RANKIN-SELBERG L-FUNCTIONS IN CYCLOTOMIC TOWERS, I

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Abstract. We formulate and prove a conjecture in the style of Mazur-Greenberg for the nonvanishing of central values of Rankin-Selberg L-functions attached to elliptic curves in abelian extensions of imaginary quadratic fields. This in particular generalizes the theorem of Rohrlich on L-functions of elliptic curves in cyclotomic towers to the setting of abelian extensions of imaginary quadratic fields, corresponding to families of degree-four L-functions given by $\mathrm{GL}(2) \times \mathrm{GL}(2)$ Rankin-Selberg L-functions. It also generalizes the theorems of Rohrlich, Greenberg, Vatsal, and Cornut for L-functions of elliptic curves in $\mathbf{Z}_p^2$-extensions of imaginary quadratic fields.

Contents

1. Introduction

We formulate and prove a conjecture in the style of Mazur-Greenberg for central values of Rankin-Selberg L-functions in the non self-dual setting, motivated by applications to bounding Mordell-Weil ranks in the setting of two-variable main conjectures of Iwasawa theory for elliptic curves without complex multiplication (see e.g. [7], [28], [34], [46]). Such applications are explained in the sequel work [44], along with how stronger results can be deduced from the existence of a suitable p-adic L-function (such as [19] and [34]) to generalize the theorems of Greenberg [16], Rohrlich [38] [37], Vatsal [47] and Cornut [11]. The main purpose of this work is to consider the problem from an analytic point of view, and to derive estimates which should be of independent interest. In particular, we develop spectral decompositions of the shifted convolution sums in ways that should be applicable to study average central values of arbitrary GL_4-automorphic L-functions.

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That is, we develop spectral analogues of the shifted convolution problem to estimate the off-diagonal terms appearing in first moments of a general class of Rankin-Selberg L-functions, together with averages over primitive Dirichlet characters of these L-functions (see also remark (3) below). We expect that this double averaging could be developed in the broader setup described in [43] to estimate central values of GL_4-automorphic L-functions in these families, possibly through development of similar summation methods for Eisenstein series on GL_4.

Let f be a holomorphic cuspidal eigenform of squarefree level N and trivial nebentype character[1]. We shall assume that f is a newform of integral weight $l \geq 2$, with Fourier series expansion at infinity

$$f(z) = \sum_{n \geq 1} n^{\frac{(l-1)}{2}} \lambda(n) e^{2\pi i n z}, \ z \in \mathfrak{H}$$

normalized so that $\lambda(1) = 1$. Let K be an imaginary quadratic field of discriminant $D < 0$ prime to N and associated quadratic Dirichlet character $\eta = \eta_D$. Fix an odd prime p which is coprime to the product ND. Let $\mathcal{W}$ be a finite order Hecke character of K of the form

$$\mathcal{W} = \rho\chi \circ \mathbf{N}, \tag{1}$$

where ρ is a primitive ring class character of K of some p-power conductor, and $\chi \circ \mathbf{N}$ is induced via composition with the norm $\mathbf{N}$ from a primitive Dirichlet character χ of p-power conductor. Note that as functions on nonzero ideals $\mathfrak{a} \subset \mathcal{O}_K$, the ring class Hecke character ρ in this decomposition (1) can be characterized by the condition $\rho(\overline{\mathfrak{a}}) = \overline{\rho(\mathfrak{a})}$, i.e. ring class characters are equivariant under complex conjugation. Given such a Hecke character $\mathcal{W}$ of K as in (1), let us write $c(\mathcal{W}) = c(\rho)c(\chi) \in \mathbf{Z}$ to denote its conductor. A classical construction due to Hecke (see e.g. [18] or [17, § I (5.2) and § IV]) associates to $\mathcal{W}$ a holomorphic theta series $\Theta(\mathcal{W})$ of weight 1, level $|D|c(\mathcal{W})^2$ and character $\eta\mathcal{W}|_{\mathbf{Q}} = \eta\chi^2$. We consider the Rankin-Selberg L-function $L(s, f \times \mathcal{W})$ of f times $\Theta(\mathcal{W})$, whose completed L-function $\Lambda(s, f \times \mathcal{W})$ satisfies a functional equation of the form

$$\Lambda(s, f \times \mathcal{W}) = \epsilon(1/2, f \times \mathcal{W})\Lambda(1 - s, f \times \overline{\mathcal{W}}). \tag{2}$$

Here, $\epsilon(1/2, f \times \mathcal{W}) \in \mathbf{S}^1$ is a complex number of modulus one known as the root number. If $\mathcal{W} = \rho$ is a ring class character of K, then it is easy to see that the coefficients in the Dirichlet series expansion of $L(s, f \times \rho)$ are real-valued, and hence that $\epsilon(1/2, f \times \mathcal{W})$ takes values in the set $\{\pm 1\}$. In this setting, the L-function $L(s, f \times \mathcal{W})$ is said (for representation theoretic reasons) to be self-dual. Moreover, when $\mathcal{W} = \rho$ is a ring class character, it is easy to see that the functional equation is symmetric, relating values of the same L-function $\Lambda(s, f \times \rho)$ on either side (via equivariance of ρ under complex conjugation). In particular, when $\mathcal{W} = \rho$ is a ring class character, if $\epsilon(1/2, f \times \rho)$ equals -1 (as opposed to $+1$), then the central value $\Lambda(1/2, f \times \rho)$ is forced to vanish by the functional equation. Let us for future reference distinguish this particular case of forced vanishing from all others as follows:

Definition We refer to a pair $(f, \mathcal{W})$ as *exceptional* if the Hecke character $\mathcal{W} = \rho$ is a ring class character (including products of genus class group characters) and $\epsilon(1/2, f \times \mathcal{W}) = -1$, and as *generic* otherwise.

Remark This characterization of forced vanishing is stable in the following sense. As we explain below, it is well-known that there exists an integer $\nu = \nu(f, D, p) \in \{0, 1\}$ for which the root number $\epsilon(1/2, f \times \rho) = (-1)^\nu$ for all but finitely many ring class characters ρ of K of p-power conductor. Our generic case consists of the setting where either (i) this ring class root number is parametrized by $(-1)^\nu = 1$ for $\nu = 0$ in this way, or more generally (ii) the Hecke character $\mathcal{W} = \rho\chi \circ \mathbf{N}$ contains a nontrivial cyclotomic part $\chi \circ \mathbf{N}$ irrespective of the characterization of ν, in particular so that there is no forced vanishing of central values via the functional equation (2). We then study the values $L^{(k)}(1/2, f \times \mathcal{W})$ with $k = 0$ when $(f, \mathcal{W})$ is generic in this sense.

We establish nonvanishing estimates for averages of central derivative values $L'(1/2, f \times \rho)$ in the exceptional setting on $(f, \mathcal{W})$, and averages of central values $L(1/2, f \times \mathcal{W})$ in the generic setting on $(f, \mathcal{W})$. In particular, we derive estimates in the latter setting for $\mathcal{W} = \rho\chi \circ \mathbf{N}$ having both nontrivial ring class part ρ and cyclotomic part $\chi \circ \mathbf{N}$. This fills a gap in the literature, and has various arithmetic applications such as to bounding Mordell-Weil ranks via Iwasawa main conjectures, as we explain in the sequel note [44], which can be viewed as an appendix to this paper. In particular, pairing together the analytic estimates shown in

[1] In fact, we can work with an arbitrary Hecke-Maass eigenform f of level prime to pD for most of our arguments, but restrict to this setting (which is relevant to Shimura's rationality theorems and Iwasawa theory) for simplicity.

this paper with the investigation of the structure of some associated p-adic L-function in the sequel paper [44], we prove this conjecture in many cases. Our results here also generalize the well-known theorems of Cornut [11] and Vatsal [47] in the self-dual setting, as well as the theorems of Greenberg [16] and Rohrlich [38], [37]. Note that the aforementioned proofs of Cornut and Vatsal rely on completely different methods, starting with the special value formulae of Gross [16], Waldspuger [48], Gross-Zagier [17], Zhang [50], and Yuan-Zhang-Zhang [49] which cannot be extended to the generic non self-dual setting (ii) we consider here.

The point of departure in the works of Greenberg [16], Rohrlich [38] [37] and Vatsal [47] for central values is the following rationality theorem of Shimura [39, Theorem 4]. To recall this briefly, let F denote the Hecke field of f, i.e. the finite extension of the rational number field $\mathbf{Q}$ obtained by adjoining the eigenvalues of f. Hence, F is shown by Shimura to be a number field; in fact, F is totally real if f is not dihedral, and otherwise imaginary quadratic. Fix a Hecke character $\mathcal{W} = \rho\chi \circ \mathbf{N}$ as in (1) above, with ρ a primitive ring class character of conductor p^{α}, and χ a primitive Dirichlet character of conductor p^{β}, for some integers $\alpha, \beta \geq 0$. Let $F(\mathcal{W})$ denote the cyclotomic extension of F obtained by adjoining the values of $\mathcal{W}$. Given a complex embedding of $F(\mathcal{W})$ fixing F, let $\mathcal{W}^{\sigma}$ denote the Hecke character defined on ideals $\mathfrak{a}$ of K by $\mathfrak{a} \mapsto \mathcal{W}(\mathfrak{a})^{\sigma}$. Let $\langle f, f \rangle$ denote the Petersson norm of f. Shimura [39, Theorem 4] shows that the values

$$\mathcal{L}(1/2, f \times \mathcal{W}) = \frac{L(1/2, f \times \mathcal{W})}{8\pi^2 \langle f, f \rangle}$$

are algebraic, in fact that they lie in the number field $F(\mathcal{W})$, and moreover that these values are Galois conjugate in the sense that any automorphism σ of $F(\mathcal{W})$ acts on the value $\mathcal{L}(1/2, f \times \mathcal{W})$ by the rule

$$\mathcal{L}(1/2, f \times \mathcal{W})^{\sigma} = \mathcal{L}(1/2, f^{\tau} \times \mathcal{W}^{\sigma}),$$

where τ denotes the restriction of σ to F. It follows $L(1/2, f \times \mathcal{W})$ vanishes if any only if $L(1/2, f \times \mathcal{W}^{\sigma})$ vanishes for each automorphism σ of $F(\mathcal{W})$ fixing F. A similar notion of Galois conjugacy can be established in the exceptional setting via the formulae of Gross-Zagier [17] and more generally Zhang [50] or Yuan-Zhang-Zhang [49] for the central derivative values $L'(1/2, f \times \rho)$ (cf. [37, p. 385]), at least assuming f has weight $l = 2$. Thus, it can also be deduced in the exceptional setting on $(f, \mathcal{W})$ that the value $L'(1/2, f \times \rho)$ vanishes if any only if the value $L'(1/2, f \times \rho^{\sigma})$ vanishes for each complex embedding σ of $F(\rho)$ fixing F. Let us therefore define $k = k(f, \mathcal{W}) \in \{0, 1\}$ by the condition

$$k = k(f, \mathcal{W}) = \begin{cases} 0 & \text{if the pair } (f, \mathcal{W}) \text{ is generic} \\ 1 & \text{if the pair } (f, \mathcal{W}) \text{ is exceptional, i.e. } \mathcal{W} = \rho \text{ ring class with } \epsilon(1/2, f \times \rho) = -1. \end{cases}$$

We then define the associated k-th Galois average

$$G_{[\mathcal{W}]}^{(k)} = \frac{1}{[F(\mathcal{W}) : F]} \sum_{\sigma} L^{(k)}(1/2, f \times \mathcal{W}^{\sigma}). \tag{3}$$

Here, the sum runs over embeddings $\sigma : F(\mathcal{W}) \rightarrow \mathbf{C}$ fixing F. As well, $L^{(0)}(1/2, f \times \mathcal{W})$ denotes the central value $L(1/2, f \times \mathcal{W})$, and $L^{(1)}(1/2, f \times \mathcal{W})$ the derivative value $L'(1/2, f \times \mathcal{W})$. We make the following conjecture in the spirit of Mazur, Greenberg [16] and Coates-Fukaya-Kato-Sujatha [7] for these averages, of which the theorems of Greenberg [16], Rohrlich [38], [37], Vatsal [47] and Cornut [11] would be special cases.

Conjecture 1.1. *Let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ be a Hecke character of K of finite order as in* (1) *above. Let $k = 1$ if the pair $(f, \mathcal{W})$ is exceptional, or else $k = 0$ if the pair $(f, \mathcal{W})$ is generic. If $c(\mathcal{W})$ is sufficiently large, then $G_{[\mathcal{W}]}^{(k)} \neq 0$. Equivalently, for all but finitely many such $\mathcal{W}$, the value $L^{(k)}(1/2, f \times \mathcal{W})$ does not vanish.*

We prove this conjecture in many cases, starting with analytic estimates for the following coarser averages. Fix a Hecke character $\mathcal{W} = \rho\chi \circ \mathbf{N}$ of the form described in (1) above, with ρ a primitive ring class character of conductor p^{α} for some integer $\alpha \geq 0$, and χ a primitive even[2] Dirichlet character of conductor p^{β} for some integer $\beta \geq 0$. Let $\#C^{\star}(\alpha)$ denote the number of primitive ring class characters of conductor p^{α}. Hence, writing $\#C(\alpha)$ to denote the number of ring class characters of K of conductor p^{α}, and hence $\#C(\alpha - 1)$ the number of ring class characters of conductor $p^{\alpha-1}$, we have that $\#C^{\star}(\alpha) = \#C(\alpha) - \#C(\alpha - 1)$. Note that

[2]We average over even Dirichlet characters so that the archimedean component of the corresponding idele class character is trivial, and so the archimedean components of the L-function do not depend on the choice of character. That is, we average over a family of wide ray class characters of K, which is simpler for our approximate functional equation below.

by Dedekind's theorem (see e.g. [12, Theorem 7.24]), the cardinality $\#C(\alpha)$ is given by an integer multiple of the class number $h_K = \#C(0) = \#\operatorname{Pic}(\mathcal{O}_K)$. To be more precise, we have Dedekind's formula

$$\#C(\alpha) = \frac{h_K \cdot p^\alpha}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} \cdot \left(1 - \frac{\eta(p)}{p}\right) = \begin{cases} \frac{h_K p^\alpha}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} \cdot \left(\frac{p-1}{p}\right) & \text{if } p \text{ splits in } K \\ \frac{h_K p^\alpha}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} \cdot \left(\frac{p+1}{p}\right) & \text{if } p \text{ is inert in } K \\ \frac{h_K p^\alpha}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} & \text{if } p \text{ ramifies in } K. \end{cases} \tag{4}$$

Let $\varphi^\star(p^\beta)$ the number of primitive Dirichlet characters $\chi \bmod p^\beta$, so that $\varphi^\star(p^\beta)/2$ counts the number of primitive even Dirichlet characters $\chi \bmod p^\beta$. Writing $k \in \{0,1\}$ as above according as to whether the pair $(f, \mathcal{W})$ is generic or exceptional, we first consider the weighted average over primitive ring class characters ρ of conductor p^α and primitive even Dirichlet characters $\chi \bmod p^\beta$ of the corresponding L-values $L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N})$:

$$\mathcal{H}^{(k)}(\alpha, \beta) = \frac{1}{\#C^\star(\alpha)} \sum_{\substack{\rho \in \operatorname{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N}).$$

Here, $\operatorname{Pic}(\mathcal{O}_{p^\alpha})$ denotes the class group of the order $\mathcal{O}_{p^\alpha} = \mathbf{Z} + p^\alpha \mathcal{O}_K$ of conductor p^α in $\mathcal{O}_K$, so that the character group of $\operatorname{Pic}(\mathcal{O}_{p^\alpha})^\vee$ of $\operatorname{Pic}(\mathcal{O}_{p^\alpha})$ contains all of the ring class class characters ρ of conductor p^α.

To describe the estimates we derive for these averages, we first need to recall the Dirichlet series expansion of the symmetric square L-function $L(s, \operatorname{Sym}^2 f)$ of f: Given $s \in \mathbf{C}$ (first with $\Re(s) > 1$), we can then define $L(s, \operatorname{Sym}^2 f)$ by the Dirichlet series expansion

$$L(s, \operatorname{Sym}^2 f) = \zeta^{(N)}(2s) \sum_{n \geq 1} \frac{\lambda(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{1}{m^{2s}} \sum_{n \geq 1} \frac{\lambda(n^2)}{n^s}. \tag{5}$$

In the non-self-dual setting with χ a primitive even Dirichlet character $\chi \bmod p^\beta$ for some $\beta \geq 1$, we also consider the twisted symmetric square L-functions $L(s, \operatorname{Sym}^2 f \otimes \chi)$ defined for $\Re(s) \geq 1$ by

$$L(s, \operatorname{Sym}^2 f \otimes \chi) = L^{(Np)}(2s, \eta\chi^2) \sum_{n \geq 1} \frac{\lambda(n^2)\chi(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ (m,pN)=1}} \frac{\eta\chi^2(m)}{m^{2s}} \sum_{n \geq 1} \frac{\lambda(n^2)\chi(n^2)}{n^s}.$$

We refer to [21, §5], [8], and [15] for background on these L-functions. Let $\gamma = \lim_{x \to \infty} \left(\sum_{x \leq n} \frac{1}{n} - \log x \right)$ denote the Euler-Mascheroni constant. Hence, $\gamma \approx 0.577$ appears as the constant term in the Laurent series expansion of $\zeta(s)$ around $s = 1$. Given an integer $M \geq 1$ and an L-function $L(s)$, we also write $L^{(M)}(s)$ to denote the L-function determined by $L(s)$ after removal of the Euler factors at primes dividing M. Let $w = w_K$ denote the number of roots of unity in K. Let us also use the word "constant" to mean any nonzero complex number that does not depend on any of the parameters $f, p, D, \alpha \geq 0, \beta \geq 0$ unless stated otherwise, using the corresponding subscripts in the big O notation. We give these constants as precisely as possible throughout to spell out all dependencies. While these constants often depend on the eigenform f, the prime p, or the fundamental discriminant D, they are typically independent of the ring class exponent $\alpha \geq 1$ or the cyclotomic exponent $\beta \geq 1$. We first prove the following main result.

Theorem 1.2. *Let f be a Hecke eigenform of even weight $l \geq 2$, squarefree level N and trivial character. If the cyclotomic exponent $\beta = 0$ is trivial, then assume that f is not dihedral, in other words not given by a theta series corresponding to some Hecke character of a quadratic field. Fix K an imaginary quadratic field of discriminant $D < 0$ and quadratic Dirichlet character $\eta = \eta_D$. Let $p \geq 3$ be prime number, and assume that $(p, ND) = (N, D) = 1$. We prove the following estimates in either case on the generic root number $k \in \{0,1\}$:*

(i) *(Theorem 3.4 (i)). If $k=0$ and $\alpha \geq 0$ is any integer, then we have for any $\varepsilon > 0$ the estimate*

$$\mathcal{H}^{(0)}(\alpha,0) = \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) \cdot \frac{4}{w} \cdot L(1,\eta) \cdot \frac{L^{(p^\alpha)}(1, \mathrm{Sym}^2 f)}{\zeta^{(Np^\alpha)}(2)} + O_{f,\varepsilon}\left(|D|^{-\frac{1}{8}+\varepsilon} p^{-\frac{\alpha}{2}}\right) + O_{f,\varepsilon}\left((|D|p^{2\alpha})^{-\frac{1}{16}+\varepsilon}\right).$$

In particular, if α is sufficiently large, then the average $\mathcal{H}^{(0)}(\alpha,0)$ does not vanish.

(ii) *(Theorem 3.4 (ii)). If $k=1$ and $\alpha \geq 0$ is any integer, then we have for any $\varepsilon > 0$ the estimate*

$$\begin{aligned}\mathcal{H}^{(1)}(\alpha,0) &= \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) \\ &\times \frac{4}{w} \cdot L(1,\eta) \cdot \frac{L^{(p^\alpha)}(1, \mathrm{Sym}^2 f)}{\zeta^{(Np^\alpha)}(2)} \cdot \left[\log\left(N|D|p^{2\alpha}\right) + \frac{L'}{L}(1,\eta) + \frac{L^{(p^\alpha)\prime}}{L^{(p^\alpha)}}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)) - \frac{\zeta^{(Np^\alpha)\prime}}{\zeta^{(Np^\alpha)}}(2)\right] \\ &+ O_{f,\varepsilon}\left(|D|^{-\frac{1}{8}+\varepsilon} p^{-\frac{\alpha}{2}}\right) + O_{f,\varepsilon}((|D|p^{2\alpha})^{-\frac{1}{16}+\varepsilon}).\end{aligned}$$

In particular, if α (or $p^\alpha|D|$) is sufficiently large, then the average $\mathcal{H}^{(1)}(\alpha,0)$ does not vanish.

(iii) *(Theorem 4.6). Fix an integer $\beta \geq 1$. We have for each integer $\alpha \geq 1$ the estimate*

$$\begin{aligned}\mathcal{H}^{(0)}(\alpha,\beta) &= \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) \\ &\times \frac{2}{w} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive } \chi(-1)=1}} \left(L(1,\eta\chi^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \chi)}{L^{(N)}(2,\chi)} + \frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \cdot L(1,\eta\overline{\chi}^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \overline{\chi})}{L^{(N)}(2,\overline{\chi})}\right) \\ &+ O_{f,\epsilon}\left(\left(|D|p^\beta\right)^{\frac{3}{16}+\varepsilon}\left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}}\right) + O_{f,\beta,\varepsilon}\left(\left(|D|p^{2\alpha}\right)^{-\frac{1}{4}+\delta_0}\right).\end{aligned}$$

In particular, if $\alpha \gg \beta$ is sufficiently large, then the average $\mathcal{H}^{(0)}(\alpha,\beta)$ converges to the constant

$$\begin{aligned}&\left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) \\ &\times \frac{2}{w} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive } \chi(-1)=1}} \left(L(1,\eta\chi^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \chi)}{L^{(N)}(2,\chi)} + \frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \cdot L(1,\eta\overline{\chi}^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \overline{\chi})}{L^{(N)}(2,\overline{\chi})}\right).\end{aligned}$$

We also show in the proof of Theorem 4.6 that this constant cannot vanish, using the nonvanishing of each of the values $L(1,\eta\chi^2)L(1, \mathrm{Sym}^2 f \otimes \chi)$ to derive an argument by contradiction.

Some remarks. Let us make the following immediate comments; we give a high-level sketch of the proofs (and explain the provenance of the exponents) in this final subsection of this introduction.

(1) We assume for simplicity that f is non-dihedral when $\beta = 0$, since otherwise there would be a more complicated main term in our self-dual estimates (i) and (ii) coming from the residual spectrum of the space of L^2-automorphic forms on the metaplectic cover of $\mathrm{GL}_2(\mathbf{A})$ (see [42, Theorem 1]). As well, there would be a similar residual contribution in the error term of (iii). These residual contributions however present only technical complications to our main argument. In principle, a similar set of nonvanishing estimates can be established in the dihedral setting, consistent with the

nonvanishing theorems of Rohrlich [37] and Greenberg [16]. We leave this for a subsequent work.

(2) It is easy to deduce from Theorem 1.2 that $\mathcal{H}^{(k)}(\alpha, 0) \neq 0$ for sufficiently large ring class exponent $\alpha \gg 1$, and in the case of $k = 1$ also for sufficiently large absolute discriminant $|D| \gg 1$ (cf. [40]). Here, we use the well-known nonvanishing of $L(1, \mathrm{Sym}^2 f)$. In fact, these values are in fact bounded below by 1 via the theorem of Goldfeld-Hoffstein-Lieman [15], cf. [8, Lemma 4.2]. Moreover, since η is an odd Dirichlet character, we could also use Colmez [10, Proposition 5] to derive the lower bound

$$\frac{1}{2}\log(N|D|p^{2\max(\alpha,\beta)}) + \frac{L'}{L}(1,\eta) \gg \frac{1}{2}\log\left(N|D|^2 p^{2\max(\alpha,\beta)}\right) \gg \log p^{\alpha} + O_{f,D}(1).$$

We refer to Lemma 3.2 for more on the nonvanishing and bounds for the residual terms appearing in Theorem 1.2 (i) and (ii), as well as the more general residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ defined below.

(3) In the more general setting of $\mathcal{H}^{(0)}(\alpha, \beta)$ in (iii), we proceed in the same way as for the self-dual case (see Theorem 3.3 and Theorem 4.6), and note that a finer result can be derived without taking the average over Dirichlet characters (as detailed in the second sequel work [45]). In any case, we only derive nonvanishing estimates this way for a fixed cyclotomic exponent β (or character), with sufficiently large ring class exponent $\alpha \gg \beta$.

Using variations of the ideas appearing in Theorems 3.4 and 4.6, we obtain the following refinement for the Galois averages $G_{[\mathcal{W}]}^{(k)}$ of Conjecture 1.1. Let us for any integer $\alpha \geq 0$ write $C(\alpha) = \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ to denote the class group of the order $\mathcal{O}_{p^\alpha}$, with $C(\alpha)^\vee$ its character group, and $\#C(\alpha)$ its cardinality. Recall that for $\alpha \geq 1$, a ring class character ρ of $C(\alpha)$ is said to be primitive if it does not factor through $C(\alpha - 1)$, and that we write $\#C^\star(\alpha) = \#C(\alpha) - \#C(\alpha-1)$ to denote the number of such characters. Writing $x = \mathrm{ord}_p(\#C(\alpha))$ for the exponent of p in $\#C(\alpha)$, we consider subaverages over primitive ring class characters of conductor p^α and exact order p^x. To fix ideas, recall that by Dedekind's formula (4), the cardinality $\#C(\alpha)$ is given by a precise integer multiple of the class number h_K of K. This formula (4) gives us the relation

$$x = \mathrm{ord}_p\left(\#C(\alpha)\right) = \alpha + \mathrm{ord}_p\left(\frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} \cdot \left(1 - \frac{\eta(p)}{p}\right)\right).$$

From this, we can write $\alpha = x + \delta$, where

$$\delta = -\,\mathrm{ord}_p\left(\frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}_{p^\alpha}^\times]} \cdot \left(1 - \frac{\eta(p)}{p}\right)\right)$$

is an integer depending on p. If the exponent of the ring class conductor $\alpha \geq 0$ is sufficiently large so that $x = \mathrm{ord}_p(\#C(\alpha)) \geq 1$, (for instance if $\alpha \geq 1$), then we consider averages over ring class characters of exact order p^x. Writing $\#C(\alpha, y)$ for each integer $0 \leq y \leq x$ to denote the index $\#C(\alpha, y) = [C(\alpha) : C(\alpha)^{p^y}]$, with

$$\#C^\star(\alpha, x) = \#C(\alpha, x) - \#C(\alpha, x-1) = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}]$$

the difference, we consider the corresponding averages over primitive ring class character of exact order p^x,

$$G^{(k)}(\alpha; x) = \frac{1}{\#C^\star(\alpha, x)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1}\ \forall 0 \leq y \leq x-1}} L^{(k)}(1/2, f \times \rho)$$

in the self-dual setting with $\beta = 0$, and more generally the double average

$$G^{(0)}(\alpha, \beta; x) = \frac{1}{\#C^\star(\alpha, x)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1}\ \forall 0 \leq y \leq x-1}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \mathrm{primitive}}} L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N})$$

in the generic, non-self-dual setting with $\beta \geq 2$.

Given a class $A \in C(\alpha)$, let $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ be any[3] binary quadratic form class representative in the corresponding class group $Q(\alpha) \cong C(\alpha)$ of binary quadratic forms of discriminant $Dp^{2\alpha}$. Let us for each divisor $q \mid \gamma_A$ write $f^{(q)}$ to denote the shift of f defined on $z = x+iy \in \mathfrak{H}$ by $f^{(q)}(z) = f(q^{-1}z)$. We then write $\lambda^{(q)}$ to denote its corresponding shift of the Fourier coefficient by q. We consider the corresponding congruence symmetric square Dirichlet series $L_q(s, \mathrm{Sym}^2 f^{(q)})$ defined for $\Re(s) > 1$ by the expansion

$$L_q(s, \mathrm{Sym}^2 f^{(q)}) = \zeta_q^{(N)}(2s) \sum_{\substack{n \geq 1 \\ n \equiv 0 \bmod q}} \frac{\lambda^{(q)}(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{1}{m^{2s}} \sum_{\substack{n \geq 1 \\ m^2 n \equiv 0 \bmod q}} \frac{\lambda(n^2 q^{-1})}{n^s},$$

and similarly

$$L_q^{(p^\alpha)}(s, \mathrm{Sym}^2 f^{(q)}) = \zeta_q^{(Np^\alpha)}(2s) \sum_{\substack{n \geq 1 \\ (n,p^\alpha)=1 \\ n \equiv 0 \bmod q}} \frac{\lambda^{(q)}(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ m^2 \equiv 0 \bmod q}} \frac{1}{m^{2s}} \sum_{\substack{n \geq 1 \\ (n,p^\alpha)=1 \\ n \equiv 0 \bmod q}} \frac{\lambda(n^2 q^{-1})}{n^s}.$$

These congruence series appear in the corresponding residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ for the self-dual setting,

$$\mathfrak{L}_{0,f,\gamma_A}(1) := \frac{4}{w} \cdot \sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot L(1,\eta) \cdot \frac{L_q^{(p^\alpha)}(1, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)}$$

and

$\mathfrak{L}_{1,f,\gamma_A}(1)$

$$:= \frac{4}{w} \cdot L(1,\eta) \cdot \frac{L^{(p^\alpha)}(1, \mathrm{Sym}^2 f)}{\zeta^{(Np^\alpha)}(2)} \cdot \left[\left(N|D|p^{2\alpha}\right) + \frac{L'}{L}(1,\eta) + \frac{L^{(p^\alpha)\prime}}{L^{(p^\alpha)}}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)) - \frac{\zeta^{(Np^\alpha)\prime}}{\zeta^{(Np^\alpha)}}(2) \right].$$

We give estimates for both residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ in Lemma 3.2 below. To describe these briefly, let us first recall following [8, §4], [20], and [15] that there exists a constant $C = C(A_2) > 0$ depending on the size of the largest known zero-free region $[1 - A_2/\log(N), 1]$ of the symmetric square L-function $L(s, \mathrm{Sym}^2 f)$ for which we have the lower bound

$$L(1, \mathrm{Sym}^2 f) = \frac{(4\pi)^l}{\Gamma(l)} \cdot \frac{\langle f, f \rangle}{\mathrm{Vol}(\Gamma_0(N) \backslash \mathfrak{H})} \gg \log(N)^{-C}.$$

Putting this together with upper bounds that can be derived using the automorphy of $L(s, \mathrm{Sym}^2 f)$ as in [8, (1.3), Lemma 4.1], we then have

$$\log(N)^{-C} \ll L(1, \mathrm{Sym}^2 f) \ll \log(N)^3.$$

Let us also define the constant

$$\mathfrak{K}_{f,\gamma_A}(1) = \sum_{\substack{q \mid \gamma_A \\ q \geq 1, \mathrm{squarefree}}} \frac{\mu(q)}{q} \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{\mu(d)}{d} \lambda\left(\frac{r''}{d}\right).$$

Here, the iterated sum over divisors $d \mid r'' \mid \cdots \mid r \mid q$ terminates with $d = 1$, and $\mathfrak{K}_{f,1}(1) = 1$ when $\gamma_A = 1$. Note that for the principal class $A = \mathbf{1}$, taking $q_A(x,y) = q_{\mathbf{1}}(x,y)$ to be the reduced quadratic form representative with leading coefficient $\gamma_A = 1$, this is simply $\mathfrak{K}_{f,\gamma_A}(1) = 1$. Writing $\epsilon_p(1)^{-1}$ to denote the Euler factor at p of $\frac{L(1,\mathrm{Sym}^2 f)}{\zeta(2)}$ so that

$$\epsilon_p(1) = \frac{\left(1 - \frac{\lambda(p^2)}{p} + \frac{\lambda(p^2)}{p^2} - \frac{1}{p^3}\right)}{(1 - \frac{1}{p^2})} > 0, \tag{6}$$

and writing

$$\kappa_{D,N}(1) = \left(\frac{48 \cdot h_K}{\pi w^2 \sqrt{|D|}} \right) \cdot \prod_{l \mid N} \frac{1}{(1 - \frac{1}{l^2})} = \frac{4}{w} \cdot \prod_{l \mid N} \frac{1}{(1 - \frac{1}{l^2})} \cdot \frac{L(1,\eta)}{\zeta^{(N)}(2)} = \frac{4}{w} \cdot \frac{L(1,\eta)}{\zeta^{(N)}(2)} > 0 \tag{7}$$

[3]In the first part of the body of the text, we shall take $q_A(x,y)$ to be the reduced binary quadratic form representative for simplicity. This choice is not necessary, however.

to denote the number obtained from Dirichlet's class number formula $L(1,\eta) = \frac{2\pi h_K}{w\sqrt{|D|}}$ and Euler's formula $\zeta(2) = \frac{\pi^2}{6}$, we can show that for each $\alpha \geq 1$ we have the estimates

$$\log(N)^{-C} \cdot \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll \mathfrak{L}_{0,f,\gamma_A}(1) \ll \log(N)^3 \cdot \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot \mathfrak{K}_{f,\gamma_A}(1)$$

for the residual terms corresponding to ring class averages of central values. The same estimates hold with the inverse Euler factors $\epsilon_p(1)$ removed if $\alpha = 0$. We also obtain the bounds

$$\log\left(\frac{|D|^{\frac{3}{2}}p^{2\alpha}}{\gamma_A}\right) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll_{f,p,\gamma_A} \mathfrak{L}_{1,f,\gamma_A}(1) \ll_{p,D,f,\gamma_A,\varepsilon} \cdot \log\left(\frac{p^{2\alpha}}{\gamma_A}\right) \cdot N^{\varepsilon} \cdot |D|^{\varepsilon} \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1)$$

for the residual terms corresponding to the central derivative values. In particular, we have the lower bound

$$\mathfrak{L}_{1,f,\gamma_A}(1) \gg_{f,p,\gamma_A} \log\left(\frac{|D|^{\frac{3}{2}}p^{2\alpha}}{\gamma_A^{\frac{1}{2}}}\right) \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

We refer to Lemma 3.2 for details.

We also consider the following residual terms for $\beta \geq 1$, given in terms of the corresponding congruence symmetric square L-values defined for each divisor $q \mid \gamma_A$ by

(8)

$$L_q(s, \mathrm{Sym}^2 f^{(q)} \otimes \chi) = L_q^{(Np)}(2s, \eta\chi^2) \sum_{\substack{n \geq 1 \\ n \equiv 0 \bmod q}} \frac{\lambda_\chi(n^2q^{-1})}{n^s} = \sum_{\substack{m \geq 1 \\ (m, Np^\alpha)=1}} \frac{\eta\chi^2(m)}{m^{2s}} \sum_{\substack{n \geq 1 \\ m^2 n \equiv 0 \bmod q}} \frac{\lambda(n^2q^{-1})\chi(n^2q^{-1})}{n^s}$$

That is, we consider the residual terms defined by

(9)

$$\mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) = \frac{2}{w}\sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \mathrm{primitive}}} L(1, \eta\chi^2) \cdot \frac{L_q(1, \mathrm{Sym}^2 f^{(q)} \otimes \chi)}{L_q^{(Np^\alpha)}(2,\chi)}$$
$$+ \frac{2}{w}\sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \mathrm{primitive}}} \frac{\eta\chi^2(-N)\chi(\gamma_A)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \cdot L(1, \eta\overline{\chi}^2) \cdot \frac{L_q(1, \mathrm{Sym}^2 f^{(q)} \otimes \chi)}{L_q^{(Np^\alpha)}(2,\chi)},$$

or equivalently

(10)

$$\mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) = \frac{2}{w}\sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \sum_{m \geq 1} \frac{\eta(m)}{m} \left(\sum_{\substack{a \geq 1 \\ a \equiv 0 \bmod q \\ m^2\gamma_A a^2 \equiv \pm 1 \bmod p^\beta}} \frac{\lambda^{(q)}(a^2)}{a} - \frac{1}{\varphi(p)} \sum_{\substack{a \geq 1, a \equiv 0 \bmod q \\ m^2\gamma_A a^2 \equiv \pm 1 \bmod p^{\beta-1} \\ m^2\gamma_A a^2 \not\equiv \pm 1 \bmod p^\beta}} \frac{\lambda^{(q)}(a^2)}{a} \right)$$
$$+ \frac{2}{w}\sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{a \geq 1 \\ a \equiv 0 \bmod q}} \frac{\lambda^{(q)}(a^2)}{a} \mathrm{Kl}_4(\pm(m^2\gamma_A a^2 \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p^\beta).$$

Theorem 1.3. *(Theorem 5.2) Keep the setup of Theorem 1.2. We derive the following estimates for the subaverages $G^{(k)}(\alpha;x)$ and $G^{(0)}(\alpha,\beta;x)$.*

(i) *If $k=0$, then we have for any $\varepsilon > 0$ the estimate*

$$G^{(0)}(\alpha;x) = \sum_{A \in C(\alpha)^{p^x}} \left(\mathfrak{L}_{0,f,\gamma_A}(1) + O_{f,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right)$$
$$- \frac{\#C(\alpha,x-1)}{\#C^\star(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \left(\mathfrak{L}_{0,f,\gamma_A}(1) + O_{f,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right).$$

(ii) *If $k=1$, then we have for any $\varepsilon>0$ the estimate*

$$G^{(1)}(\alpha;x) = \sum_{A\in C(\alpha)^{p^x}} \left(\mathfrak{L}_{1,f,\gamma_A}(1) + O_{f,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right)$$
$$- \frac{\#C(\alpha,x-1)}{\#C^\star(\alpha,x)} \sum_{\substack{A\in C(\alpha)^{p^{x-1}} \\ A\notin C(\alpha)^{p^x}}} \left(\mathfrak{L}_{1,f,\gamma_A}(1) + O_{f,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right).$$

(iii) *Fix an integer $\beta\geq 2$. We have for each anticyclotomic exponent $\alpha\geq 1$ the estimate*

$$G^{(0)}(\alpha,\beta;x)$$
$$= \sum_{A\in C(\alpha)^{p^x}} \mathfrak{L}_{f,A}^{(\beta)}(1) + O_{f,\beta,\varepsilon}\left(\gamma_A\cdot(|D|p^{2\max(\alpha,\beta)})^{\frac{1}{4}+\varepsilon}\cdot(|D|p^{2\alpha})^{\delta_0-\frac{\theta_0}{2}+\varepsilon}\cdot|\epsilon_A|^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon}\right)$$
$$- \frac{\#C(\alpha,x-1)}{\#C^\star(\alpha,x)} \sum_{\substack{A\in C(\alpha)^{p^{x-1}} \\ A\notin C(\alpha)^{p^x}}} \mathfrak{L}_{f,A}^{(\beta)}(1) + +O_{f,\beta,\varepsilon}\left(\gamma_A\cdot(|D|p^{2\max(\alpha,\beta)})^{\frac{1}{4}+\varepsilon}\cdot(|D|p^{2\alpha})^{\delta_0-\frac{\theta_0}{2}+\varepsilon}\cdot|\epsilon_A|^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon}\right).$$

We discuss the asymptotic behaviour of these averages, particularly the residual terms, in Corollary 5.3. Roughly speaking, if we know that each class $A\in C(\alpha)^{p^x}, C(\alpha)^{p^{x-1}}\backslash C(\alpha)^{p^x}$ has a binary quadratic form representative $q_A(x,y)=\gamma_A x^2+\delta_A xy+\epsilon_A y^2$ with a large last coefficient $|\epsilon_A|$ relative to the leading coefficient γ_A, then we can deduce that the corresponding averages converge in the limit with α to the sums over residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ and $\mathfrak{L}_{f,\gamma_A}^{(\beta)}(1)$. This condition on the coefficients seems to define the limits of the current technology on spectral decompositions of shifted convolution sums that we use here. Although it remains unclear whether this condition on the coefficients can be met for the classes $A\in C(\alpha)^{p^{x-1}}\backslash C(\alpha)^{p^x}$, we do show that the sums over residues converge to nonzero constants in Corollary 5.3.

Taken together with the sequel work [44], our estimates for Theorem 1.2 go some way towards proving Conjecture 1.1. This combined approach allows us to deduce the nonvanishing of the Galois averages described in Theorem 5.2 for $\alpha\gg 1$ sufficiently large as well. In particular, we obtain the following application via Shimura's rationality theorem [39] in the case of central values corresponding to $k=0$ here, together with the central derivative value formulae of Gross-Zagier [17], Zhang [50], and Yuan-Zhang-Zhang [49] in the case of central derivative values corresponding to our case of $k=1$.

Theorem 1.4. *Let us retain the setup of Theorem 1.2 above. Let $\alpha\gg 1$ be a sufficiently large integer, and write $x=\mathrm{ord}_p(\#C(\alpha))$ again to denote the exponent of p in the cardinality of the class group of the order $\mathcal{O}_{p^\alpha}$.*

(i) *For each sufficiently large integer $\alpha\gg 1$, there exists a primitive ring class character ρ of conductor p^α for which the corresponding Galois averages $G_{[\rho]}^{(k)}$ (for either $k=0,1$) do not vanish (cf.* [47], [11], [37]*).*

(ii) *Fix a cyclotomic exponent $\beta\geq 1$. For each sufficiently large anticyclotomic exponent $\alpha\gg\beta$, there exists a primitive ring class character ρ of conductor p^α and a primitive even Dirichlet character $\chi \bmod p^\beta$ for which the corresponding Galois average $G_{[\rho\chi\circ\mathbf{N}]}^{(0)}$ does not vanish. Hence by Shimura's rationality theorem, for each primitive Dirichlet character $\chi \bmod p^\beta$, and for each primitive ring class character ρ of conductor p^α the central value $L(1/2,f\times\rho\chi\circ\mathbf{N})$ does not vanish (cf.* [38]*).*

Proof. We deduce the result from Theorem 1.2. For (i), the claim follows from Theorem 1.2 (i) for $k=0$ and (ii) for $k=1$. This is because for each fixed $\alpha\gg 1$, we show that $L(1/2,f\times\rho)\neq 0$ for some primitive ring class character ρ of conductor p^α. We can then use Shimura's algebraicity theorem, in the style of the arguments of Rohrlich [38], [37], to deduce that each summand in the average $G_{[\rho]}^{(k)}$ cannot vanish. For (ii), we argue in the same way using Theorem 1.2 (iii). This result shows that for each sufficiently large ring class exponent $\alpha\gg\beta$ there exist both a primitive ring class character ρ of conductor p^α and a primitive even Dirichlet character $\chi \bmod p^\beta$ for which the central value $L(1/2,f\times\rho\chi\circ\mathbf{N})$ does not vanish. Using Shimura's algebraicity theorem again, we deduce that the corresponding Galois average $G_{[\rho\chi\circ\mathbf{N}]}^{(0)}$ cannot vanish. We show the following stronger result in the proof of Theorem 4.6 (and more generally in [45]): For each primitive even Dirichlet character χ mod p^{β}, there exists for each $\alpha \gg \beta$ a primitive ring class character ρ of conductor p^{α} for which the corresponding central value $L(1/2, f \times \rho\chi \circ \mathbf{N})$ does not vanish, and hence for which $G^{(0)}_{[\rho\chi\circ\mathbf{N}]}$ does not vanish. ☐

As well, the approach with spectral decompositions of shifted convolution sums can be developed to estimate averages over ring class characters of Rankin-Selberg L-functions associated to cuspidal automorphic representations of GL_2 over a totally real number field, and this is taken up in the sequel [45]. Finally, we also explain the applications of these estimates to Iwasawa main conjectures and Mordell-Weil ranks in the sequel note [44], which can be viewed as an appendix to this work.

1.1. **Outline of proof.** We now give a high-level sketch of the proof of Theorem 1.2 for the expert reader. Fix an integer $\alpha \geq 0$. Let us also fix a class $A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$. Given an integer $n \geq 1$, let $r_A(n)$ denote the corresponding counting function for the number of proper ideals in A of norm n. Note that for any quadratic form class representative $q_A(x, y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ corresponding to A, we can parametrize this function (non-uniquely) as

$$r_A(n) = \frac{1}{w} \cdot \# \{a, b \in \mathbf{Z} : q_A(a, b) = n\},$$

where $w = w_k$ denotes the number of automorphs of $q_A(x, y)$. We shall also consider the corresponding binary theta series θ_{q_A} defined on $z \in \mathfrak{H}$ by

$$\theta_{q_A}(z) = \sum_{n \geq 0} r_A(n) e(nz) = \frac{1}{w} \sum_{a,b \in \mathbf{Z}} e(q_A(a, b)z).$$

We know classically that this theta series determines a holomorphic modular form of weight 1, level $|D|p^{2\alpha}$, and character $\eta = \eta_D$. Moreover, the sum over classes $A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ gives the inverse Mellin transform of the Dedekind zeta function of the ring class field $K[p^{\alpha}]$ of conductor p^{α} over K. In particular, the level of θ_{q_A} is the absolute discriminant $|\operatorname{disc}(\mathcal{O}_{p^{\alpha}})| = |D|p^{2\alpha}$ of the $\mathbf{Z}$-order $\mathcal{O}_{p^{\alpha}} = \mathbf{Z} + p^{\alpha}\mathcal{O}_K$. We consider the following sums associated to such a class $A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$, for any choice of real parameter $Z > 0$. In the self-dual cases of Theorem 1.2 (i) and (ii), with A corresponding to the principal class of conductor p^{α}, it will suffice for any $Z > 0$ to estimate the more general sums

$$H_A^{(k)}(\alpha, 0) = H_{A,1}^{(k)}(\alpha, 0; Z) + H_{A,2}^{(k)}(\alpha, 0; Z)$$

with

$$H_{A,1}^{(k)}(\alpha, 0; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n, p^{\alpha})=1}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right)$$

and

$$H_{A,2}^{(k)}(\alpha, 0; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n, p)=1}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\alpha}}\right).$$

In the generic case of Theorem 1.2 (iii) with $\beta \geq 2$, it will suffice for any $Z > 0$ to estimate the sums

$$H_A^{(0)}(\alpha, \beta) = H_{A,1}^{(0)}(\alpha, \beta; Z) + H_{A,2}^{(0)}(\alpha, \beta; Z)$$

with

$$\begin{aligned} H_{A,1}^{(k)}(\alpha, \beta; Z) = &\sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1, (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^{\beta}}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(Z m^2 n\right) \\ &- \frac{1}{\varphi(p)} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1, (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^{\beta-1} \\ m^2 n \not\equiv \pm 1 \bmod p^{\beta}}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(Z m^2 n\right) \end{aligned}$$

and

$$
\begin{aligned}
&H_{A,2}^{(k)}(\alpha,\beta;Z)\\
&=\frac{(-1)^{k+1}\eta(N)}{(|D|p^{\beta})^{\frac{1}{2}}}\frac{p}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right)\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^{\beta}).
\end{aligned}
$$

Here, the cutoff functions $V_{k+1}(y)$ defined on $y\in\mathbf{R}_{>0}$ come from our choice of approximate functional equation (Lemma 2.2), and are smooth and rapidly decaying. As well, $\mathrm{Kl}_4(c,p^{\beta})$ denotes the standard hyper-Kloosterman sum of dimension 4 and modulus p^{β} evaluated at a coprime residue class $c \bmod p^{\beta}$, with $\mathrm{Kl}_4(\pm c,p^{\beta})=\mathrm{Kl}_4(c,p^{\beta})+\mathrm{Kl}_4(-c,p^{\beta})$ (see (22) and (23) below). Given a coprime residue class $c \bmod p^{\beta}$, we also write $\overline{c} \bmod p^{\beta}$ to denote the inverse class, and $c^{\frac{1}{2}}$ a square root of $c \bmod p^{\beta}$ when such a root exists. Let us for each class $A\in\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ take $q_A(x,y)=\gamma_Ax^2+\delta_Axy+\epsilon_Ay^2$ to be the unique reduced[4] positive definite quadratic form representing the class. Hence, $\gamma_A\geq 1$ is a positive integer, $|\delta_A|\leq\gamma_A\leq\epsilon_A$, and $\delta_A^2-4\gamma_A\epsilon_A=Dp^{2\alpha}$. Let us then for each divisor $q\mid\gamma_A$ write $f^{(q)}$ to denote the eigenform defined on $z=x+iy\in\mathfrak{H}$ by $f^{(q)}(z)=f(q^{-1}z)$, with $\lambda^{(q)}$ its corresponding Hecke eigenvalues. We can then consider the corresponding partial symmetric square L-function $L(s,\mathrm{Sym}^2 f^{(q)})$, defined first for $\Re(s)>1$ by the Dirichlet series expansion

$$L(s,\mathrm{Sym}^2 f^{(q)})=\zeta^{(N)}(2s)\sum_{n\geq 1}\lambda^{(q)}(n^2)n^{-s}=\sum_{\substack{m\geq 1\\(m,N=1)}}m^{-2s}\sum_{n\geq 1}\lambda(n^2q^{-1})n^{-s}.$$

Again, we also define the congruence series

$$L_q(s,\mathrm{Sym}^2 f^{(q)})=\zeta_q^{(N)}(2s)\sum_{\substack{n\geq 1\\n\equiv 0 \bmod q}}\lambda^{(q)}(n^2)n^{-s}=\sum_{\substack{m\geq 1\\(m,N)=1}}m^{-2s}\sum_{\substack{n\geq 1\\m^2n\equiv 0 \bmod q}}\lambda(n^2q^{-1})n^{-s}.$$

We use the same notations as above to denote sums over integers coprime to a given integer M, and write μ to denote the Möbius function. In the self-dual cases of Theorem 1.2 (i) and (ii) with $\beta=0$, we develop a relatively standard approach to estimating the sums $H_{A,j}^{(k)}(\alpha,0;Z)$ for a balanced choice of parameter $Z=(N|D|p^{2\alpha})^{-1}$, i.e. so that each sum has a length equal to the square root of the conductor. This gives us the simpler expression

$$
\begin{aligned}
H_A^{(k)}(\alpha,0)&=2\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{n\geq 1}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{N|D|p^{2\alpha}}\right)\\
&=2\sum_{m\geq 1}\frac{\eta(m)}{m}\frac{1}{w}\sum_{a,b\in\mathbf{Z}}\frac{\lambda(\gamma_Aa^2+\delta_Aab+\epsilon_Ab^2)}{(\gamma_Aa^2+\delta_Aab+\epsilon_Ab^2)^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2(\gamma_Aa^2+\delta_Aab+\epsilon_Ab^2)}{N|D|p^{2\alpha}}\right).
\end{aligned}
$$

In this approach, the contributions from $b=0$ terms are estimated separately, and by a contour argument (Lemma 3.1) approximated in terms of the residual sums $\mathfrak{L}_{k,f,\gamma_A}(1)$ defined above by

$$\mathfrak{L}_{k,f,\gamma_A}(1)+O_{\varepsilon}\left(|D|^{\frac{3}{16}+\varepsilon}(|D|p^{2\alpha}\gamma_A^{-1})^{-\frac{1}{8}}\right)+O\left((|D|p^{2\alpha})^{-\frac{1}{8}}\right).$$

Here, we can assume without loss of generality that $\gamma_A<|D|p^{2\alpha}$, since otherwise the contributions can be estimated trivially. Now, each remaining contribution from $b\neq 0$ in the region of moderate decay for V_{k+1} can then be bounded using a standard application of the shifted convolution problem. To be more precise, using a standard dyadic subdivision argument to reduce to the setting where V_{k+1} is a smooth and compactly supported function (cf. [4, §5.1], [3, §2.9] or [42, §1.2]), assuming that $\delta_A=0$, and writing $D_A=(p^{2\alpha}D-\delta_A^2)/4=-\gamma_A\epsilon_A$ for simplicity, each a-sum corresponding to each remaining $b\neq 0$ contribution can be described essentially as the Fourier coefficient at b^2D_A of some genuine automorphic form Φ on the two-fold metaplectic cover of $\mathrm{GL}_2(\mathbf{A})$ corresponding to a modular form of half-integral weight. Although this automorphic form Φ is not K-finite, it has convergent Sobolev norm, and we may therefore decompose

[4]As noted above, this choice is arbitrary. In the proof of Corollary 5.3, we make a distinct choice for the binary quadratic form representative for each class $A\in C(\alpha)^{p^x}$.

it spectrally to derive our stated bounds. To be more precise, we obtain for any $\varepsilon > 0$ via [42, Theorem 1] the organic estimate

$$\frac{4}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{b\geq 1\\ mb\leq\left(\frac{N|D|p^{2\alpha}}{D_A}\right)^{\frac{1}{2}}}}\frac{\lambda(a^2-b^2D_A)}{(a^2-b^2D_A)^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2(a^2-b^2D_A)}{N|D|p^{2\alpha}}\right)\ll_{f,k,\varepsilon}\left(|D|p^{2\alpha}\right)^{\frac{1}{4}+\delta_0+\varepsilon}(|D|p^{2\alpha})^{-\frac{1}{2}},$$

where $0 < \delta_0 < 1/4$ denotes the exponent in the best approximation for the size of Fourier coefficients of half-integral weight forms, which by the theorem of Kohnen-Zagier [25] is equivalent to the best exponent approximation towards the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms in the level aspect (with $\delta_0 = 0$ conjectured in either interpretation). Hence, the admissible exponent of $\delta_0 = 3/16$ shown by the theorem of Blomer-Harcos [2] gives us the effective upper bound

$$O_{f,k,\varepsilon}\left((|D|p^{2\alpha})^{-\frac{1}{16}+\varepsilon}\right).$$

This bound depends on the level of f and the choice of cutoff function V_{k+1}. In fact, for the general case on the quadratic form representative, we prove a stronger version of this bound by generalizing the theorem of Blomer [1] to this setting, using decompositions into Poincaré series to derive a distinct integral presentation for each of the shifted convolution sums with direct appear to the metaplectic theta series. We refer to Theorem 3.3, where we take for granted the general key bound (50) for Fourier coefficients of metaplectic forms shown in [42], deriving exact integral presentations for our sums to reduce to this estimate. Let us also remark that most of the proof of Theorem 1.2 works more generally for any Maass form.

A similar approach can be taken in the non self-dual case after deriving an explicit balanced approximate functional equation formula for the average. The leading sum is estimated in the same way, with the $b = 0$ contributions giving a twisted linear combination of symmetric square L-functions at $s = 1$, and the $b \neq 0$ terms estimated by a distinct application of the spectral decompositions of shifted convolution sums. Here, more care needs to be taken to ensure the nonvanishing of the residual term, and our estimates need to be interpreted as requiring the cyclotomic exponent β to be fixed, with ring class exponent $\alpha \gg \beta$ sufficiently large. A similar analysis applies to the twisted sum, which after unraveling is seen by inspection to correspond to the "contragredient" for the leading sum. Finally, although we do not develop the idea here, this approach does not require taking an average over primitive even Dirichlet characters $\chi \bmod p^{\beta}$. Indeed, we could fix a nontrivial nebentype or central character χ associated to the underlying eigenform f_{χ}, take an average over primitive ring class twists, then proceed in the same way as outlined above. Again, taking care with the residual terms coming from the $b = 0$ contributions, a similar estimate can be established in this way with $\chi \bmod p^{\beta}$ fixed and $\alpha \gg \beta$ varying. We refer to the more general sequel work [45] for more details.

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Notations. Throughout, we fix an imaginary quadratic field K of discriminant $D < 0$ and ring of integers $\mathcal{O}_K$. Given an integer $\alpha \geq 0$, we write $\mathcal{O}_{p^{\alpha}} = \mathbf{Z} + p^{\alpha}\mathcal{O}_K$ to denote the $\mathbf{Z}$-order of conductor p^{α} in K, with $\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ its corresponding class group. We write ρ to denote a ring class character of K of some prime-power conductor p^{α} for any integer $\alpha \geq 0$. Hence ρ can be viewed as a character of $\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$. We write

$$\sum_{\substack{\rho\in\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee}\\ \text{primitive}}}$$

to denote the sum over all such primitive ring class characters ρ of $\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$. Hence, such characters factor through $\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ but not $\mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$ in the even that $\alpha \geq 1$. We shall take for granted the well-known classical fact that ring class characters of K correspond to wide ray class characters of K, or equivalently to idele class characters of K having trivial archimedean component. On the cyclotomic side, we write χ to denote a primitive Dirichlet character of conductor p^β for $\beta \geq 1$ an integer, with $\varphi^\star(p^\beta)$ the number of such characters (hence $\varphi^\star(p^\beta) = (p-1)^2 p^{\beta - 2}$ if $\beta \geq 2$). We also write

$$\sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}}$$

to denote the sum over primitive even Dirichlet characters $\chi \bmod p^\beta$. Let us remark that we must in fact take such a subaverage here, as the primitive even Dirichlet characters correspond to the wide ray class characters or idele class characters of $\mathbf{Q}$ having trivial archimedean component. In particular, the corresponding archimedean local factors of the completed L-functions $\Lambda(s, f \times \rho\chi \circ \mathbf{N})$ we consider below then do not depend on the choice of Dirichlet character χ (or ring class character ρ). This is an important simplifying assumption for our approximate functional equation arguments throughout.

2. Some background

We now give some background on the Rankin-Selberg L-functions $L(s, f \times \mathcal{W}) = L(s, f \times \theta(\mathcal{W}))$. Fix a prime p which is coprime to the product DN, and let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ denote a Hecke character of the form described above, with ρ a ring class character of K of conductor p^α for some integer $\alpha \geq 0$, and χ a primitive even Dirichlet character of conductor p^β for some integer $\beta \geq 0$. Note that ρ denotes a class group character if $\alpha = 0$, and $\beta = 0$ in our notations means that there is no Dirichlet character. A classical construction due to Hecke (again see [18], cf. [17, (5.2)]) associates to any such character $\mathcal{W}$ a theta series $\theta(\mathcal{W})$ of weight one, level $|D| p^{2 \max(\alpha, \beta)}$, and nebentype character $\mathcal{W}|_{\mathbf{Q}}^\times = \eta\chi^2$, and we shall henceforth take this for granted.

2.1. **Dirichlet series expansions.**

We consider the Rankin-Selberg L-function $L(s, f \times \mathcal{W})$ of f times $\theta(\mathcal{W})$ in the setup described above, which for $\Re(s) > 1$ has the Dirichlet series expansion

$$L(s, f \times \mathcal{W}) = L(2s, \eta\chi^2) \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \mathcal{W}(\mathfrak{a}) \lambda(\mathbf{N}\mathfrak{a}) \mathbf{N}\mathfrak{a}^{-s}. \tag{11}$$

Here, the sum runs over nonzero integral ideals $\mathfrak{a} \subset \mathcal{O}_K$, with the convention that $\mathcal{W}(\mathfrak{a}) = 0$ for ideals $\mathfrak{a}$ which are not coprime to the conductor $c(\mathcal{W})$, and $L(s, \eta\chi^2)$ denotes the Dirichlet L-function of $\eta\chi^2$. Since ρ is a ring class character of conductor p^α, we can identify ρ as a character of the class group $\mathrm{Pic}(\mathcal{O}_{p^\alpha})$ of the $\mathbf{Z}$-order $\mathcal{O}_{p^\alpha} = \mathbf{Z} + p^\alpha \mathcal{O}_K$ of conductor p^α in K.

Given an integer $n \geq 1$ and a class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$, let $r_A(n)$ denote the number of ideals of norm n in A. Expanding out the second sum in (11) with respect to these counting functions, we have the well-known equivalent Dirichlet series expansion over integers

$$L(s, f \times \mathcal{W}) = \sum_{m \geq 1} \frac{\eta\chi^2(m)}{m^{2s}} \sum_{n \geq 1} \left(\sum_A \rho(A) r_A(n) \right) \frac{\lambda(n)\chi(n)}{n^s}, \tag{12}$$

where the third sum runs over all classes $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ of the order $\mathcal{O}_{p^\alpha}$. Note that to justify the equivalence of (11) and (12) properly, one can compare Euler factors in each of the corresponding completed L-functions (the former coming from an L-function of degree 2 over K, and the latter an L-function of degree 4 over $\mathbf{Q}$).

2.1.1. *Counting functions and binary quadratic forms.*

Let us now consider the counting functions $r_A(n)$ for class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ with $\alpha \geq 0$, and in particular the following (non-unique) parametrizations we shall use. Fixing a quadratic form class representative $q_A(x, y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$, and noting that the proper integral ideal with $\mathbf{Z}$-basis given by $[\gamma_A, (-\delta_A + \sqrt{Dp^{2\alpha}})/2]$ is a representative for the corresponding class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ (see [12, Theorem 7.7]), the counting function $r_A(n)$ can be parametrized as

$$r_A(n) = \frac{1}{w} \cdot \# \left\{ a, b \in \mathbf{Z} : q_A(a, b) = n \right\}. \tag{13}$$

Here again, w denotes the number of automorphs of $q_A(x, y)$, which is the same as the number of roots of unity in K. We shall often take $q_A(x, y)$ to be the unique reduced class representative, so with $|\delta_A| \leq \gamma_A \leq \epsilon_A$, and with $\delta_A \geq 0$ if either $|\delta_A| = \gamma_A$ or $\gamma_A = \epsilon_A$. Let us also note that in the special case where $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ is the principal class, we then know that $\gamma_A = 1$. In any case, we have the relation $\delta_A^2 - 4\gamma_A \epsilon_A = \mathrm{disc}(\mathcal{O}_{p^\alpha}) = Dp^{2\alpha}$.

We shall write $r_{\mathbf{1}}(n)$ to denote the counting function corresponding the principal class $\mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$, taking for granted that the choice of integer exponent $\alpha \geq 0$ will be clear from the context. Hence, this counting function can be parametrized via the quadratic form $q_{\mathbf{1}}$ associated to the principal class in $\mathcal{O}_{p^\alpha}$ as

$$r(n) = \frac{1}{w} \cdot \# \left\{ (a,b) \in \mathbf{Z}^2 : \; q_{\mathbf{1}}(a,b) = n \right\}.$$

Here again, w denotes the number of automorphs of q_1, equivalently the number of roots of unity in K. To be more explicit, writing $\Delta = Dp^{2\alpha} = \mathrm{disc}(\mathcal{O}_{p^\alpha})$ for to denote the discriminant, the reduced binary quadratic form $q_{\mathbf{1}}(x,y)$ associated to the principal class in $\mathrm{Pic}(\mathcal{O}_{p^\alpha})$ is given by

$$q_{\mathbf{1}}(x,y) = \begin{cases} x^2 - \frac{\Delta}{4} y^2 & \text{if } \Delta \equiv 0 \bmod 4 \\ x^2 + xy + \left(\frac{1-\Delta}{4} \right) y^2 & \text{if } \Delta \equiv 1 \bmod 4. \end{cases}$$

Note as well that each of these functions can be used to parametrize the corresponding theta series θ_A, e.g. defined with respect to any fixed $\mathcal{O}_{p^\alpha}$-ideal representative $\mathfrak{a}$ of $A^{-1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ on $z \in \mathfrak{H}$ by

$$\theta_{q_A}(z) = \sum_{n \geq 0} r_A(n) e(nz) = \frac{1}{w} \sum_{\lambda \in \mathfrak{a}} e\left(\frac{\mathbf{N}(\lambda)}{\mathbf{N}\mathfrak{a}} z \right), \quad e(z) = \exp(2\pi i z).$$

Again, this parametrization of the theta series is not unique, and the definition on the right hand side is independent of the choice of $\mathcal{O}_{p^\alpha}$-ideal representative $[\mathfrak{a}] = A^{-1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$. The definition can also be given equivalently in terms of any representative $q_A(x,y)$ of the class of positive definite binary quadratic forms of discriminant $\mathrm{disc}(\mathcal{O}_{p^\alpha}) = p^{2\alpha}D$ as

$$\theta_{q_A}(z) = \sum_{n \geq 0} r_A(n) e(nz) = \frac{1}{w} \sum_{a,b \in \mathbf{Z}} e\left(q_A(a,b) z \right).$$

In any case, θ_A determines a modular form of weight one, level $|D|p^{2\alpha}$, and character η. Hence, the level of θ_A is equal to the absolute value of the discriminant $\mathrm{disc}(\mathcal{O}_{p^\alpha}) = p^{2\alpha}D$ of the order $\mathcal{O}_{p^\alpha} = \mathbf{Z} + p^\alpha \mathcal{O}_K$. Let us also write $\theta_A = \theta_{q_A}$ in the event that we parametrize this theta series in terms of our fixed quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ for the class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$.

2.2. **Functional equations.** We shall take for granted the following important result throughout.

Theorem 2.1. *Fix f a normalized newform of weight 2, level N, and trivial character. Fix K an imaginary quadratic field of discriminant $-D$ prime to N and character $\eta = \eta_D$. Fix a prime p coprime to ND with integers $\alpha, \beta \geq 0$. Let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ be a Hecke character of K of the form described above, with ρ a primitive ring class character of conductor p^α, and χ a primitive even Dirichlet character of conductor p^β. Writing $\Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$ and $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$, let us define the archimedean local factor*

$$\begin{aligned} L_\infty(s) &= (2\pi)^{-2s} \Gamma\left(s - \frac{1-l}{2}\right) \Gamma\left(s + \frac{1+l}{2}\right) \\ &= \left(\frac{1}{2} \cdot \Gamma_{\mathbf{C}}\left(s + \frac{l-1}{2}\right) \right)^2 = \left(\frac{1}{2} \cdot \Gamma_{\mathbf{R}}\left(s + \frac{l-1}{2}\right) \Gamma_{\mathbf{R}}\left(s + \frac{l+1}{2}\right) \right)^2. \end{aligned}$$

The Rankin-Selberg L-function $L(s, f \times \mathcal{W}) = L(s, f \times \theta(\mathcal{W}))$ of f times the theta series $\theta(\mathcal{W}) \in M_1(|D|p^{2\max(\alpha,\beta)}, \eta\chi^2)$ has an analytic continuation to $\mathbf{C}$, and its completed L-function

$$\Lambda(s, f \times \mathcal{W}) := (N|D|p^{2\max(\alpha,\beta)})^s L_\infty(s) L(s, f \times \mathcal{W})$$

satisfies the functional equation

$$\Lambda(s, f \times \mathcal{W}) = \epsilon(1/2, f \times \mathcal{W}) \Lambda(1-s, f \times \overline{\mathcal{W}}).$$

Here, the root number $\epsilon(1/2, f \times \mathcal{W}) = \epsilon(1/2, f \times \rho\chi \circ \mathbf{N}) \in \mathbf{S}^1$ is given explicitly by the formula

$$\epsilon(1/2, f \times \mathcal{W}) = \eta\chi^2(-N) \left(\frac{\tau(\eta\chi^2)^2}{|D|p^\beta} \right)^2,$$

where

$$\tau(\eta\chi^2) = \sum_{x \bmod |D|p^\beta} \eta\chi^2(x) e\left(\frac{x}{|D|p^\beta}\right)$$

denotes the Gauss sum of the Dirichlet character $\eta\chi^2$. Hence, we have for a given Hecke character $\mathcal{W} = \rho\chi\circ\mathbf{N}$ as above the explicit functional equation

$$L_\infty(s)L(s, f\times\rho\chi\circ\mathbf{N}) = \eta\chi^2(-N)\cdot\frac{\tau(\eta\chi^2)^4}{(|D|p^\beta)^2}\cdot(N|D|p^{2\max(\alpha,\beta)})^{1-2s}L_\infty(1-s)L(1-s, f\times\rho\overline{\chi}\circ\mathbf{N}). \tag{14}$$

Proof. The result is attributed to Rankin (and Selberg) and Shimura, but the exact form we use here has only been established more recently thanks to the works of Jacquet-Langlands [23] and [22], and Li [27]. For the description of the root number in particular, see [27, Theorem 2.2, and Example 2]. Here, we have used the automorphic normalization, so that $s = 1/2$ is the point of symmetry for the functional equation, and that $\overline{f} = f$. We have also used that $\eta\chi^2$ has conductor $|D|p^\beta$, that the Hecke L-function $L(s,\mathcal{W}) = L(s,\theta(\mathcal{W}))$ has root number $\tau(\eta\chi^2)^2/(|D|p^\beta)$, and that ring class characters are equivariant under complex conjugation (cf. the discussion in [37]). ☐

2.3. **Approximate functional equations.** Fix an index $k\in\{0,1\}$, together with integers $\alpha,\beta\geq 0$. Let $\mathcal{W} = \rho\chi\circ\mathbf{N}$ be any Hecke character of K of the form described above, with ρ a primitive ring class character of conductor p^α, and χ a primitive even Dirichlet character of conductor p^β. We now use the functional equation described in Theorem 2.1 to give a contour integral description of the central (derivative) value $L^{(k)}(1/2, f\times\mathcal{W}) = L^{(k)}(1/2, f\times\rho\chi\circ\mathbf{N})$, i.e. outside of the range or absolute convergence for the corresponding Dirichlet series. Recall that this Dirichlet series expansion of $L(s, f\times\mathcal{W}) = L(s, f\times\rho\chi\circ\mathbf{N})$ is given for $\Re(s) > 1$ by the formula (12) above.

Fix $g\in\mathcal{C}_c^\infty(\mathbf{R}_{>0})$ any smooth and compactly supported function on $y\in\mathbf{R}_{>0}$ whose Mellin transform $g(s) := \int_0^\infty g(y)y^s\frac{dy}{y}$ satisfies the condition $g^*(0) = 1$. Let us for a given integer $m\geq 1$ write G_m to denote the meromorphic function defined on $s\in\mathbf{C}$ by $G_m(s) = g^*(s)s^{-m}$. Hence, $G_m(s)$ is holomorphic except at $s = 0$, where it behaves like s^{-m}. Note that this function is bounded in vertical strips. Let us also suppose additionally that $g^*(s)$ is even, so that $G_m(-s) = (-1)^m G_m(s)$ for all $s\neq 0$. Write $V_m\in\mathcal{C}^\infty$ to denote the smooth and rapidly decaying function defined on $y\in\mathbf{R}_{>0}$ by

$$V_m(y) = \int_{\Re(s)=2}\widehat{V}_m(s)y^{-s}\frac{ds}{2\pi i}, \tag{15}$$

where $\widehat{V}_m(s)$ denotes the function defined on $s\in\mathbf{C}-\{0\}$ by

$$\widehat{V}_m(s) = \frac{L_\infty(s+1/2)}{L_\infty(1/2)}G_m(s). \tag{16}$$

Again, $L_\infty(s)$ denotes the archimedean component of the completed L-function $\Lambda(s, f\times\mathcal{W}) = \Lambda(s, f\times\rho\chi\circ\mathbf{N})$ defined in Theorem 2.1 above, and this component remains invariant as we vary over wide ray class characters $\mathcal{W}$ of K (as considered here).

Lemma 2.2. *Fix $k\in\{0,1\}$, together with integers $\alpha,\beta\geq 0$. Fix $\mathcal{W} = \rho\chi\circ\mathbf{N}$ any wide ray class Hecke character of K as above, given by the product of a ring class character ρ of conductor p^α with the composition with the norm of a primitive even Dirichlet character $\chi \bmod p^\beta$. Then, for any choice of real unbalancing parameter $Z>0$, the value $L^{(k)}(1/2, f\times\mathcal{W}) = L^{(k)}(1/2, f\times\rho\chi\circ\mathbf{N})$ is given by the formula*

$$\begin{aligned}
&L(1/2, f\times\mathcal{W}) = L(1/2, f\times\rho\chi\circ\mathbf{N})\\
&= \sum_{m\geq 1}\frac{\eta\chi^2(m)}{m}\sum_{n\geq 1}\left(\sum_{A\in\operatorname{Pic}(\mathcal{O}_{p^\alpha})}\rho(A)r_A(n)\right)\frac{\lambda(n)\chi(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2 nZ\right)\\
&+(-1)^{k+1}\epsilon(1/2, f\times\rho\chi\circ\mathbf{N})\sum_{m\geq 1}\frac{\eta\overline{\chi}^2(m)}{m}\sum_{n\geq 1}\left(\sum_{A\in\operatorname{Pic}(\mathcal{O}_{p^\alpha})}\rho(A)r_A(n)\right)\frac{\lambda(n)\overline{\chi}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right).
\end{aligned}$$

Note that in the exceptional case of $k = 1$, the root number is $\epsilon(1/2, f \times \rho) = \eta(-N) = -\eta(N)$ with $\eta(N) = 1$, the formula in our setup is never identically zero. Finally, we note again that this root number $\epsilon(1/2, f \times \mathcal{W}) = \epsilon(1/2, f \times \rho\chi \circ \mathbf{N})$ can be given explicitly according to Theorem 2.1 above.

Proof. The proof is standard, see e.g. [21, §5.2] or [40, §7.2]. Let $m = k+1$, and consider the contour integral

$$\int_{\Re(s)=2} \Lambda(s+1/2, f \times \mathcal{W}) G_m(s) Z^{-s} \frac{ds}{2\pi i}.$$

Note that by Stirling's formula, the completed L-function $\Lambda(s, f \times \mathcal{W})$ decays rapidly as $\Im(s) \to \infty$, so we are justified in using Cauchy's theorem to shift contours. Shifting the contour to $\Re(s) = -2$, we cross a pole at $s = 0$ of residue

$$R := \mathrm{Res}_{s=0} \left(\Lambda(s+1/2, f \times \mathcal{W}) G_m(s) Z^{-s} \right) = \Lambda(1/2, f \times \mathcal{W}) = (N|D|p^{2\max(\alpha,\beta)})^{\frac{1}{2}} L_\infty(1/2) L(1/2, f \times \mathcal{W}),$$

which gives us the identification

$$\int_{\Re(s)=2} \Lambda(s+1/2, f \times \mathcal{W}) G_m(s) Z^{-s} \frac{ds}{2\pi i} = R + \int_{\Re(s)=-2} \Lambda(s+1/2, f \times \mathcal{W}) G_m(s) Z^{-s} \frac{ds}{2\pi i}. \tag{17}$$

Now, the integral on the right hand side of (17) is the same as

$$\int_{\Re(s)=2} \Lambda(-s+1/2, f \times \mathcal{W}) G_m(-s) Z^{s} \frac{ds}{2\pi i}.$$

Applying the functional equation (2.1) to $\Lambda(-s+1/2, f \times \mathcal{W})$ in this latter expression, and using that $G_m(-s) = (-1)^m G_m(s)$, we then obtain

$$-(-1)^m \epsilon(1/2, f \times \mathcal{W}) \int_{\Re(s)=2} \Lambda(s+1/2, f \times \overline{\mathcal{W}}) G_m(s) Z^{s} \frac{ds}{2\pi i}.$$

Expanding out the absolutely convergent Dirichlet series, it is easy to see that (17) is equivalent to

$$\begin{aligned} R = (N|D|p^{2\max(\alpha,\beta)})^{\frac{1}{2}} L_\infty(1/2) \Bigg(&\sum_{m \geq 1} \frac{\eta\chi^2(m)}{m} \sum_{n \geq 1} \Bigg(\sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})} \rho(A) r_A(n) \Bigg) \frac{\lambda(n)\chi(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right) \\ &+ (-1)^m \epsilon(1/2, f \times \rho\chi \circ \mathbf{N}) \sum_{m \geq 1} \frac{\eta\overline{\chi}^2(m)}{m} \sum_{n \geq 1} \Bigg(\sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})} \rho(A) r_A(n) \Bigg) \frac{\lambda(n)\overline{\chi}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\max(\alpha,\beta)}} \right) \Bigg). \end{aligned}$$

Dividing out by $(N|D|p^{2\max(\alpha,\beta)})^{\frac{1}{2}} L_\infty(1/2)$ on each side, we then obtain the stated formula. □

It is also easy to see that the cutoff functions appearing in the formula (2.2) decay rapidly as follows:

Lemma 2.3. *Let $k \in \{0, 1\}$ be an integer. Then for each integer $j \geq 0$, we have the estimates*

$$V_{k+1}^{(j)}(y) = \begin{cases} F_k^{(j)}(y) + O_j(y^{\frac{1}{2}-j}) & \text{for } 0 < y \leq 1, \text{ where } F_k(y) := (-1)^k (\log y)^k \\ O_{C,j}(y^{-C}) & \text{for } y \geq 1, \text{ for any choice of constant } C > 0. \end{cases}$$

Here, given a function F of $y \in \mathbf{R}_{>0}$ and an integer $j \geq 0$, we write $F^{(j)}(y)$ to denote the j-th derivative.

Proof. The result follows from a standard contour argument; see [40, Lemma 7.1] or [21, Proposition 5.4]. To estimate the behaviour as $y \to 0$ for the first estimate(s), we move the line of integration in (15) to the left, crossing a pole at $s = 0$ of residue

$$\mathrm{Res}_{s=0}\left(\widehat{V}_{k+1}(y) y^{-s} \right) = \lim_{s \to 0} \frac{1}{k!} \frac{d^k}{ds^k} \left(\pi L_\infty(s) y^{-s} \right).$$

Note that $\frac{d^k}{ds^k} y^{-s} = (-1)^k y^{-s} (\log y)^k$. Using Stirling's formula to estimate the remaining integral, we derive the stated bound(s). To estimate the behaviour of as $y \to \infty$, we move the line of integration right to $\Re(s) = C$ to obtain the second estimate(s). □

2.4. **Derivation of formulae.** Fix integers $\alpha, \beta \geq 0$. Let us now consider a (wide ray class) Hecke character $\mathcal{W} = \rho\chi \circ \mathbf{N}$ of K of the form described in (1) above, with ρ a primitive ring class character of some conductor p^α, and χ a primitive even Dirichlet character of some conductor p^β. Note that if $\alpha = 0$, then ρ denotes a character of the ideal class group $\mathrm{Pic}(\mathcal{O}_0) = \mathrm{Pic}(\mathcal{O}_K)$. Note as well that if $\beta = 0$, then our convention is that the Hecke character $\mathcal{W} = \rho$ is a ring class character with no cyclotomic component. Now recall that for either choice of index $k \in \{0, 1\}$ parametrizing the generic root number, we define the corresponding average

$$\mathcal{H}^{(k)}(\alpha, \beta) = \frac{1}{\# C^\star(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N}),$$

where $\# C^\star(\alpha) = \# C(\alpha) - \# C(\alpha - 1)$ is the number of primitive ring class characters of conductor p^α, and

$$\varphi^\star(p^\beta) = p^\beta \prod_{p || p^\beta} \left(1 - \frac{2}{p}\right) \prod_{p^2 | p^\beta} \left(1 - \frac{1}{p}\right)^2,$$

the number of primitive Dirichlet characters $\chi \bmod p^\beta$, i.e. where the factor of $(1 - 2/p)$ is omitted if $\beta \geq 2$ (as we shall usually assume). We now use the formula of Lemma 2.2 to derive formulae for these averages in Proposition 2.6. Given a primitive ring class character ρ of some conductor p^α, we also compute the average

$$D(\rho, \beta) = \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive } \chi(-1)=1}} L(1/2, f \times \rho\chi \circ \mathbf{N})$$

in Proposition 2.7.

2.4.1. *Calculations via orthogonality.* We start with the average $\mathcal{H}^{(k)}(\alpha, \beta)$, which by Lemma 2.2 above can be described for any choice of unbalancing parameter $Z > 0$ by the preliminary formula

$$\frac{1}{\# C^\star(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N}) = \mathcal{H}_1^{(k)}(\alpha, \beta; Z) + \mathcal{H}_2^{(k)}(\alpha, \beta; Z),$$

where we define

$$\begin{aligned}\mathcal{H}_1^{(k)}(\alpha, \beta; Z) &:= \frac{1}{\# C^\star(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \\ &\times \sum_{m \geq 1} \frac{\eta\chi^2(m)}{m} \sum_{\substack{n \geq 1 \\ (n, p^\alpha)=1}} \left(\sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})} \rho(A) r_A(n) \right) \frac{\lambda(n)\chi(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_2^{(k)}(\alpha, \beta; Z) &:= (-1)^{k+1} \frac{1}{\# C^\star(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \\ &\times \epsilon(1/2, f \times \rho\chi \circ \mathbf{N}) \sum_{m \geq 1} \frac{\eta\overline{\chi}^2(m)}{m} \sum_{\substack{n \geq 1 \\ (n, p^\alpha)=1}} \left(\sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})} \rho(A) r_A(n) \right) \frac{\lambda(n)\overline{\chi}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\max(\alpha,\beta)}} \right).\end{aligned}$$

Let us first consider the average over primitive ring class characters ρ of conductor p^α, this being equivalent to the average over characters of $\mathrm{Pic}(\mathcal{O}_{p^\alpha})$ which do not factor through the class group $\mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$. In the special case where $\alpha = 0$, we simply take the average over characters in the class group of the maximal order $\mathcal{O}_K$, and all of the discussion about the differences arising from characters in $\mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$ can be disregarded. In the general case, applying the inclusion-exclusion principle to the basic orthogonality relation

$$\sum_{\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee} \rho(A) = \begin{cases} \# \mathrm{Pic}(\mathcal{O}_{p^\alpha}) & \text{if } A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha}) \text{ is the principal class} \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the following orthogonality relation for primitive ring class characters of conductor p^{α}. Let us write $j : \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \rightarrow \mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$ to denote the natural surjective morphism, with $X(\alpha)$ its kernel. Applying inclusion exclusion (or Möbius inversion) then gives us the relation

$$\sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee} \\ \text{primitive}}} \rho(A) = \sum_{\rho \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee}} \rho(A) - \sum_{\rho' \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})^{\vee}} \rho'(j(A))$$

$$= \begin{cases} \#C(\alpha) & \text{if } A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \text{ is principal} \\ 0 & \text{otherwise} \end{cases} - \begin{cases} \#C(\alpha-1) & \text{if } A \in X(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \#C(\alpha) - \#C(\alpha-1) & \text{if } A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \text{ is principal} \\ -\#C(\alpha-1) & \text{if } A \in X(\alpha) \text{ but } A \neq \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \text{ is not the principal class of conductor } p^{\alpha} \\ 0 & \text{otherwise} \end{cases},$$

so that

$$\frac{1}{\#C^{\star}(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee} \\ \text{primitive}}} \rho(A) = \begin{cases} 1 & \text{if } A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \text{ is principal} \\ -\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)} & \text{if } A \in X(\alpha) \text{ but } A \neq \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \\ 0 & \text{otherwise} \end{cases}, \tag{18}$$

Note that the sum or union over all classes $A \in X(\alpha) = \ker(j : \mathrm{Pic}(\mathcal{O}_{p^{\alpha}}) \rightarrow \mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}}))$ can be identified with the principal class in $\mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$. Writing $r_{\mathbf{1}}^{\star} = r_{\mathbf{1},p^{\alpha-1}}$ to denote the counting function for the principal class in $\mathrm{Pic}(\mathcal{O}_{p^{\alpha-1}})$, i.e. so that $r_{\mathbf{1}}^{\star}(n)$ for any integer $n \geq 1$ denotes the number of ideals in the principal class of $\mathcal{O}_{p^{\alpha-1}}$ of norm n, we deduce from (19) that we have the following relation for each integer $n \geq 1$:

$$\begin{aligned} &\frac{1}{\#C^{\star}(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee} \\ \text{primitive}}} \sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})} \rho(A) r_A(n) = \frac{1}{\#C^{\star}(\alpha)} \sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})} r_A(n) \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})^{\vee} \\ \text{primitive}}} \rho(A) \\ &= \left(1 - \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) r_{\mathbf{1}}(n) - \left(\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) r_{\mathbf{1}}^{\star}(n). \end{aligned} \tag{19}$$

Here again, $r_{\mathbf{1}}(n) = r_{\mathbf{1},p^{\alpha}}(n)$ denotes the function counting the number of ideals in the principal class $\mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ having norm equal to n. Switching the order of summation and using the relation (19) to evaluate the inner A-sums, we then have for each of $j = 1, 2$ the relation

$$\mathcal{H}_j^{(k)}(\alpha, \beta; Z) = \left(1 - \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) H_j^{(k)}(\alpha, \beta; Z) - \left(\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right) H_{j,\star}^{(k)}(\alpha, \beta; Z),$$

where we define[5]

$$H_1^{(k)}(\alpha, \beta; Z) = \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \text{primitive}, \chi(-1)=1}} \sum_{m \geq 1} \frac{\eta\chi^2(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p^{\alpha})=1}} \frac{\lambda(n)\chi(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right),$$

$$H_{1,\star}^{(k)}(\alpha, \beta; Z) = \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \text{primitive}, \chi(-1)=1}} \sum_{m \geq 1} \frac{\eta\chi^2(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p^{\alpha})=1}} \frac{\lambda(n)\chi(n) r_{\mathbf{1}}^{\star}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right),$$

$$\begin{aligned} H_2^{(k)}(\alpha, \beta; Z) = (-1)^{k+1} \frac{2}{\varphi^{\star}(p^{\beta})} &\sum_{\substack{\chi \bmod p^{\beta} \\ \text{primitive}, \chi(-1)=1}} \epsilon(1/2, f \times \rho\chi \circ \mathbf{N}) \sum_{m \geq 1} \frac{\eta\overline{\chi}^2(m)}{m} \\ &\times \sum_{\substack{n \geq 1 \\ (n,p^{\alpha})=1}} \frac{\lambda(n)\overline{\chi}(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\max(\alpha,\beta)}}\right), \end{aligned}$$

[5]Note that the unbalancing parameter $Z > 0$ can be chosen separately for each L-value in the average, and in particular that we may make separate choices for the leading sums $H_j^{(k)}(\alpha, \beta; Z)$ and the difference sums $H_{j,\star}^{(k)}(\alpha, \beta; Z)$.

and

$$H_{2,\star}^{(k)}(\alpha,\beta;Z) = (-1)^{k+1}\frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive},\chi(-1)=1}} \epsilon(1/2, f\times\rho\chi\circ\mathbf{N}) \sum_{m\geq 1}\frac{\eta\overline{\chi}^2(m)}{m}$$
$$\times \sum_{\substack{n\geq 1 \\ (n,p^\alpha)=1}} \frac{\lambda(n)\overline{\chi}(n) r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2p^{4\max(\alpha-1,\beta)}}\right).$$

Let us now consider the averages over primitive even Dirichlet characters $\chi \bmod p^\beta$ in these expressions. Recall that, after using orthogonality with Möbius inversion, we have for any integer $m\geq 1$ prime to p that

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1,\text{primitive}}} \chi(m) = \begin{cases} \frac{1}{2}\varphi^\star(p^\beta) & \text{if } m\equiv \pm 1 \bmod p^\beta \\ -\frac{1}{2}\varphi(p^{\beta-1}) & \text{if } m\equiv\pm 1 \bmod p^{\beta-1} \text{ but } m\not\equiv \pm 1 \bmod p^\beta \\ 0 & \text{otherwise}\end{cases} \tag{20}$$

if $\beta\geq 2$, and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1,\text{primitive}}} \chi(m) = \begin{cases} 0 & \text{if } m\equiv 0 \bmod p \\ \frac{1}{2}\varphi(p)-1 & \text{if } m\equiv\pm 1 \bmod p \\ -1 & \text{otherwise}\end{cases} \tag{21}$$

if $\beta=1$. Using these relations, the first sum over primitive even Dirichlet characters $\chi \bmod p^\beta$ is given by

$$H_1^{(k)}(\alpha,\beta;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^{\beta-1} \\ m^2n\not\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

and

$$H_{1,\star}^{(k)}(\alpha,\beta;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^{\beta-1} \\ m^2n\not\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

if $\beta\geq 2$, and by

$$H_1^{(k)}(\alpha,\beta;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\left(\frac{2}{p-3}\right)\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\not\equiv\pm 1 \bmod p}} \frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

and

$$H_{1,\star}^{(k)}(\alpha,\beta;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\left(\frac{2}{p-3}\right)\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1,(n,p^\alpha)=1 \\ m^2n\not\equiv\pm 1 \bmod p}} \frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

if $\beta=1$. To compute the twisted sums $H_2^{(k)}(\alpha,\beta;Z)$ and $H_{2,\star}^{(k)}(\alpha,\beta;Z)$ this way when $\beta\geq 1$, we shall have to compute the sum over root numbers $\epsilon(1/2, f\times\rho\chi\circ\mathbf{N})$, and hence a sum over fourth powers of the Gauss sums

$\tau(\eta\chi^2)$. We therefore first give the following more general calculation. Fix an integer $n \geq 2$. Recall for for a given integer $\beta \geq 1$ and a coprime residue class $c \bmod p^\beta$, we define the n-dimensional hyper-Kloosterman sum of modulus p^β evaluated at c by

$$\mathrm{Kl}_n(c, p^\beta) = \sum_{\substack{x_1, \ldots, x_n \bmod p^\beta \\ x_1 \cdots x_n \equiv c \bmod p^\beta}} e\left(\frac{x_1 + \ldots + x_n}{p^\beta}\right). \tag{22}$$

Here (as usual), we write $e(x) = \exp(2\pi i x)$. Let us also introduce the convenient shorthand notation

$$\mathrm{Kl}_n(\pm c, p^\beta) := \mathrm{Kl}_n(c, p^\beta) + \mathrm{Kl}_n(-c, p^\beta) = \sum_{\substack{x_1, \ldots, x_n \bmod p^\beta \\ x_1 \cdots x_n \equiv \pm c \bmod p^\beta}} e\left(\frac{x_1 + \ldots + x_n}{p^\beta}\right). \tag{23}$$

Proposition 2.4. *Assume (as we do throughout) that D is prime to p, so that $\eta\chi^2$ has conductor $|D|p^\beta$. Given a class $c \bmod p\beta$, we write $\overline{c}$ to denote its multiplicative inverse $\bmod p^\beta$, so that $c\overline{c} \equiv 1 \bmod p^\beta$. We write $c^{\frac{1}{2}}$ to denote a square root of $c \bmod p^\beta$ in the event that such a class exists. In particular, for the sums of hyper-Kloosterman sums defined in (23) above, we shall write*

$$\mathrm{Kl}_n(\pm c^{\frac{1}{2}}, p^\beta) = \begin{cases} \mathrm{Kl}_n(\pm c^{\frac{1}{2}}, p^\beta) & \text{if } c \text{ admits a square root } c^{\frac{1}{2}} \bmod p^\beta \\ 0 & \text{otherwise.} \end{cases}$$

(i) Fix an integer $n \geq 2$. If $\beta \geq 2$, then we have for each coprime residue class $c \bmod p^\beta$ summation formula

$$\frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \text{primitive}}} \chi(c)\tau(\eta\chi^2)^n = \tau(\eta)^n \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_n(\pm(\overline{c}\overline{D}^{2n})^{\frac{1}{2}}, p^\beta).$$

(ii) If $\beta \geq 2$, then the twisted sum $H_2^{(k)}(\alpha, \beta; Z)$ is given by

$$H_2^{(k)}(\alpha, \beta; Z) = (-1)^{k+1} \frac{\eta(N)}{|D|^2 p^{2\beta}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_4(\pm(m^2 n \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p^\beta)$$

and $H_{2,\star}^{(k)}(\alpha, \beta; Z)$ by

$$H_{2,\star}^{(k)}(\alpha, \beta; Z) = (-1)^{k+1} \frac{\eta(N)}{|D|^2 p^{2\beta}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{\lambda(n) r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_4(\pm(m^2 n \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p^\beta).$$

Proof. Let us start with (i). Consider the standard twisted multiplicativity relation

$$\tau(\eta\chi^2) = \eta(p^\beta)\chi^2(D)\tau(\eta)\tau(\chi^2).$$

Using this relation in our initial expression, we find that

$$\begin{aligned}
\sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(c)\tau(\eta\chi^2)^n &= \eta(p^\beta)^n \tau(\eta)^n \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(cD^{2n})\tau(\chi^2)^n \\
&= (\eta(p^\beta)\tau(\eta))^n \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \sum_{x_1, \ldots, x_n \bmod p^\beta} \chi(x_1^2 \cdots x_n^2 c D^{2n}) e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right) \\
&= \tau(\eta)^n \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \sum_{x_1, \ldots, x_n \bmod p^\beta} \chi(x_1^2 \cdots x_n^2 c D^{2n}) e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right)
\end{aligned}$$

since $\eta(p^\beta)^n = 1$ (as $\beta \geq 2$ and η is quadratic). To evaluate the inner double sum, we switch the order of summation and use the orthogonality relation (20) to obtain

$$\sum_{x_1,\dots,x_n \bmod p^\beta} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(x_1^2\cdots x_n^2 cD^{2n})$$

$$= \frac{\varphi^\star(p^\beta)}{2} \sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1^2\cdots x_n^2 cD^{2n} \equiv \pm 1 \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) - \frac{\varphi(p^{\beta-1})}{2} \sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1^2\cdots x_n^2 cD^{2n} \equiv \pm 1 \bmod p^{\beta-1} \\ x_1^2\cdots x_n^2 cD^{2n} \not\equiv \pm 1 \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right),$$

so that our initial expression is seen to be equivalent to

(24)

$$\tau(\eta)^n \left(\sum_{\substack{x_1,\cdots x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) - \frac{\varphi(p^{\beta-1})}{2} \sum_{\substack{x_1,\cdots x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) \right).$$

Let us consider the second inner sum in this latter expression, which can be decomposed as

$$\sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right)$$

$$= \sum_{x_1,\dots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}}{p^\beta}\right) \sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv \pm\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^{\beta-1} \\ x_n \not\equiv \pm\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right).$$

Observe that we can express each class $x_n \bmod p^\beta$ in the second sum as $\pm\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}} + lp^{\beta-1}$ for some integer $1 \leq l \leq p-1$, so that

$$\sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv \pm\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^{\beta-1} \\ x_n \not\equiv \pm\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right)$$

$$= \left(e\left(\frac{\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}}}{p^\beta}\right) + e\left(\frac{-\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}}}{p^\beta}\right) \right) \sum_{l=1}^{p-1} e\left(\frac{lp^{\beta-1}}{p^\beta}\right)$$

$$= -\left(e\left(\frac{\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}}}{p^\beta}\right) + e\left(\frac{-\overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}}}{p^\beta}\right) \right)$$

by the well-known identity $\sum_{1\leq l\leq p-1} e\left(\frac{l}{p}\right) = -1$. Hence, we derive the relation

$$\sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) = \sum_{x_1,\dots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1} \pm \overline{x_1\cdots x_{n-1}}(\overline{cD}^{2n})^{\frac{1}{2}}}{p^\beta}\right)$$

$$= -\sum_{\substack{x_1,\dots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right),$$

from which it follows that (24) is equivalent to the expression

$$\tau(\eta)^n \left(\frac{\varphi^\star(p^\beta)}{2} \sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{c}\overline{D}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) + \frac{\varphi(p^{\beta-1})}{2} \sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{c}\overline{D}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) \right).$$

Using that $\varphi^\star(p^\beta) = \varphi(p^\beta) - \varphi(p^{\beta-1})$, we then derive the identity

$$\sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(c)\tau(\eta\chi^2)^n = \tau(\eta)^n \left(\frac{\varphi(p^\beta)}{2}\right) \sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv \pm(\overline{c}\overline{D}^{2n})^{\frac{1}{2}} \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right).$$

Dividing out by $2/\varphi^\star(p^\beta)$ on each side then gives the stated identity.

To show (ii), we start with the expansion

$$H_2^{(k)}(\alpha,\beta;Z) = (-1)^{k+1} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \epsilon(1/2, f\times\rho\chi\circ\mathbf{N})$$
$$\times \sum_{m\geq 1} \frac{\eta\overline{\chi}^2(m)}{m} \sum_{\substack{n\geq 1 \\ (n,p^{\max(\alpha,\beta)})=1}} \frac{\lambda(n)\overline{\chi}(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}}\right),$$

which after opening up the expression for the root number and switching the order of summation equals

$$= \frac{(-1)^{k+1}}{(|D|p^\beta)^2} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{n\geq 1 \\ (n,p)=1}} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}}\right) \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(\overline{m}^2\overline{n}N^2)\tau(\eta\chi^2)^4.$$

Using (i) to evaluate the inner χ-sum for each pair of integers m, n via the identity

$$\frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \tau(\eta\chi^2)^4 \chi(\overline{m}^2\overline{n}N^2) = \tau(\eta)^4 \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_4(\pm(m^2 n\overline{N}^2)^{\frac{1}{2}}\overline{D}^4, p^\beta),$$

and using the elementary identity $\varphi^\star(p^\beta) = (p-1)^2 p^{\beta-2}$ which holds for $\beta \geq 2$, we then obtain

$$(-1)^{k+1}\left(\frac{p}{\varphi(p)}\right) \frac{\eta(N)\tau(\eta)^4}{(|D|p^\beta)^2} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{n\geq 1 \\ (n,p)=1}} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}}\right) \mathrm{Kl}_4(\pm(m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}}, p^\beta).$$

The stated identity then follows after noting that $(\tau(\eta)|D|^{-\frac{1}{2}})^4 = 1$ as η is quadratic. The calculation for the second twisted sum $H_{2,\star}^{(k)}(\alpha,\beta;Z)$ works in the same way. □

Corollary 2.5. *Assume again that the discriminant D is coprime to p so that $\eta\chi^2$ has conductor $|D|p^\beta$.*

(i) Fix an integer $n \geq 2$. If $\beta = 1$, then we have for each coprime class $c \bmod p$ the summation formula

$$\sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} \chi(c)\tau(\eta\chi^2)^n = \tau(\eta)^n \left(\left(\frac{p-3}{2}\right) \mathrm{Kl}_n(\pm(\overline{c}\overline{D}^{2n})^{\frac{1}{2}}, p) - (-1)^n\right).$$

Again, $\overline{c}^{\frac{1}{2}}$ denotes a square root of $\overline{c} \bmod p^\beta$ if it exists, otherwise there is no contribution to the sum.

(ii) If $\beta = 1$, then the sum $H_2^{(k)}(\alpha,\beta;Z)$ is given by

$$H_2^{(k)}(\alpha,\beta;Z) = (-1)^{k+1} \frac{\eta(N)}{|D|^2 p^{2\beta}} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{n\geq 1, \\ (n,p)=1}} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}}\right)$$
$$\times \left(\frac{p-3}{p-2}\right)\left(\mathrm{Kl}_4(\pm(m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}}, p) - \left(\frac{2}{p-3}\right)\right),$$

and the sum $H_{2,\star}^{(k)}(\alpha,\beta;Z)$ *by*

$$H_{2,\star}^{(k)}(\alpha,\beta;Z)=(-1)^{k+1}\frac{\eta(N)}{|D|^2p^{2\beta}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1}}\frac{\lambda(n)r_{\mathbf{1}}^{\star}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right)$$
$$\times\left(\frac{p-3}{p-2}\right)\left(\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p)-\left(\frac{2}{p-3}\right)\right).$$

Proof. The calculations are given by a minor variation of those for Proposition 2.4 above as follows. For (i), we open up the sum over primitive even Dirichlet characters χ mod p,

$$\sum_{\substack{\chi \bmod p\\ \text{primitive},\chi(-1)}}\chi(c)\tau(\eta\chi^2)^n=\tau(\eta)^n\sum_{\substack{\chi \bmod p\\ \text{primitive},\chi(-1)=1}}\chi(cD^{2n})\sum_{x_1,\cdots,x_n \bmod p}\chi^2(x_1\cdots x_n)e\left(\frac{x_1+\cdots+x_n}{p}\right)$$
$$=\sum_{x_1,\cdots,x_n \bmod p}e\left(\frac{x_1+\cdots+x_n}{p}\right)\sum_{\substack{\chi \bmod p\\ \text{primitive},\chi(-1)=1}}(x_1^2\cdots x_n^2cD^{2n}).$$

Applying the relation (21) to the inner χ-sum in latter expression, we then obtain

$$\left(\frac{\varphi(p)}{2}-1\right)\sum_{\substack{x_1,\cdots,x_n \bmod p\\ x_1^2\cdots x_n^2cD^{2n}\equiv\pm 1 \bmod p}}e\left(\frac{x_1+\cdots+x_n}{p}\right)-\sum_{\substack{x_1,\cdots,x_n \bmod p\\ x_1^2\cdots x_n^2cD^{2n}\not\equiv\pm 1 \bmod p}}e\left(\frac{x_1+\cdots+x_n}{p}\right),$$

which gives the stated formula

$$\tau(\eta)^n\left(\left(\frac{\varphi(p)}{2}-1\right)\sum_{\substack{x_1,\cdots,x_n \bmod p\\ x_1\cdots x_n\equiv\pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p}}e\left(\frac{x_1+\cdots+x_n}{p}\right)-\sum_{\substack{x_1,\cdots,x_n \bmod p\\ x_1\cdots x_n\not\equiv\pm(\overline{cD}^{2n})^{\frac{1}{2}} \bmod p}}e\left(\frac{x_1+\cdots+x_n}{p}\right)\right)$$
$$=\tau(\eta)^n\left(\left(\frac{\varphi(p)}{2}-1\right)\mathrm{Kl}_n(\pm(\overline{cD}^{2n})^{\frac{1}{2}},p)-(-1)^n\right).$$

To derive the stated formulae for (ii), we simply use this formula to compute of the corresponding χ-sums in the previous discussion (in the proof of Proposition 2.4 (ii)), with all other steps being the same. ☐

Hence, we have derived the following explicit average formula.

Proposition 2.6. *Fix integers $\alpha\geq 0$ and $\beta\geq 0$. Let $\mathcal{W}=\rho\chi\circ\mathbf{N}$ be a Hecke character of K as in (1) above, with ρ a primitive ring class character of conductor p^α, and χ a primitive even Dirichlet character of conductor p^β. We have for either choice of $k\in\{0,1\}$ the following formula for the corresponding average*

$$\mathcal{H}^{(k)}(\alpha,\beta)=\frac{1}{\#C^{\star}(\alpha)}\sum_{\substack{\rho\in\mathrm{Pic}(\mathcal{O}_{p^\alpha})^\vee\\ \text{primitive}}}\frac{2}{\varphi^{\star}(p^\beta)}\sum_{\substack{\chi \bmod p^\beta\\ \text{primitive},\chi(-1)=1}}L^{(k)}(1/2,f\times\rho\chi\circ\mathbf{N}).$$

Namely, we have for any choice of real (unbalancing) parameter $Z>0$ the formula

(25)
$$\mathcal{H}^{(k)}(\alpha,\beta)=\left(1-\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right)\left(H_1^{(k)}(\alpha,\beta;Z)+H_2^{(k)}(\alpha,\beta;Z)\right)-\left(\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right)\left(H_{1,\star}^{(k)}(\alpha,\beta;Z)+H_{2,\star}^{(k)}(\alpha,\beta;Z)\right).$$

Here (as above),

$$H_1^{(k)}(\alpha,\beta;Z)=\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm 1 \bmod p^{\beta-1}\\ m^2n\not\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

and

$$H_{1,\star}^{(k)}(\alpha,\beta;Z)=\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\equiv\pm 1 \bmod p^{\beta-1}\\m^2n\not\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

if $\beta\geq 2$, with

$$H_{1}^{(k)}(\alpha,\beta;Z)=\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\left(\frac{2}{p-3}\right)\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\not\equiv\pm 1 \bmod p}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

and

$$H_{1,\star}^{(k)}(\alpha,\beta;Z)=\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\equiv\pm 1 \bmod p^\beta}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$
$$-\left(\frac{2}{p-3}\right)\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1\\m^2n\not\equiv\pm 1 \bmod p}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

if $\beta=1$. As well, the dual sums are given by

$$H_{2}^{(k)}(\alpha,\beta;Z)=(-1)^{k+1}\frac{\eta(N)}{|D|^2p^{2\beta}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times\left(\frac{p}{\varphi(p)}\right)\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^\beta)$$

and

$$H_{2,\star}^{(k)}(\alpha,\beta;Z)=(-1)^{k+1}\frac{\eta(N)}{|D|^2p^{2\beta}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times\left(\frac{p}{\varphi(p)}\right)\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^\beta)$$

if $\beta\geq 2$, with

$$H_{2}^{(k)}(\alpha,\beta;Z)=(-1)^{k+1}\frac{\eta(N)}{|D|^2p^{2\beta}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1}}\frac{\lambda(n)r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right)$$
$$\times\left(\frac{p-3}{p-2}\right)\left(\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p)-\left(\frac{2}{p-3}\right)\right)$$

and

$$H_{2,\star}^{(k)}(\alpha,\beta;Z)=(-1)^{k+1}\frac{\eta(N)}{|D|^2p^{2\beta}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\(n,p)=1}}\frac{\lambda(n)r_{\mathbf{1}}^\star(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right)$$
$$\times\left(\frac{p-3}{p-2}\right)\left(\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p)-\left(\frac{2}{p-3}\right)\right)$$

if $\beta = 1$. If $\beta = 0$, then the hyper-Kloosterman sums $\mathrm{Kl}_4(\pm c, p^\beta)$ are trivial, and the corresponding dual sums $H_2^{(k)}(\alpha, 0)$ and $H_{2,\star}(\alpha, 0)$ can be described equivalently by the formulae for the $\alpha \geq 0$, i.e. without any congruences coming from orthogonality relations for the averages over primitive even Dirichlet characters.

Remark Note that the unbalancing parameters $Z > 0$ are chosen implicitly for each $L^{(k)}(1/2, f \times \rho\chi \circ \mathbf{N})$ in the average $\mathcal{H}^{(k)}(\alpha, \beta)$. In particular, separate choices can be made for the main sums $H_j(\alpha, \beta; Z)$ and the difference sums $H_{j,\star}(\alpha, \beta; Z)$, relative to the square root of the conductor for the underlying L-values.

An easy variation of the same calculations also gives the following explicit average formula.

Proposition 2.7. *Fix integers $\alpha \geq 0$ and $\beta \geq 1$. Fix a primitive ring class character ρ of conductor p^α. Given an integer $n \geq 1$ prime to p, let us then write*

$$c_\rho(n) = \sum_{A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})} r_A(n)\rho(A)$$

to denote the corresponding coefficient in the Dirichlet series expansion (12). The one-variable average

$$D(\rho, \beta) := \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive}, \chi(-1)=1}} L(1/2, f \times \rho\chi \circ \mathbf{N})$$

over primitive even Dirichlet characters $\chi \bmod p^\beta$ is given for any choice of real parameter $Z > 0$ by

$$D(\rho, \beta) = D_1(\rho, \beta; Z) + D_2(\rho, \beta; Z).$$

Here, the leading sum is defined by

$$D_1(\rho, \beta; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^\beta}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(m^2 n Z\right)$$
$$- \frac{1}{\varphi(p)} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^{\beta-1} \\ m^2 n \not\equiv \pm 1 \bmod p^\beta}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(m^2 n Z\right)$$

if $\beta \geq 2$, and

$$D_1(\rho, \beta; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(m^2 n Z\right)$$
$$- \frac{2}{p-3} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1 \\ m^2 n \not\equiv \pm 1 \bmod p}} \frac{r(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(m^2 n Z\right)$$

if $\beta = 1$; the second (twisted) sum is defined by

$$D_2(\rho, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^{2\beta}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(\frac{m^2 n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_4(\pm(m^2 n \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p^\beta)$$

if $\beta \geq 2$, and by

$$D_2(\rho, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^{2\beta}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1\left(\frac{m^2 n}{Z(N|D|p^{2\max(\alpha,\beta)})^2}\right)$$
$$\times \left(\frac{p-3}{p-2}\right) \left(\mathrm{Kl}_4(\pm(m^2 n \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p) - \left(\frac{2}{p-3}\right)\right)$$

if $\beta = 1$.

Proof. The proof works in the same way as for Proposition 2.6, using the the approximate functional equation (Lemma 2.2) with the orthogonality relations of (20) and (21), but without taking the average over all ring class characters of conductor p^{α}. $\square$

3. Self-dual estimates

Let us first estimate the average $H^{(k)}(\alpha, 0)$ for any $\alpha \geq 0$, taking the cyclotomic part to be trivial $\beta = 0$. Hence, we average over primitive ring class characters of conductor of a given conductor p^{α}. Note that we could fix any integer $\beta \geq 1$ here, and allow for $\alpha \geq 0$ to vary using the same method. We treat this simpler setting to keep the exposition clear, leaving the more general setting for the sequel work [46].

3.1. **Strategy.** We see by inspection of (25) that it will suffice for any choice of $Z > 0$ to estimate the sum

$$H^{(k)}(\alpha, 0) := H_1^{(k)}(\alpha, 0; Z) + H_2^{(k)}(\alpha, 0; Z),$$

with

$$H_1^{(k)}(\alpha, 0; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right)$$

and

$$H_2^{(k)}(\alpha, 0; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{\lambda(n) r_{\mathbf{1}}(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\alpha}}\right).$$

Here again, $r_{\mathbf{1}}(n)$ denotes the function that counts the number of ideals of norm n in the principal class $\mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$. We parametrize this counting function in terms of the reduced quadratic form class representative $q_{\mathbf{1}}(x, y) = \gamma_{\mathbf{1}} x^2 + \delta_{\mathbf{1}} xy + \epsilon_{\mathbf{1}} y^2$ with $\gamma_{\mathbf{1}} = 1$ as in (13) above. The difference sums $H_{j,\star}^{(k)}(\alpha, 0; Z)$ can be estimated in the same way, and we omit explicit reference to them in some of the discussion that follows. We can also consider for any class $A \in \mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ the more general sums defined by

$$H_{A,1}^{(k)}(\alpha, 0; Z) := \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(m^2 n Z\right)$$

and

$$H_{A,2}^{(k)}(\alpha, 0; Z) := \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Z N^2 |D|^2 p^{4\alpha}}\right).$$

Here, we can take any binary quadratic form representative $q_A(x, y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ corresponding to A to parametrize the counting function $r_A(n)$ in these expressions in terms of $q_A(x, y)$, as in (13) above.

To estimate these sums $H_{A,j}^{(k)}(\alpha, 0; Z)$, we shall expand the counting functions accordingly. In all cases, we first estimate the contributions from the $b = 0$ terms via a contour argument, these being the canonical residue terms for each of our self-dual estimates. The remaining contributions from $b \neq 0$ terms can be identified as Fourier coefficients of certain automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ or its metaplectic cover, then estimated via spectral decompositions. Note that to derive nonvanishing estimates with this approach, we shall require the best known approximations towards both the generalized Ramanujan conjecture for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms, as well as the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms in the level aspect. To this end, we shall use the theorems of Kim-Sarnak [24] and Blomer-Harcos [2, Theorem 2] respectively.

3.2. **Calculation of residues.** Let $Z = (N|D|p^{2\alpha})^{-1}$ so that the the approximate function equation is balanced. In particular, we then have $H_{A,1}^{(k)}(\alpha, 0; (N|D|p^{2\alpha})^{-1}) = H_{A,2}^{(k)}(\alpha, 0; (N|D|p^{2\alpha})^{-1})$. Recall that since the level N of f is assumed to be squarefree, we can define the symmetric square L-function $L(s, \mathrm{Sym}^2 f)$ of f by the Dirichlet series (5) above (first for $\Re(s) > 1$). Recall too that given an integer $M \geq 2$, we write $L^{(M)}(s, \mathrm{Sym}^2 f)$ to denote the L-function $L(s, \mathrm{Sym}^2 f)$ with the Euler factors at primes dividing M removed. Let us for each divisor q of the positive leading integer coefficient γ_A of $q_A(x, y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ write $f^{(q)}$ to denote the shift of f defined on $z = x + iy \in \mathfrak{H}$ by $f^{(q)}(z) = f(q^{-1} z)$. We then write $\lambda^{(q)}$ to denote its corresponding shift of the Hecke eigenvalue (or equivalently Fourier coefficient) by q.[6] Again, we consider the corresponding symmetric square Dirichlet series $L(s, \mathrm{Sym}^2 f^{(q)})$, defined (first for $\Re(s) > 1$) by

$$L(s, \mathrm{Sym}^2 f^{(q)}) = \zeta^{(N)}(2s) \sum_{n \geq 1} \frac{\lambda^{(q)}(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{1}{m^{2s}} \sum_{n \geq 1} \frac{\lambda(n^2 q^{-1})}{n^s}.$$

We also consider the corresponding congruence series for each divisor $q \mid \gamma_A$ defined (first for $\Re(s) > 1$) by

$$L_q(s, \mathrm{Sym}^2 f^{(q)}) = \zeta_q^{(N)}(2s) \sum_{\substack{n \geq 1 \\ n \equiv 0 \bmod q}} \frac{\lambda^{(q)}(n^2)}{n^s} = \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{1}{m^{2s}} \sum_{\substack{n \geq 1 \\ m^2 n \equiv 0 \bmod q}} \frac{\lambda(n^2 q^{-1})}{n^s}.$$

Lemma 3.1. *Let $\alpha \geq 0$ be any exponent, and $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ any class with corresponding binary quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$. We have the following estimates for the corresponding $b = 0$ contributions in sums*

$$\begin{aligned} H_A^{(k)}(\alpha, 0) &= H_{A,1}^{(k)}(\alpha, 0; (N|D|p^{2\alpha})^{-1}) + H_{A,2}^{(k)}(\alpha, 0; (N|D|p^{2\alpha})^{-1}) \\ &= 2 \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{N|D|p^{2\alpha}}\right) \\ &= 2 \sum_{\substack{m \geq 1 \\ (m,N)=1}} \frac{\eta(m)}{m} \sum_{a,b \in \mathbf{Z}} \frac{\lambda(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{N|D|p^{2\alpha}}\right), \end{aligned} \tag{26}$$

(i) *If the pair (f, ρ) is generic (so $k = 0$), then the $b = 0$ terms in (26) are estimated for any $\varepsilon > 0$ by*

$$\frac{4}{w} \cdot \sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot L(1,\eta) \cdot \frac{L_q^{(p^\alpha)}(1, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)} + O_{f,\varepsilon}\left(|D|^{\frac{3}{16}+\varepsilon}\left(\frac{\gamma_A}{|D|p^{2\alpha}}\right)^{\frac{1}{4}}\right).$$

(ii) *If the pair (f, ρ) is exceptional (so $k = 1$), then the $b = 0$ terms in (26) are estimated for any $\varepsilon > 0$ by*

$$\begin{aligned} &\frac{4}{w} \cdot \sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot L(1,\eta) \cdot \frac{L_q^{(p^\alpha)}(1, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)} \\ &\times \left[\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right) + \frac{L'}{L}(1,\eta) + \frac{L_q'^{(p^\alpha)}}{L_q^{(p^\alpha)}}(1, \mathrm{Sym}^2 f^{(q)}) - 2(\gamma + \log(2\pi)) - \frac{\zeta_q'^{(Np^\alpha)}}{\zeta_q^{(Np^\alpha)}}(2)\right] \\ &+ O_{f,\varepsilon}\left(|D|^{\frac{3}{16}+\varepsilon}\left(\frac{\gamma_A}{|D|p^{2\alpha}}\right)^{\frac{1}{4}}\right). \end{aligned}$$

Proof. In either case on $k \in \{0, 1\}$, the contribution from the $b = 0$ terms is given by the expression

$$\frac{4}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{a \geq 1 \\ (a^2, p^\alpha)=1}} \frac{\lambda(\gamma_A a^2)}{(\gamma_A a^2)^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 \gamma_A a^2}{N|D|p^{2\alpha}}\right),$$

which after using the Hecke relation

$$\lambda(\gamma_A a^2) = \sum_{q \mid \gcd(\gamma_A, a)} \mu(q) \lambda\left(\frac{\gamma_A}{q}\right) \lambda\left(\frac{a^2}{q}\right) = \sum_{q \mid \gcd(\gamma_A, a)} \mu(q) \lambda^{(q)}(\gamma_A) \lambda^{(q)}(a^2),$$

[6]Note that in terms of the cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbf{A})$ generated by f, this corresponds to taking the shift by right multiplication q of a new vector $\phi \in V_\pi$, $\phi \in V_\pi, g \in \mathrm{GL}_2(\mathbf{A}), \phi(g) \longmapsto \phi\left(g \begin{pmatrix} q^{-1} & \\ & 1 \end{pmatrix}\right) \in V_\pi$.

for each integer a in the sum is the same (after switching the order of summation) as

$$\begin{aligned}&\frac{4}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a\geq 1\\(a,p^{\alpha})=1}}\sum_{q|\gcd(\gamma_A,a)}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\cdot\frac{\lambda^{(q)}(a^2)}{(a^2)^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2\gamma_A a^2}{N|D|p^{2\alpha}}\right)\\&=\frac{4}{w}\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a\geq 1\\(a,p^{\alpha})=1\\a\equiv 0 \bmod q}}\frac{\lambda^{(q)}(a^2)}{a}V_{k+1}\left(\frac{m^2\gamma_A a^2}{N|D|p^{2\alpha}}\right),\end{aligned}$$

and which after opening up the contour defining $V_{k+1}(y)$ is seen to be given by

$$\begin{aligned}&\frac{4}{w}\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\int_{\Re(s)=2}\sum_{m\geq 1}\frac{\eta(m)}{m^{2s+1}}\sum_{\substack{a\geq 1\\(a,p^{\alpha})=1\\a\equiv 0 \bmod q}}\frac{\lambda^{(q)}(a^2)}{a^{2s+1}}\cdot\widehat{V}_{k+1}(s)\left(\frac{\gamma_A}{N|D|p^{2\alpha}}\right)^{-s}\frac{ds}{2\pi i}\\&=\frac{4}{w}\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\int_{\Re(s)=2}L(2s+1,\eta)\frac{L_q^{(p^{\alpha})}(2s+1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^{\alpha})}(4s+2)}\widehat{V}_{k+1}(s)\left(\frac{\gamma_A}{N|D|p^{2\alpha}}\right)^{-s}\frac{ds}{2\pi i}.\end{aligned}\tag{27}$$

Now, recall that we define

$$\widehat{V}_{k+1}(s)=\frac{L_{\infty}(s+1/2)}{L_{\infty}(1/2)}G_{k+1}(s),$$

where $G_{k+1}(s)=g^*(s)s^{-(k+1)}$ (with $g^*(0)=1$). Let us write

$$\mathfrak{L}_{f,\gamma_A}(s)=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\cdot L(s,\eta)\cdot\frac{L_q^{(p^{\alpha})}(s,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^{\alpha})}(2s)}.$$

Moving the line of integration leftward to $\Re(s)=-1/4$, we cross a pole at $s=0$ of residue

$$\begin{aligned}\mathfrak{L}_{k,f,\gamma_A}(1)&:=\mathrm{Res}_{s=0}\left(\mathfrak{L}_{f,\gamma_A}(2s+1)\cdot\hat{V}_k(s)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\&=\lim_{s\to 0}\frac{d^k}{ds^k}\left(s^{k+1}\cdot\mathfrak{L}_{f,\gamma_A}(2s+1)\cdot\hat{V}_k(s)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right).\end{aligned}$$

If $k=0$, then we see immediately that we get the stated residual term

$$\mathfrak{L}_{0,f,\gamma_A}(1)=\mathfrak{L}_{f,\gamma_A}(1)=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\cdot L(1,\eta)\cdot\frac{L_q^{(p^{\alpha})}(1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^{\alpha})}(2)}.$$

If $k=1$, then we follow the approach of [40, Lemma 7.2], using that the function $\widehat{V}_2(s)$ behaves as

$$\widehat{V}_2(s)=\frac{1}{s^2}-2\cdot\frac{\gamma+\log 2\pi}{s}+O(1)\ \text{ as } s\to 0.\tag{28}$$

To compute the corresponding residue $\mathfrak{L}_{1,f,\gamma_A}(1)$ using this approximation, let us first observe that

$$\mathfrak{L}_{1,f,\gamma_A}(1)=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\cdot\mathrm{Res}_{s=0}\left(L(2s+1,\eta)\cdot\frac{L_q^{(p^{\alpha})}(2s+1)}{\zeta_q^{(Np^{\alpha})}(4s+2)}\cdot\hat{V}_2(s)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right).$$

We also write

$$\mathfrak{L}_q(s)=L(s,\eta)\cdot\frac{L_q^{(p^{\alpha})}(s,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^{\alpha})}(2s)},$$

so that

$$\mathfrak{L}_{1,f,\gamma_A}(1)=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}}\cdot\mathrm{Res}_{s=0}\left(\mathfrak{L}_q(2s+1)\cdot\hat{V}_2(s)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right).$$

We can then use the asymptotic (28) to compute each of the residues appearing in the inner sum as

(29)

$$\begin{aligned}
&\mathrm{Res}_{s=0}\left(\mathfrak{L}_q(2s+1)\cdot\hat{V}_2(s)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\
&=\mathrm{Res}_{s=0}\left(\frac{1}{s^2}\cdot\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\
&-2(\gamma+\log(2\pi))\cdot\mathrm{Res}_{s=0}\left(\frac{1}{s}\cdot\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)+O(1)\cdot\mathrm{Res}_{s=0}\left(\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\
&=\lim_{s\to0}\frac{d}{ds}\left(\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\
&-2(\gamma+\log(2\pi))\cdot\lim_{s\to0}\frac{d}{ds}\left(s\cdot\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)+O(1)\cdot\lim_{s\to0}\frac{d}{ds}\left(s^2\cdot\mathfrak{L}_q(2s+1)\cdot\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)^s\right)\\
&=\mathfrak{L}_q(1)\cdot\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)-2(\gamma+\log(2\pi))\right)+\mathfrak{L}_q'(1)\\
&=L(1,\eta)\cdot\frac{L_q^{(p^\alpha)}(1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)}\cdot\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)-2(\gamma+\log(2\pi))+\frac{L'}{L}(1,\eta)+\frac{L_q^{(p^\alpha)\prime}}{L_q^{(p^\alpha)}}(1,\mathrm{Sym}^2 f^{(q)})-\frac{\zeta_q^{(Np^\alpha)\prime}}{\zeta_q^{(Np^\alpha)}}(2)\right)
\end{aligned}$$

so that we get the stated residual term

$$\begin{aligned}
\mathfrak{L}_{1,f,\gamma_A}(1)&=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac12}}\cdot\left[\mathfrak{L}_q(1)\cdot\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)-2(\gamma+\log(2\pi))\right)+\mathfrak{L}_q'(1)\right]\\
&=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac12}}\cdot L(1,\eta)\cdot\frac{L_q^{(p^\alpha)}(1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)}\\
&\times\left[\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)+\frac{L'}{L}(1,\eta)+\frac{L_q^{(p^\alpha)\prime}}{L_q^{(p^\alpha)}}(1,\mathrm{Sym}^2 f^{(q)})-2(\gamma+\log(2\pi))-\frac{\zeta_q^{(Np^\alpha)\prime}}{\zeta_q^{(Np^\alpha)}}(2)\right].
\end{aligned}$$

The remaining integral in either case $k=0,1$ is bounded above by

$$O_f\left(\left(|D|p^{2\alpha}\gamma_A^{-1}\right)^{-\frac18}\right)$$

using the Stirling approximation formula to estimate $\widehat{V}_{k+1}(s)$ as $\Im(s)\to\pm\infty$, and the Burgess subconvexity bound to estimate $L(s,\eta)$ on the line $\Re(s)=1/2$ as $\ll_\varepsilon |D|^{\frac{3}{16}+\varepsilon}$. The error terms in each case depend on the best existing subconvexity estimates for the symmetric square L-functions $L(s,\mathrm{Sym}^2 f)$ in the level aspect, or more generally the best approximations towards the generalized Lindelöf hypothesis for $\mathrm{GL}_3(\mathbf{A})$-automorphic L-functions in the level aspect; these quantities depend only on the level N of f. ☐

Lemma 3.2. *We have the following estimates for the residual sums*

$$\mathfrak{L}_{0,f,\gamma_A}(1)=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac12}}\cdot L(1,\eta)\cdot\frac{L_q^{(p^\alpha)}(1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)}$$

and

$$\begin{aligned}
\mathfrak{L}_{1,f,\gamma_A}(1)&=\frac{4}{w}\cdot\sum_{q|\gamma_A}\mu(q)\cdot\frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac12}}\cdot L(1,\eta)\cdot\frac{L_q^{(p^\alpha)}(1,\mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2)}\\
&\times\left[\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)+\frac{L'}{L}(1,\eta)+\frac{L_q^{(p^\alpha)\prime}}{L_q^{(p^\alpha)}}(1,\mathrm{Sym}^2 f^{(q)})-2(\gamma+\log(2\pi))-\frac{\zeta_q^{(Np^\alpha)\prime}}{\zeta_q^{(Np^\alpha)}}(2)\right]
\end{aligned}$$

appearing in Lemma 3.1. Each of these residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ is nonvanishing. To be more precise, there exists a constant $C=C(A_2)$ depending on the size of the known zero-free region $[1-A_2/\log(N),1]$ of the symmetric square L-function $L(s,\mathrm{Sym}^2 f)$ for which $L(1,\mathrm{Sym}^2 f)\gg\log(N)^{-C}$. Let $\epsilon_p(1)$ be the inverse of *the Euler factor at p of $\frac{L(1,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2)}$, as defined explicitly in (6) above. Let $\kappa_{D,N}(1) = \frac{4}{w}\frac{L(1,\eta)}{\zeta^{(N)}(2)}$ denote the quantity defined explicitly in (7) above. Let us also define the constant*

$$\mathfrak{K}_{f,\gamma_A}(1) = \sum_{\substack{q \mid \gamma_A \\ q \geq 1, \mathrm{squarefree}}} \frac{\mu(q)}{q} \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{\mu(d)}{d} \lambda\left(\frac{r''}{d}\right).$$

Here, the iterated sum over divisors $d \mid r'' \mid \cdots \mid r \mid q$ terminates with $d = 1$, and $\mathfrak{K}_{f,1}(1) = 1$ when $\gamma_A = 1$.

(i) *For $k = 0$, we have the lower and upper bounds*

$$\log(N)^{-C} \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll_{p,D} \mathfrak{L}_{0,f,\gamma_A}(1) \ll_{p,D} \log(N)^3 \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

To be more precise, if $\alpha = 0$, then we have the bounds

$$\log(N)^{-C} \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll \mathfrak{L}_{0,f,\gamma_A}(1) \ll_{\varepsilon} \log(N)^3 \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

If $\alpha \geq 1$, then we have the bounds

$$\log(N)^{-C} \cdot \epsilon_p(1) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll \mathfrak{L}_{0,f,\gamma_A}(1) \ll_{\varepsilon} \log(N)^3 \cdot \epsilon_p(1) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

Here, the implied constants do not depend on the coefficient γ_A. Nor do they depend on the ring class exponent α. In particular, we have the lower bound $\mathfrak{L}_{0,f,\gamma_A}(1) \gg_{p,D,f} \mathfrak{K}_{f,\gamma_A}(1)$.

(ii) *For $k = 1$, we have the lower and upper bounds*

$$\mathfrak{L}_{0,f,\gamma_A}(1) \cdot \log\left(\frac{|D|^{\frac{3}{2}} p^{2\alpha}}{\gamma_A}\right) \ll_{f,p,\gamma_A} \mathfrak{L}_{1,f,\gamma_A}(1) \ll_{p,D,f,\gamma_A,\varepsilon} \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \log\left(\frac{p^{2\alpha}}{\gamma_A}\right) \cdot N^{\varepsilon} \cdot |D|^{\varepsilon}.$$

To be more precise, we have the bounds

$$\log\left(\frac{|D|^{\frac{3}{2}} p^{2\alpha}}{\gamma_A}\right) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \ll_{f,p,\gamma_A} \mathfrak{L}_{1,f,\gamma_A}(1) \ll_{p,D,f,\gamma_A,\varepsilon} \cdot \log\left(\frac{p^{2\alpha}}{\gamma_A}\right) \cdot N^{\varepsilon} \cdot |D|^{\varepsilon} \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

In particular, we have the lower bound $\mathfrak{L}_{1,f,\gamma_A}(1) \gg_{f,p,\gamma_A} \log(|D|^{\frac{3}{2}} p^{2\alpha} \gamma_A^{-\frac{1}{2}}) \cdot \mathfrak{K}_{f,\gamma_A}(1)$.

Proof. We consider the function defined on $s \in \mathbf{C}$ with $\Re(s) \geq 1$ by

$$\mathfrak{L}_{f,\gamma_A}(s) = \frac{4}{w} \cdot \sum_{q \mid \gamma_A} \mu(q) \cdot L(s,\eta) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{L_q^{(p^\alpha)}(s, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np^\alpha)}(2s)}.$$

Again, we write

$$L(s, \mathrm{Sym}^2 f) = \zeta(2s) \sum_{a \geq 1} \frac{\lambda(a^2)}{a^s} = \prod_{l < \infty} L(s, \mathrm{Sym}^2 \pi(f)_l)$$

for the Dirichlet series expansion of the symmetric square L-function of the self-dual $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f)$ associated to the eigenform f, with each $L(s, \mathrm{Sym}^2 \pi(f)_l)$ denoting the local Euler factor at l. Hence if l does not divide the level N of f, we know that this Euler factor is given by

$$L(s, \mathrm{Sym}^2 \pi(f)_l) = \left(1 - \frac{\lambda(l^2)}{l^s} + \frac{\lambda(l^2)}{l^{2s}} - \frac{1}{l^{3s}}\right)^{-1}.$$

See [21, §5.12] for details. Let us henceforth assume without loss of generality that $\alpha \geq 1$, noting that we can ignore the Euler factors at p in the special case of $\alpha = 0$ corresponding to moments of class group characters. We shall later remove the Euler factor $\epsilon_p(s)^{-1}$ at p of $L(s, \mathrm{Sym}^2 f)/\zeta^{(N)}(2s)$. Hence, separating out the $q = 1$ term in the $\alpha \geq 1$ case, we see that

$$\mathfrak{L}_{f,\gamma_A}(s) = \frac{4}{w} \cdot L(s,\eta) \cdot \frac{\lambda(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{L^{(p)}(s, \mathrm{Sym}^2 f)}{\zeta^{(Np)}(2s)} + \frac{4}{w} \cdot L(s,\eta) \sum_{\substack{q \mid \gamma_A \\ q > 1}} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{L_q^{(p)}(s, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np)}(2s)}$$

Let us now examine the inner sum of this latter expression, which by definition equals

$$\sum_{\substack{q \mid \gamma_A \\ q>1}} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{L_q^{(p)}(s, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(p)}(2s)} = \sum_{\substack{q \mid \gamma_A \\ q>1}} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \sum_{\substack{n \geq 1, (n,p)=1 \\ n \equiv 0 \bmod q}} \frac{\lambda\left(\frac{n^2}{q}\right)}{n^s}$$

$$= \sum_{\substack{q \mid \gamma_A \\ q>1}} \mu(q) \cdot \frac{\lambda^{(q)}(\gamma_A)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{1}{q^s} \sum_{\substack{j \geq 1 \\ (jq,p)=1}} \frac{\lambda(j^2 q)}{j^s}.$$

We can apply the Hecke relation

$$\lambda(j^2 q) = \sum_{r \mid \gcd(j,q)} \mu(r) \lambda\left(\frac{j^2}{r}\right) \lambda\left(\frac{q}{r}\right)$$

to each term in the inner sum to get

$$\frac{1}{q^s} \sum_{\substack{j \geq 1 \\ (j,p)=1}} \frac{\lambda(j^2 q)}{j^s} = \frac{1}{q^s} \sum_{\substack{j \geq 1 \\ (j,p)=1}} \sum_{r \mid \gcd(j,q)} \mu(r) \lambda\left(\frac{j^2}{r}\right) \lambda\left(\frac{q}{r}\right) \frac{1}{j^s}$$

$$\text{(30)} \qquad = \frac{1}{q^s} \sum_{r \mid q} \mu(r) \lambda\left(\frac{q}{r}\right) \sum_{\substack{j \geq 1, (j,p)=1 \\ j \equiv 0 \bmod r}} \frac{\lambda\left(\frac{j^2}{r}\right)}{j^s}$$

$$= \frac{1}{q^s} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \sum_{\substack{j \geq 1 \\ (jr,p)=1}} \frac{\lambda(j^2 r)}{j^s}.$$

Observe that we may apply the same argument to evaluate the inner sum on the right-hand side of (30) as

$$\text{(31)} \qquad \sum_{\substack{j \geq 1 \\ (jr,p)=1}} \frac{\lambda(j^2 r)}{j^s} = \sum_{r' \mid r} \frac{\mu(r')}{r'^s} \lambda\left(\frac{r}{r'}\right) \sum_{\substack{j \geq 1 \\ (jr',p)}} \frac{\lambda(j^2 r')}{j^s}.$$

We may then apply this procedure iteratively to the inner sum on the right-hand side of (31) to get

$$\frac{L_q^{(p)}(s, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np)}(2s)} = \frac{1}{q^s} \sum_{\substack{j \geq 1 \\ (jq,p)=1}} \frac{\lambda(j^2 q)}{q^s} = \frac{1}{q^s} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \sum_{\substack{j \geq 1 \\ (jr,p)=1}} \frac{\lambda(j^2 r)}{j^s}$$

$$= \frac{1}{q^s} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \sum_{r' \mid r} \frac{\mu(r')}{r'^s} \lambda\left(\frac{r}{r'}\right) \quad \cdots \quad \sum_{d \mid r''} \frac{\mu(d)}{d^s} \lambda\left(\frac{r''}{d}\right) \sum_{\substack{j \geq 1 \\ (j,p)=1}} \frac{\lambda(j^2 d)}{j^s}$$

$$= \frac{1}{q^s} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \sum_{r' \mid r} \frac{\mu(r')}{r'^s} \lambda\left(\frac{r}{r'}\right) \quad \cdots \quad \sum_{\substack{d=1 \\ \text{terminal divisor of } r''}} \lambda\left(r''\right) \sum_{\substack{j \geq 1 \\ (j,p)=1}} \frac{\lambda(j^2)}{j^s}.$$

Here, the sum runs over sequences of divisors $1 \mid r'' \mid \cdots \mid r' \mid r \mid q$. Each sequence ends when we reach the terminal divisor $d = 1$ of predecessor r'' (say). We obtain from this the expression

(32)

$$\frac{L_q^{(p)}(s, \mathrm{Sym}^2 f^{(q)})}{\zeta_q^{(Np)}(2s)} = \frac{1}{q^s} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \sum_{r' \mid r} \frac{\mu(r')}{r'^s} \lambda\left(\frac{r}{r'}\right) \quad \cdots \quad \sum_{\substack{d=1 \\ \text{terminal divisor of } r''}} \lambda\left(r''\right) \cdot \frac{L^{(p)}(s, \mathrm{Sym}^2 f)}{\zeta^{(Np)}(2s)}.$$

Dividing out by the Euler factor $\epsilon_p(s)^{-1}$ at p, or rather multiplying by $\epsilon_p(s)$ to obtain the relation

$$\frac{L^{(p)}(s, \mathrm{Sym}^2 f)}{\zeta^{(Np)}(2s)} = \epsilon_p(1) \cdot \frac{L(s, \mathrm{Sym}^2 f)}{\zeta^{(N)}(2s)} = \frac{\left(1 - \frac{\lambda(p^2)}{p^s} + \frac{\lambda(p^2)}{p^{2s}} - \frac{1}{p^{3s}}\right)}{\left(1 - \frac{1}{p^{2s}}\right)} \cdot \frac{L(s, \mathrm{Sym}^2 f)}{\zeta^{(N)}(2s)},$$

we then derive the formula

$$\mathfrak{L}_{f,\gamma_A}(s) = \frac{4}{w} \cdot L(s,\eta) \cdot \epsilon_p(s) \cdot \sum_{\substack{q \mid \gamma_A \\ q \geq 1}} \frac{\mu(q)}{q^s} \cdot \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \cdot \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \quad \cdots \quad \lambda(r'') \cdot \frac{L(s,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2s)}$$

$$(33) \qquad = \frac{4}{w} \cdot L(s,\eta) \cdot \epsilon_p(s) \cdot \frac{L(s,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2s)} \cdot \sum_{\substack{q \mid \gamma_A \\ q \geq 1}} \frac{\mu(q)}{q^s} \cdot \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \cdot \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \quad \cdots \quad \lambda(r'')$$

$$= \frac{4}{w} \cdot L(s,\eta) \cdot \epsilon_p(s) \cdot \frac{L(s,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2s)} \cdot \mathfrak{K}_{f,\gamma_A}(s)$$

where

$$\mathfrak{K}_{f,\gamma_A}(s) = \sum_{\substack{q \mid \gamma_A \\ q \geq 1}} \frac{\mu(q)}{q^s} \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \sum_{r \mid q} \frac{\mu(r)}{r^s} \lambda\left(\frac{q}{r}\right) \cdots \lambda(r'').$$

Let us now return to the residual sums $\mathfrak{L}_{k,f,\gamma_A}(1)$. In the special case of central values corresponding to $k=0$, we have that $\mathfrak{L}_{0,f,\gamma_A}(1) = \mathfrak{L}_{f,\gamma_A}(1)$ in the discussion above. Hence, using Euler's formula $\zeta(2) = \frac{\pi^2}{6}$ and Dirichlet's class number formula $L(1,\eta) = \frac{h_K 2\pi}{w\sqrt{|D|}}$, we derive from (32) the formula

(34)

$$\begin{aligned}\mathfrak{L}_{0,f,\gamma_A}(1) &= \frac{4}{w} \cdot L(1,\eta) \cdot \epsilon_p(1) \cdot \frac{L(1,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2)} \cdot \mathfrak{K}_{f,\gamma_A}(1) \\ &= \frac{48 \cdot h_K}{\pi w^2 \sqrt{|D|}} \cdot \prod_{l \mid N} \frac{1}{\left(1-\frac{1}{l^2}\right)} \cdot \epsilon_p(1) \cdot L(1,\mathrm{Sym}^2 f) \cdot \left(\sum_{\substack{q \mid \gamma_A \\ q \geq 1}} \frac{\mu(q)}{q} \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \lambda(r'') \right) \\ &= \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1,\mathrm{Sym}^2 f) \cdot \mathfrak{K}_{f,\gamma_A}(1).\end{aligned}$$

Here, we use the simplifying notations (7) and (6) defined above. To derive bounds, let us first recall that $L(1,\mathrm{Sym}^2 f) \neq 0$. This can be deduced from the prime number theorem for $\mathrm{GL}_3(\mathbf{A})$-automorphic L-functions, the lower bound of Hoffstein-Lockhart [20] and Goldfeld-Hoffstein-Lieman [15], as well as by comparison with the adjoint L-function (see e.g. [21, §5.12, (5.101)]) to deduce the relation

$$L(1,\mathrm{Sym}^2 f) = \frac{(4\pi)^l}{\Gamma(l)} \cdot \frac{\langle f,f \rangle}{\mathrm{Vol}(\Gamma_0(N)\backslash\mathfrak{H})} > 0.$$

As explained in [8, Lemmas 4.1, 4.2] also have upper and lower bounds for this quantity. That is, there exists a constant $C = C(A_2)$ depending on the size of the region $[1 - A_2/\log(N), 1]$ on which we know $L(s,\mathrm{Sym}^2 f)$ does not vanish for which

$$\log(N)^{-C} \ll L(1,\mathrm{Sym}^2 f) \ll \log(N)^3.$$

Hence we have the bounds

$$\log(N)^{-C} \cdot \epsilon_p(1) \cdot \mathfrak{K}_{f,\gamma_A}(1) \cdot \kappa_{D,N}(1) \ll \mathfrak{L}_{0,f,\gamma_A}(1) \ll \log(N)^3 \cdot \epsilon_p(1) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).$$

Note that the implied constants in these bounds do not depend on the coefficient γ_A.

In the case of central derivative values corresponding to $k=1$, we see from Lemma 3.1 with (33) that

$$\begin{aligned}\mathcal{L}_{1,f,\gamma_A}(1) &= \mathrm{Res}_{s=0} \left(\mathfrak{L}_{f,\gamma_A}(2s+1) \cdot \hat{V}_2(s) \cdot \left(\frac{N|D|p^{2\alpha}}{\gamma_A} \right)^s \right) \\ &= \mathrm{Res}_{s=0} \left(\frac{4}{w} \cdot L(2s+1,\eta) \cdot \epsilon_p(2s+1) \cdot \frac{L(2s+1,\mathrm{Sym}^2 f)}{\zeta^{(N)}(4s+2)} \cdot \mathfrak{K}_{f,\gamma_A}(2s+1) \cdot \hat{V}_2(s) \cdot \left(\frac{N|D|p^{2\alpha}}{\gamma_A} \right)^s \right)\end{aligned}$$

can be computed following the residue calculation (29) with (33) as

(35)

$$\begin{aligned}
&\mathfrak{L}_{1,f,\gamma_A}(1) = \mathfrak{L}_{f,\gamma_A}(1)\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right) - 2(\gamma+\log(2\pi))\right) + \mathfrak{L}'_{f,\gamma_A}(1)\\
&= \frac{4}{w}\cdot L(1,\eta)\cdot\epsilon_p(1)\cdot\frac{L(1,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2)}\cdot\mathfrak{K}_{f,\gamma_A}(1)\cdot\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)-2(\gamma+\log(2\pi))\right)\\
&+\frac{4}{w}\cdot L(1,\eta)\cdot\epsilon_p(1)\cdot\frac{L(1,\mathrm{Sym}^2 f)}{\zeta^{(N)}(2)}\cdot\mathfrak{K}_{f,\gamma_A}(1)\cdot\left[\frac{L'}{L}(1,\eta)+\frac{\epsilon_p'}{\epsilon_p}(1)+\frac{L'}{L}(1,\mathrm{Sym}^2 f)-\frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2)+\frac{\mathfrak{K}'_{f,\gamma_A}}{\mathfrak{K}_{f,\gamma_A}}(1)\right]\\
&= \mathfrak{L}_{0,f,\gamma_A}(1)\cdot\left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right)+\frac{L'}{L}(1,\eta)+\frac{L'}{L}(1,\mathrm{Sym}^2 f)-2(\gamma+\log(2\pi))-\frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2)+\frac{\epsilon_p'}{\epsilon_p}(1)+\frac{\mathfrak{K}'_{f,\gamma_A}}{\mathfrak{K}_{f,\gamma_A}}(1)\right)
\end{aligned}$$

To estimate this expression, we consider the various logarithmic derivative terms. Let us first consider $\frac{L'}{L}(1,\mathrm{Sym}^2 f)$. Here, we can use an approximate functional equation argument as in [41, Proposition 2.1] to derive the following estimate. Let us write the gamma factor of the symmetric square L-function as

$$\gamma(s) := \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{R}}(s+l-1)\Gamma_{\mathbf{R}}(s+l), \quad \Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$

so that we have the functional equation

$$\Lambda(s) := N^s\gamma(s)L(s,\mathrm{Sym}^2 f) = \Lambda(1-s). \tag{36}$$

Fix $G(z)$ any even meromorphic function of $z\in\mathbf{C}$ having a single pole at $z=0$ and Laurent series expansion $G(z)=\frac{1}{z^2}+O(1)$, which is bounded in the strip $-4\le\Re(z)\le 4, \Im(z)\ge 1$. Consider the integral

$$I(s) := \int\limits_{\Re(s)=3}\Lambda(s+z)G(z)\frac{dz}{2\pi i}.$$

Shifting the line of integration to $\Re(s)=-3$, we cross a pole at $z=0$ of residue

$$\Lambda'(1) = N\cdot\gamma(1)\cdot L(1,\mathrm{Sym}^2 f)\cdot\left(\log(N)+\frac{\gamma'}{\gamma}(1)+\frac{L'}{L}(1,\mathrm{Sym}^2 f)\right).$$

Applying the functional equation (36) to the remaining integral, we obtain the identity $I(1)=\Lambda'(1)+I(0)$. Note that opening up the Dirichlet series of $L(s,\mathrm{Sym}^2 f)$, we have the expansion

$$I(s) = N^s\sum_{\substack{m\ge1\\(m,N)=1}}\frac{1}{m^s}\sum_{n\ge1}\frac{\lambda(n^2)}{n^s}\int\limits_{\Re(z)=3}\gamma(z+s)G(z)\left(\frac{m^2n}{N}\right)^{-z}\frac{dz}{2\pi i}.$$

Note as well that we can already derive from this discussion the crude estimate

$$\frac{L'}{L}(1,\mathrm{Sym}^2 f) = \frac{(I(1)-I(0))}{\gamma(1)L(1,\mathrm{Sym}^2 f)} - \log(N) - \frac{\gamma'}{\gamma}(1).$$

We can refine this estimate following the argument of [41, Proposition 2.1]. First, observe that we can write

$$I(s) = NI_f^\star(s) + \int\limits_{\Re(z)=3}N^{s+z}\zeta^{(N)}(2s+2z)\gamma(s+z)G(z)\frac{dz}{2\pi i},$$

where

$$I_f^\star(s) := \frac{1}{N}\sum_{\substack{m\ge1\\(m,N)=1}}\sum_{n\ge1}\lambda(n^2)\left(\frac{N}{m^2n}\right)^sJ_s\left(\frac{m^2n}{N}\right), \quad J_s(y) := \int\limits_{\Re(z)=3}y^{-z}\gamma(z+s)G(z)\frac{dz}{2\pi i}.$$

Now for $s=0$, we shift the contour $I_f^\star(0)$ to $\Re(z)=\varepsilon$, crossing a simple pole at $z=1/2$ to obtain

$$I(0) = NI_f^\star(0) + O_\varepsilon(N^\varepsilon).$$

For $s=1$, we shift the contour $I_f^\star(1)$ to $\Re(z)=-1$, crossing a simple pole at $z=0$ to obtain the estimate

$$I(1) = N I_f^\star(1) + \left(N - \frac{1}{N}\right) \cdot \zeta^{(N)}(2) \cdot \gamma(1) \cdot \left(\frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2) + \frac{\gamma'}{\gamma}(1) + \frac{N+N^{-1}}{N-N^{-1}} \log(N) \right) + O(1).$$

In this way, we derive the estimate

$$\begin{aligned} \frac{L'}{L}(1, \mathrm{Sym}^2 f) &= \frac{\left(I_f^\star(1) - I_f^\star(0)\right)}{\gamma(1) L(1, \mathrm{Sym}^2 f)} - \log(N) - \frac{\gamma'}{\gamma}(1) \\ &+ \frac{\zeta(2)}{L(1, \mathrm{Sym}^2 f)} \cdot \left(\frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2) + \frac{\gamma'}{\gamma}(1) + \log(N) + O\left(\frac{\log(N)}{N}\right)\right). \end{aligned} \tag{37}$$

To estimate the first term $I_f^\star(1) - I_f^\star(0)$ in this expression, observe that the cutoff functions $J_s(y)$ decay rapidly as $y \to \infty$. We can then bound the remaining truncated sums via the bounds of Molteni [31] for the $\mathrm{GL}_3(\mathbf{A})$-automorphic L-function $L(s, \mathrm{Sym}^2 f)$. That is, for any $\varepsilon > 0$ and any real number $x \geq 1$, we have

$$\sum_{1 \leq n \leq x} \frac{|\lambda(n^2)|}{n} \ll_\varepsilon (Nx)^\varepsilon.$$

In this way, we deduce that for any $\varepsilon > 0$, we can estimate

$$I_f^\star(1) - I_f^\star(0) = \frac{N}{N} \cdot (N^2)^\varepsilon + \frac{1}{N} \cdot N^{1+2\varepsilon} = O_\varepsilon(N^\varepsilon).$$

We then use (37) to derive the crude estimate

$$\frac{L'}{L}(1, \mathrm{Sym}^2 f) \ll_\varepsilon \log(N)^C N^\varepsilon. \tag{38}$$

Now, let us note that the quantity

$$\frac{L'}{L}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi))$$

is related to the self-intersection number $\langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle$ of the relative dualizing sheaf $\omega_{X_0(N)}$ of $X_0(N)$,

$$\langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle = \frac{3}{\pi} \sum_{f \in S_l^{\mathrm{new}}(\Gamma_0(N))} \frac{L'}{L}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)).$$

Here, the sum runs over a basis of newforms for $S_l(\Gamma_0(N))$ and we have the lower bound on this self-intersection number $\langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle \geq -g(X_0(N))$ by a theorem of Faltings. Writing $\langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle_f$ to denote the projection to the f-isotypical component, we deduce that the logarithmic derivative term

$$\frac{L'}{L}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)) = \frac{\pi}{3} \cdot \langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle_f$$

will satisfy some lower bound determined by the relation with $\langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle$, and in any case that this logarithmic derivative term contributes a quantity which depends only on the eigenform f. Similarly, the logarithmic derivative term $-\frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2)$ contributes a quantity which depends only on f, and $\frac{\epsilon_p'}{\epsilon_p}(1)$ contributes a quantity which depends only on f and p. Let us now consider the logarithmic derivative term $\frac{L'}{L}(1, \eta)$, which under the Riemann Hypothesis can be bounded as $\frac{L'}{L}(1, \eta) = O(\log\log(|D|))$. We know that

$$\mathcal{L}_D := \frac{1}{2} \cdot \log(|D|) + \frac{L'}{L}(1, \eta)$$

can be bounded above by Siegel's theorem, and below by Colmez [10, Proposition 5] to get

$$\log|D| \ll \mathcal{L}_D \ll_\varepsilon |D|^\varepsilon.$$

We refer to [40, §1.8] for more details. We deduce from this that

$$\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right) + \frac{L'}{L}(1, \eta) = \log\left(\frac{N|D|^{\frac{1}{2}} p^{2\alpha}}{\gamma_A}\right) + \mathcal{L}_D \gg \log\left(\frac{N|D|^{\frac{3}{2}} p^{2\alpha}}{\gamma_A}\right). \tag{39}$$

Putting the pieces together, we then deduce from the formula (35) and the lower bound (39) that we have

$$\begin{aligned}
&\mathfrak{L}_{1,f,\gamma_A}(1)\\
&= \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A} \right) + \frac{L'}{L}(1,\eta) + \frac{L'}{L}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)) - \frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2) + \frac{\epsilon_p'}{\epsilon_p}(1) + \frac{\mathfrak{K}_{f,\gamma_A}'}{\mathfrak{K}_{f,\gamma_A}}(1) \right)\\
&= \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \left(\log\left(\frac{N|D|^{\frac{1}{2}}p^{2\alpha}}{\gamma_A} \right) + \mathcal{L}_D + \frac{\pi}{3} \cdot \langle \omega_{X_0(N)}, \omega_{X_0(N)} \rangle_f - \frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2) + \frac{\epsilon_p'}{\epsilon_p}(1) + \frac{\mathfrak{K}_{f,\gamma_A}'}{\mathfrak{K}_{f,\gamma_A}}(1) \right)\\
&\gg_{f,p,\gamma_A} \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \log\left(\frac{|D|^{\frac{3}{2}}p^{2\alpha}}{\gamma_A} \right).
\end{aligned}$$

Hence, via our bounds for $\mathfrak{L}_{0,f,\gamma_A}(1)$, we can see that we have the more explicit lower bound

$$\begin{aligned}
\mathfrak{L}_{1,f,\gamma_A}(1) &\gg_{f,p,\gamma_A} \log(N)^{-C} \cdot \log\left(\frac{N|D|^{\frac{3}{2}}p^{2\alpha}}{\gamma_A} \right) \cdot \epsilon_p(1) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1)\\
&\gg_{f,p} \log\left(\frac{|D|^{\frac{3}{2}}p^{2\alpha}}{\gamma_A} \right) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).
\end{aligned}$$

We also derive from the formula (35) with our discussion above the upper bound

$$\begin{aligned}
\mathfrak{L}_{1,f,\gamma_A} &\ll_{f,p,\gamma_A,\varepsilon} \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \log\left(\frac{N|D|^{\frac{1}{2}}p^{2\alpha}}{\gamma_A} \right) \cdot |D|^{\varepsilon} \cdot \log(N)^C \cdot N^{\varepsilon}\\
&\ll_{f,p,\gamma_A,\varepsilon} \log(N)^{C+3} N^{\varepsilon} \cdot \log\left(\frac{N|D|^{\frac{1}{2}}p^{2\alpha}}{\gamma_A} \right) \cdot |D|^{\varepsilon} \cdot \epsilon_p(1) \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1)\\
&\ll_{f,p,\gamma_A,\varepsilon} N^{\varepsilon} \cdot \log\left(\frac{p^{2\alpha}}{\gamma_A} \right) \cdot |D|^{\varepsilon} \cdot \kappa_{D,N}(1) \cdot \mathfrak{K}_{f,\gamma_A}(1).
\end{aligned}$$

$\square$

3.3. **Main estimates.** Fix an integer $\alpha \geq 0$, and a class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$. Let us retain the setup of Lemma 3.1. and the corresponding parametrization of the counting function (i.e. (13)). Let $Z > 0$ be any real parameter. Fixing such a parameter, we can then use the decay properties of the cutoff functions V_{k+1} described in Lemma 2.3 above to reduce to estimating the truncated sums defined for any $\varepsilon > 0$ by

$$H_{A,1}^{(k),\dagger}(\alpha, 0; Z) := \frac{4}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{b \geq 1\\ m^2b^2 \leq \left(\frac{1}{|\epsilon_A|Z}\right)^{1+\varepsilon}}} \sum_{a \in \mathbf{Z}} \frac{\lambda(q_A(a,b))}{(q_A(a,b))^{\frac{1}{2}}} V_{k+1}\left(m^2 q_A(a,b) Z\right)$$

and

$$H_{A,2}^{(k),\dagger}(\alpha, 0; Z) := \frac{4}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{b \geq 1\\ m^2b^2 \leq \left(\frac{ZN^2|D|^2p^{4\alpha}}{|\epsilon_A|}\right)^{1+\varepsilon}}} \sum_{a \in \mathbf{Z}} \frac{\lambda(q_A(a,b))}{q_A(a,b)^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 q_A(a,b)}{ZN^2|D|^2p^{4\alpha}} \right)$$

Note that we shall later often take $Z = Y^{-1}$ with $Y = (N|D|p^{2\alpha})$ the square root of the conductor.

Theorem 3.3. *Let f be any non-dihedral cuspidal modular form of arbitrary weight l, level N, and nebentype character ξ, and whose Fourier coefficients we denote by $\lambda(n) = \lambda_f(n)$. Fix an integer $\alpha \geq 0$ and a class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ with corresponding quadratic form representative $q_A(x,y) = \gamma_A xy + \delta_A xy + \epsilon_A y^2$. Let $0 \leq \delta_0 < 1/4$ denote the best known approximation to the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms in the level aspect, i.e. with $\delta_0 = 0$ is conjectured, and $\delta_0 = 3/16$ known thanks to the theorem of Blomer-Harcos* [2]. *Similarly, let $0 \leq \theta_0 < 1/2$ denote the best approximation the generalized Ramanujan conjecture for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms in the level aspect, i.e. with $\theta_0 = 0$ is conjectured,* *and $\theta_0 = 7/64$ known thanks to the theorem of Kim-Sarnak* [24].

(i) *Suppose $\gamma_A = 1$ and $\delta_A = 0$, as is the case for the principal class $A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ when $D \equiv 0 \bmod 4$ and we take $q_A(x,y) = q_{\mathbf{1}}(x,y)$ to be the reduced quadratic form representative. We have for any choices of real parameters $Y > 1$ and $\varepsilon > 0$ the bounds*

$$H_{A,1}^{(k),\dagger}(\alpha, 0; Y^{-1}) \ll_{f,k,\varepsilon} Y^{\frac{1}{4}+\delta_0+\varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2}}$$

and

$$H_{A,2}^{(k),\dagger}(\alpha, 0; Y^{-1}) \ll_{f,k,\varepsilon} \left(\frac{N^2 |D|^2 p^{4\alpha}}{Y} \right)^{\frac{1}{4}+\delta_0+\varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2}}.$$

(ii) *Suppose now that the form f is holomorphic. Then, without conditions on the reduced binary quadratic form representative $q_A(x,y) = \gamma_A xy + \delta_A xy + \epsilon_A y^2$, we have by a variation of the proof for (i) the following improved bounds: For any $Y > 1$ and any $\varepsilon > 0$,*

$$H_{A,1}^{(k),\dagger}(\alpha, 0; Y^{-1}) \ll_{f,k,\varepsilon} \gamma_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot |p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2} - \delta_0 + \frac{\theta_0}{2} + \varepsilon}$$

and

$$H_{A,2}^{(k),\dagger}(\alpha, 0; Y^{-1}) \ll_{f,k,\varepsilon} \gamma_A \cdot \left(\frac{N^2 |D|^2 p^{4\alpha}}{Y} \right)^{\frac{1}{4}+\delta_0} \cdot |p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2} - \delta_0 + \frac{\theta_0}{2} + \varepsilon}.$$

Here, the bounds depend on the choice of form f (and hence the character ξ and level N), as well as the choice of cutoff function V_{k+1} (and hence also depend on the generic root number parametrized by $k \in \{0,1\}$).

Proof. We divide the proof into a handful steps as follows.

(1) Setup of main estimates (i) and (ii). For (i), we use standard estimates for the shifted convolution problem for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms via spectral decompositions of automorphic forms on the two-fold metaplectic cover $\overline{G}$ of $\mathrm{GL}_2(\mathbf{A})$. Here, for each nonzero integer b, we shall explain for (i) how an adaptation of the proof of Templier-Tsimerman [42, Theorem 1] gives us for any choices of real numbers $Y > 1$ and $\varepsilon > 0$ the uniform bounds

$$\sum_{a \in \mathbf{Z}} \frac{\lambda(a^2 + \epsilon_A b^2)}{(a^2 + \epsilon_A b^2)^{\frac{1}{2}}} V_{k+1} \left(\frac{a^2 + \epsilon_A b^2}{Y} \right) \ll_{f,k,\varepsilon} Y^{\frac{1}{4}} \cdot |b^2 \epsilon_A|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|b^2 \epsilon_A|}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}. \tag{40}$$

For (ii), we explain how a stronger bound can be derived without restriction on the coefficients of the quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ by a generalization of the theorem of Blomer [1]. This will give us for any choices of $Y > 1$ and $\varepsilon > 0$ the uniform bound

$$\sum_{a \in \mathbf{Z}} \frac{\lambda(q_A(a,b))}{q_A(a,b)^{\frac{1}{2}}} V_{k+1} \left(\frac{q_A(a,b)}{Y} \right) \ll_{f,k,\varepsilon} \gamma_A \cdot |b^2 \cdot p^{2\alpha} D|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|b^2 \cdot p^{2\alpha} D|}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}. \tag{41}$$

(2) Reduction to local sums via smooth partitions of unity and dyadic decompositions. To show (40) and (41), we first explain a standard reduction via smooth partition of unitary and dyadic decomposition to "local" sums over compactly supported weight functions. Here, we refer the reader to similar discussions or setups in [42, p. 7], [4, § 5.1], [3, § 2.9], or [33, §2.1]. In this way, the estimates (40) and (41) can be reduced to their local analogues, replacing the cutoff functions V_{k+1} with some smooth and compactly supported function $W \in \mathcal{C}^\infty(\mathbf{R}_{>0})$ with the decay condition $W^{(i)} \ll 1$ for all integers $i \geq 0$. That is, it will suffice to show for any such weight function W the respective bounds

$$\sum_{a \in \mathbf{Z}} \frac{\lambda(a^2 + \epsilon_A b^2)}{(a^2 + \epsilon_A b^2)^{\frac{1}{2}}} W \left(\frac{a^2 + \epsilon_A b^2}{Y} \right) \ll_{f,\varepsilon} Y^{\frac{1}{4}} \cdot |\epsilon_A b^2|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|\epsilon_A| b^2}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} \tag{42}$$

and

(43)
$$\sum_{a \in \mathbf{Z}} \frac{\lambda(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)^{\frac{1}{2}}} W\left(\frac{\gamma_A a^2 + \delta_A ab + \epsilon_A b^2}{Y}\right) \ll_{f,\varepsilon} \gamma_A \cdot |b^2 \cdot p^{2\alpha} D|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|b^2 \cdot p^{2\alpha} D|}{Y}\right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

To justify this reduction, consider the class $\mathcal{C}^+(\mathbf{R}_{>0})$ of functions $U : \mathbf{R}_{>0} \to \mathbf{C}$ which are supported on the closed interval $[1, 2]$ and satisfy $W^{(i)} \ll 1$ for all $i \geq 0$. A standard result in analysis shows there exists a sequence $\{(U, R)\}$ consisting of functions $U \in \mathcal{C}^+(\mathbf{R}_{>0})$ and ranges $R \in \mathbf{R}_{>0}$ such that

$$\sum_{(U,R)} U\left(\frac{r}{R}\right) = 1 \quad \text{for any } r \in (0, \infty). \tag{44}$$

Moreover, for any integer $l \in \mathbf{Z}$, at most finitely many of the pairs (U, R) (independent of the choice of l) in this sequence have the property that $R \in [2^l, 2^{l+1}]$. Fixing such a partition of unity (44) once and for all, the corresponding dyadic subdivision of any sum

$$\sum_{r \geq 1} A(r)$$

is then defined to be the subdivision

$$\sum_{r \geq 1} A(r) = \sum_{(U,R)} \sum_{r \geq 1} A(r) U\left(\frac{r}{R}\right).$$

As explained in [33], we can moreover fix the weight function $U \in \mathcal{C}^+(\mathbf{R}_{>0})$ and obtain such a decomposition (varying only over the ranges $R \in \mathbf{R}_{>0}$). Hence, rearranging and relabelling as necessary, this reduces to

$$\sum_{r \geq 1} A(r) = \sum_{l \in \mathbf{Z}} \sum_{\substack{R \in \{R\} \\ R \cap [2^l, 2^{l+1}] \neq \emptyset}} \sum_{r \geq 1} A(r) U\left(\frac{r}{R}\right). \tag{45}$$

Now, we can apply such a decomposition (45) to our sums of the form

$$\sum_{m,n \geq 1} \frac{\eta \xi^2(m)}{m} \frac{\lambda(n) r(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Y}\right)$$

as follows. Fixing a weight function $U \in \mathcal{C}^+(\mathbf{R}_{>0})$, the corresponding decomposition (45) takes the form

$$\sum_{l \in \mathbf{Z}} \sum_{\substack{R \in \{R\} \\ R \cap [2^l, 2^{l+1}] \neq \emptyset}} \sum_{m,n \geq 1} \frac{\eta \xi^2(m)}{m} \frac{\lambda(n) r(n)}{n^{\frac{1}{2}}} V_{k+1}\left(\frac{m^2 n}{Y}\right) U\left(\frac{m^2 n}{R}\right). \tag{46}$$

Note that the second sum in this latter expression refers simply to the range R in the set of all ranges $\{R\}$ making up partition of unity which happens to be contained in the interval $[2^l, 2^{l+1}]$. This gives us for each integer $l \in \mathbf{Z}$ a compactly supported function $V_{k+1,l}(y)$ defined on $y \in \mathbf{R}_{>0}$ by

$$V_{k+1,l}(y) = V_{k+1}(y) \sum_{\substack{R \in \{R\} \\ R \cap [2^l, 2^{l+1}] \neq \emptyset}} U\left(\frac{yY}{R}\right),$$

so that the dyadic subdivision (46) takes the simpler form

$$\sum_{l \in \mathbf{Z}} \sum_{m,n \geq 1} \frac{\eta \xi^2(m)}{m} \frac{\lambda(n) r(n)}{n^{\frac{1}{2}}} V_{k+1,l}\left(\frac{m^2 n}{Y}\right). \tag{47}$$

In particular, since U is supported on $[1, 2]$, it follows that each local weight function $V_{k+1,l}(y)$ is supported on $2^{l+1} \leq yY \leq 2^{l+2}$. Hence, the region of moderate decay $y \leq Y$ (equivalently $yY \leq Y^2$) corresponding to our global cutoff function V_{k+1} in the truncated sum $H_A^{(k),\dagger}(\alpha, 0; Y^{-1})$ can be reparametrized in terms of

$yY \leq 2^x$ for $x = 2\log Y/\log 2$. In particular, to estimate the truncated sum $H_{A,j}^{(k),\dagger}(\alpha, 0; Y^{-1})$ for each of $j = 1, 2$ as stated, it will do to estimate the sum over roughly $\log Y$ many local sums:

$$H_{A,j}^{(k),\dagger}(\alpha, 0; Y^{-1}) \ll \sum_{0 \leq l \leq \log Y} H_{A,j,l}^{(k),\dagger}(\alpha, 0; Y^{-1});$$
$$H_{A,j,l}^{(k),\dagger}(\alpha, 0; Y^{-1}) := \frac{4}{w} \sum_{m \geq 1} \frac{\eta\xi^2(m)}{m} \sum_{\substack{b \geq 1 \\ mb \leq \left(\frac{Y}{|\epsilon_A|}\right)^{\frac{1}{2}}}} \sum_{a \in \mathbf{Z}} \frac{\lambda(q_A(a,b))}{(q_A(a,b))^{\frac{1}{2}}} V_{k+1,l}\left(\frac{m^2 q_A(a,b)}{Y}\right).$$

The arguments used to prove [42, Theorem 1] and variations of those in [1, §4] apply directly to each inner l-sum in the latter expression. To be more precise, for any smooth function W supported on $[1/2, 1]$ and any real number $R \geq 1$, we have for (i) that

$$\left| \sum_{m \geq 1} \frac{\eta\xi^2(m)}{m} \sum_{\substack{a,b \in \mathbf{Z} \\ q_A(a,b) \neq 0}} \frac{\lambda(a^2 + \epsilon_A b^2)}{(a^2 + \epsilon_A b^2)^{\frac{1}{2}}} W\left(\frac{m^2(a^2 + \epsilon_A b^2)}{R}\right) \right|$$
$$\ll \sum_{m \geq 1} \frac{1}{m} \sum_{b \neq 0} \left| \sum_{a \in \mathbf{Z}} \frac{\lambda(a^2 + \epsilon_A b^2)}{(a^2 + \epsilon_A b^2)^{\frac{1}{2}}} W\left(\frac{m^2(a^2 + \epsilon_A b^2)}{R}\right) \right|,$$

from which we derive from (42) the uniform bound

$$\ll_{\varepsilon} \sum_{\substack{b \neq 0 \\ b \leq \left(\frac{R}{|\epsilon_A|}\right)^{\frac{1}{2}}}} R^{-\frac{1}{4} + \frac{\theta_0}{2} + \varepsilon} \cdot |\epsilon_A b^2|^{\delta_0 - \frac{\theta_0}{2}}$$
$$\ll |\epsilon_A|^{\delta_0 - \frac{\theta_0}{2}} \cdot R^{-\frac{1}{4} + \frac{\theta_0}{2} + \varepsilon} \cdot \left(\frac{R}{|\epsilon_A|}\right)^{\frac{1}{2} + \delta_0 - \frac{\theta_0}{2} + \varepsilon} = R^{\frac{1}{4} + \delta_0 + \varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2}}.$$

Taking the sum over $1 \leq R \leq \log Y$ then gives the bound for (i). Similarly for (ii), we have for any range $R \geq 1$ that

$$\left| \sum_{m \geq 1} \frac{\eta\xi^2(m)}{m} \sum_{\substack{a,b \in \mathbf{Z} \\ q_A(a,b) \neq 0}} \frac{\lambda(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)^{\frac{1}{2}}} W\left(\frac{m^2(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{R}\right) \right|$$
$$\ll \sum_{m \geq 1} \frac{1}{m} \sum_{b \neq 0} \left| \sum_{a \in \mathbf{Z}} \frac{\lambda(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)^{\frac{1}{2}}} W\left(\frac{m^2(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{R}\right) \right|,$$

from which we derive from (42) the uniform bound

$$\ll_{\varepsilon} \sum_{\substack{b \neq 0 \\ b \leq \left(\frac{R}{|\epsilon_A|}\right)^{\frac{1}{2}}}} \gamma_A \cdot R^{-\frac{1}{2} + \frac{\theta_0}{2} + \varepsilon} \cdot |b^2 p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon}$$
$$\ll \gamma_A \cdot R^{-\frac{1}{2} + \frac{\theta_0}{2} + \varepsilon} \cdot |p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \sum_{\substack{b \neq 0 \\ b \leq \left(\frac{R}{|\epsilon_A|}\right)^{\frac{1}{2}}}} b^{2\left(\delta_0 - \frac{\theta_0}{2} - \varepsilon\right)}$$
$$\ll \gamma_A \cdot R^{-\frac{1}{2} + \frac{\theta_0}{2} + \varepsilon} \cdot |p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot \left(\frac{R}{|\epsilon_A|}\right)^{\frac{1}{2} + \delta_0 - \frac{\theta_0}{2} - \varepsilon} = \gamma_A \cdot R^{\delta_0} \cdot |p^{2\alpha} D|^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2} - \delta_0 + \frac{\theta_0}{2} + \varepsilon}.$$

Taking the sum over $1 \leq R \leq \log Y$ then gives the stated bound for (ii).

(3) Bound for the local sums in (i) via spectral decompositions of shifted convolution sums. Let us now explain how to derive the local bound (42) from the argument given in [42, Theorem 1, § 6], and then that of (43) from a variation of the argument given in [1, §4]. Here, we first set up the problem adelically, then work semi-classically (at the archimedean component). The first step is to relate the sums in question to certain Fourier coefficients of automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ or its two-fold metaplectic cover $\overline{G}(\mathbf{A})$, so as to reduce the problem to one of bounding Fourier coefficients (via spectral decomposition). Let $\psi = \otimes_v \psi_v$ denote the standard additive character on $\mathbf{A}/\mathbf{Q}$. The archimedean factor ψ_∞ then coincides with the function $x \mapsto e(x) = \exp(2\pi i x)$. Recall that a decomposable new vector $\phi = \otimes_v \phi_v \in V_\pi$ can be viewed as a cuspidal $\mathrm{GL}_2(\mathbf{A})$-automorphic form[7]. It has the following Fourier-Whittaker expansion: Given $x \in \mathbf{A}$ an adele, and $y = y_f y_\infty \in \mathbf{A}^\times$ an idele split into finite component $y_f \in \mathbf{A}_f^\times$ times real component $y_\infty \in \mathbf{R}^\times$, and writing $|y|$ to denote the idele norm of y, we have

$$\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{r \neq 0 \in \mathbf{Z}} \frac{\lambda(r y_f)}{|r y_f|^{\frac{1}{2}}} W_\phi(r y_\infty) \psi(rx).$$

Here, the archimedean Whittaker coefficient $W_\phi(y_\infty)$ is defined on $y_\infty \in \mathbf{Q}_\infty^\times \cong \mathbf{R}^\times$ by

$$W_\phi(y_\infty) = W_\phi\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = \int_{\mathbf{A}/\mathbf{Q}} \phi\left(\begin{pmatrix} y_\infty & x \\ & 1 \end{pmatrix}\right) \psi(-x) dx.$$

Note that by the surjectivity of the archimedean local Kirillov map $V_{\pi_\infty} \cong \mathcal{W}(\pi_\infty), \phi_\infty \mapsto W_{\phi_\infty}$, given any smooth function of compact support $W \in \mathcal{C}_c^\infty(\mathbf{R}^\times)$, there exists a new vector $\phi = \otimes_v \phi_v \in V_\pi$ with $W_{\phi_\infty} = W$, that is, with $W_\phi(y) = W(y)$ as functions of $y \in \mathbf{R}^\times \cong \mathbf{Q}_\infty^\times$. Let us also remark that this discussion can be given equivalently in classical terms. That is, instead of choosing a suitable pure tensor $\phi = \otimes_v \phi_v \in V_\pi$ with specified archimedean component ϕ_∞, we could instead construct a Maass form ϕ on $z = x + iy \in \mathfrak{H}$ with the same Fourier-Whittaker expansion as some convergent infinite linear combination of some initial weight zero Maass form ϕ_0 under weight raising operators, in the style of Motohashi [32] (cf. [3]). We omit the details of this classical version for simplicity, but keep in mind that we can also think of ϕ as a smooth Maass form which is not K-finite. Let us now consider the metaplectic theta series θ_Q attached to the quadratic form $Q(r) = r^2$, which in the classical setting corresponds to a modular form of half-integral weight. Viewed as a genuine automorphic form on the two-fold metaplectic cover of $\mathrm{GL}_2(\mathbf{R})$, this theta series has the following expansion: For $x \in \mathbf{R}$ and $y \in \mathbf{R}^\times$

$$\theta_Q\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = |y|^{\frac{1}{4}} \sum_{r \in \mathbf{Z}} e\left(Q(r)(x+iy)\right),$$

and its image $\overline{\theta}_Q$ under the Hecke operator T_{-1} corresponding to the classical Hecke operator acting on $z \in \mathfrak{H}$ via $z \mapsto -\overline{z}$ has the expansion

$$\overline{\theta}_Q\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = |y|^{\frac{1}{4}} \sum_{r \in \mathbf{Z}} e\left(Q(r)(-x+iy)\right).$$

Using the orthogonality of additive characters on the compact abelian group $\mathbf{R}/\mathbf{Z} \cong [0,1]$, it is then easy to derive for any $\phi \in V_\pi$ (or Maass form ϕ with the same Fourier-Whittaker expansion) the relation

$$\begin{aligned}
&\int_0^1 \phi\overline{\theta}_Q\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) e\left(-b^2\epsilon_A x\right) dx = \int_0^1 \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \overline{\theta}_Q\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) e\left(-b^2\epsilon_A x\right) dx \\
&= |y|^{\frac{1}{4}} \sum_{r_1 \neq 0 \in \mathbf{Z}} \frac{\lambda(r_1)}{|r_1|^{\frac{1}{2}}} W_\phi(r_1 y) \sum_{r_2 \in \mathbf{Z}} e\left(iQ(r_2)y\right) \int_0^1 e\left(r_1 x - Q(r_2)x - b^2\epsilon_A x\right) dx \\
&= |y|^{\frac{1}{4}} \sum_{\substack{r \in \mathbf{Z} \\ Q(r)+b^2\epsilon_A \neq 0}} \frac{\lambda(Q(r)+b^2\epsilon_A)}{|Q(r)+b^2\epsilon_A|^{\frac{1}{2}}} W_\phi\left((Q(r)+b^2\epsilon_A)y\right) \psi(iyQ(r)).
\end{aligned} \tag{48}$$

[7]To be more precise, one can choose a decomposable vector $\phi = \otimes_v \phi_v \in V_\pi$ whose nonarchimedean local components ϕ_v are each essential Whittaker vectors. The corresponding decomposable vector ϕ is then known to be a new vector, in the sense that its Fourier-Whittaker coefficients are related directly via Mellin transform to the L-function coefficients of π.

Note that in the more general case with $q_A(x,y)$ having leading coefficient $\gamma_A \neq 1$ and middle coefficient $\delta_A = 0$, we may replace the metaplectic theta series θ_A with the scaled metaplectic series $\theta_{\gamma_A, Q}$ defined by

$$\theta_{\gamma_A, Q}\left(\left(\begin{array}{cc} y & x \\ & 1 \end{array} \right) \right) = |y|^{\frac{1}{4}} \sum_{r \in \mathbf{Z}} e\left(\gamma_A Q(r)(x+iy) \right),$$

which has conductor of size roughly γ_A, to derive the more general integral presentation

$$\begin{aligned} &\int_0^1 \phi \overline{\theta}_{\gamma_A, Q}\left(\left(\begin{array}{cc} y & x \\ & 1 \end{array} \right) \right) e\left(-b^2 \epsilon_A x \right) dx \\ &= |y|^{\frac{1}{4}} \sum_{\substack{r \in \mathbf{Z} \\ Q(r) + b^2 \epsilon_A \neq 0}} \frac{\lambda(\gamma_A Q(r) + b^2 \epsilon_A)}{|\gamma_A Q(r) + b^2 \epsilon_A|^{\frac{1}{2}}} W_\phi\left((\gamma_A Q(r) + b^2 \epsilon_A) y \right) \psi(iyQ(r)). \end{aligned} \tag{49}$$

The significance of the unipotent integral identities (48) and (49) is that they allow us to realize the shifted convolution sums on the right-hand side as the Fourier-Whittaker coefficients at the nonzero integer $b^2 \epsilon_A$ of the metapletic forms defined by $\phi \overline{\theta}_Q$ and $\phi \overline{\theta}_{\gamma_A, Q}$ respectively. In this way, the corresponding bounds (42) and the special case of (43) with $\delta_A = 0$ can be derived after taking the spectral decompositions of the forms

$$\phi \overline{\theta}_Q\left(\left(\begin{array}{cc} \frac{1}{Y} & \\ & 1 \end{array} \right) \right) \quad \text{and} \quad \phi \overline{\theta}_{\gamma_A, Q}\left(\left(\begin{array}{cc} \frac{1}{Y} & \\ & 1 \end{array} \right) \right),$$

then passing to the respective unipotent integrals on the left-hand sides of (48) and (49) to derive bounds from existing estimates for Fourier coefficients of half-integral weight forms and Whittaker functions near zero, as derived in [42, § 4-6] (for instance). Note that the argument of [42, §6, Theorem 1] shows more generally for Φ any genuine automorphic form on the two-fold metaplectic cover $\overline{G}(\mathbf{A})$ of $\mathrm{GL}_2(\mathbf{A})$, with α any nonzero integer and $Y \gg |\alpha|$ any real parameter, we have for any $\varepsilon > 0$ the uniform bound

$$\begin{aligned} W_\Phi\left(\left(\begin{array}{cc} \frac{\alpha}{Y} & \\ & 1 \end{array} \right) \right) &:= \int_0^1 \Phi\left(\left(\begin{array}{cc} 1 & x \\ & 1 \end{array} \right) \left(\begin{array}{cc} \frac{1}{Y} & \\ & 1 \end{array} \right) \right) e(-\alpha x) dx \\ &\ll_\varepsilon |\alpha|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|\alpha|}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} \end{aligned} \tag{50}$$

for the Fourier-Whittaker coefficient of Φ at α. Again, such bounds are derived via the spectral decomposition of Φ, after passing to unipotent integrals, then using a standard Sobolev norms argument to reduce to the best existing approximations for Fourier coefficients of half-integral weight forms,[8] together with bounds for classical Whittaker functions $W_{p,ir}(y)$ for $y \to 0$ (as derived in [42, §7]). We then derive the stated bounds from the integral presentations (48) and (49) described above with $y = 1/Y$, multipliying in the factor of $Y^{\frac{1}{4}}$ from the metaplectic theta series. In this way, we can derive the bound (42) and hence (40) for (i).

(4) Bounds for local sums in (ii) via spectral decompositions of shifted convolution sums. We now consider the following variation for (ii). Although not necessary for our main theorem with the principal class $A = \mathbf{1}$ when $D \equiv 0 \bmod 4$, we derive this so that we can give more general estimates which apply to any of the sums $H_{A,j}^{(k)}(\alpha, \beta; Z)$ introduced above. To this end, we generalize the argument of Blomer [1, § 4] to derive a bound for the local sum (43), and hence for the global sum (41). Here, we first argue that we can decompose our new vector $\phi \in V_\pi$ into a linear combination of Poincaré series. Decomposing into such a linear combination, we then open up Fourier-Whittaker coefficients and apply Poisson summation to relate to linear combinations of certain distinct genuine Poincaré series on the metaplectic cover $\overline{G}(\mathbf{A})$. As in the previous case (i), this calculation will reduce us after spectral decomposition to existing bounds for half-integral weight forms and Whittaker functions near zero.

Let us explain this reduction to metaplectic Poincaré series in several steps. We again fix a smooth weight function $W \in \mathcal{C}_c^\infty(\mathbf{R}_{>0})$ supported on $[1/2, 1]$ with $W^{(i)} \ll 1$ for each $i \geq 0$, and a real parameter $R \geq 1$. We also fix a nonzero integer $1 \leq b \leq (R/|\epsilon_A|)^{\frac{1}{2}}$ for our sum, and consider the corresponding quadratic polynomial defined by $f_{A,b}(x) := q_A(x,b) = \gamma_A x^2 + \delta_A b x + \epsilon_A b^2$, which we write in simpler notations as $f_{A,b}(x) = \gamma_A x^2 + \delta_A' x + \epsilon_A'$. Note that the discriminant of this quadratic polynomial is given

[8] which by theorems of Kohnen-Zagier and Waldspurger is equivalent to the best approximation towards the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A})$-automorphic forms

by $\delta_A'^2 - 4\gamma_A \epsilon_A' = b^2(\delta_A^2 - 4\gamma_A\epsilon_A) = b^2 \Delta$. Note as well that if we choose a new vector $\phi = \otimes_v \phi_v \in V_\pi$ whose archimedean local component ϕ_∞ satisfies the constraint that $W_\infty(y) = W(y)$ as functions of $y \in \mathbf{R}^\times \cong \mathbf{Q}_\infty^\times$, as we may thanks to the surjectivity of the archimedean Kirillov map, then we have the natural presentation

$$\begin{aligned}(51)\qquad &\sum_{r \in \mathbf{Z}} \frac{\lambda(f_{A,b}(r))}{|f_{A,b}(r)|^{\frac{1}{2}}} W\left(\frac{f_{A,b}(r)}{R}\right) \\ &= \sum_{r \in \mathbf{Z}} W_\phi\left(\begin{pmatrix} \frac{f_{A,b}(r)}{R} & \\ & 1 \end{pmatrix}\right) = \sum_{r \in \mathbf{Z}} \int_0^1 \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right)(-f_{A,b}(r)x)dx.\end{aligned}$$

Let us also explain for later how this choice of ϕ can be realized in a more explicit way via absolutely convergent linear combinations of Maass weight-raising operators, in the style of the arguments of Motohashi (see e.g. [32]). Recall that in the notations we define above, a cuspidal form φ on $\mathrm{GL}_2(\mathbf{A})$ of weight k and spectral parameter ν has the more explicit Fourier-Whittaker expansion

$$(52)\quad \varphi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in \mathbf{Q}^\times} W_\varphi\left(\begin{pmatrix} \gamma y & \\ & 1 \end{pmatrix}\right) e(\gamma x) = \sum_{\gamma \in \mathbf{Q}^\times} \frac{\lambda_\varphi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot W_{\frac{\mathrm{sgn}(k)}{2}, \nu - \frac{1}{2}}(4\pi|\gamma|y_\infty) \cdot e(\gamma x)$$

where each $W_{\kappa,\mu}$ denotes the classical Whittaker function defined as in [42, §7] (for instance). Hence, for $s \in \mathbf{C}$ with $\Re(s) > \frac{1}{2} \pm \nu$, this $W_{\kappa,\mu}$ can be viewed as the function of $y_\infty \in \mathbf{Q}_\infty^\times \cong \mathbf{R}^\times$ defined implicitly by the Mellin transform relation

$$\int_0^\infty W_{\kappa,\mu}(y_\infty) y_\infty^s \frac{dy_\infty}{y_\infty} = \frac{\Gamma\left(\frac{1}{2} + s + \mu\right)\Gamma\left(\frac{1}{2} + s - \mu\right)}{\Gamma(1 + s - \kappa)}.$$

Note that a suitable normalization of these classical Whittaker functions supplies an orthonormal basis of the Hilbert space $L^2(\mathbf{R}^\times)$; we refer to [3, (23)-(25)] or [6, § 4] for more details. Now, recall that we also have the first Maass weight-raising operator R_k, defined on $y_\infty \in \mathbf{R}^\times$ and $x_\infty \in \mathbf{R}$ by

$$R_k = i y_\infty \cdot \frac{\partial}{\partial x_\infty} + y_\infty \cdot \frac{\partial}{\partial y_\infty} + \frac{k}{2}.$$

It is well-known that this operator R_k raises the weight of a form φ of weight k by 2, so that the cusp form defined by $R_k \varphi$ has weight $k+2$. In fact, it is also well-known that for any $\gamma \in \mathbf{Q}^\times$, we have

$$R_k\left(W_{\frac{k}{2},\mu}(4\pi|\gamma|y_\infty)e(\gamma x_\infty)\right) = c_{\kappa,\mu} \cdot W_{\frac{k+2}{2},\mu}(4\pi|\gamma|y_\infty) \cdot e(\gamma x_\infty),$$

where

$$c_{\kappa,\mu} := \begin{cases} -1 & \text{if } \gamma > 0 \\ -\left(\mu^2 - \left(\frac{k+1}{2}\right)\right)^2 & \text{if } \gamma < 0. \end{cases}$$

Observe that this operator R_k does not affect the finite Fourier-Whittaker coefficient of the cuspidal form φ. In particular, for φ as defined in (52), the image $R_k\varphi$ has the Fourier-Whittaker expansion

$$R_k\varphi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in \mathbf{Q}^\times} \frac{\lambda_\varphi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot \left(c_{\frac{sgn(k)+2}{2}, \nu - \frac{1}{2}} \cdot W_{\frac{\mathrm{sgn}(k)+2}{2}, \nu - \frac{1}{2}}(4\pi|\gamma|y_\infty)\right) \cdot e(\gamma x).$$

Note that if $\varphi = \widetilde{f}$ is the lift to $\mathrm{GL}_2(\mathbf{A})$ of a holomorphic cusp form f of weight k, then we have that $W_{\frac{k}{2}, \frac{k-1}{2}}(y_\infty) = y_\infty^{\frac{k}{2}} e^{-\frac{y_\infty}{2}}$ (see e.g. [42, §3]). Let us now consider the operator $\mathcal{R}_k$ defined for a sequence of complex coefficients $\{\mathfrak{K}_{k+2j}\}_{j \geq 0}$ by

$$\mathcal{R}_k = \mathcal{R}_k(\{\mathfrak{K}_{k+2j}\}_i) = \sum_{j \geq 0} \mathfrak{K}_{k+2j} \cdot R_{k+2j}.$$

Applying this operator $\mathcal{R}_k$ to our fixed cusp form φ of weight k as above gives us some non-K-finite cuspidal form with the explicit Fourier-Whittaker expansion

$$\mathcal{R}_k\varphi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in \mathbf{Q}^\times} \frac{\lambda_\varphi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot \left(\sum_{j \geq 0} \mathfrak{K}_{k+2j} \cdot R_{k+2j} W_{\frac{\mathrm{sgn}(k)}{2}, \nu}(4\pi|\gamma|y_\infty)\right) \cdot e(\gamma x).$$

In particular, our choice of pure tensor $\phi = \otimes_v \phi_v$ with specified archimedean component ϕ_∞ above can be realized explicitly as such a form $\phi = \mathcal{R}_k \varphi$, where the condition on the corresponding archimedean local Whittaker function W_{ϕ_∞} with respect to the chosen weight function W is equivalent to the condition that

$$\begin{aligned} W_\phi(y_\infty) &:= W_\phi\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) \\ &= W(y_\infty) = \mathcal{R}_k\left(W_{\frac{sgn(k)}{2}, \nu - \frac{1}{2}}(4\pi y_\infty)\right) = \sum_{j \geq 0} \mathfrak{K}_{k+2j} \cdot R_{k+2j}\left(W_{\frac{\mathrm{sgn}(k)}{2}, \nu}(4\pi y_\infty)\right) \end{aligned} \tag{53}$$

as functions of $y_\infty \in \mathbf{Q}_\infty^\times \cong \mathbf{R}^\times$. In particular, we deduce from the decay conditions imposed on our chosen weight function W that the expansion on the right hand side of (53) is absolutely convergent.

Taking for granted this more explicit setup, let us now explain how to reduce the estimate for this sum (51) to a special case of the argument for (i). Following [1, §4], we first explain how decompose this chosen vector ϕ into a linear combination of Poincaré series, then open up the coefficients and apply Poisson summation to identify a sum of (parts of) Fourier-Whittaker coefficients at $b^2\Delta$ of some Poincaré series on the metaplectic cover $\overline{G}(\mathbf{A})$. To make this precise, and to avoid any direct use of Kuznetsov formulae, let us first include a few words about generalized Poincaré series and their Fourier expansions following [9, Proposition 2.5] and [35]. Let us for simplicity write G to denote either GL_2 or its two-fold metaplectic cover $\overline{G}$, with the context making the choice of G clear. We fix a congruence subgroup $\Gamma \in \mathrm{GL}_2(\mathbf{A})$, and write Γ_∞ to denote the stabilizer of the cusp at infinity. Given $\xi = \otimes_v \xi_v$ an idele class character, and $\gamma \in \Gamma$ a matrix, we define $\xi(\gamma)$ in the usual way via evaluation at the lower left entry. Let us also fix a theta multiplier $\nu : \mathrm{GL}_2 \to \mathbf{C}$ (see [35, (3), (4)] and [14, § 2]). Writing $N_2 \subset G$ again to denote the unipotent subgroup of upper triangular matrices, and taking $\psi = \otimes_v \psi_v$ to be any nontrivial additive character of $\mathbf{A}/\mathbf{Q}$, we consider the space $\mathcal{S}\left(N_2(\mathbf{A})\backslash G(\mathbf{A}); \psi\right)$ of decomposable Schwartz functions $\varphi = \otimes_v \varphi_v : G(\mathbf{A}) \to \mathbf{C}$ which are smooth and compactly supported, as well as compact and rapidly decreasing modulo $N_2(\mathbf{R})$ (see [9, §2]), and whose left action by $N_2(\mathbf{A})$ is given through the chosen additive character ψ, i.e. $\varphi(ng) = \psi(n)\varphi(g)$ for all $n \in N_2(\mathbf{A})$ and $g \in G(\mathbf{A})$. Given such a function φ, we can then consider the corresponding Poincaré series on $g \in G(\mathbf{A})$ given by

$$P_{\varphi, \xi, \nu}(g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\xi(\gamma)} \cdot \overline{\nu(\gamma)} \cdot \varphi(\gamma g),$$

dropping the subscripts ξ and ν when the respective idele class character or theta multiplier is trivial. Let us now write $\psi = \otimes_v \psi_v$ to denote the standard additive character, whose real component ψ_∞ is given on $x \in \mathbf{Q}_\infty \cong \mathbf{R}$ by the familiar additive character $\psi_\infty(x) = e(x) := \exp(2\pi i x)$. In the event that $G = \mathrm{GL}_2$ and hence the multiplier ν is trivial, and we take a Schwartz function $\varphi \in \mathcal{S}(N_2(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A}); \psi_1)$ which transforms under the additive character defined on $x \in \mathbf{A}$ by $\psi_1(x) = \psi_\infty(lx) = e(lx)$ for some integer $l \in \mathbf{Z}$, we shall write the corresponding Poincaré series $P_{\varphi, \xi, \nu} = P_{\varphi, \xi, \mathbf{1}}$ simply as P_l.

We argue as follows that we can decompose our smooth new vector $\phi \in V_\pi$ into a linear combination

$$\phi = \sum_{l \neq 0} c_l(\phi) \cdot P_l \tag{54}$$

for some complex numbers $c_l(\phi) \in \mathbf{C}^\times$. To be more precise, we shall later assume following (53) above that

$$\phi = \mathcal{R}_l \widetilde{f} = \mathcal{R}_l(\{\mathfrak{K}_{l+2j}\}_j)\widetilde{f} \tag{55}$$

for $\widetilde{f}$ the lift to $\mathrm{GL}_2(\mathbf{A})$ of a holomorphic cusp form f of weight k. However, we can decompose any pure tensor $\phi = \otimes_v \phi_v$ into a linear combination of Poincaré series as follows. To be more precise, we can give a generalization of the argument for the classical case (see e.g. [21, Lemma 14.3 and Corollary 14.4]). Taking for granted all of the notations defined above, we first observe that each Petersson inner product $\langle \phi, P_l \rangle = \langle \phi, P_{\varphi, \xi, \mathbf{1}} \rangle$ recovers the Fourier-Whittaker coefficient at l of ϕ. To see this, we use the automorphy of ϕ and collapse the summation after expanding the definition of $\overline{P_l}(g)$ to compute

$$\begin{aligned}\langle \phi, P_l \rangle &= \int_{Z_2(\mathbf{A}) \operatorname{GL}_2(\mathbf{Q}) \backslash \operatorname{GL}_2(\mathbf{A})} \phi(g) \overline{P_l}(g) dg = \int_{Z_2(\mathbf{A}) \operatorname{GL}_2(\mathbf{Q}) \backslash \operatorname{GL}_2(\mathbf{A})} \phi(g) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \xi(\gamma) \overline{\varphi(\gamma g)} dg \\ &= \int_{Z_2(\mathbf{A}) \operatorname{GL}_2(\mathbf{Q}) \backslash \operatorname{GL}_2(\mathbf{A})} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\gamma g) \overline{\varphi(\gamma g)} dg = \int_{Z_2(\mathbf{A}) \operatorname{GL}_2(\mathbf{Q}) \backslash \operatorname{GL}_2(\mathbf{A})} \phi(g) \overline{\varphi(g)} dg,\end{aligned}$$

which after passing to the standard fundamental domain is the same as

$$\langle \phi, P_l \rangle = \int_{\mathbf{R}_{>0}} \int_0^1 \phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \varphi\left(\begin{pmatrix} y & -x \\ & 1 \end{pmatrix}\right) \frac{dxdy}{y^2},$$

and which after opening up the Fourier-Whittaker expansion of ϕ and using the unipotent transformation property of the chosen Schwartz function φ is the same as

$$\begin{aligned}\langle \phi, P_l \rangle &= \int_{\mathbf{R}_{>0}} \varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \sum_{\gamma \in \mathbf{Q}^\times} W_\phi\left(\begin{pmatrix} \gamma y & \\ & 1 \end{pmatrix}\right) \int_0^1 e(\gamma x - lx) dx \frac{dy}{y^2} \\ &= \int_{\mathbf{R}_{>0}} \varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) W_\phi\left(\begin{pmatrix} ly & \\ & 1 \end{pmatrix}\right) \frac{dy}{y^2} = \frac{\rho_\phi(l)}{|l|^{\frac{1}{2}}} \int_{\mathbf{R}_{>0}} \varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) W_\phi(ly) \frac{dy}{y^2}.\end{aligned} \tag{56}$$

Again, we use the orthogonality of additive characters on the compact abelian group $[0,1] \cong \mathbf{R}/\mathbf{Z}$, and factorize the Whittaker coefficient of ϕ into nonarchimedean and archimedean components. Using this identity (56), we deduce that ϕ can be expressed as a linear combination of such Poincaré series $\{P_l\}_l$ as claimed for (54). That is, we deduce that the closure of the span of $\langle P_l \rangle_l$ equals the cuspidal spectrum $L^2_{\text{cusp}}(\operatorname{GL}_2(\mathbf{Q}) \backslash \operatorname{GL}_2(\mathbf{A}), \xi)$. Suppose now that we assume $\phi = \mathcal{R}_k \widetilde{f}$ is constructed from the lift to $\operatorname{GL}_2(\mathbf{A})$ of a holomorphic cusp form f of weight k as above. Then, a better-known classical argument such as that used in [1] implies that such a $\widetilde{f}$ can be decomposed into a *finite* linear combination of Poincaré series,

$$\widetilde{f} = \sum_m c_l(\widetilde{f}) \cdot P_m. \tag{57}$$

Suppose now that we apply the operator $\mathcal{R}_k$ for which $\phi = \mathcal{R}_k \widetilde{f}$ to this expansion. Since the decomposition (57) of $\widetilde{f}$ into Poincaré series is finite, and the sum defining $\mathcal{R}_k \widetilde{f}$ absolutely convergent, we see that

$$\begin{aligned}\phi = \mathcal{R}_k \widetilde{f} = \mathcal{R}_k \left(\sum_m c_m(\widetilde{f}) \cdot P_m \right) &= \sum_{j \geq 0} \mathfrak{K}_{k+2j} \cdot R_{k+2j} \left(\sum_m c_m(\widetilde{f}) \cdot P_m \right) \\ &= \sum_m c_m(\widetilde{f}) \left(\sum_{j \geq 0} \mathfrak{K}_{k+2j} \cdot R_{k+2j} \cdot P_m \right) = \sum_m c_m(\widetilde{f}) \cdot \mathcal{R}_j P_m\end{aligned}$$

with Fourier series expansion

$$\begin{aligned}\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) &= \sum_m c_m(\widetilde{f}) \cdot \mathcal{R}_k P_m \left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) \\ &= \sum_m c_m(\widetilde{f}) \cdot \mathcal{R}_k \sum_{\gamma \in \mathbf{Q}} W_{P_m}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) e(\gamma x) \\ &= \sum_m c_m(\widetilde{f}) \cdot \mathcal{R}_k \sum_{\gamma \in \mathbf{Q}} \frac{\rho_{P_m}(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot W_{P_m}(\gamma y_\infty) \cdot e(\gamma x) \\ &= \sum_m c_m(\widetilde{f}) \cdot \sum_{\gamma \in \mathbf{Q}} \frac{\rho_{P_m}(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot (\mathcal{R}_k W_{P_m}(\gamma y_\infty)) \cdot e(\gamma x).\end{aligned}$$

Hence, writing $P_m(\phi)$ to denote the Poincaré series constructed from P_m by taking the archimedean Whittaker coefficient to be $W_{P_m(\phi)}(y_\infty) = \mathcal{R}_k W_{P_m}(y_\infty)$ (again as functions of $y_\infty \in \mathbf{Q}_\infty^\times \cong \mathbf{R}^\times$), we can take our decomposition of the chosen pure tensor $\phi = \mathcal{R}_k \widetilde{f}$ into Poincaré series to be the corresponding finite sum

$$\phi = \sum_l c_l(\phi) \cdot P_l(\phi) = \sum_m c_m(\widetilde{f}) \cdot \mathcal{R}_k P_m. \tag{58}$$

That is, since we now assume that $\phi = \widetilde{f}$ arises from the lift of a holomorphic eigenform f of some weight k, we can also deduce as in the argument of Blomer [1] that the sum over Poincaré series is finite, and in particular ignore the contributions of the coefficients $c_l(\phi)$ in our subsequent calculations. Hence, we now use this decomposition (58) of ϕ to derive bounds via the following argument, writing $P_l = P_l(\phi)$ to simplify notations, and treating these non-K-finite Poincaré series abstractly as we may. Here, we follow the general derivation of Fourier-Whittaker expansions given above, again writing φ to denote the underlying section in the space of Schwarz functions – taking for granted the context will distinguish this from the discussion of cuspidal forms above. Taking unipotent integrals in the expansion (58), we obtain the corresponding expansion of Fourier-Whittaker coefficients

$$\begin{aligned} W_{P_l}\left(\begin{pmatrix} \frac{f_{A,b}(r)}{R} & \\ & 1 \end{pmatrix}\right) &:= \int_{\mathbf{A}/\mathbf{Q}} P_l\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) e(-f_{A,b}(r)x)dx \\ &= \int_0^1 P_l\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) e(-f_{A,b}(r)x)dx \end{aligned}$$

for each integer $r \neq 0$. It follows that we have the corresponding decomposition of the sum (51) as

$$\sum_{r \in \mathbf{Z}} W_\phi\left(\begin{pmatrix} \frac{f_{A,b}(r)}{R} & \\ & 1 \end{pmatrix}\right) = \sum_{l \neq 0} c_l(\phi) \cdot \sum_{r \in \mathbf{Z}} W_{P_l}\left(\begin{pmatrix} \frac{f_{A,b}(r)}{R} & \\ & 1 \end{pmatrix}\right). \tag{59}$$

Now, to bound this latter expression in (51), we must first consider the following general result about Fourier-Whittaker coefficients of Poincaré series (see e.g. [9, Proposition 2.5]). To describe this, we first set up some extra notations following [9]. Hence, let us write

$$\Omega(\Gamma) = \left\{ c \in \mathbf{R}^\times : N_2(\mathbf{R}) \cdot w \cdot \underline{c} \cdot N_2(\mathbf{R}) \cap \Gamma \neq \emptyset \right\}, \quad w := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad \underline{c} := \begin{pmatrix} c & \\ & 1 \end{pmatrix}.$$

Given $c \in \Omega(\Gamma)$, we then consider the congruence subgroup $\Gamma_c \subset \Gamma$ defined by $\Gamma_c = N_2(\mathbf{R}) \cdot w \cdot \underline{c} \cdot N_2(\mathbf{R}) \cap \Gamma$. We also write the Bruhat decomposition of any matrix $\gamma \in \Gamma_c$ as $\gamma = n_1(\gamma) \cdot w \cdot \underline{c} \cdot n_2(\gamma)$. To be more explicit, using the elementary matrix decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} c & \\ & c^{-1}(ad-bc) \end{pmatrix}\begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix}$$

we have that

$$n_1(\gamma) = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix}, \quad n_2(\gamma) = \begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_c.$$

Now, we have the following general description of Fourier-Whittaker coefficients of Poincaré series $P_{\varphi,\xi,\nu}$ as defined above on $g \in G(\mathbf{A})$, i.e. with $\varphi \in \mathcal{S}(N_2(\mathbf{A}) \backslash G(\mathbf{A}); \psi_1)$ for ψ_1 some fixed additive character of $\mathbf{A}/\mathbf{Q}$. Let ψ_2 be any additive character of $\mathbf{A}/\mathbf{Q}$. Then, the corresponding Fourier-Whittaker coefficient

$$W_{P_{\varphi,\xi,\nu},\psi_2}(g) := \int_{\mathbf{A}/\mathbf{Q}} P_{\varphi,\xi,\nu}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi_2(-x)dx = \int_0^1 P_{\varphi,\xi,\nu}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi_2(-x)dx$$

is given by the formula

$$\begin{aligned} W_{P_{\varphi,\xi,\nu},\psi_2}(g) &= \int_{\mathbf{A}/\mathbf{Q}} \varphi(g)\psi_1(x)\psi_2(-x)dx \\ &+ \sum_{c \in \Omega(\Gamma)} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(\gamma) \cdot \overline{\nu(\gamma)} \cdot \psi_1(n_1(\gamma)) \cdot \psi_2(n_2(\gamma)) \right) \int_{\mathbf{A}/\mathbf{Q}} \varphi\left(w \cdot \underline{c} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot g\right)\psi_2(-x)dx. \end{aligned} \tag{60}$$

Indeed, this identity (60) can be checked easily after opening the expansion of the Poincaré series and using the Bruhat decomposition $\Gamma = \Gamma_\infty \cup \bigcup_{c \in \Omega(\Gamma)} \Gamma_c$ as in the proof of [9, Proposition 2.5]; we omit details for brevity. In particular, if for nonzero integers r and l we take the additive characters defined on $x \in \mathbf{A}/\mathbf{Q}$ by $\psi_1(x) = \psi_\infty(lx) = e(lx)$ and $\psi_2(x) = \psi_\infty(rx) = e(rx)$, then we obtain

$$W_{P_{\varphi,\xi,\nu},\psi_2}(g) = \varphi(g) \cdot \int_{\mathbf{A}/\mathbf{Q}} e(lx - rx)dx + \sum_{c \in \Omega(\Gamma)} \mathrm{Kl}_{\Gamma,\xi,\nu}(l, m; c) \cdot \mathcal{F}_{\varphi,r,c}(g), \tag{61}$$

where $\mathrm{Kl}_{\Gamma,\xi,\nu}(l,r;c)$ denotes the Kloosterman sum defined by

$$\begin{aligned}\mathrm{Kl}_{\Gamma,\xi,\nu}(l,r;c) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(\gamma) \cdot \overline{\nu(\gamma)} \cdot \psi_1(n_1(\gamma)) \cdot \psi_2(n_2(\gamma)) \\ &= \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(d) \cdot \overline{\nu(\gamma)} \cdot e\left(\frac{al}{c}\right) e\left(\frac{dr}{c}\right)\end{aligned}$$

and $\mathcal{F}_{\varphi,r,c}(g)$ the intertwining integral defined by

$$\mathcal{F}_{\varphi,r,c}(g) = \int_{\mathbf{R}} \varphi\left(w \cdot \underline{c} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot g\right) e(-rx)dx.$$

Using this description of the Fourier coefficients of each P_l in the decomposition (59), we then derive

$$\begin{aligned}&\sum_{l \neq 0} c_l(\phi) \sum_{r \in \mathbf{Z}} W_{P_l}\left(\begin{pmatrix} \frac{f_{A,b}(r)}{R} & \\ & 1 \end{pmatrix}\right) \\ &= \sum_{l \neq 0} c_l(\phi) \sum_{r \in \mathbf{Z}} \left(\varphi\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) \int_0^1 e(lx - f_{A,b}(r)x)dx + \sum_{c \in \Omega(\Gamma)} \mathrm{Kl}_{\Gamma,\xi,\mathbf{1}}(l, f_{A,b}(r); c) \cdot \mathcal{F}_{\varphi,f_{A,b}(r),c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right)\right).\end{aligned} \tag{62}$$

Now, observe that the first summand in this latter expression in negligible, and vanishes unless $l = f_{A,b}(r)$. In particular, it is enough to estimate the second term

$$\begin{aligned}&\sum_{l \neq 0} c_l(\phi) \sum_{r \in \mathbf{Z}} \sum_{c \in \Omega(\Gamma)} \mathrm{Kl}_{\Gamma,\xi,\mathbf{1}}(l, f_{A,b}(r); c) \cdot \mathcal{F}_{\varphi,f_{A,b}(r),c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) \\ &= \sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(d) e\left(\frac{al}{c}\right) \sum_{r \in \mathbf{Z}} e\left(\frac{df_{A,b}(r)}{c}\right) \mathcal{F}_{\varphi,f_{A,b}(r),c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right)\end{aligned}$$

as $R \to \infty$, which after partitioning the r-sum into congruence classes $u \bmod c$ is the same as

$$\sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(d) e\left(\frac{al}{c}\right) \sum_{u \bmod c} \sum_{\substack{r \in \mathbf{Z} \\ r \equiv u \bmod c}} e\left(\frac{df_{A,b}(r)}{c}\right) \mathcal{F}_{\varphi,f_{A,b}(r),c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right). \tag{63}$$

We now apply the Poisson summation formula (cf. [1, Lemma 1], [21, (4.25)]), i.e.

$$\sum_{\substack{r \in \mathbf{Z} \\ r \equiv u \bmod c}} e\left(\frac{df_{A,b}(r)}{c}\right) \cdot \mathcal{F}_{\varphi,f_{A,b}(r),c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) = \frac{1}{|c|} \sum_{h \in \mathbf{Z}} \widehat{\mathcal{F}}_{\varphi,\frac{f_{A,b}(h)}{c},c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) e\left(\frac{df_{A,b}(u) + hu}{c}\right),$$

with Fourier transform

$$\begin{aligned}\widehat{\mathcal{F}}_{\varphi,\frac{f_{A,b}(h)}{c},c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) &= \int_{\mathbf{R}} \mathcal{F}_{\varphi,\frac{f_{A,b}(t)}{c},c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) e\left(-ht\right) dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi\left(w \cdot \underline{c} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix}\right) e\left(\frac{f_{A,b}(t)}{c} \cdot x\right) dxe(-ht)dt,\end{aligned}$$

to each class u mod c (i.e. where the R variable is constant) to obtain from (63) the expression

(64)

$$\sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \frac{1}{|c|} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(d) e\left(\frac{al}{c}\right) \sum_{u \bmod c} \sum_{h \in \mathbf{Z}} \widehat{\mathcal{F}}_{\varphi, \frac{f_{A,b}(h)}{c}, c} \left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix} \right) e\left(\frac{df_{A,b}(u) + hu}{c}\right)$$

$$= \sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \frac{1}{|c|} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \xi(d) e\left(\frac{al}{c}\right) \sum_{h \in \mathbf{Z}} \widehat{\mathcal{F}}_{\varphi, \frac{f_{A,b}(h)}{c}, c} \left(\begin{pmatrix} \frac{1}{R} & \\ & 1 \end{pmatrix} \right) \sum_{u \bmod c} e\left(\frac{df_{A,b}(u) + hu}{c}\right).$$

Let us now consider the inner quadratic Gauss sum in this latter expression (64),

$$\sum_{u \bmod c} e\left(\frac{df_{A,b}(u) + hu}{c}\right) = \sum_{u \bmod c} e\left(\frac{d(\gamma_A u^2 + \delta_A' u + \epsilon_A') + hu}{c}\right) = e\left(\frac{d\epsilon_A'}{c}\right) \sum_{u \bmod c} e\left(\frac{d\gamma_A u^2 + (d\delta_A' + h)u}{c}\right).$$

Here, we can evaluate the inner Gauss sum via Poisson summation as in [29, Theorem 9.15] and quadratic reciprocity as in [1, Lemma 7] (for instance). That is, let us for positive integers $r, c > 0$ with at least one being even ($rc \equiv 0 \bmod 2$) and another integer $s \in \mathbf{Z}$ consider the quadratic Gauss sum defined by

$$T(r, s; c) := \sum_{u \bmod c} e\left(\frac{ru^2 + su}{2c}\right).$$

Completing the square via the elementary calculation

$$\frac{r}{2c}\left(x^2 + \frac{s}{r}\right) = \frac{r}{2c}\left(x + \frac{s}{2r}\right)^2 - \frac{s^2}{8c},$$

we get

$$T(r, s; c) = e\left(-\frac{s^2}{8c}\right) \sum_{u \bmod c} e\left(\frac{r}{2c}\left(u + \frac{s}{2r}\right)^2\right) = e\left(-\frac{s^2}{8c}\right) \sum_{u \bmod c} e\left(\frac{ru^2}{2c}\right) = e\left(-\frac{s^2}{8c}\right) T(r, 0; c).$$

On the other hand, we know by the Poisson summation calculation derived in [29, Theorem 9.15] that

$$T(r, 0; c) = e\left(\frac{1}{8}\right) \cdot \left(\frac{c}{r}\right)^{\frac{1}{2}} \cdot \overline{T(c, 0; r)},$$

and hence

$$T(r, s; c) = e\left(\frac{1}{8}\right) \cdot \left(\frac{c}{r}\right)^{\frac{1}{2}} \cdot e\left(-\frac{s^2}{8c}\right) \cdot \overline{T(c, 0; r)}. \tag{65}$$

Applying this formula (65) to evaluate our inner quadratic Gauss sum

$$T(2d\gamma_A, 2(d\delta_A' + h); c) = \sum_{u \bmod c} e\left(\frac{d\gamma_A u^2 + (d\delta_A' + h)u}{c}\right),$$

we then get

$$T(2d\gamma_A, 2(d\delta_A' + h); c) = e\left(-\frac{(d\delta_A' + h)^2/4}{c}\right) \cdot e\left(\frac{1}{8}\right) \cdot \left(\frac{c}{2d\gamma_A}\right)^{\frac{1}{2}} \cdot \overline{T(c, 0; 2d\gamma_A)}. \tag{66}$$

Now, we can evaluate $\overline{T(c, 0; 2d\gamma_A)}$ via the quadratic reciprocity law. Given an integer q, define

$$\epsilon_q = \begin{cases} 1 & \text{if } q \equiv 1 \bmod 4 \\ i & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

By quadratic reciprocity, we have that

$$T(c, 0, 2d\gamma_A) = \sum_{u=1}^{2d\gamma_A} e\left(\frac{c/4 \cdot u^2}{2d\gamma_A}\right) \quad = \begin{cases} 0 & \text{if } 2d\gamma_A \equiv 2 \bmod 4 \text{ or } (c/4, 2d\gamma_A) \neq 1 \\ (1+i) \cdot \epsilon_{c/4}^{-1} \cdot \left(\frac{2d\gamma_A}{c/4}\right) & \text{if } 2d\gamma_A \equiv 0 \bmod 4 \end{cases}$$

and hence

$$\overline{T(c,0,2d\gamma_A)} = \begin{cases} 0 & \text{if } 2d\gamma_A \equiv 2 \bmod 4 \text{ or } (c/4, 2d\gamma_A) \neq 1 \\ (1-i)\cdot \overline{\epsilon}_{c/4}^{-1} \cdot \left(\frac{2d\gamma_A}{c/4}\right) & \text{if } 2d\gamma_A \equiv 0 \bmod 4. \end{cases}$$

Thus, we can evaluate (66) more explicitly as

$$\begin{aligned} &T(2d\gamma_A, 2(d\delta_A' + h); c) \\ &= e\left(-\frac{(d\delta_A'+h)^2/4}{c}\right)\cdot e\left(\frac{1}{8}\right)\cdot\left(\frac{c}{2d\gamma_A}\right)^{\frac{1}{2}}\cdot \begin{cases} 0 & \text{if } 2d\gamma_A \equiv 2 \bmod 4 \text{ or } (c/4, 2d\gamma_A) \neq 1 \\ (1-i)\cdot \overline{\epsilon}_{c/4}^{-1} \cdot \left(\frac{2d\gamma_A}{c/4}\right) & \text{if } 2d\gamma_A \equiv 0 \bmod 4. \end{cases} \end{aligned}$$

In this way, we can evaluate the inner quadratic Gauss sum in the previous expression (64) as
(67)

$$\begin{aligned} &\sum_{u \bmod c} e\left(\frac{df_{A,b}(u)+hu}{c}\right) = e\left(\frac{d\epsilon_A'}{c}\right)\sum_{u \bmod c} e\left(\frac{d\gamma_A u^2 + (d\delta_A'+h)u}{c}\right) \\ &= e\left(\frac{d\epsilon_A'}{c}\right)\cdot e\left(\frac{-(d\delta_A'+h)^2/4}{c}\right)\cdot e\left(\frac{1}{8}\right)\cdot\left(\frac{c}{2d\gamma_A}\right)^{\frac{1}{2}}\cdot \begin{cases} 0 & \text{if } 2d\gamma_A \equiv 2 \bmod 4 \text{ or } (c/4, 2d\gamma_A) \neq 1 \\ (1-i)\cdot \overline{\epsilon}_{c/4}^{-1} \cdot \left(\frac{2d\gamma_A}{c/4}\right) & \text{if } 2d\gamma_A \equiv 0 \bmod 4. \end{cases}. \end{aligned}$$

Substituting back into (64) and switching the order of summation, we obtain the expression

$$\begin{aligned} e\left(\frac{1}{8}\right)\cdot\sum_{l\neq 0} c_l(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)}{|c|^{\frac{1}{2}}} \sum_{\substack{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty \\ 2d\gamma_A\equiv 0 \bmod 4}} \xi(d)\cdot e\left(\frac{al}{c}\right) e\left(\frac{d\epsilon_A'}{c}\right)\cdot\overline{\epsilon}_{c/4}^{-1}\cdot(2d\gamma_A)^{-\frac{1}{2}}\cdot\left(\frac{2d\gamma_A}{c/4}\right) \\ \times \sum_{\substack{h\in\mathbf{Z} \\ 2|\overline{c}_\nu(d\delta_A'+h)}} \widehat{\mathcal{F}}_{\varphi,\frac{f_{A,b}(h)}{c},c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1\end{pmatrix}\right) e\left(\frac{-(d\delta_A'+h)^2/4}{c}\right). \end{aligned}$$

Note that we can write the latter exponential term more explicitly as

$$e\left(\frac{-(d\delta_A'+h)^2/4}{c}\right) = e\left(\frac{-d^2\delta_A'^2/4}{c}\right) e\left(\frac{-d\delta_A' h/2}{c}\right) e\left(\frac{-h^2/4}{c}\right),$$

which after using that $ad \equiv 1 \bmod c$ leads us to the simpler expression

$$\begin{aligned} &e\left(\frac{1}{8}\right)\cdot\sum_{l\neq 0} c_l(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)}{|c|^{\frac{1}{2}}} \\ &\times \sum_{\substack{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty \\ 2d\gamma_A\equiv 0 \bmod 4}} \xi(d)\cdot e\left(\frac{al}{c}\right)\cdot\overline{\epsilon}_{c/4}^{-1}\cdot(2d\gamma_A)^{-\frac{1}{2}}\cdot\left(\frac{2d\gamma_A}{c/4}\right)\cdot e\left(\frac{d(\epsilon_A' - \overline{a}\delta_A'/4)}{c}\right) \\ &\times \sum_{h\in\mathbf{Z}} \widehat{\mathcal{F}}_{\varphi,\frac{f_A(h)}{c},c}\left(\begin{pmatrix} \frac{1}{R} & \\ & 1\end{pmatrix}\right) e\left(\frac{-h^2/4 - d\delta_A' h/2}{c}\right), \end{aligned} \tag{68}$$

which after opening up the Fourier transform and switching the order of summation is the same as

$$
\begin{aligned}
&e\left(\frac{1}{8}\right)\cdot\sum_{l\neq 0}c_l(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)}{|c|^{\frac{1}{2}}}\\
&\times\sum_{\substack{\gamma=\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty\\ 2d\gamma_A\equiv 0 \bmod 4}}\xi(d)\cdot e\left(\frac{al}{c}\right)\cdot\overline{\epsilon}_{c/4}^{-1}\cdot(2d\gamma_A)^{-\frac{1}{2}}\cdot\left(\frac{2d\gamma_A}{c/4}\right)\cdot e\left(\frac{d(\epsilon'_A-\overline{a}\delta'_A/4)}{c}\right)\\
&\times\sum_{h\in\mathbf{Z}}e\left(\frac{-h^2/4-\delta'_Ah/2}{c}\right)\int_{x\in\mathbf{R}}\varphi\left(w\cdot\underline{c}\cdot\left(\begin{array}{cc}1&x\\&1\end{array}\right)\cdot\left(\begin{array}{cc}\frac{1}{R}&\\&1\end{array}\right)\right)\int_{t\in\mathbf{R}}e\left(\frac{-f_{A,b}(t)x-hct}{c}\right)dtdx.
\end{aligned}
\tag{69}
$$

Now, for each $x\neq 0$, we can evaluate the inner integral

$$
\begin{aligned}
\int_{\mathbf{R}}e\left(\frac{-f_{A,b}(t)x-hct}{c}\right)dt&=\int_{-\infty}^{\infty}e\left(-\left(\frac{\gamma_Ax}{c}\right)t^2-\left(\frac{\delta'_Ax+hc}{c}\right)t-\left(\frac{\epsilon'_A}{c}\right)\right)dt\\
&=\int_{-\infty}^{\infty}e^{-2\pi i\left(\left(\frac{\gamma_Ax}{c}\right)t^2+\left(\frac{\delta'_Ax-hc}{c}\right)t+\left(\frac{\epsilon'_A}{c}\right)\right)}dt
\end{aligned}
$$

as

$$
\int_{-\infty}^{\infty}e^{-2\pi i\left(\left(\frac{\gamma_Ax}{c}\right)t^2+\left(\frac{\delta'_Ax+hc}{c}\right)t+\left(\frac{\epsilon'_A}{c}\right)\right)}dt=\sqrt{\frac{c}{2i\gamma_Ax}}\cdot e\left(\frac{(\delta'^2_A-4\gamma_A\epsilon'_A)}{4\gamma_Ac}\cdot x\right)\cdot e\left(\frac{2\delta'_Ahc}{4\gamma_Ac}\right)\cdot e\left(\frac{h^2c^2}{4\gamma_Axc}\right)
$$

via the integral formula

$$
\int_{-\infty}^{\infty}e^{-\left(A(x)t^2+B(x)t+C(x)\right)}dt=\sqrt{\frac{\pi}{A(x)}}\cdot e^{\frac{B(x)^2-4A(x)C(x)}{4A(x)}}
$$

with

$$
A(x)=2\pi i\left(\frac{\gamma_Ax}{c}\right),\quad B(x)=2\pi i\left(\frac{\delta'_Ax+hc}{c}\right),\quad C(x)=2\pi i\left(\frac{\epsilon'_Ax}{c}\right).
$$

In this way, we argue that for some related Schwartz function $\varphi'\in\mathcal{S}(N_2(\mathbf{A})\backslash\overline{G}(\mathbf{A});\psi'')$ modulo $N_2(\mathbf{R})$ with $\psi''(x)=\psi'(x/4\gamma_Ac)$, we can reduce to estimating the simpler sum

$$
\begin{aligned}
&e\left(\frac{1}{8}\right)\cdot\sum_{l\neq 0}c_l(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)}{|c|^{\frac{1}{2}}}\\
&\times\sum_{\substack{\gamma=\left(\begin{array}{cc}a&b\\c&d\end{array}\right)\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty\\ 2d\gamma_A\equiv 0 \bmod 4}}\xi(d)\cdot e\left(\frac{al}{c}\right)\cdot\overline{\epsilon}_{c/4}^{-1}\cdot(2d\gamma_A)^{-\frac{1}{2}}\cdot\left(\frac{2d\gamma_A}{c/4}\right)\cdot e\left(\frac{d(\epsilon'_A-\overline{a}\delta'_A/4)}{c}\right)\\
&\times\mathcal{F}_{\varphi',-\Delta,4\gamma_Ac}\left(\left(\begin{array}{cc}\frac{1}{R}&\\&1\end{array}\right)\right).
\end{aligned}
\tag{70}
$$

To be clear, we argue that we can approximate the expression (69) in terms of the nonzero contributions $x \neq 0$ in the integral, which after evaluating and switching the order of summation of the h-sum equals

$$\begin{aligned} & e\left(\frac{1}{8}\right) \cdot \sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \frac{(1-i)}{|c|^{\frac{1}{2}}} \\ & \times \sum_{\substack{\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty \\ 2d\gamma_A \equiv 0 \bmod 4}} \xi(d) \cdot e\left(\frac{al}{c}\right) \cdot \overline{\epsilon}_{c/4}^{-1} \cdot (2d\gamma_A)^{-\frac{1}{2}} \cdot \left(\frac{2d\gamma_A}{c/4}\right) \cdot e\left(\frac{d(\epsilon_A' - \overline{a}\delta_A'/4)}{c}\right) \\ & \times \int_{\substack{x \in \mathbf{R} \\ x \neq 0}} \varphi\left(w \cdot \underline{c} \cdot \left(\begin{array}{cc} 1 & x \\ & 1 \end{array}\right) \cdot \left(\begin{array}{cc} \frac{1}{R} & \\ & 1 \end{array}\right)\right) \\ & \times \left\{ \sqrt{\frac{c}{2i\gamma_A x}} \sum_{h \in \mathbf{Z}} e\left(\frac{-h^2/4 - \delta_A' h/2}{c}\right) e\left(\frac{2\delta_A' hc}{4\gamma_A c}\right) e\left(\frac{h^2 c^2}{4\gamma_A x c}\right) \right\} e\left(\frac{(\delta_A'^2 - 4\gamma_A \epsilon_A')}{4\gamma_A c} x\right) dx. \end{aligned}$$

We then argue that the inner h-sum in this latter expression can be approximated as a Gaussian integral, and hence as a constant. Now, using the description of the theta multiplier ν for half-integral weight forms given in [14], and making a change of variables $c \to c'' = 4\gamma_A c$, we then argue in the style of [1, §4] that the sum (70) can be approximated by the even simpler expression

$$\begin{aligned} & e\left(\frac{1}{8}\right) \sum_{l \neq 0} c_l(\phi) \sum_{c \in \Omega(\Gamma)} \frac{(1-i)}{|c|^{\frac{1}{2}}} \\ & \times \sum_{\gamma = \left(\begin{array}{cc} a'' & b'' \\ c'' & d'' \end{array}\right) \in \Gamma_\infty \backslash \Gamma_{c''} / \Gamma_\infty} \xi(d'') \left(\frac{d''}{c''}\right) \epsilon_{d''}^{-1} e\left(\frac{a'' l}{c''}\right) e\left(-\frac{d'' b^2 \Delta}{c''}\right) \mathcal{F}_{\varphi', -b^2\Delta, c''}\left(\left(\begin{array}{cc} \frac{1}{R} & \\ & 1 \end{array}\right)\right). \end{aligned} \tag{71}$$

Now, we observe from the description of Fourier-Whittaker coefficients of Poincaré series above that each inner sum over $c \in \Omega(\Gamma)$ in (71) can be described in terms of the Fourier-Whittaker coefficient at $b^2\Delta$ of some genuine metaplectic Poincaré series $\mathcal{P}_l = P_{\varphi', \xi, \nu}$ of half-integral weight. In particular, to estimate the shifted convolution sum (41), it is enough to estimate the sum of Whittaker coefficients

$$\sum_{l \neq 0} c_l(\phi) \cdot W_{\mathcal{P}_l}\left(\left(\begin{array}{cc} -\frac{b^2\Delta}{R} & \\ & 1 \end{array}\right)\right),$$

i.e. where the coefficients on the right-hand side are defined by the usual unipotent integrals

$$\begin{aligned} & W_{\mathcal{P}_l}\left(\left(\begin{array}{cc} -\frac{b^2\Delta}{R} & \\ & 1 \end{array}\right)\right) = W_{\mathcal{P}_l}\left(\left(\begin{array}{cc} -\frac{b^2\Delta}{R} & \\ & 1 \end{array}\right), 1\right) \\ & := \int_{\mathbf{A}/\mathbf{Q}} \mathcal{P}_l\left(\left(\begin{array}{cc} 1 & x \\ & 1 \end{array}\right)\left(\begin{array}{cc} \frac{1}{R} & \\ & 1 \end{array}\right), 1\right) e(-\Delta x) dx = \int_0^1 \mathcal{P}_l\left(\left(\begin{array}{cc} 1 & x \\ & 1 \end{array}\right)\left(\begin{array}{cc} \frac{1}{R} & \\ & 1 \end{array}\right), 1\right) e(-\Delta x) dx \end{aligned}$$

This latter sum can be approximated via the general bound (50) the same way as for (42) via spectral decompositions of genuine metaplectic forms as given e.g. in [42, §6], generalizing the proof given in the classical setting (with $\gamma_A = 1$) in [1, §4]. That is, we find by using the same argument – multiplying in the factor $Y^{\frac{1}{4}}$ to compensate for the fact that the Fourier coefficients of the half-integral weight Poincaré series in this latter decomposition are necessarily proportional to $Y^{-\frac{1}{4}}$ – that we have for each $\varepsilon > 0$ the bound

$$\sum_{r \in \mathbf{Z}} \frac{\lambda(f_{A,b}(r))}{|f_{A,b}(r)|^{\frac{1}{2}}} W\left(\frac{|f_{A,b}(r)|}{R}\right) \ll \left| \sum_{l \neq 0} c_l(\phi) \cdot W_{\mathcal{P}_l}\left(\left(\begin{array}{cc} -\frac{\Delta_b}{R} & \\ & 1 \end{array}\right)\right) \right| \ll_{\pi, \varepsilon} Y^{\frac{1}{4}} \cdot |\Delta_b|^{\delta_0 - \frac{1}{2}} \cdot \left(\frac{|\Delta_b|}{R}\right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

Here again, we write $\Delta_b = b^2\Delta$ for the discriminant of $f_{A,b}$ to simplify notations. In this way, we justify the bounds (43) and (41), and hence the second claim (ii). □

Putting together the bounds of Theorem 3.3 (i) and (ii) with Lemma 3.1, we obtain the following estimates.

Theorem 3.4. *Let $\alpha \geq 0$ be any integer. Let A be any class in $\operatorname{Pic}(\mathcal{O}_{p^\alpha})$, with corresponding binary quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$. We have the following estimates for the balanced sums*

$$H_A^{(k)}(\alpha, 0) = \frac{4}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{a, b \in \mathbf{Z}} \frac{\lambda(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{(\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)^{\frac{1}{2}}} V_{k+1} \left(\frac{m^2 (\gamma_A a^2 + \delta_A ab + \epsilon_A b^2)}{N |D| p^{2\alpha}} \right).$$

Let us again consider the nonvanishing residual terms $\mathfrak{L}_{k,f,\gamma_A}(1)$ defined in Lemma 3.2 and Lemma 3.1 above. We have in either case on the generic root number $k = 0, 1$ the estimate

$$H_A^{(k)}(\alpha, 0) = \mathfrak{L}_{k,f,\gamma_A}(1) + O_{f,p,\varepsilon} \left(|D|^{\frac{3}{16}+\varepsilon} \left(\frac{\gamma_A}{|D| p^{2\alpha}} \right)^{\frac{1}{4}} \right) + O_{f,k,p,\varepsilon} \left(\gamma_A \cdot (|D| p^{2\alpha})^{\frac{7}{16}+\varepsilon} |\epsilon_A|^{-\frac{1}{2}} \right).$$

Proof. The result is a direct consequence of Lemma 3.1 with Theorem 3.3 (ii). A similar albeit weaker bound can be derived using Theorem 3.3 (i) in the special case where $\gamma_A = 1$ and $\delta_A = 0$ (via a simpler proof). ☐

Remark Observe that if γ_A is sufficiently small relative to ϵ_A and hence the discriminant $\delta_A^2 - 4\gamma_A \epsilon_A = p^{2\alpha} D$, then $H^{(k)}(\alpha, 0)$ converges to the nonvanishing residual term $\mathfrak{L}_{k,f,\gamma_A}(1)$ with $\alpha \gg 1$. Note that for the principal class $\mathbf{1} \in \operatorname{Pic}(\mathcal{O}_{p^\alpha})$, we have $\gamma_{\mathbf{1}} = 1$ for $q_{\mathbf{1}}(x,y) = \gamma_{\mathbf{1}} x^2 + \delta_{\mathbf{1}} xy + \epsilon_{\mathbf{1}} y^2$ the reduced representative, and so we can always derive this nonvanishing consequence. Hence, we take $q_{\mathbf{1}}(x,y)$ to be the reduced representative for the principal class $A = \mathbf{1} \in \operatorname{Pic}(\mathcal{O}_{p^\alpha})$ for each $\alpha \geq 0$ to deduce Theorem 1.2 (i) and (ii).

4. Non self-dual estimates

Fix integers $\alpha \geq 0$ and $\beta \geq 4$, as well as a primitive ring class character ρ of K of conductor p^α. We now estimate the averages $\mathcal{H}^{(0)}(\alpha, \beta)$ of Proposition 2.6, as well as the averages $D(\rho, \beta)$ of Proposition 2.7. Let us first work in some more generality than required for these estimates, fixing a class $A \in \operatorname{Pic}(\mathcal{O}_{p^\alpha})$, with $r_A(n)$ its corresponding counting function parametrized as above via the corresponding reduced quadratic form representative q_A. We consider for any choice of unbalancing parameter $Z > 0$ the corresponding sums

$$H_A^{(0)}(\alpha, \beta) = H_{A,1}^{(0)}(\alpha, \beta; Z) + H_{A,2}^{(0)}(\alpha, \beta; Z),$$

where the first sum is defined by

We also consider the sums in the average formula of Proposition 2.7, which recall is given by

$$D(\rho, \beta) = D_1(\rho, \beta; Z) + D_2(\rho, \beta; Z).$$

for any choice of real parameter $Z > 0$, where

$$\begin{aligned} D_1(\rho, \beta; Z) = \sum_{m \geq 1} \frac{\eta(m)}{m} &\sum_{\substack{n \geq 1 \\ m^2 n \equiv \pm 1 \bmod p^\beta}} \frac{c_\rho(n) \lambda(n)}{n^{\frac{1}{2}}} V_1 \left(m^2 n Z \right) \\ &- \frac{1}{\varphi(p)} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ m^2 n \equiv \pm 1 \bmod p^{\beta - 1} \\ m^2 n \not\equiv \pm 1 \bmod p^\beta}} \frac{c_\rho(n) \lambda(n)}{n^{\frac{1}{2}}} V_1 \left(m^2 n Z \right) \end{aligned}$$

and

$$\begin{aligned} &D_2(\rho, \beta; Z) \\ &= \frac{\eta(-N)}{|D|^2 p^{2\beta}} \left(\frac{p}{\varphi(p)} \right) \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{n \geq 1} \frac{c_\rho(n) \lambda(n)}{n^{\frac{1}{2}}} V_1 \left(\frac{m^2 n}{Z N^2 |D|^2 p^{4 \max(\alpha, \beta)}} \right) \operatorname{Kl}_4(\pm (m^2 n \overline{N}^2 \overline{D}^8)^{\frac{1}{2}}, p^\beta). \end{aligned}$$

4.1. **Strategy.** We first describe some background for the twisted sums, including how to evaluate the hyper-Kloosterman $\mathrm{Kl}_4(\pm c, p^\beta)$, together with presentations in terms of Weyl sums. We also present some preliminary ("trivial") estimates for the leading sums to illustrate the context. Finally, taking the balanced approximate functional equation formula corresponding to the parameter $Z = Y^{-1} = (N|D|p^{2\max(\alpha,\beta)})^{-\frac{1}{2}}$, we explain how to derive nonvanishing estimates in a style similar to the self-dual case (with $\beta = 0$) above. That is, we shall open up the counting functions $r_A(n)$ according to the parametrization given in terms of the reduced quadratic form representative $q_A(x, y)$ for the class A. We then derive separate estimates for the $b = 0$ and $b \neq 0$ terms in the style of the arguments above. Although we require some more argument to show that the $b = 0$ terms in this estimate is nonvanishing for $\beta \gg \alpha$ sufficiently large (and $A = \mathbf{1}$ principal), the estimates for the $b \neq 0$ terms are estimated by a variation of the same argument given above.

4.2. **The twisted sums.** Recall that for any integer $n \geq 2$, we introduce the hyper-Kloosterman sum $\mathrm{Kl}_n(c, p^\beta)$ in (22) above, as well as the shorthand notation $\mathrm{Kl}_n(\pm c, p^\beta) = \mathrm{Kl}_n(c, p^\beta) + \mathrm{Kl}_n(-c, p^\beta)$ in (23). If $\beta \geq 4$ and p does not divide the dimension n, then these sums can be evaluated explicitly as follows.

Proposition 4.1 ("Salié")**.** *Fix $n \geq 2$ an integer, and p a prime which does not divide n. Assume that $\beta \geq 4$, and without loss of generality that β is even, say $\beta = 2b$ for some integer $b \geq 2$. Then, for any coprime residue class $c \bmod p^\beta$, we have the formula*

$$\mathrm{Kl}_n(c, p^\beta) = (p^\beta)^{\frac{(n-1)}{2}} \sum_{\substack{w \bmod p^b \\ w^n \equiv c \bmod p^b}} e\left(\frac{(n-1)w + c\overline{w}}{p^\beta}\right).$$

Here, the sum runs over n-th roots of $c \bmod p^b$ (if these exist).

Proof. The result, attributed to Salié, is considered to be classical. However, the main written reference seems to be [5, Theorem C.1, Lemma C.2], where there is a minor error with the formulation of the final statement (which needs to be given in terms of liftings of the roots $r^{1/n}$). We therefore indicate a proof of the stated formula for the convenience of the reader. Let us lighten notation by writing $x = (x_1, \ldots, x_{n-1})$ to denote the 3-tuple of classes $\bmod p^{2b}$. We then write $h = h_c$ denote the function defined on x by

$$h(x) = x_1 + \cdots + x_{n-1} + c\overline{x_1 \cdots x_{n-1}},$$

and $\nabla h(x)$ the column vector determined by $\nabla h(x) = 1 - c\overline{x}^2 = (1 - c\overline{(x_1 \cdots x_{n-1})x_1}, \ldots, 1 - c\overline{(x_1 \cdots x_{n-1})x_{n-1}})$. It is easy to show (cf. [5, Lemme C4]) that we have the expansion $h(x) = h(y) + p^b \nabla h(y) \cdot z$, where $\cdot$ denotes the dot product. Substituting this expansion into the definition of $\mathrm{Kl}_n(c, p^\beta)$ then gives the relation

$$\mathrm{Kl}_n(c, p^\beta) = \sum_{\substack{x \bmod p^{2b} \\ (x, p^{2b})=1}} e\left(\frac{h(x)}{p^{2b}}\right) = \sum_{\substack{y \bmod p^b \\ (y, p^b)=1}} \sum_{z \bmod p^b} e\left(\frac{h(y) + p^b \nabla h(y) \cdot z}{p^{2b}}\right)$$
$$= \sum_{\substack{y \bmod p^{2b} \\ (y, p^{2b})=1}} e\left(\frac{h(y)}{p^{2b}}\right) \sum_{z \bmod p^b} e\left(\frac{\nabla h(y) \cdot z}{p^b}\right).$$

Since the inner sum runs over all $(n-1)$-tuples of classes $z \bmod p^b$, we can use orthogonality of additive characters to evaluate the inner sum, and hence to obtain the relation

$$\mathrm{Kl}_n(c, p^{2b}) = p^{(n-1)b} \sum_{\substack{y \bmod p^{2b} \\ (y, p^{2b})=1 \\ \nabla h(y) \equiv 0 \bmod p^b}} e\left(\frac{h(y)}{p^{2b}}\right).$$

Now, it is easy to see that the solutions to the congruence $\nabla h(y) \equiv 0 \bmod p^b$ take the form of invertible classes $y \bmod p^b$ for which $yy_j \equiv c \bmod p^b$ for each of $j = 1, \ldots, n-1$. This proves the stated formula. $\square$

Corollary 4.2. *Fix an integer $\alpha \geq 0$, together with a class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$. Fix a primitive ring class character ρ of K of conductor p^α. Assume $p > 2$ (and that p hence does not divide the dimension $n = 4$),* *and that $\beta \geq 4$ is even, say $\beta = 2b$ for $b \geq 2$. Then, we have for any $Z > 0$ the equivalent expressions*

$$H_{A,2}^{(0)}(\alpha, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \left(\frac{p}{\varphi(p)} \right) \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{r_A(n)\lambda(n)}{n^{\frac{1}{2}}} V_1 \left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right)$$
$$\times \sum_{\substack{w \bmod p^b \\ w^4 \equiv \pm(m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}} \bmod p^b}} e\left(\frac{3w + (m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}}\overline{w}}{p^\beta} \right)$$

and

$$D_2(\rho, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \left(\frac{p}{\varphi(p)} \right) \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{n \geq 1 \\ (n,p)=1}} \frac{c_\rho(n)\lambda(n)}{n^{\frac{1}{2}}} V_1 \left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right)$$
$$\times \sum_{\substack{w \bmod p^b \\ w^4 \equiv \pm(m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}} \bmod p^b}} e\left(\frac{3w + (m^2 n\overline{N}^2\overline{D}^8)^{\frac{1}{2}}\overline{w}}{p^\beta} \right).$$

4.2.1. *Weyl sums and preliminary estimates.* Corollary 4.2 allows us to express the twisted sums $H_{2,A}^{(0)}(\alpha, \beta; Z)$ and $D_2(\rho, \beta; Z)$ in terms of Weyl sums of p-adic phase as follows, from which it is easy to derive a preliminary (trivial) estimate. We include this description for illustration only, as it might be of independent interest, but note that it does not play a role in our subsequent proofs. Let us retain all of the setup of Corollary 4.2, and simplify notations by writing $e(\pm x) = e(x) + e(-x) = \exp(2\pi i x) + \exp(-2\pi i x)$.

Proposition 4.3. *Fix any integer $1 \leq s \leq \beta$. Given a class $u \bmod p^s$, let $(\frac{u}{p})_8$ denote the octic residue symbol. Given an integer $m \geq 1$, let μ_m denote the coprime residue class $\mu_m \equiv m^2\overline{N}^2\overline{D}^8 \bmod p^s$. Let us also write $F_{u,m,s}(t)$ denote the polynomial in t defined by*

$$F_{u,m,s}(t) = \frac{1}{p^\beta} \left(\sum_{j=0}^{\lfloor \beta/s \rfloor} \binom{\frac{1}{8}}{j} \left[3\xi_{u,m}(\overline{u}\mu_m)^j + \overline{\xi}_{u,m}(u\overline{\mu}_m)^j \right] p^{sj} t^j \right),$$

where

$$\binom{\frac{1}{n}}{j} = \frac{\frac{1}{n}(\frac{1}{n} - 1)\cdots(\frac{1}{n} - j + 1)}{j!}$$

for each integer $n \geq 2$, with $\xi_{u,m}$ some fixed eight root of the class $(m\overline{ND})u\overline{\mu}_m \bmod p^\beta$, and $\lfloor \beta/s \rfloor$ denotes the largest positive integer $j < \beta/s$. We then have for any choice of $Z > 0$ the equivalent expressions

$$H_{A,2}^{(0)}(\alpha, \beta; Z)$$
$$= \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^s \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{t \geq 0} \frac{\lambda(u\overline{\mu}_m + p^s t) r_A(u\overline{\mu}_m + p^s t)}{(u\overline{\mu}_m + p^s t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^s t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,s}(t)\right)$$

and

$$D_2(\rho, \beta; Z)$$
$$= \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^s \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{t \geq 0} \frac{\lambda(u\overline{\mu}_m + p^s t) c_\rho(u\overline{\mu}_m + p^s t)}{(u\overline{\mu}_m + p^s t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^s t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,s}(t)\right).$$

Proof. We divide mn-sums of into congruence classes modulo p^s to obtain

$$H_{A,2}^{(0)}(\alpha, \beta; Z)$$
$$= \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^s \\ (\frac{u}{p})_8 = 1}} \sum_{\substack{m,n \geq 1 \\ (n,p)=1 \\ u \equiv m^2 n\overline{N}^2\overline{D}^8 \bmod p^s}} \frac{\eta(m)\lambda(n)r_A(n)}{mn^{\frac{1}{2}}} V_1 \left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) \sum_{\substack{w \bmod p^b \\ w^8 \equiv u \bmod p^b}} e\left(\frac{3w \pm u\overline{w}}{p^\beta} \right)$$

and

$$D_2(\rho,\beta;Z)$$
$$= \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}}\frac{p}{\varphi(p)} \sum_{\substack{1\leq u\leq p^s \\ (\frac{u}{p})_8=1}} \sum_{\substack{m,n\geq 1 \\ (n,p)=1 \\ u\equiv m^2 n\overline{N}^2\overline{D}^8 \bmod p^s}} \frac{\eta(m)\lambda(n)c_\rho(n)}{mn^{\frac{1}{2}}} V_1\left(\frac{m^2 n}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}}\right) \sum_{\substack{w \bmod p^b \\ w^8\equiv u \bmod p^b}} e\left(\frac{3w\pm u\overline{w}}{p^\beta}\right),$$

where the condition $(\frac{u}{p})_8 = 1$ in the u-sum comes from the w-sum via Hensel's lemma. Note that here, each of the w-sums consists one a single pair of eighth roots $w \bmod p^b$. To give a more explicit description of the eighth roots $w^8 \equiv \pm u \bmod p^b$ appearing in these expressions, fix a class $u \bmod p^s$ for which $(\frac{u}{p})_8 = 1$, and consider the corresponding inner mn-sum. Let $\mu_m \equiv m^2\overline{N}^2\overline{D}^8 \bmod p^s$. Hence $n\mu_m \equiv m^2\overline{N}^2\overline{D}^8 n \bmod p^s$, from which it follows that $n\mu_m \equiv u \bmod p^s$. Hence, we can expand each integer $n \geq 1$ in the second sum in terms of the congruence condition $u \equiv n\mu_m \bmod p^s$ as $n = u\overline{\mu}_m + p^s t$, with $t \geq 0$ varying over positive integers. This in turn gives us the expansion

$$w^8 \equiv \pm m^2\overline{N}^2\overline{D}^8(u\overline{\mu}_m + p^s t) \equiv \pm (m\overline{ND}^4)^2(u\overline{\mu}_m + p^s t) \equiv \pm (m\overline{ND}^4)^2 u\overline{\mu}_m(1+\kappa p^s t) \bmod p^\beta,$$

where $\kappa = \overline{u}\mu_m$ denotes the multiplicative inverse of $u\overline{\mu}_m \bmod p^\beta$. Now, Hensel's lemma ensures that we can find an integer $\xi = \xi_{u,m}$ such that $\xi^8 \equiv (m\overline{ND}^4)u\overline{\mu}_m \bmod p^\beta$. (In fact, there exist $O_p(1)$ many such roots, and we choose one implicitly). Hence, we can express the roots appearing in the corresponding w-sum as

$$w^8 \equiv \pm\xi^8(1+\kappa p^s t) \bmod p^\beta, \tag{72}$$

so that $w \equiv \xi_u(1+\kappa p^s t)^{\frac{1}{8}}$ and $w^4 \equiv \xi_u^4(1+\kappa p^s t)^{\frac{1}{2}}$. Now, to give an even more explicit description of these classes w and w^4, we can use the classical fact that for any $x \in p\mathbf{Z}_p$, the power series

$$\sum_{j\geq 0}\binom{\frac{1}{n}}{j}x^j \ \in \mathbf{Z}_p[[x]]$$

converges in the p-adic norm to the n-th root $(1+x)^{\frac{1}{n}}$ for any integer $n \geq 2$ (see e.g. [36, p. 173]) to obtain

$$(1+\kappa p^s t)^{\frac{1}{n}} = \sum_{j\geq 0}\binom{\frac{1}{n}}{j}\kappa^j p^{sj}t^j,$$

so that

$$w \equiv \xi_u \sum_{j\geq 0}\binom{\frac{1}{8}}{j}\kappa^j p^{sj}t^j \bmod p^\beta.$$

Hence, we derive the more explicit presentation

$$3w + u\overline{w} = 3\xi(1+\kappa p^s t)^{\frac{1}{8}} + u\overline{\xi}(1+\overline{\kappa}p^s t)^{\frac{1}{8}} = \sum_{j\geq 0}\binom{\frac{1}{8}}{j}[3\kappa^j + u\overline{\kappa}^j]p^{sj}t^j. \tag{73}$$

Writing $\Phi_{u,m,s}(t)$ to denote the polynomial in $\mathbf{Q}[t]$ obtained by reducing (73) modulo p^β, we can then write our expression in the obvious way in terms of the Weyl polynomial $F_{u,m,s}(t) := p^{-\beta}\Phi_{u,m,s}(t)$. This gives the stated expressions for the exponential sums described at the start of the proof. □

Corollary 4.4. *Assume $\beta \geq 4$. We have for any choices of $Z > 0$ and $\varepsilon > 0$ the upper bounds*

$$H_{2,A}^{(0)}(\alpha,\beta;Z) = O_{f,D,p,\varepsilon}\left(\left(Zp^{4\max(\alpha,\beta)}\right)^{\frac{1}{2}+\varepsilon}\right)$$

and

$$D_2(\rho,\beta;Z) = O_{f,D,p,\rho,\varepsilon}\left(\left(Zp^{4\max(\alpha,\beta)}\right)^{\frac{1}{2}+\varepsilon}\right).$$

Proof. Taking $s = b = \frac{\beta}{2}$ in Proposition 4.3 above, we have the identities

$$H_{A,2}^{(0)}(\alpha, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^b} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^b \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{t \geq 0} \frac{\lambda(u\overline{\mu}_m + p^b t) r_A(u\overline{\mu}_m + p^b t)}{(u\overline{\mu}_m + p^b t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^b t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,b}(t)\right)$$

and

$$D_2(\rho, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^b} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^b \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{t \geq 0} \frac{\lambda(u\overline{\mu}_m + p^b t) c_\rho(u\overline{\mu}_m + p^b t)}{(u\overline{\mu}_m + p^b t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^b t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,b}(t)\right).$$

Let us now put $T = ZN^2|D|^2 p^{4\max(\alpha,\beta)} p^{-\frac{\beta}{2}}$. Observe that by the rapid decay of the cutoff function V_1, it will suffice to estimate the truncated sums defined by

$$H_{A,2}^{(0);\dagger}(\alpha, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^b} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^b \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{t \geq 0 \\ m^2 t \leq T}} \frac{\lambda(u\overline{\mu}_m + p^b t) r_A(u\overline{\mu}_m + p^b t)}{(u\overline{\mu}_m + p^b t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^b t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,b}(t)\right)$$

and

$$D_2^{\dagger}(\rho, \beta; Z) = \frac{\eta(-N)}{|D|^2 p^b} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^b \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{t \geq 0 \\ m^2 t \leq T}} \frac{\lambda(u\overline{\mu}_m + p^b t) c_\rho(u\overline{\mu}_m + p^b t)}{(u\overline{\mu}_m + p^b t)^{\frac{1}{2}}} V_1 \left(\frac{m^2(u\overline{\mu}_m + p^b t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right) e\left(\pm F_{u,m,b}(t)\right).$$

Recall that this function $V_1(y)$ on is defined on $y \in \mathbf{R}_{>0}$ by the contour integral

$$V_1(y) = \int_{\Re(s)=2} \frac{L_\infty(s+1/2)}{L_\infty(s)} \frac{g^*(s)}{s} y^{-s} \frac{ds}{2\pi i},$$

where the shifted local archimedean L-function $L_\infty(s+1/2)$ has poles at $s = -1$ and $s = -2$. To estimate the behaviour as $y \to 0$, we can therefore shift the contour leftward to $\Re(s) = -B$ for some $0 < B < 1$, crossing only a simple pole at $s = 0$ to derive $V_1(y) = 1 + O_B(y^B)$ as $y \to 0$. Applying this estimate directly to the first truncated sum $H_{A,2}^{(0),\dagger}(\alpha, \beta; Z)$, we find

$$\begin{aligned} H_{A,2}^{(0),\dagger}(\alpha, \beta; Z) = &\frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1 \leq u \leq p^{\frac{\beta}{2}} \\ (\frac{u}{p})_8 = 1}} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{t \geq 0 \\ m^2 t \leq T}} \frac{\lambda(\overline{u}\mu_m + p^{\frac{\beta}{2}} t) r_A(\overline{u}\mu_m + p^{\frac{\beta}{2}} t)}{(\overline{u}\mu_m + p^{\frac{\beta}{2}} t)^{\frac{1}{2}}} e\left(F_{u,m,\frac{\beta}{2}}(t)\right) \\ &+ O_{D,p,\varepsilon}\left(\sum_{m \geq 1} \frac{1}{m} \sum_{\substack{t \geq 0 \\ m^2 t > T}} (\overline{u}\mu_m + p^{\frac{\beta}{2}} t)^{\varepsilon - \frac{1}{2}} \left(\frac{m^2(\overline{u}\mu_m + p^{\frac{\beta}{2}} t)}{ZN^2|D|^2 p^{4\max(\alpha,\beta)}} \right)^B \right) \end{aligned}$$

for any $\varepsilon > 0$ and $0 < B < 1$. It is easy to see that the error term here is bounded above as

$$\ll_{f,D,p,\varepsilon} (Zp^{4\max(\alpha,\beta)})^{-B} (p^{\frac{\beta}{2}})^{\varepsilon - \frac{1}{2} + B} \sum_{m \geq 1} m^{-1+2B} \sum_{\substack{t \geq 0 \\ m^2 t > T}} t^{\varepsilon - \frac{1}{2} + B} \ll (ZN^2|D|^2 p^{4\max(\alpha,\beta)})^{\frac{1}{2}+\varepsilon},$$

so that

$$H_{A,2}^{(0),\dagger}(\alpha,\beta;Z) = \frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1\leq u \leq p^{\frac{\beta}{2}} \\ (\frac{u}{p})_8=1}} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{t\geq 0 \\ m^2 t \leq T}} \frac{\lambda(\overline{u}\mu_m + p^{\frac{\beta}{2}}t) r_A(\overline{u}\mu_m + p^{\frac{\beta}{2}}t)}{(\overline{u}\mu_m + p^{\frac{\beta}{2}}t)^{\frac{1}{2}}} e\left(F_{u,m,\frac{\beta}{2}}(t)\right)$$

$$+ O_{f,D,p,\varepsilon}\left((Zp^{4\max(\alpha,\beta)})^{\frac{1}{2}+\varepsilon}\right).$$

The residual term in this latter estimate is also seen easily to be bounded above by

$$\frac{\eta(-N)}{|D|^2 p^{\frac{\beta}{2}}} \frac{p}{\varphi(p)} \sum_{\substack{1\leq u \leq p^{\frac{\beta}{2}} \\ (\frac{u}{p})_8=1}} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{t\geq 0 \\ m^2 t \leq T}} \frac{\lambda(\overline{u}\mu_m + p^{\frac{\beta}{2}}t) r_A(\overline{u}\mu_m + p^{\frac{\beta}{2}}t)}{(\overline{u}\mu_m + p^{\frac{\beta}{2}}t)^{\frac{1}{2}}} e\left(F_{u,m,\frac{\beta}{2}}(t)\right)$$

$$\ll_{f,D,p,\varepsilon} (p^{\frac{\beta}{2}})^{\varepsilon-\frac{1}{2}} (\log T) T^{\frac{1}{2}+\varepsilon} \ll (Zp^{4\max(\alpha,\beta)})^{\frac{1}{2}+\varepsilon}.$$

Hence we derive the estimate $H_{A,2}^{(0),\dagger}(\alpha,\beta;Z) \ll_{f,D,p,\varepsilon} (Zp^{4\max(\alpha,\beta)})^{\frac{1}{2}+\varepsilon}$, from which the claim follows. Replacing r_A by c_ρ, a completely analogous argument works to bound the twisted sum $D_2(\rho,\beta;Z)$. $\square$

4.3. **The leading sums.** Let us again fix an integer $\alpha \geq 1$, together with a class $A \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$. Let us also fix a primitive ring class character ρ of conductor p^α. We now consider the leading sums $H_{A,1}^{(0)}(\alpha,\beta;Z)$ and $D_1(\rho,\beta;Z)$ above. Here we shall consider standard arguments leading to trivial preliminary estimates for these sums for varying choice of $Z > 0$. Let us first describe the trivial estimate.

4.3.1. *Preliminary estimates.* Recall that the function $V_1(y)$ is defined on $y \in \mathbf{R}_{>0}$ by the contour integral

$$V_1(y) = \int_{\Re(s)=2} \widehat{V}_1(s) y^{-s} \frac{ds}{2\pi i} = \int_{\Re(s)=2} \frac{L_\infty(s+1/2)}{L_\infty(s)} \frac{g^*(s)}{s} y^{-s} \frac{ds}{2\pi i},$$

where the shifted archimedean local L-function $L_\infty(s+1/2) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{R}}(s+3/2)$ has poles at $s=-1$ and $s=-2$. Shifting the contour to the left, we then have the estimate $V_1(y) = 1 + O_B(y^B)$ for any $0 < B < 1$ as $y \to 0$. Shifting to the left, we have that $V_1(y) = O_C(y^{-C})$ for any $C > 0$ as $y \to \infty$.

Lemma 4.5 (Trivial estimate)**.** *Let Z be any choice of real parameter so that $1 < 1/Z < p^{\beta-1}$. We have for any choices of constants $0 < B < 1/2$, $C > 1/2$, and $\varepsilon > 0$ the estimate*

$$H_{A,1}^{(0)}(\alpha,\beta;Z) = 1 + O_B\left(Z^B\right) + O_{C,\varepsilon}\left(Z^{-C}(p^\beta)^{-(\frac{1}{2}+C)+\varepsilon}\right).$$

In particular, taking $Z = p^{-\gamma}$ for $0 < \gamma < \beta - 1$ with $B = 1/4$ and $C = 1/2+\varepsilon$ gives us the estimate

$$H_{A,1}^{(0)}(\alpha,\beta;p^{-\gamma}) = 1 + O(p^{-\frac{\gamma}{4}}) + O_\varepsilon\left((p^\gamma)^{\frac{1}{2}+\varepsilon}(p^\beta)^{-1+\varepsilon}\right).$$

Both assertions remain true after replacing $H_{A,1}^{(0)}(\alpha,\beta;Z)$ by the sum $D_1(\rho,\beta;Z)$.

Proof. Cf. [43, Lemma 5.2]. Put $Z = Y^{-1}$. Expanding out the definition, we have that

$$H_{A,1}^{(0)}(\alpha,\beta;Y^{-1}) = \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{n\geq 1 \\ (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^\beta}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_1\left(\frac{m^2 n}{Y}\right)$$

$$- \frac{1}{\varphi(p)} \sum_{m\geq 1} \frac{\eta(m)}{m} \sum_{\substack{n\geq 1 \\ (n,p)=1 \\ m^2 n \equiv \pm 1 \bmod p^{\beta-1} \\ m^2 n \not\equiv \pm 1 \bmod p^\beta}} \frac{\lambda(n) r_A(n)}{n^{\frac{1}{2}}} V_1\left(\frac{m^2 n}{Y}\right).$$

Observe that the contribution of $m = n = 1$ in the first sum can be estimated for any $0 < B < 1/2$ as

$$V_1\left(\frac{1}{Y}\right) = \int_{\Re(s)=2} G_1(s) \frac{L_\infty(1/2+s)}{L_\infty(s)} Y^s \frac{ds}{2\pi i} = 1 + O_B(Y^{-B}) \qquad \forall \quad 0 < B < 1/2.$$

To deal with the remaining contributions, let us first recall that $V_1(y) = O_C(y^{-C})$ for any choice of constant $C>0$ when $y>1$. Recall as well that we have by Deligne's theorem the bound $\lambda(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon>0$, and the classical bound $r_A(n) \ll_\varepsilon n^\varepsilon$, for any $\varepsilon>0$. The point is that we choose $Y>1$ in such a way that all additional terms $m^2n/Y>1$ lie in the region of rapid decay for the cutoff function $V_1(y)$, so that each of the corresponding coefficients is bounded as

$$\frac{\eta(m)}{m}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_1\left(\frac{m^2n}{Y}\right) \ll_{C,\varepsilon} \frac{n^\varepsilon}{mn^{\frac{1}{2}}}\left(\frac{m^2n}{Y}\right)^{-C} = O_{C,\varepsilon}\left(Y^C(m^2n)^{-\frac{1}{2}-C+\varepsilon}\right).$$

Writing $\overline{m}^2$ to denote the least integer representative of the class $\overline{m}^2 \bmod p^{\beta-1}$ (so that $m^2\overline{m}^2 \equiv 1 \bmod p^{\beta-1}$), we expand the second sum as

$$\begin{aligned}
&\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm1 \bmod p^{\beta-1}\\ m^2n\not\equiv\pm1 \bmod p^\beta}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_1\left(\frac{m^2n}{Y}\right)\\
&= \frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{l=1}^{p-1}\frac{\lambda(\overline{m}^2+lp^{\beta-1})r_A(\overline{m}^2+lp^{\beta-1})}{(\overline{m}^2+lp^{\beta-1})^{\frac{1}{2}}}V_1\left(\frac{m^2(\overline{m}^2+lp^{\beta-1})}{Y}\right)\\
&+ \frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{l=1}^{p-1}\frac{\lambda(-\overline{m}^2+lp^{\beta-1})r_A(-\overline{m}^2+lp^{\beta-1})}{(-\overline{m}^2+lp^{\beta-1})^{\frac{1}{2}}}V_1\left(\frac{m^2(-\overline{m}^2+lp^{\beta-1})}{Y}\right)
\end{aligned}$$

That is, the congruence condition in the second sum can be parametrized simply by $\pm\overline{m}^2+lp^{\beta-1}$ for integers $1\leq l\leq p-1$. Since we assume that $1<Y<p^{\beta-1}$, each term in the latter expression is bounded above by

$$\frac{\eta(m)}{m}\frac{\lambda(\pm\overline{m}^2+lp^{\beta-1})r_A(\pm\overline{m}^2+lp^{\beta-1})}{(\pm\overline{m}^2+lp^{\beta-1})^{\frac{1}{2}}}V_1\left(\frac{m^2(\pm\overline{m}^2+lp^{\beta-1})}{Y}\right) \ll_{C,\varepsilon} (m^2(\pm\overline{m}^2+lp^{\beta-1}))^{\varepsilon-\frac{1}{2}-C}Y^C,$$

so that the second sum is bounded above by

$$\begin{aligned}
&\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{l=1}^{p-1}\frac{\lambda(\pm\overline{m}^2+lp^{\beta-1})r_A(\pm\overline{m}^2+lp^{\beta-1})}{(\pm\overline{m}^2+lp^{\beta-1})^{\frac{1}{2}}}V_1\left(\frac{m^2(\pm\overline{m}^2+lp^{\beta-1})}{Y}\right)\\
&\ll_{C,\varepsilon} \frac{1}{\varphi(p)}\sum_{m\geq 1}\sum_{l=1}^{p}Y^C(m^2(\pm\overline{m}^2+lp^{\beta-1}))^{\varepsilon-\frac{1}{2}-C} = O_{C,\varepsilon}\left(Y^C(p^\beta)^{-(\frac{1}{2}+C)+\varepsilon}\right).
\end{aligned}$$

We use a similar argument to bound the remaining terms in the first sum. That is, writing $\overline{m}^2$ now to denote the least integer representative of the class $\overline{m}^2 \bmod p^\beta$ (so that $m^2\overline{m}^2\equiv 1 \bmod p^\beta$), we expand out as

$$\begin{aligned}
&\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm1 \bmod p^\beta}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_1\left(\frac{m^2n}{Y}\right)\\
&= \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{t\geq 0}\frac{\lambda(\overline{m}^2+tp^\beta)r_A(\overline{m}^2+tp^\beta)}{(\overline{m}^2+tp^\beta)^{\frac{1}{2}}}V_1\left(\frac{m^2(\overline{m}^2+tp^\beta)}{Y}\right)\\
&+ \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{t\geq 0}\frac{\lambda(-\overline{m}^2+tp^\beta)r_A(-\overline{m}^2+tp^\beta)}{(-\overline{m}^2+tp^\beta)^{\frac{1}{2}}}V_1\left(\frac{m^2(-\overline{m}^2+tp^\beta)}{Y}\right)
\end{aligned}$$

It is easy to see from the argument for the second sum that the $t\geq 1$ contributions are bounded above by

$$\begin{aligned}
&\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{t\geq 1}\frac{\lambda(\pm\overline{m}^2+tp^\beta)r_A(\overline{m}^2+tp^\beta)}{(\pm\overline{m}^2+tp^\beta)^{\frac{1}{2}}}V_1\left(\frac{m^2(\pm\overline{m}^2+tp^\beta)}{Y}\right)\\
&\ll_{C,\varepsilon} Y^C\sum_{m,t\geq 1}(m^2(\pm\overline{m}^2+tp^\beta))^{-(\frac{1}{2}+C)+\varepsilon} = O_{C,\varepsilon}\left(Y^C(p^\beta)^{-(\frac{1}{2}+C)+\varepsilon}\right).
\end{aligned}$$

Here, we assume that $C > 1/2$ so that the t-sum converges. We can then treat this sum as a constant. Finally, we consider the contributions from $t = 0$ which do not arise from the leading term $m = n = 1$, so corresponding to $m = 1$ and $t = 0$ in our congruence expansion above. That is, it remains to consider

$$\sum_{m \geq 2} \frac{\eta(m)}{m} \frac{\lambda(\overline{m}^2) r_A(\overline{m}^2)}{(\overline{m}^2)^{\frac{1}{2}}} V_1\left(\frac{m^2\overline{m}^2}{Y}\right) + \sum_{m \geq 2} \frac{\eta(m)}{m} \frac{\lambda(-\overline{m}^2) r_A(-\overline{m}^2)}{(-\overline{m}^2)^{\frac{1}{2}}} V_1\left(\frac{m^2(-\overline{m})^2}{Y}\right).$$

Since $m \geq 2$, the congruence condition $m^2\overline{m}^2 \equiv 1 \bmod p^\beta$ implies that $m^2\overline{m^2} = 1 + up^\beta$ for some integer $u \geq 1$, and the condition $m^2(-\overline{m}^2) \equiv 1 \bmod p^\beta \equiv 1 \bmod p^\beta$ implies that $m^2(-\overline{m}^2) = 1 + u'p^\beta$ for some integer $u' \geq 1$. The hypothesis that $1 < Y < p^{\beta-1}$ implies that each of the terms in this latter expression lies in the region of rapid decay for $V_1(y)$, and can be bounded in the same way as described above. Replacing the coefficients r_A with the (averaged) coefficients $c(\rho)$, we derive the same estimate for the sum $D_1(\rho, \beta)$. $\square$

4.4. **Main estimate.** Let us now return to the averages

$$\mathcal{H}^{(0)}(\alpha, \beta) = \frac{1}{\#C^\star(\alpha)} \sum_{\substack{\rho \in \mathrm{Pic}(\mathcal{O}_p^\alpha)^\vee \\ \text{primitive}}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \text{ primitive}}} L(1/2, f \times \rho\chi \circ \mathbf{N})$$

described in Proposition 2.6 above. In particular, taking the unbalancing parameter $Z = Y^{-1} = D|N|p^{2\max(\alpha,\beta)}$, and working only with the principal class $A = \mathbf{1} \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ (hence dropping the class from the notation) we have the explicit expression

$$\begin{aligned}\mathcal{H}^{(0)}(\alpha, \beta) = &\left(1 - \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \cdot \left(H_1^{(0)}(\alpha, \beta; Y^{-1}) + H_2^{(0)}(\alpha, \beta; Y^{-1})\right) \\ &- \left(\frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \cdot \left(H_{1,\star}^{(0)}(\alpha, \beta; Y^{-1}) + H_{2,\star}^{(0)}(\alpha, \beta; Y^{-1})\right).\end{aligned}$$

Theorem 4.6. *Fix an integer $\beta \geq 2$. We have for any anticyclotomic exponent $\alpha \geq 1$ the estimate*

$$\begin{aligned}\mathcal{H}^{(0)}(\alpha, \beta) = &\left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \frac{1}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \left(\sum_{\substack{a \neq 0 \in \mathbf{Z} \\ (a,p)=1 \\ m^2a^2 \equiv \pm 1 \bmod p^\beta}} \frac{\lambda(a^2)}{a} - \frac{1}{\varphi(p)} \sum_{\substack{a \neq 0 \in \mathbf{Z} \\ (a,p)=1 \\ m^2a^2 \equiv \pm 1 \bmod p^{\beta-1} \\ m^2a^2 \not\equiv \pm 1 \bmod p^\beta}} \frac{\lambda(a^2)}{a} \right) \\ &+ \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \frac{\eta(-N)}{|D|^2p^{2\beta}} \frac{1}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \left(\sum_{\substack{a \neq 0 \in \mathbf{Z} \\ (a,p)=1}} \frac{\lambda(a^2)}{a} \cdot \left(\frac{p}{\varphi(p)}\right) \mathrm{Kl}_4(\pm(m^2a^2\overline{N}^2\overline{D}^8)^{\frac{1}{2}}, p^\beta) \right) \\ &+ O_{f,\epsilon}\left(\left(|D|p^\beta\right)^{\frac{3}{16}+\varepsilon} \left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}} \right) + O_{f,\beta,\varepsilon}\left(\left(|D|p^{2\alpha}\right)^{-\frac{1}{4}+\delta_0} \right),\end{aligned}$$

or equivalently

(74)

$$\begin{aligned}\mathcal{H}^{(0)}(\alpha, \beta) = &\left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \\ &\times \frac{2}{w} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive } \chi(-1)=1}} \left(L(1, \eta\chi^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \chi)}{L^{(Np)}(2, \chi)} + \frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2p^{2\beta}} \cdot L(1, \eta\overline{\chi}^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \overline{\chi})}{L^{(Np)}(2, \overline{\chi})} \right) \\ &+ O_{f,\epsilon}\left(\left(|D|p^\beta\right)^{\frac{3}{16}+\varepsilon} \left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}} \right) + O_{f,\beta,\varepsilon}\left(\left(|D|p^{2\alpha}\right)^{-\frac{1}{4}+\delta_0} \right).\end{aligned}$$

In particular, we can deduce that for $\alpha \gg \beta$ sufficiently large, the average converges to the constant term (75)

$$\left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)}\right)$$
$$\times \frac{2}{w} \sum_{\substack{\chi \bmod p^{\beta} \\ \text{primitive } \chi(-1)=1}} \left(L(1, \eta\chi^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \chi)}{L^{(Np)}(2, \chi)} + \frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \cdot L(1, \eta\overline{\chi}^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \overline{\chi})}{L^{(Np)}(2, \overline{\chi})} \right).$$

Using the nonvanishing of each $L(1, \mathrm{Sym}^2 f \otimes \chi)$, we can also show that this term (75) *does not vanish.*

Proof. We see from the explicit formulae of Proposition 2.6 that it is enough to estimate the main sum

$$H_1^{(0)}(\alpha, \beta; Y) + H_2^{(0)}(\alpha, \beta; Y^{-1}) = H_{\mathbf{1},1}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1}) + H_{\mathbf{1},2}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1}).$$

Here, for reduced form representative $q_{\mathbf{1}}(x,y) = x^2 + \delta_{\mathbf{1}} xy + \epsilon_{\mathbf{1}} y^2$ as described above (with $\epsilon_{\mathbf{1}} \approx Dp^{2\alpha}$)

$$H_{\mathbf{1},1}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1}) = \frac{1}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{a,b \in \mathbf{Z} \\ q_{\mathbf{1}}(a,b) \neq 0, (q_{\mathbf{1}}(a,b),p)=1 \\ m^2 q_{\mathbf{1}}(a,b) \equiv \pm 1 \bmod p^{\beta}}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right)$$
$$- \frac{1}{w}\frac{1}{\varphi(p)} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{a,b \in \mathbf{Z} \\ q_{\mathbf{1}}(a,b) \neq 0, (q_{\mathbf{1}}(a,b),p)=1 \\ m^2 q_{\mathbf{1}}(a,b) \equiv \pm 1 \bmod p^{\beta-1} \\ m^2 q_{\mathbf{1}}(a,b) \not\equiv \pm 1 \bmod p^{\beta}}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right)$$

and

$$H_{\mathbf{1},2}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1})$$
$$= \frac{\eta(-N)}{|D|^2 p^{2\beta}} \left(\frac{p}{\varphi(p)}\right) \frac{1}{w} \sum_{m \geq 1} \frac{\eta(m)}{m} \sum_{\substack{a,b \in \mathbf{Z} \\ q_{\mathbf{1}}(a,b) \neq 0 \\ (q_{\mathbf{1}}(a,b),p)=1}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right) \mathrm{Kl}_4(\pm(m^2 q_{\mathbf{1}}(a,b)\overline{N}^2\overline{D}^8)^{\frac{1}{2}}, p^{\beta}).$$

Let us first estimate the $b=0$ terms, following a minor variation of the argument of Lemma 3.1 above. In this way, we find that the $b=0$ terms in the leading sum are estimated as

(76)

$$\begin{aligned}
&H_{\mathbf{1},1}^{(0)}(\alpha,\beta;(N|D|p^{2\max(\alpha,\beta)})^{-1})|_{b=0} \\
&= \frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\left(\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a^2,p^{\alpha})=1\\ m^2a^2\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(a^2)}{a}V_1\left(\frac{m^2a^2}{N|D|p^{2\max(\alpha,\beta)}}\right)-\frac{1}{\varphi(p)}\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a^2,p^{\alpha})=1\\ m^2a^2\equiv\pm 1 \bmod p^{\beta-1}\\ m^2a^2\not\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(a^2)}{a}V_1\left(\frac{m^2a^2}{N|D|p^{2\max(\alpha,\beta)}}\right)\right) \\
&= \frac{2}{w}\frac{2}{\varphi^{\star}(p^{\beta})}\sum_{\substack{\chi \bmod p^{\beta}\\ \text{primitive}\,\chi(-1)=1}}\sum_{m\geq 1}\frac{\eta(m)\chi^2(m)}{m}\sum_{\substack{a\geq 1\\ (a,p^{\alpha})=1}}\frac{\lambda(a^2)\chi(a^2)}{a}V_1\left(\frac{m^2a^2}{N|D|p^{2\max(\alpha,\beta)}}\right) \\
&= \frac{2}{w}\sum_{\substack{\chi \bmod p^{\beta}\\ \text{primitive}\,\chi(-1)=1}}\int_{\Re(s)=2}L(1+2s,\eta\chi^2)\cdot\frac{L(1+2s,\mathrm{Sym}^2 f\otimes\chi)}{L^{(Np)}(2(1+2s),\chi)}\widehat{V}_1(s)\left(\frac{1}{N|D|p^{2\max(\alpha,\beta)}}\right)^{-s}\frac{ds}{2\pi i} \\
&= \frac{2}{w}\sum_{\substack{\chi \bmod p^{\beta}\\ \text{primitive}\,\chi(-1)=1}}L(1,\eta\chi^2)\cdot\frac{L(1,\mathrm{Sym}^2 f\otimes\chi)}{L^{(Np)}(2,\chi)}+O_{f,\epsilon}\left((|D|p^{\beta})^{\frac{3}{16}+\varepsilon}\left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}}\right) \\
&= \frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\left(\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a,p)=1\\ m^2a^2\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(a^2)}{a}-\frac{1}{\varphi(p)}\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a,p)=1\\ m^2a^2\equiv\pm 1 \bmod p^{\beta-1}\\ m^2a^2\not\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(a^2)}{a}\right)+O_{f,\epsilon}\left((|D|p^{\beta})^{\frac{3}{16}+\varepsilon}\left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}}\right).
\end{aligned}$$

Similarly, using the calculations of Lemma 3.1 with those of Proposition 2.6, as well as the relation

$$\frac{2}{\varphi^{\star}(p^{\beta})}\sum_{\substack{\chi \bmod p^{\beta}\\ \chi(-1)=1\ \ \text{primitive}}}\chi(\overline{m}^2\overline{a}^2N^2)\tau(\eta\chi^2)^4=\left(\frac{p}{\varphi(p)}\right)\mathrm{Kl}_4(\pm(m^2a^2\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^{\beta})$$

implied by Proposition 2.4, we find the that the $b=0$ terms in the twisted sum can be estimated as

$$\begin{aligned}
&H_{\mathbf{1},2}^{(0)}(\alpha,\beta;(N|D|p^{2\max(\alpha,\beta)})^{-1})|_{b=0} \\
&= \frac{\eta(-N)}{|D|^2p^{2\beta}}\left(\frac{p}{\varphi(p)}\right)\frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a,p)=1}}\frac{\lambda(a^2)}{a}V_1\left(\frac{m^2a^2}{N|D|p^{2\max(\alpha,\beta)}}\right)\mathrm{Kl}_4(\pm(m^2a^2\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^{\beta}) \\
&= \frac{\eta(-N)}{|D|^2p^{2\beta}}\left(\frac{p}{\varphi(p)}\right)\frac{1}{w}\int_{\Re(s)=2}\sum_{m\geq 1}\frac{\eta(m)}{m^{1+2s}}\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a,p)=1}}\frac{\lambda(a^2)}{a^{1+2s}}\left(\frac{1}{N|D|p^{2\max(\alpha,\beta)}}\right)\widehat{V}_1(s)\frac{ds}{2\pi i} \\
&= \frac{\eta(-N)}{|D|^2p^{2\beta}}\left(\frac{p}{\varphi(p)}\right)\frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a\neq 0\in\mathbf{Z}\\ (a,p)=1}}\frac{\lambda(a^2)}{a}\cdot\mathrm{Kl}_4(\pm(m^2a^2\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^{\beta}) \qquad (77) \\
&+O_{f,\epsilon}\left(p^{-\frac{\beta}{2}}|D|^{-2}\cdot|D|^{\frac{3}{16}+\varepsilon}\left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}}\right) \\
&= \frac{2}{w}\cdot\frac{2}{\varphi^{\star}(p^{\beta})}\sum_{\substack{\chi \bmod p^{\beta}\\ \chi(-1)=1\ \ \text{primitive}}}\frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2p^{2\beta}}\cdot L(1,\eta\overline{\chi}^2)\cdot\frac{L(1,\mathrm{Sym}^2 f\otimes\overline{\chi})}{L^{(Np)}(2,\overline{\chi})} \\
&+O_{f,\epsilon}\left((|D|p^{\beta})^{\frac{3}{16}+\varepsilon}\left(|D|p^{2\max(\alpha,\beta)}\right)^{-\frac{1}{4}}\right).
\end{aligned}$$

Here, we use different applications of the argument of Lemma 3.1 in the last step according to whether we unwind the expression to get a twisting linear combination of L-values (which gives a worse error term).

Let us now consider the $b \neq 0$ contributions. For the leading sum, we deduce from Theorem 3.3 that

(78)

$$\begin{aligned} &H_{\mathbf{1},1}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1})|_{b\neq 0} \\ &= \frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a,b\in\mathbf{Z} \\ q_{\mathbf{1}}(a,b), b\neq 0, (a^2,p^\alpha)=1 \\ m^2 q_{\mathbf{1}}(a,b)\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right) \\ &- \frac{1}{w}\frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a,b\in\mathbf{Z} \\ q_{\mathbf{1}}(a,b), b\neq 0, (a^2,p^\alpha)=1 \\ m^2 q_{\mathbf{1}}(a,b)\equiv\pm 1 \bmod p^{\beta-1} \\ m^2 q_{\mathbf{1}}(a,b)\not\equiv\pm 1 \bmod p^\beta}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right) \\ &= \frac{2}{w}\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \text{ primitive}}}\sum_{m\geq 1}\frac{\eta\chi^2(m)}{m}\sum_{\substack{a,b\in\mathbf{Z} \\ q_{\mathbf{1}}(a,b), b\neq 0, (a^2,p^\alpha)=1}} \frac{\lambda(q_{\mathbf{1}}(a,b))\chi^2(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right) \\ &\ll_{f,\beta,\varepsilon} \left(|D|p^{2\max(\alpha,\beta)}\right)^{\frac{1}{4}+\delta_0}\cdot\left(|D|p^{2\alpha}\right)^{\delta_0-\frac{\theta_0}{2}-\varepsilon}\cdot|\epsilon_{\mathbf{1}}|^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon} = O_{f,\beta,\varepsilon}\left(\left(|D|p^{2\max(\alpha,\beta)}\right)^{\frac{1}{4}+\delta_0}\cdot\left(|D|p^{2\alpha}\right)^{-\frac{1}{2}}\right). \end{aligned}$$

Note that from this point, we shall take the cyclotomic estimate $\beta \geq 2$ to be fixed, and let $\alpha \gg \beta$ become large for our estimates. Similarly, for the twisted sum (estimating the root number contributions trivially), we deduce from Theorem 3.3 that

(79)

$$\begin{aligned} &H_{\mathbf{1},2}^{(0)}(\alpha, \beta; (N|D|p^{2\max(\alpha,\beta)})^{-1})|_{b\neq 0} \\ &= \frac{\eta(-N)}{|D|^2 p^{2\beta}}\left(\frac{p}{\varphi(p)}\right)\frac{1}{w}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{a,b\in\mathbf{Z} \\ q_{\mathbf{1}}(a,b), b\neq 0 \\ (q_{\mathbf{1}}(a,b),p^\alpha)=1}} \frac{\lambda(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right)\mathrm{Kl}_4(\pm(m^2 q_{\mathbf{1}}(a,b)\overline{N}^2\overline{D}^8)^{\frac{1}{2}}, p^\beta) \\ &= \frac{1}{w}\cdot\frac{2}{\varphi^\star(p^\beta)}\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \text{ primitive}}}\frac{\eta\chi^2(N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}}\cdot\sum_{m\geq 1}\frac{\eta\chi^2(m)}{m}\sum_{\substack{a,b\in\mathbf{Z} \\ b\neq 0 \\ (q_{\mathbf{1}}(a,b),p^\alpha)=1}} \frac{\lambda(q_{\mathbf{1}}(a,b))\overline{\chi}(q_{\mathbf{1}}(a,b))}{q_{\mathbf{1}}(a,b)^{\frac{1}{2}}} V_1\left(\frac{m^2 q_{\mathbf{1}}(a,b)}{N|D|p^{2\max(\alpha,\beta)}}\right) \\ &\ll_{f,\beta,\varepsilon} \left(|D|p^{2\max(\alpha,\beta)}\right)^{\frac{1}{4}+\delta_0}\cdot\left(|D|p^{2\alpha}\right)^{\delta_0-\frac{\theta_0}{2}-\varepsilon}\cdot|\epsilon_{\mathbf{1}}|^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon} = O_{f,\beta,\varepsilon}\left(\left(|D|p^{2\max(\alpha,\beta)}\right)^{\frac{1}{4}+\delta_0}\cdot\left(|D|p^{2\alpha}\right)^{-\frac{1}{2}}\right). \end{aligned}$$

Again, we note that the cyclotomic exponent $\beta \geq 2$ is fixed, and we take the anticyclotomic exponent $\alpha \gg \beta$ to be sufficiently large. Hence, putting pieces together, we derive the stated estimate.

To deduce the nonvanishing for $\beta \geq 2$ fixed and $\alpha \gg \beta$ sufficiently large, we now argue as follows. We start with the estimate (74). Fixing $\beta \geq 2$, the error terms tend to zero as $\alpha \gg \beta$ tends to infinity. Hence, the average converges to the constant term (75). We now argue that this constant term (75) does not vanish. Let us for each primitive even Dirichlet character $\chi \bmod p^\beta$ in the sum defining (75) write the corresponding product of L-values as

$$\mathfrak{L}_\chi(1) = L(1, \eta\chi^2)\cdot\frac{L(1, \mathrm{Sym}^2 f\otimes\chi)}{L^{(Np)}(2,\chi)}, \quad \mathfrak{L}_{\overline{\chi}}(1) = L(1, \eta\overline{\chi}^2)\cdot\frac{L(1, \mathrm{Sym}^2 f\otimes\overline{\chi})}{L^{(Np)}(2,\overline{\chi})}.$$

Let us also write

$$\epsilon(\chi) = \frac{\eta\chi^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} = \epsilon(1/2, f\times\rho\chi\circ\mathbf{N}) \in \mathbf{S}^1$$

to denote the root number of the underlying Rankin-Selberg L-function $L(s, f\times\rho\chi\circ\mathbf{N})$. We claim that

$$\mathfrak{L}_\chi(1) + \epsilon(\chi)\cdot\mathfrak{L}_{\overline{\chi}}(1) \neq 0 \tag{80}$$

for any primitive Dirichlet character $\chi \mod p^{\beta}$. To see why, assume otherwise that the sum (80) vanishes. Since each L-value $\mathfrak{L}_{\chi}(1)$ is known to be nonvanishing[9], we can then deduce that $\mathfrak{L}_{\chi}(1)/\mathfrak{L}_{\overline{\chi}}(1) = -\epsilon(\chi)$. On the other hand, it is apparent that the root number $\epsilon(\chi)$ determines an algebraic number, and hence admits a natural action by the absolute Galois group. In particular, we obtain for each $\sigma \in \mathrm{Aut}(\overline{\mathbf{Q}}/\mathbf{Q})$ the relation $\mathfrak{L}_{\chi^{\sigma}}(1)/\mathfrak{L}_{\overline{\chi}^{\sigma}}(1) = -\epsilon(\chi^{\sigma})$. To examine the consequences of this, let us first describe this Galois orbit of characters more precisely. We first express the chosen primitive character χ in terms of the exponential function, fixing an integer $1 \leq a < p^{\beta}$ corresponding to the class in $(\mathbf{Z}/p^{\beta}\mathbf{Z})^{\times} \backslash (\mathbf{Z}/p^{\beta-1}\mathbf{Z})^{\times}$ for which χ is given by the rule sending an integer $n \in \mathbf{Z}$ to the root of unity $e(an/p^{\beta})$. That is, the primitive character $\chi \bmod p^{\beta}$ is given by the rule

$$n \longmapsto \chi(n) = e\left(\frac{an}{p^{\beta}}\right) = \exp\left(2\pi i \cdot \frac{an}{p^{\beta}}\right).$$

Let us also write $\mathbf{Q}(\chi) = \mathbf{Q}(\zeta_{p^{\beta}})$ to denote the cyclotomic field of degree $\varphi(p^{\beta})$ obtained by adjoining the values of χ to $\mathbf{Q}$, equivalently by adjoining any primitive p^{β}-th root of unity $\zeta_{p^{\beta}}$ to $\mathbf{Q}$. Note that for each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$, the character χ^{σ} is given by the rule $n \mapsto \chi(n)^{\sigma}$ on $n \in \mathbf{Z}$. Fixing any isomorphism

$$\mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q}) \cong (\mathbf{Z}/p^{\beta}\mathbf{Z})^{\times}, \quad \sigma \longmapsto b_{\sigma},$$

and identifying each class $b_{\sigma} \in (\mathbf{Z}/p^{\beta}\mathbf{Z})^{\times}$ with its corresponding integer representative $1 \leq b_{\sigma} < p^{\beta}$, we can then describe the Galois orbit $G(\chi)$ of the primitive character $\chi \bmod p^{\beta}$ more explicitly as

$$G(\chi) := \{\chi^{\sigma} : \sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})\} = \left\{ n \mapsto e\left(\frac{ab_{\sigma}n}{p^{\beta}}\right) : b_{\sigma} \in (\mathbf{Z}/p^{\beta}\mathbf{Z})^{\times} \right\}. \tag{81}$$

Notice that while this set $G(\chi)$ does not contain the principal character $\chi_0 = \mathbf{1} \bmod p^{\beta}$, it does contain the unique inverse $(\chi^{\sigma})^{-1} = \overline{\chi^{\sigma}} \in G(\chi)$ of each character $\chi^{\sigma} \in G(\chi)$. Let us now return to the implications of $\mathfrak{L}_{\chi}(1) + \epsilon(\chi)\mathfrak{L}_{\overline{\chi}}(1) = 0$, which as we argued above would imply that $\mathfrak{L}_{\chi^{\sigma}}(1)/\mathfrak{L}_{\overline{\chi^{\sigma}}}(1) = -\epsilon(\chi^{\sigma})$ for each $\chi^{\sigma} \in G(\chi)$. Taking the product over Galois conjugate characters would then give us the relation

$$\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} -\epsilon(\chi^{\sigma}) = -\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\chi^{\sigma}) = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^{\sigma}}(1)}{\mathfrak{L}_{\overline{\chi^{\sigma}}}(1)}. \tag{82}$$

Observe that we have $\epsilon(\chi^{\sigma})\epsilon(\overline{\chi^{\sigma}}) = \epsilon(\chi^{\sigma})\overline{\epsilon(\chi^{\sigma})} = |\epsilon(\chi^{\sigma})|^2 = 1$ for each automorphism $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$. Hence, pairing together each $\chi^{\sigma} \in G(\chi)$ with its inverse $\overline{\chi^{\sigma}} \in G(\chi)$, we see that the left-hand side of (82) must equal -1. On the other hand, we see from the description of the Galois orbit (81) that

$$\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \mathfrak{L}_{\chi^{\sigma}}(1) = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \mathfrak{L}_{\overline{\chi^{\sigma}}}(1),$$

and hence that the right-hand side of (82) must equal 1. In other words, (82) is equivalent to $-1 = 1$, which gives us a contradiction. Hence, we deduce that for $\chi \bmod p^{\beta}$ any primitive Dirichlet character,

$$\mathfrak{L}_{\chi}(1) + \epsilon(\chi)\mathfrak{L}_{\overline{\chi}}(1) \neq 0.$$

To see why the sum over all primitive even Dirichlet characters $\chi \bmod p^{\beta}$ of the nonvanishing sums of L-values $\mathfrak{L}_{\chi}(1) + \epsilon(\chi)\mathfrak{L}_{\overline{\chi}}(1)$ does not vanish, let us suppose otherwise that it does. This would imply that

$$\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1, \text{primitive}}} \mathfrak{L}_{\chi}(1) = -\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1, \text{primitive}}} \epsilon(\chi) \cdot \mathfrak{L}_{\overline{\chi}}(1) = \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1, \text{primitive}}} -\epsilon(\overline{\chi}) \cdot \mathfrak{L}_{\chi}(1). \tag{83}$$

Exponentiating each side of (83), we obtain

$$\prod_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1, \text{ primitive}}} \exp\left(\mathfrak{L}_{\chi}(1)\right) = \prod_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1, \text{ primitive}}} \exp\left(-\epsilon(\overline{\chi}) \cdot \mathfrak{L}_{\chi}(1)\right). \tag{84}$$

[9]For instance, by the prime number theorem for $\mathrm{GL}_3(\mathbf{A})$-automorphic L-functions. See also [8, Lemma 4.2], and the discussion of Lemma 3.2 above, both of which apply to this setting to give $L(1, \mathrm{Sym}^2 f \otimes \chi) \neq 1$ for any Dirichlet character χ.

Comparing products in (84), we then deduce that for each primitive even Dirichlet character $\chi \bmod p^\beta$ on the left-hand side, there exists a unique primitive even Dirichlet character $\chi' \bmod p^\beta$ for which

$$\exp\left(\mathfrak{L}_\chi(1)\right) = \exp\left(-\epsilon(\overline{\chi}') \cdot \mathcal{L}_{\chi'}(1)\right) \quad \iff \quad \mathfrak{L}_\chi(1) = -\epsilon(\overline{\chi}') \cdot \mathcal{L}_{\chi'}(1).$$

Here, we take the logarithm to deduce the second identification, which is the same as

$$-\epsilon(\overline{\chi}') = \frac{\mathfrak{L}_\chi(1)}{\mathfrak{L}_{\chi'}(1)}. \tag{85}$$

Again, we use that the negative root number $-\epsilon(\overline{\chi}')$ on the left-hand side of (85) is an algebraic number. Hence, we can apply any automorphism $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$ to (85) to get the corresponding identity

$$-\epsilon(\overline{\chi}'^\sigma) = \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\chi'^\sigma}(1)}.$$

Taking the product over all of the automorphisms $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$, we then obtain the identity

$$-\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\overline{\chi}'^\sigma) = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\chi'^\sigma}(1)}. \tag{86}$$

Pairing together each character with its inverse for the root number term, we argue again that

$$\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\overline{\chi}'^\sigma) = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\chi'^\sigma}(1)} = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\chi^\sigma}(1)} = 1$$

to deduce that the identity (86) is equivalent to $-1 = 1$. This gives us a contradiction. Hence, the sum

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \text{primitive}}} \mathfrak{L}_\chi(1) + \epsilon(\chi)\mathfrak{L}_{\overline{\chi}}(1)$$

does not vanish. This implies that the sum defining the constant term (75)

$$\begin{aligned} &\left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \cdot \frac{2}{w} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1, \text{primitive}}} \mathfrak{L}_\chi(1) + \epsilon(\chi)\mathfrak{L}_{\overline{\chi}}(1) \\ &= \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \\ &\times \frac{2}{w} \sum_{\substack{\chi \bmod p^\beta \\ \text{primitive } \chi(-1)=1}} \left(L(1, \eta\chi^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \chi)}{L^{(Np)}(2, \chi)} + \frac{\eta\overline{\chi}^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \cdot L(1, \eta\overline{\chi}^2) \cdot \frac{L(1, \mathrm{Sym}^2 f \otimes \overline{\chi})}{L^{(Np)}(2, \overline{\chi})} \right) \end{aligned}$$

does not vanish, as claimed. □

Remark Notice that we show a stronger statement in the proof given above. Namely, without taking an average over primitive even Dirichlet characters $\chi \bmod p^\beta$, we can still prove a nonvanishing estimate for fixed primitive even Dirichlet character $\chi \bmod p^\beta$, then consider the average over twists by primitive ring class characters ρ of conductor p^α. In this way, we obtain a simpler nonvanishing estimate for $\alpha \gg \beta$ is sufficiently large. This approach is developed in greater generality in the sequel [45].

5. Galois conjugate ring class characters

We now derive the following refinement of Theorems 3.4 and Theorem 4.6 for Galois conjugate ring class characters, or more simply ring class characters of a given exact order, analogous to the underlying averages considered in the works of Rohrlich [37], [38] Vatsal [47] and Cornut [11].

Fix an integer $\alpha \geq 0$. Again, we write $C(\alpha) = \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ for simplicity to denote the class group of $\mathcal{O}_{p^\alpha} = \mathbf{Z} + p^\alpha \mathcal{O}_K$. We then define $x = \mathrm{ord}_p(\#C(\alpha))$. Hence, p^x divides the order of the group $C(\alpha)$. Recall that an element $A \in C(\alpha)$ is said to have exponent p^x if A^{p^x} is the identity, equivalently if $A^{p^x} = \mathbf{1} \in C(\alpha)$

is the principal class. The characters of exponent or exact order p^x of $C(\alpha)$ are precisely those which are trivial on the subgroup of p^x-th powers in $C(\alpha)$,

$$C(\alpha)^{p^x} = \left\{ g^{p^x} : g \in C(\alpha) \right\}.$$

Equivalently, writing $C(\alpha)^\vee$ to denote the character group of $C(\alpha)$, the characters $\rho \in C(\alpha)^\vee$ of exact order p^x are those for which $\rho^{p^x} = \mathbf{1}$, where $\mathbf{1}$ denotes the trivial character of $C(\alpha)$. It is classical (see [21, §3.1]) that such characters detect p^x-th powers via the orthogonality relation

$$\sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} \equiv \mathbf{1}}} \rho(A) = \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] & \text{if } A \in C(\alpha)^{p^x} \\ 0 & \text{if } A \notin C(\alpha)^{p^x}. \end{cases}$$

Let us now fix $\alpha \geq 1$, which is sufficiently large so that $x = \mathrm{ord}_p(\#C(\alpha)) \geq 1$. We consider primitive ring class characters ρ of conductor of exact order p^x and conductor p^α. Since the ring class characters of exact order p^x are necessarily primitive (as they cannot factor through $C(\alpha-1)$), it is easy to see via inclusion-exclusion that we have the corresponding orthogonality relation

$$\begin{aligned} \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1} \forall 0 \leq y \leq x-1}} \rho(A) &= \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1}}} \rho(A) - \sum_{\substack{\rho' \in C(\alpha)^\vee \\ (\rho')^{p^{x-1}} = \mathbf{1}}} \rho'(A) \\ &= \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] & \text{if } A \in C(\alpha)^{p^x} \\ 0 & \text{if } A \notin C(\alpha)^{p^x} \end{cases} - \begin{cases} [C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^{x-1}} \\ 0 & \text{if } A \notin C(\alpha)^{p^{x-1}} \end{cases} \\ &= \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^x} \\ -[C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{87}$$

Hence, writing $R(\alpha)^{p^x} = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}]$, we now define in either case $k \in \{0, 1\}$ on the generic root number the corresponding average over ring class characters $\rho \in \mathrm{Pic}(\mathcal{O}_{p^\alpha})$ of exact order p^x:

$$G^{(k)}(\alpha; x) = \frac{1}{R(\alpha)^{p^x}} \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1} \forall 0 \leq y \leq x-1}} L^{(k)}(1/2, f \times \rho). \tag{88}$$

Given an integer $\beta \geq 2$, we also define (for $k = 0$) the corresponding double average over primitive even Dirichlet characters modp^β (of exact order p^β) of the central values $L(1/2, f \times \rho\chi \circ \mathbf{N})$,

$$G^{(0)}(\alpha, \beta; x) = \frac{1}{R(\alpha)^{p^x}} \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1} \forall 0 \leq y \leq x-1}} \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1) = 1, \text{primitive}}} L(1/2, f \times \rho\chi \circ \mathbf{N}). \tag{89}$$

We retain all of the notations and conventions of Proposition 2.6, Theorem 3.3, and Lemma 4.5. Let us for each integer $0 \leq y \leq x$ write $\#C(\alpha, y)$ for simplicity to denote the index $\#C(\alpha, y) = [C(\alpha) : C(\alpha)^{p^y}]$, with $\#C^\star(\alpha, x) = \#C(\alpha, x) - \#C(\alpha, x-1) = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}]$.

Lemma 5.1. *Fix a sufficiently large anticyclotomic exponent $\alpha \gg 1$ so that $x = \mathrm{ord}_p(\#C(\alpha)) \geq 1$. Let us also fix a cyclotomic exponent $\beta \geq 2$ as above. We have the following formulae for the averages $G^{(k)}(\alpha; x)$ and $G^{(0)}(\alpha, \beta; x)$, given in terms of the Dirichlet series expansion* (12)*:*

(i) *We have for any choice of real parameter $Z > 0$ the self-dual average formula*

$$\begin{aligned} &G^{(k)}(\alpha; x) \\ &= \sum_{A \in C(\alpha)^{p^x}} \left(H_{A,1}^{(k)}(\alpha, 0; Z) + H_{A,2}^{(k)}(\alpha, 0; Z) \right) - \frac{\#C(\alpha, x-1)}{\#C^\star(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \left(H_{A,1}^{(k)}(\alpha, 0; Z) + H_{A,2}^{(k)}(\alpha, 0; Z) \right), \end{aligned}$$

where (as above)

$$H_{A,1}^{(k)}(\alpha,0;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p^{\alpha})=1}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(m^2nZ\right)$$

and

$$H_{A,2}^{(k)}(\alpha,0;Z) = \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p^{\alpha})=1}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\alpha}}\right).$$

(ii) *If $\beta \geq 2$, then we have for any choice of real parameter $Z>0$ the non-self-dual average formula*

$$\begin{aligned}&G^{(0)}(\alpha,\beta;x)\\ &= \sum_{A\in C(\alpha)^{p^x}}\left(H_{A,1}^{(0)}(\alpha,\beta;Z)+H_{A,2}^{(0)}(\alpha,\beta;Z)\right) - \frac{\#C(\alpha,x-1)}{\#C^{\star}(\alpha,x)}\sum_{\substack{A\in C(\alpha)^{p^{x-1}}\\ A\notin C(\alpha)^{p^x}}}\left(H_{A,1}^{(0)}(\alpha,\beta;Z)+H_{A,2}^{(0)}(\alpha,\beta;Z)\right),\end{aligned}$$

where (as above)

$$\begin{aligned}H_{A,1}^{(k)}(\alpha,\beta;Z) &= \sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(Zm^2n\right)\\ &\quad - \frac{1}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1\\ m^2n\equiv\pm 1 \bmod p^{\beta-1}\\ m^2n\not\equiv\pm 1 \bmod p^{\beta}}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(Zm^2n\right)\end{aligned}$$

and

$$\begin{aligned}&H_{A,2}^{(k)}(\alpha,\beta;Z)\\ &= \frac{(-1)^{k+1}\eta(N)}{(|D|p^{\beta})^{\frac{1}{2}}}\frac{p}{\varphi(p)}\sum_{m\geq 1}\frac{\eta(m)}{m}\sum_{\substack{n\geq 1\\ (n,p)=1}}\frac{\lambda(n)r_A(n)}{n^{\frac{1}{2}}}V_{k+1}\left(\frac{m^2n}{ZN^2|D|^2p^{4\max(\alpha,\beta)}}\right)\mathrm{Kl}_4(\pm(m^2n\overline{N}^2\overline{D}^8)^{\frac{1}{2}},p^{\beta}).\end{aligned}$$

Again, we write $r_A(n)$ to denote the number of ideals in the class $A\in C(\alpha)=\mathrm{Pic}(\mathcal{O}_{p^{\alpha}})$ of norm equal to n, and also $\mathrm{Kl}_4(\pm c,p^{\beta})=\mathrm{Kl}_4(c,p^{\beta})+\mathrm{Kl}_4(-c,p^{\beta})$ the sum of hyper-Kloosterman sums of dimension $n=4$ and modulus p^{β} evaluated at a coprime residue class $c \bmod p^{\beta}$.

Proof. Both formulae follow from a variation of the proof of Proposition 2.6, using (87) in place of (19). □

Note that we can deduce an estimate for the ring class Galois averages $G^{(k)}(\alpha;x)$ from Theorem 3.4, and for the double Galois averages $G^{(0)}(\alpha,\beta;x)$ by a minor technical variation of the calculations used to show Theorem 4.6. This allows us to derive the following estimates.

Theorem 5.2. *We have the following estimates for the Galois averages introduced above.*

(i) We have for either case on the generic root number $k\in\{0,1\}$ the following estimate for the corresponding ring class Galois average $G^{(k)}(\alpha,0;x)$: In the setup of Theorem 3.4 above, with $q_A(x,y)=\gamma_Ax^2+\delta_Axy+\epsilon_Ay^2$ any choice of binary quadratic form representative for each class $A\in C(\alpha)$, we have for any $\varepsilon>0$ that

$$\begin{aligned}G^{(k)}(\alpha;x) &= \sum_{A\in C(\alpha)^{p^x}}\left(\mathfrak{L}_{k,f,\gamma_A}(1)+O_{f,k,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right)\\ &\quad - \frac{\#C(\alpha,x-1)}{\#C^{\star}(\alpha,x)}\sum_{\substack{A\in C(\alpha)^{p^{x-1}}\\ A\notin C(\alpha)^{p^x}}}\left(\mathfrak{L}_{k,f,\gamma_A}(1)+O_{f,k,p,\varepsilon}\left(\gamma_A(|D|p^{2\alpha})^{\frac{7}{16}+\varepsilon}|\epsilon_A|^{-\frac{1}{2}}\right)\right).\end{aligned}$$

(ii) Fix an integer $\beta \geq 2$, and retain all of the setup of Lemma 3.1, Theorem 3.3, and Theorem 4.6. We have for each ring class exponent $\alpha \geq 1$ and class $A \in C(\alpha)$ corresponding to any choice of binary quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ the following more general estimate for the corresponding sum

$$H_A^{(0)}(\alpha,\beta) = H_{A,1}^{(0)}(\alpha,\beta;(N|D|p^{2\max(\alpha,\beta)})^{-1}) + H_{A,2}^{(0)}(\alpha,\beta;(N|D|p^{2\max(\alpha,\beta)})^{-1}).$$

That is, we have for each class $A \in C(\alpha)$ the more general estimate

$$\begin{aligned} H_A(\alpha,\beta) &= \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) + O_{f,\varepsilon}\left((|D|p^{\beta})^{\frac{3}{16}+\varepsilon} \left(\frac{\gamma_A}{|D|p^{2\max(\alpha,\beta)}} \right)^{\frac{1}{4}} \right) \\ &+ O_{f,\beta,\varepsilon}\left(\gamma_A \cdot (|D|p^{2\max(\alpha,\beta)})^{\frac{1}{4}+\varepsilon} \cdot (|D|p^{2\alpha})^{\delta_0 - \frac{\theta_0}{2}+\varepsilon} \cdot |\epsilon_A|^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon} \right), \end{aligned}$$

where each $\mathfrak{L}_{f,\gamma_A}^{(\beta)}(1)$ denotes the residual sum defined in (9) and (10) above. Hence, after direct substitution, we derive the corresponding estimate for the Galois average

$$G^{(0)}(\alpha,\beta;x) = \sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) - \frac{\#C(\alpha,x-1)}{\#C^{\star}(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1).$$

Proof. The estimate for (i) is a direct consequence of Theorem 3.4 above with Lemma 5.1 (i). Taking balanced parameter $Z = Y^{-1} = N|D|p^{2\max(\alpha,\beta)}$, the estimate for (ii) is deduced by a minor technical variation of Theorem 4.6, using the Hecke relation as in Lemma 3.1 to describe the Fourier coefficients when $\gamma_A > 1$ for the $b = 0$ contributions, and using the estimates of Theorem 3.3 (i) and (ii) for the $b \neq 0$ contributions. ☐

Finally, let us say something about the asymptotic behaviour of the residual terms in these averages.

Corollary 5.3. *If for each of the classes $A \in C(\alpha)^{p^x}, C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x}$ contributing to the averages $G^{(k)}(\alpha;x)$ and $G^{(0)}(\alpha,\beta)$ we can choose a binary quadratic form representative $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ with last coefficient $|\epsilon_A|$ large relative to the leading coefficient γ_A, then the averages converge to the sums of the corresponding residues. That is,*

(i) *If $|\epsilon_A| \gg |\gamma_A|^2$ for each $A \in C(\alpha)^{p^x}, C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x}$, then we have*

$$\lim_{\alpha \to \infty} G^{(k)}(\alpha;x) = \sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{k,f,\gamma_A}(1) - \frac{\#C(\alpha,x-1)}{\#C^{\star}(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{k,f,\gamma_A}(1). \tag{90}$$

(ii) *If $|\epsilon_A| \gg |\gamma_A|^2$ for each $A \in C(\alpha)^{p^x}, C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x}$, then for any fixed $\beta \geq 2$ we have*

$$\lim_{\alpha \to \infty} G^{(0)}(\alpha,\beta) = \sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) - \frac{\#C(\alpha,x-1)}{\#C^{\star}(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1). \tag{91}$$

In any case, the quantities on the right-hand sides of (90) and (91) converge to nonzero quantities.

Proof. We first argue that for $\alpha \gg 1$ sufficiently large, the leading coefficients γ_A of any choice of system of binary quadratic form representatives $q_A(x,y) = \gamma_A x^2 + \delta_A xy + \epsilon_A y^2$ for the non-principal classes $A \in C(\alpha)$ must carry some dependence on the conductor Dp^{α} of the order $\mathcal{O}_{p^{\alpha}} = \mathbf{Z} + p^{\alpha}\mathcal{O}_K$, and specifically on the exponent α. Indeed, we deduce this from the constraint

$$\delta_A^2 - 4\gamma_A\epsilon_A = Dp^{2\alpha} \implies \gamma_A = \frac{(\delta_A^2 - Dp^{2\alpha})}{4\epsilon_A} \tag{92}$$

on the coefficients of each of these quadratic forms $q_A(x,y)$ of discriminant $Dp^{2\alpha}$. Here, we use Dedekind's formula (4) to ensure the existence of sufficiently many classes. Let us also consider the indices $C(\alpha,x-1)$ and $C^{\star}(\alpha,x)$. Let $C(\alpha)[p] \subset C(\alpha)$ denote the subgroup of elements of order p, so the elements $A \in C(\alpha)$ with order p^k for some $0 \leq k \leq x$. Let $C(\alpha)' = C(\alpha)/C(\alpha)[p]$ denote the subgroup of elements $A \in C(\alpha)$ of orders prime to p. Given any $A \in C(\alpha)[p]$ of order p^k say, we see that $A^{p^x} = (A^{p^k})^{p^{x-k}} = \mathbf{1}^{p^{x-k}} = \mathbf{1}$. It is then easy to see that we can identify the subgroup of p^x-th powers $C(\alpha)^{p^x}$ with the quotient $C(\alpha)/C(\alpha)[p] = C(\alpha)'$, and hence calculate the index $[C(\alpha) : C(\alpha)^{p^x}]$ as $\#C(\alpha)/\#C(\alpha)^{p^x} = \#C(\alpha)/\#C(\alpha)' = p^x$. Similarly, we can identify the subgroup $C(\alpha)^{p^{x-1}}$ with the subgroup $C(\alpha)'$ of elements of orders prime to p along with the subgroup $C(\alpha)[p^x] \subset C(\alpha)[p]$ of elements of exact order p^x. That is, for $A \in C(\alpha)[p]$ of order p^k with $0 \leq k \leq x-1$, we have that $A^{p^{x-1}} = (A^{p^k})^{p^{x-1-k}} = (\mathbf{1})^{p^{x-1-k}} = \mathbf{1}$. Hence, we can identify $C(\alpha)^{p^{x-1}} \cong C(\alpha)/(C(\alpha)[p]/C(\alpha)[p^x]) \cong C(\alpha)'C(\alpha)[p^x]$ in this way to compute $\#C(\alpha)^{p^{x-1}} = \#C(\alpha)' \cdot p$ so that $[C(\alpha) : C(\alpha)^{p^{x-1}}] = \#C(\alpha)/\#C(\alpha)^{p^{x-1}} = p^x\#C(\alpha)'/(p\#C(\alpha)') = p^{x-1}$. In this way, we compute

$$\#C(\alpha, x-1) = [C(\alpha) : C(\alpha)^{p^{x-1}}] = p^{x-1}$$

and

$$\#C^{\star}(\alpha, x) = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}] = p^x - p^{x-1} = p^{x-1}(p-1) = \varphi(p^x)$$

so that

$$\frac{\#C(\alpha, x-1)}{\#C^{\star}(\alpha, x)} = \frac{p^{x-1}}{p^{x-1}(p-1)} = \frac{1}{p-1} = \frac{1}{\varphi(p)}.$$

Observe that we can identify the first sum over classes $A \in C(\alpha)^{p^x}$ with the sum over classes $A \in C(\alpha)'$, and the second sum over classes $A \in C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x}$ with the sum over classes $A \in C(\alpha)[p^x]$. In particular, the first sum stabilizes with the ring class exponent in the sense that $C(\alpha)' \cong C(0)'$ for all $\alpha \geq 1$. Let us now make the following choice of quadratic form representatives for these classes. For each class $A_0 \in C(0)'$, we consider the reduced binary quadratic form representative $q_{A_0}(x,y) = \gamma_{A_0}x^2 + \delta_{A_0}xy + \epsilon_{A_0}y^2$ of discriminant $\delta_{A_0}^2 - 4\gamma_{A_0}\epsilon_{A_0} = D$. We know classically that $C(\alpha)$ can be identified canonically with the class group $Q(\alpha)$ of positive definite binary quadratic forms of discriminant $Dp^{2\alpha}$. We also know that we can consider the quotient $Q(\alpha)' = Q(\alpha)/Q(\alpha)[p]$ of elements of orders prime to p (with respect to the composition law). Hence, we can consider the reduced quadratic form representative $q_{A_0}(x,y)$ as an element $\mathfrak{Q}(A_0) = [q_{A_0}(x,y)]$ of the quotient $Q'(0)$. Let us write $\mathfrak{Q}_\alpha(A_0)$ to denote the image of $\mathfrak{Q}(A_0)$ under the corresponding isomorphism $Q(0)' \cong Q(\alpha)'$. We know there exists a unique reduced quadratic form representative $q_{A_0,\alpha}(x,y)$ for $\mathfrak{Q}_\alpha(A_0)$. Hence, $q_{A_0,\alpha}(x,y) = \gamma_{A_0,\alpha}x^2 + \delta_{A_0,\alpha}xy + \epsilon_{A_0,\alpha}y^2$ with $\delta_{A_0,\alpha}^2 - 4\gamma_{A_0,\alpha}\epsilon_{A_0,\alpha} = Dp^{2\alpha}$ and $|\delta_{A_0,\alpha}| \leq \gamma_{A_0,\alpha} \leq \epsilon_{A_0,\alpha}$ (with $\delta_{A_0,\alpha} \geq 0$ if either $|\delta_{A_0,\alpha}| = \gamma_{A_0,\alpha}$ or $\gamma_{A_0,\alpha} \leq \epsilon_{A_0,\alpha}$). At the same time, we argue that any quadratic form $f_{A_0,\alpha}(x,y) \in \mathfrak{Q}_\alpha(A_0) \in Q(\alpha)'$ will have to represent the first coefficient γ_{A_0} nontrivially. It is then well-known (see [12, Lemma 2.3, p. 23]) that $f_{A_0,\alpha}(x,y) = \gamma_{A_0}x^2 + B_{0,\alpha}xy + C_{0,\alpha}y^2$ for integers $B_{0,\alpha}$ and $C_{0,\alpha}$, which have to satisfy the constraint $B_{0,\alpha}^2 - 4\gamma_{A_0}C_{0,\alpha} = Dp^{2\alpha} = p^{2\alpha}(\delta_{A_0}^2 - 4\gamma_{A_0}\epsilon_{A_0})$. Indeed, if $q_{A_0,\alpha}(p,q) = \gamma_{A_0}$ for coprime integers p and q, then we can find integers r and s for which $ps - qr = 1$, and consider the equivalent form defined by

$$q_{A_0,\alpha}(px+ry, qx+sy) = q_{A_0,\alpha}(p,q)x^2 + (2\gamma_{A_0,\alpha}pr + \delta_{A_0,\alpha}ps + \delta_{A_0,\alpha}rq + 2\epsilon_{A_0,\alpha}qs)\,xy + q_{A_0,\alpha}(r,s)y^2.$$

Taking $f_{A_0,\alpha}(x,y) = q_{A_0,\alpha}(px+ry, qx+sy) = \gamma_{A_0}x^2 + B_{0,\alpha}xy + C_{0,\alpha}y^2$, with $B_{0,\alpha} = (Dp^{2\alpha} + 4\gamma_{A_0}C_{0,\alpha})^{\frac{1}{2}}$ and $C_{0,\alpha} = (B_{0,\alpha}^2 - Dp^{2\alpha})/4\gamma_{A_0}$ as we may, and assuming we choose input integers so that the middle coefficient $B_{0,\alpha}$ is minimized (or equivalently so that the last coefficient $C_{0,\alpha}$ is maximized), we find for each class $A_0 \in C'(0)$ a non-reduced binary quadratic form representative $f_{A_0,\alpha}$ for the image of A_0 under the isomorphism $C(0)' \cong C(\alpha)'$ whose leading coefficient is γ_{A_0}. Let us henceforth take[10] this quadratic form representative $f_{A_0,\alpha}(x,y)$ for each class in $C(\alpha)' \cong C(0)'$, for each ring class exponent $\alpha \geq 1$, as we may. Hence, writing $A_{0,\alpha}$ to denote the image of $A_0 \in C(0)'$ under the natural isomorphism $C(0)' \cong C(\alpha)'$ for any ring class exponent ≥ 1, we parametrize the corresponding counting function $r_{A_{0,\alpha}}(n)$ via this non-reduced from $f_{A_0,\alpha}(x,y)$. Using this choice in all of our arguments for each of these classes $A_{0,\alpha}$ (for all $\alpha \geq 1$), we see immediately that the leading coefficients γ_{A_0} do not depend on the ring class exponent $\alpha \geq 1$. In this

[10] Note that the choice of reduced representative was somewhat arbitrary here. We have the freedom to take any choice of representative for each class $A \in C(\alpha)$.

way, we deduce via Deligne's bound $\lambda(n) \ll_\varepsilon n^\varepsilon$ that for any $\alpha \geq 1$,

$$\begin{aligned}
&\sum_{A \in C(\alpha)^{p^x}} \mathfrak{K}_{f,\gamma_A}(1) = \sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{K}_{f,\gamma_{A_0}}(1) \\
&= \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{-\frac{1}{2}} \sum_{q \mid \gamma_{A_0}} \frac{\mu(q)}{q} \lambda\left(\frac{\gamma_A}{q}\right) \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{\mu(d)}{d} \lambda\left(\frac{r''}{d}\right) \\
&\ll_\varepsilon \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{\varepsilon-\frac{1}{2}} \left(\sum_{q \mid \gamma_{A_0}} \frac{1}{q} \left(\sum_{r \mid q} \frac{1}{r} \left(\cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{1}{d} \right) \cdots \right) \right) \\
&= \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{\varepsilon-\frac{1}{2}} \prod_{\substack{i=1 \\ q_{0,i} \mid \gamma_{A_0}}}^{k} \sum_{d_{0,i} \mid q_{0,i}} \frac{1}{d_{0,i}} = \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{\varepsilon-\frac{1}{2}} \prod_{\substack{i=1 \\ q_{0,i} \mid \gamma_{A_0}}}^{k} \frac{\sigma(q_{0,i})}{q_{0,i}} \\
&\ll \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{\varepsilon-\frac{1}{2}} \cdot d(\gamma_{A_0}) \cdot \frac{\sigma(\gamma_{A_0})}{\gamma_{A_0}}.
\end{aligned}$$

Here, the product on the right-hand-side runs over all divisors $q_{0,i}$ of γ_{A_0}; we write $\sigma(n) = \sigma_1(n) = \sum_{d \mid n} d$ to denote the sum-over-divisors function and $d(n) = \sigma_0(n) = \sum_{d \mid n} 1$ the divisor function. Using the standard upper bounds $d(n) \ll_\varepsilon n^\varepsilon$ and $\sigma(n) \leq n \log(n+1)$, we then deduce that

$$\sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{K}_{f,\gamma_{A_0}}(1) \ll_\varepsilon \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{\varepsilon-\frac{1}{2}} \cdot d(\gamma_{A_0}) \cdot \frac{\sigma(\gamma_{A_0})}{\gamma_{A_0}} \ll_\varepsilon \sum_{A_{0,\alpha} \in C(\alpha)'} \gamma_{A_0}^{2\varepsilon-\frac{1}{2}} \cdot \log(\gamma_{A_0}+1) = O_D(1). \tag{93}$$

On the other hand, the corresponding sum over classes $A \in C(\alpha)[p^x]$ satisfies a different bound. Here, since the classes by definition cannot come from lifts of classes in $C(\alpha-1)$, we deduce from the constraint (92) that the coefficients γ_A of any set of binary quadratic form representatives $q_A(x,y)$ of the classes $A \in C(\alpha)[p^x]$ carry a dependence on the ring class exponent $\alpha \geq 1$. In particular, we deduce that for each class $A \in C(\alpha)[p^x]$, there exists a constant $0 < \kappa(A) \leq 1$ such that $\mathrm{ord}_p(\gamma_A) = \kappa(A)x$. Taking

$$\kappa = \kappa(\alpha) = \max_{A \in C(\alpha)[p^x]} \{\kappa(A)\},$$

we use the same argument to derive the corresponding bound

$$\begin{aligned}
\sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) &\ll_\varepsilon \sum_{A \in C(\alpha)[p^x]} \gamma_A^{\varepsilon-\frac{1}{2}} \cdot d(\gamma_A) \cdot \frac{\sigma(\gamma_A)}{\gamma_A} \\
&\ll_\varepsilon \sum_{A \in C(\alpha)[p^x]} \gamma_A^{2\varepsilon-\frac{1}{2}} \cdot \log(\gamma_A+1) \ll_D p \cdot (p^{\kappa x})^{\varepsilon-\frac{1}{2}} \cdot \log(p^{\kappa x}).
\end{aligned} \tag{94}$$

To deduce that the limiting constant in (90) is nonvanishing, we can apply the calculations and lower bounds of Lemma 3.2 directly in each case. Thus for $k=0$, we obtain that

$$\begin{aligned}
&\sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{0,f,\gamma_A}(1) - \frac{\# C(\alpha, x-1)}{\# C^\star(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{0,f,\gamma_A}(1) \\
&= \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{A \in C(\alpha)^{p^x}} \mathfrak{K}_{f,\gamma_A}(1) - \frac{\# C(\alpha, x-1)}{\# C^\star(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{K}_{f,\gamma_A}(1) \right) \\
&= \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{A \in C(\alpha)/C(\alpha)[p]} \mathfrak{K}_{f,\gamma_A}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \right).
\end{aligned}$$

Let us now consider the inner sum

$$\begin{aligned}
&\sum_{A \in C(\alpha)^{p^x}} \mathfrak{K}_{f,\gamma_A}(1) - \frac{\#C(\alpha, x-1)}{\#C^{\star}(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{K}_{f,\gamma_A}(1) \\
&= \sum_{A \in C(\alpha)/C(\alpha)[p]} \mathfrak{K}_{f,\gamma_A}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \qquad (95) \\
&= 1 + \sum_{\substack{A \in C(\alpha)/C(\alpha)[p] \\ A \neq \mathbf{1}}} \mathfrak{K}_{f,\gamma_A}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1)
\end{aligned}$$

in this latter expression, recalling that

$$\begin{aligned}
\mathfrak{K}_{f,\gamma_A}(1) &= \gamma_A^{-\frac{1}{2}} \sum_{q \mid \gamma_A} \frac{\mu(q)}{q} \lambda\left(\frac{\gamma_A}{q}\right) \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{\mu(d)}{d} \lambda\left(\frac{r''}{d}\right) \\
&= \frac{\lambda(\gamma_A)}{\gamma_A^{\frac{1}{2}}} + \sum_{\substack{q \mid \gamma_A \\ q > 1}} \frac{\mu(q)}{q} \lambda\left(\frac{\gamma_A}{q}\right) \sum_{r \mid q} \frac{\mu(r)}{r} \lambda\left(\frac{q}{r}\right) \cdots \sum_{\substack{d \mid r'' \\ d=1}} \frac{\mu(d)}{d} \lambda\left(\frac{r''}{d}\right).
\end{aligned}$$

If the constant term (95) were to vanish, then we would have that

$$\frac{\sum_{A \in C(\alpha)'} \mathfrak{K}_{f,\gamma_A}(1)}{\sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1)} = \frac{1}{\varphi(p)} \qquad (96)$$

for any ring class exponent $\alpha \gg 1$ and holomorphic eigenform f, where the right-hand-size is completely independent of α and f. However, the left-hand side of (96) cannot satisfy this property. As discussed above, the numerator on the left-hand-side can be viewed as a constant depending on f and D which is bounded above by $O_D(1)$ according to (93), while the denominator depends on α and is bounded above by $\ll_{\varepsilon} (p^x)^{-\frac{\kappa}{2}+\varepsilon}$ for some constant $\kappa = \kappa(\alpha) > 0$ according to (94). In this way, we see that the left-hand side of (96) diverges in the limit with $\alpha \to \infty$, so that (96) cannot hold in general. Thus, the inner sum (95) cannot vanish. We view it as a nonvanishing constant that depends on f and α. We then proceed to bound

$$\begin{aligned}
&\sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{0,f,\gamma_A}(1) - \frac{\#C(\alpha, x-1)}{\#C^{\star}(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{0,f,\gamma_A}(1) = \sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{L}_{0,f,\gamma_{A_0}}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{L}_{0,f,\gamma_A}(1) \\
&= \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{K}_{f,\gamma_{A_0}}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \right) \\
&\gg_D \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \gg \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot \log(N)^{-C} \gg 1.
\end{aligned}$$

Here, we use the bounds (93) and (94) to estimate the contributions from each part of the inner sum (95).

For $k=1$, we argue in the same way, using the calculation

$$\begin{aligned}
&\mathfrak{L}_{1,f,\gamma_A}(1) \\
&= \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \left(\log\left(\frac{N|D|p^{2\alpha}}{\gamma_A}\right) + \frac{L'}{L}(1,\eta) + \frac{L'}{L}(1, \mathrm{Sym}^2 f) - 2(\gamma + \log(2\pi)) - \frac{\zeta^{(N)\prime}}{\zeta^{(N)}}(2) + \frac{\epsilon_p'}{\epsilon_p}(1) + \frac{\mathfrak{K}_{f,\gamma_A}'}{\mathfrak{K}_{f,\gamma_A}}(1) \right)
\end{aligned}$$

from Lemma 3.2 to deduce that

$$\begin{aligned}
&\sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{1,f,\gamma_A}(1) - \frac{\# C(\alpha, x-1)}{\# C^\star(\alpha, x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{1,f,\gamma_A}(1) = \sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{L}_{1,f,\gamma_{A_0}}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{L}_{1,f,\gamma_A}(1) \\
&\gg_{f,p} \sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{L}_{0,f,\gamma_{A_0}}(1) \cdot \left(\log \left(\frac{N|D|p^{2\alpha}}{\gamma_{A_0}} \right) + \frac{\mathfrak{K}'_{f,\gamma_{A_0}}}{\mathfrak{K}_{f,\gamma_{A_0}}}(1) \right) \\
&- \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{L}_{0,f,\gamma_A}(1) \cdot \left(\log \left(\frac{N|D|p^{2\alpha}}{\gamma_A} \right) + \frac{\mathfrak{K}'_{f,\gamma_A}}{\mathfrak{K}_{f,\gamma_A}}(1) \right) \\
&= \log(|D|^{\frac{3}{2}} p^{2\alpha}) \cdot \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{K}_{f,\gamma_{A_0}}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \right) \\
&- \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{\substack{A_{0,\alpha} \in C(\alpha)' \\ A \neq \mathbf{1}}} \mathfrak{K}_{f,\gamma_{A_0}}(1) \cdot \log(\gamma_{A_0}) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \cdot \log(\gamma_A) \right) \\
&+ \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \cdot \left(\sum_{\substack{A_{0,\alpha} \in C(\alpha)' \\ A \neq \mathbf{1}}} \mathfrak{K}_{f,\gamma_{A_0}}(1) \cdot \frac{\mathfrak{K}'_{f,\gamma_{A_0}}}{\mathfrak{K}_{f,\gamma_{A_0}}}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{K}_{f,\gamma_A}(1) \cdot \frac{\mathfrak{K}'_{f,\gamma_A}}{\mathfrak{K}_{f,\gamma_A}}(1) \right) \\
&\gg_{f,D} \log(|D|^{\frac{3}{2}} p^{2\alpha}) \cdot \kappa_{D,N}(1) \cdot \epsilon_p(1) \cdot L(1, \mathrm{Sym}^2 f) \gg 1.
\end{aligned}$$

Again, we use the bounds (93) and (94) to estimate the contributions from each part of the inner sum (95), as well as the variations of this inner sum appearing in the expression above – which can be estimated similarly.

To deduce that the limiting constant in (91) is nonvanishing, we first express each summand as

$$\mathfrak{L}^{(\beta)}_{f,\gamma_A}(1) = \frac{2}{w} \sum_{q \mid \gamma_A} \mu(q) \cdot \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \cdot \frac{2}{\varphi^\star(p^\beta)} \sum_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}^q_\chi(1) + \chi(\gamma_A)\epsilon(\chi) \mathfrak{L}^q_{\overline{\chi}}(1) \right),$$

where

$$\mathfrak{L}^q_\chi(1) = L(1, \eta\chi^2) \cdot \frac{L_q(1, \mathrm{Sym}^2 f^{(q)} \otimes \chi)}{L^{(Np)}(1, \chi)}, \quad \mathfrak{L}^q_{\overline{\chi}}(1) = L(1, \eta\overline{\chi}^2) \cdot \frac{L_q(1, \mathrm{Sym}^2 f^{(q)} \otimes \overline{\chi})}{L^{(Np)}(1, \overline{\chi})}$$

and

$$\epsilon(\chi) = \frac{\eta\chi^2(-N)\tau(\eta\chi^2)^4}{|D|^2 p^{2\beta}} \in \mathbf{S}^1$$

as before denotes the root number for the ambient family of Rankin-Selberg L-functions $L(s, f \times \rho\chi \circ \mathbf{N})$, with the twisted congruence symmetric square L-values $L^q(1, \mathrm{Sym}^2 f^{(q)} \otimes \chi)$ defined via the Dirichlet series

expansions in (8) above. Hence, we have the more explicit expression
(97)

$$\sum_{A \in C(\alpha)^{p^x}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) - \frac{\#C(\alpha, x-1)}{\#C^{\star}(\alpha)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1) = \sum_{A_{0,\alpha} \in C(\alpha)'} \mathfrak{L}_{f,\gamma_{A_0}}^{(\beta)}(1) - \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \mathfrak{L}_{f,\gamma_A}^{(\beta)}(1)$$

$$= \sum_{A_{0,\alpha} \in C(\alpha)'} \frac{2}{w} \sum_{q \mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_{A_0}) \mathfrak{L}_{\overline{\chi}}^{q}(1) \right)$$

$$- \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \frac{2}{w} \sum_{q \mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_A) \mathfrak{L}_{\overline{\chi}}^{q}(1) \right).$$

Using the nonvanishing of $L(1, \mathrm{Sym}^2 f \otimes \chi)$, we deduce that each of the congruence symmetric square L-values $\mathfrak{L}_{\chi}^{q}(1)$ does not vanish. Here, we can deduce this using the same contour argument as given in [8, Lemma 4.2] (for instance) applied to the Dirichlet series $L_q(s, \mathrm{Sym}^2 f^{(q)} \otimes \chi)$. We could also calculate $L_q(1, \mathrm{Sym}^2 f^{(q)} \otimes \chi)$ in terms of $L(1, \mathrm{Sym}^2 f \otimes \chi)$ in the style of Lemma 3.2 above to reach the same conclusion. Since $\chi(\gamma_A)\epsilon(\chi)$ is an algebraic number for each constant coefficient γ_A we consider here, we can use a minor variation of the argument given in the proof of Theorem 4.6 to deduce that each summand $\mathfrak{L}_{\chi}^{q}(1) + \chi(\gamma_A)\epsilon(\chi)\mathfrak{L}_{\overline{\chi}}^{q}(1)$ in (97) does not vanish. To be sure, suppose otherwise that any of these summands vanishes, equivalently that

$$\frac{\mathfrak{L}_{\chi}^{q}(1)}{\mathfrak{L}_{\overline{\chi}}^{q}(1)} = -\epsilon(\chi)\chi(\gamma_A). \tag{98}$$

Since the right-hand-side of (98) is an algebraic number, we can act on each side by $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$. Taking the product of such Galois conjugates on each side of (98), we then obtain the identity

$$\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^{\sigma}}^{q}(1)}{\mathfrak{L}_{\overline{\chi}^{\sigma}}^{q}(1)} = - \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\chi^{\sigma})\chi^{\sigma}(\gamma_A),$$

which after pairing together each character χ^{σ} with its inverse $\overline{\chi^{\sigma}} = \chi^{\sigma^{-1}}$ becomes the impossible identity $1 = -1$. Hence, the identity (98) cannot hold, and so $\mathfrak{L}_{\chi}^{q}(1) + \chi(\gamma_A)\epsilon(\chi)\mathfrak{L}_{\overline{\chi}}^{q}(1)$ cannot vanish. Similarly, we can show that the corresponding sum over all primitive even Dirichlet characters $\chi \bmod p^{\beta}$

$$\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \chi(\gamma_A)\epsilon(\chi)\mathfrak{L}_{\overline{\chi}}^{q}(1) \right)$$

does not vanish. To be clear, if we suppose otherwise that it does vanish, then we must have the identity

$$\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \mathfrak{L}_{\chi}^{q}(1) = - \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1),$$

which after exponentiating each side gives

$$\prod_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \exp\left(\mathfrak{L}_{\chi}^{q}(1)\right) = \prod_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \exp\left(-\epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1)\right).$$

Comparing products in this latter expression, we deduce that for each primitive even Dirichlet character $\chi \bmod p^{\beta}$, there exists a unique primitive even Dirichlet character $\chi' \bmod p^{\beta}$ such that

$$\exp\left(\mathfrak{L}_{\chi}^{q}(1)\right) = \exp\left(-\epsilon(\chi')\chi'(\gamma_A)\mathfrak{L}_{\overline{\chi}'}^{q}(1)\right),$$

which after taking logarithms gives

$$\mathfrak{L}_{\chi}^{q}(1) = -\epsilon(\chi')\chi'(\gamma_A)\mathfrak{L}_{\overline{\chi}'}^{q}(1),$$

equivalently

$$\frac{\mathfrak{L}_{\chi}^{q}(1)}{\mathfrak{L}_{\overline{\chi}'}^{q}(1)} = -\epsilon(\chi')\chi'(\gamma_A). \tag{99}$$

Since the right-hand-side of (98) is again an algebraic number, we can consider the action by an automorphism $\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})$. Taking the product of (99) over all such Galois conjugates, we then obtain

$$\prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^{\sigma}}^{q}(1)}{\mathfrak{L}_{\overline{\chi}'^{\sigma}}^{q}(1)} = \prod_{\sigma \in \mathrm{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} -\epsilon(\chi'^{\sigma})\chi'^{\sigma}(\gamma_A),$$

which after pairing each character χ^{σ} with its inverse $\overline{\chi^{\sigma}} = \chi^{\sigma^{-1}}$ gives the impossible identity $1 = -1$. Hence, we have shown that for any class $A \in C(\alpha)$ and any divisor $q \mid \gamma_A$ of the first coefficient γ_A of the chosen binary quadratic form representative for A, the sum

$$\frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1) \right)$$

does not vanish. It is then simple to see that the first sum

$$\begin{aligned}
&\sum_{A_{0,\alpha} \in C(\alpha)'} \frac{2}{w} \sum_{q \mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}^{q}(1) \right) \\
&= \frac{2}{w} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}(1) \right) \\
&+ \sum_{\substack{A_{0,\alpha} \in C(\alpha)' \\ A_{0,\alpha} \neq \mathbf{1}}} \frac{2}{w} \sum_{q \mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1) \right)
\end{aligned}$$

in (97) does not vanish identically. Here, we write $\mathfrak{L}_{\chi}^{1}(1) = \mathfrak{L}_{\chi}(1)$, as in the proof of Theorem 4.6 above. We now consider the full residual sum (97). We claim that this cannot vanish. For supposing otherwise that it did, we would obtain that

$$\begin{aligned}
&\sum_{A_{0,\alpha} \in C(\alpha)'} \frac{2}{w} \sum_{q \mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}^{q}(1) \right) \\
&= \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \frac{2}{w} \sum_{q \mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \frac{2}{\varphi^{\star}(p^{\beta})} \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1) \right),
\end{aligned}$$

which after canceling out identical extra scalar terms and switching the order of summation is the same as

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \sum_{A_{0,\alpha} \in C(\alpha)'} \sum_{q \mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}^{q}(1) \right) \\
&= \sum_{\substack{\chi \bmod p^{\beta} \\ \chi(-1)=1 \\ \text{primitive}}} \frac{1}{\varphi(p)} \sum_{A \in C(\alpha)[p^x]} \sum_{q \mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \left(\mathfrak{L}_{\chi}^{q}(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^{q}(1) \right).
\end{aligned}$$

Exponentiating both sides, we obtain

$$\prod_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \\ \text{primitive}}} \exp\left(\sum_{A_{0,\alpha}\in C(\alpha)'} \sum_{q\mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \left(\mathfrak{L}_\chi^q(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}^q(1)\right)\right)$$

$$= \prod_{\substack{\chi \bmod p^\beta \\ \chi(-1)=1 \\ \text{primitive}}} \exp\left(\frac{1}{\varphi(p)} \sum_{A\in C(\alpha)[p^x]} \sum_{q\mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \left(\mathfrak{L}_\chi^q(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^q(1)\right)\right).$$

Comparing products again, it follows that for each primitive even Dirichlet character χ mod p^β, there exists a unique primitive even Dirichlet character χ' mod p^β for which

$$\exp\left(\sum_{A_{0,\alpha}\in C(\alpha)'} \sum_{q\mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \left(\mathfrak{L}_\chi^q(1) + \epsilon(\chi)\chi(\gamma_{A0})\mathfrak{L}_{\overline{\chi}}^q(1)\right)\right)$$

$$= \exp\left(\frac{1}{\varphi(p)} \sum_{A\in C(\alpha)[p^x]} \sum_{q\mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \left(\mathfrak{L}_{\chi'}^q(1) + \epsilon(\chi')\chi'(\gamma_A)\mathfrak{L}_{\overline{\chi'}}^q(1)\right)\right),$$

which after taking logarithms gives

$$\sum_{A_{0,\alpha}\in C(\alpha)'} \sum_{q\mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \left(\mathfrak{L}_\chi^q(1) + \epsilon(\chi)\chi(\gamma_{A_0})\mathfrak{L}_{\overline{\chi}}^q(1)\right)$$

$$= \frac{1}{\varphi(p)} \sum_{A\in C(\alpha)[p^x]} \sum_{q\mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \left(\mathfrak{L}_{\chi'}^q(1) + \epsilon(\chi')\chi'(\gamma_A)\mathfrak{L}_{\overline{\chi'}}^q(1)\right),$$

equivalently

$$\frac{\sum_{A_{0,\alpha}\in C(\alpha)'} \sum_{q\mid \gamma_{A_0}} \mu(q) \frac{\lambda\left(\frac{\gamma_{A_0}}{q}\right)}{\gamma_{A_0}^{\frac{1}{2}}} \left(\mathfrak{L}_\chi^q(1) + \epsilon(\chi)\chi(\gamma_A)\mathfrak{L}_{\overline{\chi}}^q(1)\right)}{\sum_{A\in C(\alpha)[p^x]} \sum_{q\mid \gamma_A} \mu(q) \frac{\lambda\left(\frac{\gamma_A}{q}\right)}{\gamma_A^{\frac{1}{2}}} \left(\mathfrak{L}_{\chi'}^q(1) + \epsilon(\chi')\chi'(\gamma_A)\mathfrak{L}_{\overline{\chi'}}^q(1)\right)} = \frac{1}{\varphi(p)}. \tag{100}$$

We claim that this identity cannot hold. To be more precise, a simpler variation of the arguments used to derive the bounds (93) and (94) to estimate (95) shows that the limit on the right-hand side of (100) diverges with $\alpha \to \infty$. This gives a contradiction, from which we deduce the nonvanishing of the residual term (97). $\square$

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