p-ADIC INTERPOLATION OF GL_n-AUTOMORPHIC PERIODS OVER CM FIELDS

JEANINE VAN ORDER

Abstract. Given $n \geq 2$ an even integer and K a CM field, we construct anticyclotomic p-adic interpolation series for the central and critical values of a conjugate self-dual cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_K)$ which arises via cyclic basechange from a cuspidal automorphism π' of $\mathrm{GL}_n(\mathbf{A}_F)$, where F denotes the maximal totally real subfield of K. To derive the interpolation properties, we use both the classical theory of Eulerian integral presentations for $\mathrm{GL}_n(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$-automorphic L-functions, as well as new progress on the Ichino-Ikeda Gan-Gross-Prasad conjecture. Here, we also pose some open questions about the comparison of periods inherent in these distinct approaches.

Nous construisons pour $n \geq 2$ un entier pair et K un corps CM deux fonctionnes L p-adiques anticyclotomiques pour les valeurs centrales et critiques d'une représentation automorphe cuspidale conjuguée auto-duale π de $\mathrm{GL}_n(\mathbf{A}_K)$ qui provient, par changement de base cyclique, d'un automorphisme cuspidal π' de $\mathrm{GL}_n(\mathbf{A}_F)$, où F et le sous-champ maximal totalement réel de K. pour dériver les propriétés d'interpolation, nous utilisons à la fois la théorie classique des présentations intégrales eulériennes pour les fonctions L automorphes de $\mathrm{GL}_n(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$ ainsi que de nouveaux progrès sur la conjecture d'Ichino-Ikeda Gan-Gross-Prasad. Nous posons également quelques questions ouvertes sur la comparaison des périodes inhérentes à ces approches distinctes.

Contents

1. Introduction

Fix $n \geq 2$ an even integer. Let K be a CM field. Hence, K is a totally imaginary quadratic extension of its maximal totally real subfield $F = K^+$. Let $\pi = \otimes_v \pi_v$ be a cohomological automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$. We shall assume that π arises via cyclic basechange from a cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbf{A}_F)$ in the sense of Arthur-Clozel [5], this relation often written as

$$\pi = \mathrm{BC}_{K/F}(\pi'). \tag{1}$$

Writing χ to denote a ring class character of K, we shall consider the standard L-function

$$\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty) L(s, \pi \otimes \chi)$$

of π twisted by K, which has a well-known analytic continuation by Godemont-Jacquet [26], and which is equivalent to the $\mathrm{GL}_n(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L-function

$$\Lambda(s, \pi' \times \pi(\chi)) = L(s, \pi'_\infty \times \pi(\chi)_\infty) L(s, \pi' \times \pi(\chi))$$

of π' times the representation $\pi(\chi)$ of $\mathrm{GL}_2(\mathbf{A}_F)$ induced from χ. In particular, we look at critical values $s_0 \in \frac{n-1}{2} + \mathbf{Z}$ of these L-functions, which recall are those values s_0 for which neither the archimedean component $L(s, \pi_\infty)$ nor its contragedient $L(1-s, \widetilde{\pi}_\infty)$ has a pole at s_0. Of particular interest for conjectures in the direction of Birch and Swinnerton-Dyer is the special case where $s_0 = 1/2$ is the central point. In any case, the far-reaching conjectures of Deligne [21] predict the existence of periods $\Omega_{s_0}(\pi, \chi) \in \mathbf{C} \backslash \{0\}$ (i.e. "Langlands quotients") depending on the cohomological representation π, the idele class character χ, and the critical value s_0 for which the normalized value

$$\mathfrak{L}(s_0, \pi \otimes \chi) = \frac{\Lambda(s_0, \pi \otimes \chi)}{\Omega_{s_0}(\pi, \chi)} = \frac{L(s_0, \pi_\infty) L(s_0, \pi \otimes \chi)}{\Omega_{s_0}(\pi, \chi)} \in \overline{\mathbf{Q}} \tag{2}$$

is some algebraic number. Various recent advances towards this conjecture for automorphic motives due to Harris et al. (and others) [33], [28], [30], [31], and [29] (cf. [35]) make it possible to derive many cases of this property (2). Here, we propose to look at these results from the classical and somewhat neglected viewpoint of Eulerian integral presentations (cf. e.g. [16, Lecture 5]), which allows us to view the critical values $\Lambda(s_0, \pi \otimes \chi)$ we wish to consider as toric periods integrals. As we explain in more detail below, for certain pure tensors $\varphi \in V_\pi$, we have certain Eulerian integral representations of the form

$$\Lambda(s_0, \pi \otimes \chi) = \int_{\mathbf{A}_K^\times} W_\varphi \left(\begin{pmatrix} y & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) \chi(y) |y|^{s_0 - \left(\frac{n-1}{2}\right)} dy,$$

where W_φ denotes the Whittaker coefficient of φ, and $\mathbf{1}_{n-1}$ the $(n-1) \times (n-1)$ identity matrix. On the other hand, recent advances in progress on the Ichino-Ikeda Gan-Gross-Prasad conjecture for unitary groups $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$ – as formulated by Liu in [48] – allow us to derive a distinct toric integral period formula for the central critical values $\Lambda(s_0, \pi \otimes \chi) = \Lambda(1/2, \pi \otimes \chi)$ in the special case where $\pi \cong \widetilde{\pi}$ is self-dual. Here, we shall assume that the cuspidal automorphic representation $\pi' = \otimes_v \pi'_v$ of $\mathrm{GL}_n(\mathbf{A}_F)$ descends via stable basechang to a cuspidal automorphic representation π'^U of $U_n(\mathbf{A}_F) = U(V)(\mathbf{A}_F)$ associated to a Hermitian vector space V over K of even dimension n. In this case, we have an identification of the standard L-functions

$$\Lambda(s, \pi') := L(s, \pi'_\infty) L(s, \pi') = \Lambda(s, \pi'^U) := L(s, \pi'^U_\infty) L(s, \pi'^U),$$

and hence for the corresponding Rankin-Selberg L-functions $\Lambda(s, \pi' \times \pi(\chi)) = \Lambda(s, \pi'^U \times \pi(\chi))$ for any Hecke character χ of K. We also know thanks to theorems of Harris-Taylor [36], Caraiani [1], [2], and Clozel [14] in this case that the representation π is (everywhere locally) tempered, as predicted by the generalized Ramanujan conjecture. Writing $\pi^U = \mathrm{BC}_{K/F}(\pi'^U)$ to denote the corresponding quadratic basechange of π'^U to the unitary group $U_n(\mathbf{A}_K) = U(V)(\mathbf{A}_K)$, we then have the corresponding identification of standard L-functions $\Lambda(s, \pi \otimes \chi) = \Lambda(s, \pi^U \otimes \chi)$ for any Hecke character χ of K. Roughly speaking, the Ichino-Ikeda conjecture would allow us to realize the central critical values $\Lambda(1/2, \pi \otimes \chi) = \Lambda(1/2, \pi^U \otimes \chi)$ in this setting as toric period integrals of the form

$$\frac{\Lambda(1/2, \pi^U \otimes \chi)}{\Lambda(1, \pi^U, \mathrm{Ad})} \cdot \prod_{v \leq \infty} P_{\chi_v}(\phi_v, \phi_v) = |P_\chi(\phi)|^2, \quad P_\chi(\phi) := \int\limits_{\substack{U(L)(\mathbf{A}_F)/U(L)(F) \\ \cong \mathbf{A}_K^\times / K^\times \mathbf{A}_F^\times}} \mathcal{P}_\psi \phi(t) \chi(t) dt,$$

where $\Lambda(s, \pi^U, \mathrm{Ad}) = \Lambda(s, \pi, \mathrm{Ad})$ is the L-function of the adjoint representation, $\phi = \otimes_v \phi_v$ is some decomposable vector in the representation space of π'^U, and $\mathcal{P}\phi$ is some projection operator defined on $t \in U_1(\mathbf{A}_F) \cong \mathbf{A}_K^\times$ by an integral over a certain unipotent subgroup $N \subset U_n$, and with respect to a certain automorphic additive character ψ on N (defined in [24]) by

$$\mathcal{P}\phi(t) = \int_{N(F) \backslash N(\mathbf{A}_F)} \phi(nt) \psi^{-1} n(dn).$$

As well, the formula includes for each place v of K a certain local sesquilinear pairing $P_{\chi_v} : \pi_v^U \times \pi_v^U \longrightarrow \mathbf{C}$. We state this expected formula in Conjecture 2.10, and assume it to derive a second construction of p-adic interpolation series. This latter setup then allows us to give a generalization of the construction in

[61] for rank $n = 2$. More generally, taking for granted these theorems about rationality with the toric integral period presentations (Eulerian or à la Ichino-Ikeda) we can fix a prime number p together with an embedding $\iota_p = \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ to consider the values (2) as p-adic algebraic numbers. It is then natural to consider the problem of constructing p-adic interpolation series or p-adic L-functions, for instance as a first step to approaching the Iwasawa-Greenberg main conjecture in the style of Bertolini-Darmon [9], [10] via the theory of level raising congruences, or ergodic theoretic Galois averaging arguments in the style of Vatsal [62] [63] and Cornut-Vatsal [19], to derive unconditional results in the direction of Birch and Swinnerton-Dyer.

To describe what we show here, let $\mathfrak{P} \subset \mathcal{O}_K$ be a prime ideal in the ring of integers $\mathcal{O}_K$, with underlying F-rational prime $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$ in the ring of integers $\mathcal{O}_F$, and underlying rational prime $p = \mathfrak{P} \cap \mathbf{Z} = \mathfrak{p} \cap \mathbf{Z}$ in $\mathbf{Z}$. We consider for each integer $m \geq 1$ the $\mathcal{O}_F$-order $\mathcal{O}_{\mathfrak{p}^m} = \mathcal{O}_F + \mathfrak{p}^m \mathcal{O}_K$ of conductor $\mathfrak{p}^m$ in $\mathcal{O}_K$, together with its corresponding class group

$$X_m = \operatorname{Pic}(\mathcal{O}_{\mathfrak{p}^m}) = \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times, \quad \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times = \prod_{v < \infty} \mathcal{O}_{\mathfrak{p}^m, v}^\times.$$

Hence, the primitive ring class characters χ of K of conductor $\mathfrak{p}^m$ factor through these finite abelian groups X_m, and we can consider the profinite limit $X = \varprojlim_m X_m$. Fix a discrete valuation ring $\mathcal{O}$ large enough to contain the algebraic values (2) as χ varies over all ring class characters factoring through X. We can then consider the $\mathcal{O}$-Iwasawa algebra

$$\mathcal{O}[[X]] = \varprojlim_m \mathcal{O}[X_m],$$

whose elements we can and do view as $\mathcal{O}$-valued measures on the profinite abelian group X. We also consider bounded distributions on X which gives rise to $\mathcal{O}$-valued measures after suitably rescaling, and hence to elements of $\mathcal{O}[[X]]$ in this way. Assuming that π is $\mathfrak{P}$-ordinary in the sense that its eigenvalues under suitable "trace" Hecke operators $T_{\mathfrak{P}}$ at $\mathfrak{P}$ are p-adic units under our fixed embedding $\iota_p : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$, we construct (1) bounded distributions $\mathcal{L}_{\mathfrak{P}}(\pi, s_0) = d\mathcal{L}_{\mathfrak{P}}(\pi, s_0)$ on X whose specializations

$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi, s_0)) = \int_X \chi(\sigma) d\mathcal{L}_{\mathfrak{P}}(\pi, s_0)(\sigma) = \sum_{A \in X_m} \chi(A) d\mathcal{L}_{\mathfrak{P}}(\pi, s_0)|_{X_m}(A)$$

to primitive ring class characters χ of conductor $\mathfrak{p}^m$ for each $m \geq 1$ interpolate the algebraic critical values $\mathfrak{L}(s_0, \pi \otimes \chi) \in \overline{\mathbf{Q}}_p$, and (2) bounded distributions $\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2) = d\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)$ on X whose specializations

$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)) = \int_X \chi(\sigma) d\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)(\sigma) = \sum_{A \in X_m} \chi(A) d\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)|_{X_m}(A)$$

to primitive ring class characters χ of conductor $\mathfrak{p}^m$ for each $m \geq 1$ interpolate the squares of automorphic periods $|P_\chi(\phi)|^2$ introduced above. In particular, assuming the Ichino-Ikeda conjecture (see Conjecture 2.10), we obtain for (2) a p-adic interpolation measure for the central critical values $\Lambda(1/2, \pi \otimes \chi) = \Lambda(1/2, \pi^U \otimes \chi)$.

Our main strategy for either construction rests on a careful choice of vectors in the ambient representation space. For (1), we use the theory of Eulerian integral presentations (e.g. [16], [17]), and make crucial use of both the theory of essential Whittaker vectors (as developed by Matringe [51]) and the ability to vary vectors in the archimedean local Whittaker model (see [42]) to find a suitably normalized cuspidal pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ for our main construction. For (2), we take for granted the Ichino-Ikeda Gan-Gross-Prasad conjecture (Conjecture 2.10, now a theorem of Beuzart-Plessis-Chaudouard [11, Theorem 1.3.6.1]), then use the existence of integral models for unitary Shimura varieties due to [45] and [44] to deduce that we find a suitable normalization $\phi = \otimes_v \pi \in V_{\pi'^U}$, for π^U the transfer of π to an automorphic representation on some unitary group U. Choosing vectors suitably in each case, we can then construct p-adic interpolation series in the style developed for $n = 2$ in the author's previous work [61]. Broadly, after identifying suitable "toric period formulae" as described above, the idea is to work on the level of these decomposable vectors $\varphi = \otimes_v \varphi_v \in V_\pi$ and $\phi = \otimes_v \phi_v \in V_{\pi'^U}$. In particular, after identification of the Hecke operators in the Bruhat-Tits building of $\mathrm{SL}_n(K_{\mathfrak{P}})$ which resemble the underlying "trace" operators corresponding to sums over elements of the kernels defined by the transition maps $\ker(\pi_{m+1,m} : X_{m+1} \longrightarrow X_m)$, we proceed by looking at the action of these operators on the corresponding vectors to construct natural distributions.

Let us now describe these constructions. Recall that for a number field L, a $\mathrm{GL}_n(\mathbf{A}_L)$-representation $\pi = \otimes_v \pi_v$ is said to be *cohomological* if there exists a finite dimensional irreducible algebraic representation

$\mathcal{E}_\mu$ of the Lie group $\mathrm{GL}_n(L \otimes_{\mathbf{Q}} \mathbf{R})$ whose archimedean component π_∞ has nontrivial relative Lie cohomology with respect to $\mathcal{E}_\mu$. When this occurs, the highest weight vector μ for $\mathcal{E}_\mu$ can be parametrized as $\mu = (\mu_v)_{v \mid \infty}$ with each $\mu_v = (\mu_{v,j})_{j=1}^n$ an n-tuple of integers $\mu_{v,j} \in \mathbf{Z}$ arranged so that $\mu_{v,1} \geq \cdots \geq \mu_{v,n}$. There exists an integer w known as the *purity weight* for which $w = \mu_{v,j} + \mu_{v,n+1-j}$ for any $1 \leq j \leq n$, independently of the choice of place $v \mid \infty$. As well, π_∞ is said to be *regular* if $\mu_{v,i} \neq \mu_{v,j}$ for any $i \neq j$ and $v \mid \infty$. Let $\tau(\mathcal{O}_{\mathfrak{p}^m})$ denote the volume of $K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$ in $\mathbf{A}_K^\times$ with respect to a fixed Haar measure. Again, we fix a prime ideal $\mathfrak{P} \subset \mathcal{O}_K$, with underlying F-rational prime $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$ and rational prime $p = \mathfrak{P} \cap \mathbf{Z}$, as well as an embedding $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. We write $K_{\mathfrak{P}}$ to denote the localization of K at $\mathfrak{P}$, with $\mathcal{O}_{K_{\mathfrak{P}}}$ its ring of integers; we fix a uniformizer $\varpi_{\mathfrak{P}} \in \mathcal{O}_{K_{\mathfrak{P}}}$, and write $q_{\mathfrak{P}}$ to denote the cardinality of the residue field $\mathcal{O}_{K_{\mathfrak{P}}}/\varpi_{\mathfrak{P}}$ at $\mathfrak{P}$.

Theorem 1.1. *Fix $n \geq 2$ an even integer, and K a CM field. Let $\pi = \otimes_v \pi_v$ be a cohomological automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$. Assume that $\pi = \mathrm{BC}_{K/F}(\pi)$ arises via cyclic basechange from an irreducible cuspidal automorphic representation $\pi' = \otimes_v \pi'_v$ of $\mathrm{GL}_n(\mathbf{A}_F)$. Let us write the conductor of the local representation $\pi_{\mathfrak{P}}$ at $\mathfrak{P}$ as $q_{\mathfrak{P}}^\delta$ for $\delta = 0$ or $\delta = 1$, with the case of $\delta = 1$ corresponding to some specific parahoric structure. Assume that image under $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ of the eigenvalue $\alpha_{\mathfrak{P}}$ of the so-called "trace" Hecke operator $T_{\mathfrak{P}}$ (described below) acting on $\pi_{\mathfrak{P}}$ is a p-adic unit.*

(1) *(Theorem 3.4 (1), Corollary 3.4 (1)). Assume the archimedean component $\pi'_\infty = \otimes_{v \mid \infty} \pi'_v$ is regular of highest weight $\mu = (\mu_v)_{v \mid \infty}$, so that the $\mu_{v,j} \neq \mu_{v,i}$ for any archimedean place v of F and $j \neq i$, and also that $2\mu_{v,j}$ is not equal to the purity weight w for any $1 \leq j \leq n$. Let $s_0 \in \frac{n-1}{2} + \mathbf{Z}$ be any critical value. There exists a bounded distribution $\mathfrak{L}_{\mathfrak{P}}(\pi, s_0) = d\mathfrak{L}_{\mathfrak{P}}(\pi, s_0)$ on X, constructed from a particular choice of decomposable vector $\varphi = \varphi^{\mathrm{int}} \in V_\pi$, which satisfies the following interpolation property: For χ any primitive ring class character of conductor $\mathfrak{p}^m$ for any integer $m \geq 1$, we have*
$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi, s_0)) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot \mathfrak{L}(s_0, \pi \otimes \chi)^2 \in \mathcal{O}.$$

(2) *(Theorem 3.4 (2), Corollary 3.5 (2)). Assume $\pi = \widetilde{\pi}$ is self-contragredient, and that π descends to a cuspidal automorphic representation π^U on a unitary group $U_n(\mathbf{A}_K) = U(V)(\mathbf{A}_K)$ associated to a hermitian vector space V over K of even dimension n, as described above. There exists a bounded distribution $\mathfrak{L}_{\mathfrak{P}}(\pi^U, 1/2) = d\mathfrak{L}_{\mathfrak{P}}(\pi^U, 1/2) \in \mathcal{O}[[X]]$, constructed from a particular choice of decomposable vector $\phi = \phi^{\mathrm{int}} \in V_{\pi'^U}$, which satisfies the following interpolation property: For χ any primitive ring class character of conductor $\mathfrak{p}^m$ for any integer $m \geq 1$, we have*
$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot |P_\chi(\phi)|^2 \in \mathcal{O}.$$
Moreover, if we assume the Ichino-Ikeda conjecture for $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$ (Conjecture 2.10 below), then this latter interpolation formula is given equivalently by
$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot \prod_{v \leq \infty} P_{\chi_v}(\phi_v, \phi_v) \cdot \frac{\Lambda(1/2, \pi^U \otimes \chi)}{\Lambda(1, \mathrm{Ad}, \pi^U)} \in \mathcal{O}.$$

We argue that this construction does indeed give a bounded[1] distribution, and not merely a distribution. Again, this construction can be viewed a generalization of that given for rank $n = 2$ in [61], building on the works of Bertolini-Darmon [9] and [10]) for central values $s_0 = 1/2$, but using test vector refinements [23] of Waldspurger's theorem [64] or Gross-Zagier type formulae [65] to prove the interpolation formulae. Our key new input here is to use two novel "toric period formula" – Eulerian integral presentations and the Ichino-Ikeda Gan-Gross-Prasad conjecture – together with careful choices of decomposable vectors in each case to derive the most natural possible distributions. Note that the unitary construction (2) gives the same interpolation formulae up to periods given by suitable normalization of vectors. As explained in Corollary 2.13, this comparison of periods or normalizations appears to suggest the existence of congruence numbers for the cohomological representation π, and ultimately should be related to the theory of level raising

[1] Let us also remark that there seems to be some subtlety in the setting we consider. For instance, Loeffler [49] suggests that bounded distributions should only exist in the setting of spherical varieties he describes there, of which this is not an example.

congruences in this setting. We discuss this setup in more detail for Conjecture 2.14 below. Our construction also coincides with special cases of the constructions of Harris-Li-Skinner [34] and Eischen-Harris-Li-Skinner [22, Theorem 9.2.2] obtained in a completely different way via the doubling method. It would be interesting to compare with this construction, as well as related constructions such as [47], with the view towards studying congruence ideals and their generators in this constellation. It would also be interesting to study the elements we construct in Hida families, or their images $\bmod \mathfrak{P}$ with a view towards Euler system constructions in the style of [10] and [37], using higher-rank analogues of the theory of level-raising congruences and "Ihara's lemma"(see [4] and [15] in this connection). Finally, we note that this construction, like that of [61], has its origins in looking at the distribution relations underlying the nonvanishing theorems of Vatsal [62], [63], Cornut [18] and Cornut-Vatsal [20], [19] (cf. [57]) – as a step to reducing to the theorems of Ratner [55] and Margulis-Tomanov [50] on p-adic unipotent flows. We intend to pursue this connection in a separate work.

Outline. We first review background on essential Whittaker vectors (Theorem 2.1, Corollaries 2.2 and 2.3), then Eulerian integral representations (Proposition 2.4, Theorem 2.6), followed by the relevant version of the Ichino-Ikeda Gan-Gross-Prasad conjecture for this setting (Conjecture 2.10, Theorem 2.11). We then explain how to choose decomposable vectors in each case so that the corresponding automorphic periods take values in the algebraic integers (Proposition 2.8, Corollary 2.9, and Proposition 2.12). We also discuss ratios of L-values and relations to congruence numbers in Corollary 2.13 and Conjecture 2.14. In the second section, we explain how to identify trace operators with a certain "trace" local Hecke operator (see (27)). We then explain how the eigenfunctions of these operators with p-adic unit eigenvalues can be used to derive a natural construction of a p-adic interpolation series (Lemma 3.2, Corollary 3.3, and Theorem 3.4).

Notations and conventions. Given a number field K, we write D_K to denote absolute discriminant, $\mathcal{O}_K$ the ring of integers, $\mathbf{A}_K$ the ring of adeles, and $\mathbf{A}_K^\times$ the group of ideles. Given a place v of K, we write K_v to denote the localization at v. When v is nonarchimedean, we use the same notation to denote the underlying prime ideal, and write $\mathcal{O}_{K_v}$ to denote the local ring of integers. We write $| \cdot |$ to denote idele norm on $\mathbf{A}_K^\times$, and $\mathbf{N}$ the absolute norm on $\mathcal{O}_K$ (with the choice of K being clear from the context). We also fix measures on local unipotent quotients $N_n(K_v)\backslash \mathrm{GL}_n(K_v)$ as in Matringe [51].

Acknowledgements. I am grateful to Raphaël Beuzart-Plessis, Pierre-Henri Chaudouard, Laurent Clozel, Christophe Cornut, Michael Harris, Nadir Matringe, Vincent Pilloni, and Eric Urban for helpful discussions.

2. Integral presentations of central values

2.1. **Ring class twists of $\mathrm{GL}_n(\mathbf{A}_K)$-automorphic L-functions.** We first record some basics about ring class characters over K and $\mathrm{GL}_n(\mathbf{A}_K)$-automorphic L-functions twisted by such characters for later.

2.1.1. *Ring class characters.* Recall that we fix a CM field K. Let $F = K^+$ denote its maximal totally real subfield, so that K is a totally imaginary quadratic extension of F. Given a nonzero integral ideal $\mathfrak{c} \subset \mathcal{O}_F$, let $\mathcal{O}_\mathfrak{c} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ denote the $\mathcal{O}_F$-order of conductor $\mathfrak{c}$ in K, with $\mathrm{Pic}(\mathcal{O}_\mathfrak{c})$ its corresponding class group

$$\mathrm{Pic}(\mathcal{O}_\mathfrak{c}) = \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times, \quad \widehat{\mathcal{O}}_\mathfrak{c}^\times = \prod_{v<\infty} \mathcal{O}_{\mathfrak{c},v}^\times.$$

We call a finite order idele class character $\chi = \otimes_v \chi_v$ of $\mathbf{A}_K^\times / K^\times$ a *ring class character of conductor* $\mathfrak{c}$ if it factors properly though such a class group $\mathrm{Pic}(\mathcal{O}_\mathfrak{c})$, i.e. so that there is no divisor $\mathfrak{c}' \mid \mathfrak{c}$ for which χ factors through the corresponding class group $\mathrm{Pic}(\mathcal{O}_{\mathfrak{c}'})$. Let us remark that this definition of ring class character (following [20]) is somewhat more general than the classical definition. In particular, the restriction of χ to $\mathbf{A}_F^\times$ need not be trivial, but rather an everywhere unramified (Hilbert class) character of $\mathbf{A}_F^\times$. We shall refer to the various discussions in [20, §1.1, 2, 6.1] for more background on these characters and the orders that parametrize them. Let us note that these characters are equivariant under complex conjugation, and hence their L-functions $L(s,\chi)$ satisfy the relations $L(s,\chi) = L(s,\overline{\chi}) = L(s,\chi^{-1})$. Let us also note that any ring class character χ of K has trivial archimedean component $\chi_\infty = \mathbf{1}$.

2.1.2. *Automorphic L-functions on* $\mathrm{GL}_n(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$. Let us fix $\pi = \otimes_v \pi_v$ an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ of level $c(\pi) \subset \mathcal{O}_K$ and central character $\omega = \otimes_v \omega_v$. We consider the standard L-function $\Lambda(s, \pi) = L(s, \pi_\infty)L(s, \pi)$ of π, with $L(s, \pi) = \prod_{v<\infty} L(s, \pi_v)$ denoting the Euler product over finite places, which has a well-known analytic continuation by Godement-Jacquet [26]. Thus $\Lambda(s, \pi)$ satisfies a functional equation of the form $\Lambda(s, \pi) = \epsilon(s, \pi)L(1-s, \widetilde{\pi})$, where $\epsilon(s, \pi) = q(\pi)^{\frac{1}{2}-s}\epsilon(1/2, \pi)$ denotes the epsilon factor, and $\widetilde{\pi}$ the contragredient representation. Here, $q(\pi)$ denotes the conductor of $\Lambda(s, \pi)$, and $\epsilon(1/2, \pi) \in \mathbf{S}^1$ the root number.

Let χ be any ring class character of K of conductor $\mathfrak{c} \subset \mathcal{O}_F$. We shall consider the standard L-function $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty)L(s, \pi \otimes \chi)$, where $L(s, \pi \otimes \chi) = \prod_{v<\infty} L(s, \pi_v \otimes \chi_v)$ again denotes the Euler product over finite primes, and which again has an analytic continuation by [26] with functional equation

$$\Lambda(s, \pi \otimes \chi) = \epsilon(s, \pi \otimes \chi)\Lambda(1-s, \widetilde{\pi} \otimes \chi^{-1}). \tag{3}$$

Here, $\epsilon(s, \pi \otimes \chi) = q(\pi \otimes \chi)^{\frac{1}{2}-s}\epsilon(1/2, \pi \otimes \chi)$ denotes the corresponding epsilon factor. Again writing D_K to denote the absolute discriminant of K, $q(\pi \otimes \chi) = D_K^n \mathbf{N}c(\pi)\mathbf{N}(\mathfrak{c}\mathcal{O}_K)^n = D_K^n \mathbf{N}c(\pi)\mathbf{N}\mathfrak{c}^{2n}$ denotes the conductor of $\Lambda(s, \pi \otimes \chi)$, and $\epsilon(1/2, \pi \otimes \chi) \in \mathbf{S}^1$ the root number. If the conductors $c(\pi)$ and $\mathfrak{c}\mathcal{O}_K$ are mutually coprime, then the root number $\epsilon(1/2, \pi \otimes \chi)$ can be computed according to [7, Proposition 4.1] as

$$\epsilon(1/2, \pi \otimes \chi) = \omega(\mathfrak{c}\mathcal{O}_K) \cdot \chi(c(\pi)) \cdot \epsilon(1/2, \pi) \cdot \epsilon(1/2, \chi)^n, \tag{4}$$

where $\epsilon(1/2, \chi) \in \{\pm 1\}$ denotes the root number of $L(s, \chi)$. Hence if $n \geq 2$, then this term does not contribute. If we assume that π has trivial central character ω_π, then the functional equation (3) is symmetric, relating the same L-function $\Lambda(s, \pi \otimes \chi)$ on either side. As the coefficients are real-valued in this case, we then have that $\epsilon(1/2, \pi \otimes \chi) \in \{\pm 1\}$. Moreover, we deduce in this setting that $\chi(c(\pi)) = \overline{\chi}(c(\pi)) = \chi^{-1}(c(\pi))$; the first identification implies that $\chi(c(\pi)) \in \{\pm 1\}$. Moreover, if the central character $\omega = \omega_\pi$ is trivial, then

$$\varepsilon(1/2, \pi \otimes \chi) = \chi(c(\pi))\epsilon(1/2, \pi) \in \{\pm 1\}. \tag{5}$$

Now, it is easy to see by inspection that the contribution of $\chi(c(\pi))$ will be determined by the location of the ideal $c(\pi)$ in the corresponding ring class group $\mathrm{Pic}(\mathcal{O}_\mathfrak{c})$ through which χ factors. On the other hand, if χ factors through the profinite limit $\varprojlim_m \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m})$, then it is easy to see by inspection of either formula (4) or (5) that the root number $\epsilon(1/2, \pi \otimes \chi)$ stabilizes, and in particular does not depend on the choice of ring class character χ if conductor $\mathfrak{c} = \mathfrak{p}^m$ is sufficiently large. Moreover, if we assume that central character $\omega = \omega_\pi = \mathbf{1}$ is trivial so that the formula is given by (5), then it is easy to deduce that as we vary over all ring class characters χ of K of $\mathfrak{p}$-power conductor that there exists an integer $k \in \{0, 1\}$ for which $\epsilon(1/2, \pi \otimes \chi) = (-1)^k$ for all such χ of sufficiently large conductor.

2.2. **Eulerian integral presentations.** We now introduce essential Whittaker vectors for each of our local representations π_v, leading to useful Eulerian integral presentations for the completed L-functions $\Lambda(s, \pi \times \chi) = L(s, \pi)L(s, \pi \otimes \chi)$ we study (Theorem 2.6). Roughly speaking, this classical but little-known setup gives us the following "toric integral" presentation. For $\varphi = \otimes_v \varphi_v \in V_\pi$ a certain choice of pure tensor, and $\chi = \otimes_v \chi_v$ any idele class character of K, we have that

$$\Lambda(s, \pi \otimes \chi) = \int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} y & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \chi(y)|y|^{s-\left(\frac{n-1}{2}\right)} dy,$$

where $W_\varphi = W_{\varphi,\psi}$ denotes the Whittaker coefficient defined on $g \in \mathrm{GL}_n(\mathbf{A}_K)$ with respect to the standard additive character $\psi = \otimes_v \psi_v$ of $\mathbf{A}_K/K$ by

$$W_\varphi(g) = \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi(ng)\psi^{-1}(n)dn.$$

Here, the additive character ψ is extended in the usual way to the standard unipotent subgroup of upper triangular matrices $N_n \subset \mathrm{GL}_n$. We also write $\mathbf{1}_m$ to denote the $m \times m$ identity matrix for any integer $m \geq 2$, and again $|\cdot|$ to denote the idele norm.

To describe this in more detail, we first introduce essential Whittaker vectors $\varphi_v \in V_{\pi_v}$ at nonarchimedean local places $v < \infty$. This allows us to construct an analogue of a "new vector" $\varphi = \otimes_v \varphi_v \in V_\pi$ in this setting. We then introduce the classical projection operator $\mathbb{P}_1^n$ taking cuspidal automorphic forms on $\mathrm{GL}_n(\mathbf{A}_K)$ to cuspidal L^2-automorphic forms on the mirabolic subgroup $P_2(\mathbf{A}_K) \subset \mathrm{GL}_2(\mathbf{A}_K)$. In particular, we describe how basic properties of its Fourier-Whittaker expansion can be used to derive the stated integral presentation by inspection of the Fourier-Whittaker expansion of the projected pure tensor $\mathbb{P}_1^n \varphi$. We then describe some relevant rationality theorems for critical values in this setting, and how to view them in this rubric.

2.2.1. *Essential Whittaker vectors.* Let v be any nonarchimedean place of K, which we identify with its corresponding prime ideal. We then write K_v to denote the localization of K at v, $\mathcal{O}_{K_v}$ its ring of integers, $\varpi_v \in \mathcal{O}_{K_v}$ a fixed uniformizer, and $q_v = \# \mathcal{O}_{K_v}/\varpi_v \mathcal{O}_{K_v}$ the cardinality of the residue field at v. Let us fix a nontrivial additive character ψ_v of K_v, and write ν_v to denote the character defined on $g \in \mathrm{GL}_n(K_v)$ by the rule $\nu_v(g) = |\det(g)|_v$. Recall that given a character ξ_v of $K_v^\times$, the real part $\Re(\xi_v)$ is defined to be the real number r for which $|\xi_v(t)|_{\mathbf{C}} = |t|_v^r$ as functions of $t \in K_v^\times$, i.e. where for $z \in \mathbf{C}$, we write $|z|_{\mathbf{C}} = z\overline{z}$ to denote the complex absolute value.

Let π_v be any generic representation of $\mathrm{GL}_n(K_v)$, and $\mathcal{W}(\pi_v; \psi_v)$ its corresponding Whittaker model with respect to the additive character ψ_v. As explained in [51, §1.4], such a representation π_v can be decomposed into a product of *segments*

$$\pi_v = \Delta_{v,1} \times \cdots \times \Delta_{v,m}. \tag{6}$$

Here, for each index $1 \leq j \leq m$, the corresponding segment $\Delta_{v,j}$ takes the form

$$\Delta_{v,j} = [\nu_v^{-(k_j(\pi_v)-1)} \rho_j(\pi_v), \cdots, \rho_j(\pi_v)]$$

for some real number $k_j(\pi_v)$ and cuspidal representation $\rho_j(\pi_v)$ of $\mathrm{GL}_{a_j}(K_v)$ for some integer $a_j \geq 1$. Note that this decomposition (6) is unique up to permutation of factors. Let us now retain only those cuspidal representations $\rho_j(\pi_v)$ which occur as unramified characters $\rho_j(\pi_v) = \xi_{v,j}$ of $K_v^\times / \mathcal{O}_{K_v}^\times$ for the subsequent construction, writing $0 \leq r \leq n-1$ to denote the cardinality of this set of unramified characters $\{\rho_j(\pi_v) = \xi_{v,j}\}$ occurring in the decomposition (6). We shall refer these elements simply as $\xi_{v,1}, \ldots, \xi_{v,r}$, and assume they are ordered as $\Re(\xi_{1,v}) \geq \ldots \Re(\xi_{v,r})$ whenever $r \geq 1$. We then define $\pi_{v,u}$ to be either

(i) the trivial representation of $\{1\}$ if $r = 0$, or else

(ii) the unramified representation of Langlands type defined by $\xi_{v,1} \times \cdots \times \xi_{v,m}$ if $r \geq 1$.

In case (ii), the product $\pi_{v,u} = \xi_{v,1} \times \cdots \times \xi_{v,m}$ determines an unramified generic[2] representation of $\mathrm{GL}_r(K_v)$, and we write $W_{v,u}^0$ to denote its normalized unramified Whittaker function in the Whittaker model $\mathcal{W}(\pi_{v,u}; \psi_v^{-1})$ of $\pi_{v,u}$ with respect to the additive character ψ_v^{-1}. We shall use the following result of Matringe [51, Theorem 3.1], generalizing earlier work of Casselman [12] for $n = 2$ ("new vectors"), and Jacquet-Piatetski-Shapiro-Shalika [40] (cf. [39] and [51, Remark, pp. 3-4]) for the general case on the rank $n \geq 2$ when π_v is unramified.

Theorem 2.1. *Fix $n \geq 2$ an integer, and let π be any generic representation of $\mathrm{GL}_n(K_v)$ as described above. There exists a $\mathrm{GL}_{n-1}(\mathcal{O}_{K_v})$-invariant function $W_v^{\mathrm{ess}} \in \mathcal{W}(\pi_v; \psi_v)$ whose restriction to the subgroup of diagonal matrices $A_{n-1}(K_v) \subset \mathrm{GL}_{n-1}(K_v)$ (when parametrized by simple roots) is given by*

$$W_v^{\mathrm{ess}} \left(\begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{n-1} & \\ & & & 1 \end{pmatrix} \right) = \prod_{j=1}^{n-1} \mathbf{1}_{\mathcal{O}_{K_v}^\times}(t_j)$$

[2]The representation $\pi_{v,u}$ may be reducible, however a theorem of Jacquet-Shalika [43] shows that it has a unique injective Whittaker model.

if $r = 0$, and by

$$W_v^{\mathrm{ess}} \left(\begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{n-1} & \\ & & & 1 \end{pmatrix} \right)$$

$$= W_{v,u}^0 \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \right) \nu_v \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_r \end{pmatrix} \right)^{\frac{n-r}{2}} \mathbf{1}_{\mathcal{O}_{K_v}}(t_r) \prod_{j=r+1}^{n-1} \mathbf{1}_{\mathcal{O}_{K_v}^\times}(t_j)$$

if $1 \leq r \leq n-1$. Here, we write $\mathbf{1}_{\mathcal{O}_{K_v}}$ to denote the characteristic function of the integer ring $\mathcal{O}_{K_v}$, and $\mathbf{1}_{\mathcal{O}_{K_v}^\times}$ that of the local unit group $\mathcal{O}_{K_v}^\times$. This function W_v^{ess} is unique up to nonzero scalar multiple, and moreover equal to 1 on $\mathrm{GL}_{n-1}(\mathcal{O}_{K_v})$.

Proof. See [51, §1.4, Theorem 3.1], as well as [51, Corollaries 3.2 and 3.3]. □

Remark We expect that the function W_v^{ess} defined in Theorem 2.1 can be normalized so that its values all have real part equal to zero (although we do not require such a normalization for our construction).

Specializing to diagonal matrices that are trivial away from the upper left entry (as we shall), we derive the following useful characterization of these essential Whittaker vectors as functions of $t \in K_v^\times$. Given any integer $m \geq 1$, let us write $\mathbf{1}_m$ to denote the identity matrix in GL_m, with $\mathbf{1}_0 = 0$ for $m = 0$.

Corollary 2.2. *For any $t \in K_v^\times$, the unique $\mathrm{GL}_{n-1}(\mathcal{O}_{K_v})$-invariant function $W_v^{\mathrm{ess}} \in \mathcal{W}(\pi_v; \psi_v)$ satisfies*

$$W_v^{\mathrm{ess}} \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) = \begin{cases} \mathbf{1}_{\mathcal{O}_{K_v}^\times}(t) & \text{if } r = 0 \\ W_{v,u}^0 \left(\begin{pmatrix} t & \\ & \mathbf{1}_{r-1} \end{pmatrix} \right) \nu_v \left(\begin{pmatrix} t & \\ & \mathbf{1}_{r-1} \end{pmatrix} \right)^{\frac{n-r}{2}} & \text{if } r \geq 1. \end{cases} \tag{7}$$

Moreover, this function is trivial on integral elements $t \in A_1(\mathcal{O}_{K_v}) \cong \mathcal{O}_{K_v} \backslash \{0\}$.

Proof. The statement is a direct consequence of Theorem 2.1, taking into account that the function is known to be trivial on $A_{n-1}(\mathcal{O}_{K_v}) \subset \mathrm{GL}_{n-1}(\mathcal{O}_{K_v})$. □

We also have the following important consequence of Theorem 2.1, as explained in [51, Corollary 3.3]. Writing $B_m \subset \mathrm{GL}_m$ for any integer $m \geq 1$ to denote the standard Borel subgroup of upper triangular matrices, let δ_{B_m} denote the positive character on B_m defined so that if μ is a right Haar measure on $B_m(K_k)$ and int denotes the action on smooth functions f with compact support on $b \in B_m(K_v)$ given by $(\mathrm{int}(b)f)(x) = f(b^{-1}xb)$, then $\mu \circ \mathrm{int}(b) = \delta_{B_m}(b)\mu$ for any $b \in B_m(K_v)$ (which is trivial for $m = 1$).

Corollary 2.3. *For each integer $1 \leq m \leq n-1$ and each representation π_v' of Langlands type of $\mathrm{GL}_m(K_v)$ with normalized spherical Whittaker vector $W_{\pi_v'}^0 \in \mathcal{W}(\pi_v', \psi_v^{-1})$, we have the identification*

$$I(W_v^{\mathrm{ess}}, W_{\pi_v'}, s) := \int_{A_m(K_v)} W_v^{\mathrm{ess}} \begin{pmatrix} t & \\ & \mathbf{1}_{n-m} \end{pmatrix} W_{\pi_v'}^0(t) \delta_{B_m}^{-1}(t) \nu_v(t)^{s - \left(\frac{n-m}{2}\right)} dt = L(s, \pi_v \times \pi_v').$$

Here, $s \in \mathbf{C}$ is a complex parameter (first with $\Re(s) \gg 1$), $N_m \subset \mathrm{GL}_m$ the unipotent subgroup of upper triangular matrices, and $L(s, \pi_v \times \pi_v')$ the local Rankin-Selberg L-function of π_v times π_v'. As well, we normalize the Haar measure on $N_m(K_v) \backslash \mathrm{GL}_m(K_v)$ in the same way as for [51, §3].

This latter result gives us a direct relation between the the local L-factors $L(s, \pi_v \otimes \chi_v)$ in the L-functions $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty) L(s, \pi \otimes \chi)$ we wish to study, and in particular allows us to relate any nonarchimedean local factor $L(s, \pi_v \otimes \chi_v)$ to a shifted adelic Mellin transform of W_v^{ess}, again viewed as a function of $t \in K_v^\times$ as in (7). We explain this in more detail as follows, using the theory of Eulerian integral presentations.

2.2.2. *Eulerian integral presentations.* Recall that we fix $\pi = \otimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ for some dimension $n \geq 2$. Let us now fix a pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$. Again, we write $N_n \subset \mathrm{GL}_n$ to denote the unipotent subgroup of upper triangular matrices. Let us write $\psi = \otimes_v \psi_v$ to denote the standard additive character on $\mathbf{A}_K/K$, so that the archimedean component ψ_∞ coincides with the function sending $z_\infty = (z_{\infty,j}, \overline{z}_{\infty,j})_{j=1}^d \in K_\infty^\times \cong \mathbf{C}^{2d}$ to $\psi_\infty(z_\infty) = \exp(2\pi i (z_{\infty,1} + \overline{z}_{\infty,1} \cdots + x_{\infty,d} + \overline{z}_{\infty,d}))$, and at each nonarchimedean local place v of K is trivial on the local inverse different $\mathfrak{d}_{K_v}^{-1}$ but nontrivial on $v^{-1}\mathfrak{d}_{K_v}^{-1}$. Recall that the Fourier-Whittaker of φ evaluated at any matrix $g \in \mathrm{GL}_n(\mathbf{A}_K)$ is given by

$$W_\varphi(g) = \sum_{\gamma \in N_{n-1}(K) \backslash \mathrm{GL}_{n-1}(K)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right), \tag{8}$$

where again $W_\varphi = W_{\varphi,\psi}$ denotes the Whittaker coefficient defined on $g \in \mathrm{GL}_n(\mathbf{A}_K)$ by

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi(ng)\psi^{-1}(n) dn.$$

Note that for a pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ as we consider, this Whittaker function has the corresponding decomposition into local components $W_\varphi = \otimes_v W_{\varphi_v}$. Moreover, at each place v of K, the corresponding Kirillov map $\mathfrak{K}_v : \varphi_v \mapsto W_{\varphi_v}$ gives us the isomorphism

$$\mathfrak{K}_v : V_{\pi_v} \cong \mathcal{W}(\pi_v; \psi_v), \quad \varphi_v \longmapsto W_{\varphi_v} \tag{9}$$

between the space V_{π_v} of the representation (π_v, V_{π_v}) and its corresponding Whittaker model $\mathcal{W}(\pi_v; \psi_v)$ with respect to our fixed additive character ψ_v, i.e. the component at v of the standard additive character $\psi = \otimes_v \psi_v$ of $\mathbf{A}_K/K$. Let us for future reference record the following important consequence of this fact.

Proposition 2.4 (Jacquet-Shalika, Bernstein-Zelevinsky)**.** *Fix $\pi = \otimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ of any dimension $n \geq 2$, and let $\psi = \otimes_v \psi_v$ as described above denote the standard additive character of $\mathbf{A}_K/K$. Given a place v of K, let $W_v : K_v^\times \to \mathbf{C}^\times$ be any smooth function which is square integrable (or more generally summable) and compactly supported. There exists a vector $\varphi_v \in V_{\pi_v}$ in the local representation π_v such that, as functions of $t_v \in K_v^\times$, we have the identification*

$$W_{\varphi_v}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) = W_v(t_v).$$

Proof. Again, this property is a formal consequence of the isomorphism (9). The result was first shown for v nonarchimedean by Bernstein-Zelevinsky [8], then for v archimedean by Jacquet-Shalika [42, e.g. (3.8)]. ☐

Caveat. We note that in the archimedean setting, the cited theorem of Jacquet-Shalika [42] shows that the restriction of the unitary representation σ of $\mathrm{GL}_n(K_\infty)$ to the mirabolic subgroup $P_n(K_\infty) \subset \mathrm{GL}_n(K_\infty)$ is equivalent to the unitary representation τ of $P_n(K_\infty)$. However, smooth vectors in this latter representation space V_τ need not correspond to smooth vectors in that of the former V_σ. Although this does not pose a problem for the choice of normalization of the archimedean local pure tensor $\varphi_\infty = \otimes_{v|\infty} \varphi_v$ we make below, we record this subtle point explicitly here as it is not so well-known. In other words, one cannot assume a priori that the chosen archimedean local vector φ_v in the statement of Proposition 2.4 above is smooth.

Let us now consider the unipotent radical $Y_{n,1} \subset \mathrm{GL}_n$ of the parabolic subgroup associated to the partition $(2, 1, \ldots, 1)$ of n. Hence, $Y_{n,1}$ is normalized by $\mathrm{GL}_2 \subset \mathrm{GL}_n$, and we have the decomposition $N_n \cong N_2 \ltimes Y_{n,1}$. Let $P_2 \subset \mathrm{GL}_2$ denote the mirabolic subgroup corresponding to the stabilizer of ψ; its adelic points $P_2(\mathbf{A}_K)$ can be described simply as

$$P_2(\mathbf{A}_K) = \left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix} : y \in \mathbf{A}_K^\times,\ x \in \mathbf{A}_K \right\} \subset \mathrm{GL}_2(\mathbf{A}_K).$$

Corresponding to any pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ in the cuspidal $\mathrm{GL}_n(\mathbf{A}_K)$-representation π, we can define the classical projection $\mathbb{P}_1^n \varphi$ on $\varphi \in V_\pi$ by the rule sending a matrix $p \in P_2(\mathbf{A}_K)$ to the integral

$$\mathbb{P}_1^n \varphi(p) = |\det(p)|^{-\left(\frac{n-2}{2}\right)} \int_{Y_{n,1}(K)\backslash Y_{n,1}(\mathbf{A}_K)} \varphi\left(y \begin{pmatrix} p & \\ & \mathbf{1}_{n-2} \end{pmatrix} \right) \psi^{-1}(y) dy.$$

Note that since this integral is defined over a compact domain, it converges absolutely. As explained in Cogdell [16, Lemma 5.2] (cf. also [17, § 2.2.1]), this function $\mathbb{P}_1^n\varphi$ is left $P_2(K)$-invariant, and moreover can be viewed as a cuspidal L^2-automorphic form on $p \in P_2(\mathbf{A}_F)$ having the Fourier-Whittaker expansion

$$\mathbb{P}_1^n\varphi(p) = |\det(p)|^{-\left(\frac{n-2}{2}\right)} \sum_{\gamma \in K^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & \mathbf{1}_{n-1}\end{pmatrix}\begin{pmatrix} p & \\ & \mathbf{1}_{n-2}\end{pmatrix}\right). \tag{10}$$

Note that, somewhat remarkably, this latter expansion (10) on $P_2(\mathbf{A}_K)$ carries all of the information about specializations to upper left diagonal elements of the Fourier-Whittaker coefficients appearing in (8) as needed to construct the L-function of π. Cogdell[3] used this observation to give the following Eulerian integral presentation of the L-functions $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty)L(s, \pi \otimes \chi)$ we consider above, although we remark that this theory of Eulerian integral presentations applies more generally to Rankin-Selberg L-functions for $\mathrm{GL}_n(\mathbf{A}_K) \times \mathrm{GL}_m(\mathbf{A}_K)$ for any dimensions $n \geq 2$ and $1 \leq m \leq n$, and any number field K. To describe this special case, let us for $\varphi = \otimes_v \varphi_v \in V_\pi$ any pure tensor for π, $\chi = \otimes_v \chi_v$ any ring class character of K, and $s \in \mathbf{C}$ any complex parameter (first with $\Re(s) \gg 1$) consider the integral

$$I(s, \varphi, \chi) := \int_{\mathbf{A}_K^\times/K^\times} \mathbb{P}_1^n\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi(t)|t|^{s-\frac{1}{2}}dt. \tag{11}$$

Here, we write $|\cdot| = |\cdot|_{\mathbf{A}_K^\times}$ to denote the idele norm on $\mathbf{A}_K^\times$.

Proposition 2.5. *The integral $I(s, \varphi, \chi)$ of (11), defined for any complex number $s \in \mathbf{C}$, cuspidal pure tensor $\varphi = \otimes_v \varphi_v$ of $\mathrm{GL}_n(\mathbf{A}_K)$, and idele class character $\chi = \otimes_v \chi_v$ of K, satisfies the following properties:*

(i) *It is absolutely convergent for each $s \in \mathbf{C}$, and hence entire.*
(ii) *It is bounded in vertical strips.*
(iii) *It satisfies the functional equation $I(s, \varphi, \chi) = I(1-s, \widetilde{\varphi}, \chi^{-1})$, where $\widetilde{\varphi}$ denotes the contragredient vector defined on $g \in \mathrm{GL}_n(\mathbf{A}_K)$ by $\widetilde{\varphi}(g) = \varphi({}^tg^{-1})$.*

Proof. These analytic properties are shown in [16, Lecture 5, Proposition 5.3] (with $n' = 1$), for instance. □

Opening up the Fourier-Whittaker expansion (10) of $\mathbb{P}_1^n\varphi$, then switching the order of summation and integration to collapse the integral, we obtain

$$\begin{aligned} I(s, \varphi, \chi) &= \int_{\mathbf{A}_K^\times/K^\times} \sum_{\gamma \in K^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & \mathbf{1}_{n-1}\end{pmatrix}\begin{pmatrix} t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi(t)|t|^{s-\frac{1}{2}-\left(\frac{n-2}{2}\right)}dt \\ &= \int_{\mathbf{A}_K^\times/K^\times} \sum_{\gamma \in K^\times} W_\varphi\left(\begin{pmatrix} \gamma t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\xi(t)|t|^{s-\left(\frac{n-1}{2}\right)}dt \\ &= \int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi(t)|t|^{s-\left(\frac{n-1}{2}\right)}dt. \end{aligned}$$

Now, since the Whittaker function W_φ decomposes into local factors as $W_\varphi = \otimes_v W_{\varphi_v}$, we can decompose these latter integrals as

$$\int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi(t)|t|^{s-\left(\frac{n-1}{2}\right)}dt = \prod_{v \leq \infty}\int_{K_v^\times} W_{\varphi_v}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi_v(t_v)|t_v|_v^{s-\left(\frac{n-1}{2}\right)}dt_v$$

and in particular

$$\int_{\mathbf{A}_{K,f}^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi(t)|t|^{s-\left(\frac{n-1}{2}\right)}dt = \prod_{v < \infty}\int_{K_v^\times} W_{\varphi_v}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right)\chi_v(t_v)|t_v|_v^{s-\left(\frac{n-1}{2}\right)}dt_v.$$

These integrals in fact give presentations of the completed L-function $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty)L(s, \pi \otimes \chi)$ and its finite part $L(s, \pi \otimes \chi)$ respectively. Putting this together with the existence and characterization of essential Whittaker vectors as described above, we obtain the following integral presentations.

[3] in the course of proving converse theorems for $\mathrm{GL}_n(\mathbf{A}_K)$

Theorem 2.6. *Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ for any dimension $n \geq 2$, and let $\varphi = \otimes_v \varphi_v \in V_\pi$ be a pure tensor. Suppose as we may by Theorem 2.1 and Proposition 2.4 that for each nonarchimedean place v of K, we choose the local vector $\varphi_v \in V_{\pi_v}$ in such a way that its corresponding Whittaker function W_{φ_v} as a function of $t_v \in K_v^\times$ is given by the specialization of the essential Whittaker function W_v^{ess} described in (7) above, i.e. so that we have the identification(s)*

$$W_{\varphi_v}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) = W_v^{\mathrm{ess}}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \qquad \forall\ t_v \in K_v^\times.$$

Let $\chi = \otimes_v \chi_v$ be any ring class character of K. We have for any $s \in \mathbf{C}$ (first with $\Re(s) \gg 1$) the following presentation for the finite part $L(s, \pi \otimes \chi)$ of the completed L-function $\Lambda(s, \pi\otimes\chi) = L(s, \pi_\infty)L(s, \pi \otimes \chi)$:

$$L(s, \pi \otimes \chi) = \int_{\mathbf{A}_{K,f}^\times} W_\varphi\left(\begin{pmatrix} t_f & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(t_f)|t_f|^{s-\left(\frac{n-1}{2}\right)}dt_f.$$

We also have that

$$\Lambda(s, \pi \otimes \chi) = \int_{\mathbf{A}_{K}^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(t)|t|^{s-\left(\frac{n-1}{2}\right)}dt.$$

Proof. Let us suppose first that the idele class character $\chi = \otimes_v \chi_v$ is everywhere unramifed. Keeping the setup described above, we then have that

$$\begin{aligned} L(s, \pi\otimes\chi) &= \prod_{v<\infty} L(s, \pi_v \otimes \chi_v) \\ &= \prod_{v<\infty} \int_{A_1(K_v)} W_v^{\mathrm{ess}}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi_v(t_v)\nu_v(t_v)^{s-\left(\frac{n-1}{2}\right)}dt_v \\ &= \prod_{v<\infty} \int_{K_v^\times} W_v^{\mathrm{ess}}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi_v(t_v)|t_v|_v^{s-\left(\frac{n-1}{2}\right)}dt_v \\ &= \prod_{v<\infty} \int_{K_v^\times} W_{\varphi_v}\left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi_v(t_v)|t_v|_v^{s-\left(\frac{n-1}{2}\right)}dt_v \\ &= \int_{\mathbf{A}_{K,f}^\times} W_\varphi\left(\begin{pmatrix} t_f & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(t_f)|t_f|^{s-\left(\frac{n-1}{2}\right)}dt_f. \end{aligned}$$

Here, the first identification follows from the definition of the L-function $L(s, \pi \otimes \chi)$, and the second is shown in [41, §5] (cf. [51, comment after Proposition 1.4]). The third identification follows from making the identification of Corollary 2.3 at each finite place v, using that the positive character δ_{B_1} is trivial. The fourth identification is a simplification of notations. The fifth identification follows from our choice of pure tensor $\varphi = \otimes_v \varphi_v$ at each finite place v, and the sixth is simply the global integral corresponding to the product. Now, taking χ to be the trivial character, and comparing Dirichlet series expansions first for $\Re(s) > 1$, we deduce that we have the identification

$$\sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{c_\pi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s} = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\rho_\varphi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^{s-\left(\frac{n-1}{2}\right)}} = \int_{\mathbf{A}_K^\times/K^\times} \mathbb{P}_1^n\varphi\left(\begin{pmatrix} y_f & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)|y_f|^{s-\frac{1}{2}}dy_f.$$

Via inspection of the Fourier-Whittaker expansion on the right hand side, we then claim that for χ any idele class character of K we have the corresponding twisted relation

$$\sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{c_\pi(\mathfrak{a})\chi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s} = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\rho_\varphi(\mathfrak{a})\chi(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^{s-\left(\frac{n-1}{2}\right)}} = \int_{\mathbf{A}_K^\times/K^\times} \mathbb{P}_1^n\varphi\left(\begin{pmatrix} y_f & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(y_f)|y_f|^{s-\frac{1}{2}}dy_f.$$

This implies the first claim. The second identification then is easy to deduce from the discussion above with the projection operator $\mathbb{P}_1^n\varphi$ and its Fourier-Whittaker expansion; we refer to [16, § 3, 5] for more details. $\square$

2.3. **Rationality theorems and normalizations.** We now record some relevant rationality theorems in the direction of Deligne's conjectures and explain the connection with the setup described above. Let us now suppose that $\pi = \otimes_v \pi_v$ is a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ for any dimension $n \geq 2$. That is, we suppose now that there exists a finite dimensional irreducible algebraic representation $\mathcal{E}_\mu$ of the real Lie group $G_{n,\infty} = \mathrm{GL}_n(K \otimes_{\mathbf{Q}} \mathbf{R})$ for which the archimedean component π_∞ has nontrivial relative Lie cohomology with respect to $\mathcal{E}_\mu$. Here, μ denotes the highest weight for $\mathcal{E}_\mu$; this highest weight can be parametrized as $(\mu_v)_{v \in \infty}$ for v ranging over (pairs of) archimedean places of K, with each entry $\mu_v = (\mu_{v,j})_{j=1}^n$ arranged so that $\mu_{v,1} \geq \cdots \geq \mu_{v,n}$ (see e.g. [29, §1]). Recall that the purity weight $w \in \mathbf{Z}$ is a constant for which $w = \mu_{v,j} + \mu_{v,n+1-j}$ for any $1 \leq j \leq n$ (and any archimedean place v).

Recall that an automorphic L-function $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty) L(s, \pi \otimes \chi)$ of the type we consider here, i.e. with χ corresponding to a wide class character so that $\chi_\infty = 1$ in all cases, an argument $s_0 \in \frac{n-1}{2} + \mathbf{Z}$ is said to be *critical* in the sense of Deligne [21] if neither $L(s, \pi_\infty)$ nor the corresponding contragredient factor $L(1-s, \widetilde{\pi}_\infty)$ has a pole at s_0. Here, we use the normalization for automorphic motives appearing in various works such as [29], [28], and [33], and note that Deligne [21] uses the arithmetic normalization so that a critical value s_0 can only ever be an integer. Note as well that when $n \geq 2$ is even, the central value $s_0 = 1/2$ with respect to the functional equation (3) is always critical. We have the following special cases of Deligne's conjectures in the setting we consider for this work, which for our purposes generalize the well-known theorems of Shimura for Rankin-Selberg L-functions on $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ (e.g. [60]).

Theorem 2.7. *Let $n \geq 2$ be any even integer. Let $\pi = \otimes_v \pi_v$ be a cohomological automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$. Let us assume that $\pi = \mathrm{BC}_{K/F}(\pi')$ arises via cyclic basechange in the sense of Arthur-Clozel [5] from an irreducible cuspidal cohomological representation π' of $\mathrm{GL}_n(\mathbf{A}_F)$, and moreover that π' descends to a cuspidal automorphic representation $\pi'^U = \otimes_v \pi_v'^U$ on some unitary group $U(V)(\mathbf{A}_F)$ associated to a hermitian vector space V of dimension n over K. Again, we write $\pi^U = \mathrm{BC}_{K/F}(\pi'^U)$ to denote the quadratic basechange of π'^U to $U(V)(\mathbf{A}_K)$. Let us also assume that $\pi'_\infty = \otimes_{v \mid \infty} \pi'_v$ is regular, hence with distinct Hodge-Tate weights $(\lambda_j)_{j=1}^n = (\lambda_{j,v})_{j,v}^n$ for each real place $v \mid \infty$ in F; writing $w \in \mathbf{Z}$ to denote the purity weight so that $\lambda_j + \lambda_{n+1-j} = w$ for each $1 \leq j \leq n$, we also assume that $\lambda_j \neq w/2$ for each $1 \leq j \leq n$. Let χ be any ring class character of K. Then for $s_0 \in \frac{n-1}{2} + \mathbf{Z}$ any critical value of $\Lambda(s, \pi \otimes \chi) = L(s, \pi_\infty) L(s, \pi \otimes \chi)$, there exists a nonzero complex number $\Omega_{s_0}(\pi, \chi) \in \mathbf{C} \backslash \{0\}$ depending on both π and χ such that the ratio*

$$\mathfrak{L}(s_0, \pi \otimes \chi) = \frac{\Lambda(s_0, \pi \otimes \chi)}{\Omega_{s_0}(\pi, \chi)} = \frac{L(s_0, \pi_\infty)}{\Omega_{s_0}(\pi, \chi)} \cdot L(s_0, \pi \otimes \chi) \in \overline{\mathbf{Q}}$$

is an algebraic number, contained in the number field $E(\pi, \chi) = \mathbf{Q}(\pi)\mathbf{Q}(\chi)K^{\mathrm{Gal}}$ given by the compositum of the Hecke field $\mathbf{Q}(\pi)$ of π with the cyclotomic field $\mathbf{Q}(\chi)$ generated by the values of χ with the Galois closure K^{Gal} of K in $\overline{\mathbf{Q}}$. Moreover, there is a natural action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on these values of the form

$$\sigma\left(\mathfrak{L}(s_0, \pi \otimes \chi)\right) = \mathfrak{L}(s_0, \pi^\gamma \otimes \chi^\sigma),$$

where $\gamma = \sigma|_{\mathbf{Q}(\pi)}$ denotes the restriction of σ to the Hecke field $\mathbf{Q}(\pi)$. Here, γ acts on π in a natural way via the Hecke eigenvalues, and χ^σ is defined by the rule sending $t \in \mathbf{A}_K^\times / K^\times$ to the complex value $\sigma(\chi(t))$.

Proof. See Harris-Lin [35, Theorem 4.7]. The result can be deduced from the main theorem of Harris [33], as well as those of Guerberoff [30] and Guerberoff-Lin [31], albeit with an additional nonvanishing hypothesis to deal with special case where n is even and $s_0 = 1/2$ is the central value.

Note that the result for the underlying $\mathrm{GL}_n(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L-function $L(s, \pi' \times \pi(\chi))$ is also addressed in Harder-Raghuram [32, Theorem 7.21]. See also Raghuram [54] for $\mathrm{GL}_n(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$ families. The period $\Omega_{s_0}(\pi \otimes \chi)$ implicit in these latter results takes the form of a Langlands quotient

$$\Omega_{s_0}(\pi \otimes \chi) = \frac{\Lambda(s_0 + 1, \widetilde{\pi} \otimes \chi)}{i^{n[F:\mathbf{Q}]} \tau(\eta_{K/F})} = \frac{L(s_0 + 1, \widetilde{\pi}_\infty) L(s_0 + 1, \widetilde{\pi} \otimes \chi)}{i^{n[F:\mathbf{Q}]} \tau(\eta_{K/F})}$$

in our notations, with $i = \sqrt{-1}$ a fixed square root of -1, and $\tau(\eta_{K/F})$ the Gauss sum of the idele class character $\eta_{K/F}$ of F associated to the quadratic extension K/F[4]. Here, we write $\widetilde{\pi}$ to denote the contragredient representation associated to π, which arises via cyclic basechange from the corresponding contragredient $\widetilde{\pi}'$ of π'. In particular, this period carries some dependence on the choice of ring class character χ. ☐

[4]which is of course independent of the choice of ring class character χ of K.

2.3.1. *Normalizations of vectors.* Let us now return to the setup of Theorem 2.6 above, taking for granted the results of Theorems 2.1 and 2.7. We can make the following useful choice(s) of normalization(s) for the archimedean component of the pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ appearing in Theorem 2.6.

Proposition 2.8. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal cohomological automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ as in Theorem 2.7 above, with $s_0 \in \frac{n-1}{2} + \mathbf{Z}$ any critical value for the standard L-function $\Lambda(s, \pi) = L(s, \pi_\infty)L(s, \pi)$. Let $\varphi = \otimes_v \varphi_v \in V_\pi$ be any pure tensor whose nonarchimedean local components are chosen to be essential Whittaker vectors as in Theorems 2.1 and 2.6 above. We can find a normalization $\varphi_\infty^{\mathrm{int}} = \otimes_{v\mid\infty} \varphi_v^{\mathrm{int}}$ of the archimedean component of any such pure tensor so that the Eulerian integral*

$$\begin{aligned} \mathfrak{L}(s_0, \pi) := \frac{\Lambda(s_0, \pi)}{\Omega_{s_0}(\pi)} &= \int_{\mathbf{A}_K^\times} W_{\varphi^{\mathrm{int}}} \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |t|^{s_0 - \left(\frac{n-1}{2}\right)} dt \\ &= \frac{L(s_0, \pi_\infty)}{\Omega_{s_0}(\pi)} \cdot \int_{\mathbf{A}_{K,f}^\times} W_{\varphi^{\mathrm{int}}} \left(\begin{pmatrix} t_f & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |t_f|^{s_0 - \left(\frac{n-1}{2}\right)} dt_f \end{aligned} \tag{12}$$

corresponding to the normalized vector $\varphi^{\mathrm{int}} = \otimes_{v<\infty} \varphi_v \otimes \varphi_\infty^{\mathrm{int}}$ is an algebraic integer, contained in the ring of integers $\mathcal{O}_{E_\pi}$ of the number field $E_\pi = \mathbf{Q}(\pi)K^{\mathrm{Gal}}$.

Proof. Given complex numbers A and $B \neq 0$ and a number field $E \subset \overline{\mathbf{Q}}$, let us write $A \sim_E B$ if there exists an algebraic number $\alpha \in E$ for which $A = \alpha B$, or equivalently if $A/B = \alpha \in \overline{\mathbf{Q}}$. Taking Theorem 2.7 with trivial class group character $\chi = \mathbf{1}$ (which we henceforth drop from the notations), we have the relation

$$L(s_0, \pi) \sim_{E_\pi} \frac{\Omega_{s_0}(\pi)}{L(s_0, \pi_\infty)}. \tag{13}$$

Using Theorem 2.6 to describe the left hand side of (13) as an Eulerian integral presentation, it follows that for $\varphi = \otimes_v \varphi_v \in V_\pi$ any pure tensor whose nonarchimedean local components are chosen to be essential Whittaker vectors as in Theorems 2.1 and 2.6 above, we have the rationality relation

$$\int_{\mathbf{A}_{K,f}^\times} W_\varphi \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |t|^{s_0 - \left(\frac{n-1}{2}\right)} dt \sim_{E_\pi} \frac{\Omega_{s_0}(\pi)}{L(s_0, \pi)}.$$

Hence, writing

$$A = \int_{\mathbf{A}_{K,f}^\times} W_\varphi \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |t|^{s_0 - \left(\frac{n-1}{2}\right)} dt \quad \text{and} \quad B = \frac{\Omega_{s_0}(\pi)}{L(s_0, \pi)},$$

we can find some $\alpha \in E_\pi$ for which $A = \alpha B$. Now, put $\alpha = \alpha_1/\alpha_2$ for E_π-integers $\alpha_1, \alpha_2 \in \mathcal{O}_{E_\pi}$ with $\alpha_2 \neq 0$. It follows that $\alpha_2 A = \alpha_1 B$, hence $\alpha_2 A/B = \alpha_1$ is an integer in E_π. We can now use Proposition 2.4 to justify normalizing the achimedean component $\varphi_\infty = \otimes_{v\mid\infty} \varphi_v$ in this way to prove the result. $\square$

Corollary 2.9. *Let us retain the setup of Proposition 2.8 above, as well as Theorems 2.1, 2.6, and 2.7. Again taking the trivial class group character $\chi = \mathbf{1}$, we write $\varphi^{\mathrm{int}} \in V_\pi$ for the corresponding normalization of (the archimedean component of) the pure tensor $\varphi \in V_\pi$, so that the algebraic part of the critical value*

$$\begin{aligned} \mathfrak{L}(s_0, \pi) = \frac{\Lambda(s_0, \pi)}{\Omega_{s_0}(\pi)} &= \int_{\mathbf{A}_K^\times} W_{\varphi^{\mathrm{int}}} \left(\begin{pmatrix} y & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |y|^{s_0 - \left(\frac{n-1}{2}\right)} dy \\ &= \frac{L(s_0, \pi_\infty)}{\Omega_{s_0}(\pi, \mathbf{1})} \cdot \int_{\mathbf{A}_{K,f}^\times} W_\varphi \left(\begin{pmatrix} y_f & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) |y_f|^{s_0 - \left(\frac{n-1}{2}\right)} dy_f \end{aligned}$$

is an algebraic integer. We consider the following function on certain finite quotients of the idele group $\mathbf{A}_K^\times$. Hence, let us fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underling rational prime p. For each integer $m \geq 0$, we put

$$X_m = \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m}) := \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times, \quad \mathcal{O}_{\mathfrak{p}^m} := \mathcal{O}_F + \mathfrak{p}^m \mathcal{O}_K.$$

For each class $A \in X_m$, we fix an idele representative $t_A \in \mathbf{A}_K^\times$ so that $A = [t_A] \in X_m = \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$. We can then consider the function $\mathfrak{W}_{s_0,m}$ defined on any class A in any of these finite quotients X_m of the *idele class group $\mathbf{A}_K^\times/K^\times$ by taking the sum over ideles in the class $[t_A]$:*

$$\mathfrak{W}_{s_0,m}(A) = \mathfrak{W}_{s_0,m}([t_A]) := \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} W_{\varphi^{\mathrm{int}}}\left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1}\end{pmatrix}\right) \cdot |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}. \tag{14}$$

Note that the sum defining (14) *is absolutely convergent by Proposition 2.5 (i). By construction, this function takes values in the field of rationality $E = E_\pi$. More precisely, using the surjectivity of the archimedean local Kirillov map to chose the archimedean local vector φ_∞ as in Proposition 2.8 above, we deduce that the sums* (14) *take values in the ring of integers $\mathcal{O}_E$. Hence, each of the coefficient functions* (14) *associated to the normalized vector φ^{int} takes values in the algebraic integers.*

2.4. **The Ichino-Ikeda formula for $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$.** Let us now consider the special case of central values $s_0 = 1/2$, and in particular the following form of the Ichino-Ikeda conjecture for unitary groups $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F) = U(V)(\mathbf{A}_F) \times U(L)(\mathbf{A}_F)$ as formulated by Liu [48], and whose proof was announced recently in important special cases by Beuzart-Plessis and Chaudouard [11, Theorem 1.3.6.1].

Fix $n = 2m \geq 2$ an even integer. Let V be a hermitian space of dimension n over K, with $U_n = U(V)$ its unitary group. We assume there exists a nondegenerate line $L \subset V$ whose orthogonal complement in V admits an isotropic subspace Z of the maximal possible dimension $m-1$, and write $U_1 = U(L)$ to denote its unitary group. Write $P \subset U(V)$ to denote the parabolic subgroup which stabilizes a complete flag of subspaces in Z, with Levi factor $M_P \cong \mathrm{GL}_1(K)^{m-1} \times U_1$, and unipotent radical $N = N_P$. Note that P contains $U_1 = U(L)$. Let $\psi = \psi_N$ denote the automorphic additive character on $N(F)\backslash N(\mathbf{A}_F)$ defined in Gan-Gross-Prasad [24] and [11, §1.3]. Note that this additive character is invariant under conjugation by $U_1(\mathbf{A}_F)$. Let us also remark that this choice of automorphic character $\psi = \psi_N$ is similar in spirit to the choice of unipotent radical subgroup $Y_{n,1} \subset N_n$ used to define the projection operator $\mathbb{P}_1^n\varphi$, in that it ensures the corresponding integral over the unipotent radical $N(F)\backslash N(\mathbf{A}_F)$ is automorphic.

Let us now take $\pi'^U = \otimes_v \pi_v'^U$ to be a cuspidal automorphic representation of $U_n(F)\backslash U_n(\mathbf{A}_F)$, which we shall assume has a basechange to a cuspidal automorphic representation $\pi' = \otimes_v \pi_v'$ of $\mathrm{GL}_n(\mathbf{A}_F)$ as above. Following Beuzart-Plessis-Chaudouard [11, Theorem 1.3.6.1], we shall also assume that π'^U us everywhere locally tempered. Recall that we write $\pi = \mathrm{BC}_{K/F}(\pi')$ to denote the quadratic basechange of π' to $\mathrm{GL}_n(\mathbf{A}_K)$. Let us also write π^U to denote the corresponding quadratic basechange of the unitary representation π'^U. This set of conditions implies that we have the identification of completed L-functions $\Lambda(s, \pi \otimes \chi) = \Lambda(s, \pi^U \otimes \chi)$ for any idele class character χ of K. Consider the corresponding projection operator $\mathcal{P}_\psi$ defined for $t \in U_1(\mathbf{A}_F)$ and $\phi = \otimes_v \phi_v \in V_{\pi'^U}$ a decomposable vector as the integral

$$\mathcal{P}_\psi \phi(t) = \int_{N(F)\backslash N(\mathbf{A}_F)} \phi(nt)\psi^{-1}(n) dn. \tag{15}$$

Again, we note that the choice of automorphic character $\psi = \otimes_v \psi_v$ ensures that the integral (15) is left $U_1(F)$-invariant, analogous to the definition of projection operator $\mathbb{P}_1^n\varphi$ for $\mathrm{GL}_n(\mathbf{A}_K)$. Let χ be a ring class character of K, which we now identify as an automorphic form on $U_1(F)\backslash U_1(\mathbf{A}_F) \cong \mathbf{A}_K^\times/K^\times\mathbf{A}_F^\times$. Consider the linear form defined in the natural way via the corresponding (Bessel) automorphic period integral

$$\phi \in \pi'^U \longmapsto P_\chi(\phi) := \int_{U_1(F)\backslash U_1(\mathbf{A}_F)} \mathcal{P}_\psi \phi(t) \cdot \chi(t) dt. \tag{16}$$

Conjecture 2.10 (Ichino-Ikeda for $U_n(\mathbf{A}_F) \times U_1(\mathbf{A}_F)$, [48])**.** *There exists for each place v of K a local sesquilinear form*[5] *$P_{\chi_v} : \pi_v'^U \times \pi_v'^U \to \mathbf{C}$ such that for any decomposable vector $\phi = \otimes_v \phi_v \in \pi'^U$,*

$$|P_\chi(\phi)|^2 = \frac{\Lambda(1/2, \pi^U \otimes \chi)}{\Lambda(1, \pi^U, \mathrm{Ad})} \cdot \prod_v P_{\chi_v}(\phi_v, \phi_v). \tag{17}$$

Here, $\Lambda(s, \pi^U, \mathrm{Ad}) = \Lambda(s, \pi, \mathrm{Ad})$ is the completed adjoint L-function of π^U, and the identity (17) *is given up to special values of abelian L-functions which cancel out to one given suitable choices of Haar measures.*

[5] essentially integrals of matrix coefficients

Theorem 2.11 (Beuzart-Plessis-Chaudouard). *Assume $n = 2r \geq 2$ is even, and note that the unitary representation π'^U we consider is everywhere locally tempered as a consequence of the works* [36], [1], [2], *and* [14]. *Conjecture 2.10 holds if the weak basechange π' of π'^U is a hermitian Arthur parameter. To be more precise, define for $s \in \mathbf{C}$ (first with $\Re(s) \gg 1$) the abelian L-factor*

$$\begin{aligned}\mathcal{L}(s, \pi \otimes \chi) = \mathcal{L}(s, \pi^U \otimes \chi) &= \prod_{i=1}^{n+1} L(s - 1/2 + i, \eta^i) \cdot \frac{L(s, \pi \otimes \chi)}{L(s + 1/2, \pi \otimes \chi, \mathrm{As})} \\ &= \prod_{i=1}^{n+1} L(s - 1/2 + i, \eta^i) \cdot \frac{L(s, \pi' \times \pi(\chi))}{L(s + 1/2, \pi' \times \pi(\chi), \mathrm{As})}.\end{aligned}$$

Here, $\eta = \eta_{K/F}$ denotes the idele class character of F associated to the totally imaginary quadratic extension K/F, and $L(s, \pi \otimes \chi, \mathrm{As})$ the L-function associated to $\mathrm{As}^{+1} \boxtimes \mathrm{As}^{-1}$. Let us also write $\mathcal{L}_v(s, \pi \otimes \chi)$ to denote the corresponding Rankin-Selberg local factor at each place v of F, so that

$$\mathcal{L}(s, \pi \otimes \chi) = \prod_v \mathcal{L}_v(s, \pi \otimes \chi).$$

Define for each place v of F the local Bessel period $\mathcal{P}_{\mathcal{B},v} = \mathcal{P}_{N,v}$ on $f_v \in C_c^\infty(U(V \oplus L)(F_v))$ by

$$\mathcal{P}_{\mathcal{B},v}(f_v) = \mathcal{P}_{N,v}(f_v) = \int_{\{1\} \times N(F_v)} f_v(g_v) \psi_{\mathcal{B},v}(g_v) dg_v = \int_{N(F_v)} f_v(g_v) \psi_{N,v}(g_v) dg_v.$$

Here, we write $\psi_{\mathcal{B}} = \otimes_v \psi_{\mathcal{B},v} \equiv \psi_N = \otimes \psi_{N,v}$ to denote the extension of the automorphic additive character ψ_N to the Bessel subgroup $\mathcal{B}(\mathbf{A}_F) = \{1\} \times N(\mathbf{A}_F) \cong N(\mathbf{A}_F)$ of $U(V \oplus L)(\mathbf{A}_F)$. Viewing each $\sigma_v = \pi_v^{U\prime} \boxtimes \chi_v$ as a tempered irreducible representation of $U(V \oplus L)(F_v)$ equipped with an invariant inner product $(\cdot, \cdot)_v$, we can then take f_v to be the matrix coefficient defined by $f_{\varphi_v, \varphi_v'} = (\sigma_v(g)\varphi_v, \varphi_v')_v$ for all $g \in U(V \oplus L)(F_v)$, with $\varphi_v, \varphi_v' \in \sigma_v$. We then obtain from this a local pairing

$$\mathcal{P}_{\mathcal{B},v}(\varphi_v, \varphi_v') = \mathcal{P}_{\mathcal{B},v}(f_{\varphi_v, \varphi_v'}),$$

as well as a normalized local pairing

$$\mathcal{P}^{\#}_{\mathcal{B},v}(\varphi_v, \varphi_v') = \mathcal{L}_v(1/2, \pi \otimes \chi)^{-1} \mathcal{P}_{\mathcal{B},v}(f_{\varphi_v, \varphi_v'}).$$

Consider the global Bessel period $\mathcal{P}_{\mathcal{B}}$ defined on $\varphi \in \sigma = \pi^{U\prime} \boxtimes \chi$ by

$$\mathcal{P}_{\mathcal{B}}(\varphi) = \mathcal{P}_{\psi_N}(\varphi) = \int_{\mathcal{B}(F) \backslash \mathcal{B}(\mathbf{A}_F)} \varphi(g) \psi_{\mathcal{B}}(g) dg = c \int_{N(F) \backslash N(\mathbf{A}_F)} \varphi(g) \psi_N(g) dg,$$

and let $(\varphi, \varphi)_{\mathrm{Pet}}$ denote the corresponding Petersson inner product. Let us normalize all local Haar measures so that the products determine Tamagawa measures. Given any decomposable vector $\varphi = \otimes_v \varphi_v \in \sigma$, we have

$$|\mathcal{P}_{\mathcal{B}}(\varphi)|^2 = \mathcal{L}(1/2, \pi \otimes \chi) \cdot \prod_v \frac{\mathcal{P}^{\#}_{\mathcal{B},v}(\varphi_v, \varphi_v)}{(\varphi_v, \varphi_v)_v} \cdot (\varphi, \varphi)_{\mathrm{Pet}}. \tag{18}$$

Proof. See [11, Theorem 1.3.6.1], taking $m = 0$ and $r = n/2$ in their Bessel periods setup [11, §1.3]. As explained in [11, Remarks 1.3.3.2], the group $U(L \oplus V) = U(h_{n+1})$ can be identified with the quasi-split unitary group U_{2r+1} of rank $2r + 1$, and the Bessel period $\mathcal{P}_{\mathcal{B}} = \mathcal{P}_{\psi_N}$ with a Fourier-Whittaker coefficient. The implicit nonvanishing equivalence was known already thanks to Ginzburg-Rallis-Soudry [25], and the refined statement shown here was conjectured earlier by Lapid-Mao [46]. ☐

2.4.1. *Normalizations of vectors.* Taking for granted Conjecture 2.10, we make the following choice of vector.

Proposition 2.12. *Assume Conjecture 2.10, or the conditions of Theorem 2.11. We can choose the decomposable vector $\phi \in \pi'^U$ or $\varphi = \otimes_v \varphi_v \in \sigma$ to be normalized so that $|P_\chi(\phi)|^2$ or $|\mathcal{P}_{\mathcal{B}}(\varphi)|^2$ takes algebraic* *integer values. We thus fix such a choice ϕ^{int} or φ^{int}. Moreover, fixing an idele representative t_A for each class $A \in X_m$ (for each integer $m \geq 0$), we can choose this vector so that each idele class sum*

$$\sum_{\substack{\gamma \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \mathcal{P}_\psi \phi^{\mathrm{int}}(\gamma) = \sum_{\substack{\gamma \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \mathcal{P}_{\psi_N} \phi^{\mathrm{int}}(\gamma) \tag{19}$$

takes algebraic integer values.

Proof. Let us first remark that [61, Lemma 4.1] would carry over if the hermitian space V were compact at infinity, this is if the corresponding unitary group $U_n(F_\infty) = U(V)(F_\infty)$ were compact.[6] Let us also remark that since the representations π^U and π'^U are all cohomological, we deduce from the corresponding rational structure of the $(\mathfrak{g}, K)$-cohomology spaces that we can normalize decomposable vectors to take algebraic values. Moreover, we can realize vectors in the representation spaces π^U and π'^U as classes in the coherent cohomology of the associated unitary Shimura variety $\mathrm{Sh}(U(V))$. We know that each of these Shimura varieties $\mathrm{Sh}(U(V))$ has an integral model by the work of [45] and [44]. Hence, using the induced integral structure on the sections of the relative dualizing sheaves of these integral models, we deduce that we can even normalize these decomposable vectors to take algebraic integer values.

Assuming the respective Ichino-Ikeda formulae (17) or (18), we use the sesquilinear pairings that appear on the right-hand sides to deduce more directly that we can choose decomposable vectors $\phi^{\mathrm{int}} \in V_\pi$ or $\phi^{\mathrm{int}} \in V_{\pi'^U}$ so that the left-hand side is an algebraic integer. It is then easy to see that the corresponding Bessel period $P_\chi(\phi^{\mathrm{int}})$ or $\mathcal{P}_{\mathcal{B}}(\varphi^{\mathrm{int}}) = \mathcal{P}_{\psi_N}(\varphi^{\mathrm{int}}) = \mathcal{P}_\psi(\varphi^{\mathrm{int}})$ will take algebraic integer values. We then deduce that each idele class sum(mand) of the form (19) must also take algebraic integer values, since otherwise taking finite sums over classes $A \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m})$ (with $\chi \in \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m})^\vee$) would give a contradiction. □

2.5. **Ratios of periods and congruences.** We now gather make extra notes on congruences and comparisons of periods inherent in choosing automorphic pure tensors in the discussion above. Taking Conjecture 2.10 for granted, we can make the following comparison with the Eulerian integral presentations considered above, as well as the rationality theorems as described in Theorem 2.7 above.

Corollary 2.13. *Assuming Conjecture 2.10, and the conditions of Theorem 2.7 on the cohomological cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_K)$ met, Theorem 2.6 gives us the following identification(s): For decomposable vectors $\varphi = \otimes_v \varphi_v \in \pi$ and $\phi = \otimes_v \phi_v \in \pi'^U$ and χ any ring class character of K, we have*

$$\Lambda(1/2, \pi \otimes \chi) = \int_{\mathbf{A}_K^\times / K^\times} \mathbb{P}_1^n \varphi \left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} \right) \chi(t) dt = \frac{|P_\chi(\phi)|^2 \cdot L(1, \pi^U, \mathrm{Ad})}{\prod_v P_{\chi_v}(\phi_v, \phi_v)} = \Lambda(1/2, \pi^U \otimes \chi).$$

In particular, we have the identification of toric period integrals

$$\int_{\mathbf{A}_K^\times / K^\times} \mathbb{P}_1^n \varphi \left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} \right) \chi(t) dt = \left| \int_{U_1(F) \backslash U_1(\mathbf{A}_F)} \mathcal{P}_\psi \phi(t) \chi(t) dt \right|^2 \cdot \frac{L(1, \pi^U, \mathrm{Ad})}{\prod_v P_{\chi_v}(\phi_v, \phi_v)}.$$

Moreover, writing $\Omega(\pi, \chi) = \Omega_{\frac{1}{2}}(\pi, \chi)$ to denote the period(s) described in Theorem 2.7, we have that

$$\begin{aligned} \frac{\Lambda(1/2, \pi \otimes \chi)}{\Omega(\pi, \chi)} &= \frac{L(1/2, \pi_\infty)}{\Omega(\pi, \chi)} \cdot \int_{\mathbf{A}_{K,f}^\times / K^\times} \mathbb{P}_1^n \varphi \left(\begin{pmatrix} t_f & \\ & 1 \end{pmatrix} \right) \chi(t_f) dt_f \\ &= \frac{1}{\Omega(\pi, \chi)} \cdot \left| \int_{U_1(F) \backslash U_1(\mathbf{A}_F)} \mathcal{P}_\psi \phi(t) \chi(t) dt \right|^2 \cdot \frac{L(1, \pi^U, \mathrm{Ad})}{\prod_v P_{\chi_v}(\phi_v, \phi_v)} \end{aligned} \tag{20}$$

is an algebraic number, lying in some proscribed Hecke field according to Theorem 2.7, and admitting an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. On the other hand, if we normalize the decomposable vector $\phi = \otimes_v \phi_v \in \pi'^U$ as in Proposition 2.12 above, the corresponding toric period

$$\left| \int_{U_1(F) \backslash U_1(\mathbf{A}_F)} \mathcal{P}_\psi \phi(t) \chi(t) dt \right|^2 \tag{21}$$

[6] In this simpler case, the vector ϕ can be viewed as a function on a finite set, and hence normalized trivially to take values in the algebraic integers.

is seen easily to be an algebraic number (or even an algebraic integer). Finally, using this normalized choice of vector $\phi \in \pi^U$ in (20) *leads to two comparable rational structures for the central values. To be more precise, normalizing $\varphi = \otimes_v \varphi_v \in \pi$ according to Proposition 2.8 and $\phi = \otimes_v \phi_v \in \pi^U$ to take algebraic integer values, the corresponding ratio of periods*

$$\frac{1}{\Omega(\pi,\chi)} \cdot \frac{L(1,\pi^U,\mathrm{Ad})}{\prod_v P_{\chi_v}(\phi_v,\phi_v)} = \frac{1}{\Omega(\pi,\chi)} \cdot \frac{L(1,\pi,\mathrm{Ad})}{\prod_v P_{\chi_v}(\phi_v,\phi_v)} \tag{22}$$

is an algebraic integer.

We expect the algebraic integers (22) to convey subtle information about level-raising congruences for π. To be more precise, we expect the following relations to hold, in the style of the work of Prasanna [53], [52] on ratios of Petersson inner products, as well as the theorem of Ribet-Takehasi [56]. Let $\mathcal{O}$ be a finite extension of $\mathbf{Z}_p$ containing all of the Hecke eigenvalues of the cuspidal cohomological representation π of $\mathrm{GL}_n(\mathbf{A}_K)$, as well as those of its corresponding unitary transfer π^U. Let ρ_π denote the Galois representation associated to π by the construction of [3], via the boundary cohomology of the Borel-Serre compactification of a Shimura variety $\mathrm{Sh}(U(n,n))$ attached to the unitary group $U(n,n)$. Let ρ_{π^U} denote the Galois representation associated to π^U via the etale cohomology of a Shimura variety $\mathrm{Sh}(U(V))$ attached to the unitary group $U(V)$ (see e.g. [36]). We write $\overline{\rho}_\pi = \rho_\pi \bmod \mathfrak{P}$ and $\overline{\rho}_{\pi^U} \bmod \mathfrak{P}$ to denote the corresponding residual Galois representations modulo the prime ideal $\mathfrak{P} \subset \mathcal{O}$. Let $\mathfrak{m}_\pi \subset \mathbb{T} \otimes \mathcal{O}$ denote the maximal ideal of the Hecke algebra $\mathbb{T} \otimes \mathcal{O}$ corresponding to ρ_π, with $\mathbb{T}_{\mathfrak{m}_\pi}$ the corresponding localization at $\mathfrak{m}_\pi$, and $\xi_\pi : \mathbb{T}_{\mathfrak{m}_\pi} \longrightarrow \mathcal{O}$ the corresponding eigencharacter. We then consider the corresponding congruence ideal for π defined by

$$\mathcal{C}(\pi) := \xi_\pi\left(\mathrm{Ann}\left(\ker\left(\xi_\pi : \mathbb{T}_{\mathfrak{m}_\pi} \longrightarrow \mathcal{O}\right)\right)\right) \subset \mathcal{O}$$

Similarly, we let $\mathfrak{m}_{\pi^U} \subset \mathbb{T} \otimes \mathcal{O}$ denote the maximal ideal of the Hecke algebra $\mathbb{T} \otimes \mathcal{O}$ corresponding to ρ_{π^U}, with $\mathbb{T}_{\mathfrak{m}_{\pi^U}}$ the corresponding localization at $\mathfrak{m}_{\pi^U}$, and $\xi_{\pi^U} : \mathbb{T}_{\mathfrak{m}_{\pi^U}} \longrightarrow \mathcal{O}$ the corresponding eigencharacter. We then consider the corresponding congruence ideal for π^U defined by

$$\mathcal{C}(\pi^U) := \xi_{\pi^U}\left(\mathrm{Ann}\left(\ker\left(\xi_{\pi^U} : \mathbb{T}_{\mathfrak{m}_{\pi^U}} \longrightarrow \mathcal{O}\right)\right)\right) \subset \mathcal{O}.$$

Again using the existence of integral models for these Shimura varieties $\mathrm{Sh}(U(n,n))$ and $\mathrm{Sh}(U(V))$ due to [45] and [44], we argue as the in the proof of Proposition 2.12 that we can choose arithmetically normalized decomposable vectors $\varphi^{\mathrm{int}} \in V_\pi$ and $\phi^{\mathrm{int}} \in V_{\pi^U}$ which take algebraic integer values – and whose corresponding respective Whittaker and Bessel periods take algebraic integer values.

Conjecture 2.14. *Let us retain the setup of Corollary 2.13, and all of the discussion above. Let $u_\pm(\pi)$ denote the canonical periods associated to the Galois representation ρ_π constructed from the l-adic étale cohomology of the boundary of the Borel-Serre compactification of the unitary Shimura variety* $\mathrm{Sh}(U(n,n))$ *by* [3] *(see also* [59] *and* [13]*) in the theory of Deligne* [21, (0.4.1), Proposition 1.4, (1.7.1)], *and $u_\pm(\pi^U)$ the periods associated to the Galois representation ρ_{π^U} constructed from the l-adic étale cohomology of the Shimura variety* $\mathrm{Sh}(U)$ *attached to the unitary group $U = U(V)$ from our discussion above. That is, we let $u_\pm(\pi)$ denote the periods defined with respect to the isomorphism*

$$I : H_B^*(\mathrm{Sh}(U(n,n))(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{C} \longrightarrow H_{\mathrm{dR}}^*(\mathrm{Sh}(U(n,n))(\mathbf{C})) \otimes_{\mathbf{R}} \mathbf{C},$$

and $u_\pm(\pi^U)$ the periods defined with respect to the isomorphism

$$I : H_B^*(\mathrm{Sh}(U(V))(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{C} \longrightarrow H_{\mathrm{dR}}^*(\mathrm{Sh}(U(V))(\mathbf{C})) \otimes_{\mathbf{R}} \mathbf{C}.$$

We expect to have the following relations:

(i) *Writing $(\cdot,\cdot)_{\mathrm{Pet}}$ again to the denote the Petersson inner product, the ratios*

$$\delta_\pi = \frac{(\varphi^{\mathrm{int}}, \varphi^{\mathrm{int}})_{\mathrm{Pet}}}{u_+(\pi) \cdot u_-(\pi)} \quad \text{and} \quad \delta_{\pi^U} = \frac{(\phi^{\mathrm{int}}, \phi^{\mathrm{int}})_{\mathrm{Pet}}}{u_+(\pi^U) \cdot u_-(\pi^U)}$$

are algebraic numbers.

(ii) *We have the equalities $\mathcal{C}(\pi) = (\delta_\pi)$ and $\mathcal{C}(\pi^U) = (\delta_{\pi^U})$ of ideals in $\mathcal{O}$.*

(iii) *The ratio* $(\varphi^{\mathrm{int}}, \varphi^{\mathrm{int}})_{\mathrm{Pet}}/(\phi^{\mathrm{int}}, \phi^{\mathrm{int}})_{\mathrm{Pet}}$ *is an algebraic number, and moreover lies in the Hecke field* $\mathbf{Q}(\pi)$ *obtained by adjoining the Hecke eigenvalues of* π *to* $\mathbf{Q}$*, or equivalently in the Hecke field* $\mathbf{Q}(\pi^U)$ *obtained by adjoining the Hecke eigenvalues of* π^U *to* $\mathbf{Q}$*. If* $\overline{\rho}_\pi$ *is irreducible, then this algebraic number counts the number of automorphic representations of* $\mathrm{GL}_n(\mathbf{A}_K)$ *whose eigenvalues are congruent to those of* $\pi \bmod \mathfrak{P}$*, but which do not arise via transfer to the unitary group* $U(V)$*.*

(iv) *If* π *and* π^U *are* $\mathfrak{P}$*-ordinary, then the algebraic integer ratio* (22) *is a* p*-adic unit, i.e. lies in* $\mathcal{O}^\times$*.*

We can deduce Conjecture 2.14 (iv) from our construction of p-adic interpolation measures below. While we do not address Conjecture 2.14 (i) and (ii) here, we expect they can be deduced from a subsequent discussion of Iwasawa main conjectures (including those for the adjoint L-function), which would be interesting to explore in a subsequent work. Conjecture 2.14 (iii) seems subtler, and is related to the still-nascent theory of level-raising congruences in this setting. In brief, we expect there to be some analogue of the theorem of Ribet-Takehashi [56] for $n = 2$, given in terms of the geometry of the ambient Shimura variety $\mathrm{Sh}(U(V))$.

3. Main construction

We now give the main construction, leading to a proof of Theorem 1.1.

3.1. Distribution relations.

Fix a prime ideal $\mathfrak{P} \subset \mathcal{O}_K$ with underlying F-rational prime $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_F$, and underlying rational prime $p = \mathfrak{P} \cap \mathbf{Z}$. We fix a uniformizer $\varpi_{\mathfrak{P}}$, and write $k_{\mathfrak{P}}$ to denote the cardinality of the residue field at $\mathfrak{P}$. Recall that for each integer $m \geq 0$, we write

$$X_m = \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m}) = \mathbf{A}_K^\times / K^\times K_\infty^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$$

to denote the class group of the $\mathcal{O}_F$-order $\mathcal{O}_F + \mathfrak{p}^m \mathcal{O}_K$ of conductor $\mathfrak{p}^m$ in $\mathcal{O}_K$. Consider the profinite limit

$$X = \varprojlim_m X_m.$$

As is well-known (e.g. [20, §2.1]), X has a finite torsion subgroup X_{tors}, and the quotient X/X_{tors} is isomorphic as a topological group to $\mathbf{Z}_p^{\delta_{\mathfrak{p}}}$, where $\delta_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbf{Q}_p]$ denotes the residue degree of the prime $\mathfrak{p} \subset \mathcal{O}_F$. We shall now consider only ring class characters χ of each of these finite abelian groups X_m as $m \geq 0$ varies.

3.1.1. *Iwasawa algebras.*

Let us fix $\pi = \otimes_v \pi_v$ an irreducible cuspidal cohomological representation of $\mathrm{GL}_n(\mathbf{A}_K)$ as in Theorem 2.7 above, together with a critical value $s_0 \in \frac{n-1}{2} + \mathbf{Z}$. Let us consider the infinite tower of number fields defined by the compositum

$$E = \bigcup_{m \geq 0} \bigcup_{\chi \in X_m^\vee} E_{\pi,\chi} = \bigcup_{m \geq 0} \bigcup_{\chi \in X_m^\vee} \mathbf{Q}(\pi)\mathbf{Q}(\chi)K^{\mathrm{Gal}} \subset \overline{\mathbf{Q}},$$

where each $E_{\pi,\chi} = \mathbf{Q}(\pi)\mathbf{Q}(\chi)K^{\mathrm{Gal}} \subset \overline{\mathbf{Q}}$ is the Galois extension of $\mathbf{Q}$ defined in Theorem 2.7. Note that E will be some cyclotomic extension of the number field defined by $E_\pi = \mathbf{Q}(\pi)K^{\mathrm{Gal}}$. Fixing an embedding $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ once and for all, we consider the union $L = \cup_{w \mid p} E_w$ over all primes $w \mid p$ above p in E of the corresponding localizations E_w of E at w, and write $\mathcal{O} = \mathcal{O}_L$ to denote the corresponding ring of integers. Hence, $\mathcal{O}$ is a discrete valuation ring, and we can consider the corresponding $\mathcal{O}$-Iwasawa algebra of X:

$$\mathcal{O}[[X]] = \varprojlim_m \mathcal{O}[X_m] = \varprojlim_m \mathcal{O}[\mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m})].$$

Now, recall that the elements of $\mathcal{O}[[X]]$ can be parametrized as $\mathcal{O}$-valued measures on the profinite group $X = \varprojlim_m X_m$. That is, each element $\vartheta \in \mathcal{O}[[X]]$ can be parametrized as a collection $d\vartheta$ of maps

$$\{\vartheta_m\}_{m \geq 1}, \quad \vartheta_m : X_m \to \mathcal{O}$$

satisfying the following distribution relation: For each $A \in X_m$, we have that

$$\vartheta_m(A) = \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \vartheta_{m+1}(\gamma A) = \sum_{\gamma \in \mathcal{O}_{\mathfrak{p}^m, \mathfrak{P}^\times} / \mathcal{O}_{\mathfrak{p}^{m+1}, \mathfrak{P}}^\times} \vartheta_{m+1}(\gamma A). \tag{23}$$

Note that this can be viewed more simply as the compatibility relation required for the collection of group ring elements of $\mathcal{O}[X_m]$ to form an element of the inverse limit. As well, we write

$$Z_m = \mathcal{O}_{\mathfrak{p}^m, \mathfrak{p}} = \mathcal{O}_{\mathfrak{p}^m, \mathfrak{P}} = \mathcal{O}_{F_\mathfrak{p}} + \mathfrak{p}^m \mathcal{O}_{K_\mathfrak{P}} \subset \mathcal{O}_{K_\mathfrak{P}}$$

for any integer $m \geq 0$ here (and henceforth) to denote the localization at $\mathfrak{P}$ of the order $\mathcal{O}_{\mathfrak{p}^m} = \mathcal{O}_F + \mathfrak{p}^m \mathcal{O}_K$. Note as well that we have a natural notion of integration or *specialization* via such a collection of maps $d\vartheta = \{\vartheta_m\}_m$. That is, for any character χ of the profinite limit X factoring through some X_m, we consider

$$\chi(\vartheta) := \int_X \chi(\sigma) d\vartheta(\sigma) = \sum_{\sigma \in X_m} \chi(\sigma)\vartheta_m(\sigma).$$

3.1.2. *Strategy.* The idea is to interpret the trace operator inherent in the distribution relation (23) as some local "trace" Hecke operator at $\mathfrak{P}$. Taking for granted either (1) the setup with Eulerian integral presentations (Theorem 2.6, Theorem 2.7, and Corollary 2.9) or (2) the Ichino-Ikeda formula of Conjecture 2.10 and Proposition 2.12, we fix decomposable vectors $\varphi = \varphi^{\mathrm{int}} \in V_\pi$ or $\phi = \phi^{\mathrm{int}} \in V_{\pi^U}$ accordingly, and use that that these are eigenvectors for the trace Hecke operators to construct natural bounded distributions on X in the style of [61]. Thus, we shall assume henceforth that π is $\mathfrak{P}$-ordinary, so that the image under $\iota_p : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ of the eigenvalue of $\varphi \in V_\pi$ under this local trace Hecke operator at $\mathfrak{P}$ is a p-adic unit. In this way, we shall construct p-interpolation series for the algebraic integer values (12) under our fixed embedding $\iota_p : \mathcal{O}_E \rightarrow \mathcal{O}_L$ for (1), or for the toric periods $P_\chi(\phi)$ appearing in Conjecture 2.10 for (2). Such a construction gives at least a partial generalization of the construction of [61], as well as that from the Euler system construction of Bertolini-Darmon (e.g. [10], [9]), and others for the special case of $\mathrm{GL}_2(\mathbf{A}_K)$. Note that each of these works relies on the Jacquet-Langlands correspondence for $\mathrm{GL}_2(\mathbf{A}_F)$, together with relations to central values via theta correspondence as established by theorems of Waldspurger [64] and others (e.g. [23]). Here, we rely instead on the theories of essential Whittaker vectors and Eulerian integral presentations as described above, and are constrained by what is known in this setting towards Deligne's rationality conjecture (e.g. Theorem 2.7) for the first construction (1), or else what is known towards the Ichino-Icheda variant of the Gan-Gross-Prasad conjecture (Conjecture 2.10) for (2).

3.2. **Local Hecke operators.** Fix an irreducible cuspidal cohomological representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_n(\mathbf{A}_K)$ as above for (1), or $\pi^U = \otimes_v \pi_v^U$ a cuspidal automorphic representation on $U_n(\mathbf{A}_K)$ for (2). In the latter case, we shall assume that π is a basechange of π^U to $\mathrm{GL}_n(\mathbf{A}_K)$, and moreover that the prime $\mathfrak{p} \subset \mathcal{O}_F$ splits in K so that $U_n(K_\mathfrak{P})$ splits, in which case we can and do fix an isomorphism $U_n(K_\mathfrak{P}) \cong \mathrm{GL}_n(K_\mathfrak{P})$. Consequently, we have an isomorphism $\pi_\mathfrak{P} \cong \pi_\mathfrak{P}^U$. Let $q_\mathfrak{P}^\delta$ denote the conductor of the local representation $\pi_\mathfrak{P} \cong \pi_\mathfrak{P}^U$ of $\mathrm{GL}_n(K_\mathfrak{P})$. Thus when $\delta = 0$, the local representation $\pi_\mathfrak{P} \cong \pi_\mathfrak{P}^U$ is right $\mathrm{GL}_n(\mathcal{O}_{K_\mathfrak{P}})$-invariant, and when $\delta = 1$ it is right invariant by some parahoric subgroup $I_{\mathfrak{P},1} \subset \mathrm{GL}_n(\mathcal{O}_{K_\mathfrak{P}})$.

In this preliminary section, we describe the action of some suitable local Hecke operator acting on vectors $\pi_\mathfrak{P} \cong \pi_\mathfrak{P}^U$, and in particular an operator that takes the form of the trace operator appearing in the distribution relation (23). Characterizing such an operator $T_\mathfrak{P}$, we then impose the natural condition that $\pi_\mathfrak{P} \cong \pi_\mathfrak{P}^U$ is $\mathfrak{P}$-ordinary with respect to this "trace" Hecke operator $T_\mathfrak{P}$, with p-adic unit eigenvalue $\alpha_\mathfrak{P}$ with respect to our fixed embedding $\iota_p : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$. We can then construct in either case on $\delta = 0, 1$ some sequences of local vectors $(T_\mathfrak{P}(x_m \cdot \varphi_\mathfrak{P}(\cdot)))_{m \geq 1}$ and $(T_\mathfrak{P}(x_m \cdot \phi_\mathfrak{P}(\cdot)))_{m \geq 1}$ which satisfy the distribution relation (23), in the style of the construction given in [61] (cf. [20, §6]). Our subsequent construction of distributions is then built naturally from such a sequence, i.e. viewed in terms of the component at $\mathfrak{P}$ our chosen decomposable vectors $\varphi = \varphi^{\mathrm{int}} \in V_\pi$ for (1) or $\phi = \phi^{\mathrm{int}} \in V_{\pi'^U}$ for (2).

3.2.1. *Local orders.* Recall that for a given integer $m \geq 0$, we write $Z_m = \mathcal{O}_{F_\mathfrak{p}} + \mathfrak{p}^m \mathcal{O}_{K_\mathfrak{P}}$ to denote the local order of conductor $\mathfrak{p}^m$ in $\mathcal{O}_{K_\mathfrak{P}}$. It is not hard to show that given any $\mathcal{O}_{F_\mathfrak{p}}$-order Z of $\mathcal{O}_{K_\mathfrak{P}}$, there exists a unique integer $m \geq 0$ for which $Z = Z_m$ (see e.g. [20, §6.1, p. 62]).

3.2.2. *Spaces of lattices and elementary divisors.* Fix a $K_\mathfrak{P}$-vector space V of dimension n over $K_\mathfrak{P}$, so $V \cong K_\mathfrak{P}^n$. Let $\mathcal{L} = \mathcal{L}(V)$ denote the space of $\mathcal{O}_{K_\mathfrak{P}}$-lattices V of rank n. Hence, fixing a base lattice $L_0 \in \mathcal{L}$ stabilized by $\mathrm{GL}_n(\mathcal{O}_{K_\mathfrak{P}})$, we have the natural bijection

$$\mathrm{GL}_n(K_\mathfrak{P}) / \mathrm{GL}_n(\mathcal{O}_{K_\mathfrak{P}}) \longrightarrow \mathcal{L}, \quad g \longmapsto gL_0. \tag{24}$$

We consider the action of $K_{\mathfrak{P}}^\times$ via the upper diagonal embedding

$$K_{\mathfrak{P}}^\times \longrightarrow \mathrm{GL}_n(K_{\mathfrak{P}}), \quad \gamma \longmapsto \begin{pmatrix} \gamma & \\ & \mathbf{1}_{n-1} \end{pmatrix}. \tag{25}$$

Given a lattice $L \in \mathcal{L}$, let us write $[L] = \{\alpha L : \alpha \in K_{\mathfrak{P}}^\times\}$ to denote the corresponding class under this action. We shall denote the action of $\gamma \in K_{\mathfrak{P}}^\times$ on a given lattice $L \in \mathrm{GL}_n(K_{\mathfrak{P}})/\mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}}) \cong \mathcal{L}$ by the symbol $\gamma \star L$. Let us also record the following standard result. Recall that we fix a uniformizer $\varpi_{\mathfrak{P}} \in \mathcal{O}_{K_{\mathfrak{P}}}$.

Lemma 3.1. *Write $\Gamma = \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}})$. Given $\xi \in \mathrm{GL}_n(K_{\mathfrak{P}})$, let us also write out the corresponding double coset decomposition into right cosets as $\Gamma\xi\Gamma = \coprod_\nu \Gamma\xi_\nu$. Again, we fix V a vector space over $K_{\mathfrak{P}}$ of dimension n, and write $\mathcal{L} = \mathcal{L}(V)$ to denote the space of rank n lattices in V.*

(i) *For each $\xi \in \mathrm{GL}_n(K_{\mathfrak{P}})$, the double coset $\Gamma\xi\Gamma$ has a unique representative*

$$\Gamma\xi\Gamma = \Gamma \begin{pmatrix} \varpi_{\mathfrak{P}}^{a_1} & & \\ & \ddots & \\ & & \varpi_{\mathfrak{P}}^{a_n} \end{pmatrix} \Gamma$$

for rational integers $a_j \in \mathbf{Z}$ with $a_j \leq a_{j+1}$ for each $1 \leq j \leq n-1$.

(ii) *We have that $\mathrm{Stab}_{\mathrm{GL}_n(K_{\mathfrak{P}})}(L) = \Gamma$ for some distinguished base lattice $L := L_0 \in \mathcal{L}$, and consequently for any pair $\xi_1, \xi_2 \in \mathrm{GL}_n(K_{\mathfrak{P}})$ that $\xi_1\Gamma = \xi_2\Gamma$ if and only if $\xi_1 L = \xi_2 L$.*

(iii) *Given $M \in \mathcal{L}$, let us (for $L = L_0$) write $(L : M) = (\varpi_{\mathfrak{P}}^{a_j})_{j=1}^n$ to denote the set of elementary divisors. Thus L admits an $\mathcal{O}_{K_{\mathfrak{P}}}$-basis $(e_j)_{j=1}^n$ such that M admits an $\mathcal{O}_{K_{\mathfrak{P}}}$-basis $(\varpi_{\mathfrak{P}}^{a_j} e_j)_{j=1}^n$. Given lattices $M, N, L \in \mathcal{L}$, we have that $(L : M) = (L : N)$ if and only if there exists $\xi \in \Gamma$ for which $M = \xi N$.*

(iv) *Given $\xi \in \mathrm{GL}_n(K_{\mathfrak{P}})$ with double coset $\Gamma\xi\Gamma = \coprod_\nu \Gamma\xi_\nu$ represented as in (1) above, there is a one-to-one correspondence between the right cosets $\Gamma\xi_\nu$ in $\Gamma\xi\Gamma$ and pairs of lattices $L, M \in \mathcal{L}$ with $(L : M) = (\varpi_{\mathfrak{P}}^{a_j})_{j=1}^n$.*

Proof. The result is standard, see e.g. [6, §2.1]. □

Given $\xi \in \mathrm{GL}_n(K_{\mathfrak{P}})$ with $\Gamma\xi\Gamma = \coprod_\nu \Gamma\xi_\nu$ having the representation described in Lemma 3.1 (1), we can now consider the corresponding Hecke operator defined on a lattice $L \in \mathcal{L}$ by

$$T_{\mathfrak{P}}(\xi)(L) = T_{\mathfrak{P}}(\varpi_{\mathfrak{P}}^{a_1}, \ldots, \varpi_{\mathfrak{P}}^{a_n})(L) := \sum_{\xi_\nu} \xi_\nu L = \sum_{(L:M) = (\varpi_{\mathfrak{P}}^{a_1}, \ldots, \varpi_{\mathfrak{P}}^{a_n})} M. \tag{26}$$

These operators appear in the classical literature (see e.g. [6], [58]), and can be viewed as operators on the Bruhat-Tits building of $\mathrm{SL}_n(K_{\mathfrak{P}})$, i.e. on a certain $(n-1)$-dimension complex whose vertices correspond to lattices in $\mathcal{L}$. We shall consider Hecke operators of this form associated to the diagonal matrices $(\varpi_{\mathfrak{P}}^{-1}, 1, \ldots, 1)$, and their relations to the induced action $\star$ of $K_{\mathfrak{P}}^\times$ via the embedding (25). To be more precise, recall that we write $\delta \in \{0, 1\}$ to denote the exponent of the conductor $q_{\mathfrak{P}}^\delta$ of the local representation $\pi_{\mathfrak{P}}$. Let us for simplicity write $\Gamma = \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}})$ when $\delta = 0$, and $\Gamma = I_{\mathfrak{P}}$ for $I_{\mathfrak{P}} \subset \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}})$ any suitable parahoric subgroup when $\delta = 1$. We shall consider the Hecke operators corresponding to the double coset operators

$$T_{\mathfrak{P}} := \Gamma \begin{pmatrix} \varpi_{\mathfrak{P}}^{-1} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \Gamma \tag{27}$$

in each case for our construction.

3.2.3. *Hecke operators describing the trace and distributions.* We seek to characterize those local Hecke operators on $x = L \in \mathrm{GL}_n(K_{\mathfrak{P}})/\mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}}) \cong \mathcal{L}$ which describe or emulate the trace operators

$$\mathrm{Tr}_{m+1,m}(L) := \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \gamma \star L$$

appearing implicitly in the distribution relations (23) described above. That is, we consider the $\mathcal{O}$-Iwasawa algebra $\mathcal{O}[[X]]$ of the profinite abelian group $X = \varprojlim_m X_m$. Let us write the natural transition maps as

$$\pi_{m+1,m} : X_{m+1} \longrightarrow X_m.$$

Let us now fix an integer $m \geq 0$. Writing $\kappa_{\mathfrak{p}} = \mathcal{O}_{F_{\mathfrak{p}}}/\varpi_{\mathfrak{p}}$ as above to denote the residue field at $\mathfrak{p}$, we claim that for any $A = [t_A] \in X_m$ with $t_A = t_{A,m} \in \mathbf{A}_K^\times$ any fixed idele representative, we have the relation

$$\pi_{m+1,m}^{-1}(A) = \left\{ A(1 + \varpi_{\mathfrak{p}}^m v) : v \in \kappa_{\mathfrak{p}} \right\} = \left\{ t_A(1 + \varpi_{\mathfrak{p}}^m v) : v \in \kappa_{\mathfrak{p}} \right\}. \tag{28}$$

Let also also define

$$\Gamma = \begin{cases} \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}}) & \text{if } \delta = 0 \\ I_{\mathfrak{P}} & \text{if } \delta = 1. \end{cases}$$

For each integer $m \geq 0$, we define the matrix representative

$$x_m = \begin{pmatrix} \varpi_{\mathfrak{P}}^{-m} & \\ & \mathbf{1}_{n-1} \end{pmatrix}. \tag{29}$$

Here, we embed $x_m \in \mathrm{GL}_n(K_{\mathfrak{P}})$ into $\mathrm{GL}_n(\mathbf{A}_K)$ in the natural way, sending it to the idele having trivial component at each place $v \neq \mathfrak{P}$ of K, and component x_m at $v = \mathfrak{P}$. Recall that given an idele $t \in \mathbf{A}_K^\times$ and a matrix $g \in \mathrm{GL}_n(\mathbf{A}_K)$, we write $t \star g$ to denote the action of t by left multiplication on g via the embedding

$$\mathrm{GL}_1(\mathbf{A}_K) \cong \mathbf{A}_K^\times \longrightarrow \mathrm{GL}_n(\mathbf{A}_K), \quad t \longmapsto \begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}.$$

Let us for each class $A \in X_m = \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^m}) = \mathbf{A}_{K,f}^\times / K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$ fix a representative $t_A \in \mathbf{A}_{K,f}^\times$, so that the sum over ideles in the class A can be represented as the sum over $t \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times = [t_A]$. Given a decomposable vector $\varphi = \otimes_v \varphi_v \in V_\pi$, we then define the basic theta element

$$\Theta_m(\varphi) := \sum_{\substack{A \in X_m \\ A = [t_A]}} \left(\sum_{t \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \varphi\left(t \star x_m\right) \right) A. \tag{30}$$

Similarly, for a decomposable vector $\phi = \otimes_v \phi_v \in V_{\pi^U}$, we make the analogous definition

$$\Theta_m(\phi) := \sum_{\substack{A \in X_m \\ A = [t_A]}} \left(\sum_{t \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \phi\left(t \star x_m\right) \right) A. \tag{31}$$

Note that although the $\star$ notation is redundant[7] in this case, as the action is induced by the natural subspace embedding $L \longrightarrow V$ described above. It is easy to construct the following distributions from these elements.

Lemma 3.2. *Let $\varphi = \otimes_v \varphi_v \in V_\pi$ and $\phi = \otimes_v \phi_v \in V_{\pi^U}$ be any pure tensors, and define the corresponding theta elements as in* (30) *and* (31) *above. Writing $T_{\mathfrak{P}}$ in each of the exponent $\delta \in \{0, 1\}$ of the local conductor $q_{\mathfrak{P}}^\delta$ of $\pi_{\mathfrak{P}} \cong \pi_{\mathfrak{P}}^U$ to denote the Hecke operator corresponding to the double coset operator defined above in* (27)*, we have for each integer $m \geq 0$ that*

$$\pi_{m+1,m}(\Theta_{m+1}(\varphi)) = \Theta_m(T_{\mathfrak{P}}\varphi)$$

and similarly

$$\pi_{m+1,m}(\Theta_{m+1}(\phi)) = \Theta_m(T_{\mathfrak{P}}\phi).$$

[7]We retain it for aesthetic reasons, namely so that we can state the general construction in a uniform way.

Proof. We give details for the first claim with the theta elements associated to a pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$, the analogous claim with the pure tensor $\phi = \otimes_v \phi_v \in V_{\pi^U}$ following by a minor variation of the same argument. Using the identification (28), we find as a formal consequence that

$$\begin{aligned}
\pi_{m+1,m}(\Theta_{m+1}(\varphi)) &= \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \sum_{\substack{A \in X_m \\ A = [t_A]}} \left(\sum_{t \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \varphi(\gamma t \star x_{m+1}) \right) A \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \left(\sum_{t \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \varphi(\gamma t \star x_{m+1}) \right) A \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left((1 + \varpi_{\mathfrak{p}}^m v) t \star x_{m+1} \right) [t_A].
\end{aligned}$$

We now want to identify the inner sum in the latter expression with the sum defining the Hecke operator $T_{\mathfrak{P}}$ in either case, i.e. so that the right hand side can be identified with the theta element $\Theta_m(T_{\mathfrak{P}}\varphi)$. Hence,

$$\begin{aligned}
&\sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left((1 + \varpi_{\mathfrak{p}}^m v) t \star x_{m+1} \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left(\begin{pmatrix} 1 + \varpi_{\mathfrak{p}}^m v & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-(m+1)} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left(\begin{pmatrix} t(1 + \varpi_{\mathfrak{p}}^m v) & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-(m+1)} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left(\begin{pmatrix} t(1 + \varpi_{\mathfrak{p}}^m v) & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-1} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-m} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left(\begin{pmatrix} t(1 + \varpi_{\mathfrak{p}}^m v) & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} \varpi_{\mathfrak{p}}^{-1} & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left(\begin{pmatrix} \varpi_{\mathfrak{p}}^{-1}(1 + \varpi_{\mathfrak{p}} v) & \\ & \mathbf{1}_{n-1} \end{pmatrix} \begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) [t_A].
\end{aligned}$$

Now, the inner sum in the latter expression can be identified with the Hecke operator $T_{\mathfrak{P}}$ corresponding to the double coset operator $\Gamma \operatorname{diag}(\varpi_{\mathfrak{P}}^{-1}, 1 \ldots, 1)\Gamma$, as the matrices $\operatorname{diag}(\varpi_{\mathfrak{p}}^{-1}(1 + \varpi_{\mathfrak{p}}^m v), 1, \ldots, 1)$ in the summand (with $v \in \kappa_{\mathfrak{p}}$ varying) run over a full set of right coset representatives. Hence, we check the desired claim

$$\begin{aligned}
\pi_{m+1,m}(\Theta_{m+1}(\varphi)) &= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \sum_{v \in \kappa_{\mathfrak{p}}} \varphi\left((1 + \varpi_{\mathfrak{p}}^m v) t \star x_{m+1} \right) [t_A] \\
&= \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} T_{\mathfrak{P}} \varphi\left(t \star x_m \right) [t_A] = \Theta_m(T_{\mathfrak{P}}\varphi).
\end{aligned}$$

The same calculation works for second element $\Theta_m(\phi)$ defined for the unitary case. ☐

Corollary 3.3. *Let us in each case write $\alpha_{\mathfrak{P}}$ to denote the eigenvalue of the Hecke operator $T_{\mathfrak{P}}$ acting on $\varphi \in V_\pi$ or $\phi \in V_{\pi^U}$, and moreover assume that the image $\iota_p(\alpha)$ under our fixed embedding $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ is a p-adic unit, then the weighted elements defined by*

$$\theta_m(\varphi) := \alpha_{\mathfrak{P}}^{-m} \cdot \Theta_m(\varphi) = \alpha_{\mathfrak{P}}^{-m} \sum_{\substack{A \in X_m \\ A = [t_A]}} \sum_{t \in [t_A]} \varphi(t \star x_m) A \tag{32}$$

and

$$\theta_m(\phi) := \alpha_{\mathfrak{P}}^{-m} \cdot \Theta_m(\phi) = \alpha_{\mathfrak{P}}^{-m} \sum_{\substack{A \in X_m \\ A=[t_A]}} \sum_{t \in [t_A]} \phi(t \star x_m) A \tag{33}$$

form distributions on the profinite group $X = \varprojlim_m X_m$. *That is, for each integer* $m \geq 0$, *we have*

$$\pi_{m+1,m}\left(\theta_m(\varphi)\right) = \pi_{m+1,m}\left(\alpha_{\mathfrak{P}}^{-(m+1)} \cdot \Theta_m(\varphi)\right) = \alpha_{\mathfrak{P}}^{m-1} \cdot \pi_{m+1,m}\left(T_{\mathfrak{P}}\varphi\right) = \alpha_{\mathfrak{P}}^{-m} \cdot \Theta_m(\varphi) = \theta_m(\varphi)$$

and

$$\pi_{m+1,m}\left(\theta_m(\phi)\right) = \pi_{m+1,m}\left(\alpha_{\mathfrak{P}}^{-(m+1)} \cdot \Theta_m(\phi)\right) = \alpha_{\mathfrak{P}}^{m-1} \cdot \pi_{m+1,m}\left(T_{\mathfrak{P}}\phi\right) \alpha_{\mathfrak{P}}^{-m} \cdot \Theta_m(\varphi) = \theta_m(\phi).$$

3.3. **Interpolation properties.** Let us for each integer $m \geq 0$ write $\tau(\mathcal{O}_{\mathfrak{p}^m})$ to denote the volume of $K^\times K_\infty^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$ in $\mathbf{A}_K^\times$ with respect to our fixed Haar measure. Let us also fix normalized pure tensors

$$\varphi^{\mathrm{int}} = \varphi = \otimes_v \varphi_v \in V_\pi$$

and

$$\phi^{\mathrm{int}} = \phi = \otimes_v \phi_v \in V_{\pi'^U}$$

as described in Corollary 2.9 and Corollary 2.13 respectively. Recall that for each integer $m \geq 0$, we define matrices $x_m \in \mathrm{GL}_n(K_{\mathfrak{P}})$ to construct distributions via Lemma 3.2 above. We also fix an idele representative $t_A \in \mathbf{A}_{K,f}^\times$ for each class in $A \in X_m$, so that $A = [t_A] \in X_m$. Let us modify the construction of elements given in Corollary 3.3 by introducing the following Whittaker-like class functions

$$\mathfrak{W}_{s_0}(A \star x_m) = \sum_{\substack{\gamma \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A \in X_m}} W_{\varphi^{\mathrm{int}}}\left(\begin{pmatrix} \gamma & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m\right) |\gamma|^{s_0 - \left(\frac{n-1}{2}\right)}$$

and

$$\mathcal{W}(A \star x_m) = \sum_{\substack{\gamma \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A \in X_m}} \mathcal{P}_\psi \phi^{\mathrm{int}}(\gamma \star x_m).$$

Theorem 3.4. *We have the following constructions of interpolation distributions and measures on the profinite* $X = \varprojlim_m X_m$ *in either case* $\delta \in \{0,1\}$ *on exponent of the conductor* $q_{\mathfrak{P}}^{\delta}$ *on the local representation* $\pi_{\mathfrak{P}} \cong \pi_{\mathfrak{P}}^U$. *Again, we fix normalized decomposable vectors* $\varphi = \varphi^{\mathrm{int}} \in V_\pi$ *and* $\phi = \phi^{\mathrm{int}} \in V_{\pi'^U}$ *as in Propositions 2.8 and 2.9 above, in particular so that we can take for granted both the algebraicity of the values taken by these vectors, as well as the integrality of the specializations.*

(1) *Assume that the conditions of Theorem 2.7, Proposition 2.8, and Corollary 2.9 are met. Then, the collection* $(\vartheta_m)_{m\geq 1} = (\vartheta_m(\varphi^{\mathrm{int}}, s_0))_{m \geq 1}$ *of maps* ϑ_m *defined on each* $A \in X_m$ *by*

$$\vartheta_m(A) = \alpha_{\mathfrak{P}}^{-m} \cdot \mathfrak{W}_{s_0}(A \star x_m)$$

determines a bounded distribution on $X = \varprojlim_m X_m$ *satisfying the following interpolation property: For any primitive ring class character* χ *of conductor* $\mathfrak{p}^m$ *factoring through* X_m *with* $m \geq 1$, *we have*

$$\chi(\vartheta) := \int_X \chi(\sigma) d\vartheta(\sigma) = \sum_{A \in X_m} \chi(A)\vartheta_m(A) = \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})}\right) \cdot \mathfrak{L}(s_0, \pi \otimes \chi) \in \mathcal{O}.$$

Here, $\mathfrak{L}(s_0, \pi \otimes \chi)$ *denotes image of the algebraic integer*

$$\begin{aligned}\mathfrak{L}(s_0, \pi \otimes \chi) = \frac{\Lambda(s_0, \pi \otimes \chi)}{\Omega_{s_0}(\pi, \chi)} &= \int_{\mathbf{A}_K^\times} W_{\varphi^{\mathrm{int}}}\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \chi(t)|t|^{s_0 - \left(\frac{n-1}{2}\right)} dt \\ &= \frac{L(s_0, \pi_\infty)}{\Omega_{s_0}(\pi, \chi)} \cdot \int_{\mathbf{A}_{K,f}^\times} W_\varphi\left(\begin{pmatrix} t_f & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \chi(t_f)|t_f|^{s_0 - \left(\frac{n-1}{2}\right)} dt_f \in \mathcal{O}\end{aligned}$$

described in Proposition 2.8 and Corollary 2.8 under our fixed embedding $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$.

(2) *Let us take for granted the setup leading to the statement of Conjecture 2.10 above, and assume that the F-rational prime $\mathfrak{p} \subset \mathcal{O}_F$ splits in K. Then, the collection $(\vartheta_m^U)_{m\geq 1} = (\vartheta_m^U(\phi^{\mathrm{int}}))_{m\geq 1}$ of maps ϑ_m^U defined on each $A \in X_m$ by*

$$\vartheta_m^U(A) = \alpha_{\mathfrak{P}}^{-m} \cdot \mathcal{W}(A \star x_m)$$

determines a bounded distribution on $X = \varprojlim_m X_m$ satisfying the following interpolation property: For any primitive ring class character χ of conductor $\mathfrak{p}^m$ factoring through X_m with $m \geq 1$, we have

$$\chi(\vartheta^U) := \int_X \chi(\sigma) d\vartheta^U(\sigma) = \sum_{A \in X_m} \chi(A)\vartheta_m^U(A) = \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot P_\chi(\phi^{\mathrm{int}}).$$

Here, $P_\chi(\phi)$ for a given decomposable vector $\phi \in V_{\pi'^U}$ denotes the image of the (normalized) automorphic period

$$P_\chi(\phi) = \int\limits_{U_1(\mathbf{A}_F)/U_1(F) \cong \mathbf{A}_K^\times / K^\times} \mathcal{P}_\psi \phi(t) \chi(t) dt = \int\limits_{\mathbf{A}_K^\times / K^\times} \left(\int_{N(F)\backslash N(\mathbf{A}_F)} \phi^{\mathrm{int}}(ut) \psi^{-1}(u) du \right) \chi(t) dt \in \mathcal{O}$$

described in Conjecture 2.10 under our fixed embedding $\iota_p : \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$.

Proof. Let us start with (1). We first show the distribution relation. That is, we want to show for each integer $m \geq 1$ and class $A \in X_m$ that

$$\sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \vartheta_{m+1}(\gamma A) = \vartheta_m(A). \tag{34}$$

Opening up the definition on the left hand side of (34), we have that

$$\sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \vartheta_{m+1}(\gamma A) = \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \alpha_{\mathfrak{P}}^{-(m+1)} \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} W_\varphi \left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_{m+1} \right) |\gamma\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}$$

$$= \sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \alpha_{\mathfrak{P}}^{-(m+1)} \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} \left(\int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi \left(u \begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_{m+1} \right) \psi^{-1}(u) du \right) |\gamma\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}.$$

Writing φ_u for a given pure tensor $\varphi \in V_\pi$ and unipotent matrix $u \in N_n(\mathbf{A}_K)$ to denote the function defined on $g \in \mathrm{GL}_n(\mathbf{A}_K)$ by $\varphi(ug)$ (for simplicity), this latter expression is equivalent to

$$\sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \alpha_{\mathfrak{P}}^{-(m+1)} \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi_u \left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_{m+1} \right) \psi^{-1}(u) du \cdot |\gamma\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}$$

$$= \alpha_{\mathfrak{P}}^{-(m+1)} \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \left(\sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} |\gamma|^{s_0 - \left(\frac{n-1}{2}\right)} \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} \varphi_u \left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_{m+1} \right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)} \right) \psi^{-1}(u) du$$

$$= \alpha_{\mathfrak{P}}^{-(m+1)} \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \left(\sum_{\gamma \in Z_m^\times / Z_{m+1}^\times} \sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} \varphi_u \left(\lambda \star x_{m+1} \right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)} \right) \psi^{-1}(u) du,$$

which after applying Lemma 3.2 to the inner sum gives us

$$\alpha_{\mathfrak{P}}^{-(m+1)} \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \left(\sum_{\substack{\lambda \in t_A K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} T_{\mathfrak{P}} \varphi_u \left(\lambda \star x_m\right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)} \right) \psi^{-1}(u) du$$

$$= \alpha_{\mathfrak{P}}^{-(m+1)} \cdot \alpha_{\mathfrak{P}} \int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \left(\sum_{\substack{\lambda \in t_A \cdot K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \varphi_u \left(\lambda \star x_m\right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)} \right) \psi^{-1}(u) du$$

$$= \alpha_{\mathfrak{P}}^{-m} \sum_{\substack{\lambda \in t \cdot K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t]=A}} \left(\int_{N_n(K)\backslash N_n(\mathbf{A}_K)} \varphi_u \left(\lambda \star x_m\right) \psi^{-1}(u) du \right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}$$

$$= \alpha_{\mathfrak{P}}^{-m} \sum_{\substack{\lambda \in t_A \cdot K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} W_\varphi \left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)} = \alpha_{\mathfrak{P}}^{-m} \cdot \mathfrak{W}_{s_0}(A \star x_m) = \vartheta_m(A).$$

Hence, we have shown the distribution relation (34) for (1). To show the corresponding interpolation formula, let $\chi \in X_m^\vee$ be any primitive ring class character of conductor $\mathfrak{p}^m$ factoring through X_m. Then, we have

$$\chi(\vartheta_m) = \chi(\vartheta_m(\varphi)) = \alpha_{\mathfrak{P}}^{-m} \sum_{A \in X_m} \mathfrak{W}_{s_0}(A \star x_m) \chi(A)$$

$$= \alpha_{\mathfrak{P}}^{-m} \sum_{A \in \mathbf{A}_{K,f}^\times / K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \sum_{\substack{\lambda \in t_A \cdot K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} W_{\varphi^{\mathrm{int}}} \left(\begin{pmatrix} \lambda & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) |\lambda|^{s_0 - \left(\frac{n-1}{2}\right)}$$

$$= \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot \int_{\mathbf{A}_K^\times} W_{\varphi^{\mathrm{int}}} \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) \chi(t) |t|^{s_0 - \left(\frac{n-1}{2}\right)} dt$$

$$= \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot \frac{L(s_0, \pi_\infty)}{\Omega_{s_0}(\pi, \chi)} \cdot \int_{\mathbf{A}_{K,f}^\times} W_\varphi \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) \chi(t) |t|^{s_0 - \left(\frac{n-1}{2}\right)} dt.$$

Let us now decompose the Whittaker functions W_φ in each of the latter integrals. Using our choice of pure tensor $\varphi^{(m)} = \otimes_v \varphi_v^{(m)} \in V_\pi$ from Theorem 2.8, we have for any class $t = (t_v)_v \in \mathbf{A}_K^\times / K_\infty^\times K^\times$ that

$$W_\varphi \left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) = W_{\mathfrak{P}}^{\mathrm{ess}} \left(\begin{pmatrix} t_{\mathfrak{P}} & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m \right) \bigotimes_{v \nmid \mathfrak{P}\infty} W_{\varphi_v} \left(\begin{pmatrix} t_v & \\ & \mathbf{1}_{n-1} \end{pmatrix} \right).$$

Hence, it remains to consider the contributions of the Whittaker function $W_{\mathfrak{P}}^{\mathrm{ess}}$ of the essential Whittaker vector $\varphi_{\mathfrak{P}} = \varphi_{\mathfrak{P}}$ we chose at $\mathfrak{P}$. Recall from Theorem 2.1 that these functions are right invariant by the action of $\mathrm{GL}_{n-1}(\mathcal{O}_{K_{\mathfrak{P}}})$ via the embedding

$$\mathrm{GL}_{n-1}(\mathcal{O}_{K_{\mathfrak{P}}}) \longrightarrow \mathrm{GL}_n(\mathcal{O}_{K_{\mathfrak{P}}}), \quad g \longmapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

On the other hand, each class x_m is given by the natural embedding into $\mathrm{GL}_n(\mathbf{A}_{K,f})$ of the matrix

$$\begin{pmatrix} \gamma_{\mathfrak{P}} & \\ & \mathbf{1}_{n-1} \end{pmatrix} = \begin{pmatrix} \varpi_{\mathfrak{P}}^{-m} & \\ & \mathbf{1}_{n-1} \end{pmatrix} \in \mathrm{GL}_n(K_{\mathfrak{P}}),$$

with $\gamma_{\mathfrak{P}} \in K_{\mathfrak{P}}^\times$. We can then find some matrix

$$\begin{pmatrix} \beta_{\mathfrak{P}} & \\ & \mathbf{1}_{n-2} \end{pmatrix} \in \mathrm{GL}_{n-1}(\mathcal{O}_{K_{\mathfrak{P}}}) \tag{35}$$

for which $\mu_{\mathfrak{P}} = \gamma_{\mathfrak{P}}\beta_{\mathfrak{P}}$ is contained in $Z_m^\times = (\mathcal{O}_{F_\mathfrak{p}} + \mathfrak{p}^m\mathcal{O}_{K_\mathfrak{P}})^\times$. Since $W_{\mathfrak{P}}^{\text{ess}}$ is right invariant by any matrix of the form (35) via the embedding

$$\mathrm{GL}_{n-1}(\mathcal{O}_{K_\mathfrak{P}}) \longrightarrow \mathrm{GL}_n(\mathcal{O}_{K_\mathfrak{P}}), \quad \begin{pmatrix} \beta_\mathfrak{P} & \\ & \mathbf{1}_{n-2} \end{pmatrix} \longmapsto \begin{pmatrix} \beta_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix}$$

we can then deduce after right multiplication by some matrix (35) that we have identifications of the form

$$\begin{aligned} W_{\mathfrak{P}}^{\text{ess}}\left(\begin{pmatrix} t_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m\right) &= W_{\mathfrak{P}}^{\text{ess}}\left(\begin{pmatrix} t_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix}\begin{pmatrix} \gamma_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \\ &= W_{\mathfrak{P}}^{\text{ess}}\left(\begin{pmatrix} t_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix}\begin{pmatrix} \mu_\mathfrak{P} & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right) \end{aligned}$$

for any class $t \in \mathbf{A}_K^\times$, with $\mu_\mathfrak{P} \in Z_m^\times \subset \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times$ (fixed). It is then easy to deduce that we have

$$\int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix} x_m\right)\chi(t)|t|^{s_0-\left(\frac{n-1}{2}\right)}dt = \int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(t)|t|^{s_0-\left(\frac{n-1}{2}\right)}dt.$$

Hence, we derive the desired interpolation formula

$$\begin{aligned} \chi(\vartheta_m) &= \alpha_\mathfrak{P}^{-m}\cdot\tau(\mathcal{O}_{\mathfrak{p}^m})^{-1}\cdot\frac{L(s_0,\pi_\infty)}{\Omega_{s_0}(\pi,\chi)}\cdot\int_{\mathbf{A}_{K,f}^\times} W_\varphi\left(\begin{pmatrix} t & \\ & \mathbf{1}_{n-1} \end{pmatrix}\right)\chi(t)|t|^{s_0-\left(\frac{n-1}{2}\right)}dt \\ &= \alpha_\mathfrak{P}^{-m}\cdot\tau(\mathcal{O}_{\mathfrak{p}^m})^{-1}\cdot\mathfrak{L}(s_0,\pi\otimes\chi). \end{aligned}$$

Let us now consider (2). To show the distribution relation, we proceed in a similar way. Thus for any integer $m \geq 0$ and class $A \in X_m$, we check that

$$\sum_{\gamma\in Z_m^\times/Z_{m+1}^\times} \vartheta_{m+1}^U(\gamma A) = \vartheta_m^U(A). \tag{36}$$

Writing ϕ_n for a given decomposable vector $\phi \in V_{\pi'^U}$ and unipotent matrix $n \in N(\mathbf{A}_F)$ to denote the function defined on $u \in U(V)(\mathbf{A}_F)$ by $\phi_n(u) = \phi(nu)$, and using Lemma 3.2 in the same way, we find that

$$\begin{aligned} \sum_{\gamma\in Z_m^\times/Z_{m+1}^\times} \vartheta_{m+1}^U(\gamma A) &= \alpha_\mathfrak{P}^{-(m+1)} \sum_{\gamma\in Z_m^\times/Z_{m+1}^\times}\ \sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \mathcal{P}_\psi\phi^{\text{int}}(\lambda\star x_{m+1}) \\ &= \alpha_\mathfrak{P}^{-(m+1)} \sum_{\gamma\in Z_m^\times/Z_{m+1}^\times}\ \sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \int_{N(F)\backslash N(\mathbf{A}_F)} \phi^{\text{int}}(n\lambda\star x_{m+1})\psi^{-1}(n)dn \\ &= \alpha_\mathfrak{P}^{-(m+1)}\cdot\int_{N(F)\backslash N(\mathbf{A}_F)}\left(\sum_{\gamma\in Z_m^\times/Z_{m+1}^\times}\ \sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \phi_n^{\text{int}}(\lambda\star x_{m+1})\right)\psi^{-1}(n)dn \\ &= \alpha_\mathfrak{P}^{-(m+1)}\int_{N(F)\backslash N(\mathbf{A}_F)}\left(\sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} T_\mathfrak{P}\phi_n^{\text{int}}(\lambda\star x_m)\right)\psi^{-1}(n)dn \\ &= \alpha_\mathfrak{P}^{-(m+1)}\cdot\alpha_\mathfrak{P} \sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \int_{N(F)\backslash N(\mathbf{A}_F)} \phi^{\text{int}}(n\lambda\star x_m)\psi^{-1}(n)dn \\ &= \alpha_\mathfrak{P}^{-m} \sum_{\substack{\lambda\in t_A K^\times\widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A]=A}} \mathcal{P}_\phi\phi^{\text{int}}(\lambda\star x_m) = \vartheta_m^U(A), \end{aligned}$$

as required. To show the interpolation property, we use the same style of argument as given for (1), together with the identification $\pi_{\mathfrak{P}} \cong \pi_{\mathfrak{P}}^u$. Hence, for $\chi \in X_m^\vee$ any primitive ring class character of conductor $\mathfrak{p}^m$ factoring through X_m, we have that

$$\begin{aligned}\chi(\vartheta_m^U) &= \alpha_{\mathfrak{P}}^{-m} \sum_{A \in X_m} \mathcal{W}(A \star x_m)\chi(A) \\ &= \alpha_{\mathfrak{P}}^{-m} \sum_{A \in \mathbf{A}_K^\times / K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times} \sum_{\substack{\lambda \in t_A \cdot K_\infty^\times K^\times \widehat{\mathcal{O}}_{\mathfrak{p}^m}^\times \\ [t_A] = A}} \mathcal{P}_{\phi^{\mathrm{int}}}(\lambda \star x_m)\chi(\lambda) \\ &= \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot \int_{\mathbf{A}_K^\times / K^\times \mathbf{A}_F^\times} \mathcal{P}_\psi \phi^{\mathrm{int}}(t \star x_m)\chi(t)dt,\end{aligned}$$

which after making a change of variables $t \mapsto \varpi_{\mathfrak{P}}^m t$ is the same as

$$\begin{aligned}&\alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot \int_{\mathbf{A}_K^\times / K^\times \mathbf{A}_F^\times} \mathcal{P}_\psi \phi^{\mathrm{int}}(t)\chi(\varpi_{\mathfrak{P}}^m t)dt \\ &= \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot \chi(\varpi_{\mathfrak{P}}^m) \cdot \int_{\mathbf{A}_K^\times / K^\times \mathbf{A}_F^\times} \mathcal{P}_\psi \phi^{\mathrm{int}}(t)\chi(t)dt = \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right) \cdot P_\chi(\phi^{\mathrm{int}})\end{aligned}$$

. □

Now, as noted above, the distributions $\vartheta = (\vartheta_m(\varphi, s_0))_{m \geq 1}$ and $\vartheta^U = (\vartheta_m^U(\phi))_{m \geq 1}$ on $X = \varprojlim_m X_m$ constructed in Theorem 3.4 above both depend on our choice of sequence of local matrices $x_m \in \mathrm{GL}_n(K_{\mathfrak{P}})/\Gamma$. In fact, it is easy to see as in the analogous construction for $\mathrm{GL}_2(\mathbf{A}_K)$ (cf. [10], [61]) that a different choice of system of idele representatives $t = t_m$ and local matrices x_m in this construction has the effect of giving an element which is equal to $(\vartheta_m(\varphi, s_0))_{m \geq 1}$ in $\mathrm{Frac}(\mathcal{O})[[X]]$ or $(\vartheta_m^U(\phi))_{m \geq 1}$ in $\mathrm{Frac}(\mathcal{O})[[X]]$ respectively times some automorphism $\sigma \in X$. To get around this, let us write ϑ^* to denote the image of any element $\vartheta \in \mathrm{Frac}(\mathcal{O})[[X]]$ under the involution $\mathrm{Frac}(\mathcal{O})[[X]] \rightarrow \mathrm{Frac}(\mathcal{O})[[X]]$ given by inversion on group-like elements $\sigma \mapsto \sigma^{-1} \in X$. We then define elements $\mathcal{L}_{\mathfrak{P}}(\pi, s_0)$ and $\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)$ by the products

$$\mathcal{L}_{\mathfrak{P}}(\pi, s_0) := \vartheta\vartheta^* = \vartheta(\varphi, s_0)\vartheta(\varphi, s_0)^* \in \mathrm{Frac}(\mathcal{O})[[X]] \tag{37}$$

and

$$\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2) := \vartheta^U \vartheta^{U*} = \vartheta^U(\phi)\vartheta^U(\phi)^* \in \mathrm{Frac}(\mathcal{O})[[X]]. \tag{38}$$

These elements (37) and (38) are seen easily to be independent of the choice of local matrices $x_m \in \mathrm{GL}_n(K_{\mathfrak{P}})$.

Corollary 3.5. *Fix $e = (e_j)_{j=1}^n$ any $K_{\mathfrak{P}}$-basis of the vector space $V \cong K_{\mathfrak{P}}^n$. Let $\vartheta = \vartheta(\varphi, s_0, e)$ be the corresponding distribution on X as constructed in Theorem 3.4 (1) above, and $\vartheta^U = \vartheta^U(\phi, e)$ the corresponding measure on X as constructed in Theorem 3.4 (2) above. Then, the corresponding elements defined in* (37) *and* (38) *above are is independent of this choice. Moreover, for any primitive ring class character χ of conductor $\mathfrak{p}^m$ factoring through X_m with $m \geq 1$, these elements satisfy the following interpolation properties:*

(1)

$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi, s_0)) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot \mathfrak{L}(s_0, \pi \otimes \chi)^2 \in \mathcal{O}.$$

(2)

$$\chi\left(\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)\right) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot |P_\chi(\phi^{\mathrm{int}})| \in \mathcal{O},$$

which if Conjecture 2.10 is known is the same as

$$\chi\left(\mathcal{L}_{\mathfrak{P}}(\pi^U, 1/2)\right) = \alpha_{\mathfrak{P}}^{-2m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})} \right)^2 \cdot \prod_{v \leq \infty} P_{\chi_v}(\phi_v, \phi_v) \cdot \frac{\Lambda(1/2, \pi^U \otimes \chi)}{L(1, \pi^U, \mathrm{Ad})} \in \mathcal{O}.$$

Proof. The result follows in a direct way from Theorem 3.4 and the discussion above (cf. [61, §4.1.3]). To derive the stated interpolation formula from Theorem 3.4 for (1), observe that we can extend the given ring class character χ to an Iwasawa algebra homomorphism $\chi : \mathcal{O}[[X]] \longrightarrow \mathcal{O}$ to derive the relation

$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi, s_0)) = \chi(\vartheta\vartheta^*) = \chi(\vartheta)\chi(\vartheta^*) = \chi(\vartheta)\chi^{-1}(\vartheta) = \chi(\vartheta)\overline{\chi}(\vartheta),$$

and hence

$$\chi(\mathcal{L}_{\mathfrak{P}}(\pi, s_0)) = \alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})}\right) \cdot \mathfrak{L}(s_0, \pi \otimes \chi) \cdot \overline{\alpha_{\mathfrak{P}}^{-m} \cdot \left(\frac{1}{\tau(\mathcal{O}_{\mathfrak{p}^m})}\right) \cdot \mathfrak{L}(s_0, \pi \otimes \chi)}.$$

The proof for (2) is derived formally in the same way, spelling out the connection to special values of L-functions after taking for granted the statement of Conjecture 2.10. □

References

[1] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** no. 12 (2012), 2311-2413.

[2] A. Caraiani, *Local-global compatibility for $l = p$*, Algebra Number Theory **8** no. 7 (2014), 1597-1646.

[3] P. Allen, F. Calegari, A. Caraiani, T. Gee, D. Helm, B.V. Le Hung, J. Newton, P. Scholze, and R. Taylor, *Potential Automorphy over CM fields*, Ann. of Math. **197** (2023), 897-1113.

[4] C. Anastassiades and J. Thorne, *Raising the level of automorphic representations of* GL_{2n} *of unitary type*, J. Inst. Math. Jussieu (2021), 1-24.

[5] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Ann. of Math. Stud. **120** Princeton Univ. Press, Princeton, NJ (1989).

[6] C. Ballantine, J.A. Rhoades, and T.R. Shemanske, *Hecke operators for* GL_n *and Buildings*, Acta. Arith. **112** (2004), 131-140.

[7] L. Barthel and D. Ramakrishnan, *A nonvanishing result for twists of L-functions of* $\mathrm{GL}(n)$, Duke Math. J. **74** No. 3 (1994), 681-700.

[8] J. Bernstein and A. Zelevinsky, *Representations of* $\mathrm{GL}(n, F)$ *where F is a non-archimedean local field*, Russian Math. Surveys **31** (1976), 1-68.

[9] M. Bertolini and H. Darmon, *Heegner points on Mumford-Tate curves*, Invent. math. **126** (1996), 413-456.

[10] M. Bertolini and H. Darmon, *Iwasawa's Main Conjecture for elliptic curves over anticyclotomic* $\mathbf{Z}_p$*-extensions*, Ann. of Math. **162** (2005), 1-64.

[11] R. Beuzart-Plessis and P.-H. Chaudouard, *The global Gan-Gross-Prasad conjecture for unitary groups. II. From Eisenstein series to Bessel periods*, preprint (2023), available at `https://arxiv.org/abs/2302.12331v1`.

[12] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301-314.

[13] L. Clozel, *On the Eisenstein functoriality in cohomology for maximal parabolic subgroups*, Selecta Math. (N.S.) **28** no. 80 (2022).

[14] L. Clozel, *Purity Reigns Surpreme*, Int. Math. Res. Not. IMRN no. 2 **2013**, 328-346.

[15] L. Clozel, M. Harris, and R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes. Études. Sci. **108** (2008), 1-181, with Appendix A, summarizing work of Russ Mann, and Appendix B by M.-F. Vignéras.

[16] J. Cogdell, *Lectures on L-functions, Converse Theorems, and Functoriality for* $\mathrm{GL}(n)$, in "Lectures on Automorphic L-functions", Eds. J. Cogdell, H. Kim, and M.R. Murty, Fields Inst. Monogr., Amer. Math. Soc., Providence RI (2004).

[17] J. Cogdell, *Notes on L-functions for* $\mathrm{GL}(n)$, Lectures given at the School on Automorphic Forms on $\mathrm{GL}(n)$, Trieste, 31 July-18 August 2000.

[18] C. Cornut, *Mazur's conjecture on higher Heegner points*, Invent. math. **148** No. 3 (2002), 495-523.

[19] C. Cornut and V. Vatsal, *CM points and quaternion algebras*, Doc. Math. **10** (2005), 263-309.

[20] C. Cornut and V. Vatsal, *Nontriviality of Rankin-Selberg L-functions and CM points*, in "L-functions and Galois Representations", Ed. Burns, Buzzard and Nekovář, Cambridge University Press (2007), 121-186.

[21] P. Deligne, *Valeurs des fonctions L et périodes d'intégrales*, Proc. Symp. Pure Math. **33.2** (1979), 313-346.

[22] E. Eischen, M. Harris, J.-S. Li, and C. Skinner, *p-adic L functions for unitary Shimura varieties, II : zeta integral calculations; III : ordinary families and p-adic L-functions*, Forum of Mathematics, Pi **8** no. 9 (2020)

[23] D. File, K. Martin, and A. Pitale, *Test vectors and central L-values for* $\mathrm{GL}(2)$, Algebra Number Theory **11** 2 (2017), 253-318.

[24] W.T. Gan, B.H. Gross, and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, in "Sur les conjectures de Gross et Prasad, I", Astérisque **346** (2012), 111-170.

[25] D. Ginzburg, S. Rallis, and D. Soudry, *The descent map from automorphic representations of* $\mathrm{GL}(n)$ *to classical groups*, World Scientific Publishing Co. (2011), Pte. Ltd. Hackensack, NJ.

[26] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math. **260**, Springer, Berlin-New York (1972).

[27] D. Goldfeld and J. Hundley, *Automorphic Representations and L-Functions for the General Linear Group, Volume II*, Cambridge Stud. Adv. Math. **130**, Cambridge University Press (2011).

[28] H. Grobner and M. Harris, *Whittaker periods, motivic periods, and special values of tensor product L-functions*, J. Inst. Math. Jussieu **15** (2016), 711-769.

[29] H. Grobner and G. Sachdeva, *Relations of rationality for special values of Rankin-Selberg L-functions*, preprint (2019).

[30] L. Guerberoff, *Period relations for automorphic forms on unitary groups and critical values of L-functions*, Doc. Math. **21** (2016), 1397-1458.

[31] L. Guerberoff and J. Lin, *Galois equivariance of critical values of L-functions for unitary groups*, preprint (2016).

[32] G. Harder and A. Raghuram, *Eisenstein cohomology for* GL(N) *and ratios of critical values of Rankin-Selberg L-functions*, Ann. of Math. Stud. **203**, (2019).

[33] M. Harris, *L-functions and periods of polarized regular motives*, J. reine angew. Math. **483** (1997), 75-161.

[34] M. Harris, J.-S. Li, and C. Skinner, *p-adic L functions for unitary Shimura varieties, I : Construction of the Eisenstein measure*, Doc. Math., John H. Coates' Sixtieth Birthday (2006) 393-464.

[35] M. Harris and J. Lin, *Period relations and special values of Rankin-Selberg L-functions*, in J. Cogdell, J.-L. Kim, C.-B. Zhu, eds., "Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe on the Occasion of His 70th Birthday", Progress in Math. **323** (2017).

[36] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Ann. of Math. Stud. **151**, Princeton University Press (2001), `https://www.jstor.org/stable/j.ctt1287kfd`.

[37] B. Howard, *Bipartite Euler systems*, J. reine angew. Math. **597** (2006).

[38] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** no. 5 (2010), 1378-1425.

[39] H. Jacquet, *A correction to "Conducteur des repreésentations du groupe linéaire*, Pacific J. Math. **260** no. 2 (2012), 515-525.

[40] H. Jacquet, II.I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** no. 2 (1981), 199-214.

[41] H. Jacquet, II.I. Piatetski-Shapiro, and J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), 367-464.

[42] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic forms*, Amer. J. Math. **103** no. 3 (1981), 499-558.

[43] H. Jacquet and J. Shalika, *The Whittaker models of induced representations*, Pacific J. Math. **109** no. 1 (1983), 107-120.

[44] W. Kim and K. Madapusi Pera, 2*-adic integral canonical models and the Tate conjecture in characteristic* 2, Forum Math. Sigma **4** (2016), e28.

[45] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), 967-1012.

[46] E. Lapid and B. Mao, *A conjecture on Whittaker-Fourier coefficients of cusp forms*, J. Number Theory **146** (2015), 448-505.

[47] Y. Liu, *Anticyclotomic p-adic L-function for Rankin-Selberg product*, preprint (2023), available at `https://arxiv.org/abs/2306.07039`

[48] Y. Liu, *Refined global Gan-Gross-Prasad conjecture for Bessel periods*, J. reine angew. Math. **717** (2016), 133-194.

[49] D. Loeffler, *Spherical varieties and norm relations in Iwasawa theory* (2019), to appear in J. Th. Nombres Bordeaux (Iwasawa 2019 special issue)

[50] G. Margulis and G.M. Tomanov, *Invariant measures for actions of unipotent groups over local fields on homogeneous spaces*, Invent. math. **116** (1994), 347-392.

[51] N. Matringe, *Essential Whittaker Functions for* GL(n), Doc. Math. **18** (2013), 1191-1214.

[52] K. Prasanna, *Arithmetic aspects of the theta correspondence and periods of modular forms*, in "Eisenstein series and Applications", Progress in Math. **258**, Birkhäuser (2008), 251-269.

[53] K. Prasanna, *Integrality of a ratio of Petersson norms and level-lowring congruences*, Ann. of Math. **163** (2003), 901-967.

[54] A. Raghuram, *Special values of L-functions for* GL(n) *over a CM field*, preprint (2020).

[55] M. Ratner, *Raghunathan's conjectures for Cartesian products of real and p-adic Lie groups*, Duke Math. J. **77** (2) (1995), 275-382.

[56] K.A. Ribet and S. Takehashi, *Parametrizations of elliptic curves by Shimura curves and by classical modular curves*, Elliptic curves and modular forms (Washington D.C. 1996), Proc. Nat. Acad. Sci. U.S.A. **94** no. 21 (1997), 11110-11114.

[57] D. Rohrlich, *On L-functions of elliptic curves and anticyclotomic towers*, Invent. math. **75** No. 3 (1984), 383-408.

[58] I. Satake, *Theory of spherical functions on algebraic reductive groups over* $\mathfrak{P}$*-adic fields*, Publ. Math. Inst. Hautes. Études. Sci. **18** (1963), 5-59.

[59] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, Ann. of Math. (2) **182** (2015) no. 3, 945-1066.

[60] G. Shimura, *The special values of zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), 637-679.

[61] J. Van Order, *p-adic interpolation of automorphic periods for* GL_2, Doc. Math. **22** (2017), 1467-1499.

[62] V. Vatsal, *Uniform distribution of Heegner points*, Invent. math. **148** 1-46 (2002).

[63] V. Vatsal *Special values of anticyclotomic p-adic L-functions*, Duke Math. J. **116** (2) (2003), 219 - 261.

[64] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compos. Math. **54** (1985), 173-242

[65] X. Yuan, S.-W. Zhang, and W. Zhang, *The Gross-Zagier Formula for Shimura Curves*, Ann. of Math. Studies **184**, Princeton University Press 2013.