The restriction $\mathfrak{s}_v$ is ordinary at all primes $v \mid \mathfrak{p}\mathfrak{N}^-$.
(iii) The restriction $\mathfrak{s}_v$ is arbitrary at all primes $v \mid \mathfrak{S}\mathfrak{N}^+$.

Admissible sets. Let us also for future reference define the notion of an n-admissible set with respect to $\mathbf{f}$.

Definition A finite set of primes $\mathfrak{S}$ of $\mathcal{O}_F$ is said to be *n-admissible with respect to* $\mathbf{f}$ if

(i) Each prime $v \in \mathfrak{S}$ is n-admissible with respect to $\mathbf{f}$.
(ii) The natural map $\mathrm{Sel}_{\mathbf{f},n}(K) \longrightarrow \bigoplus_{v|\mathfrak{S}} H^1_{\mathrm{unr}}(K_v, A_{\mathbf{f},n})$ is injective.

Theorem 5.5. *If $\mathfrak{S}$ in an n-admissible set of primes of $\mathcal{O}_F$ with respect to $\mathbf{f}$, then $\widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ is free of rank $|\mathfrak{S}|$ over $\Lambda/\mathfrak{P}^n$.*

Proof. See [3, Theorem 3.3]. The proof in the more general setting follows in the same way from [1, Theorem 3.2], which is given for arbitrary abelian extensions over K.

□

Now, recall that the Galois group $G_{\mathfrak{p}^\infty}$ is topologically isomorphic to $\mathbf{Z}_p^\delta$ with $\delta = [F_\mathfrak{p} : \mathbf{Q}_p]$, and hence pro-$p$. Hence, the Iwasawa algebra Λ is a local ring of dimension $\delta + 1$. Let $\mathfrak{m}_\Lambda$ denote the maximal ideal of Λ. We have the following result.

Theorem 5.6.

(i) *The natural map $H^1(K, A_{\mathbf{f},1}) \longrightarrow H^1(K_{\mathfrak{p}^\infty}, A_{\mathbf{f},1})[\mathfrak{m}_\Lambda]$ induced by restriction is an isomorphism.*
(ii) *If $\mathfrak{S}$ is an n-admissible set of primes with respect to $\mathbf{f}$, then the natural map $H^1(K, T_{\mathbf{f},1}) \longrightarrow H^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},1})/\mathfrak{m}_\Lambda$ induced by corestriction is an injection.*

Proof. See [3, Theorem 3.4]. The proof given there carries over here with the same argument by using Proposition 5.5 above. □

Relations between Selmer groups. Let us start with some motivation. Our method of approach to dihedral main conjectures generalizes the Euler system argument of Bertolini-Darmon [3]. More precisely, it generalizes the refinement of this argument given by Pollack-Weston in [50]. As such, it requires the construction of classes in $\widehat{H}^1_v(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ indexed by n-admissible primes $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$. The residues of these classes can be related to their corresponding group ring elements $\mathcal{L}_\Phi \in \Lambda/\mathfrak{P}^n$ via the first and second explicit reciprocity laws introduced above (Theorems 7.1 and 7.3 below). Here, Φ denotes the mod $\mathfrak{P}^n$ quaternionic eigenform corresponding to $\mathbf{f}$ mod $\mathfrak{P}^n$ under the Jacquet-Langlands correspondence. Recall that we write $L_\mathfrak{p}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ to denote the associated $\mathfrak{p}$-adic L-function $\mathcal{L}_\mathfrak{p}(\Phi, K) = \mathcal{L}_\Phi \mathcal{L}_\Phi^* \in \Lambda/\mathfrak{P}^n$. The explicit reciprocity laws, which a priori only give relations in the compactified Selmer group $\widehat{H}^1_v(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$, in fact give relations in the dual residual Selmer group

$$\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^\vee = \mathrm{Hom}(\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty}), \mathbf{Q}_p/\mathbf{Z}_p)$$

thanks to the following result.

Proposition 5.7. *If $\mathfrak{s} \in \widehat{H}^1_v(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$, then for all $s \in \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$,*

$$\langle \partial_v(\mathfrak{s}), \vartheta_v(s) \rangle_v = 0.$$

Proof. By direct generalization of [3, Proposition 2.10]. That is, fix classes $\mathfrak{s}$ and s as above. The global reciprocity law of class field theory implies that

$$\sum_v \langle \partial_v(\mathfrak{s}), \vartheta_v(s) \rangle_v = 0. \tag{23}$$

Here, the sum runs over all finite primes $v \subset \mathcal{O}_F$. Let $\mathfrak{S} \subset \mathcal{O}_F$ be any integral ideal prime to $\mathfrak{N}$. If $\mathfrak{s} \in \widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ and $s \in \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$, then $\langle \partial_v(\mathfrak{s}), \vartheta_v(s) \rangle_v = 0$

for all $v \nmid \mathfrak{S}$ by local conditions defining these groups, and $\partial_v(s) = 0$ for all $v \mid \mathfrak{S}$. It follows from (23) that

$$\sum_{v \mid \mathfrak{S}} \langle \partial_v(\mathfrak{s}), \vartheta_v(s) \rangle_v = 0.$$

Taking $\mathfrak{S} = v$ then proves the claim. □

Finally, we make the following

Definition Let

$$\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty}) = \varinjlim_n \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty}), \tag{24}$$

where the limits are taken with respect to those in (17). We claim as before that these identifications can be justified, for instance by [59, Theorem 2.2] (cf. also [50, Proposition 3.6]).

6. Control theorems

We shall use the following results to prove the main conjecture divisibility (2).

The Fitting ideals criterion. Given R a ring, and X a finitely-presented R-module, let $\mathrm{Fitt}_R(X)$ denote the Fitting ideal of X over R. We refer the reader to [45, Appendix] for instance for definitions and background on Fitting ideals.

Proposition 6.1. *Suppose that X is a finitely-generated Λ-module, and that $\mathcal{L}$ is an element of Λ. If $\varphi(\mathcal{L}) \subset \mathrm{Fitt}_{\mathcal{O}'}(X \otimes_\varphi \mathcal{O}')$ for all homomorphisms $\varphi : \Lambda \longrightarrow \mathcal{O}'$ with $\mathcal{O}'$ any discrete valuation ring, then $\mathcal{L} \subset \mathrm{char}_\Lambda(X)$.*

Proof. See [3, Proposition 3.1], which proves the claim for the case of $F = \mathbf{Q}$ (i.e. with $\delta = 1$), and [42, Proposition 7.4] for the general case. □

Control of Selmer. Recall that for each finite prime $v \nmid \mathfrak{N} \subset \mathcal{O}_F$ not dividing the residue characteristic of $\mathfrak{p}$, we have a natural residue map $\partial_v : H^1(K_v, A_{\mathbf{f},1}) \longrightarrow H^1_{\mathrm{sing}}(K_v, A_{\mathbf{f},1})$. Recall as well that we commit a minor abuse of notation in also writing ∂_v to denote the composition of maps

$$H^1(K, A_{\mathbf{f},1}) \longrightarrow H^1(K_v, A_{\mathbf{f},1}) \longrightarrow H^1_{\mathrm{sing}}(K_v, A_{\mathbf{f},1}).$$

Theorem 6.2. *Given a nonzero class $s \in H^1(K, A_{\mathbf{f},1})$, there exist infinitely many n-admissible primes $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ in K such that $\partial_v(s) = 0$ and $\vartheta_v(s) \neq 0$.*

Proof. By direct generalization of [3, Theorem 3.2]. That is, fix a class $s \in H^1(K, A_{\mathbf{f},n})$. Let $F(A_{\mathbf{f},n})$ denote the extension of F fixed by the kernel of the G_F-representation $A_{\mathbf{f},n}$. Let L denote the compositum extension $KF(A_{\mathbf{f},n})$. Since we assume that the relative discriminant $\mathfrak{D}_{K/F}$ is prime to the level $\mathfrak{N}$, we claim that the extensions $F(A_{\mathbf{f},n})$ and K are linearly disjoint over F. Granted this property, we obtain the following description of the Galois group of L over F:

$$\begin{aligned} \mathrm{Gal}(L/F) &= \mathrm{Gal}(K/F) \times \mathrm{Gal}(F(A_{\mathbf{f},n})/F) \\ &\subseteq \{\mathbf{1}, \tau\} \times \mathrm{Aut}_{\mathcal{O}/\mathfrak{P}^n}(A_{\mathbf{f},n}). \end{aligned}$$

Here, $\tau \in \mathrm{Gal}(K/F)$ denotes the complex conjugation automorphism. Hence, any element of $\mathrm{Gal}(L/F)$ can be written as a pair (τ^j, T), with $j \in \{0, 1\}$ and $T \in$

$\mathrm{Aut}_{\mathcal{O}_F/\mathfrak{P}^n}(A_{\mathbf{f},n})$. Let $\overline{s}$ denote the image of s under restriction to the cohomology group

$$H^1(L, A_{\mathbf{f},1}) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbf{Q}}/L), A_{\mathbf{f},1}).$$

Let L_s denote the extension of L cut out by this class. Assume without loss of generality that s belongs to a fixed eigenspace for the action of the complex conjugation automorphism τ. Let us then write ϖ to denote the eigenvalue of τ acting on s, so that we have the relation $\tau \cdot s = \varpi s$, where $\varpi \in \{\pm 1\}$. It follows from this assumption that L_s, a priori only Galois over K, is in fact Galois over F. Moreover, since $A_{\mathbf{f},1}$ is an irreducible G_F-module by Hypothesis 1.1 (iii), we can and will make the following identification:

$$\mathrm{Gal}(L_s/F) = A_{\mathbf{f},1} \rtimes \mathrm{Gal}(L/F).$$

Here, $\mathrm{Gal}(L/F)$ acts on the normal abelian subgroup $A_{\mathbf{f},1}$ by the rule

$$(\tau^j, T)(a) = \varpi^j \overline{T} a, \tag{25}$$

where a denotes an element of $A_{\mathbf{f},1}$, and $\overline{T}$ denotes the image of T in $\mathrm{Aut}_{\mathcal{O}/\mathfrak{P}}(A_{\mathbf{f},1})$. Since the image of $\overline{\rho}_{\mathbf{f}}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$ by Hypothesis 1.1, we can and will identify $\mathrm{Aut}_{\mathcal{O}/\mathfrak{P}}(A_{\mathbf{f},1})$ with $\mathrm{SL}_2(\mathbf{F}_p)$. We deduce from this description that $\mathrm{Gal}(L_s/F)$ contains at least one element (a, τ, T) such that the following conditions hold:

1. The automorphism T has distinct eigenvalues ϖ and λ, where the eigenvalue λ lies in $(\mathcal{O}/\mathfrak{P})^{\times}$, has order prime to p, and satisfies the property that $\mathbf{N}(\lambda)$ is not congruent to $\pm 1 \bmod p$.
2. The vector $a \in A_{\mathbf{f},1}$ belongs to the ϖ-eigenspace for the action of $\overline{T}$.

Let us now take $v \nmid \mathfrak{N}$ to be *any* prime of F that is unramified in the extension L_s, with the additional condition that

$$\mathrm{Frob}_v(L_s/F) = (a, \tau, T). \tag{26}$$

Observe that infinitely many such primes exist by the Cebotarev density theorem. We deduce from (26) that $\mathrm{Frob}_v(L/F) = (\tau, T)$, and in particular that v is n-admissible with respect to $\mathbf{f}$. We now argue that $\vartheta_v(s) \neq 0$. To see this, fix a prime $\mathfrak{v}$ above v in L. Let e denote the degree of the corresponding residue field. Note that e is necessarily even, as $L_{\mathfrak{v}}$ contains the quadratic unramified extension of F_v. Using (25) along with condition 2. for (a, τ, T), we find that

$$\mathrm{Frob}_{\mathfrak{v}}(L_s/L) = (a, \tau, T)^e = a + \varpi \overline{T} a + \overline{T}^2 a + \ldots \varpi \overline{T}^{e-1} a = ea.$$

Here, the addition symbol denotes group multiplication. Recall that we let $\overline{s}$ denote the image of s in $H^1(L, A_{\mathbf{f},1}) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbf{Q}}/L), A_{\mathbf{f},1})$ under restriction. Since e is prime to p by Hypothesis 1.1 (i), we find that

$$\overline{s}\left(\mathrm{Frob}_v(L_s/L)\right) = e \cdot \overline{s}(a) \neq 0.$$

Hence, the restriction at $\mathfrak{v}$ of $\overline{s}$ does not vanish. Hence, $\vartheta_v(s)$ does not vanish, as required. $\square$

7. The Euler system argument

Let first describe the Euler system that we shall construct in the subsequent sections. This construction and subsequent argument will generalize those of Bertolini-Darmon [3], or more specifically the refinements of these due to Pollack-Weston [50]. Fix an integer $n \geq 1$. Recall that we write $\mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ to denote the subspace of $\mathcal{S}_2(\mathfrak{N}^+\mathfrak{N}^-)$ consisting of cuspforms that are new at all primes $v \subset \mathcal{O}_F$ dividing $\mathfrak{N}^-$. Recall as well that we write $\mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$ to denote the algebra of Hecke operators acting faithfully on $\mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$, with $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$ its p-adic completion. Let us now fix an eigenform $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$. We shall use the theories of level raising congruences and CM points on Shimura curves over totally real fields to construct for each n-admissible prime $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ a class

$$\zeta(v) \in \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n}). \tag{27}$$

Observe that since v is n-admissible, we have the decompositions

$$\begin{aligned}\widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) &= \widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \oplus \widehat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \\ &= \widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}).\end{aligned}$$

Hence, we may view the homomorphism

$$\vartheta_v : \widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \longrightarrow \widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) / \widehat{H}^1_{\mathrm{ord}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$$

as a projection onto the first component of the first decomposition, and the homomorphism

$$\partial_v : \widehat{H}^1(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \longrightarrow \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$$

as a projection onto the second component of the second decomposition. We shall deduce in subsequent sections the following explicit reciprocity laws for the classes (27).

Theorem 7.1. (The first explicit reciprocity law). *Let $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ be a $\mathfrak{p}$-ordinary eigenform, as defined above. Assume that the conditions of Theorem 11.4 and Corollary 11.6 below are satsified. If $v \subset \mathcal{O}_F$ is an n-admissible prime with respect to $\mathbf{f}$, then $\vartheta_v(\zeta(v)) = 0$. Moreover, the equality*

$$\partial_v(\zeta(v)) = \mathcal{L}_{\mathbf{f}} \tag{28}$$

holds in $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \cong \Lambda/\mathfrak{P}^n$, up to multiplication by elements of $\mathcal{O}^\times$ or $G_{\mathfrak{p}^\infty}$.

Proof. See Theorem 13.1 below. □

To state the second reciprocity law for these classes (27), we require the following weak level-raising result at two primes. That is, let v_1 and v_2 be two distinct n-admissible primes with respect to $\mathbf{f}$ such that

$$\mathbf{N}(v_i) + 1 - \varepsilon_i \cdot a_{v_i}(\mathbf{f}) \equiv 0 \mod \mathfrak{P}_n \tag{29}$$

for each of $i = 1, 2$, where $\varepsilon_i \in \{\pm 1\}$.

Proposition 7.2. *Let $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ be a $\mathfrak{p}$-ordinary eigenform, as defined above. Assume that the conditions of Theorem 11.4 and Corollary 11.6 below are satisfied, and moreover that F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$. There exists a mod $\mathfrak{P}_n$ eigenform $\mathbf{g}$ with respect to the Hecke algebra $\mathbf{T}_0(\mathfrak{N}^+, v_1 v_2 \mathfrak{N}^-)$ such that the following congruences hold:*

(i) $T_w(\mathbf{g}) \equiv a_w(\mathbf{f}) \cdot \mathbf{g} \mod \mathfrak{P}_n$ *for all primes* $w \nmid v_1 v_2 \mathfrak{N}^+ \mathfrak{N}^-$ *of* $\mathcal{O}_F$.
(ii) $U_w(\mathbf{g}) \equiv a_w(\mathbf{f}) \cdot \mathbf{g} \mod \mathfrak{P}_n$ *for all primes* $w \mid \mathfrak{N}^+ \mathfrak{N}^-$ *of* $\mathcal{O}_F$.
(iii) $U_{v_i}(\mathbf{g}) \equiv \varepsilon_i \cdot \mathbf{g} \mod \mathfrak{P}_n$ *for* $i = 1, 2$.

Proof. See Proposition 11.9 below. □

We then use this result to deduce the following

Theorem 7.3. (The second explicit reciprocity law). *Keep the notations and hypotheses of Proposition 7.2. The equality*

$$\vartheta_{v_1}\left(\zeta(v_2)\right) = \mathcal{L}_{\mathbf{g}} \tag{30}$$

holds in $\widehat{H}^1(K_{\mathfrak{p}^\infty, v_2}, T_{\mathbf{f},n}) \cong \Lambda/\mathfrak{P}^n$, *up to multiplication by elements of* $\mathcal{O}^\times$ *or* $G_{\mathfrak{p}^\infty}$.

Proof. See Theorem 13.2 below. □

Observe that since the choice of n-admissible primes v_1 and v_2 is symmetric in Theorem 7.3, we obtain the following immediate

Corollary 7.4. *The equality*

$$\vartheta_{v_1}\left(\zeta(v_2)\right) = \vartheta_{v_2}\left(\zeta(v_1)\right) \tag{31}$$

holds in $\Lambda/\mathfrak{P}^n$, *up to multiplication by elements of* $\mathcal{O}^\times$ *or* $G_{\mathfrak{p}^\infty}$.

The inductive argument. We now prove the main conjecture divisibility, assuming the existence of an Euler system of classes (27) that satisfy the first and second explicit reciprocity laws (Theorems 7.1 and 7.3). The arguments in this section are essentially the same as those of [50] (based on those of [3, §4] but removing the unneccesary p-isolatedness hypothesis), which extend without much trouble to this setting.

Recall that $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$ denotes the p-adic completion of the Hecke algebra $\mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$ acting on the space of cusp forms $\mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$. Fix an integer $n \geq 1$. Let us now always $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ view as a homomorphism

$$\theta_{\mathbf{f}} : \mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$$

in the natural way, by sending Hecke operators to their associated eigenvalues. We shall often commit an abuse of notation in writing $\mathbf{f}$ to denote this homomorphism $\theta_{\mathbf{f}}$.

Definition Fix an $\mathcal{O}$-algebra homomorphism $\varphi : \Lambda_{\mathcal{O}} \longrightarrow \mathcal{O}'$. Here, $\mathcal{O}'$ is any discrete valuation ring, with maximal ideal denoted by $\mathfrak{P}'$. Let $s_{\mathbf{f}}$ denote the $\mathcal{O}'$-length of $\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\vee \otimes_\Lambda \mathcal{O}'$. Let $2t_{\mathbf{f}}$ denote the $\mathcal{O}'$-valuation of $\varphi\left(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})\right)$ in $\mathcal{O}'/\varphi(\mathfrak{P}')^n$, setting $2t_{\mathbf{f}} = \infty$ if $\varphi\left(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})\right) = 0$.

Proposition 7.5. *Fix integers* $n \geq 1$ *and* $t_0 \geq 0$. *Let* $\widetilde{\mathbf{f}}$ *be an* $\mathcal{O}_0/\mathfrak{P}_{n+t_0}$*-valued eigenform for the completed Hecke algebra* $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$, *with* $\mathbf{f}$ *its projection onto* $\mathcal{O}_0/\mathfrak{P}_n$. *Assume that*

(i) *The homomorphism* $\theta_{\mathbf{f}} : \mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$ *is surjective.*
(ii) *The first and second explicit reciprocity laws (Theorems 7.1 and 7.3) hold.*
(iii) *We have the inequality* $2t_{\mathbf{f}} < 2t_0$.

Then, we have the inequality $s_{\mathbf{f}} \leq 2t_{\mathbf{f}}$.

Before getting to the proof, let us give the following

Corollary 7.6. *Keep the notations and hypotheses of Proposition 7.5. Then, the dual Selmer group* $\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\vee$ *is a torsion* Λ*-module, hence has a characteristic power series* $\mathrm{char}_{\Lambda_{\mathcal{O}}} \mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\vee$. *Moreover, there is an inclusion of ideals*

$$(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})) \subseteq (\mathrm{char}_\Lambda \mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\vee) \text{ in } \Lambda. \tag{32}$$

Proof. Let $X = \mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})^\vee$. Observe that to show the divisibility (32), it suffices by Proposition 6.1 to show the containment $\varphi(\mathcal{L}_{\mathbf{f}}) \in \mathrm{Fitt}_{\mathcal{O}'}(X)$, where $\varphi : \Lambda \longrightarrow \mathcal{O}'$ is any homomorphism, and $\mathcal{O}'$ any discrete valuation ring. Fix a such a ring $\mathcal{O}'$ and homomorphism $\varphi : \Lambda \longrightarrow \mathcal{O}'$. Observe that if $\varphi(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K)) = 0$, then $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K) \in \mathrm{Fitt}_{\mathcal{O}'}(X)$ trivially. If $\varphi(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K)) \neq 0$, then let us take t_0 to be larger than the $\mathcal{O}'$-valuation of $\varphi(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K))$. Using Proposition 7.5 for all $n \geq 0$, it follows that $\varphi(\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K)) \in \mathrm{Fitt}_{\mathcal{O}'}(X)$. Now, observe that once (32) is shown, the nonvanishing of the $\mathfrak{p}$-adic L-function $\mathcal{L}_{\mathfrak{p}}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ (deduced from [16, Theorem 1.4]) implies that $\mathrm{Sel}(\mathbf{f}, K_{\mathfrak{p}^\infty})$ is Λ-cotorsion. The result follows. ☐

Proof of Proposition 7.5. Let us keep all of the notations defined above. We start by defining the following classes. Fix an $(n + t_{\mathbf{f}})$-admissible prime $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$. Define from this an $(n+t_{\mathbf{f}})$-admissible set $\mathfrak{S} = \{v\}$ with respect to $\mathbf{f}$, and a cohomology class

$$\zeta(v) \in \widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}, n+t_{\mathbf{f}}})$$

as in (27). Let $\zeta'_\varphi(v)$ denote the image of $\zeta(v)$ in

$$H^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}, t_{\mathbf{f}}+n}) \otimes_\varphi \mathcal{O}'. \tag{33}$$

Note that (33) is free of rank 1 over $\mathcal{O}'/\varphi(\mathfrak{P})^{n+t_{\mathbf{f}}}$ by Theorem 5.5. Let

$$t = \mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v)\right).$$

Since the residue map ∂_v is a homomorphism, Theorem 7.1 implies that

$$t < \mathrm{ord}_{\mathfrak{P}'}\left(\partial_v(\zeta'_\varphi(v))\right) = t_{\mathbf{f}}. \tag{34}$$

Let us now write $\xi'_\varphi(v)$ to denote an element of the module (33) such that

$$(\mathfrak{P}')^{t_{\mathbf{f}}} \cdot \xi'_\varphi(v) = \zeta'_\varphi(v).$$

Let $\xi''_\varphi(v)$ denote the image of this element $\xi'_\varphi(v)$ in $H^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n}) \otimes_\varphi \mathcal{O}'$.

Lemma 7.7. *The element* $\xi''_\varphi(v)$ *satisfies the following properties.*

1. $\mathrm{ord}_{\mathfrak{P}'}(\xi''_\varphi(v)) = 0$.
2. $\partial_w\left(\xi''_\varphi(v)\right) = 0$ *for all primes* $w \mid v\mathfrak{N}^+$ *in* $\mathcal{O}_F$.
3. $\vartheta_v\left(\xi''_\varphi(v)\right) = 0$.
4. $\mathrm{ord}_{\mathfrak{P}'}\left(\partial_v\left(\xi''_\varphi(v)\right)\right) = t_{\mathbf{f}} - t$.
5. $\partial_v\left(\xi''_\varphi(v)\right)$ *lies in the kernel of the natural surjection*

$$\pi_v : \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty, v}, T_{\mathbf{f},n}) \longrightarrow \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^\vee \otimes_\varphi \mathcal{O}'.$$

Proof. See [3, Lemmas 4.5 and 4.6]. Property (1) follows from the definition of $\xi''_\varphi(v)$, along with the fact that $\mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v)\right) = t$. Property (2) follows from the fact that $\zeta'_\varphi(v) \in \widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}, n+t_{\mathbf{f}}})$, using explicit definitions. Properties (3) and (4) follow from Theorem 7.1. Property (5) follows from the same argument of [3, Lemma 4.6], which uses the global reciprocity law of class field theory. . ☐

Using Lemma 7.7, we can show the following

Proposition 7.8. *If $t_{\mathbf{f}} = 0$, then $\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^\vee$ is trivial.*

Proof. See [3, Proposition 4.7]. If $t_{\mathbf{f}} = 0$, then $\varphi(\mathcal{L}_{\mathbf{f}})$ is a unit. Theorem 7.1 then implies that the residue $\partial_v(\zeta(v))$ generates $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n})$ for $v \subset \mathcal{O}_F$ any n-admissible prime with respect to $\mathbf{f}$. Observe that this renders the projective map π_v in Lemma 7.7(5) trivial. Let us now suppose that $\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^\vee$ were not trivial. Nakayama's lemma would then imply that

$$(\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})[\mathfrak{m}_\Lambda])^\vee = \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})/\mathfrak{m}_\Lambda \neq 0$$

Here, as above, $\mathfrak{m}_\Lambda$ denotes the maximal ideal of Λ. We could then choose a class $s \neq 0$ in $\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})/\mathfrak{m}_\Lambda$. By Theorem 5.6, we could then identify this class s with an element of $H^1(K, A_{\mathbf{f},1})$. By Theorem 6.2, we could then choose another n-admissible prime $q \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ such that $\vartheta_q(s) \neq 0$. But observe then that the projection π_v cannot be trivial, by the nondegeneracy of the local Tate pairing $\langle\ ,\ \rangle_v$. This supplies the desired contradiction. ∎

We are now ready to begin a proof of Proposition 7.5 by induction on $t_{\mathbf{f}}$. By Proposition 7.8, we may assume without loss of generality that $t_{\mathbf{f}} > 0$. Let us write Π_0 to denote the set of $(n + t_0)$-admissible primes for which the valuation $\mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v)\right)$ is minimal.

Lemma 7.9. *Suppose that $t = \mathrm{ord}_{\mathfrak{P}'}(\zeta_\varphi(v))$ with $v \in \Pi_0$. Then, $t < t_{\mathbf{f}}$.*

Proof. See [3, Lemma 4.8]. Suppose otherwise that the claim did not hold. Then, by (34), it would follow that

$$\mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v)\right) = \mathrm{ord}_{\mathfrak{P}'}(\varphi(\mathcal{L}_{\mathbf{f}}))$$

for all $(n + t_{\mathbf{f}})$-admissible primes $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$. By Theorem 5.6, we could then find a nontrivial class s in $H^1(K, A_{\mathbf{f},1}) \cap \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$. By Theorem 6.2, we could then choose an $(n+t_{\mathbf{f}})$-admissible prime v such that $\vartheta_v(s) \neq 0$. Property (4) of Lemma 7.7 implies that the natural image of $\vartheta_v\left(\zeta'_\varphi(v)\right)$ in $H^1(K_v, T_{\mathbf{f},1}) \otimes_\varphi \mathcal{O}'$ does not vanish. Property (5) of Lemma 7.7 implies that this image is orthogonal under the local Tate pairing $\langle\ ,\ \rangle_v$ to the nonvanishing class $\vartheta_s(v)$, contradicting the fact that $\langle\ ,\ \rangle_v$ is a perfect, nondegenerate pairing between the $\mathcal{O}'/\mathfrak{P}'$-vector spaces to which these classes belong. ∎

Let us now fix a prime $v_1 \in \Pi_0$. Let $s \in H^1(K, T_{\mathbf{f}}) \otimes \mathcal{O}'/\mathfrak{P}'$ denote the image of $\zeta'_\varphi(v)$ in

$$\begin{aligned}\widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}}) \otimes \mathcal{O}'/\mathfrak{P}' &\subset \widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty}, T_{\mathbf{f}})/\mathfrak{m}_\Lambda \otimes \mathcal{O}'/\mathfrak{P}' \\ &\subset \widehat{H}^1_{\mathfrak{S}}(K, T_{\mathbf{f}}) \otimes \mathcal{O}'/\mathfrak{P}'.\end{aligned}$$

By Theorem 6.2, there exists an $(n+t_0)$-admissible prime v_2 such that $\vartheta_{v_2}(s) \neq 0$. Here, $\vartheta_{v_2} : \widehat{H}^1_{\mathfrak{S}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{f}}) \longrightarrow \widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{f}})$. Now, observe that we have the relations

$$t = \mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v_1)\right) \leq \mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v_2)\right) \leq \mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_1}(\zeta_\varphi(v_2))\right). \tag{35}$$

The first inequality follows from the definition of Π_0. The second inequality follows from the fact that ϑ_{v_2} is a homomorphism. Corollary 7.4 to the second explicit reciprocity law then gives us the relation

$$\mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_1}\left(\zeta'_\varphi(v_2)\right)\right) = \mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_2}\left(\zeta'_\varphi(v_1)\right)\right). \tag{36}$$

Now, since $\vartheta_{v_2}(s) \neq 0$, we find that

$$\mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_2}\left(\zeta'_\varphi(v_2)\right)\right) = \mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v_1)\right).$$

It follows that the inequalities of (35) are equalities. In particular,

$$\mathrm{ord}_{\mathfrak{P}'}\left(\zeta'_\varphi(v_2)\right) = t.$$

Hence, we find that $v_2 \in \Pi_0$.

Let $\mathbf{g}$ denote the $\mathcal{O}_0/\mathfrak{P}_{(n+t_0)}$-valued eigenform associated to $\mathbf{f}$ and the pair of n-admissible primes (v_1, v_2) with respect to $\mathbf{f}$ by Theorem 7.2. By Theorem 7.1, we have that

$$\vartheta_{v_2}\left(\zeta'_\varphi(v_1)\right) = \mathcal{L}_{\mathbf{g}}.$$

It follows that $t_{\mathbf{g}} = t < t_{\mathbf{f}}$. Now, since $\mathbf{g}$ satisfies all of the hypotheses of Proposition 7.5, we may apply the inductive hypothesis to deduce that $s_{\mathbf{g}} \leq 2t_{\mathbf{g}}$. We may now argue in the same way as [3, pp. 34-35] to conclude the argument. That is, let us write $\mathrm{Sel}_{[v_1v_2]}(K_{\mathfrak{p}^\infty}) \subseteq \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})$ to denote the subgroup of classes that are trivial at primes dividing v_1v_2. Let $\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty})$ denote the group defined by the exactness of the sequence

$$0 \longrightarrow \mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \longrightarrow \mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^\vee \longrightarrow \mathrm{Sel}_{[v_1v_2]}(K_{\mathfrak{p}^\infty})^\vee \longrightarrow 0. \tag{37}$$

Observe that by applying local Tate duality (Proposition 5.1) to the natural inclusion

$$\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty})^\vee \subseteq H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v_1}, A_{\mathbf{f},n}) \oplus H^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty,v_2}, A_{\mathbf{f},n}),$$

we obtain a natural surjection

$$\eta_{\mathbf{f}} : \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_1}, T_{\mathbf{f},n}) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{f},n}) \longrightarrow \mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}).$$

Let $\eta^\varphi_{\mathbf{f}}$ denote the map obtained from $\eta_{\mathbf{f}}$ after tensoring with $\mathcal{O}'$ via φ. By Lemma 5.4, the domain of $\eta_{\mathbf{f}}$ is isomorphic to $(\mathcal{O}'/\varphi(\mathfrak{P})^n)^2$. Property (5) of Lemma 7.7 implies that $\ker(\eta^\varphi_{\mathbf{f}})$ contains the vectors $\left(\partial_{v_1}\left(\xi''_\varphi(v_1)\right), 0\right)$ and $\left(0, \partial_{v_2}\left(\xi''_\varphi(v_2)\right)\right)$ in

$$\left(\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_1}, T_{\mathbf{f},n}) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{f},n})\right) \otimes_\varphi \mathcal{O}' \cong \left(\mathcal{O}'/\varphi(\mathfrak{P})^n\right)^2.$$

Observe that by property (3) of Lemma 7.7, we have the equalities

$$t_{\mathbf{f}} - t_{\mathbf{g}} = \mathrm{ord}_{\mathfrak{P}'}\left(\partial_{v_1}\left(\xi''_\varphi(v_1)\right)\right) = \mathrm{ord}_{\mathfrak{P}'}\left(\partial_{v_2}\left(\xi''_\varphi(v_2)\right)\right).$$

Thus, we obtain the inclusion

$$(\mathfrak{P}')^{t_{\mathbf{f}}-t_{\mathbf{g}}} \in \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_\varphi \mathcal{O}'\right). \tag{38}$$

Let us now repeat the same argument for the eigenform $\mathbf{g}$. That is, consider the short exact sequence

$$0 \longrightarrow \mathrm{Sel}^{\mathbf{g}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \longrightarrow \mathrm{Sel}_{\mathbf{g},n}(K_{\mathfrak{p}^\infty})^\vee \longrightarrow \mathrm{Sel}_{[v_1v_2]}(K_{\mathfrak{p}^\infty})^\vee \longrightarrow 0, \tag{39}$$

and the natural surjective map induced by local Tate duality

$$\eta_{\mathbf{g}} : \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_1}, T_{\mathbf{g},n}) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{g},n}) \longrightarrow \mathrm{Sel}^{\mathbf{g}}_{v_1v_2}(K_{\mathfrak{p}^\infty}).$$

Let $\eta^\varphi_{\mathbf{g}}$ denote the map obtained from $\eta_{\mathbf{g}}$ after tensoring with $\mathcal{O}'$ via φ. The global reciprocity law of class field theory implies that $\ker\left(\eta^\varphi_{\mathbf{g}}\right)$ contains the vectors $\left(\vartheta_{v_1}\left(\xi''_\varphi(v_2)\right), 0\right)$ and $\left(\vartheta_{v_1}\left(\xi''_\varphi(v_1)\right), \vartheta_{v_2}\left(\xi''_\varphi(v_1)\right)\right) = \left(0, \vartheta_{v_2}\left(\xi''_\varphi(v_1)\right)\right)$ in

$$\left(\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_1}, T_{\mathbf{g},n}) \oplus \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v_2}, T_{\mathbf{g},n})\right) \otimes_\varphi \mathcal{O}' \cong \left(\mathcal{O}'/\varphi(\mathfrak{P})^n\right)^2.$$

By Corollary 7.4,

$$\mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_1}\left(\xi''_{\varphi}(v_2)\right)\right) = \mathrm{ord}_{\mathfrak{P}'}\left(\vartheta_{v_2}\left(\xi''_{\varphi}(v_1)\right)\right) = t_{\mathbf{g}} - t = 0.$$

It follows that $\mathrm{Sel}^{\mathbf{g}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_{\varphi} \mathcal{O}'$ is trivial, in which case the natural surjective map of (39) defines an isomorphism

$$\mathrm{Sel}_{\mathbf{g},n}(K_{\mathfrak{p}^\infty})^{\vee} \longrightarrow \mathrm{Sel}_{[v_1v_2]}(K_{\mathfrak{p}^\infty})^{\vee}. \tag{40}$$

Now, Lemma 7.9 implies that $t_{\mathbf{g}} < t_{\mathbf{f}}$. Recall that since $\mathbf{g}$ satisfies the hypotheses of Proposition 7.5, we may invoke the inductive hypothesis to conclude that

$$\varphi\left(\mathcal{L}_{\mathbf{g}}\right)^2 \in \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{\mathbf{g},n}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right). \tag{41}$$

Now,

$$\begin{aligned}(\mathfrak{P}')^{2t_{\mathbf{f}}} &= (\mathfrak{P}')^{2(t_{\mathbf{f}} - t_{\mathbf{g}})} \cdot (\mathfrak{P}')^{2t_{\mathbf{g}}} \\ &\in \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_{\varphi} \mathcal{O}'\right) \cdot \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{\mathbf{g},n}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right)\end{aligned}$$

by (38) and (41). The isomorphism (40) gives an inclusion

$$\begin{aligned}&\mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_{\varphi} \mathcal{O}'\right) \cdot \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{\mathbf{g},n}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right) \\ &\subseteq \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_{\varphi} \mathcal{O}'\right) \cdot \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{[v_1,v_2]}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right).\end{aligned}$$

The short exact sequence (37) and the theory of Fitting ideals then give

$$\begin{aligned}&\mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}^{\mathbf{f}}_{v_1v_2}(K_{\mathfrak{p}^\infty}) \otimes_{\varphi} \mathcal{O}'\right) \cdot \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{[v_1v_2]}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right) \\ &\subseteq \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^{\vee}\right).\end{aligned}$$

In particular, we may conclude that

$$\varphi\left(\mathcal{L}_{\mathbf{f}}\right)^2 \in \mathrm{Fitt}_{\mathcal{O}'}\left(\mathrm{Sel}_{\mathbf{f},n}(K_{\mathfrak{p}^\infty})^{\vee} \otimes_{\varphi} \mathcal{O}'\right),$$

which proves Proposition 7.5.

8. Integral models of Shimura curves

We collect here some facts about integral models of Shimura curves over totally real fields, following the works of Carayol [9], Cerednik [11], Drinfeld [20] and Varshavsky [64], [65], as required for the Euler system construction. The reader should note that in some places, in particular where we describe the work(s) of Carayol [9], we assume for simplicity that the degree d of the totally real field is greater than 1. This however does not affect the validity of the results stated below, for which the $d = 1$ cases have already been established (see also the article of Buzzard [8]).

Reduction at split primes. Fix an indefinite quaternion algebra B over F as above, ramified at all the real places of F save a fixed real place τ_1. Fix a finite prime $v \subset \mathcal{O}_F$ where B is split. Hence, we may fix an isomorphism $B_v \cong \mathrm{M}_2(F_v)$.

Integral models. Fix a compact open subgroup $H \subset \widehat{B}^{\times}$. Let us assume that H factorizes as $H_v \times H^v$, with $H_v \subset B_v^{\times}$ assumed to be maximal, i.e. isomorphic to $\mathrm{GL}_2(\mathcal{O}_{F_v})$. The following theorem was first proved by Morita [47], then subsequently generalized by Carayol in [9]. Recall that let M_H denote the quaternionic Shimura curve associated to the complex manifold $M_H(\mathbf{C}) = M_H(B, X)(\mathbf{C})$.

Theorem 8.1 (Morita-Carayol)**.** *Fix a finite prime $v \subset \mathcal{O}_F$ that splits the quaternion algebra B. Let $H \subset \widehat{B}^\times$ be any compact open subgroup admitting the factorization $H_v \times H^v$, with $H_v \cong \mathrm{GL}_2(\mathcal{O}_{F_v})$. Then, the Shimura curve M_H has good reduction at v. In particular, there exists a smooth, proper model $\mathbf{M}_H$ of M_H over $\mathcal{O}_{(v)}$. This model is unique up to isomorphism.*

Proof. See [47] and [9], where the result is proved for H^v "sufficiently small". If H^v is not "sufficiently small", then it is still possible to obtain an integral model $\mathbf{M}_H$ of M_H over $\mathcal{O}_{(v)}$, as explained in [33, §12] or [15, §3.1.3] (cf. [37, p. 508]). That is, let $H^{'v} \subset H^v$ be any sufficiently small, compact open normal subgroup, and put $H' = H^{'v} \times H_v$. We can then define $\mathbf{M}_H$ to be the quotient of $\mathbf{M}_{H'}$ by the $\mathcal{O}_{(v)}$-linear right action of H/H'. It is then possible to show that this model $\mathbf{M}_H$ is proper and regular if H_v is maximal. Moreover, this construction does not depend on the choice of auxiliary $H^{'v}$. □

Supersingular points. Recall that we fix an isomorphism $B_v \cong \mathrm{M}_2(F_v)$. To be consistent with the notations of Carayol [9], let us write H_v^0 to denote the compact open subgroup of $B_v^\times$ corresponding to $\mathrm{GL}_2(\mathcal{O}_{F_v})$ under this isomorphism. Given an integer $n \geq 1$, let $H_v^n \subset H_v^0$ denote the subgroup corresponding to matrices that are congruent to $1 \mod v^n$. Given any integer $n \geq 0$, we then write

$$M_{n,H^v} = M_{H_v^n \times H^v}$$

to denote the associated Shimura curve. Let $\mathbf{M}_{0,H^v}$ denote the integral model of M_{0,H^v} over $\mathcal{O}_{(v)}$. Consider the right action of the quotient group $H_v^0/H_v^n \cong \mathrm{GL}_2(\mathcal{O}_{F_v}/v^n)$ on the $\mathcal{O}_{F_v}$-module $\left(v^{-n}/\mathcal{O}_{F_v}\right)^2$ given by the rule

$$g \in \mathrm{GL}_2(\mathcal{O}_{F_v}/v^n) \text{ sends } h \in \left(v^{-n}/\mathcal{O}_{F_v}\right)^2 \text{ to } g^{-1}\cdot h.$$

This same group H_v^0/H_v^n acts on the Shimura curve M_{n,H^v} via the quotient M_{0,H^v}. If H^v is "sufficiently small" in the sense of [9], then this action is free. One can then define a scheme of $\mathcal{O}_{F_v}$-modules over M_{0,H^v}:

$$E_n = \left(M_{n,H^v} \times \left(v^{-n}/\mathcal{O}_{F_v}\right)^2\right)/\mathrm{GL}_2(\mathcal{O}_{F_v}/v^n).$$

These E_n form a compatible system with respect to the indices n and H^v, and the inductive limit

$$E_\infty := \lim_{n} E_n$$

is the Barsotti-Tate group associated to the projective limit

$$M_\infty := \varprojlim_{H^v} M_{0,H^v}.$$

A variant of the main method of [9] can be used to find unique extensions of the groups E_n to finite, locally free group schemes $\mathbf{E}_n$ over the smooth, proper $\mathcal{O}_{(v)}$-schemes $\mathbf{M}_{0,H^v}$. As before, the inductive limit

$$\mathbf{E}_\infty := \lim_{n} \mathbf{E}_n$$

is the Barsotti-Tate group associated to the projective limit

$$\mathbf{M}_\infty := \varprojlim_{H} \mathbf{M}_{0,H^v}.$$

The group $\mathbf{E}_\infty$ has also been studied by Drinfeld [21] (cf. also [9, Appendice]) as a "divisible $\mathcal{O}_{F_v}$-module of height 2". In particular, this description gives the following classification of points. Let x be a point in the special fibre $\mathbf{M}_{0,H^v} \otimes \kappa_v$. Consider the covering

$$\mathbf{M}_\infty \longrightarrow \mathbf{M}_{0,H^v},$$

and choose a lift y of x. Consider the pullback of $\mathbf{E}_\infty/\mathbf{M}_\infty$ with respect to the map $y : \text{Spec } (\overline{\kappa}_v) \longrightarrow \mathbf{M}_\infty$. The resulting $\mathcal{O}_{F_v}$-module, written here as $\mathbf{E}_\infty|x$, does not depend on choice of y. Drinfeld's theory shows that there are only two possibilities for this module:

(i) $\mathbf{E}_\infty|x \cong (F_v/\mathcal{O}_{F_v}) \times \Sigma_1$. Here, $(F_v/\mathcal{O}_{F_v})$ is the constant divisible $\mathcal{O}_{F_v}$-module, and Σ_1 the unique formal $\mathcal{O}_{F_v}$ module of height 1.

(ii) $\mathbf{E}_\infty|x \cong \Sigma_2$. Here, Σ_2 is the unique formal $\mathcal{O}_{F_v}$-module of height 2.

Hence, we can make the following

Definition A geometric point x in the special fibre $\mathbf{M}_{0,H^v} \otimes \kappa_v$ is *ordinary* if $\mathbf{E}_\infty|x \cong (F_v/\mathcal{O}_{F_v}) \times \Sigma_1$, and *supersingular* if $\mathbf{E}_\infty|x \cong \Sigma_2$.

Carayol [9, § 11] shows that the set of supersingular points $\mathbf{M}_{0,H^v}^{ss} \otimes \kappa_v$ of $\mathbf{M}_{0,H^v} \otimes \kappa_v$ is finite and nonempty. That is, let D denote the quaternion algebra obtained from B by switching invariants at τ_1 and v. Hence,

$$\text{Ram}(D) = \text{Ram}(B) \cup \{\tau_1, v\}.$$

Proposition 8.2 (Carayol)**.** *Let $\mathbf{M}_{0,H^v}^{ss} = \mathbf{M}_{0,H^v} \otimes \kappa_v$ denote the set of supersingular points of $\mathbf{M}_{0,H^v} \otimes \kappa_v$. There are bijections of finite sets*

$$\begin{aligned}\mathbf{M}_{0,H^v}^{ss} &\cong D^\times \backslash \widehat{B}^{v\times} \times F_v^\times / H^v \times \mathcal{O}_{F_v}^\times \\ &\cong D^\times \backslash \widehat{D}^\times / H^v \times \mathcal{O}_{D_v}^\times.\end{aligned}$$

Proof. See [9, § 11.2] for the case of $d > 1$. The result for $d = 1$ is well known, see for instance the paper of Ribet [53]. □

Geometric connected components. Let $\mathcal{M}_{n,\text{nrd}(H)}$ denote the set of geometric connected components of $M_{n,H}$, viewed as a finite F-scheme. Let $\mathbf{M}_{n,H^v}$ denote the normalization of $\mathbf{M}_{0,H^v}$ in M_{n,H^v}. (Carayol in [9], using the theory of Drinfeld bases with an analogue of the Serre-Tate theorem, shows that $\mathbf{M}_{n,H^v}$ is an integral model of M_{n,H^v} over $\mathcal{O}_{(v)}$. Moreover, it is a regular scheme, finite and flat over $\mathbf{M}_{0,H^v}$). The reciprocity law for canonical models gives an isomorphism

$$\mathcal{M}_{n,\text{nrd}(H)} \longrightarrow \text{Spec } (F'). \tag{42}$$

Here, F' is a certain finite abelian extension of F. Hence, $\mathcal{M}_{n,\text{nrd}(H)}$ extends in a natural way to a normal $\mathcal{O}_{(v)}$-scheme

$$\mathcal{M}_{n,\text{nrd}(H)} \longrightarrow \text{Spec } (\mathcal{O}'_{(v)}),$$

with $\mathcal{O}'_{(v)}$ the ring of v integers of F'. The structural morphism of $M_{n,H}$ in $\mathcal{M}_{n,\text{nrd}(H)}$ is then shown by Carayol [9] to extend to a morphism

$$\mathbf{M}_{n,H} \longrightarrow \mathcal{M}_{n,\text{nrd}(H)}. \tag{43}$$

This morphism is smooth outside of the finite set of supersingular points. Moreover, if x is a geometric point in the special fibre $\mathcal{M}_{n,\text{nrd}(H)} \otimes \kappa$, then the fibre over x in (43) is given by a union of smooth, irreducible curves indexed by $\mathbf{P}^1(\mathcal{O}_{F_v}/v^n)$ that intersect transversally at each supersingular point, and nowhere else.

Reduction at ramified primes. We now consider the reduction of a Shimura curve modulo a prime that divides the discriminant of the underlying quaternion algebra.

Admissible curves. Let us for future reference establish the notion of an *admissible curve*, following Jordan-Livné [35, § 3]. Let R be the ring of integers of any local field, with κ the residue field, and π a uniformizer.

Definition A curve C defined over R is said to be *admissible* if

1. C is proper and flat over R, with a smooth generic fibre.
2. The special fibre of C is reduced. The normalization of each of its irreducible components is isomorphic to $\mathbf{P}^1_\kappa$. The only singular points on the special fibre of C are κ-rational, ordinary double points.
3. The completion of the local ring of C at any one of its singular points x is isomorphic as an R-algebra to $R[[X,Y]]/(XY-\pi^{m(x)})$ for some uniquely determined integer $m(x) \geq 1$.

The special fibre of an admissible curve C/R can be described as a graph, following [39]. In general, a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ here consists of a vertex set $\mathcal{V}$ and an edgeset $\mathcal{E}$. We fix an orientation of $\mathcal{G}$, i.e. a pair of maps $s, t : \mathcal{E} \longrightarrow \mathcal{V}$ that associates to each edge $e \in \mathcal{E}$ a *source* $s(e)$ and a *target* $t(e)$. Each edge e then has an associated opposite edge $\overline{e}$ such that $s(\overline{e}) = t(e)$ with $\overline{\overline{e}} = e$. (The possibility that $\overline{e} = e$ is allowed). A graph $\mathcal{G}$ is said to have a *length* if there exists a function

$$l = l_{\mathcal{G}} : \mathcal{E} \longrightarrow \mathbf{N} = \{1, 2, 3, \ldots\}$$

with $l(e) = l(\overline{e})$. Such a graph has the following standard representation: a marked point corresponds to a vertex; a line joining two marked points corresponds to a pair of edges $\{e, \overline{e}\}$, and has "length" $l(e) = l(\overline{e})$.

Definition Let C be an admissible curve over R, and C_0 its special fibre. The *dual graph* $\mathcal{G}(C) = (\mathcal{V}(C), \mathcal{E}(C))$ *of* C is the following graph:

(i) The vertex set $\mathcal{V}(C)$ consists of the components of C_0.
(ii) The edge set $\mathcal{E}(C)$ consists of the branches of C_0 through each double point of C_0.
(iii) If an edge $e \in \mathcal{E}(C)$ passes through a double point $x \in C_0$, then $\overline{e}$ is the other branch of C_0 passing through x. Moreover, $s(e)$ is the component of C_0 containing x, and $t(e) = s(\overline{e})$.
(iv) The lenth $l(e)$ of an edge $e \in \mathcal{E}(C)$ passing through a double point x is the uniquely determined integer $m(x)$ defined above.

Mumford-Kurihara uniformization. Fix a finite prime v of F. Let $\mathbf{C}_v$ denote the completion of a fixed algebraic closure of F_v. Let $\widehat{\Omega} = \widehat{\Omega}_{F_v}$ denote the v-adic upper half plane over F_v, viewed as a formal scheme over $\mathcal{O}_{F_v}$. Hence, $\widehat{\Omega}$ is flat and locally of finite type over $\mathcal{O}_{F_v}$. It is regular and irreducible, and supports a natural action of $\mathrm{PGL}_2(F_v)$. The generic fibre Ω of $\widehat{\Omega}$, which also supports a natural action of $\mathrm{PGL}_2(F_v)$, is a rigid analytic space with $\mathbf{C}_v$-points given by

$$\Omega(\mathbf{C}_v) = \mathbf{P}^1_{\mathbf{C}_v} - \mathbf{P}^1_{F_v} = \mathbf{C}_v - F_v.$$

We refer the reader to [7], [35] or [48] for further background on this construction. Let us just collect the following crucial facts:

(i) The special fibre of $\widehat{\Omega}$ is reduced and geometrically connected. Its components are smooth, projective, κ_v-rational curves that intersect transversally.

(ii) The dual graph of $\widehat{\Omega}_v$ equipped with its natural $\mathrm{PGL}_2(F_v)$-action is identified canonically with the Bruhat-Tits tree $\Delta = (\mathcal{V}(\Delta), \mathcal{E}(\Delta))$ of $\mathrm{SL}_2(F_v)$ (as constructed for instance in [48, § 1]).
(iii) If $\Gamma \subset \mathrm{PGL}_2(F_v)$ is a discrete, cocompact subgroup, then the quotient $\Gamma \backslash \widehat{\Omega}$ is a formal scheme over $\mathcal{O}_{F_v}$, identified canonically with the completion of some scheme Ω_Γ over $\mathcal{O}_{F_v}$ along its closed fibre.

Theorem 8.3 (Mumford-Kurihara). *If $\Gamma \subset \mathrm{PGL}_2(F_v)$ is any discrete, cocompact subgroup, then the associated scheme Ω_Γ is an admissible curve over $\mathcal{O}_{F_v}$ whose dual graph is canonically isomorphic to $\Gamma\backslash\Delta$ minus loops.*

Proof. See [48] for the torsionfree case, with [39] for the general case. □

Cerednik-Varshavsky uniformization. Fix a finite prime $v \subset \mathcal{O}_F$. Let F_v^{unr} denote the maximal unramified extension of F_v, with $\mathcal{O}_{F_v}^{\mathrm{unr}}$ its ring of integers. Let

$$\widehat{\Omega}^{\mathrm{unr}} = \widehat{\Omega} \times_{\mathrm{Spf}(\mathcal{O}_{F_v})} \mathrm{Spf}(\mathcal{O}_{F_v}^{\mathrm{unr}}).$$

Following Drinfeld [20], we define a natural action of $\mathrm{GL}_2(F_v)$ on $\widehat{\Omega}^{\mathrm{unr}}$ as follows: for any $\gamma \in \mathrm{GL}_2(F_v)$ and $(x, u) \in \widehat{\Omega}^{\mathrm{unr}}$,

$$\gamma \cdot (x, u) = ([\gamma]x, \mathrm{Frob}_v^{n(\gamma)} u).$$

Here, $[\gamma]$ denotes the class of γ in $\mathrm{PGL}_2(F_v)$, and $n(\gamma) = -\mathrm{ord}_v(\det(\gamma))$. Suppose now that $\Gamma \subset \mathrm{GL}_2(F_v)$ is a discrete, cocompact subgroup containing some power of the matrix

$$\begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix}.$$

Then, the quotient $\Gamma\backslash\widehat{\Omega}^{\mathrm{unr}}$ exists, and is given canonically by the completion of a scheme $\Omega_\Gamma^{\mathrm{unr}}$ along its closed fibre. This scheme $\Omega_\Gamma^{\mathrm{unr}}$ is moreover an admissible curve over $\mathcal{O}_{F_v}$.

Let $\mathfrak{N}^+$ and $\mathfrak{N}^-$ be relatively coprime ideals of $\mathcal{O}_F$. Let $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$. Suppose that $\mathfrak{N}^-$ is the squarefree product of a number of primes congruent to $d \mod 2$. Fix a prime divisor $\mathfrak{q}$ of $\mathfrak{N}^-$. Let v be a finite prime of F that does not divide $\mathfrak{N}$. Let B, B', and D be the quaternion algebras over F with ramification sets given by

$$\begin{aligned} \mathrm{Ram}(B) &= \{\tau_2, \ldots \tau_d\} \cup \{w : w \mid \mathfrak{N}^-/\mathfrak{q}\} \\ \mathrm{Ram}(B') &= \mathrm{Ram}(B) \cup \{v, \mathfrak{q}\} \\ \mathrm{Ram}(D) &= \mathrm{Ram}(B) \cup \{\tau_1, \mathfrak{q}\}. \end{aligned}$$

Hence, B is indefinite with $\mathrm{disc}(B) = \mathfrak{N}^-/\mathfrak{q}$, B' is indefinite with $\mathrm{disc}(B') = v\mathfrak{N}^-$, and D totally definite with $\mathrm{disc}(D) = \mathfrak{N}^-$. Note that we have isomorphisms $\widehat{B}^{v\mathfrak{q}} \cong \widehat{B}'^{v\mathfrak{q}} \cong \widehat{D}^{v\mathfrak{q}}$. Let us fix compatible isomorphisms $\widehat{B}^{v\mathfrak{q}} \cong \widehat{B}'^{v\mathfrak{q}}$, $\widehat{B}^{\mathfrak{q}} \cong \widehat{D}^{\mathfrak{q}}$, $\widehat{B}'^{v} \cong \widehat{D}^{v}$. In particular, let us fix an isomorphism

$$\varphi : \widehat{D}^v \cong \widehat{B}'^v. \tag{44}$$

Fix a compact open subgroup $U \subset \widehat{D}^\times$ of level $\mathfrak{N}^+$. Let us assume that

$$U^v = U_S \prod_{w \notin S \cup \{v\}} U_w, \tag{45}$$

where $S \supset \mathrm{Ram}(D)$ is any finite set of places of F. Let us then define

$$H' = \varphi(U^v) \times \mathcal{O}_{B'_v}^\times. \tag{46}$$

Theorem 8.4 (Cerednik-Varshavsky)**.** *Let $M_{H'}$ be a Shimura curve as defined above. Suppose that H' admits the factorization $H' = H'_v \times H'^v$, with H'_v maximal. Then there exists an integral model $\mathbf{M}_{H'}$ of $M_{H'}$ over $\mathcal{O}_{(v)}$ whose completion along its closed fibre is canonically isomorphic to*

$$V_{H'} = \mathrm{GL}_2(F_v)\backslash\widehat{\Omega}^{\mathrm{unr}} \times D^\times\backslash\widehat{D}^\times/U^v. \tag{47}$$

This canonical isomorphism is $\widehat{B}'^{\times v}$-equivariant, where $\widehat{B}'^{\times v}$ acts on $\mathbf{M}_{H'}$ in the natural way, and on $V_{H'}$ via its action on the finite set $D^\times\backslash\widehat{D}^\times/U^v$.

Proof. See [51, § 3.1]. Existence of the integral model $\mathbf{M}_{H'}$ follows from Varshavsky [65, Theorem 5.3], taking $r = 1$, $v_1 = v$, $D = D$, $D^{\mathrm{int}} = B'$, and $\mathcal{G}' = \widehat{D}^{\times v}$. Note that the conditions of [65, Theorem 5.3] are satisfied by [64, 1.5.2]. Identification of the completion of $\mathbf{M}_{H'}$ along its closed fibre with $V_{H'}$ is then a consequence of Cerednik [11, Theorem 2.2]. □

By Theorem 8.3, $\mathbf{M}_{H'}$ is seen easily to be an admissible (hence semistable) curve over $\mathcal{O}_{F_v}$, with dual graph $\mathcal{G}(\mathbf{M}_{H'})$ given canonically by

$$\mathcal{G}(\mathbf{M}_{H'}) = \mathrm{GL}_2(F_v)^+\backslash\Delta \times D^\times\backslash\widehat{D}^\times/U^v. \tag{48}$$

Here, $\mathrm{GL}_2(F_v)^+ \subset \mathrm{GL}_2(F_v)$ denotes the subset of matrices whose determinants have even v-adic valuation, and $\Delta = (\mathcal{V}(\Delta), \mathcal{E}(\Delta))$ is the Bruhat-Tits tree of $\mathrm{SL}_2(F_v)$.

Corollary 8.5. *Let $\mathcal{G}(\mathbf{M}_{H'}) = (\mathcal{V}(\mathbf{M}_{H'}), \mathcal{E}(\mathbf{M}_{H'}))$ denote the dual graph of the special fibre of $\mathbf{M}_{H'}$. We have the following identifications:*

$$\mathcal{V}(\mathbf{M}_{H'}) \cong D^\times\backslash\widehat{D}^\times/U \times \mathbf{Z}/2\mathbf{Z}$$
$$\mathcal{E}(\mathbf{M}_{H'}) \cong D^\times\backslash\widehat{D}^\times/U(v).$$

Here, $U = U_v \times U^v$ with $U_v \cong \mathrm{GL}_2(\mathcal{O}_{F_v})$, and

$$U(v) = \{u \in U : u_v \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \varpi_v\}.$$

Proof. See [51, 3.2]. The result is easy to deduce from the standard identifications $\mathcal{V}(\Delta) \cong \mathrm{PGL}_2(F_v)/\mathrm{PGL}_2(\mathcal{O}_{F_v})$ and $\mathcal{E}(\Delta) \cong \mathrm{GL}_2(F_v)^+/V_0(v)F^\times$, where $V_0(v) \subset \mathrm{GL}_2(\mathcal{O}_{F_v})$ denotes the matrices congruent to $0 \mod v$. □

Orientation of the dual graph. Let us from now on fix the following orientation of the dual graph $\mathcal{G}(\mathbf{M}_{H'}) = (\mathcal{V}(\mathbf{M}_{H'}), \mathcal{E}(\mathbf{M}_{H'}))$ via (8.5). That is, let us call a vertex in $\mathcal{V}(\Delta)$ *even* or *odd* according to its distance from the origin vertex corresponding to the local maximal order $\mathrm{M}_2(\mathcal{O}_{F_v})$ (see [66, § II.2]). Since $\mathrm{GL}_2(F_v)^+$ consists of matrices having even v-adic valuation, its action by conjugation on maximal orders is seen to send even vertices to even vertices, and odd vertices to odd vertices. In particular, the notions of even and odd vertices on the quotient graph $\mathrm{GL}_2(F_v)^+\backslash\Delta$ are well defined. Hence, the notions are also well defined on the dual graph $\mathcal{G}(\mathbf{M}_{H'})$. We then chose an orientation $s, t : \mathcal{E}(\mathbf{M}_{H'}) \longrightarrow \mathcal{V}(\mathbf{M}_{H'})$ such that for any edge $e \in \mathcal{E}(\mathbf{M}_{H'})$, the source $s(e)$ is even, and the target $t(e)$ is odd.

9. Character groups and connected components

Fix a Shimura curve $M = M_H$ associated to an indefinite quaternion algebra B, as above. Fix a prime $v \subset \mathcal{O}_F$. Let F_{v^2} denote the quadratic unramified extension of F_v. Assume that the level H factorizes as $H = H_v \times H^v$, with $H_v \subset B_v^\times$ maximal. Let $\mathbf{M} = \mathbf{M}_H$ denote the integral model of M over $\mathcal{O}_{F_v}$ basechanged to $\mathcal{O}_{F_{v^2}}$. Hence, $\mathbf{M}$ is the basechange of the integral model of Theorem 8.1 if v does not divide the discriminant of B, or else the basechange of the integral model of Theorem 47 if v does divide the discriminant of B. Write

J for the Jacobian of M,
$\mathbf{J}$ for the Néron model of $J \otimes_F F_{v^2}$ over $\mathcal{O}_{F_{v^2}}$.
$\mathbf{J}_v$ for the special fibre $\mathbf{J} \otimes \kappa_{v^2}$,
$\mathbf{J}_v^0$ for the component of the identity of $\mathbf{J}_v$,
Φ_v for the group of geometric connected components $\mathbf{J}_v/\mathbf{J}_v^0$.

Definition Let $\mathrm{Tor}(\mathbf{J}_v^0)$ denote the maximal subtorus of $\mathbf{J}_v^0$. The group $\mathcal{X}_v = \mathrm{Hom}(\mathrm{Tor}(\mathbf{J}_v^0), \mathbf{G}_m)$ is the *character group associated to* $\mathbf{J}_v$.

We have two different descriptions of the character group $\mathcal{X}_v$ and the group of connected components Φ_v: a combinatorial one due to Raynaud [52], and a cohomological one due to Grothendieck [27]. Following Edixhoven [22], we combine these to obtain a third description, which we shall use later to describe the specialization of divisors in the group of connected components Φ_v.

Dual graph description. Let $\mathcal{G}_v = (\mathcal{V}(\mathcal{G}_v), \mathcal{E}(\mathcal{G}_v))$ be the dual graph associated to the special fibre $\mathbf{M} \otimes \kappa_{v^2}$. (In the case where v does not divide the discriminant of B, the dual graph of $\mathbf{M}$ is defined in the same way as for admissible curves). Let $\mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]$ denote the module of formal divisors supported on $\mathcal{V}(\mathcal{G}_v)$ with coefficients in $\mathbf{Z}$, and $\mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0$ the submodule of divisors having degree zero on each connected component of $\mathcal{V}(\mathcal{G}_v)$. Let $\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]$ denote the module of formal divisors supported on $\mathcal{E}(\mathcal{G}_v)$ with coefficients in $\mathbf{Z}$. Fixing an orientation $s, t : \mathcal{E}(\mathcal{G}_v) \longrightarrow \mathcal{V}(\mathcal{G}_v)$, we then define boundary and coboundary maps respectively by

$$d_* = t_* - s_* : \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] \longrightarrow \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)],$$
$$d^* = t^* - s^* : \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] \longrightarrow \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)].$$

Theorem 9.1 (Raynaud). *There is a canonical short exact sequence*

$$0 \longrightarrow \mathcal{X}_v \longrightarrow \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] \xrightarrow{d_*} \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \longrightarrow 0, \tag{49}$$

as well as a canonical isomorphism $\mathcal{X}_v \cong H_1(\mathcal{G}_v, \mathbf{Z})$. *In particular, there is an isomorphism* $\mathcal{X}_v \cong \ker(d_*)$.

Proof. See [52, Proposition 8.1.2] or [5, Theorem 9.6/1] with the discussion in [35]. The result is also described in [22, § 1]. □

Corollary 9.2. *Assume that v does not divide the discriminant of B, and that H has the factorization $H_v \times H^v$ with H_v maximal. Then,* $\mathcal{X}_v \cong \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]^0$.

Proof. The result follows from Carayol's description of singular (= supersingular) points of $\mathbf{M}_{0,H^v}$, cf. [3, Proposition 5.3].

□

Corollary 9.3. *Assume that v does not divide the discriminant of B, and that H has the factorization $H_v \times H^v$ with H_v maximal. Choose an orientation* s, t :

$\mathcal{E}(\mathcal{G}_v) \longrightarrow \mathcal{V}(\mathcal{G}_v)$ such that for any edge $e \in \mathcal{E}(\mathcal{G}_v)$, the source $s(e)$ is even, and the target $t(e)$ is odd. Then, writing δ_* to denote the restriction of the coboundary map d_* to $\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]^0$, the character group $\mathcal{X}_v$ fits into the short exact sequence

$$0 \longrightarrow \mathcal{X}_v \longrightarrow \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]^0 \xrightarrow{\delta_*} \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \longrightarrow 0.$$

Proof. We claim that with this choice of orientation, the elements of $H_1(\mathcal{G}_v, \mathbf{Z})$ belong to $\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]^0$, cf. [3, Proposition 5.5]. □

Vanishing cycles description. The character group $\mathcal{X}_v$ can also be described in language of vanishing cycles of [27, § XIII and XV] to give the following main result.

Theorem 9.4 (Grothendieck)**.** *There is a canonical short exact sequence*

$$0 \longrightarrow \mathcal{X}_v \xrightarrow{\lambda} \widehat{\mathcal{X}}_v \longrightarrow \Phi_v \longrightarrow 0. \tag{50}$$

Here, $\widehat{\mathcal{X}}_v$ denotes the $\mathbf{Z}$*-dual of $\mathcal{X}_v$, and λ denotes the canonical injection induced by the monodromy pairing of* [27, §9]*. In particular, there is a canonical isomorphism* $\operatorname{coker}(\lambda) \cong \Phi_v$.

Proof. See [27, Théorème 11.5]. The result is also described in [22, §1]. □

Comparison description (Edixhoven). Following [22, (1.6)], we may then compare the descriptions of Raynaud and Grothendieck via the following commutative diagram, whose rows and columns are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{X}_v & \xrightarrow{\mathrm{id}} & \mathcal{X}_v & & \\
 & & & & \downarrow & & \downarrow \lambda & & \\
0 & \longrightarrow & \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] & \xrightarrow{d^*} & \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] & \longrightarrow & \widehat{\mathcal{X}}_v & \longrightarrow & 0 \\
 & & \downarrow -\mathrm{id} & & \downarrow d_* & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] & \xrightarrow{\mu_0} & \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 & \longrightarrow & \Phi_v & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here, the composition of μ_0 with the natural inclusion $\mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \longrightarrow \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]$ is given by the map

$$\mu : \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] \longrightarrow \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)], \ \mu(C) = \sum_{C'} (C \cdot C')C',$$

where $C, C' \in \mathcal{V}(\mathcal{G}_v)$ are irreducible components of the special fibre $\mathbf{M} \otimes \kappa_{v^2}$, and $(C \cdot C') \in \mathbf{Z}$ is their intersection product on $\mathbf{M} \otimes \mathcal{O}_{F_{v^2}}$. We refer the reader to [22, §1] or [49, 1.6.5] for more details.

Specialization to connected components. Fix a Shimura curve $M = M_H$ as above, associated to an indefinite quaternion algebra B over F. Fix a prime $v \subset \mathcal{O}_F$ that divides the discriminant of B.

Proposition 9.5. *There is a natural map*

$$\omega_v : \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \longrightarrow \Phi_v. \tag{51}$$

Proof. See the argument of [3, Corollary 5.12] (or that of [43, § 4.4], where it is applied to each connected component of $\mathcal{G}_v$). Let us write the short exact sequence (49) as

$$0 \longrightarrow \mathcal{X}_v \xrightarrow{\gamma} \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] \xrightarrow{d_*} \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] \xrightarrow{\deg} \mathbf{Z} \longrightarrow 0,$$

where deg denotes the degree map. Taking distinguished bases for $\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]$ and $\mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]$, we can then consider the dual exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\text{diag}} \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)] \xrightarrow{d^*} \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] \xrightarrow{\widehat{\gamma}} \widehat{\mathcal{X}}_v \longrightarrow 0,$$

where diag denotes the diagonal map. Let $\lambda_0 : \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] \longrightarrow \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]$ denote the map induced by the monodromy pairing of [27, § 9]. We then deduce that the map λ in the short exact sequence

$$0 \longrightarrow \mathcal{X}_v \xrightarrow{\lambda} \widehat{\mathcal{X}}_v \xrightarrow{c_v} \Phi_v \longrightarrow 0,$$

of Theorem 9.4 must be given by the composition $\widehat{\gamma} \circ \lambda_0 \circ \gamma$. The sought after map ω_v can then be defined as follows:

$$\begin{aligned} \omega_v : \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 &\longrightarrow \Phi_v \\ x &\longmapsto (c_v \circ \widehat{\gamma} \circ \lambda_0)(y), \end{aligned}$$

where y is chosen such that $d_*(y) = x$. □

Specialization of divisors. Let $\mathrm{Div}(M)$ denote the group of divisors on $M \otimes_F \overline{F}$ having coeffients in $\mathbf{Z}$. Let $\mathrm{Div}^0(M)$ denote subgroup of divisors having degree 0 on each connected component of $M \otimes_F \overline{F}$. Hence, the class of a divisor $D \in \mathrm{Div}^0(M)$ under linear equivalence corresponds to an element $[D]$ of $J(\overline{F})$. Given a divisor $D \in \mathrm{Div}(M)$, let $\mathrm{Supp}(D)$ denote its support. Let

$$\mathrm{red}_v : M \otimes_F \overline{F} \longrightarrow \mathcal{V}(\mathcal{G}_v) \cup \mathcal{E}(\mathcal{G}_v)$$

denote the map that sends a point P to either the connected component containing its image in $\mathbf{M} \otimes \kappa_{v^2}$ if P does not reduce to a singular point, or else to its image in $\mathbf{M} \otimes \kappa_{v^2}$ (a singular point). We consider divisors

$$D = \sum_P n_P P \in \mathrm{Div}^0(M)$$

for which the following conditions hold:

(i) Each $P \in \mathrm{Supp}(D)$ is defined over F_{v^2}.
(ii) The image of each $P \in \mathrm{Supp}(D)$ under red_v goes to a vertex in $\mathcal{V}(\mathcal{G}_v)$, i.e. the image of each P in $\mathbf{M} \otimes \kappa_{v^2}$ is a nonsingular point.

Let us for future reference call any such divisor F_{v^2}*-nonsingular*. The reduction mod v of such a divisor D then takes the form

$$\mathrm{red}_v(D) = \sum_P n_P \cdot \mathrm{red}_v(D) \in \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0.$$

Now, consider the specialization map

$$\partial_v : J(F_{v^2}) \longrightarrow \Phi_v.$$

Proposition 9.6. *Let $D \in \mathrm{Div}^0(M)$ be an F_{v^2}-nonsingular divisor, with $[D]$ its class in $J(F_{v^2})$. Then,*

$$\partial_v\left([D]\right) = \omega_v\left(\mathrm{red}_v(D)\right). \tag{52}$$

Proof. The image $\partial_v([D])$ can be described in terms of intersection numbers via Raynaud's description of Φ_v, following the argument of [22, § 2] (cf. [3, Proposition 5.14], [49, 1.6.6]). The result is then simple to deduce. □

10. Hecke module correspondences

Let us return to the setup of Theorem 8.4, keeping all of the notations and hypotheses as above. We must first introduce some more precise notations. To this end, suppose we are given coprime ideals $\mathfrak{M}^+$ and $\mathfrak{M}^-$ of $\mathcal{O}_F$, where $\mathfrak{M}^-$ is the squarefree product of a number of primes congruent to $d-1 \mod 2$. We shall henceforth write $M(\mathfrak{M}^+, \mathfrak{M}^-)$ to denote the Shimura curve of level $\mathfrak{M}^+$ associated to the indefinite quaternion algebra of discriminant $\mathfrak{M}^-$. (Here "indefinite" means that the underlying quaternion algebra is ramified at all but one of the real places of F, hence the condition on $\mathfrak{M}^-$). Fix a prime $v \subset \mathcal{O}_F$ that does not divide $\mathfrak{M}^+\mathfrak{M}^-$. We assume that the level of $M(\mathfrak{M}^+, \mathfrak{M}^-)$ is maximal at v, in which case there exists an integral model $\mathbf{M}(\mathfrak{M}^+, \mathfrak{M}^-)$ of $M(\mathfrak{M}^+, \mathfrak{M}^-)$ over $\mathcal{O}_{F_v}$ (due to Carayol in the case that $v \nmid \mathfrak{M}^-$, or Cerednik-Varshavsky in the case that $v \mid \mathfrak{M}^-$). Let $J(\mathfrak{M}^+, \mathfrak{M}^-)$ denote the jacobian of $M(\mathfrak{M}^+, \mathfrak{M}^-)$, with $\mathbf{J}(\mathfrak{M}^+, \mathfrak{M}^-)$ its Néron model over $\mathcal{O}_{F_v}$, and $\mathbf{J}_v^0(\mathfrak{M}^+, \mathfrak{M}^-)$ the component of the identity of its special fibre. Let $\mathcal{X}_v(\mathfrak{M}^+, \mathfrak{M}^-)$ denote the character group of the maximal torus of $\mathbf{J}_v^0(\mathfrak{M}^+, \mathfrak{M}^-)$. Given an ideal $\mathfrak{m} \subset \mathcal{O}_F$ that does not divide $\mathfrak{M}^+\mathfrak{M}^-$, let $M(\mathfrak{m}; \mathfrak{M}^+, \mathfrak{M}^-)$ denote the Shimura curve $M(\mathfrak{M}^+, \mathfrak{M}^-)$ with maximal level structure at primes dividing $\mathfrak{m}$ inserted. Hence, $M(\mathfrak{m}\mathfrak{M}^+, \mathfrak{M}^-)$ is the Shimura curve of $\mathfrak{m}\mathfrak{M}^+$-level structure associated to the indefinite Shimura curve of discriminant $\mathfrak{M}^-$, with the extra condition that the level be maximal at primes dividing $\mathfrak{m}$.

Suppose now that we are given two coprime ideals $\mathfrak{N}^+$ and $\mathfrak{N}^-$ of $\mathcal{O}_F$ such that $\mathfrak{N}^-$ is the squarefree product of a number of primes congruent to $d \mod 2$. Given a ring $\mathcal{O}$, recall that we let $\mathbb{S}_2(\mathfrak{N}^+, \mathfrak{N}^-; \mathcal{O})$ denote the space of $\mathcal{O}$-valued automorphic forms of weight 2 and level $\mathfrak{N}^+$ on the totally definite quaternion algebra of discriminant $\mathfrak{N}^-$ over F. Let $\mathbb{T}(\mathfrak{N}^+, \mathfrak{N}^-)$ denote the associated algebra of Hecke operators. Given an ideal $\mathfrak{n} \subset \mathcal{O}_F$ that does not divide the product $\mathfrak{N}^+\mathfrak{N}^-$, let $\mathbb{S}_2(\mathfrak{n}; \mathfrak{N}^+, \mathfrak{N}^-; \mathbf{Z})$ denote the space of forms of level $\mathfrak{n}\mathfrak{N}^+$, with the level being maximal at primes dividing $\mathfrak{n}$. Fix a prime $v \subset \mathcal{O}_F$ that does not divide the product $\mathfrak{N}^+\mathfrak{N}^-$. Let us now take $\mathfrak{M}^+ = \mathfrak{N}^+$ and $\mathfrak{M}^- = v\mathfrak{N}^-$ in the setup above. In particular, we consider the Shimura curve $M(\mathfrak{N}^+, v\mathfrak{N}^-)$, with $\mathcal{X}_v(\mathfrak{N}^+, v\mathfrak{N}^-)$ the associated character group, and $\mathcal{G}_v = (\mathcal{V}(\mathcal{G}_v), \mathcal{E}(\mathcal{G}_v))$ the associated dual graph. Putting things together, we obtain the following diagram à la Ribet [54], where the rows are exact, and the vertical arrows are isomorphisms:

(53)

$$\begin{array}{ccccc}
 & & \mathcal{X}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2 \\
 & & \uparrow & & \uparrow \\
 & & \mathrm{Div}^0\left(\mathbf{M}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right) & \longrightarrow & \mathrm{Div}^0\left(\mathbf{M}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right)^2 \\
 & & \uparrow & & \uparrow \\
\mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \longrightarrow & \mathbf{Z}[\mathcal{E}(\mathcal{G}_v)] & \xrightarrow{d_*} & \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \longrightarrow & \mathbf{Z}[D^\times\backslash\widehat{D}^\times/U(v)] & \longrightarrow & \mathbf{Z}[D^\times\backslash\widehat{D}^\times/U]^0\times\mathbf{Z}/2\mathbf{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}_2(U(v),D;\mathbf{Z})^{\text{v-new}} & \longrightarrow & \mathbb{S}_2(U(v),D;\mathbf{Z}) & \xrightarrow{\alpha_*,\beta_*} & \mathbb{S}_2(U,D;\mathbf{Z})^{\oplus 2}
\end{array}$$

Here, we start with the exact sequence of Theorem 49. The identifications

$$\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]\cong\mathbf{Z}[D^\times\backslash\widehat{D}^\times/U(v)],\ \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0\cong\mathbf{Z}[D^\times\backslash\widehat{D}^\times/U]^0\times\mathbf{Z}/2\mathbf{Z}$$

come from Corollary 8.5, making the bottom exact sequence a direct consequence of definitions. The identification

$$\mathbf{Z}[D^\times\backslash\widehat{D}^\times/U]^0\cong\mathrm{Div}^0\left(\mathbf{M}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right)$$

comes from Proposition 8.2. The identification

$$\mathbf{Z}[\mathcal{E}(\mathcal{G}_v)]\cong\mathrm{Div}^0\left(\mathbf{M}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right)$$

is deduced from Corollary 9.3. The identifications

$$\mathrm{Div}^0\left(\mathbf{M}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right)\cong\mathcal{X}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})$$
$$\mathrm{Div}^0\left(\mathbf{M}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^{ss}\otimes\kappa_{\mathfrak{q}}\right)\cong\mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})$$

come from Corollary 9.2. In particular, we use (53) deduce the following result. Recall that we write $\eta_v=\begin{pmatrix}0&1\\0&\varpi_v\end{pmatrix}$, where ϖ_v is a fixed uniformizer of v. Let us write the associated monodromy exact sequences of Theorem 9.4 as

$$\begin{array}{ccccc}
\mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \xrightarrow{\lambda} & \widehat{\mathcal{X}}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \longrightarrow & \Phi_v(\mathfrak{N}^+,v\mathfrak{N}^-) \\
\mathcal{X}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \xrightarrow{\lambda(v\mathfrak{q})} & \widehat{\mathcal{X}}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \longrightarrow & \Phi_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) \\
\mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \xrightarrow{\lambda(\mathfrak{q})} & \widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \longrightarrow & \Phi_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}).
\end{array}$$

Theorem 10.1. *We have the following diagram of* $\mathbb{T}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})$*-modules, where the rows are exact:*

$$\begin{array}{ccccc}
\widehat{\mathcal{X}}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \longleftarrow & \widehat{\mathcal{X}}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \xleftarrow{1_*\oplus\eta_{v*}} & \widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2 \\
\uparrow\lambda & & \uparrow\lambda(v\mathfrak{q}) & & \uparrow\lambda(\mathfrak{q}) \\
\mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-) & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q}) & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2.
\end{array}$$

Proof. We extract the bottom exact sequence from that of the top of (53). The top exact sequence is then induced by duality. □

Corollary 10.2 (Jacquet-Langlands)**.**

(i) *We have the following diagram of* $\mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$*-modules, where the rows are exact:*

$$\begin{array}{ccccc} \widehat{\mathcal{X}}_v(\mathfrak{N}^+, v\mathfrak{N}^-) & \longleftarrow & \widehat{\mathcal{X}}_{\mathfrak{q}}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q}) & \xleftarrow{1_* \oplus \eta_{v*}} & \widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2 \\ \uparrow \lambda & & \uparrow \lambda(v\mathfrak{q}) & & \uparrow \lambda(\mathfrak{q}) \\ \mathcal{X}_v(\mathfrak{N}^+, v\mathfrak{N}^-) & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q}) & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2. \end{array}$$

(ii) *The subring of of* $\mathrm{End}(J(\mathfrak{N}^+, v\mathfrak{N}^-))$ *generated by Hecke correspondences on* $M(\mathfrak{N}^+, v\mathfrak{N}^-)$ *is isomorphic to the Hecke algebra* $\mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$*.*

Proof. To show (i), we simply take into account the identifications induced by the vertical arrows of (53) to obtain a diagram of $\mathbb{T}(\mathfrak{N}^+, \mathfrak{N}^-)$-modules. The result then follows from the Jacquet-Langlands correspondence. Since $M(\mathfrak{N}^+, v\mathfrak{N}^-) \otimes F_v$ has a semistable model over $\mathcal{O}_{F_v}$, the general theory of Néron models (see [5][§ 9], [49][1.6.2]) shows that $J(\mathfrak{N}^+, v\mathfrak{N}^-)$ has purely toric reduction at v. Hence, the Hecke algebra $\mathbb{T}(\mathfrak{N}^+, v\mathfrak{N}^-)$ acting faithfully on $\mathcal{X}_v(\mathfrak{N}^+, v\mathfrak{N}^-)$ can be identified with the subalgebra of $\mathrm{End}(J(\mathfrak{N}^+, v\mathfrak{N}^-))$ generated by Hecke correspondences. The result then also follows from the Jacquet-Langlands correspondence. □

11. Weak level raising

Recall that given a given a quaternion algebra B and a level $H \subset \widehat{B}^\times$, we let $\mathbb{T} = \mathbb{T}(H, B)$ denote the $\mathbf{Z}$-algebra generated by the standard Hecke operators T_w and S_w for all primes $w \subset \mathcal{O}_F$ (where they are defined). Let us adopt the convention of writing U_w for the operators T_w if $w \subset \mathcal{O}_F$ is a prime that divides the level.

Definition A maximal ideal $\mathfrak{m} \subset \mathbb{T}$ is said to be *Eisenstein* if there exists an ideal $\mathfrak{f} \subset \mathcal{O}_F$ such that for all but finitely many primes $w \subset \mathcal{O}_F$ that split completely in the ray class field of $\mathfrak{f} \mod F$, $T_w - 2 \in \mathfrak{m}$ and $S_w - 1 \in \mathfrak{m}$.

Proposition 11.1 (Jarvis)**.** *A maximal ideal* $\mathfrak{m} \subset \mathbb{T}$ *associated to an eigenform* $\mathbf{f}$ *is Eisenstein if and only if the associated Galois representation* $\rho_{\mathbf{f}} : G_F \longrightarrow \mathrm{GL}_2(\mathcal{O})$ *is reducible.*

Proof. See [34, § 3], which extends to totally real fields [17][Proposition 2]. □

Level raising at one prime.

Theorem 11.2 (Rajaei)**.** *Let* $\mathfrak{m} \subset \mathbb{T}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})$ *be any non-Eisenstein maximal ideal.*

(i) *We have the following diagram of* $\mathbb{T}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})$*-modules, where the rows are exact:*

$$\begin{array}{ccccc} \widehat{\mathcal{X}}_v(\mathfrak{N}^+, v\mathfrak{N}^-)_{\mathfrak{m}} & \longleftarrow & \widehat{\mathcal{X}}_{\mathfrak{q}}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}} & \xleftarrow{1_* \oplus \eta_{v*}} & \widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}} \\ \uparrow \lambda & & \uparrow \lambda(v\mathfrak{q}) & & \uparrow \lambda(\mathfrak{q}) \\ \mathcal{X}_v(\mathfrak{N}^+, v\mathfrak{N}^-)_{\mathfrak{m}} & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}} & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}. \end{array}$$

(ii) *We have an isomorphism of* $\mathbb{T}(v\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})$*-modules*

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}} / \left(U_v^2 - S_v\right) \cong \Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-)_{\mathfrak{m}}.$$

Proof. For (i), see [51, Theorem 3], which is a generalization to totally real fields of the method of Ribet [54]. For (ii), see [51, Corollary 4], which shows that there is an isomorphism of $\mathbb{T}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})$-modules

$$\widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}/\left(U_v^2-S_v\right)\cong\widehat{\mathcal{X}}_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}}/\mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}}.$$

By [51, Proposition 5], there is an isomorphism

$$\widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}}\cong\mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}}.$$

The result then follows from the identification of Theorem 9.4. □

Corollary 11.3 (Jacquet-Langlands)**.** *Let* $\mathfrak{m}\subset\mathbf{T}(\mathfrak{N}^+,\mathfrak{N}^-)$ *be any non-Eisenstein maximal ideal.*

(i) *We have the following diagram of* $\mathbf{T}(\mathfrak{N}^+,\mathfrak{N}^-)$*-modules, where the rows are exact:*

$$\begin{array}{ccccc}\widehat{\mathcal{X}}_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}} & \longleftarrow & \widehat{\mathcal{X}}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}} & \xleftarrow{1_*\oplus\eta_{v*}} & \widehat{\mathcal{X}}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}\\ \uparrow\lambda & & \uparrow\lambda(v\mathfrak{q}) & & \uparrow\lambda(\mathfrak{q})\\ \mathcal{X}_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}} & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(v\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}} & \longrightarrow & \mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}.\end{array}$$

(ii) *We have an isomorphism of* $\mathbf{T}(\mathfrak{N}^+,\mathfrak{N}^-)$*-modules*

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}/\left(U_v^2-S_v\right)\cong\Phi_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}}.$$

Proof. The result follows directly from Theorem 11.2 with Corollary 10.2. □

Theorem 11.4. *Fix an* $\mathbf{f}\in\mathcal{S}_2(\mathfrak{N}^+,\mathfrak{N}^-)$ *an eigenform,* $v\subset\mathcal{O}_F$ *a prime, and* $n\geq 1$ *an integer. Assume that the associated Galois representation* $\rho_{\mathbf{f}}$ *is residually irreducible, and that the prime* $v\subset\mathcal{O}_F$ *is* n*-admissible with respect to* $\mathbf{f}$*. Then, there exists a mod* $\mathfrak{P}_n$ *eigenform* $\mathbf{f}_v$ *associated to surjective homomorphism*

$$\mathbf{T}_0(\mathfrak{N}^+,v\mathfrak{N}^-)\longrightarrow\mathcal{O}_0/\mathfrak{P}_n$$

such that the following properties hold:

(i) $T_w\left(\mathbf{f}_v\right)\equiv T_w\left(\mathbf{f}\right)\mod\mathfrak{P}_n$ *for all primes* $w\nmid v\mathfrak{N}$ *of* $\mathcal{O}_F$*.*
(ii) $U_w\left(\mathbf{f}_v\right)\equiv U_w\left(\mathbf{f}\right)\mod\mathfrak{P}_n$ *for all primes* $w\mid\mathfrak{N}$ *of* $\mathcal{O}_F$*.*
(iii) $U_v\left(\mathbf{f}_v\right)\equiv\varepsilon\cdot\mathbf{f}_v\mod\mathfrak{P}_n$*.*

Here, $\varepsilon\in\{\pm1\}$ *is the integer for which* $\mathfrak{P}_n$ *divides* $\mathbf{N}(v)+1-\varepsilon\cdot a_v(\mathbf{f})$*.*

We describe two proofs of this result.

Proof 1. (Rajaei, Ribet, Taylor) Ribet first proved the result for $d=1$ and $n=1$ in [54], where the $n>1$ case follows by a simple inductive argument. The general case of $d\geq1$ and $n=1$ is shown in Rajaei [51, Main Theorem 3 and Corollary 4], granted certain technical hypotheses on F that always hold in our setting. (That is, if $\mathbf{Q}(\zeta_p)^+\subset F$, then it is assumed that $\overline{\rho}_{\mathbf{f}}$ is not induced from a character. If $d\equiv 0\mod 2$, then it is assumed that the associated automorphic representation $\pi_{\mathbf{f}}$ is either special or supercuspidal at some finite prime $w\nmid\mathfrak{p}v\subset\mathcal{O}_F$). The result is also proved by Taylor [60, Theorem 1] for d even with $\mathfrak{N}^-=\mathcal{O}_F$. The general case with $n>1$ can be deduced from the methods of Ribet [54, § 7] developed by Rajaei [51, §4]. That is, in the setup above, one looks at the Hecke module structure(s) of

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q};\mathfrak{N}^+,\mathfrak{N}^-/\mathfrak{q})^2_{\mathfrak{m}}/\left(U_v^2-S_v\right)\cong\Phi_v(\mathfrak{N}^+,v\mathfrak{N}^-)_{\mathfrak{m}}.$$

to deduce the result. □

Proof 2. (Kisin) If $\mathbf{f}$ is associated via Jacquet-Langlands to an eigenform on a totally definite quaternion algebra over F, then Kisin [38, § 3.1] gives a different proof for arbitrary totally real fields with minor hypotheses on the level structure [38, (3.1.1) and (3.1.2)]. Note that by our hypotheses imposed on the integer factorization of $\mathfrak{N}$, the eigenform $\mathbf{f}$ is always associated via Jacquet-Langlands to an eigenform on a totally definite quaternion algebra. So, the results of [38, § 3.1] apply. □

We now impose the following crucial hypothesis on the Galois representation $\rho_{\mathbf{f}}$ associated to $\mathbf{f}$, following the approach of Pollack-Weston [50]:

Hypothesis 11.5 (Multiplicity one for character groups.)**.** *Given* $\mathfrak{m} \subset \mathbf{T}(\mathfrak{N}^+, \mathfrak{N}^-)$ *a non-Eisenstein maximal ideal, the completed character group*

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}_{\mathbf{f}}} \otimes \mathcal{O}$$

is free of rank one over completed Hecke algebra $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$*.*

Remark The result is well known for $F = \mathbf{Q}$, see for instance the explanation given in [50, Theorem 6.2]. The general case is treated in Cheng [13], granted suitable technical hypotheses. Rather than state these here explicitly, we shall just assume Hypothesis 11.5 in what follows to reveal the mechanism of the proof.

Let $\mathcal{I}_{\mathbf{f}}$ denote the kernel of the natural homomorphism $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$ associated to $\mathbf{f}$, and $\mathcal{I}_{\mathbf{f}_v}$ to denote that of $\mathbf{T}_0(\mathfrak{N}^+, v\mathfrak{N}^-) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$ associated to $\mathbf{f}_v$. We obtain the following crucial result.

Corollary 11.6. *If* $\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}_{\mathbf{f}}} \otimes \mathcal{O}$ *is free of rank* 1 *over* $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$*, then*

(i) *There is an isomorphism of groups* $\Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-)/\mathcal{I}_{\mathbf{f}_v} \cong \mathcal{O}_0/\mathfrak{P}_n$.
(ii) *There is an isomorphism of* G_F*-modules* $\mathrm{Ta}_p\left(J(\mathfrak{N}^+, v\mathfrak{N}^-)\right)/\mathcal{I}_{\mathbf{f}_v} \cong T_{\mathbf{f},n}$.

Proof. For (i), see [3, Theorem 5.15 (2)], or the generalization to totally real fields given in [42, Theorem 3.3] (neither of which requires $\mathfrak{P}$-isolatedness). The idea in either case is to observe that the freeness condition implies that

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})/\mathcal{I}_{\mathbf{f}} \cong \mathcal{O}_0/\mathfrak{P}_n.$$

It can then be deduced via the Ribet exact sequence (Theorem 10.1) that we have isomorphisms

$$\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2/\left(\mathcal{I}_{\mathbf{f}}, U_v - \varepsilon\right) \cong \mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})^2/\left(\mathcal{I}_{\mathbf{f}}, U_v^2 - 1\right) \cong \mathcal{O}_0/\mathfrak{P}_n.$$

The result then follows from the isomorphism of Theorem 11.2 (ii). For (ii), we use the argument of Pollack-Weston [50, Proposition 4.4]. That is, a straightforward generalization of the second part of the proof [3, Lemma 5.16] shows that the following property is satisfied: for each element $z \in \Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-)/\mathcal{I}_{\mathbf{f}_v}$, there exists an element $t \in J(\mathfrak{N}^+, v\mathfrak{N}^-)[p^{n'}](\overline{F}_{v^2})/\mathcal{I}_{\mathbf{f}_v}$ for some integer $n' \geq 1$ that maps to z under the natural map

$$J(\mathfrak{N}^+, v\mathfrak{N}^-)[p^{n'}](\overline{F}_{v^2})/\mathcal{I}_{\mathbf{f}_v} \longrightarrow \Phi(\mathfrak{N}^+, v\mathfrak{N}^-)/\mathcal{I}_{\mathbf{f}_v}.$$

We may then use the same argument as given in [50, Proposition 4.4] to show the result. □

Recall that given an integer $m \geq 1$, we write $K_{\mathfrak{p}^m}$ to denote the m-th layer of the dihedral $\mathbf{Z}_p^{\delta}$-extension of K. Let us define the corresponding m-th level component group to be the direct sum of component groups

$$\Phi_{v,m}(\mathfrak{N}^+, v\mathfrak{N}^-) = \bigoplus_{\mathfrak{v}|v} \Phi_{\mathfrak{v}}(\mathfrak{N}^+, v\mathfrak{N}^-).$$

Here, the sum ranges over all primes $\mathfrak{v}$ above v in $K_{\mathfrak{p}^m}$, and $\Phi_{\mathfrak{v}}(\mathfrak{N}^+, v\mathfrak{N}^-)$ denotes the component group associated to the jacobian $\mathbf{J}(\mathfrak{N}^+, v\mathfrak{N}^-)$ at $\mathfrak{v}$. Let us then write

$$\widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-) = \varprojlim_m \Phi_{v,m}(\mathfrak{N}^+, v\mathfrak{N}^-)$$

to denote the inverse limit with respect to norm maps. Recall that v must be inert in K by hypothesis (ii) of n-admissibility, and hence splits completely in $K_{\mathfrak{p}^\infty}$ by class field theory. It follows that we have an isomorphism

$$\widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-) \cong \Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-) \otimes \Lambda.$$

Here, as before, we let Λ denote the $\mathcal{O}$-Iwasawa algebra $\mathcal{O}[[G_{\mathfrak{p}^\infty}]]$. The isomorphism $\Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} \cong \mathcal{O}_0/\mathfrak{P}_n$ of Corollary 11.6 (ii) then allows us to make the identification

$$\widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} \cong \Lambda/\mathfrak{P}^n.$$

Now, write

$$\widehat{J}(\mathfrak{N}^+, v\mathfrak{N}^-)(K_{\mathfrak{p}^\infty})/I_{\mathbf{f}_v} = \varprojlim_m J(\mathfrak{N}^+, v\mathfrak{N}^-)(K_{\mathfrak{p}^m})/I_{\mathbf{f}_v}$$

to denote the inverse limit with respect to norm maps. Taking the inverse limit of the associated specialization maps to groups of connected components, we then obtain a completed specialization map

$$\widehat{\partial}_v : \widehat{J}(\mathfrak{N}^+, v\mathfrak{N}^-)(K_{\mathfrak{p}^\infty})/I_{\mathbf{f}_v} \longrightarrow \widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} \cong \Lambda/\mathfrak{P}^n. \tag{54}$$

Corollary 11.7. *If $\mathcal{X}_{\mathfrak{q}}(\mathfrak{q}; \mathfrak{N}^+, \mathfrak{N}^-/\mathfrak{q})_{\mathfrak{m}_{\mathbf{f}}} \otimes \mathcal{O}$ is free of rank 1 over $\mathbf{T}_0(\mathfrak{N}^+, \mathfrak{N}^-)$, then*

(i) *We have isomorphisms*

$$\Phi_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} \cong H^1_{\mathrm{sing}}(K_v, T_{\mathbf{f},n}) \tag{55}$$

$$\widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} \cong \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}), \tag{56}$$

both of which are canonical up to choice of isomorphism

$$\mathrm{Ta}_p\left(J(\mathfrak{N}^+, v\mathfrak{N}^-)\right)/\mathcal{I}_{\mathbf{f}_v} \cong T_{\mathbf{f},n}.$$

(ii) *We have a commutative diagram*

$$\begin{array}{ccc} \widehat{J}(\mathfrak{N}^+, v\mathfrak{N}^-)(K_{\mathfrak{p}^\infty})/I_{\mathbf{f}_v} & \xrightarrow{\mathfrak{K}} & \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n}) \\ {\scriptstyle\widehat{\partial}_v}\downarrow & & {\scriptstyle\partial_v}\downarrow \\ \widehat{\Phi}_v(\mathfrak{N}^+, v\mathfrak{N}^-)/I_{\mathbf{f}_v} & \xrightarrow{(56)} & \widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}). \end{array} \tag{57}$$

Here, $\mathfrak{K}$ denotes the Kummer map, ∂_v the residue map, and $\widehat{\partial}_v$ the induced specialization map of (54).

Proof. See [3, Corollary 5.18], the same argument applies granted the results of Corollary 11.6. ☐

Level raising at two primes. Keep all of the notations of the section above. Let us now fix two n-admissible primes $v_1, v_2 \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ such that

$$\mathbf{N}(v_i) + 1 - \varepsilon_i \cdot a_{v_1}(\mathbf{f}) \equiv 0 \mod \mathfrak{P}_n$$

for each of $i = 1, 2$. As before, we keep all of the setup and hypotheses of Theorem 8.4, taking $v = v_1$ so that the indefinite quaternion algebra B has discriminant $v_1\mathfrak{N}^-$. Consider the composition of maps

$$\begin{aligned} J(\mathfrak{N}^+, v_1\mathfrak{N}^-)/I_{\mathbf{f}_{v_1}} &\xrightarrow{\mathfrak{K}} H^1\left(K_{v_2}, \mathrm{Ta}_p\left(J(\mathfrak{N}^+, v_1\mathfrak{N}^-)\right)/I_{\mathbf{f}_{v_1}}\right) \\ &\xrightarrow{\phi} H^1(K_{v_2}, T_{\mathbf{f},n}). \end{aligned}$$

Here, $\mathfrak{K}$ denotes the Kummer map, and ϕ is induced from a fixed isomorphism $\mathrm{Ta}_p\left(J(\mathfrak{N}^+, v_1\mathfrak{N}^-)\right)/\mathcal{I}_{\mathbf{f}_{v_1}} \cong T_{\mathbf{f},n}$. Now, since the representation $T_{\mathbf{f},n}$ is unramified at v_2, we have isomorphisms

$$H^1(K_{v_2}, T_{\mathbf{f},n}) \cong H^1_{\mathrm{unr}}(K_{v_2}, T_{\mathbf{f},n}) \cong \mathcal{O}_0/\mathfrak{P}_n,$$

where the latter isomorphism comes from Lemma 5.4. Since $J(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ has good reduction at v_2, reduction mod v_2 gives the isomorphism

$$\mathrm{red}_{v_2} : J(\mathfrak{N}^+, v_1\mathfrak{N}^-)(K_{v_2})/I_{\mathbf{f}_{v_1}} \cong \mathbf{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/I_{\mathbf{f}_{v_1}}.$$

Here (as before), $\mathbf{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ denotes the Néron model over $\mathcal{O}_{F_{v_2}}$ of the Jacobian $J(\mathfrak{N}^+, v_1\mathfrak{N}^-)$. Since $\mathcal{I}_{\mathbf{f}_v}$ is not Eisenstein, the natural inclusion

$$\mathrm{Div}^0(M(\mathfrak{N}^+, v_1\mathfrak{N}^-)) \subset \mathrm{Div}(M(\mathfrak{N}^+, v_1\mathfrak{N}^-))$$

induces an isomorphism

$$\mathrm{Div}^0((\mathfrak{N}^+, v_1\mathfrak{N}^-))/\mathcal{I}_{\mathbf{f}_v} \cong \mathrm{Div}(M(\mathfrak{N}^+, v_1\mathfrak{N}^-))/\mathcal{I}_{\mathbf{f}_v}.$$

We thus obtain an injective map

$$\mathrm{Div}\left(\mathbf{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}\right) \longrightarrow \mathbf{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/I_{\mathbf{f}_{v_1}}.$$

Hence, we obtain via the composition $\gamma = \phi \circ \mathfrak{K} \circ \mathrm{red}_{v_2}^{-1}$ a map

$$\gamma : \mathrm{Div}\left(\mathbf{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}\right) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n. \tag{58}$$

Recall that $J(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ has the structure of a $\mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$-module, as explained above. Let $T_w \in \mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ denote the Hecke operator at a prime $w \nmid v_1\mathfrak{N}^+\mathfrak{N}^-$, and $U_w \in \mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ the operator at a prime $w \mid v_1\mathfrak{N}^+\mathfrak{N}^-$. For each such operator T_w, let us write $\overline{T}_w$ to denote the image of T_w in $\mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)/\mathcal{I}_{\mathbf{f}_{v_1}}$. Similarly, for each such operator U_w, let us write $\overline{U}_w$ to denote the image of U_w in $\mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)/\mathcal{I}_{\mathbf{f}_{v_1}}$. Observe that by definition of the homomorphism $\mathbf{f}_{v_1}$, we have the relations $\overline{T}_w \equiv a_w(\mathbf{f}) \mod \mathfrak{P}_n$, $\overline{U}_w \equiv a_w(\mathbf{f}) \mod \mathfrak{P}_n$, and $\overline{T}_{v_1} \equiv \varepsilon_1 \mod \mathfrak{P}_n$.

Lemma 11.8. *The following relations hold for* $x \in \mathrm{Div}\left(\mathbf{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}\right)$*:*

(i) $\gamma(T_w \cdot x) = \overline{T}_w \cdot \gamma(x)$ *for all primes* $w \nmid v_1v_2\mathfrak{N}^+\mathfrak{N}^-$ *of* $\mathcal{O}_F$*.*
(ii) $\gamma(U_w \cdot x) = \overline{U}_w \cdot \gamma(x)$ *for all primes* $w \mid v_1\mathfrak{N}^+\mathfrak{N}^-$ *of* $\mathcal{O}_F$*.*
(iii) $\gamma(T_{v_2} \cdot x) = \overline{T}_{v_2} \cdot \gamma(x)$

(iv) $\gamma(\mathrm{Frob}_{v_2} \cdot x) = \varepsilon_2 \cdot \gamma(x)$.

Proof. See [3, Lemma 9.1], the same proof carries over to this setting. That is, by Lemma 5.4, we can identify $H^1_{\mathrm{unr}}(K_{v_2}, T_{\mathbf{f},n})$ with the module

$$T_{\mathbf{f},n}/\left(\mathrm{Frob}_{v_2}^2 - 1\right)$$

of $G_{K_{v_2}}$-coinvariants of $T_{\mathbf{f},n}$. We deduce from this that γ sends a point x to the image of $\left((\mathrm{Frob}_{v_2}^2 - 1)/\mathfrak{P}_n\right) x$ in $T_{\mathbf{f},n}/\left(\mathrm{Frob}_{v_2}^2 - 1\right)$. This implies the first two relations. The second two relations then follow from the Eichler-Shimura relations (as given for instance in [9, § 10.3]), by the same argument used in [3, Lemma 9.1]. □

Recall that in the setup above, we let D denote the totally definite quaternion algebra over F obtained from B by switching invariants at τ_1 and v_1. Hence, $\mathrm{disc}(D) = \mathfrak{N}^-$. Let D' denote the totally definite quaternion algebra over F obtained from D by switching invariants at v_1 and v_2. Hence, $\mathrm{disc}(D') = v_1 v_2 \mathfrak{N}^-$. Let $U' \subset \widehat{D}'^\times$ be the compact open subgroup defined by $U' = H'^{v_2} \times \mathcal{O}^\times_{D'_{v_2}}$. Note that we have an isomorphism $U'^{v_2} \cong H'^{v_2}$. Note as well that by Proposition 8.2, we have isomorphisms

$$\mathrm{Div}\left(\mathbf{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}\right) \cong \mathbf{Z}[D'^\times \backslash \widehat{D}'^\times / U'] \cong \mathbb{S}_2(\mathfrak{N}^+, v_1 v_2 \mathfrak{N}^-; \mathbf{Z}). \tag{59}$$

Proposition 11.9. *Keep the hypotheses of Theorem 11.4 and Corollary 11.6. Assume that F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$.The map*

$$\gamma : \mathrm{Div}\left(\mathbf{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{{v_2}^2}\right) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$$

constructed in (58) *is surjective, hence can be identified with a quaternionic eigenform in the space $\mathcal{S}_2(\mathfrak{N}^+, v_1 v_2 \mathfrak{N}^-; \mathcal{O}_0/\mathfrak{P}_n)$. In particular, associated to γ by the Jacquet-Langlands correspondence is a surjective homomorphism*

$$\mathbf{g} : \mathbf{T}_0(\mathfrak{N}^+, v_1 v_2 \mathfrak{N}^-) \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$$

such that

(i) $T_w(\mathbf{g}) \equiv a_w(\mathbf{f}) \mod \mathfrak{P}_n$ *for all primes* $w \nmid v_1 v_2 \mathfrak{N}^+ \mathfrak{N}^-$ *of* $\mathcal{O}_F$.
(ii) $U_w(\mathbf{g}) \equiv a_w(\mathbf{f}) \cdot \mathbf{g} \mod \mathfrak{P}_n$ *for all primes* $w \mid \mathfrak{N}^+ \mathfrak{N}^-$ *of* $\mathcal{O}_F$.
(iii) $U_{v_i}(\mathbf{g}) \equiv \varepsilon_i \cdot \mathbf{g} \mod \mathfrak{P}_n$ *for each of* $i = 1, 2$.

Proof. Granted that γ is surjective, the result follows from Lemma 11.8, along with the identifications of (59). The surjectivity of γ is shown by Lemma 11.10 and Proposition 11.11 below. □

Lemma 11.10. *If F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$, then the subgroup of unipotent matrices mod $\mathfrak{p}$ in $M_2(F)$ is torsionfree.*

Proof. The following proof was suggested to the author by Vladimir Dokchitser. We first show that $\mathrm{M}_2(F)$ has no nontrivial matrices of order p. That is, suppose otherwise that we had a matrix $A \neq \mathbf{1}$ of order p in $\mathrm{M}_2(F)$. The eigenvalues of A would then be p-th roots of 1. Hence, the trace of A would lie in $\mathbf{Q}(\zeta_p)$, contradicting our hypotheses on F. Now to prove the claim, we show that any matrix $B \in \mathrm{M}_2(F)$ that is unipotent mod p must have p-power order. That is, suppose otherwise that such a matrix B did not have p-power order. Then, its eigenvalues would be m-th roots of unity for some integer m prime to p. But these eigenvalues cannot be congruent to 1 mod p, as the polynomial $X^m - 1$ is coprime to its own derivative mod p, hence has distinct roots over $\overline{\mathbf{F}}_p$. □

Proposition 11.11. *Assume Condition 3. of Theorem 1.2. If F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$, then the map γ constructed in* (58) *above is surjective.*

Proof. We generalize the argument of [3, Theorem 9.2], using the version of Ihara's lemma for Shimura curves shown in the main result of [13]. Hence, keep the setup of Theorem 8.4, with $v = v_1$. Let us write

$$M(\mathfrak{N}^+, v_1\mathfrak{N}^-)(\mathbf{C}) = B'^\times \backslash \widehat{B}'^\times \times X/H' = \coprod_i \Gamma_i \backslash \mathfrak{H},$$

where the subgroups $\Gamma_i \subset B'^\times$ are the associated arithmetic subgroups (see for instance the definition given in [16, § 3]). By embedding $B' \longrightarrow B_{\mathfrak{p}}^\times$, we view these arithmetic subgroups Γ_i as (discrete) subgroups of $\mathrm{GL}_2(F_{\mathfrak{p}})$. Let $J(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}$ denote the subgroup of $J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}$ generated by divisors supported on supersingular points. Since the composition of maps defining the homomorphism

$$J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\mathcal{I}_{\mathbf{f}_{v_1}} \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$$

is surjective, it suffices to show that the image of $J(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}$ in $J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\mathcal{I}_{\mathbf{f}_{v_1}}$ fills the whole group. To this end, let us define subgroups

$$\Gamma_i(v_2) = \left(\Gamma_i \left[\frac{1}{v_2}\right]^\times / \mathcal{O}_F^\times \left[\frac{1}{v_2}\right]\right)^1, \ \Gamma(v_2) = \prod_i \Gamma_i(v_2)$$

where the superscript 1 denotes elements of reduced norm 1. Let $\widetilde{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ denote the Shimura curve obtained from $M(\mathfrak{N}^+.v_1\mathfrak{N}^-)$ obtained by imposing extra $H_{\mathfrak{p}}'^1$-level structure, with $\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ its jacobian. Let us then write the corresponding arithmetic subgroups that are congruent modulo $\mathfrak{p}$ to unipotent matrices as :

$$\widetilde{\Gamma}_i(v_2) \subset \Gamma_i(v_2), \ \widetilde{\Gamma}(v_2) = \prod_i \widetilde{\Gamma}_i(v_2).$$

Since the subgroups $\widetilde{\Gamma}_i(v_2)$ are torsionfree by Lemma 11.10, a general theorem of Ihara ([31, Theorem G]) implies that there is a canonical isomorphism

$$\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2} \cong \widetilde{\Gamma}(v_2)^{\mathrm{ab}}. \tag{60}$$

Here, $\widetilde{\Gamma}(v_2)^{\mathrm{ab}}$ denotes the abelianization of $\widetilde{\Gamma}(v_2)$. Since v_2 splits the quaternion algebra B' associated to $M(\mathfrak{N}^+, v_1\mathfrak{N}^-)$, we can fix an embedding

$$\iota : B' \longrightarrow B'_{v_2} \cong M_2(F_{v_2}),$$

to obtain an induced action of B' on the Bruhat-Tits tree $\mathcal{T}_{v_2} = (\mathcal{V}(\mathcal{T}_{v_2}), \mathcal{E}(\mathcal{T}_{v_2}))$ of $B_{v_2}'^\times / F_{v_2}^\times \cong \mathrm{PGL}_2(F_{v_2})$. Let us then fix a vector of vertices $\mathfrak{v}_0 = \{\mathfrak{v}_0^i\}_{i=1}^h$ in $\mathcal{V}(\mathcal{T}_{v_2})$ such that the stabilizer

$$\widetilde{\Gamma}_{\mathfrak{v}_0^i}(v_2) := \mathrm{Stab}_{\mathfrak{v}_0^i}\left(\widetilde{\Gamma}_i(v_2)\right)$$

for each index i can be identified with the image of $\Gamma_i(v_2)$ in $\widetilde{\Gamma}_i(v_2)$ via ι, so that we have an identification of Riemann surfaces

$$\widetilde{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)(\mathbf{C}) = \coprod_i^h \widetilde{\Gamma}_{\mathfrak{v}_0^i}(v_2) \backslash \mathfrak{H}.$$

Fix a vector of oriented edges $\mathfrak{e}_0 = \{\mathfrak{e}_0^i\}_{i=1}^h$ in $\mathcal{E}^*(\mathcal{T}_{v_2})$ such that the stabilizer

$$\widetilde{\Gamma}_{\mathfrak{e}_0^i}(v_2) := \mathrm{Stab}_{\mathfrak{e}_0^i}\left(\widetilde{\Gamma}_i(v_2)\right)$$

for each index i can be identified with the subgroup of upper triangular matrices mod v_2 via ι, so that we have an identification of Riemann surfaces

$$\widetilde{M}(v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-)(\mathbf{C}) = \coprod_i^h \widetilde{\Gamma}_{\mathfrak{e}_0^i}(v_2)\backslash\mathfrak{H}.$$

Let $\mathfrak{v}_1$ denote the vector of vertices $\{\mathfrak{v}_1^i\}_{i=1}^h$ in $\mathcal{V}(\mathcal{T}_{v_2})$ such that $\mathfrak{v}_1^i = t(\mathfrak{e}_0^i)$ for each index i. Let us for ease of notation write the products as

$$\widetilde{\Gamma}_{\mathfrak{v}_0}(v_2) = \prod_{i=1}^h \widetilde{\Gamma}_{\mathfrak{v}_0^i}(v_2),\ \widetilde{\Gamma}_{\mathfrak{e}_0}(v_2) = \prod_{i=1}^h \widetilde{\Gamma}_{\mathfrak{e}_0^i}(v_2),\ \widetilde{\Gamma}_{\mathfrak{v}_1}(v_2) = \prod_{i=1}^h \widetilde{\Gamma}_{\mathfrak{v}_1^i}(v_2).$$

Hence we obtain from Serre [55, Proposition 1.3 § II.2.8] (with $i = 1$, $M = \kappa_{\mathfrak{p}}$, and $G = \widetilde{\Gamma}(v_2)$) the exact sequence

$$\begin{aligned}(61)\qquad \mathrm{Hom}(\widetilde{\Gamma}(v_2), \kappa_{\mathfrak{p}}) &\longrightarrow \mathrm{Hom}(\widetilde{\Gamma}_{\mathfrak{v}_0}(v_2), \kappa_{\mathfrak{p}}) \oplus \mathrm{Hom}(\widetilde{\Gamma}_{\mathfrak{v}_1}(v_2), \kappa_{\mathfrak{p}}) \\ &\xrightarrow{d} \mathrm{Hom}(\widetilde{\Gamma}_{\mathfrak{e}_0}(v_2), \kappa_{\mathfrak{p}}).\end{aligned}$$

Now, via duality we see that the map d in (61) is the degeneracy map of Ihara's lemma for Shimura curves, as described for instance in [13] (cf. [17, Theorem 2, p. 451] with [3, Proposition 9.2]). Roughly, Ihara's lemma is the assertion that for any non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbb{T}(v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-)$, the natural degeneracy map

$$H^1(M(v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-), \kappa_{\mathfrak{p}})^{\oplus 2} \xrightarrow{1_* \oplus \eta_{\mathfrak{p}}} H^1(M(\mathfrak{p}v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-), \kappa_{\mathfrak{p}})$$

is injective after localization at $\mathfrak{m}$. This conjecture is proved in certain cases in the unpublished manuscript [13]. We shall invoke this result in the following way. Recall that we let $\mathfrak{m}_{\mathbf{f}_{v_1}} \supset \mathcal{I}_{\mathbf{f}_{v_1}}$ denote the maximal ideal of the Hecke algebra $\mathbb{T}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ corresponding to the mod $\mathfrak{P}_n$ eigenform $\mathbf{f}_{v_1}$ of Theorem 11.4. Let us write $\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}} \supset \widetilde{\mathcal{I}}_{\mathbf{f}_{v_1}}$ to denote the corresponding maximal ideal in the Hecke algebra $\widetilde{\mathbb{T}}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ associated to $\widetilde{M}(\mathfrak{N}^+, v_1\mathfrak{N}^-)$. Since $\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}}$ is associated to an irreducible Galois representation, we know by Proposition 11.1 that $\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}}$ is not Eisenstein. Hence, by Ihara's lemma for Shimura curves, the degeneracy map d is injective after localization at $\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}}$. We can then argue following [3, Theorem 9.2, p. 59] that $\mathrm{Hom}(\widetilde{\Gamma}(v_2), \kappa_{\mathfrak{p}})[\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}}] = 0$. Hence, $\widetilde{\Gamma}(v_2)^{\mathrm{ab}}/\widetilde{\mathfrak{m}}_{\mathbf{f}_{v_1}} = 0$, in which case it follows from Nakayama's lemma that

$$\widetilde{\Gamma}(v_2)^{\mathrm{ab}}/\widetilde{\mathcal{I}}_{\mathbf{f}_{v_1}} = 0.$$

Hence, by (11.12), the image of $\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2}$ in $\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\widetilde{\mathcal{I}}_{\mathbf{f}_{v_1}}$ fills the whole group. To complete the argument, consider the natural map

$$(62)\qquad \widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2} \longrightarrow J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}.$$

A standard argument with Shimura subgroups shows that the cokernel of this map (62) has order prime to p (see [40] with [42, Lemma 7.20]). Roughly, the idea is the following. Let Π denote the kernel of this natural map. The criterion of [40] applied to each connected component of $M(\mathfrak{N}^+, v_1\mathfrak{N}^-)$ shows that there is an injective map

$$\Pi \longrightarrow \mathrm{Hom}(\Gamma(v_2)/\Gamma_{\mathfrak{v}_0}(v_2), \mathbf{S}),$$

where $\mathbf{S}$ denotes the complex numbers of modulus 1. It is then easy to see that the order of Π must be prime to p. Hence by duality, the cokernel must have over prime to p. Hence, the composition of (62) with the projection

$$J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2} \longrightarrow J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\mathfrak{P}_n$$

is surjective. Now, since we have already shown that the natural map

$$\widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2} \longrightarrow \widetilde{J}(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\widetilde{\mathcal{I}}_{\mathfrak{f}_{v_1}}$$

is surjective, we see that the natural map

$$J(\mathfrak{N}^+, v_1\mathfrak{N}^-)^{ss} \otimes \kappa_{v_2^2} \longrightarrow J(\mathfrak{N}^+, v_1\mathfrak{N}^-) \otimes \kappa_{v_2^2}/\mathcal{I}_{\mathfrak{f}_{v_1}}$$

is surjective, as required. □

Let us for the record state the version of Ihara's lemma for Shimura curves over totally real fields used in the proof of Propostion 11.11 above.

Hypothesis 11.12 (Ihara's lemma for Shimura curves.)**.** *For any non-Eisenstein maximal ideal* $\mathfrak{m} \subset \mathbb{T}(v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-)$*, the natural degeneracy map*

$$H^1(M(v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-), \kappa_{\mathfrak{p}})^{\oplus 2} \xrightarrow{1_* \oplus \eta_{\mathfrak{p}}} H^1(M(\mathfrak{p}v_2; \mathfrak{N}^+, v_1\mathfrak{N}^-), \kappa_{\mathfrak{p}})$$

is injective after localization at $\mathfrak{m}$*.*

12. Construction of the Euler system

We first review the theory of CM points on Shimura curves over totally real fields, giving a characterization of the images of these points under reduction modulo a ramified prime for the underlying quaternion algebra. We then review the notion of the Hodge embedding, then use this to define suitable classes from the images of CM points in the v-adic uniformization.

Setup. Fix a finite prime $v \subset \mathcal{O}_F$. Recall that we fixed a totally imaginary quadratic extension K of F. Let us suppose that v remains inert in K. Writing K_v to denote the completion of K at the prime above v in K, and F_{v^2} to denote the quadratic unramified extension of F_v, we have isomorphism of fields $K_v \cong F_{v^2}$. Let us then fix such an isomorphism $K_v \cong F_{v^2}$.

Complex points. Suppose in general that we have any Shimura curve M_H, as defined above. Given elements $b \in \widehat{B}^\times$ and $z \in X$, we write $[b, z]_H$ to denote the point of $M_H(\mathbf{C})$ represented by the pair (b, z).

Embeddings. Let us return to the setup of Theorem 8.4. Hence, no prime dividing $\mathrm{disc}(D)$ is split in K. Under this assumption, there exists an injective F-algebra homomorphism $\iota : K \longrightarrow D$ (see [66]). Let us fix such a homomorphism ι, writing $\iota_v : K \otimes_F F_v \longrightarrow D_v$ to denote its component at a prime v of F, and $\widehat{\iota} : \widehat{K} \longrightarrow \widehat{D}$ to denote its adelization. Let us assume that for our fixed prime v, we have the identification $\iota_v^{-1}(U_v) = \mathcal{O}_{K_v}^\times$. Under this assumption, there exists an F-algebra injection $\iota' : K \longrightarrow B'$ such that ${\iota'_v}^{-1}(\mathcal{O}_{B'_v}) = \mathcal{O}_{K_v}$. Let us fix such an embedding ι'. Since the Skolem-Noether theorem implies that any two local embeddings ι_v, ι'_v are conjugate by an element of $B_v'^\times$, we can an will assume that the homomorphisms φ, ι and ι' are compatible outside of v in the sense that ${\iota'}^v$ is given by the composition

$$\widehat{K}^v \xrightarrow{\iota^v} \widehat{D}^v \xrightarrow{\varphi} \widehat{B}'^v.$$

CM points. Fix an embedding $K \to \mathbf{C}$ that extends $\tau_1 : F \to \mathbf{R} \subset \mathbf{C}$. The action of $\iota'_{\tau_1}(K^\times) \subset \iota'_{\tau_1}(K^\times_{\tau_1}) \subset B'^\times_{\tau_1} \cong \mathrm{GL}_2(\mathbf{R})$ on $X = \mathfrak{H}^{\pm} = \mathbf{C} - \mathbf{R}$ fixes exactly two points, of which one lies in the complex upper half plane $\mathfrak{H}^+$. Let us write $z' = z'_{\iota'}$ to denote this point. We then define the set of points with complex multiplication (CM) by K on $M_{H'}(\mathbf{C})$ to be:

$$\mathrm{CM}(M_{H'}, K) = \{[b', z']_{H'} : b' \in \widehat{B}'^\times\} \subset M_{H'}(\mathbf{C}).$$

By Shimura's reciprocity law, the set $\mathrm{CM}(M_{H'}, K)$ is contained in $M_{H'}(K^{\mathrm{ab}})$, with G_K^{ab} acting on $\mathrm{CM}(M_{H'}, K)$ by the rule

$$\forall a \in \widehat{K}^\times,\ \mathrm{rec}_K(a)\,[b', z']_{H'} = [\widehat{\iota}'(a)b', z']_{H'}\,. \tag{63}$$

Here,

$$\mathrm{rec}_K : \mathbf{A}_K^\times / K^\times \cong G_K^{\mathrm{ab}}$$

denotes the reciprocity map of class field theory, normalized to send uniformizers to their corresponding geometric Frobenius elements.

CM points of a given conductor. Recall that given an ideal $\mathfrak{c} \subset \mathcal{O}_F$, we let $\mathcal{O}_{\mathfrak{c}} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ denote the $\mathcal{O}_F$-order of conductor $\mathfrak{c}$ in K. Given a maximal order $R' \subset B'$, we call an embedding $\iota : K \longrightarrow B'$ an *optimal embedding of conductor* $\mathfrak{c}$ *into* R' if

$$\iota(\mathcal{O}_{\mathfrak{c}}) = \iota(K) \cap R'.$$

Given a maximal order $R' \subset B'$ and an element $b' \in \widehat{B}'^\times$, let $R'_{b'} \subset B'$ denote the maximal order defined by $b'^{-1}R'b' \cap B'$. We say that a point $[b', z']_{H'} = [b', z'_{\iota'}]_{H'}$ in $\mathrm{CM}(M_{H'}, K)$ has *conductor* $\mathfrak{c}$ if the associated embedding $\iota' : K \longrightarrow B'$ is an optimal embedding of conductor $\mathfrak{c}$ (see [66, Ch. III]). It is then simple to see from class field theory that a CM point of conductor $\mathfrak{c}$ in $\mathrm{CM}(M_{H'}, K)$ is defined over the ring class field $K[\mathfrak{c}]$ of conductor $\mathfrak{c}$ over K.

CM points in the v-adic uniformization. We have the following description of CM points by K on $M_{H'}(\mathbf{C})$ in the v-adic uniformization. Let

$$\mathrm{CM}(M_{H'}, K)_{\text{v-unr}} = \{[b', z']_{H'} : b' \in \widehat{B}'^\times, b'_v = 1\} \subset \mathrm{CM}(M_{H'}, K)$$

to be the subset of CM points by K on $M_{H'}(\mathbf{C})$ that are unramified outside of v. The the action of G_K^{ab} on this set of points is given by the rule

$$\forall a \in \widehat{K}^\times,\ \mathrm{rec}_K(a)\,[b', z']_{H'} = [\widehat{\iota}'(a^v)b', z']_{H'}\,.$$

Here, a^v denotes the projection of a to $\widehat{K}^{v\times} = \{x \in \widehat{K}^\times : x_v = 1\}$. From this action, we deduce (cf. [49, 1.8.2]) that a point $x = [b', z']_{H'} \in \mathrm{CM}(M_{H'}, K)_{v-unr}$ is defined over the finite abelian extension $K(x)$ of K characterized by the isomorphism

$$\mathrm{rec}_K : \widehat{K}^\times / K^\times \widehat{\iota}'^{-1}(b'H'b') \cong \mathrm{Gal}(K(x)/K).$$

Moreover, as observed in [49, 1.8.2], the prime v splits completely in $K(x)$ because $\iota'_v(\mathcal{O}_{K_v}^\times) \subset H'_v = b'_v H_v b_v'^{-1}$. Fix an embedding $K_v \to F_v^{\mathrm{unr}}$, equivalently an isomorphism $K_v \cong F_{v^2}$ over F_v. This choice determines one of two fixed points for the action of $\iota_v(K^\times) \subset \iota_v(K_v^\times) \subset \mathrm{GL}_2(F_v)$ on $\mathbf{P}^1(K_v) - \mathbf{P}^1(F_v)$, call it z $(= z_\iota)$. The image of $\mathrm{CM}(M_{H'}, K)_{\text{v-unr}}$ in $M_{H'}(K_v)$ according to either Theorem 47 is then given by

$$\mathrm{CM}(M_{H'}, K)_{\text{v-unr}} = \{[d, z]_{U^v} : d \in \widehat{D}^\times, d_v = 1\} \subset M_{H'}(K_v).$$

Let us now write $\mathcal{G}_v = (\mathcal{V}(\mathcal{G}_v), \mathcal{E}(\mathcal{G}_v))$ to denote the dual graph of the special fibre $\mathbf{M}_{H'} \otimes \kappa_{v^2}$, which is just the special fibre of the basechange to $\mathcal{O}_{F_{v^2}}$ of the integral model $\mathbf{M}_{H'}$ over $\mathcal{O}_{F_v}$. Let

$$\mathrm{red}_v : M_{H'} \otimes_F F_{v^2} \longrightarrow \mathcal{V}(\mathcal{G}_v) \cup \mathcal{E}(\mathcal{G}_v)$$

denote the map that sends a point x to either the connected component containing its image in $\mathbf{M}_{H'} \otimes \kappa_{v^2}$, or else to its image in $\mathbf{M}_{H'} \otimes \kappa_{v^2}$ (a singular point).

Proposition 12.1. *We have that* $\mathrm{red}_v\left(\mathrm{CM}(M_{H'}, K)_{\text{v-unr}}\right) \subset \mathcal{V}(\mathcal{G}_v)$.

Proof. The result seems to be well known (see [3, § 5] or [43, 5.2]). That is, fix a point $x \in \mathrm{CM}(M_{H'}, K)_{\text{v-unr}}$. We saw above that x is rational over the abelian extension $K(x)$, where v splits completely. Hence, writing $K(x)_v$ to denote the localization of $K(x)$ at any fixed prime above v in $K(x)$, we have the identification $K(x)_v \cong K_v$. In particular, we may view x as a point in $M_{H'}(K_v) \cong M_{H'}(F_{v^2})$. It therefore makes sense to compute the image of x in $\mathbf{M}_{H'} \otimes \kappa_{v^2}$.

Now, recall that the image of x in the v-adic uniformization $\mathbf{M}_{H'} \otimes \kappa_{v^2}$ is parametrized by the class of a pair (d, ι), where $d \in \widehat{D}^{\times v}$, and $\iota : K \rightarrow D$ is a suitably chosen F-algebra injection. The action of $\iota(K^\times) \subset \iota_v(K_v^\times) \subset D_v^\times \cong \mathrm{GL}_2(F_v)$ on $\Omega(\mathbf{C}_v)$ fixes two distinct points, $z_1 = z_{1,\iota}$ and $z_2 = z_{2,\iota}$ say. These points are contained in $\mathbf{P}^1(K_v) - \mathbf{P}^1(F_v) \cong \mathbf{P}^1(F_{v^2}) - \mathbf{P}^1(F_v)$. Let z_3 denote the point at infinity in $\mathbf{P}^1(K_v) \cong \mathbf{P}^1(F_{v^2})$. As explained in Mumford [48, § 1], any triple of distinct points z_1, z_2, z_3 in $\mathbf{P}^1(F_{v^2})$ corresponds canonically to a unique vertex $\mathfrak{v}_{z_1,z_2,z_3}$ in the Bruhat-Tits tree of $\mathrm{SL}_2(F_{v^2})$. The inclusion $\mathrm{red}_v(\mathrm{CM}(M_{H'}, K)_{\text{v-unr}}) \subset \mathcal{V}(\mathcal{G}_v)$ can then be deduced from the Mumford-Kurihara uniformization of $\mathbf{M}_{H'} \otimes \kappa_{v^2}$ (Theorem 8.3) with (47). ☐

Note that since any n-admissible prime $v \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ splits completely in $K_{\mathfrak{p}^\infty}$, the argument of Propostion 12.1 shows that the CM points in $M(K_{\mathfrak{p}^\infty})$ also satisfy this property under reduction mod v.

Hodge classes. Let us now explain how divisors on $M_{H'}(\overline{F})$ give rise to classes in the associated jacobian $J_{H'}(\overline{F})$, following [43, §5] and [42, §3]. Recall that $J_{H'}$ has the structure of a $\mathbf{T}_0(\mathfrak{N}^+, v\mathfrak{N}^-)$-module, as explained above. Following Zhang [72, 4.1], we make the following

Definition The *Hodge class of* $M_{H'}$ is the unique class $\xi \in \mathrm{Pic}(M_{H'})$ such that:

(i) The degree on ξ on each connected component of $M_{H'}$ is one.
(ii) The action of the operator $T_{\mathfrak{q}} \in \mathbf{T}_0(\mathfrak{N}^+, v\mathfrak{N}^-)$ for each prime $\mathfrak{q} \nmid \mathfrak{N}$ is given by mutliplication by $\mathbf{N}(\mathfrak{q}) + 1$.

The existence and uniqueness of such a class are shown by Zhang in [72, 4.1]. Let $\mathrm{Pic}^{\mathrm{Eis}}(M_{H'})$ denote the subgroup of $\mathrm{Pic}(M_{H'})$ generated by divisors whose restriction to each connected component of $M_{H'}$ is given by a multiple of the Hodge class ξ. Zhang [71][6.1] shows that there is a decomposition

$$\mathrm{Pic}(M_{H'}) = \mathrm{Pic}^0(M_{H'}) \oplus \mathrm{Pic}^{\mathrm{Eis}}(M_{H'}).$$

Using an argument of Ribet [54, Theorem 5.2 (c)], or its subsequent generalization by Jarvis in [34, § 3], it can then be shown that the $\mathbf{T}_0(\mathfrak{N}^+, v\mathfrak{N}^-)$-module $\mathrm{Pic}^{\mathrm{Eis}}(M_{H'})$ is Eisenstein. Since $\mathcal{I}_{\mathbf{f}_v} \subset \mathbf{T}_0(\mathfrak{N}^+, v\mathfrak{N}^-)$ is not Eisenstein, it can then be deduced by a standard argument that the natural inclusion $\mathrm{Pic}^0(M_{H'}) \subset$

$\mathrm{Pic}(M_{H'})$ induces an isomorphism

$$\mathrm{Pic}^0(M_{H'})/\mathcal{I}_{\mathbf{f}_v} \cong \mathrm{Pic}(M_{H'})/I_{\mathbf{f}_v}.$$

See for instance [42, 2.5] (where there is a typo on line 21 of p. 15).

Construction of classes. Let us now assume that $M_{H'} = M(\mathfrak{N}^+, v\mathfrak{N}^-)$ as above, where $v \subset \mathcal{O}_F$ is an n-admissible prime with respect to $\mathbf{f}$. Fix a sequence of points $\{P_m\}_{m\geq 1}$, where each P_m is a CM point of conductor $\mathfrak{p}^m$ in $\mathrm{CM}(M_{H'}, K)$. Let us assume that this sequence is compatible in the following sense. Each point P_m is given by the class of some pair $(g'_m, z') = (g'_m, z'_{\iota'})$, where ι' is an optimal embedding of $\mathcal{O}_{\mathfrak{p}^m} \subset \mathcal{O}_K$ into $R_{g'_m} = {g'_m}^{-1} R g'_m \subset B'$. By assumptions made made above and throughout, we can fix an isomorphism $\iota_{\mathfrak{p}} : B'_{\mathfrak{p}} \cong \mathrm{M}_2(F_{\mathfrak{p}})$. Following the explanation of [41, 3.3], we can then associate to the the local (Eichler) order $(R_{g'_m})_{\mathfrak{p}} \subset B'_{\mathfrak{p}}$ a directed edge $e_{g'_m} = (s(e_{g'_m}), t(e_{g'_m}))$ in the Bruhat-Tits tree of $B_{\mathfrak{p}}^{'\times}/F_{\mathfrak{p}}^{\times} \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$. We then say that the sequence $\{P_m\}_{m\geq 1}$ is *compatible* if $t(e_{g'_m}) = s(e_{g'_{m+1}})$ for all $m \geq 1$. Let us now fix such an oriented sequence $\{P_m\}_{m\geq 1}$. For each point P_m in the sequence, let us write P_m^* to denote the image of $\alpha_{\mathfrak{p}}^{-m} P_m$ in $J_{H'}(K[\mathfrak{p}^m])/\mathcal{I}_{\mathbf{f}_v}$. The points P_m^* are norm compatible, and their images under the Kummer maps

$$J_{H'}(K[\mathfrak{p}^m])/\mathcal{I}_{\mathbf{f}_v} \xrightarrow{\ \mathfrak{K}\ } H^1(K[\mathfrak{p}^m], \mathrm{Ta}_p(J_{H'})/\mathcal{I}_{\mathbf{f}_v})$$

give rise to a sequence of classes $\zeta_m[v]$ that are compatible under corestriction maps. Under the isomorphism of Corollary 11.6 (ii), these classes give rise to an element $\zeta[v]$ in the cohomology group $\widehat{H}^1(K[\mathfrak{p}^\infty], T_{\mathbf{f},n})$. Let $\zeta(v)$ denote image of this class under corestriction from $K[\mathfrak{p}^\infty]$ to $K_{\mathfrak{p}^\infty}$. Hence, we have constructed from our compatible system of CM points $\{P_m\}_{m\geq 1}$ in $\mathrm{CM}(M_{H'}, K)$ a class

$$\zeta(v) \in \widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n}). \tag{64}$$

13. Explicit reciprocity laws

Putting everything together, we may now at last deduce the first and second explicit reciprocity laws introduced above.

The first explicit reciprocity law. Keep all of the notations and hypotheses above. Hence, $\mathbf{f} \in \mathcal{S}_2(\mathfrak{N}^+, \mathfrak{N}^-)$ is a $\mathfrak{p}$-ordinary eigenform, and $\zeta(v)$ is the class of $\widehat{H}^1(K_{\mathfrak{p}^\infty}, T_{\mathbf{f},n})$ constructed above.

Theorem 13.1. (The first explicit reciprocity law) *Keep all of the hypotheses of Theorem 11.4 and Corollary 11.6. Then, $\vartheta_v(\zeta(v)) = 0$. Moreover, the equality*

$$\partial_v(\zeta(v)) = \mathcal{L}_{\mathbf{f}}$$

holds in $\widehat{H}^1_{\mathrm{sing}}(K_{\mathfrak{p}^\infty,v}, T_{\mathbf{f},n}) \cong \Lambda/\mathfrak{P}^n$, up to multiplication by elements of $\mathcal{O}^\times$ or $G_{\mathfrak{p}^\infty}$.

Proof. See [3, §8]. By the commutative diagram of Corollary 11.7 (ii), it suffices to show that

$$\widehat{\partial}_v(\{P_m^*\}) \equiv \mathcal{L}_{\mathbf{f}} \mod \mathfrak{P}^n.$$

Let us write Q_m to denote the image of the class P_m^* under the norm map from $K[\mathfrak{p}^m]$ to $K_{\mathfrak{p}^m}$. By Proposition 12.1, the image of each class Q_m in the special fibre $\mathbf{M}^{(v)} \otimes \kappa_{v^2}$ is a nonsingular point, hence given by a vertex v_{Q_m} in the dual graph

$\mathcal{G}_v$ under the map $\mathrm{red}_v : J^{(v)}(K_{\mathfrak{p}^m})/\mathcal{I}_{\mathbf{f},v} \longrightarrow \mathcal{V}(\mathcal{G}_v) \cup \mathcal{E}(\mathcal{G}_v)$. On the other hand, recall the natural map $\omega_v : \mathbf{Z}[\mathcal{V}(\mathcal{G}_v)]^0 \longrightarrow \Phi_v$ constructed in Proposition 9.5. We know by Proposition 9.6 that

$$\omega_v \circ \mathrm{red}_v(Q_m) = \partial_v(Q_m) \in \Phi_v/\mathcal{I}_{\mathbf{f}_v} \cong \mathcal{O}_0/\mathfrak{P}_n,$$

where the isomorphism comes from Corollary 11.6 (i). The result can now be deduced from the adelic description of the vertex set $\mathcal{V}(\mathcal{G}_v)$ above, which (via Jacquet-Langlands) allows us to view the specialization map as a map

$$D^\times \backslash \widehat{D}^\times / U \longrightarrow \mathcal{O}_0/\mathfrak{P}_n$$

having the same eigenvalues as $\mathbf{f}$. That is, it can be deduced from the description above of the induced action of $G_{\mathfrak{p}^\infty}$ on this vertex set, along with the canonical bijection $\eta_{\mathfrak{p}}$ coming from strong approximation at $\mathfrak{p}$, that

$$\partial_{\mathfrak{v}_m}(\sigma Q_m) \equiv \alpha_{\mathfrak{p}}^{-m}[\sigma, \mathfrak{e}_j]_\Phi \mod \mathfrak{P}^n.$$

Here, we have fixed a prime $\mathfrak{v}_\infty$ above v in $K_{\mathfrak{p}^\infty}$ and let $\mathfrak{v}_m = \mathfrak{v}_\infty \cap K_{\mathfrak{p}^m}$. The result is now clear via the construction of $\mathcal{L}_{\mathbf{f}}$ from these elements. □

The second explicit reciprocity law. Fix two n-admissible primes $v_1, v_2 \subset \mathcal{O}_F$ with respect to $\mathbf{f}$ such that

$$\mathbf{N}(v_i) + 1 - \varepsilon_i \cdot a_{v_1}(\mathbf{f}) \equiv 0 \mod \mathfrak{P}_n$$

for each of $i = 1, 2$. As usual, we keep all of the setup and hypotheses of Theorem 8.4, taking $v = v_1$ so that the indefinite quaternion algebra B' has discriminant $v_1\mathfrak{N}^-$.

Theorem 13.2. (The second explicit reciprocity law) *Keep the hypotheses of Theorem 11.4 and Corollary 11.6. Assume additionally that F is linearly disjoint from the cyclotomic field $\mathbf{Q}(\zeta_p)$. Then, the relation*

$$\vartheta_{v_1}(\zeta(v_2)) = \mathcal{L}_{\mathbf{g}}$$

holds in $\widehat{H}^1_{\mathrm{unr}}(K_{\mathfrak{p}^\infty, v_n}, T_{\mathbf{f},n}) \cong \Lambda/\mathfrak{P}^n$, up to multiplication by elements of $\mathcal{O}^\times$ or $G_{\mathfrak{p}^\infty}$.

Proof. The proof is the same as that for Theorem 13.1, replacing $\mathbf{f}$ with the mod $\mathfrak{P}_n$ eigenform $\mathbf{g}$ of Proposition 11.9. □

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