RANKIN-SELBERG L-FUNCTIONS IN CYCLOTOMIC TOWERS, III

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Abstract. Let π be a cuspidal automorphic representation of GL_2 over a totally real number field F. Let K be a totally imaginary quadratic extension of F. We estimate central values of the $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin-Selberg L-functions associated to π times representations induced from Hecke characters of K which are ramified only at a given prime ideal $\mathfrak{p}$ of F. More specifically, we use spectral decompositions of shifted convolution sums and relations to Fourier-Whittaker coefficients of genuine and non-genuine metaplectic forms to obtain nonvanishing estimates, averaging over primitive ring class characters of a given exact order. When π corresponds to a holomorphic Hilbert modular form of arithmetic weight $k \geq 2$, we then derive finer results from the rationality theorems of Shimura, together with the existence of suitable $\mathfrak{p}$-adic L-functions. This allows us to generalize the theorems of Rohrlich, Vatsal, and Cornut-Vatsal to this setting. Finally, in a self-contained appendix, we explain how to use these results to deduce bounds for Mordell-Weil ranks of the associated GL_2-type abelian varieties via existing Iwasawa main conjecture divisibilities.

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1. Introduction

Let F be a totally real number field of degree d, ring of integers $\mathcal{O}_F$, and ring of adeles $\mathbf{A}_F$. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of conductor $c(\pi) \subset \mathcal{O}_F$ and unitary central character $\omega = \omega_\pi$. Let K be a totally imaginary quadratic extension of F of relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F} \subset \mathcal{O}_F$, absolute discriminant $D_K = D_F^2 \mathbf{N}\mathfrak{D}$ (for D_F the absolute discriminant of F), and idele class character $\eta = \eta_{K/F}$ of F. Let $\mathcal{W}$ be a finite-order Hecke character of K of the following type. We consider the product $\mathcal{W} = \rho\chi \circ \mathbf{N}$ of a ring class character ρ of K times a character $\chi \circ \mathbf{N}$ arising via composition with the norm homomorphism $\mathbf{N} = \mathbf{N}_{K/\mathbf{Q}} : K \to \mathbf{Q}$ from some Dirichlet character χ (the basechange of χ to K). The ring class character ρ is a character of the class group of the $\mathcal{O}_F$-order $\mathcal{O}_\mathfrak{c} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ of some conductor

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$\mathfrak{c} \subset \mathcal{O}_F$, and determines a finite-order idele class character[1]

$$\rho : \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_\mathfrak{c}^\times \longrightarrow \mathbf{S}^1.$$

Here, $\mathbf{A}_K$ denotes the ring of adeles of K, with archimedean local component $K_\infty = K \otimes_\mathbf{Q} \mathbf{C}$. We write $\mathbf{A}_K^\times$ to denote the ideles of K, and $\widehat{\mathcal{O}}_\mathfrak{c}^\times$ the units of the profinite completion of the order $\mathcal{O}_\mathfrak{c}$. We shall refer to the works of Cornut-Vatsal [15] (especially [15, §2,6]) for more background on ring class extensions of these CM fields K/F, and in particular how the classical theory for $F = \mathbf{Q}$ outlined in [16] (for instance) extends in a natural way to this setting. The character $\chi \circ \mathbf{N}$ on the other hand factors through some cyclotomic extension of K, and so we shall sometimes call it "cyclotomic". To each such Hecke character $\mathcal{W}$ of K, we have an induced automorphic representation $\pi(\mathcal{W}) = \otimes_v \pi(\mathcal{W})_v$ of $\mathrm{GL}_2(\mathbf{A}_F)$, and can then consider the corresponding $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin-Selberg L-function

$$L(s, \pi \times \mathcal{W}) = L(s, \pi \times \pi(\mathcal{W})) = \prod_{v \nmid \infty} L(s, \pi_v \times \pi(\mathcal{W})_v). \tag{1}$$

This degree-four L-function has an analytic continuation thanks to the theory of Jacquet [30] and Jacquet-Langlands [31]; its completed L-function $\Lambda(s, \pi \times \pi(\mathcal{W})) = L_\infty(s) L(s, \pi \times \mathcal{W})$ satisfies a functional equation

$$\Lambda(s, \pi \times \mathcal{W}) = \epsilon(s, \pi \times \mathcal{W}) \Lambda(1 - s, \widetilde{\pi} \times \mathcal{W}^{-1}),$$

where $\epsilon(s, \pi \times \mathcal{W}) = c(\pi \times \mathcal{W})^{s - \frac{1}{2}} \epsilon(1/2, \pi \times \mathcal{W})$ denotes the ϵ-factor, with $c(\pi \times \mathcal{W})$ the conductor of the L-function, and $\epsilon(1/2, \pi \times \mathcal{W}) \in \mathbf{S}^1$ the root number of $L(s, \pi \times \mathcal{W})$. If $\pi \cong \widetilde{\pi}$ is self-contragredient and $\mathcal{W} = \rho$ is a ring class character, then this functional equation (1) is symmetric in the sense that it relates the same L-function on each side,

$$\Lambda(s, \pi \times \rho) = \epsilon(s, \pi \times \rho) \Lambda(1 - s, \pi \times \rho), \tag{2}$$

and the root number $\epsilon(1/2, \pi \times \rho) \in \{\pm 1\}$ is real-valued. This is a consequence of the fact that ring class characters are equivariant with respect to complex conjugation (cf. [52]). Hence if $\epsilon(1/2, \pi \times \rho) = -1$, then the functional equation (2) forces the vanishing of the central value: $\Lambda(s, \pi \times \rho) = L(s, \pi \times \rho) = 0$.

Let us now fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. It can be seen via direct calculation (see e.g. (2.1) below) that as one varies over ring class characters ρ of K of $\mathfrak{p}$-power conductor, the root number $\epsilon(1/2, \pi \times \rho) \in \mathbf{S}^1$ is generically independent of the choice of ring class character ρ. In particular, when $\pi \cong \widetilde{\pi}$ is self-contragredient, there exists an integer $\nu = \nu(\pi, K, p) \in \{0, 1\}$ such that $\epsilon(1/2, \pi \times \rho) = (-1)^\nu$ for all but finitely many ring class characters ρ of K of $\mathfrak{p}$-power conductor. Let us call this number $(-1)^\nu$ the *generic root number* for simplicity. It allows us to characterize (and discard[2]) the setting of forced vanishing via the functional equation (1) corresponding to the case of $\nu = \nu(\pi, D, p) = 1$ when $\pi \cong \widetilde{\pi}$ is self-contragredient. In all other settings, we seek to determine how seldom the values $L(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ vanish as (i) ρ varies over ring class characters of $\mathfrak{p}$-power conductor and (ii) χ varies over Dirichlet characters of a given p-power conductor.

We begin by fixing a (possibly trivial) primitive Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. We consider the average of central values of the Rankin-Selberg L-function of $\pi \otimes \chi$ times $\pi(\rho)$, where ρ ranges over primitive ring class characters of K of exact order p^x (and some corresponding conductor $\mathfrak{p}^\alpha$). That is, for a sufficiently large integer $\alpha \gg 1$, we consider averages over primitive ring class characters of conductor $\mathfrak{p}^\alpha$, as well as certain subaverages corresponding to ring class characters with a given "tamely ramified" part (cf. [60], [15]) or a given exact p-power order. We then explain how to refine these results, using Shimura's rationality theorem and specializations of multivariable p-adic L-functions. This allows us to generalize the nonvanishing theorems of Rohrlich [53], [52], Vatsal [60], and Cornut-Vatsal [15] to the central values $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$, when π corresponds to a holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$ with each $k_j \geq 2$. As we explain, this has various applications to p-adic L-functions constructions in the Iwasawa theory literature, and has to date been treated as a standard hypothesis. This in fact forms the

[1] Composing with the reciprocity map of class field theory, such a character ρ factors through the Galois group of the ring class field $K[\mathfrak{c}]$ of conductor $\mathfrak{c}$ over K, which is of generalized dihedral type over F. For this reason, such a character ρ is often said to be *dihedral*.

[2] The methods we develop here can be applied to study the central derivative values $L'(1/2, \pi \times \rho)$ in this setting, but at the expense of clarity, and so we save this task for another work. Note that this setting has already been addressed when π corresponds to a holomorphic Hilbert modular form of parallel weight two by Cornut [14] and Cornut-Vatsal [15].

main motivation for the work. The main arguments however come from analytic number theory, using ideas from the analytic theory of automorphic forms.

We first show the following analytic result. Here, we derive integral presentations for our underlying averages we consider in terms of Fourier coefficients of certain automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ and its two-fold metaplectic cover. This allows us to use spectral decompositions of the corresponding automorphic forms to derive bounds, leading us to the following implications. Given a sufficiently large integer $\alpha \gg 1$, we consider the set $C(\alpha)^\vee$ of ring class characters of conductor $\mathfrak{p}^\alpha$, and in particular the subset $P(\alpha) \subset C(\alpha)^\vee$ of primitive characters which do not factor through $C(\alpha-1)^\vee$. We also consider the following subaverages of these primitive characters, in the style of Cornut-Vatsal [15]. Writing $C(\infty) = \varprojlim_{\alpha \geq 0} C(\alpha)$ to denote the inverse limit, let $C_0 = C(\infty)_{\mathrm{tors}}$ denote the finite torsion subgroup. Given a character ρ_0 of C_0, we also consider weighted subaverages over primitive characters $P(\alpha, \rho_0) \subset P(\alpha) \subset C(\alpha)^\vee$ of conductor $\mathfrak{p}^\alpha$ which induce this chosen character ρ_0 on the torsion subgroup C_0 of $C(\infty)$. As well, writing $x = \mathrm{ord}_p(\#C(\alpha))$, we consider the subaverages over primitive ring class characters of $C(\alpha)$ of exact order p^x.

Theorem 1.1 (Theorem 2.9, Corollary 2.10)**.** *Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation of conductor $c(\pi) \subset \mathcal{O}_F$, which we assume corresponds to a holomorphic Hilbert modular form. Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. Let K be a totally imaginary quadratic extension of F of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and associated character η of F. Assume that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Fix a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. In the special case where $\beta = 0$, let us also also assume that the generic root number $\epsilon(1/2, \pi \times \rho) = (-1)^\eta$ described above is equal to 1 (as opposed to -1).[3] If the exponent $\alpha \gg 1$ is sufficiently large relative to β, then there exists a primitive ring class character ρ of conductor $\mathfrak{p}^\alpha$, for which $L(1/2, \pi \times \rho\chi \circ \mathbf{N}) \neq 0$. Moreover, for any choice of character ρ_0 of C_0, there exists a character ρ in the set $P(\alpha, \rho_0)$ of primitive ring class characters of conductor $\mathfrak{p}^\alpha$ of K and restriction $\rho|_{C_0} = \rho_0$ to $C_0 = C(\infty)_{\mathrm{tors}}$ for which $L(1/2, \pi \times \rho\chi \circ \mathbf{N}) \neq 0$.*

More generally, using these main results as input, we then go on to show the following refinements. First, we use the rationality theorem of Shimura [49] to deduce the following generic nonvanishing properties, generalizing the theorems of Rohrlich [53], [52], [51] (and Chinta [11]), Vatsal [60], and Cornut-Vatsal [15].

Theorem 1.2 (Theorem 2.12)**.** *Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation of conductor $c(\pi) \subset \mathcal{O}_F$ which corresponds to a holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$ with each $k_j \geq 2$. Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. Let us assume for simplicity that $\mathfrak{p}$ has residue degree one[4]. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and absolute discriminant D_K. Assume that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Assume the Hecke field $\mathbf{Q}(\pi)$ of π is linearly disjoint over $\mathbf{Q}$ to the cyclotomic tower obtained by adjoining all p-power roots of unity $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbf{Q}(\zeta_{p^n})$. Fix a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. In the event that $\beta = 0$ (hence χ trivial), let us also assume that the generic root number $\epsilon(1/2, \pi \times \rho)$ for ρ ranging over primitive ring class characters characters of conductor $\mathfrak{p}^\alpha$ with $\alpha \gg 1$ sufficiently large is not equal to -1. Then for each sufficiently large integer $\beta \geq 1$, there exists a primitive ring class character ρ of conductor $\mathfrak{p}^\alpha$ for which the Galois average $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$ does not vanish, and so $L(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish for $\mathcal{W} = \rho\chi \circ \mathbf{N}$ ranging over such Hecke characters taking values in roots of unity of exact order $\mathrm{lcm}(p^\beta, \mathrm{ord}(\rho))$, i.e. where $\mathrm{ord}(\rho) \mid (\#C(\alpha) - \#C(\alpha - 1))$ denotes the exact order of the character ρ.*

We can also derive the following results via specialization of suitable multivariable $\mathfrak{p}$-adic L-functions. Recall that a for a fixed prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p, a $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation is said to be $\mathfrak{p}$-ordinary if its Hecke eigenvalue at $\mathfrak{p}$ is an algebraic number whose image under a fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ is a p-adic unit. Let $K_\infty^{(\mathfrak{p})}$ denote the compositum of the tower of all ring class extensions of K of $\mathfrak{p}$-power conductor with the cyclotomic extension obtained by adjoining all p-power roots of unity to K. Let $\mathcal{G} = \mathrm{Gal}(K_\infty^{(\mathfrak{p})}/K)$ denote its Galois group. Composing with the Artin reciprocity map, we view the characters $\mathcal{W}$ of K described above as finite order characters factoring through $\mathcal{G}$. Let us also write $\Omega \approx \mathbf{Z}_p^\delta$ to denote the Galois group of the anticyclotomic $\mathbf{Z}_p^\delta$-extension of K, where $\delta = \delta_\mathfrak{p} = [F_\mathfrak{p} : \mathbf{Q}_p]$ denotes the residue degree of $\mathfrak{p}$, and $\Gamma \cong \mathbf{Z}_p$ the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of K.

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[3] to rule out forced vanishing from the functional equation

[4] so that $\mathfrak{p}^{\alpha-1}/\mathfrak{p}^\alpha$ is cyclic for all $\alpha \gg 1$

Theorem 1.3 (Theorem 3.7, cf. Corollary 3.4). *Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation of conductor $c(\pi) \subset \mathcal{O}_F$ which corresponds to a holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$ with each $k_j \geq 2$. Fix a prime $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and absolute discriminant D_K. Assume that π is $\mathfrak{p}$-ordinary, that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$, and that the residual Galois representation associated to π by constructions Carayol* [9], *Taylor* [54], *and Wiles* [62] *is absolutely irreducible. Fix a character $\mathcal{W}_0$ of the torsion subgroup $\mathcal{G}_{\mathrm{tors}}$ of $\mathcal{G}$. There exists a minimal exponent $\alpha_0 \geq 0$ such that for all characters ρ_w of Ω of exact order p^α with $\alpha \geq \alpha_0$, the central value $L(1/2, \pi \times \mathcal{W}_0 \rho_w \psi_w)$ does not vanish for any character ψ_w of the cyclotomic Galois group $\Gamma \cong \mathbf{Z}_p$.*

Finally, we derive the following arithmetical applications of these results in a self-contained appendix. Namely, we use work of Xin Wan [61] on the Iwasawa main conjecture together with existing work on anticyclotomic main conjectures (e.g. [42]) to deduce the following results subject to standard technical hypotheses. Fix π a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation corresponding to a holomorphic Hilbert modular form of parallel weight two and trivial character. Writing $\mathbf{Q}(\pi)$ again to denote the Hecke field of π, le us assume (as is often known) that we can attach to π an abelian variety $A = A_\pi$ defined over F such that (i) the dimension of A is equal to the degree of $\mathbf{Q}(\pi)$, (ii) the ring of endomorphisms $\mathrm{End}_F(A)$ is given by the ring of integers $\mathcal{O}_{\mathbf{Q}(\pi)}$ of π, and (iii) the Hasse-Weil L-function $L(s, A/F)$ of A/F is given by that of π, in other words: $L(s-1/2, \pi) = (2/2\pi)\Gamma(s)L(s, A/F)$. Let K_∞ denote the compositum of the anticyclotomic $\mathbf{Z}_p^\delta$-extension of K with the cyclotomic $\mathbf{Z}_p$-extension of K, where $\delta = [F_\mathfrak{p} : \mathbf{Q}_p]$ again denotes the residue degree of $\mathfrak{p}$. Hence, the corresponding Galois group $G = \mathrm{Gal}(K_\infty/K)$ is isomorphic as a topological group to $\mathbf{Z}_p^{\delta+1}$. We can then consider the Mordell-Weil group $A(K_\infty)$ of K_∞-rational points of A. We can also consider the corresponding Tate-Shafarevich group $Ш(A/K_\infty)$, or more precisely its p-primary subgroup $Ш(A/K_\infty)[p^\infty]$. We deduce the following theorems for these groups, subject to the various hypotheses in [61] (as summarized below) and our discussion of p-adic L-functions:

Theorem 1.4 (Theorem B.4). *Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation corresponding to a holomorphic Hilbert modular form of parallel weight two and trivial character. Assume that π has associated to it an abelian variety $A = A_\pi$ as described above, and fix a prime $p \geq 5$. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and absolute discriminant D_K. Assume the conditions of Hypothesis B.1 and Theorem B.2 below are met, that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$, and that the Hecke field $\mathbf{Q}(\pi)$ is linearly disjoint over $\mathbf{Q}$ the cyclotomic tower obtained by adjoining all p-power roots of unity $\mathbf{Q}(\zeta_{p^\infty})$. Let $\rho = \rho_w$ be a ring class character factoring through the Galois group $G = \mathrm{Gal}(K_\infty/K)$. There exists an integer $\beta_0(\rho)$ such that for all characters $\psi = \psi_w$ of the cyclotomic Galois group $\Gamma = \mathrm{Gal}(K^{\mathrm{cyc}}/K)$ of exact order p^β with $\beta \geq \beta_0(\rho)$, the central value $L(1/2, \pi \times \rho\psi)$ does not vanish, and hence the corresponding $\rho\psi$-isotypical components of both $A(K_\infty)$ and $Ш(A/K_\infty)[p^\infty]$ are finite.*

To state the second result in a concise way, let us also write $\epsilon = \epsilon(1/2, A/K)$ to denote the sign in the functional equation of the Hasse-Weil L-function $L(s, A/K)$ of A over K.

Theorem 1.5 (Theorem B.5). *Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation corresponding to a holomorphic Hilbert modular form of parallel weight two and trivial character. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and absolute discriminant D_K. Assume that π has associated to it an abelian variety $A = A_\pi$ as described above, and that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Assume as well that the following conditions hold: (1) if A acquires CM after basechange to some quadratic extension K_π/F, then this extension K_π is not contained in K_∞ when $\epsilon = +1$, and (2) A has good ordinary reduction at all primes above p in F. Finally, if the residue degree $\delta = [F_\mathfrak{p} : \mathbf{Q}_p]$ is greater than one, let us assume additionally that the conditions of Theorems 2.12, 3.7, and B.2 below (including the vanishing of the anticyclotomic μ-invariant) hold. Then, $A(K_\infty)$ is finitely generated if $\epsilon = +1$, and $A(K_\infty)/A(D_\infty)$ is finitely generated if $\epsilon = -1$.*

1.1. **Outline of the proof of Theorem 1.1.** Let us now give a high-level sketch of how the main analytic result Theorem 1.1 is derived. We consider the weighted average $P_\alpha(\pi, \chi)$ over central values $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ with ρ varying over primitive ring class characters of conductor $\mathfrak{p}^\alpha$, as well as the subaverage $P_{\alpha,\rho}(\pi, \chi)$ with $\rho \in P(\alpha, \rho_0)$ varying over those characters inducing a given character ρ_0 on the torsion subgroup $C_0 = C(\infty)_{\mathrm{tors}}$. More precisely, we describe the values in the average using an unbalanced approximate functional equation, and this reduces us to looking at sums of the form $D_{A,1}(\pi, \chi; Z) + D_{A,2}(\pi, \chi; Z)$ for any choice of real parameter $Z > 0$, where

$$D_{A,1}(\pi, \chi; Z) = \frac{1}{w_K} \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{a,b \in \mathcal{O}_F // \mathcal{O}_F^\times} \frac{\lambda(f_A(a,b))\chi(f_A(a,b))}{\mathbf{N} f_A(a,b)^{\frac{1}{2}}} V_1 \left(\mathbf{N}(\mathfrak{m}^2 f_A(a,b)) Z \right)$$

and

$$D_{A,2}(\pi, \chi; Z) = \frac{1}{w_K} \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{a,b \in \mathcal{O}_F / \mathcal{O}_F^\times} \frac{\overline{\lambda(f_A(a,b))\chi(f_A(a,b))}}{\mathbf{N} f_A(a,b)^{\frac{1}{2}}} V_2 \left(\frac{\mathbf{N}(\mathfrak{m}^2 f_A(a,b))}{ZC} \right).$$

Here, the sums run over pairs of F-integers $a, b \in \mathcal{O}_F$ modulo the action of units $\mathcal{O}_F^\times$, and can be viewed equivalently as sums over principal ideals in $(a), (b) \subset \mathcal{O}_F$. The coefficients $\lambda = \lambda_\pi$ are the L-functions coefficients of π, so that the finite part of the L-function of π has the expansion $L(s, \pi) = \sum_{\mathfrak{n} \subset \mathcal{O}_F} \lambda(\mathfrak{n}) \mathbf{N}\mathfrak{n}^{-s}$ for $\Re(s) > 1$, writing $\mathbf{N}$ as usual to denote the absolute norm. The function $f_A(x, y) = a_A x^2 + b_A xy + c_A y^2$ denotes a fixed F-rational positive definite binary quadratic form ($a_A, b_A, c_A \in \mathcal{O}_F$) representing the class A in the class group $C(\alpha)$ of the order $\mathcal{O}_{\mathfrak{p}^\alpha} = \mathcal{O}_F + \mathfrak{p}^\alpha \mathcal{O}_K$, and w_K denotes the number of automorphs of f_A. We shall often assume that $f_A(x, y)$ is the reduced class representative, and hence that $\mathbf{N}b_A \leq \mathbf{N}a_A \leq \mathbf{N}c_A$. Let us remark as well that only the principal class contributes to the primitive average $P_\alpha(\pi, \chi)$, and only classes factoring through the image $C_0(m)$ of the torsion[5] subgroup C_0 in $C(m)$ to the corresponding subaverage $P_{\alpha, \rho_0}(\pi, \chi)$. As we explain later, there are constraints on the possible coefficients a_A we can consider with this method, although a variation which we develop via decompositions into Poincaré series in the style of [4] (see Theorem 2.5) allows us to proceed in a conceptually similar way irrespective of the relative sizes of these coefficients. The $\mathfrak{m}$-sums run over nonzero integral ideals in $\mathcal{O}_F$, and the a, b-sums over nonzero F-integers. The functions V_j are smooth and rapidly decaying cutoff functions coming from our choice of approximate functional equation (Lemma 2.2). Finally, writing $c(\pi_K) \subset \mathcal{O}_K$ to denote the conductor of the quadratic basechange representation $\pi_K = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ associated to π,

$$C = \mathbf{N}(\mathfrak{D}^2 c(\pi_K) \cdot c(\rho \cdot \chi \circ \mathbf{N})^2) = \mathbf{N}\left(\mathfrak{D}^2 c(\pi_K) \cdot \left(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F) \mathcal{O}_K \right)^2 \right) = \mathbf{N}(\mathfrak{D}^2 c(\pi_K)) \cdot p^{4d \max(\alpha, \beta)}$$

denotes the conductor of each L-function appearing in each of the averages $P_\alpha(\pi, \chi)$ and $G_\alpha(\pi, \chi; x)$.

We present two methods of estimating the sums $D_{A,j}(\pi, \chi; Z)$, both using Kirillov models to derive novel integral presentations for the sums we consider in terms of Fourier-Whittaker coefficients of some distinct (non-$\mathcal{K}$-finite) automorphic forms which can then be decomposed spectrally to derive estimates. On the one hand, we can approximate off-diagonal sums in terms of metaplectic Fourier-Whittaker coefficients after taking the unbalancing parameter Z of size approximately $\mathbf{N}(\mathfrak{D}\mathfrak{p}^{2\alpha})^{-1}$ to be the inverse of the discriminant of the order $\mathcal{O}_{\mathfrak{p}^\alpha} = \mathcal{O}_F + \mathfrak{p}^\alpha \mathcal{O}_K$, i.e. essentially the inverse of the square root of the conductor C. Here, the contributions from $b = 0$ terms are estimated in terms of a residual Dirichlet series related to the symmetric square L-function $L(s, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})$ at $s = 1$ (Proposition 2.8). These values in particular do not vanish, and moreover can be bounded from below in terms of the conductor (see [24], [12, (1.5)], and also (9) below). As we explain for Theorem 2.4 and Theorem 2.9 below (see also Theorem 2.11), the remaining coefficients in the expression for $D_{A,1}(\pi, \chi; Z)$ can be described equivalently in terms of Fourier coefficients of certain automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ and its two fold metaplectic cover $\overline{G}(\mathbf{A}_F)$. This enables us to use spectral decompositions of such forms to estimate these sums. For instance, taking the $Z = Y^{-1} = C^{-\frac{1}{2}}$ of size approximately $\mathbf{N}(\mathfrak{D}\mathfrak{p}^{2\alpha})^{-1}$ so that the length of the sums is equal to the square root of the conductor (corresponding to a balanced approximate functional equation formula), and assuming $D_K \equiv 0 \bmod 4$, we use spectral decompositions of shifted convolution sums to derive the following estimate for the average over

[5]Note that $C_0(m) \cong C_0$ for each sufficiently large integer $m \geq 1$.

primitive ring class characters (see Proposition 2.8 and Theorem 2.9 (i)): For $\alpha \gg 1$ sufficiently large,

$$\begin{aligned} &P_{\alpha}(\pi, \chi) \\ &= \left(1 - \frac{1}{\mathbf{N}\mathfrak{p}}\right) \frac{1}{w_K} \left(L(1, \omega\eta\chi^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \omega\chi^2 \circ \mathbf{N})} + \epsilon(\chi) \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot L(1, \overline{\omega}\eta\overline{\chi}^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \widetilde{\pi} \otimes \overline{\chi} \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \overline{\omega\chi}^2 \circ \mathbf{N})} \right) \\ &+ O_{\pi, \varepsilon}\left(\left(D_K p^{2d\beta}\right)^{\frac{1}{4} - \frac{(1-2\theta_0)}{16} + \varepsilon} Y^{-\frac{1}{4} - \varepsilon} \right) + O_{\pi, \chi, \varepsilon}\left(Y^{\frac{1}{4} + \delta_0 + \varepsilon} \mathbf{N}(\mathfrak{D}\mathfrak{p}^{2\alpha})^{-\frac{1}{2}} \right), \end{aligned}$$

Here, $L_{\mathbf{1}}^{\star}(s, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})$ is essentially the partial Dirichlet series expansion over principal ideals of the symmetric square L-function $L(s, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})$ (up to a convergent finite product of Euler factors), and $L_{\mathbf{1}}^{\star}(s, \omega\chi^2 \circ \mathbf{N})$ is defined similarly with respect to the Hecke L-function $L(s, \omega\chi^2 \circ \mathbf{N})$. The

$$\epsilon(\chi) = \epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N}) = \omega(\mathrm{lcm}(\mathfrak{p}^{\alpha}, p^{\beta}\mathcal{O}_F)) \cdot \eta(p^{4\beta}\mathfrak{d}c(\pi)) \cdot \epsilon(1/2, \pi) \cdot \chi(\mathbf{N}(\mathfrak{d}^2 c(\pi)^2 \mathfrak{D}^8)) \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}} \right)^{4d}$$

denotes the (stable) root number for each primitive ring class character ρ of conductor $\mathfrak{p}^{\alpha}$ in the average (see Proposition 2.1 and Lemma 2.7), $L_{\infty}(s)$ the archimedean local factor of each completed L-function $\Lambda(s, \pi \times \rho \times \chi \circ \mathbf{N})$, and $\widetilde{L}_{\infty}(s)$ that of each $\Lambda(s, \widetilde{\pi} \times \rho\overline{\chi} \circ \mathbf{N})$. As well, we write $0 \leq \theta_0 \leq 1/2$ to denote the best approximation to the generalized Ramanujan conjecture for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms, with $\theta_0 = 7/64$ an admissible choice by theorem of Blomer-Brumley [5]. The Burgess-like exponent $1/4 - (1 - 2\theta_0)/16$ of Wu [64] in the first error term can then be taken to be 206/1024. We refer to Theorems 2.4 and 2.9 for how the off-diagonal bounds are proved. The exponent in the error term here has the more organic form $-1/4 + \delta_0 + \varepsilon$, where $0 \leq \delta_0 \leq 1/4$ denotes the best approximation to the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms in the level aspect, or what is the same – by the theorem of Kohnen-Zagier [33] and more generally Baruch-Mao [2] – the best exponent approximation for the Fourier coefficients of half-integral weight forms. That is, this exponent reflects our approximation of the off-diagonal sum over $b \neq 0$ contributions (in the region of moderate decay for V_1) by Fourier coefficients of genuine metaplectic forms, which we decompose spectrally to derive such a bound. We then use the admissible approximation of $\delta_0 = 103/512$ of Blomer-Harcos [6, Corollary]. We also develop a distinct and flexible variation of this idea via Theorem 2.5 using decompositions into Poincaré series in the style of the argument of Blomer [4], which allows us to deal with the coefficients appearing in the reduced form representative $f_A(x, y)$ corresponding to $A \in C(\alpha)$. This allows us to deal with an inherent limitation in the standard setup described above which requires $a_A = 1$ or small relative to c_A and $b_A = 0$, and in particular to derive estimates for the sums corresponding to any class $A \in C(\alpha)$. This latter feature in turn allows us to estimate the tame and Galois sub-averages directly, via purely analytic methods.[6]

In all our main estimates for the sums $D_{A,j}(\pi, \chi; Z)$, the surjectivity of the archimedean local Kirillov map plays a starring role. As we explain in Proposition A.1 (cf. also (47)), this property allows us to find automorphic forms whose Fourier-Whittaker coefficients describe the sums $D_{A,1}(\pi, \chi; Z)$ and $D_{A,2}(\pi, \chi; Z)$ we consider exactly. Once such presentations are known, the door is open to using spectral decompositions of the corresponding forms, and in particular to deriving estimates for both of the sums $D_{A,1}(\pi, \chi; Z)$ and $D_{A,2}(\pi, \chi; Z)$ with a flexible choice of unbalancing parameter $Z > 0$.

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2. Mean values

2.1. **Rankin-Selberg L-functions.**

We first review some relevant background from the theory of Jacquet [30] and Jacquet-Langlands [31] for the Rankin-Selberg L-functions we consider.

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[6] When the cyclotomic character χ is trivial, then the theorems of Cornut and Vatsal prove the nonvanishing of the tame sub-averages using ergodic theory, i.e. using the theorems of Ratner and Margulis-Tomanov on p-adic unipotent flows.

2.1.1. *Setup and definitions.* Let us fix a totally imaginary quadratic extension K of F of relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F}$ and associated idele class character $\eta = \eta_{K/F}$ of F. Let us also write $D_K = D_F^2 \mathbf{N}\mathfrak{D}$ to denote the absolute discriminant of K. Let $\mathcal{W}$ denote a Hecke character of K, with $\pi(\mathcal{W}) = \otimes_v \pi(\mathcal{W})_v$ the associated induced $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation. Equivalently, $\pi(\mathcal{W})$ denotes the automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ generated by the Hilbert modular theta series $\theta(\mathcal{W})$ associated to $\mathcal{W}$. We shall consider the Rankin-Selberg L-function of π times the induced representation $\pi(\mathcal{W})$, whose Euler product over finite places as a function of $s \in \mathbf{C}$ with $\Re(s) > 1$ we express as

$$L(s, \pi \times \mathcal{W}) = L(s, \pi \times \pi(\mathcal{W})) = \prod_{v < \infty} L(s, \pi_v \times \pi(\mathcal{W})_v).$$

Here, for primes v not dividing the conductor $c(\pi \times \mathcal{W})$ of $L(s, \pi \times \mathcal{W})$, the corresponding local factor $L(s, \pi_v \times \mathcal{W}_v)$ takes the form

$$L(s, \pi_v \times \mathcal{W}_v) = \det(I - A_v \otimes B_v \mathbf{N}v^{-s})^{-1},$$

where A_v denotes the Satake parameter of π_v and B_v that of $\pi(\mathcal{W})_v$. For primes v where one of π or $\pi(\mathcal{W})$ is ramified, the corresponding local factor $L(s, \pi_v \times \pi(\mathcal{W})_v)$ takes the form of $P_v(\mathbf{N}v^{-s})^{-1}$ for $P_v(x)$ a polynomial of degree at most four such that $P_v(0) = 1$. Let us also write $L(s, \pi_\infty \times \pi(\mathcal{W})_\infty)$ to denote the archimedean component of this L-function. If $\mathcal{W} = \rho$ is a ring class character, or more generally if $\mathcal{W} = \rho\chi \circ \mathbf{N}$ is the product of a ring class character ρ with the composition $\chi \circ \mathbf{N}$ of a primitive even Dirichlet character χ with the norm homomorphism $\mathbf{N}$, then $\mathcal{W}$ determines a wide ray class character with "trivial archimedean component" $\mathcal{W}_\infty \equiv \mathbf{1}$. Consequently, the archimedean local factor of the L-function does not depend on the choice of $\mathcal{W}$, i.e. as $\pi(\mathcal{W})_\infty \equiv \mathbf{1}$ for any such character, and we are justified in dropping the $\mathcal{W}$ from the notation. Hence, we write $L_\infty(s) = L(s, \pi_\infty \times \pi(\mathcal{W})_\infty)$. The completed L-function

$$\Lambda(s, \pi \times \mathcal{W}) = L(s, \pi \times \mathcal{W}) L(s, \pi_\infty \times \pi(\mathcal{W})_\infty)$$

is entire unless $\pi(\mathcal{W}) \approx \widetilde{\pi} \otimes |\cdot|^t$ for some $t \in \mathbf{R}$, and in any case holomorphic except for simple poles at $s = 0$ and 1. It satisfies the functional equation

$$\Lambda(s, \pi \times \mathcal{W}) = \epsilon(s, \pi \times \mathcal{W}) \Lambda(1 - s, \widetilde{\pi} \times \overline{\mathcal{W}}),$$

where

$$\epsilon(s, \pi \times \mathcal{W}) = \epsilon(1/2, \pi \times \mathcal{W}) c(\pi \times \mathcal{W})^{\frac{1}{2} - s}$$

denotes the ϵ-factor of $\Lambda(s, \pi \times \mathcal{W})$. Here, $\epsilon(1/2, \pi \times \mathcal{W}) \in \mathbf{S}^1$ is the root number. Note that this root number also admits an Euler product decomposition

$$\epsilon(1/2, \pi \times \mathcal{W}) = \prod_v \epsilon(1/2, \pi_v \times \mathcal{W}_v, \psi_v) = \prod_v \epsilon(1/2, \pi_v \times \pi(\mathcal{W})_v, \psi_v) \tag{3}$$

(where the local Euler factors are defined with respect to any fixed choice of additive character $\psi = \otimes_v \psi_v$), and can be given by a more explicit formula when we assume that the conductor $c(\pi)$ is prime to that of $\pi(\mathcal{W})$ (see below). As well, we shall write

$$c(\pi \times \mathcal{W}) = \mathbf{N}(\mathfrak{D}_K^2 c(\pi_K) c(\mathcal{W})^2) \tag{4}$$

to denote the conductor of $\Lambda(s, \pi \times \mathcal{W})$, where $c(\pi_K)$ denotes that of the basechange π_K of π to $\mathrm{GL}_2(\mathbf{A}_K)$, and $c(\mathcal{W})$ that of the Hecke character $\mathcal{W}$, viewed as an ideal of $\mathcal{O}_K$ (cf. [1, (16)]). We shall sometimes work with the square root of this quantity $Y := c(\pi \times \mathcal{W})^{\frac{1}{2}}$ for our arguments below, and remind the reader that taking (relative) norms of the conductor $c(\mathcal{W}) \subset \mathcal{O}_K$ to $\mathcal{O}_F$ or $\mathbf{Z}$ leads to fourth powers of the moduli.

Relations to basechange L-functions. Note that the $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L-function $\Lambda(s, \pi \times \mathcal{W})$ is equivalent to the $\mathrm{GL}_2(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$ L-function given by $\Lambda(s, \pi_K \otimes \mathcal{W})$, where π_K denotes the basechange of π to $\mathrm{GL}_1(\mathbf{A}_K)$, and $L(s, \pi_K \otimes \mathcal{W})$ the L-function of π_K twisted by the Hecke character $\mathcal{W}$ of K. We also have the Artin decomposition $\Lambda(s, \pi_K) = \Lambda(s, \pi)\Lambda(s, \pi \otimes \eta)$. In any case, the formula (4) for the conductor $c(\pi \times \mathcal{W})$ is equivalent to that of the conductor of the basechange L-function $c(\pi_K \otimes \mathcal{W})$, and we have taken the relevant formula for the latter basechange conductor as described in [1, (16)] (cf. [51]).

2.1.2. *Explicit description of the root number.* Let us now give a more explicit description of the root number (3) defined above (cf. [37, 2.1], [1, Proposition 4.1]). We assume from now on that $\mathcal{W}$ is a Hecke character of K of the form described above, hence $\mathcal{W} = \rho\chi \circ \mathbf{N}$ with ρ a primitive ring class character of conductor $\mathfrak{p}^\alpha$ for some integer $\alpha \geq 0$, and χ some primitive even Dirichlet character of conductor p^β for some integer $\beta \geq 0$. Note that $\mathcal{W}$ is then always a wide ray class character in our setup, and the archimedean local component $L_\infty(s) = L(s, \pi_\infty \times \pi(\mathcal{W})_\infty)$ does not depend on the choice of such a character $\mathcal{W}$. That is, the archimedean local component $L(s, \pi_\infty \times \pi(\mathcal{W}))$ of the completed Rankin-Selberg L-function

$$\Lambda(s, \pi \times \mathcal{W}) = \Lambda(s, \pi \times \pi(\mathcal{W})) = \Lambda(s, \pi_\infty \times \pi(\mathcal{W})_\infty) L(s, \pi \times \pi(\mathcal{W})) =: L_\infty(s) L(s, \pi \times \mathcal{W})$$

does not change as we vary over all Hecke characters $\mathcal{W} = \rho\chi \circ \mathbf{N}$ of K with ρ a primitive ring class character and χ a primitive even Dirichlet character. Note that we shall always identify such a wide ray class character $\mathcal{W}$ with its corresponding idele class character of K.

Let us now assume that the conductor $\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F) \subset \mathcal{O}_F$ of $\mathcal{W} = \rho\chi \circ \mathbf{N}$ (viewed as an ideal of $\mathcal{O}_F$) is coprime to the conductor $c(\pi) \subset \mathcal{O}_F$ and the relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F} \subset \mathcal{O}_F$, and that $c(\pi)$ and $\mathfrak{D}$ are coprime. We have by the Rankin-Selberg theory (cf. [37, §2]) the generic root number formula

$$\epsilon(1/2, \pi \times \mathcal{W}) = \omega(c(\mathcal{W})) \cdot \mathcal{W}|_{\mathbf{A}_F^\times}(c(\pi)) \cdot \epsilon(1/2, \pi) \cdot \epsilon(1/2, \mathcal{W})^4 \in \mathbf{S}^1,$$

which in our setting is given more explicitly by

$$\epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N}) = \omega(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F)) \cdot \eta\chi^2 \circ \mathbf{N}(c(\pi)) \cdot \epsilon(1/2, \pi) \cdot \epsilon(1/2, \rho\chi \circ \mathbf{N})^4. \tag{5}$$

Here, $\epsilon(1/2, \pi)$ denotes the root number of the L-function $L(s, \pi)$ of π, which appears in the functional equation $\Lambda(s, \pi) = \epsilon(1/2, \pi)\Lambda(1-s, \widetilde{\pi})$ of the corresponding completed L-function $\Lambda(s, \pi)$ of $L(s, \pi)$. As well, $\epsilon(1/2, \mathcal{W}) = \epsilon(1/2, \rho\chi \circ \mathbf{N})$ denotes the root number of the Hecke L-function $L(s, \mathcal{W}) = L(s, \rho\chi \circ \mathbf{N})$. Now, it can be deduced from the classically-known properties of the corresponding theta series $\theta(\mathcal{W}) = \theta(\rho\chi \circ \mathbf{N})$ (a Hilbert modular form of parallel weight one, level $\mathfrak{D} \cdot \mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F) \subset \mathcal{O}_F$, and character $\eta\chi^2 \circ \mathbf{N}$) that this latter root number is given by

$$\epsilon(1/2, \rho\chi \circ \mathbf{N}) = \eta\chi^2 \circ \mathbf{N}(\mathfrak{d}) \cdot \frac{\tau(\eta\chi^2 \circ \mathbf{N})}{\mathbf{N}(\mathfrak{D} p^\beta \mathcal{O}_F)^{\frac{1}{2}}} = \eta(\mathfrak{d}) \cdot \chi^2(\mathbf{N}\mathfrak{d}) \cdot \frac{\tau(\eta\chi^2 \circ \mathbf{N})}{\mathbf{N}(\mathfrak{D} p^\beta \mathcal{O}_F)^{\frac{1}{2}}}, \tag{6}$$

where $\mathfrak{d} = \mathfrak{d}_F \subset \mathcal{O}_F$ denotes the different of F. Here, the Gauss sum $\tau(\eta\chi^2 \circ \mathbf{N})$ is defined in this generality as follows (see e.g. [51, (65)]). Let e denote the function $e(x) = \exp(2\pi i x)$. Given a primitive Hecke character ξ of F of some conductor $c(\xi) = \mathfrak{q} \subset \mathcal{O}_F$, the Gauss sum $\tau(\xi)$ of ξ is defined by

$$\tau(\xi) = \sum_{x \bmod \mathbf{N}\mathfrak{q}} \xi(x\mathcal{O}_F) e\left(\frac{x^*}{\mathbf{N}\mathfrak{q}}\right),$$

where x runs over invertible classes modulo $\mathbf{N}\mathfrak{q}$, and x^* is determined uniquely as follows: If

$$x \equiv \prod_{v | \mathbf{N}\mathfrak{q}} x_v \bmod \mathbf{N}\mathfrak{q}$$

with $x_v \equiv 1 \bmod \mathbf{N}\mathfrak{q}/v$ for each prime divisor v of $\mathbf{N}\mathfrak{q}$, then $x^* \equiv \sum_{v|\mathbf{N}\mathfrak{q}} x_v \mathbf{N}\mathfrak{q}/v \bmod \mathbf{N}\mathfrak{q}$. More generally, the root number $\epsilon(1/2, \xi)$ of the corresponding Hecke L-function $L(s, \xi)$ is then given by the formula

$$\epsilon(1/2, \xi) = \mathbf{N}\mathfrak{q}^{-\frac{1}{2}} \xi(\mathfrak{d}) \tau(\xi) = \mathbf{N}\mathfrak{q}^{-\frac{1}{2}} \xi(\mathfrak{d}) \sum_{x \bmod \mathbf{N}\mathfrak{q}} \xi(x\mathcal{O}_F) e\left(\frac{x^*}{\mathbf{N}\mathfrak{q}}\right).$$

Now, we can give the following more explicit description of the root numbers (5) we consider here.

Proposition 2.1. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of level $c(\pi) \subset \mathcal{O}_F$, central character $\omega = \omega_\pi$, and root number $\epsilon(1/2, \pi)$. Let K be a totally imaginary quadratic extension of F of relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F} \subset \mathcal{O}_F$ and associated idele class character $\eta = \eta_{K/F}$ of F. Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p. Assume that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ be a wide ray class Hecke character of K as described above, with ρ a primitive ring class character of conductor $\mathfrak{p}^\alpha$ for some integer $\alpha \geq 0$, and χ a primitive even Dirichlet character of conductor p^β for some* *integer $\beta \geq 0$. Then, the root number $\epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N})$ of the corresponding Rankin-Selberg L-function $L(s, \pi \times \rho \times \chi \circ \mathbf{N}) = L(s, \pi \times \pi(\rho\chi \circ \mathbf{N}))$ is given by*

$$\epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N}) = \omega(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F)) \cdot \eta(p^{4\beta}\mathfrak{d}c(\pi)) \cdot \epsilon(1/2, \pi) \cdot \chi(\mathbf{N}(\mathfrak{d}^2 c(\pi)^2 \mathfrak{D}^8)) \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}} \right)^{4d},$$

where

$$\tau(\chi^2) = \sum_{x \bmod p^\beta} \chi^2(x) e\left(\frac{x}{p^\beta}\right)$$

denotes the Gauss sum defined over coprime residue classes $x \bmod p^\beta$. Observe in particular that this formula does not depend on the particular choice of ring character ρ, but rather the exponent α of its conductor $\mathfrak{p}^\alpha$. Moreover, for each sufficiently large exponent $\alpha \gg 1$, we see by inspection of this formula that the root number $\epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N})$ is in fact completely independent of the choice of the ring class character ρ, as $\omega(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta\mathcal{O}_F)) = 1$ for $\alpha \gg 1$.

Proof. Let us first consider the Gauss sum in the formula (6). Using the twisted multiplicativity relation (see e.g. [29, (3.16)]), we can decompose this as

$$\tau(\eta\chi^2 \circ \mathbf{N}) = \eta(p^\beta \mathcal{O}_F) \cdot \chi^2(\mathbf{N}\mathfrak{D}) \cdot \tau(\eta) \cdot \tau(\chi^2 \circ \mathbf{N}). \tag{7}$$

On the other hand, we can unravel definitions to find the simplification of the Gauss sum

$$\begin{aligned}\tau(\chi^2 \circ \mathbf{N}) &= \sum_{x \bmod \mathbf{N}(p^\beta \mathcal{O}_F)} \chi^2(\mathbf{N}(x\mathcal{O}_F)) e\left(\frac{x^*}{\mathbf{N}(p^\beta\mathcal{O}_F)}\right) = \sum_{x \bmod p^\beta} \chi^2(\mathbf{N}(x\mathcal{O}_F)) \prod_{v \mid p^{d\beta}} e\left(\frac{x_v p^{d\beta}/v}{p^{d\beta}}\right) \\ &= \sum_{x \bmod p^\beta} \prod_{\iota: F \hookrightarrow \mathbf{C}} \chi^2(x) e\left(\frac{x}{p^\beta}\right) = \tau(\chi^2)^d.\end{aligned}$$

Note that this simplification can also be deduced more directly via the classical definition of the Gauss sum (see e.g. [43, (6.3)]), which recall for any choice of representative $y \in p^\beta \mathfrak{d}^{-1} \subset \mathcal{O}_F$ takes the form

$$\sum_{x \bmod p^\beta} \chi^2(\mathbf{N}(x\mathcal{O}_F)) e(\mathrm{Tr}(xy)) = \sum_{x \bmod p^\beta} \prod_{\iota: F \hookrightarrow \mathbf{C}} \chi^2(\iota(x\mathcal{O}_F)) e(\iota(xy)) = \chi^2(\mathbf{N}\mathfrak{d}) \prod_{\iota: F \hookrightarrow \mathbf{C}} \sum_{x \bmod p^\beta} \chi^2(x) e\left(\frac{x}{p^\beta}\right).$$

Using this simplification $\tau(\chi^2 \circ \mathbf{N}) = \tau(\chi^2)^d$ in (7) then gives us the relation

$$\tau(\eta\chi^2 \circ \mathbf{N}) = \eta(p^\beta\mathcal{O}_F) \cdot \chi^2(\mathbf{N}\mathfrak{D}) \cdot \tau(\eta) \cdot \tau(\chi^2)^d,$$

from which it follows that

$$\left(\frac{\tau(\eta\chi^2 \circ \mathbf{N})}{\mathbf{N}(\mathfrak{D}p^\beta\mathcal{O}_F)^{\frac{1}{2}}}\right)^4 = \eta(p^{4\beta}\mathcal{O}_F) \cdot \chi(\mathbf{N}\mathfrak{D}^8) \cdot \left(\frac{\tau(\eta)}{\mathbf{N}\mathfrak{D}^{\frac{1}{2}}}\right)^4 \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}}\right)^{4d}.$$

Hence, we see that (6) takes the more explicit form

$$\epsilon(1/2, \rho\chi \circ \mathbf{N}) = \eta(p^{4\beta}\mathfrak{d}) \cdot \chi(\mathbf{N}(\mathfrak{d}^2\mathfrak{D}^8)) \cdot \left(\frac{\tau(\eta)}{\mathbf{N}\mathfrak{D}^{\frac{1}{2}}}\right)^4 \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}}\right)^{4d}.$$

Since the quadratic root number $\epsilon(1/2, \eta)$ satisfies the relation $\epsilon(1/2, \eta)^4 = 1$, we deduce that

$$\epsilon(1/2, \eta)^4 = \eta(\mathfrak{d}^4) \left(\frac{\tau(\omega)}{\mathbf{N}\mathfrak{D}^{\frac{1}{2}}}\right)^4 = \left(\frac{\tau(\omega)}{\mathbf{N}\mathfrak{D}^{\frac{1}{2}}}\right)^4 = 1,$$

and so we obtain the even simpler explicit formula

$$\epsilon(1/2, \rho\chi \circ \mathbf{N}) = \eta(p^{4\beta}\mathfrak{d}) \cdot \chi(\mathbf{N}(\mathfrak{d}^2\mathfrak{D}^8)) \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}}\right)^{4d}.$$

We then substitute this expression into (5) to derive the stated formula for the root number. □

2.1.3. *Dirichlet series expansions.* Let us retain the setup described above, fixing a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p, and taking $\mathcal{W} = \rho\chi \circ \mathbf{N}$ to be a wide ray class character of K given by the product of a ring class character ρ of K of conductor $\mathfrak{p}^\alpha$ for some integer $\alpha \geq 0$ times a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$ composed with the norm homomorphism $\mathbf{N}$ on ideals of K. Let us also write the Dirichlet series expansion of the finite part $L(s,\pi)$ of the L-function $\Lambda(s,\pi) = L(s,\pi_\infty)L(s,\pi)$ of π (first for $\Re(s) > 1$) as

$$L(s,\pi) = \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^s} = \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda_\pi(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^s}.$$

We can then write the Dirichlet series expansion of $L(s, \pi \times \mathcal{W})$ (first for $\Re(s) > 1$) as a $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L-function over ideals of $\mathcal{O}_F$ by

$$L(s, \pi \times \rho\chi \circ \mathbf{N}) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}^{2s}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda(\mathfrak{n})\chi(\mathbf{N}\mathfrak{n})}{\mathbf{N}\mathfrak{n}^s} \left(\sum_{A \in C(\alpha)} r_A(\mathfrak{n})\rho(A) \right). \tag{8}$$

Here, we write $C(\alpha)$ to denote the class group of the order $\mathcal{O}_{\mathfrak{p}^\alpha} := \mathcal{O}_F + \mathfrak{p}^\alpha \mathcal{O}_K$, and $r_A(\mathfrak{n})$ the number of ideals in the class A whose image under the relative norm homomorphism $\mathbf{N}_{K/F}$ equals a given $\mathfrak{n} \subset \mathcal{O}_F$. As well, each of the sums runs over ideals of $\mathcal{O}_F$ which are coprime to the conductor of the characters which appear, although we omit this natural condition from the notations for simplicity.

2.1.4. *Partial symmetric square L-values and related Dirichlet series.* Let us also introduce the following Dirichlet series which we encounter in our later calculations, and which is related up to a convergent product of Euler factors to the following partial symmetric square L-function of π at $s = 1$. We refer to [12, §1.1] and more generally [48] for some background discussion of the analytic properties of such L-functions. Fix $\pi = \otimes_v \pi_v$ a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ as above, writing $c(\pi) \subset \mathcal{O}_F$ again to denote its conductor. Let $\xi = \otimes_v \xi_v = \chi \circ \mathbf{N}$ be a wide ray class Hecke character of F induced via composition with the norm homomorphism from a primitive even Dirichlet character χ of conductor $c(\chi)$. Let S denote the set of places of F where both π and ξ are ramified. We consider the incomplete L-function of the symmetric square of π times ξ, defined for $s \in \mathbf{C}$ (first with $\Re(s) > 1$) by the Euler product

$$L^S(s, \mathrm{Sym}^2 \pi \otimes \xi) = \prod_{v \notin S} L(s, \mathrm{Sym}^2 \pi_v \otimes \xi_v),$$

where the local Euler factors $L(s, \mathrm{Sym}^2 \pi_v \otimes \xi_v)$ are defined as follows. If v is a place where π_v is unramified, then there exist unramified quasicharacters $\mu_{1,v}$ and $\mu_{2,v}$ of $\mathrm{GL}_2(F_v)$ such that π_v arises from the induced representation of the character $\mu_v = \mu_{1,v} \otimes \mu_{2,v}$ of the torus $T_v \subset \mathrm{GL}_2(F_v)$ of diagonal matrices. Fixing a uniformizer ϖ_v of $\mathcal{O}_{F_v}$, and writing A_v to denote the diagonal matrix $\mathrm{diag}(\mu_{1,v}, \mu_{2,v})$, we then have

$$L(s, \mathrm{Sym}^2 \pi_v \otimes \xi) = \det \left(I - \mathrm{Sym}^2(A_v)\xi_v(\varpi_v)\mathbf{N}v^{-s} \right)^{-1} = \prod_{1 \leq i \leq j \leq 2} \left(1 - \mu_{i,v}\mu_{j,v}\xi_v(\varpi_v)\mathbf{N}v^{-s} \right)^{-1}.$$

It is well-known that $L^S(s, \mathrm{Sym}^2 \pi \otimes \xi)$ does not vanish at $\Re(s) = 1$ (see [48, Theorem 1.1]). The work of Goldfeld-Hoffstein-Lieman [24] on exceptional zeros in fact gives a lower bound for such L-values in the classical setting. In general, as explained in [12, §1.1], one can derive individual upper and lower bounds for $L(s, \mathrm{Sym}^2 \pi \otimes \xi)$ via the automorphy of $\mathrm{Sym}^2 \pi$ (known thanks to Gelbart-Jacquet [22]). In particular, the automorphy can be used to deduce individual upper bounds $L(1, \mathrm{Sym}^2 \pi \otimes \xi) \ll_\varepsilon \mathbf{N}(c(\pi)c(\chi)^2)^\varepsilon$, as well as individual lower bounds

$$L^S(1, \mathrm{Sym}^2 \pi \otimes \xi) \gg \left(\log(\mathbf{N}(c(\pi)c(\xi)^2)) \right)^{-C} \tag{9}$$

for some constant $C > 0$. Note that the local Euler factors $L(s, \mathrm{Sym}^2 \pi_v \otimes \xi_v)$ at primes $v \mid c(\pi)$ can be defined in a more complicated way. We shall omit the superscript S in the discussion below.

Finally, we note that $L^S(s, \mathrm{Sym}^2 \pi \otimes \xi)$ has Dirichlet series expansion (first for $\Re(s) > 1$) given up to some convergent finite product of Euler factors by the simplified Dirichlet series defined by

$$L^{S,\star}(s, \mathrm{Sym}^2 \pi \otimes \xi) := L^S(2s, \omega\xi^2) \sum_{\substack{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F \\ (\mathfrak{n}, S)=1}} \frac{\lambda_{\pi\otimes\xi}(\mathfrak{n}^2)}{\mathbf{N}\mathfrak{n}^s} = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ (\mathfrak{m}, S)=1}} \frac{\omega\xi^2(\mathfrak{m})}{\mathbf{N}\mathfrak{m}^{2s}} \sum_{\substack{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F \\ (\mathfrak{n}, S)=1}} \frac{\lambda(\mathfrak{n}^2)\xi(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^s}.$$

We shall often encounter the following corresponding partial Dirichlet series expansion over principal ideals $\mathfrak{n} = (a) \subset \mathcal{O}_F$, which we denote throughout (dropping the superscript S for simplicity) by

$$L_{\mathbf{1}}^{S,\star}(s, \mathrm{Sym}^2 \pi \otimes \xi) := L_{\mathbf{1}}^S(2s, \omega\xi^2) \sum_{\substack{a \neq 0 \in \mathcal{O}_F/\mathcal{O}_F^\times \\ (a, S)=1}} \frac{\lambda_{\pi\otimes\xi}(a^2)}{\mathbf{N}a^s} = \sum_{\substack{b \neq 0 \in \mathcal{O}_F/\mathcal{O}_F^\times \\ (b, S)=1}} \frac{\omega\xi^2(b)}{\mathbf{N}b^{2s}} \sum_{\substack{a \neq 0 \in \mathcal{O}_F/\mathcal{O}_F^\times \\ (a, S)=1}} \frac{\lambda(a^2)\xi(a)}{\mathbf{N}a^s}.$$

Here, we write $b = (b)$ and $a = (a)$ to lighten notations, and the sums are really taken over nonzero principal ideals $(a), (b) \neq 0 \subset \mathcal{O}_F$. That is, we write the sums here as the corresponding sums over F-integers a, b modulo the action of units as $a, b \in \mathcal{O}_F/\mathcal{O}_F^\times$. We shall use this shorthand notation throughout, as it simplifies notations greatly for our later calculations. Also, although it does not generally admit an Euler product (unless F has class number one), this partial Dirichlet series has the same basic analytic properties as described in [12, § 1, Lemma 1.1, Remark 2] as the sum over classes $L^S(s, \mathrm{Sym}^2 \pi \otimes \xi)$, from which we deduce in particualr that it does not vanish at $s = 1$ by the argument of Goldfeld-Hoffstein-Lieman [24].

2.2. **Approximate functional equations.** Let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ be a wide ray class character of K, as above. Recall that for $\Re(s) > 1$, we write the Dirichlet series expansion (8) of the finite part $L(s, \pi \times \mathcal{W})$ of the Rankin-Selberg L-function $\Lambda(s, \pi \times \mathcal{W}) = L_\infty(s) L(s, \pi \times \mathcal{W})$ as

$$L(s, \pi \times \mathcal{W}) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}^{2s}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda(\mathfrak{n})\chi(\mathbf{N}\mathfrak{n})}{\mathbf{N}\mathfrak{n}^s} \left(\sum_{A \in C(\alpha)} r_A(\mathfrak{n}) \rho(A) \right).$$

Recall as well that the completed L-function $\Lambda(s, \pi \times \mathcal{W})$ satisfies the functional equation

$$\Lambda(s, \pi \times \mathcal{W}) = \epsilon(1/2, \pi \times \mathcal{W}) \cdot c(\pi \times \mathcal{W})^{\frac{1}{2} - s} \cdot \Lambda(1 - s, \widetilde{\pi} \times \overline{\mathcal{W}}),$$

where $\epsilon(1/2, \pi \times \mathcal{W}) \in \mathbf{S}^1$ as described in (2.1) above denotes the root number, and $c(\pi \times \mathcal{W})$ as described in (4) above denotes the conductor of the L-function.

We now derive a suitable presentation for the finite part $L(s, \pi \times \mathcal{W})$ of this L-function at $s = 1/2$, outside of the range of absolute convergence $\Re(s) > 1$ via the following standard contour argument. We present the details for the convenience of the reader. Let us fix a holomorphic test function k on $\mathbf{C}$ such that $k(s)$ is even and bounded in vertical strips. To be more precise, let $g \in \mathcal{C}_c^\infty(\mathbf{R}_{>0})$ be any smooth and compactly supported test function, and let $k(s) = \int_0^\infty g(x) x^s \frac{dx}{x}$ denote its Mellin transform. We can and do assume that g is chosen so that $k(0) = 1$. Now, recall that the generalized Ramanujan conjecture for π at the real places of F predicts that $\max_j(\mu_\infty(j)) = 0$. If π arises from a holomorphic Hilbert modular form, then this conjecture is known by work of Blasius [3] (generalizing Deligne's theorem [18]). In general, the conjecture is not yet known, and in this level of generality we can and do assume that g is chosen so that $k(\mu_\infty(j)) = 0$ for each $1 \leq j \leq 2$. This allows us to avoid having to consider addition residual coming from poles inside the critical strip $0 < \Re(s) < 1$ in our subsequent arguments. Fixing such a test function $k(s)$ once and for all, we then define the following smooth cutoff functions $V_1(y)$ and $V_2(y)$ on $y \in \mathbf{R}_{>0}$ by

$$V_1(y) = \int_{\Re(s)=2} \frac{k(s)}{s} y^{-s} \frac{ds}{2\pi i}$$
$$V_2(y) = \int_{\Re(s)=2} \frac{k(-s)}{s} \frac{\widetilde{L}_\infty(s + 1/2)}{L_\infty(-s + 1/2)} y^{-s} \frac{ds}{2\pi i}.$$

Here, we write $\widetilde{L}_\infty(s)$ to denote the archimedean component of the contragredient L-function

$$\Lambda(s, \widetilde{\pi} \times \overline{\mathcal{W}}) = \widetilde{L}_\infty(s) L(s, \widetilde{\pi} \times \overline{\mathcal{W}}),$$

which again does not depend on the choice of wide ray class character $\overline{\mathcal{W}}$ of K.

Lemma 2.2. *Let $\mathcal{W} = \rho\chi \circ \mathbf{N}$ be any wide ray class Hecke character of K of the form described above, with ρ a ring class character of K conductor $\mathfrak{p}^\alpha \subset \mathcal{O}_F$, and $\chi \bmod p^\beta$ a primitive even Dirichlet character, and assume that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Let $c(\pi \times \rho\chi \circ \mathbf{N}) = \mathbf{N}(\mathfrak{D}^2 c(\pi_K) p^{4d\max(\alpha,\beta)})$ denote the conductor of the L-function $L(s, \pi \times \mathcal{W}) = L(s, \pi \times \rho\chi \circ \mathbf{N})$. We have for any choice of real parameter $Z > 0$ the formula*

$$L(1/2, \pi \times \mathcal{W}) = \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{\omega\eta(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ \mathfrak{n} \neq \{0\}}} \frac{\lambda(\mathfrak{n})\chi(\mathbf{N}\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \left(\sum_{A \in C(\alpha)} r_A(\mathfrak{n})\rho(A) \right) V_1\left(\mathbf{N}\mathfrak{m}^2 \mathbf{N}\mathfrak{n} Z\right)$$

$$+ \epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N}) \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{\overline{\omega}\eta(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ \mathfrak{n} \neq \{0\}}} \frac{\overline{\lambda(\mathfrak{n})\chi(\mathbf{N}\mathfrak{n})}}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} \left(\sum_{A \in C(\alpha)} r_A(\mathfrak{n})\rho(A) \right) V_2\left(\frac{\mathbf{N}\mathfrak{m}^2 \mathbf{N}\mathfrak{n}}{Z c(\pi \times \rho\chi \circ \mathbf{N})} \right).$$

Proof. The proof is standard (see e.g. [37, Lemma 3.2] or [29, §5.2]). Note that the ring class character ρ in his expression is not inverted as a consequence of the fact that such characters are equivariant with respect to complex conjugation (cf. [52, §1]), as mentioned already in our discussion of the functional equation (2). ☐

Lemma 2.3. *The smooth cutoff functions $V_j(y)$ defined above satisfy the following decay properties:*

$$V_1(y) = \begin{cases} 1 + O_A(y^A) & \text{for any choice of } A \geq 1 \text{ if } 0 < y \leq 1 \\ O_C(y^{-C}) & \text{for any constant } C > 0 \text{ if } y \geq 1. \end{cases}$$

and

$$V_2(y) = \begin{cases} \frac{\widetilde{L}_\infty(1/2)}{L_\infty(1/2)} + O_\varepsilon(y^{\frac{1}{2}-\varepsilon}) & \text{for any small } \varepsilon > 0 \text{ if } 0 < y \leq 1 \\ O_C(y^{-C}) & \text{for any constant } C > 0 \text{ if } y \geq 1. \end{cases}$$

Proof. See [37, Lemma 3.1] or [29, Proposition 5.4]). One shifts the contour defining $V_j(y)$ to the right to obtain the behaviour as $y \to \infty$, and to the left to obtain the behaviour as $y \to 0$. ☐

2.3. **Shifted convolution sum estimates.** We have the following estimates for the shifted convolution problem for the L-function coefficients of $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms. Although the proof is well-known to experts (even if the exact statement we give does not appear in the literature), we provide one for the convenience of the reader in Appendix A below, especially as we shall develop several of the ideas of this proof with spectral expansions for our subsequent estimates.

Theorem 2.4. *Let $\pi = \otimes\pi_v$ be any non-dihedral cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation. Let W be any smooth function of compact support on $\mathbf{R}_{>0}$ (or any smooth function which decays rapidly near infinity and moderately near zero), whose derivatives satisfy the decay condition $W^{(i)} \ll 1$ for all integers $i \geq 0$. Let $q \neq 0 \in \mathcal{O}_F$ be any totally positive F-integer. Assume that π is not dihedral, in other words that π does not arise via automorphic induction from a Hecke character of some quadratic extension of F. For any real number $Y > 0$ and any choice of $\varepsilon > 0$, we have the uniform upper bound*

$$\sum_{\gamma \in F^\times} \frac{\lambda(\gamma^2 + q)}{\mathbf{N}(\gamma^2 + q)^{\frac{1}{2}}} W\left(\frac{\mathbf{N}(\gamma^2 + q)}{Y} \right) \ll_{\pi, \varepsilon} Y^{\frac{1}{4}} \cdot \mathbf{N}q^{\delta_0 - \frac{1}{2}} \left(\frac{\mathbf{N}q}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

Here, the implied constant depends on the choice of weight function W. As well, $0 \leq \theta_0 \leq 1/2$ denotes the best known approximation towards the generalized Ramanujan conjecture for arbitrary $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms, so that $\theta_0 = 7/64$ is admissible by the theorem of Blomer-Brumley [5]. *On the other hand, $0 \leq \delta_0 \leq 1/4$ denotes the best exponent in the bound for Fourier coefficients of automorphic forms on the metaplectic cover of $\mathrm{GL}_2(\mathbf{A}_F)$, which by theorems of Kohnen-Zagier (for $F = \mathbf{Q}$) and Baruch-Mao (for any F) is equivalent to the best known approximation towards the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms in the level aspect. Hence (taking $\theta_0 = 7/64$), $\delta_0 = 103/512$ is admissible by Blomer-Harcos* [6, Corollary 1].

We show the following variation in Appendix A (see also [6, Theorem 2]):

Theorem 2.5. *Let $\pi = \otimes_v \pi_v$ for be a non-dihedral cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation, with Hecke eigenvalues or equivalently L-function coefficients denoted by λ_π. Let us also assume that π is a holomorphic discrete series, generated by some holomorphic Hilbert modular form. Fix W a smooth function* *and compact support (or any smooth function of rapid decay near infinity and moderate decay near zero) whose derivatives satisfy the decay condition $W^{(i)} \ll 1$ for all integers $i \geq 0$. Fix an F-rational quadratic polynomial $q(x) = rx^2 + sx + t \in \mathcal{O}_F[x]$, and suppose that the discriminant $\Delta := s^2 - 4rt$ is totally negative $\Delta \ll 0$ (hence nonzero). Fix $Y > 0$ a real number. Then, with notations as in Theorem 2.4, we have for $\mathbf{N}r \gg 1$ sufficiently large and any $\varepsilon > 0$ the uniform upper bound*

$$\sum_{\gamma \in F^\times} \frac{\lambda_\pi(q(\gamma))}{\mathbf{N}(q(\gamma))^{\frac{1}{2}}} W\left(\frac{\mathbf{N}q(\gamma)}{Y}\right) \ll_{\pi,\varepsilon} Y^{\frac{1}{4}} \cdot \mathbf{N}r \cdot \mathbf{N}\Delta^{\delta_0 - \frac{1}{2}} \left(\frac{\mathbf{N}\Delta}{Y}\right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

We shall use these bounds as follows to estimate the averages of L-functions we wish to consider.

2.4. **Averages over primitive ring class characters.** Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p, and assume $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Fix a primitive even Dirichlet character χ of conductor p^β for some integer $\beta \geq 0$. Let us for any integer $\alpha \geq 0$ write $C(\alpha)$ to denote the class group of the order $\mathcal{O}_{\mathfrak{p}^\alpha} = \mathcal{O}_F + \mathfrak{p}^\alpha \mathcal{O}_K$ of conductor $\mathfrak{p}^\alpha$ of K, with $C(\alpha)^\vee$ its character group. Hence, $C(0)$ denotes the class group of $\mathcal{O}_K$, whose cardinality we denote by $h_K = \#C(0)$. Note that by Dedekind's formula (see e.g. [16, Theorem 7.24]), we have that

$$\#C(\alpha) = \frac{h_K \cdot \mathbf{N}\mathfrak{p}^\alpha}{[\mathcal{O}_K^\times : \mathcal{O}_{\mathfrak{p}^\alpha}^\times]} \cdot \left(1 - \frac{\eta(\mathfrak{p})}{\mathbf{N}\mathfrak{p}}\right), \quad \eta(\mathfrak{p}) = \eta_{K/F}(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits in } K \\ -1 & \text{if } \mathfrak{p} \text{ is inert in } K \\ 0 & \text{if } \mathfrak{p} \text{ ramifies in } K, \end{cases}$$

and hence

$$\#C(\alpha) = \begin{cases} \frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}_{\mathfrak{p}^\alpha}^\times]} \cdot (p^d - 1) \cdot p^{d(\alpha-1)} & \text{if } \mathfrak{p} \text{ splits in } K \\ \frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}_{\mathfrak{p}^\alpha}^\times]} \cdot (p^d + 1) \cdot p^{d(\alpha-1)} & \text{if } \mathfrak{p} \text{ is inert in } K \\ \frac{h_K}{[\mathcal{O}_K^\times : \mathcal{O}_{\mathfrak{p}^\alpha}^\times]} \cdot p^{d\alpha} & \text{if } \mathfrak{p} \text{ ramifies in } K. \end{cases}$$

Moreover, since $(\mathcal{O}_{\mathfrak{p}^\alpha}^\times)_{\alpha \geq 0}$ forms a decreasing sequence of subgroups of $\mathcal{O}_K^\times$ with $\cap_{\alpha \geq 0} \mathcal{O}_{\mathfrak{p}^\alpha}^\times = \mathcal{O}_F^\times$ and $\mathcal{O}_F^\times$ has finite index (two) in $\mathcal{O}_K^\times$, we deduce that $\mathcal{O}_{\mathfrak{p}^\alpha} = \mathcal{O}_F^\times$ for all $\alpha \gg 0$ sufficiently large (cf. [15, Lemma 2.1]). Hence for all sufficiently large $\alpha \gg 0$ in this sense, we have the simpler formulae

$$\#C(\alpha) = \frac{1}{2} \cdot \begin{cases} h_K \cdot (p^d - 1) \cdot p^{d(\alpha-1)} & \text{if } \mathfrak{p} \text{ splits in } K \\ h_K \cdot (p^d + 1) \cdot p^{d(\alpha-1)} & \text{if } \mathfrak{p} \text{ is inert in } K \\ h_K \cdot p^{d\alpha} & \text{if } \mathfrak{p} \text{ ramifies in } K. \end{cases} \tag{10}$$

Recall that for $\alpha \geq 1$, the primitive ring class characters of conductor $\mathfrak{p}^\alpha$ are those characters of $C(\alpha)^\vee$ which do not factor through $C(\alpha - 1)^\vee$. We shall consider the weighted averages over such characters ρ of the corresponding central values $L(1/2, \pi \times \rho\chi \circ \mathbf{N})$, as well as the following sub-averages. First, in the style of Cornut-Vatsal [15], let us consider the profinite limit $C(\infty) = \varprojlim_{\alpha \geq 0} C(\alpha)$, writing $C_0 = C(\infty)_{\mathrm{tors}}$ to denote its finite torsion subgroup. Given an integer $\alpha \geq 1$ and a character ρ_0 of the torsion subgroup C_0, we consider the subset $P(\alpha, \rho_0)$ of primitive ring class characters of $C(\alpha)$ whose induced character on C_0 (determined by the image of C_0 on $C(\alpha)$) equals ρ_0. We also consider the sub-averages over primitive ring class characters ρ of exact order p^x of $C(\alpha)$, where $x = \mathrm{ord}_p(\#C(\alpha))$ is the order of p dividing $\#C(\alpha)$. Note (see [29, §3.1]) that the ring class characters of exponent p^x detect the p^x-th powers

$$C(\alpha)^{p^x} = \left\{ A^{p^x} : A \in C(\alpha) \right\},$$

in that we have the orthogonality relation

$$\sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1}}} \rho(A) = \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] & \text{if } A \in C(\alpha)^{p^x} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Lemma 2.6. *By Möbius inversion (inclusion-exclusion), we have the following orthogonality relations for sums over primitive ring class characters of a given conductor and primitive ring class characters of a given (maximal) p-power order. Let us for any integer $\alpha \geq 1$ write $Z(\alpha) = \ker(C(\alpha) \longrightarrow C(\alpha-1))$ to denote the kernel of the surjective morphism $j : C(\alpha) \longrightarrow C(\alpha-1)$, with $\#C^{\star}(\alpha) = \#C(\alpha) - \#C(\alpha-1)$ the number of primitive ring class characters of conductor $\mathfrak{p}^{\alpha}$. Let us also for an integer $0 \leq y \leq \mathrm{ord}_p(\#C(\alpha))$ write $\#C(\alpha, y) = [C(\alpha) : C(\alpha)^{p^y}]$ to denote the index of the classes in $C(\alpha)$ of a given exponent p^y, with $\#C^{\star}(\alpha, x) = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}] = \#C(\alpha, x) - \#C(\alpha, x-1)$ the difference.*

(i) *The weighted sum over primitive ring class characters in $C(\alpha)^{\vee}$ is given by the relation*

$$\frac{1}{\#C^{\star}(\alpha)} \sum_{\substack{\rho \in C(\alpha)^{\vee} \\ \text{primitive}}} \rho(A) = \begin{cases} 1 & \text{if } A = \mathbf{1} \in C(\alpha) \text{ is the principal class} \\ -\frac{\#C(\alpha-1)}{\#C^{\star}(\alpha)} & \text{if } A \in Z(\alpha),\ A \neq \mathbf{1} \in C(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Here, we can identify the sum over classes $A \in Z(\alpha)$ with the principal class in $C(\alpha-1)$. Moreover, for each sufficiently large $\alpha \gg 1$, we have that $\#C(\alpha-1)/\#C^{\star}(\alpha) = 1/(\mathbf{N}\mathfrak{p} - 1) = 1/(p^d - 1)$.

(ii) *Fix a character ρ_0 of $C_0 = C(\infty)_{\mathrm{tors}}$, and let us for each integer $\alpha \geq 0$ write $C_0(\alpha)$ to denote the image of C_0 in $C(\alpha)$ (so that $C_0 \cong C_0(\alpha)$ for α sufficiently large), with $\overline{C}(\alpha) = C(\alpha)/C_0(\alpha)$. The weighed sum over primitive ring class characters in $P(\alpha, \rho_0)$ inducing ρ_0 on C_0 is given by*

$$\frac{1}{\#P(\alpha, \rho_0)} \sum_{\rho \in P(\alpha, \rho_0)} \rho(A) = \rho_0(A) \cdot \begin{cases} 1 & \text{if } A \in C_0(\alpha) \\ -\frac{\#\overline{C}(\alpha-1)}{\#P(\alpha,\rho_0)} & \text{if } A \in C_0(\alpha-1) \backslash C_0(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Here, we view ρ_0 as a character on the image $C_0(\alpha)$ of C_0 in $C(\alpha)$. Also, the cardinality $\#P(\alpha, \rho_0)$ is given by the difference $\#\overline{C}(\alpha) - \#\overline{C}(\alpha-1)$, and so we can express the formula accordingly.

(iii) *The weighted sum over primitive ring class characters in $C(\alpha)^{\vee}$ of exact order p^x for $x = \mathrm{ord}_p(\#C(\alpha))$ is given by the orthogonality relation*

$$\frac{1}{\#C^{\star}(\alpha, x)} \sum_{\substack{\rho \in C(\alpha)^{\vee} \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1}\ \forall\ 0 \leq y < x}} \rho(A) = \begin{cases} 1 & \text{if } A \in C(\alpha)^{p^x} \\ -\frac{\#C(\alpha, x-1)}{\#C^{\star}(\alpha, x)} & \text{if } A \in C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For (i), we first apply Möbius inversion, then apply standard orthogonality relations for sums over characters of $C(\alpha)$ and $C(\alpha-1)$ respectively to deduce that

$$\begin{aligned} \sum_{\substack{\rho \in C(\alpha)^{\vee} \\ \text{primitive}}} \rho(A) &= \sum_{\rho \in C(\alpha)^{\vee}} \rho(A) - \sum_{\rho' \in C(\alpha-1)^{\vee}} \rho'(j(A)) \\ &= \begin{cases} \#C(\alpha) & \text{if } A = \mathbf{1} \in C(\alpha) \\ 0 & \text{otherwise} \end{cases} - \begin{cases} \#C(\alpha-1) & \text{if } j(A) = \mathbf{1} \in C(\alpha-1) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \#C(\alpha) - \#C(\alpha-1) & \text{if } A = \mathbf{1} \in C(\alpha) \\ -\#C(\alpha-1) & \text{if } A \in Z(\alpha) = \ker(j : C(\alpha) \to C(\alpha-1)) \text{ but } A \neq \mathbf{1} \in C(\alpha) \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Note that the classes $A \in C(\alpha)$ in the difference expressions are identified with their images $j(A) \in C(\alpha-1)$, as the second orthogonality relation applies only in the subquotient $C(\alpha-1)$. The stated formula (i) then follows after dividing out by the factor $\#C^{\star}(\alpha)$. Note that for $\alpha \gg 1$ sufficiently large, we can use Dedekind's formula to compute

$$\frac{\#C(\alpha-1)}{\#C^\star(\alpha)} = \frac{\#C(\alpha-1)}{\#C(\alpha)-\#C(\alpha-1)} = \frac{h_K\left(1-\frac{\eta(p)}{\mathbf{N}\mathfrak{p}}\right)\mathbf{N}\mathfrak{p}^{\alpha-1}}{h_K\left(1-\frac{\eta(p)}{\mathbf{N}\mathfrak{p}}\right)(\mathbf{N}\mathfrak{p}^\alpha - \mathbf{N}\mathfrak{p}^{\alpha-1})} = \frac{1}{\mathbf{N}\mathfrak{p}-1}.$$

For (ii), we first take for granted the argument of [15, Lemma 2.8], which shows that (1) the natural surjective map $C_0 \to C_0(\alpha)$ is an isomorphism for α sufficiently large, (2) the surjective map $C(\alpha) \to \overline{C}(\alpha)$ induces an isomorphism from $Z(\alpha) = \ker(C(\alpha) \to C(\alpha-1))$ to $\ker(\overline{C}(\alpha) \to \overline{C}(\alpha-1))$, and (3) the kernel $X(\alpha) = \ker(C(\alpha) \to \overline{C}(\alpha-1))$ has the direct sum decomposition $X(\alpha) \cong C_0(\alpha) \oplus Z(\alpha)$. We then deduce as in [15, Lemma 2.8] that the subset of primitive characters $P(\alpha,\rho_0)$ has the following more explicit description: There exists a character ρ_0' on $C_0(\alpha)$ inducing ρ_0 on C_0 and the trivial character $\mathbf{1}$ on $Z(\alpha)$ such that

$$P(\alpha,\rho_0) = \rho_0' \cdot \left(\overline{C}(\alpha)^\vee - \overline{C}(\alpha-1)^\vee\right).$$

We then deduce stated formula from Möbius inversion, as in (i). That is, we find that

$$\frac{1}{\#P(\alpha,\rho_0)} \sum_{\rho \in P(\alpha,\rho_0)} \rho(A) = \frac{\rho_0'(A)}{(\#\overline{C}(\alpha) - \#\overline{C}(\alpha-1))} \cdot \left(\sum_{\overline{\rho} \in \overline{C}(\alpha)^\vee} \overline{\rho}(A) - \sum_{\overline{\rho}' \in \overline{C}(\alpha-1)^\vee} \overline{\rho}'(A) \right),$$

which reduces us to the same argument as given for (i).

For (iii), we apply Möbius inversion again to detect the characters ρ of exact order p^x as

$$\sum_{\substack{\rho \in C(\alpha)^\vee \text{ primitive} \\ \rho^{p^x}=\mathbf{1} \\ \rho^{p^y}\neq \mathbf{1} \forall 0 \leq y \leq x-1}} \rho(A) = \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1}}} \rho(A) - \sum_{\substack{\rho' \in C(\alpha)^\vee \\ (\rho')^{p^{x-1}} = \mathbf{1}}} \rho'(A).$$

That is, we argue that the ring class characters of exact order p^x will necessarily be primitive, as they cannot factor through $C(\alpha-1)$ by definition of the exponent x. We then evaluate the difference of sums via the orthogonality relation (11) as

$$\begin{aligned} &\sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1}}} \rho(A) - \sum_{\substack{\rho'' \in C(\alpha)^\vee \\ (\rho'')^{p^{x-1}} = \mathbf{1}}} \rho''(A) \\ &= \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] & \text{if } A \in C(\alpha)^{p^x} \\ 0 & \text{otherwise} \end{cases} - \begin{cases} [C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^{x-1}} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^x} \\ -[C(\alpha) : C(\alpha)^{p^{x-1}}] & \text{if } A \in C(\alpha)^{p^{x-1}} \backslash C(\alpha)^{p^x} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

Dividing out by the factor $\#C^\star(\alpha,x) = \#C(\alpha,x) - \#C(\alpha,x-1) = [C(\alpha) : C(\alpha)^{p^x}] - [C(\alpha) : C(\alpha)^{p^{x-1}}]$ then gives the stated relations. ◻

We shall consider the corresponding averages over primitive ring class characters of conductor $\mathfrak{p}^\alpha$

$$P_\alpha(\pi,\chi) := \frac{1}{\#C^\star(\alpha)} \sum_{\substack{\rho \in C(\alpha)^\vee \\ \text{primitive}}} L(1/2, \pi \times \rho\chi \circ \mathbf{N}), \tag{12}$$

the subaverage over characters inducing a given character ρ_0 on the torsion subgroup $C_0 = C(\infty)_{\text{tors}}$,

$$P_{\alpha,\rho_0}(\pi,\chi) = \frac{1}{\#P(\alpha,\rho_0)} \sum_{\rho \in P(\alpha,\rho_0)} L(1/2, \pi \times \rho\chi \circ \mathbf{N}), \tag{13}$$

and also the subaverage over characters of exact order p^x with $x = \operatorname{ord}_p(\#C(\alpha))$,

$$G_\alpha(\pi,\chi;x) = \frac{1}{\#C^\star(\alpha,x)} \sum_{\substack{\rho \in C(\alpha)^\vee \\ \rho^{p^x} = \mathbf{1} \\ \rho^{p^y} \neq \mathbf{1} \ \forall \ 0 \leq y < x}} L(1/2, \pi \times \rho\chi \circ \mathbf{N}). \tag{14}$$

2.4.1. *Classical descriptions of the averages.* Given a class $A \in C(\alpha)$, let us fix a positive definite quadratic form $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$ corresponding to A, i.e. under the bijection between the class group $\mathcal{C}(\alpha)$ of positive definite binary quadratic forms of discriminant $\mathrm{disc}(\mathcal{O}_{\mathfrak{p}^\alpha}) = \mathfrak{D}\mathfrak{p}^{2\alpha}$ and the ideal class group $C(\alpha)$ of $\mathcal{O}_{\mathfrak{p}^\alpha}$ given by the map sending the class of a quadratic form representative $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$ in $\mathcal{C}(\alpha)$ to the class of the proper integral ideal $\mathfrak{a}_A$ in $C(\alpha)$ with the $\mathcal{O}_F$-basis $[a_A, (-b_A + \mathfrak{p}^\alpha \sqrt{\mathfrak{D}})/2]$ (cf. [16, Theorem 7.7]). Here, we also write $\mathfrak{p}^\alpha$ and $\mathfrak{D}$ to denote fixed F-integer representatives of the respective ideals $\mathfrak{p}^\alpha \subset \mathcal{O}_F$ and $\mathfrak{D} \subset \mathcal{O}_F$, and simplify notation in this way henceforth. Let r_A denote the corresponding counting function. Hence, for a nonzero integral ideal $\mathfrak{n} \subset \mathcal{O}_F$, $r_A(\mathfrak{n})$ can be defined as the number of ideals $\mathfrak{a}$ in the class $A \in C(\alpha)$ of relative norm $\mathbf{N}_{K/F}(\mathfrak{a}) = \mathfrak{a}\overline{\mathfrak{a}} = \mathfrak{n}$. Writing w_K to denote the number of automorphs of the quadratic form f_A, or equivalently the number of units in $\mathcal{O}_K^\times$ which do not factor through $\mathcal{O}_F^\times$, this function $r_A(\mathfrak{n})$ can also be parametrized in terms of any representative $f_A(x,y)$ of $\mathcal{C}(\alpha)$ by

$$r_A(\mathfrak{n}) = \frac{1}{w_K} \cdot \# \left\{ a, b \in \mathcal{O}_F : f_A(a,b) \in \mathfrak{n},\ \mathbf{N} f_A(a,b) = \mathbf{N}\mathfrak{n} \right\}. \tag{15}$$

Note that this parametrization (15) is not unique, as the choice of representative $f_A(x,y)$ is not unique. We shall later often assume that $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$ is the unique reduced representative for the class, so that the F-integers a_A and c_A are both totally positive, with $\mathbf{N} b_A \leq \mathbf{N} a_A \leq \mathbf{N} c_A$, and b_A is totally positive if either $\mathbf{N} b_A = \mathbf{N} a_A$ or $a_A = c_A$. In the special case where $A = \mathbf{1} \in C(\alpha)$ is the principal class, the reduced representative $f_{\mathbf{1}}(x,y) = a_{\mathbf{1}} x^2 + b_{\mathbf{1}} xy + c_{\mathbf{1}} y^2$ has first coefficient $a_{\mathbf{1}} = 1$, from which it follows that the last coefficient $c_{\mathbf{1}}$ has norm roughly equal to that of the discriminant $\mathfrak{D}\mathfrak{p}^{2\alpha}$ of $\mathcal{O}_{\mathfrak{p}^\alpha}$.

Recall that we write $c(\pi \times \mathcal{W}) = \mathbf{N}(\mathfrak{D}^2 c(\pi_K) c(\mathcal{W})^2)$ to denote the conductor of $L(s, \pi \times \mathcal{W})$, where $c(\mathcal{W}) \subset \mathcal{O}_K$ is the conductor of the Hecke character $\mathcal{W}$ (as an ideal of $\mathcal{O}_K$ rather than $\mathcal{O}_F$). Taking $\mathcal{W} = \rho\chi \circ \mathbf{N}$ with ρ a primitive ring class character of conductor $\mathfrak{p}^\alpha$ and χ a primitive even Dirichlet character of conductor p^β as we do, we have the more explicit formula for the conductor

$$c(\pi \times \rho\chi \circ \mathbf{N}) = \mathbf{N}\left(\mathfrak{D}^2 c(\pi_K) \left(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F) \mathcal{O}_K \right)^2 \right) = \mathbf{N}(\mathfrak{D}^2 c(\pi_K)) \cdot p^{4d \max(\alpha, \beta)}.$$

Recall as well that we introduce an unbalancing parameter $Z > 0$ in Lemma 2.2. Given any such choice of real parameter $Z > 0$, then us define the corresponding sums

$$D_{A,1}(\pi, \chi; Z) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m}) \chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda(\mathfrak{n}) \chi(\mathbf{N}\mathfrak{n}) r_A(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V_1\left(Z \mathbf{N}(\mathfrak{m}^2 \mathfrak{n}) \right). \tag{16}$$

and

$$D_{A,2}(\pi, \chi; Z) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m}) \overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\overline{\lambda(\mathfrak{n}) \chi(\mathbf{N}\mathfrak{n})} r_A(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 \mathfrak{n})}{Z \mathbf{N}(\mathfrak{D}^2 c(\pi_K)) p^{4d \max(\alpha, \beta)}} \right). \tag{17}$$

If $A \in Z(\alpha) = \ker(j : C(\alpha) \rightarrow C(\alpha - 1))$ but $A \neq \mathbf{1} \in C(\alpha)$ as in Lemma 2.6 (i), i.e. in which case it is identified with the principal class $j(A) = \mathbf{1} \in C(\alpha - 1)$ in taking sum over characters $C(\alpha-1)^\vee$ to derive the orthogonality relation, then we write $r_{\mathbf{1}}^\star$ for the corresponding counting function. We also define the sums

$$D_{\mathbf{1},1}^\star(\pi, \chi; Z) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\omega\eta(\mathfrak{m}) \chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\lambda(\mathfrak{n}) \chi(\mathbf{N}\mathfrak{n}) r_{\mathbf{1}}^\star(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V_1\left(Z \mathbf{N}(\mathfrak{m}^2 \mathfrak{n}) \right). \tag{18}$$

and

$$D_{\mathbf{1},2}^\star(\pi, \chi; Z) = \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m}) \overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\mathfrak{n} \neq \{0\} \subset \mathcal{O}_F} \frac{\overline{\lambda(\mathfrak{n}) \chi(\mathbf{N}\mathfrak{n})} r_{\mathbf{1}}^\star(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 \mathfrak{n})}{Z \mathbf{N}(\mathfrak{D}^2 c(\pi_K)) p^{4d \max(\alpha - 1, \beta)}} \right). \tag{19}$$

Recall that we write $\eta = \eta_{K/F}$ to denote the idele class character of F associated to the quadratic extension K/F, and $\mathfrak{d} = \mathfrak{d}_F$ the different of F. We find the following formula for the average (14).

Lemma 2.7. *Fix prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p for which $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{D}, \mathfrak{p}) = 1$. Fix a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. Fix an integer $\alpha \geq 1$, and let*

$x = \mathrm{ord}_p(\#C(\alpha))$ *be the exponent of* p *in the order of the class group of* $\mathcal{O}_{\mathfrak{p}^\alpha}$. *Then, the average* $P_\alpha(\pi,\chi)$ *over primitive ring class characters of conductor* $\mathfrak{p}^\alpha$ *is given for any choice of* $Z \in \mathbf{R}_{>0}$ *by*

$$\begin{aligned} &P_\alpha(\pi,\chi) \\ &= \left(1 - \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right)\left(D_{\mathbf{1},1}(\pi,\chi;Z) + \epsilon \cdot D_{\mathbf{1},2}(\pi,\chi;Z)\right) - \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\left(D^\star_{\mathbf{1},1}(\pi,\chi;Z) + \epsilon \cdot D^\star_{\mathbf{1},2}(\pi,\chi;Z)\right). \end{aligned}$$

The subaverage $P_{\alpha,\rho_0}(\pi,\chi)$ *over primitive ring class characters of conductor* $\mathfrak{p}^\alpha$ *inducing a given character* ρ_0 *on the torsion subgroup* $C_0 = C(\infty)_{\mathrm{tors}}$ *is given for any choice of* $Z \in \mathbf{R}_{>0}$ *by*

$$\begin{aligned} &P_{\alpha,\rho_0}(\pi,\chi) \\ &= \sum_{A \in C_0(\alpha)} \rho_0(A)\left(D_{A,1}(\pi,\chi;Z) + \epsilon \cdot D_{A,2}(\pi,\chi;Z)\right) \\ &- \frac{\#\overline{C}(\alpha-1)}{\#P(\alpha,\rho_0)} \sum_{\substack{A \in C_0(\alpha-1) \\ A \notin C_0(\alpha)}} \rho_0(A)\left(D_{A,1}(\pi,\chi;Z) + \epsilon \cdot D_{A,2}(\pi,\chi;Z)\right), \end{aligned}$$

and the subaverage $G_\alpha(\pi,\chi;x)$ *over primitive ring class characters of conductor* $\mathfrak{p}^\alpha$ *and order* p^x *by*

$$\begin{aligned} &G_\alpha(\pi,\chi;x) \\ &= \sum_{A \in C(\alpha)^{p^x}} \left(D_{A,1}(\pi,\chi;Z) + \epsilon \cdot D_{A,2}(\pi,\chi;Z)\right) - \frac{\#C(\alpha,x-1)}{\#C^\star(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \left(D_{A,1}(\pi,\chi;Z) + \epsilon \cdot D_{A,2}(\pi,\chi;Z)\right). \end{aligned}$$

Here,

$$\epsilon = \epsilon(1/2, \pi \times \rho\chi \circ \mathbf{N}) = \omega(\mathrm{lcm}(\mathfrak{p}^\alpha, p^\beta \mathcal{O}_F)) \cdot \eta(p^{4\beta}\mathfrak{d}c(\pi)) \cdot \epsilon(1/2,\pi) \cdot \chi(\mathbf{N}(\mathfrak{d}^2 c(\pi)^2 \mathfrak{D}^8)) \cdot \left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}}\right)^{4d}$$

is the (unique) root number associated to each ring class character ρ *of conductor* $\mathfrak{p}^\alpha$ *in the average.*

Proof. This is simple to deduce using the formula of Lemma 2.2 to express the central values, switching the order of summation, then applying the respective orthogonality relations of Lemma 2.6 (i), (ii), and (iii) to evaluate each of the coefficients. For the primitive average $P_\alpha(\pi,\chi)$, we also identify the sum over all contributions $A \in Z(\alpha)$ with the principal class in $C(\alpha-1)$ to derive a more convenient expression. We also use Proposition 2.1 to describe the root number. □

To estimate the averages $P_\alpha(\pi,\chi)$ and $G_\alpha(\pi,\chi;x)$ as described in Lemma 2.7, we open the counting functions $r_A(\mathfrak{n})$ via (15) after fixing a representative $f_A(x,y)$, which leads us to study the sums

$$D_{A,1}(\pi,\chi;Z) = \frac{1}{w_K} \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b \in \mathcal{O}_F \\ f_A(a,b) \neq 0}} \frac{\lambda(f_A(a,b))\chi(\mathbf{N}f_A(a,b))}{\mathbf{N}f_A(a,b)^{\frac{1}{2}}} V_1\left(\mathbf{N}(\mathfrak{m}^2 f_A(a,b))Z\right) \tag{20}$$

and

(21)
$$D_{A,2}(\pi,\chi;Z) = \frac{1}{w_K} \sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{\eta\overline{\omega}(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b \in \mathcal{O}_F \\ f_A(a,b) \neq 0}} \frac{\overline{\lambda(f_A(a,b))\chi(\mathbf{N}f_A(a,b))}}{\mathbf{N}f_A(a,b)^{\frac{1}{2}}} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 f_A(a,b))}{Z\mathbf{N}(\mathfrak{D}^2 c(\pi_K))p^{4d\max(\alpha,\beta)}}\right).$$

The difference sums $D^\star_{\mathbf{1},j}(\pi,\chi;Z)$ for $j = 1,2$ are estimated in the same way, and so we omit them from the main discussion, including them at the end when we assemble various estimates to describe the averages.

2.4.2. *Estimates.* To estimate the averages $P_\alpha(\pi,\chi)$, $P_{\alpha,\rho_0}(\pi,\chi)$, and $G_\alpha(\pi,\chi;x)$, we shall first consider the contributions from the $b=0$ and $a=0$ terms in (20) and (21) with the following estimate. Fix a class $A \in C(\alpha)$, together with a quadratic form class representative $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$. Let us for a given divisor q of a_A consider the shift by right multiplication by $\begin{pmatrix} q & \\ & 1 \end{pmatrix}$ of a vector $\phi \in V_\pi$,

$$\phi \in V_\pi, \phi(g) \longmapsto \phi\left(g\begin{pmatrix} q^{-1} & \\ & 1 \end{pmatrix}\right) \in V_\pi.$$

Here, we also write $q \in \mathbf{A}_{F,f}^\times$ to denote a fixed finite idele representative of the F-integer $q \in \mathcal{O}_F$. We also write $\lambda^{(q)}(a) = \lambda(q^{-1}a)$ as shorthand to denote the coefficients in the corresponding Dirichlet series expansion, and consider special values of the corresponding partial congruence symmetric square L-function, defined here (first for $\Re(s) > 1$) by the Dirichlet series expansion

$$\begin{aligned} L^\star_{\mathbf{1},q}(s, \mathrm{Sym}^2 \pi^{(q)} \otimes \chi \circ \mathbf{N}) &:= L^\star_{\mathbf{1}}(2s, \omega\chi^2 \circ \mathbf{N}) \sum_{\substack{a \in \mathcal{O}_F \\ a \equiv 0 \bmod q\mathcal{O}_F}} \frac{\lambda^{(q)}(a^2)\chi(\mathbf{N}a^2)}{\mathbf{N}a^s} \\ &= L^\star_{\mathbf{1}}(2s, \omega\chi^2 \circ \mathbf{N}) \sum_{\substack{a \in \mathcal{O}_F \\ a \equiv 0 \bmod q\mathcal{O}_F}} \frac{\lambda(a^2q^{-1})\chi(\mathbf{N}a^2)}{\mathbf{N}a^s}, \end{aligned} \tag{22}$$

i.e. where the symbol $\pi^{(q)}$ is also shorthand notation. Note that if $q=1$, then this is simply

$$L^\star_{\mathbf{1},1}(s, \mathrm{Sym}^2 \pi^{(q)} \otimes \chi \circ \mathbf{N}) = L^\star_{\mathbf{1}}(s, \mathrm{Sym}^2 \pi^{(q)} \otimes \chi \circ \mathbf{N}).$$

If $a_A = 1$, as can always be arranged for the principal class $A = \mathbf{1} \in C(\alpha)$ by taking $f_A(x,y)$ to be the reduced quadratic form representative, this $q = a_A = 1$ is the only divisor we have to consider.

Proposition 2.8. *Assume that the representation $\pi \otimes \xi$ is non-dihedral*[7]*, i.e. that π is not the induced representation of some Hecke character of a quadratic extension of F in the special case where ξ is trivial. Let us keep the setup of Lemma 2.7, fixing an integer $\alpha \geq 0$, as well as a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. Given a class $A \in C(\alpha)$, we fix $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$ to be the reduced quadratic form class representative. We again write $0 \leq \theta_0 \leq 1/2$ to denote the best approximation to the generalized Ramanujan conjecture for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms in the level aspect. Let us for simplicity write $c(\omega\eta\chi^2 \circ \mathbf{N}) = \mathbf{N}(\mathfrak{D}c(\pi)p^\beta\mathcal{O}_F)$ to denote the absolute norm of the conductor of the Hecke L-function $L(s, \omega\eta\chi^2 \circ \mathbf{N})$. Writing μ to denote the Möbius function on ideals in $\mathcal{O}_F$, let us also define for any F-integer a_A the residual quantity*

$$R(\pi,\chi,a_A) = \frac{1}{w_K} \cdot L(1, \omega\eta\chi^2 \circ \mathbf{N}) \cdot \sum_{q \mid a_A} \left(\frac{\mu(q)\lambda^{(q)}(a_A)\omega(q)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}} \right) \cdot \frac{L^\star_{\mathbf{1},q}(1, \mathrm{Sym}^2 \pi^{(q)} \otimes \chi \circ \mathbf{N})}{L^\star_{\mathbf{1}}(2, \omega\chi^2 \circ \mathbf{N})}. \tag{23}$$

Note that the sum (23) *does not vanish when $a_A = 1$ (whence there is only one divisor $q = a_A = 1$) as a consequence of the nonvanishing of the symmetric square L-function $L(s, \mathrm{Sym}^2 \pi \otimes \xi)$ at $s=1$. Let us also write $0 \leq \delta_3 \leq 1/4$ to denote the best subconvexity estimate for the twisted symmetric square L-function $L(s, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})$ towards the generalized Lindelöf hypothesis for $\mathrm{GL}_3(\mathbf{A}_F)$-automorphic L-functions in the level aspect, i.e. so that $L(1/2, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N}) \ll_\varepsilon \mathbf{N}(c(\pi)p^{3\beta}\mathcal{O}_F)^{\delta_3 + \varepsilon}$ for any choice of $\varepsilon > 0$.*

(i) The $b=0$ contributions $D_{A,1}(\pi,\chi;Z)|_{b=0}$ in the expansion (20) *of $D_{A,1}(\pi,\chi;Z)$ can be estimated for any choice of parameter $Z = Y^{-1}$ with $Y > \mathbf{N}a_A$ as follows: We have for any $\varepsilon > 0$ that*

$$D_{A,1}(\pi,\chi;Y^{-1})|_{b=0} = R(\pi,\chi,a_A) + O_\varepsilon\left(c(\omega\eta\chi^2 \circ \mathbf{N})^{\frac{1}{4} - \frac{(1-2\theta_0)}{16} + \varepsilon} \cdot \mathbf{N}(c(\pi)p^{3\beta}\mathcal{O}_F)^{\delta_3 + \varepsilon} \cdot \left(\frac{\mathbf{N}a_A}{Y}\right)^{\frac{1}{4}} \right).$$

[7]This is assumed for simplicity; a variation of this and subsequent bounds can be derived in the dihedral case, but we do not pursue it here directly.

(ii) The $b=0$ contributions $D_{A,2}(\pi,\chi;Z)|_{b=0}$ in the expansion (21) of $D_{A,2}(\pi,\chi;Z)$ can be estimated for any choice of $Z>0$ for which $Z\cdot \mathbf{N}(\mathfrak{D}^2 c(\pi_K))\cdot p^{4d\max(\alpha,\beta)} > \mathbf{N}a_A$ as follows: For any small $\epsilon>0$,

$$D_{A,2}(\pi,\chi;Z)|_{b=0}$$
$$= \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})}\cdot R(\widetilde{\pi},\overline{\chi},a_A) + O_\varepsilon\left(c(\omega\eta\chi^2\circ\mathbf{N})^{\frac{1}{4}-\frac{(1-2\theta_0)}{16}+\varepsilon}\cdot \mathbf{N}(c(\pi)p^{3\beta}\mathcal{O}_F)^{\delta_3+\varepsilon}\cdot\left(\frac{\mathbf{N}a_A}{Z\mathbf{N}(\mathfrak{D}^2c(\pi_K))p^{4d\max(\alpha,\beta)}}\right)^{\frac{1}{4}}\right).$$

If on the other hand $Z>0$ is chosen so that $0<Z\cdot\mathbf{N}(a_A\mathfrak{D}^2c(\pi_K))\cdot p^{4d\max(\alpha,\beta)}<1$, then we have

$$D_{A,2}(\pi,\chi;Z)|_{b=0} = O_B\left(\left(Z\mathbf{N}(a_A^{-1}\mathfrak{D}^2c(\pi_K)p^{4d\max(\alpha,\beta)})\right)^B\right)$$

and more generally

$$D_{A,2}(\pi,\chi;Z) = O_B\left(\left(Z\mathbf{N}(a_A^{-1}\mathfrak{D}^2c(\pi_K)p^{4d\max(\alpha,\beta)})\right)^B\right)$$

for any choice(s) of constant(s) $B\geq 1$.

Proof. For (i), we expand out the sum and open up the definition of the cutoff function V_1 to find that

$$D_{A,1}(\pi,\chi;Z)|_{b=0} = \frac{1}{w_K}\sum_{\mathfrak{m}\neq\{0\}\subset\mathcal{O}_F}\frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}}\sum_{a\neq 0\in\mathcal{O}_F}\frac{\lambda(a_Aa^2)\chi(\mathbf{N}(a_Aa^2))}{\mathbf{N}(a_Aa^2)^{\frac{1}{2}}}V_1\left(\mathbf{N}(\mathfrak{m}^2a_Aa^2)Z\right).$$

Let us first consider the Hecke relation, which for each F-integer $a\in\mathcal{O}_F$ in the sum takes the form

$$\lambda(a_Aa^2) = \sum_{q|\gcd(a_A,a^2)}\mu(q)\,\omega(q)\lambda\left(\frac{a_A}{q}\right)\lambda\left(\frac{a^2}{q}\right).$$

Hence, we find that for any $\mathfrak{m}\subset\mathcal{O}_F$ in the latter expression for $D_{A,1}(\pi,\chi;Z)|b=0$,

$$\sum_{a\neq0\in\mathcal{O}_F}\frac{\lambda(a_Aa^2)\chi(\mathbf{N}(a_Aa^2))}{\mathbf{N}(a_Aa^2)^{\frac{1}{2}}}V_1\left(\mathbf{N}(\mathfrak{m}^2a_Aa^2)Z\right)$$
$$=\sum_{a\neq0\in\mathcal{O}_F}\sum_{q|\gcd(a_A,a)}\frac{\mu(q)\,\omega(q)\lambda\left(\frac{a_A}{q}\right)\lambda\left(\frac{a^2}{q}\right)\chi(\mathbf{N}(a_Aa^2))}{\mathbf{N}(a_Aa^2)^{\frac{1}{2}}}V_1\left(\mathbf{N}(\mathfrak{m}^2a_Aa^2)Z\right)$$
$$=\sum_{q|a_A}\mu(q)\omega(q)\frac{\lambda\left(\frac{a_A}{q}\right)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}}\sum_{a'\neq0\in\mathcal{O}_F}\frac{\lambda\left(\frac{(qa')^2}{q}\right)\chi(\mathbf{N}(q^2a'^2))}{\mathbf{N}((qa')^2)^{\frac{1}{2}}}V_1\left(\mathbf{N}(\mathfrak{m}^2a_A(qa')^2)Z\right)$$
$$=\sum_{q|a_A}\mu(q)\omega(q)\frac{\lambda^{(q)}\left(a_A\right)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}}\sum_{a'\neq0\in\mathcal{O}_F}\frac{\lambda^{(q)}\left(qa'^2\right)\chi(\mathbf{N}(qa'^2))}{\mathbf{N}(qa'^2)^{\frac{1}{2}}}V_1\left(\mathbf{N}(\mathfrak{m}^2a_A(qa')^2)Z\right)$$
$$=\sum_{q|a_A}\mu(q)\omega(q)\frac{\lambda^{(q)}\left(a_A\right)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}}\sum_{\substack{a\neq0\in\mathcal{O}_F\\ a\equiv 0 \bmod q\mathcal{O}_F}}\frac{\lambda^{(q)}\left(a^2\right)\chi(\mathbf{N}a^2)}{\mathbf{N}a}V_1\left(\mathbf{N}(\mathfrak{m}^2a_Aa^2)Z\right).$$

Let us now consider any of the inner sums corresponding to a given divisor $q\mid a_A$ in this latter expression, whose contribution to $D_{A,1}(\pi,\chi;Z)|_{b=0}$ is then seen to be given explicitly by

$$\mu(q)\omega(q)\frac{\lambda^{(q)}\left(a_A\right)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}}\int_{\Re(s)=2}\frac{k(s)}{s}\sum_{\mathfrak{m}\neq\{0\}\subset\mathcal{O}_F}\frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}^{1+2s}}\sum_{\substack{a\neq0\in\mathcal{O}_F\\ a\equiv 0 \bmod q\mathcal{O}_F}}\frac{\lambda^{(q)}(a^2)\chi(\mathbf{N}a^2)}{\mathbf{N}(a^2)^{\frac{1}{2}+s}}(a_AZ)^{-s}\frac{ds}{2\pi i}$$
$$=\mu(q)\omega(q)\frac{\lambda^{(q)}\left(a_A\right)\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}}\int_{\Re(s)=2}\frac{k(s)}{s}L(2s+1,\omega\eta\chi^2\circ\mathbf{N})\frac{L^\star_{\mathbf{1},q}(2s+1,\mathrm{Sym}^2\,\pi^{(q)}\otimes\chi\circ\mathbf{N})}{L^\star_{\mathbf{1}}(4s+2,\omega\chi^2\circ\mathbf{N})}(a_AZ)^{-s}\frac{ds}{2\pi i}.$$

Shifting the line of integration to $\Re(s) = -1/4$, we then cross a simple pole at $s = 0$ of residue

$$\mu(q)\omega(q)\frac{\lambda^{(q)}(a_A)\,\chi(\mathbf{N}a_A)}{\mathbf{N}a_A^{\frac{1}{2}}} \cdot L(1, \omega\eta\chi^2 \circ \mathbf{N}) \cdot \frac{L^{\star}_{\mathbf{1},q}(1, \mathrm{Sym}^2\,\pi^{(q)} \otimes \chi \circ \mathbf{N})}{L^{\star}_{\mathbf{1}}(2, \omega\chi^2 \circ \mathbf{N})}.$$

The remaining integral is seen easily to be bounded by

$$\ll_{\varepsilon} c(\omega\eta\chi^2 \circ \mathbf{N})^{\frac{1}{4} - \frac{(1-2\theta_0)}{16} + \varepsilon} \cdot \mathbf{N}(c(\pi)p^{3\beta}\mathcal{O}_F)^{\delta_3+\varepsilon} \cdot (\mathbf{N}a_A Z)^{\frac{1}{4}}$$

using Stirling's approximation formula with a subconvexity bound to estimate the Hecke L-series. Here, we use the Burgess-like subconvexity bound $L(1/2, \omega\eta\chi^2 \circ \mathbf{N}) \ll_{\varepsilon} c(\omega\eta\chi^2 \circ \mathbf{N})^{\frac{1}{4} - \frac{(1-2\theta_0)}{16} + \varepsilon}$ shown in [63], as well as the best existing subconvexity bound towards the generalized Lindelöf hypothesis in the level aspect for the twisted symmetric square L-function $L(s, \mathrm{Sym}^2\,\pi \otimes \chi \circ \mathbf{N}) \ll_{\varepsilon} \mathbf{N}(c(\pi)p^{3\beta}\mathcal{O}_F)^{\delta_3+\varepsilon}$, viewed as a $\mathrm{GL}_3(\mathbf{A}_F)$-automorphic L-function via the Gelbart-Jacquet lift (cf. also [12, Lemma 4.1]).

For (ii), we proceed in the same way, first noting that for any $\mathfrak{m} \subset \mathcal{O}_F$ we have the decomposition

$$D_{A,2}(\pi, \chi; Z)|_{b=0} = \frac{1}{w_K}\sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{a \neq 0 \in \mathcal{O}_F} \frac{\overline{\lambda(a_A a^2)\chi(\mathbf{N}(a_A a^2))}}{\mathbf{N}(a_A a^2)^{\frac{1}{2}}} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 a_A a^2)}{ZC}\right)$$

$$= \frac{1}{w_K}\sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{q | a_A} \mu(q)\overline{\omega(q)} \cdot \frac{\overline{\lambda^{(q)}(a_A)\,\chi(\mathbf{N}a_A)}}{\mathbf{N}a_A^{\frac{1}{2}}} \sum_{\substack{a \neq 0 \in \mathcal{O}_F \\ a \equiv 0 \bmod q\mathcal{O}_F}} \frac{\overline{\lambda^{(q)}(a^2)\,\chi(\mathbf{N}a^2)}}{\mathbf{N}a} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 a_A a^2)}{ZC}\right).$$

Here, we write $C = \mathbf{N}(\mathfrak{D}^2 c(\pi_K))p^{4d\max(\alpha,\beta)}$ to denote the conductor. Let us now consider any of the inner q-sums in this latter expansion, opening up the function V_2 to find

$$\mu(q)\overline{\omega(q)} \cdot \frac{\overline{\lambda^{(q)}(a_A)\,\chi(\mathbf{N}a_A)}}{\mathbf{N}a_A^{\frac{1}{2}}} \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\eta\overline{\omega}(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a \neq 0 \in \mathcal{O}_F \\ a \equiv 0 \bmod q\mathcal{O}_F}} \frac{\overline{\lambda^{(q)}(a^2)\,\chi(\mathbf{N}a^2)}}{\mathbf{N}a} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2 a_A a^2)}{ZC}\right)$$

$$= \mu(q)\overline{\omega(q)} \cdot \frac{\overline{\lambda^{(q)}(a_A)\,\chi(\mathbf{N}a_A)}}{\mathbf{N}a_A^{\frac{1}{2}}} \int_{\Re(s)=2} \frac{k(-s)}{s} \frac{\widetilde{L}_{\infty}(s+\frac{1}{2})}{L_{\infty}(-s+\frac{1}{2})} \sum_{\mathfrak{m} \neq \{0\} \subset \mathcal{O}_F} \frac{\eta\overline{\omega}(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}^{1+2s}}$$

$$\times \sum_{\substack{a \neq 0 \in \mathcal{O}_F \\ a \equiv 0 \bmod q\mathcal{O}_F}} \frac{\overline{\lambda^{(q)}(a^2)\chi(\mathbf{N}a^2)}}{\mathbf{N}a^{1+2s}} \left(\mathbf{N}a_A^{-1} ZC\right)^s \frac{ds}{2\pi i}$$

$$= \mu(q)\overline{\omega(q)} \cdot \frac{\overline{\lambda^{(q)}(a_A)\,\chi(\mathbf{N}a_A)}}{\mathbf{N}a_A^{\frac{1}{2}}} \int_{\Re(s)=2} \frac{k(-s)}{s} \frac{\widetilde{L}_{\infty}(s+\frac{1}{4})}{L_{\infty}(-s+\frac{1}{2})}$$

$$\times L(2s+1, \overline{\omega}\eta\overline{\chi}^2 \circ \mathbf{N}) \cdot \frac{L^{\star}_{\mathbf{1},1}(2s+1, \mathrm{Sym}^2\,\widetilde{\pi}^{(q)} \otimes \overline{\chi} \circ \mathbf{N})}{L^{\star}_{\mathbf{1}}(4s+2, \overline{\omega\chi}^2 \circ \mathbf{N})} \left(\frac{\mathbf{N}a_A}{ZC}\right)^{-s} \frac{ds}{2\pi i}.$$

Now if $\mathbf{N}a_A > ZC$, then we shift the contour leftward to $\Re(s) = -1/8$ again to derive the stated estimate (again using the bounds described above for the contributions of the L-functions in the contour), noting that the same argument applies to estimate the entire sum $D_{A,2}(\pi, \chi; Z)$. Otherwise, we shift the contour to the left, and use the bound of Lemma 2.3 for $V_2(y)$ to derive the stated estimate. $\square$

It remains to estimate the contribution from $b \neq 0$ terms in the region of moderate decay for cutoff functions V_j in each each of the sums $D_{A,j}(\pi, \chi; Z)$. For all of the subsequent discussion following Proposition 2.8 above, we shall choose the unbalancing parameter to be within the interval $0 < Z < 1$, and often simply $Z = Y^{-1}$ for $Y = C^{\frac{1}{2}}$ the square root of the conductor $C = \mathbf{N}(\mathfrak{D}^2 c(\pi_K))p^{4d\max(\alpha,\beta)}$ corresponding to a balanced approximate functional equation. By the decay properties of V_j, it will then do to bound the truncated finite double sums defined for an arbitrary small $\varepsilon > 0$ by

$$D^{\dagger}_{A,1}(\pi, \chi; Z) = \frac{1}{w_K}\sum_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \mathfrak{m} \neq \{0\}}} \frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b \in \mathcal{O}_F \\ b \neq 0 \\ \mathbf{N}(\mathfrak{m}^2 f_A(a,b)) \leq Z^{-1-\varepsilon}}} \frac{\lambda_{\chi}(f_A(a,b))}{\mathbf{N}(f_A(a,b))^{\frac{1}{2}}} V_1\left(\mathbf{N}(\mathfrak{m}^2(f_A(a,b)))Z\right) \tag{24}$$

and

(25)

$$D^{\dagger}_{A,2}(\pi,\chi;Z) = \frac{1}{w_K}\sum_{\substack{\mathfrak{m}\subset\mathcal{O}_F\\ \mathfrak{m}\neq 0}} \frac{\eta\overline{\omega}(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b\in\mathcal{O}_F\\ b\neq 0\\ \mathbf{N}(\mathfrak{m}^2(f_A(a,b)))\leq (CZ)^{1+\varepsilon}}} \frac{\overline{\lambda_{\chi}(f_A(a,b))}}{\mathbf{N}(f_A(a,b))^{\frac{1}{2}}} V_2\left(\frac{\mathbf{N}(\mathfrak{m}^2(f_A(a,b)))}{ZC}\right).$$

Again, we write $\lambda_{\chi}(\mathfrak{n}) := \lambda(\mathfrak{n})\chi(\mathbf{N}\mathfrak{n})$ for $\mathfrak{n}\subset\mathcal{O}_F$. Note that these sums are only defined for certain choices of unbalancing parameter $Z>0$, and in particular only need to be considered in the event that there are contributions $a\in\mathcal{O}_F$ and $b\neq 0\in\mathcal{O}_F$ in the region of moderate decay for the functions V_j.

Let us for simplicity write $\xi=\otimes_v\xi_v$ to denote the idele class character of F determined by composition with the norm $\chi\circ\mathbf{N}$ with our chosen Dirichlet character $\chi \bmod p^{\beta}$.

Theorem 2.9. *Let π be any cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic form of level $c(\pi)$ and central character ω. Let K/F a totally imaginary quadratic extension of absolute discriminant $D_K = D_F^2\mathbf{N}\mathfrak{D}$, and $\mathfrak{p}\subset\mathcal{O}_F$ a fixed prime ideal with underlying rational prime p. Assume that $(c(\pi),\mathfrak{D}\mathfrak{p}) = (\mathfrak{p},\mathfrak{D}) = 1$. Recall we fix integers $\alpha\geq 0$ and $\beta\geq 0$ corresponding to $\mathcal{W}=\rho\chi\circ\mathbf{N}$ a Hecke character of K with primitive ring class component ρ of conductor $\mathfrak{p}^{\alpha}$ and cyclotomic component $\xi=\chi\circ\mathbf{N}$ induced from a primitive even Dirichlet character $\chi \bmod p^{\beta}$. We have the following estimates for the sums $D^{\dagger}_{A,j}(\pi,\chi;Z)$ and $D_{A,j}(\pi,\chi;Z)$ for $j=1,2$, for any class $A\in C(\alpha)$ in the class group of the order $\mathcal{O}_{\mathfrak{p}^{\alpha}}=\mathcal{O}_F+\mathfrak{p}^{\alpha}\mathcal{O}_K$. Again, we shall fix a quadratic form class representative $f_A(x,y)=a_Ax^2+b_Axy+c_Ay^2$ for each class $A\in C(\alpha)$.*

(i) Suppose $\pi\otimes\xi$ is non-dihedral (i.e. not induced from a Hecke character of a quadratic extension of F) if $\xi\neq\mathbf{1}$ is nontrivial. Fix a class $A\in C(\alpha)$, and let $f_A(x,y)=a_Ax^2+b_Axy+c_Ay^2$ be the reduced class form representative. Assume that $a_A=1$ and $b_A=0$, as is the case when A is principal and $D_K\equiv 0 \bmod 4$. We have for any choice of real numbers $Y\geq 1$ and $\varepsilon>0$ the upper bounds

$$D^{\dagger}_{A,1}(\pi,\chi;Y^{-1}) \ll_{\pi,\chi,\varepsilon} Y^{\frac{1}{4}+\delta_0+\varepsilon}\cdot\mathbf{N}c_A^{-\frac{1}{2}}$$

and

$$D^{\dagger}_{A,2}(\pi,\chi;Y^{-1}) \ll_{\pi,\chi,\varepsilon} \left(\frac{\mathbf{N}(\mathfrak{D}^2c(\pi_K))p^{4d\max(\alpha,\beta)}}{Y}\right)^{\frac{1}{4}+\delta_0+\varepsilon}\cdot\mathbf{N}c_A^{-\frac{1}{2}}.$$

Here again, $0<\delta_0<1/4$ denotes the best approximation to the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms in the level aspect, and hence we can take $\delta_0=103/512$ by [6, Corollary 1], *using the approximation $\theta_0=7/64$ to the generalized Ramanujan conjecture for $\mathrm{GL}_2(\mathbf{A}_F)$ given in* [5]. *We can therefore take $\theta_0=7/64$ and $\delta_0=103/512$ in these statements to obtain unconditional estimates with exponents $1/4-(1-2\theta_0)/16=206/1024$ and $1/4+\delta_0=231/512$. In particular, if $A=\mathbf{1}\in C(\alpha)$ is the principal class, then $D_{\mathbf{1}}(\pi,\chi):=D_{\mathbf{1},1}(\pi,\chi;Y^{-1})+\epsilon\cdot D_{\mathbf{1},2}(\pi,\chi;Y^{-1})$ converges with $\alpha\to\infty$ to the constant*

(26)

$$\frac{1}{w_K}\left(L(1,\omega\eta\chi^2\circ\mathbf{N})\cdot\frac{L^{\star}_{\mathbf{1}}(1,\mathrm{Sym}^2\pi\otimes\chi\circ\mathbf{N})}{L^{\star}_{\mathbf{1}}(2,\omega\chi^2\circ\mathbf{N})}+\epsilon(\chi)\cdot\frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})}\cdot L(1,\overline{\omega}\eta\overline{\chi}^2\circ\mathbf{N})\cdot\frac{L^{\star}_{\mathbf{1}}(1,\mathrm{Sym}^2\widetilde{\pi}\otimes\overline{\chi}\circ\mathbf{N})}{L^{\star}_{\mathbf{1}}(2,\overline{\omega\chi}^2\circ\mathbf{N})}\right),$$

where

$$\epsilon(\chi):=\omega(\mathrm{lcm}(\mathfrak{p}^{\alpha},p^{\beta}\mathcal{O}_F))\cdot\eta(p^{4\beta}\mathfrak{d}c(\pi))\cdot\epsilon(1/2,\pi)\cdot\chi(\mathbf{N}(\mathfrak{d}^2c(\pi)^2\mathfrak{D}^8))\cdot\left(\frac{\tau(\chi^2)}{p^{\frac{\beta}{2}}}\right)^{4d}\in\overline{\mathbf{Q}}$$

denotes the root number $\epsilon(1/2,\pi\times\rho\chi\circ\mathbf{N})$. Here, the sum (26) is seen by inspection to be nonvanishing in the special case where $\pi\cong\widetilde{\pi}$ is self-dual and $\chi=\mathbf{1}$ is the principal Dirichlet character. In general, we can also show that this sum of residual terms (26) with $\chi\neq\mathbf{1}$ nontrivial is nonvanishing.

(ii) If in the setup of (i) above we do not impose any condition on the absolute discriminant D_K or the coefficients a_A, b_A, and c_A of the reduced quadratic form class representative $f_A(x,y)=a_Ax^2+b_Axy+c_Ay^2$,

then we can derive for any choices of real numbers $Y \geq 1$ and $\varepsilon > 0$ the stronger estimates when π corresponds to a holomorphic discrete series:

$$D^{\dagger}_{A,1}(\pi, \chi; Y^{-1}) \ll_{\pi,\chi,\varepsilon} \mathbf{N}a_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot \mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot \mathbf{N}c_A^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon}$$

and

$$D^{\dagger}_{A,2}(\pi, \chi; Y^{-1}) \ll_{\pi\otimes\xi,\varepsilon} \mathbf{N}a_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot \mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot \mathbf{N}c_A^{-\frac{1}{2}-\delta_0+\frac{\theta_0}{2}+\varepsilon}.$$

Let us remark that while these bounds appear on the surface to be uniform in a_A, they are weaker for $\mathbf{N}a_A > 1$ as the constraints on the quadratic form $f_A(x,y)$ then force the quantity $\mathbf{N}c_A$ to be strictly smaller, i.e. so that we detect less cancellation in the corresponding shifted convolution sums.

Proof. For (i), let V be any smooth function of compact support on $[1/2, 1]$ satisfying $V^{(i)} \ll 1$ for all $i \geq 0$. Given any real number $R \in \mathbf{R}_{>1}$ and nonzero F-integer $c_A \in \mathcal{O}_F$, we have that

$$\Big| \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b \in \mathcal{O}_F \\ b \neq 0}} \frac{\lambda_\chi(a^2 + c_A b^2)}{\mathbf{N}(a^2 + c_A b^2)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2(a^2 + c_A b^2)}{R} \right) \Big|$$
$$\leq \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{1}{\mathbf{N}\mathfrak{m}} \sum_{b \neq 0 \in \mathcal{O}_F} \Big| \sum_{a \in \mathcal{O}_F} \frac{\lambda_\chi(a^2 + c_A b^2)}{\mathbf{N}(a^2 + c_A b^2)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2(a^2 + c_A b^2)}{R} \right) \Big|,$$

which after applying Theorem 2.4 to the inner sum is bounded above for any choice of $\varepsilon > 0$ by

$$\ll \sum_{\substack{b \in \mathcal{O}_F \\ \mathbf{N}b \leq (R/\mathbf{N}c_A)^{\frac{1}{2}}}} R^{-\frac{1}{4}+\frac{\theta_0}{2}+\varepsilon} \mathbf{N}(c_A b^2)^{\delta_0 - \frac{\theta_0}{2}}$$
$$\ll (\mathbf{N}c_A)^{\delta_0 - \frac{\theta_0}{2}} R^{-\frac{1}{4}+\frac{\theta_0}{2}+\varepsilon} \left(\frac{R}{\mathbf{N}c_A} \right)^{\frac{1}{2}+\delta_0-\frac{\theta_0}{2}} = R^{\frac{1}{4}+\delta_0+\varepsilon} (\mathbf{N}c_A)^{-\frac{1}{2}}.$$

In particular, using a smooth partition of unity and the rapid decay of the cutoff functions V_j $(j = 1, 2)$ in (24) and (25), we deduce the claimed bounds.

Let us now consider the sum of two residual terms (26). Note that if the primitive Dirichlet character $\chi = \mathbf{1}$ is principal and $\pi \cong \widetilde{\pi}$ is self-dual, then $\widetilde{L}_\infty(\frac{1}{2})/L_\infty(\frac{1}{2}) = 1$. The sum (26) then equals

$$\frac{2}{w_K} \cdot L(1, \omega\eta) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \pi)}{L_{\mathbf{1}}^{\star}(2, \omega)}.$$

We deduce single residual term is nonvanishing by a well-known argument (see e.g. [12, Lemma]) which establishes a lower bound for the contribution of the (partial) symmetric square L-function. In general, $\widetilde{L}_\infty(\frac{1}{2})/L_\infty(\frac{1}{2}) \neq 1$, and if we assume otherwise that (26) vanishes, then we would have to have

(27)
$$L(1, \omega\eta\chi^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \omega\chi^2 \circ \mathbf{N})} = \left(-\epsilon(\chi) \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \right) \cdot L(1, \overline{\omega}\eta\overline{\chi}^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \widetilde{\pi} \otimes \overline{\chi} \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \overline{\omega\chi}^2 \circ \mathbf{N})},$$

i.e. where the $-\epsilon(\chi) \cdot \widetilde{L}_\infty(\frac{1}{2})/L_\infty(\frac{1}{2})$ term is a constant independent of the symmetric square L values. To rule out this possibility (27), we argue as follows. Let us for for any Dirichlet character $\chi \bmod p^\beta$ write

$$\mathfrak{L}_\chi(1) = L(1, \omega\eta\chi^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \pi \otimes \chi \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \omega\chi^2 \circ \mathbf{N})}, \quad \mathfrak{L}_{\overline{\chi}}(1) = L(1, \overline{\omega}\eta\overline{\chi}^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2 \widetilde{\pi} \otimes \overline{\chi} \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \overline{\omega\chi}^2 \circ \mathbf{N})}.$$

Note that we know the nonvanishing of each of the individual L-values $\mathfrak{L}_\chi(1)$, for instance as a consequence of the prime number theorem for $\mathrm{GL}_3(\mathbf{A}_F)$-automorphic L-functions. We refer also to the stronger lower bound for the symmetric square L-values values shown in [12, Lemma 4.2]. Observe that (27) implies

(28)
$$\frac{L_\infty(\frac{1}{2})}{\widetilde{L}_\infty(\frac{1}{2})} \cdot \frac{\mathfrak{L}_\chi(1)}{\mathfrak{L}_{\overline{\chi}}(1)} = -\epsilon(\chi).$$

Since the root number $\epsilon(\chi)$ determines an algebraic number, and moreover factors through[8] the cyclotomic field $\mathbf{Q}(\chi)$ obtained by adjoining the values of χ, we can act on the algebraic values in (28). This gives us for all $\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})$ the relation

$$\frac{L_\infty(\frac{1}{2})}{\widetilde{L}_\infty(\frac{1}{2})} \cdot \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\overline{\chi^\sigma}}(1)} = -\epsilon(\chi^\sigma) \quad \forall\, \sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q}). \tag{29}$$

Taking the product of (29) over all of the Galois conjugate characters,

$$G(\chi) := \left\{ \chi^\sigma : \sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q}) \right\},$$

we would then have the relation

$$\frac{L_\infty(1/2)}{\widetilde{L}_\infty(1/2)} \cdot \prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\overline{\chi^\sigma}}(1)} = - \prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\chi^\sigma). \tag{30}$$

Now, observe that the set of conjugate characters $G(\chi)$ does not contain the principal or trivial character $\chi_0 = \mathbf{1} \bmod p^\beta$. However, we see that for each character $\chi^\sigma \in G(\chi)$, the inverse $(\chi^\sigma)^{-1} = \overline{\chi^\sigma} = \chi^{\sigma^{-1}} \in G(\chi)$ lies in the set. Using that $\epsilon(\chi^\sigma)\epsilon(\overline{\chi^\sigma}) = \epsilon(\chi^\sigma)\overline{\epsilon(\chi^\sigma)} = |\epsilon(\chi^\sigma)|^2 = 1$ for each such pair, we deduce that

$$\prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\chi^\sigma) = 1$$

and hence that

$$- \prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \epsilon(\chi^\sigma) = -1$$

after pairing together each character with its inverse. In the same way, it is easy to see that

$$\prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \mathfrak{L}_{\chi^\sigma}(1) = \prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \mathfrak{L}_{\overline{\chi^\sigma}}(1)$$

and hence that

$$\prod_{\sigma \in \mathrm{Aut}(\mathbf{Q}(\chi)/\mathbf{Q})} \frac{\mathfrak{L}_{\chi^\sigma}(1)}{\mathfrak{L}_{\overline{\chi^\sigma}}(1)} = 1,$$

from which (30) would give us the relation

$$\frac{L_\infty(1/2)}{\widetilde{L}_\infty(1/2)} = -1. \tag{31}$$

Observe that in the special case where $\pi \cong \widetilde{\pi}$ and hence $L_\infty(s) = \widetilde{L}_\infty(s)$, this latter relation (35) would imply that $1 = -1$, to give a contradiction. In the general case on π, we can also derive a contradiction via the implication of (35) that $L_\infty(1/2) = -\widetilde{L}_\infty(1/2)$, by inspection of the definition of the archimedean local Euler factors $L_\infty(s)$ and $\widetilde{L}_\infty(s)$ in terms of products of shifts of the gamma function $\Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$ or (using the quadratic basechange presentation of the L-functions) $\Gamma_{\mathbf{C}}(s) = (2\pi)^{-s}\Gamma(s)$. For instance, writing $\Pi = \otimes_w \Pi_w = \mathrm{BC}(\pi)$ to denote the quadratic basechange lifing of π to $\mathrm{GL}_2(\mathbf{A}_K)$, with $\widetilde{\Pi} = \otimes_w \widetilde{\Pi}_w = \mathrm{BC}(\widetilde{\pi})$ the corresponding contragedient representation, we know (see e.g. [38]) that we can write

$$L_\infty(s) = L(s, \Pi_\infty) = \prod_{j=1}^{2} \prod_{w | \infty} \Gamma_{\mathbf{C}}\left(s - \mu_{w,j}(\Pi_\infty)\right), \quad \widetilde{L}_\infty(s) = L(s, \widetilde{\Pi}_\infty) \quad = \prod_{j=1}^{2} \prod_{w | \infty} \Gamma_{\mathbf{C}}\left(s - \overline{\mu_{w,j}(\Pi_\infty)}\right)$$

for $\mu_{j,w}(\Pi_\infty)$ the archimedean local Satake parameters of Π_∞, and $\overline{\mu_{w,j}(\Pi_\infty)} = \mu_{w,j}(\widetilde{\Pi}_\infty)$ those of $\widetilde{\Pi}_\infty$. Note that the generalized Ramanujan conjecture predicts that $\Re(\mu_{w,j}(\Pi_\infty)) = \Re(\mu_{w,j}(\widetilde{\Pi}_\infty)) = 0$ for each of these parameters, and we know this prediction to be true if π arises from a holomorphic Hilbert modular form. In any case, it is simple to see from the definition that the equality $L_\infty(1/2) = -\widetilde{L}_\infty(1/2)$ implied by

[8]In general, it factors through the compositum $\mathbf{Q}(\pi, \chi) = \mathbf{Q}(\pi)\mathbf{Q}(\chi)$ of the Hecke field $\mathbf{Q}(\pi)$ of π and the cyclotomic field $\mathbf{Q}(\chi)$. The same argument is then made via automorphisms of $\mathbf{Q}(\pi, \chi)$ fixing the Hecke field $\mathbf{Q}(\pi)$.

(35) cannot be true, to give us the desired contradiction. In this way, we deduce that the sum of the two residual terms (26) for the primitive average is nonvanishing.

To show (ii), we argue in the same way as for (i), with Theorem 2.5 replacing Theorem 2.4. To be more precise, we consider for each nonzero F-integer $b \in \mathcal{O}_F$ the quadratic polynomial

$$q_{A,b}(x) := f_A(x,b) = a_A x^2 + b_A b x + c_A b^2$$

with discriminant $\Delta_b = (b_A b)^2 - 4a_A c_A b^2 = b^2(b_A^2 - 4a_A c_A) = b^2\Delta$, i.e. where we write $\Delta = \operatorname{disc}(f_A) = \mathfrak{p}^{2\alpha}\mathfrak{D}$. We then have for any smooth function V with support on $[1/2,1]$ satisfying $V^{(i)} \ll 1$ for all $i \geq 1$ and any real parameter $R \geq 1$ that

$$\left| \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\eta\omega(\mathfrak{m})\chi^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}} \sum_{\substack{a,b \in \mathcal{O}_F \\ b \neq 0}} \frac{\lambda_\chi(q_{A,b}(a))}{\mathbf{N}q_{A,b}(a)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2 \mathbf{N}q_{A,b}(a)}{R}\right) \right|$$
$$\ll \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{1}{\mathbf{N}\mathfrak{m}} \sum_{b \in \mathcal{O}_F} \left| \sum_{a \in \mathcal{O}_F} \frac{\lambda_\chi(q_{A,b}(a))}{\mathbf{N}q_{A,b}(a)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{m}^2 \mathbf{N}q_{A,b}(a)}{R}\right) \right|.$$

Applying Theorem 2.5 to each of the inner a-sums, we then obtain for any $\varepsilon > 0$ the bound

$$\begin{aligned}
&\ll_{\pi,\varepsilon} \sum_{\substack{b \neq 0 \in \mathcal{O}_F \\ \mathbf{N}b \leq \left(\frac{R}{\mathbf{N}c_A}\right)^{\frac{1}{2}}}} \mathbf{N}a_A \cdot R^{-\frac{1}{4}+\frac{\theta_0}{2}+\varepsilon} \cdot \mathbf{N}\Delta_b^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \\
&\ll \mathbf{N}a_A \cdot R^{-\frac{1}{4}+\frac{\theta_0}{2}+\varepsilon} \cdot \mathbf{N}\Delta^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \sum_{\mathbf{N}b \leq \left(\frac{R}{\mathbf{N}c_A}\right)^{\frac{1}{2}}} \mathbf{N}b^{2\left(\delta_0 - \frac{\theta_0}{2} - \varepsilon\right)} \\
&\ll \mathbf{N}a_A \cdot R^{-\frac{1}{4}+\frac{\theta_0}{2}+\varepsilon} \cdot \mathbf{N}\Delta^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot \left(\frac{R}{\mathbf{N}c_A}\right)^{\frac{1}{2}+\delta_0 - \frac{\theta_0}{2} - \varepsilon} \\
&= \mathbf{N}a_A \cdot R^{\frac{1}{4}+\delta_0} \cdot \mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon} \cdot \mathbf{N}c_A^{-\frac{1}{2} - \delta_0 + \frac{\theta_0}{2} + \varepsilon}.
\end{aligned} \tag{32}$$

The claimed bound follows after taking a standard partition of unity and dyadic decomposition for the corresponding off-diagonal sum, taking a sum over $\log Y$ many ranges $R \geq 1$ of these bounds (32). ☐

Corollary 2.10. *Suppose the generic root number ϵ is not -1 in the special case where $\pi \otimes \xi = \pi$ is self-dual (hence with $\xi = \chi \circ \mathbf{N}$ trivial), and that π corresponds to a holomorphic discrete series. We have the following:*

(i) Assume $\pi \otimes \xi$ is non-dihedral (i.e. π is non-dihedral if ξ is trivial). Taking $Y = \mathbf{N}(\mathfrak{D}c(\pi_K)^{\frac{1}{2}})p^{2d\max(\alpha,\beta)}$ to be the square root of the conductor, with the fundamental discriminant $\mathfrak{D}$ fixed, we have the following estimate for the average $P_\alpha(\pi,\chi)$ over primitive ring class characters of conductor $\mathfrak{p}^\alpha$:

(33)

$$\begin{aligned}
P_\alpha(\pi,\chi) &= \left(1 - 2 \cdot \frac{\#C(\alpha-1)}{\#C^\star(\alpha)}\right) \\
&\times \frac{1}{w_K}\left(L(1,\omega\eta\chi^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^\star(1, \operatorname{Sym}^2 \pi \otimes \chi \circ \mathbf{N})}{L_{\mathbf{1}}^\star(2, \omega\chi^2 \circ \mathbf{N})} + \epsilon \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \cdot L(1, \overline{\omega}\eta\overline{\chi}^2 \circ \mathbf{N}) \cdot \frac{L_{\mathbf{1}}^\star(1, \operatorname{Sym}^2 \widetilde{\pi} \otimes \overline{\chi} \circ \mathbf{N})}{L_{\mathbf{1}}^\star(2, \overline{\omega\chi}^2 \circ \mathbf{N})}\right) \\
&+ O_{\pi,\chi,\varepsilon}\left(Y^{\frac{1}{4}+\delta_0+\varepsilon}\mathbf{N}(\mathfrak{D}\mathfrak{p}^{2\alpha})^{-\frac{1}{2}}\right).
\end{aligned}$$

Hence, by the discussion above for (26), the primitive average $P_\alpha(\pi,\chi)$ converges to a nonzero constant for $\mathfrak{D}$ fixed with $\alpha \to \infty$. Consequently, for each sufficiently large $\alpha \geq 1$, there exists a primitive ring class character ρ of conductor $\mathfrak{p}^\alpha$ for which $L(1/2, \pi \times \rho\chi \circ \mathbf{N}) \neq 0$.

(ii) Assume again that $\pi \otimes \xi$ is non-dihedral, and that the fundamental discriminant $\mathfrak{D}$ is fixed. Then, the subaverage $P_{\alpha,\rho}(\pi,\chi)$ over primitive ring class characters $\rho \in C(\alpha)$ with restriction to the torsion subgroup $C_0 = C(\infty)_{\mathrm{tors}}$ given by some ρ_0 can be estimated as follows. Taking $Y = \mathbf{N}(\mathfrak{D}c(\pi_K)^{\frac{1}{2}})p^{2d\max(\alpha,\beta)}$ again to

be the square root of the conductor (with $\mathfrak{D}$ fixed), and using the residues notation defined in (23)*, we have we have for any $\varepsilon > 0$ that*

$$\begin{aligned} &P_{\alpha,\rho_0}(\pi,\chi) \\ &= \sum_{A \in C_0(\alpha)} \rho_0(A) \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) + O_{\pi,\chi,\varepsilon}\left(\mathbf{N}a_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot \frac{\mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon}}{\mathbf{N}(c_A)^{\frac{1}{2}+\delta_0-\frac{\theta_0}{2}-\varepsilon}} \right) \right) \\ &- \frac{\#\overline{C}(\alpha-1)}{\#P(\alpha,\rho_0)} \sum_{\substack{A \in C_0(\alpha-1) \\ A \notin C_0(\alpha)}} \rho_0(A) \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) \right). \end{aligned}$$

Here, we can ignore the second sum over classes $A \in C_0(\alpha-1)\backslash C_0(\alpha)$, since $C_0(\alpha) \cong C_0$ for all sufficiently large $\alpha \gg 1$. In particular, we can deduce that the average converges with $\alpha \to \infty$ to the constant

$$\sum_{A \in C_0(\alpha) \cong C_0} \rho_0(A) \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) \right), \tag{34}$$

and moreover that this constant does not vanish.

(iii) Assume again that $\pi \otimes \xi$ is non-dihedral, and that the fundamental discriminant $\mathfrak{D}$ is fixed. Taking $Y = \mathbf{N}(\mathfrak{D}c(\pi_K)^{\frac{1}{2}})p^{2d\max(\alpha,\beta)}$ again to be the square root of the conductor, the Galois subaverage $G_{\alpha}(\pi,\chi;x)$ can be estimated in a similar way for any $\varepsilon > 0$ as

$$\begin{aligned} G_{\alpha}(\pi,\chi;x) = &\sum_{A \in C(\alpha)^{p^x}} \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) + O_{\pi,\chi,\varepsilon}\left(\mathbf{N}a_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot \frac{\mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon}}{\mathbf{N}(c_A)^{\frac{1}{2}+\delta_0-\frac{\theta_0}{2}-\varepsilon}} \right) \right) \\ - \frac{\#\overline{C}(\alpha,x-1)}{\#C^{\star}(\alpha,x)} &\sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) + O_{\pi,\chi,\varepsilon}\left(\mathbf{N}a_A \cdot Y^{\frac{1}{4}+\delta_0} \cdot \frac{\mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})^{\delta_0 - \frac{\theta_0}{2} - \varepsilon}}{\mathbf{N}(c_A)^{\frac{1}{2}+\delta_0-\frac{\theta_0}{2}-\varepsilon}} \right) \right). \end{aligned}$$

In particular, if $\mathbf{N}c_A \gg \mathbf{N}a_A$ is sufficiently large relative to the discriminant $\mathbf{N}(\mathfrak{p}^{2\alpha}\mathfrak{D})$ for each class A in the sum, then this Galois average converges with the ring class exponent $\alpha \to \infty$ to the constant

$$\begin{aligned} &\sum_{A \in C(\alpha)^{p^x}} \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) \right) \\ &- \frac{\#\overline{C}(\alpha,x-1)}{\#C^{\star}(\alpha,x)} \sum_{\substack{A \in C(\alpha)^{p^{x-1}} \\ A \notin C(\alpha)^{p^x}}} \left(R(\pi,\chi,a_A) + \epsilon \cdot \frac{\widetilde{L}_{\infty}(\frac{1}{2})}{L_{\infty}(\frac{1}{2})} \cdot R(\widetilde{\pi},\overline{\chi},a_A) \right). \end{aligned}$$

Proof. Taking for granted Lemma 2.7 and the residual estimates from Proposition 2.8, which also apply in a natural way to the sums $D^{\star}_{\mathbf{1},j}(\pi,\chi;Z)$, the first claim (i) follows directly from Theorem 2.9 (i) if $D_K \equiv 0 \bmod 4$, and more generally from Theorem 2.9.

To deduce the stated estimate for (ii), we put together the estimates of Proposition 2.8 and Theorem 2.9 (ii) in our average formula. Let us now describe the leading coefficients a_A of each contributing class $A \in C_0(\alpha) \cong C_0(\alpha-1) \cong C_0$. Recall that each class group $C_0(\alpha) = \mathrm{Pic}(\mathcal{O}_{\mathfrak{p}^{\alpha}})$ is isomorphic to the class group of positive definite binary quadratic forms $Q_0(\alpha)$ of discriminant $\mathfrak{D}\mathfrak{p}^{2\alpha}$. Note that while the group law for classes in $C(\alpha)$ is given in the usual way (though classes of representatives, i.e. $[\mathfrak{a}][\mathfrak{b}] = [\mathfrak{a}\mathfrak{b}]$), the group law in $Q_0(\alpha)$ is given via composition laws for quadratic forms. Recall we consider the profinite limit(s)

$$C(\infty) := \varprojlim_j C(\alpha) \cong Q(\infty) := \varprojlim_j Q(\alpha),$$

and that we have isomorphism(s) of topological groups

$$C(\infty) \cong Q(\infty) \cong \mathbf{Z}_p^{[F_{\mathfrak{p}}:\mathbf{Q}_p]} \times C_0 \cong \mathbf{Z}_p^{[F_{\mathfrak{p}}:\mathbf{Q}_p]} \times, Q_0,$$

where $Q_0 := Q(\infty)_{\mathrm{tors}} \cong C_0 := C(\infty)_{\mathrm{tors}}$ is a finite torsion subgroup. For each integer $\alpha \geq 0$, we write $C_0(\alpha)$ to denote the image of C_0 in $C_0(\alpha)$, and similarly $Q_0(\alpha)$ to denote the image of Q_0 in $Q_0(\alpha)$. It is easy to see (cf. [15, Lemma 2.8]) that for all $\alpha \gg 1$ sufficiently large, we have natural isomorphisms $C_0(\alpha) \cong C_0$ and $Q_0(\alpha) \cong Q_0$ (with $C_0(\alpha) \cong Q_0(\alpha)$ for all $\alpha \geq 0$). We see from the discussion above that only these classes in these generic liftings of the finite torsion subgroup $C_0(\alpha) \cong C_0 \cong Q_0(\alpha) \cong Q_0$ contribute to the average. On the other hand, the image of the class $[f_{A_0}] \in Q_0$ of any reduced binary quadratic form representative $f_{A_0}(x,y)$ under the isomorphism $Q_0 \cong Q_0(\alpha)$ has a corresponding reduced binary quadratic form representative $f_{A_{0,\alpha}}(x,y)$. To be more precise, let A_0 be a class in the finite torsion subgroup $C_0 = C(\infty)_{\mathrm{tors}}$. Write $f_{A_0}(x,y) = a_{A_0}x^2 + b_{A_0}xy + c_{A_0}y^2$ to denote its corresponding reduced binary quadratic form representative of discriminant $\mathfrak{D}\mathfrak{p}^{2\gamma_{A_0}} = b_{A_0}^2 - 4a_{A_0}c_{A_0}$. Hence, $\gamma_{A_0} \geq 0$ is some integer that does not depend on ring class exponent α. Given any integer $\alpha \geq 1$ which is sufficiently large so that $C_0 \cong C_0(\alpha)$, let us write $A_{0,\alpha}$ to denote the image of an element $A_0 \in C_0$ in $C_0(\alpha)$. We then write $f_{A_{0,\alpha}}(x,y) = a_{A_{0,\alpha}}x^2 + b_{A_{0,\alpha}}xy + c_{A_{0,\alpha}}y^2$ to denote the reduced binary quadratic form representative for this class $A_{0,\alpha} \in C_0(\alpha) \cong Q_0(\alpha)$. Hence, the coefficients of this lifted form $f_{A_{0,\alpha}}(x,y)$ must satisfy the constraints

- $b_{A_{0,\alpha}}^2 - 4a_{A_{0,\alpha}}c_{A_{0,\alpha}} = \mathfrak{D}\mathfrak{p}^{2\alpha}$
- $\mathbf{N}b_{A_{0,\alpha}} \leq \mathbf{N}a_{A_{0,\alpha}} \leq \mathbf{N}c_{A_{0,\alpha}}$.

On the other hand, it is apparent that the class of $f_{A_{0,\alpha}}(x,y)$ must represent the F-integer a_{A_0}. It is then a well-known consequence (see e.g. [16, Lemma 2.3, p. 23]) that this class $[f_{A_{0,\alpha}}] \in Q_0(\alpha)$ must contain a binary quadratic form representative $a_{A_0}x^2 + B_{A_0,\alpha}xy + C_{A_0,\alpha}y^2$ for some F-integers $B_{A_0,\alpha}$ and $C_{A_0,\alpha}$. Indeed, if $f_{A_{0,\alpha}}(x,y)$ represents a_{A_0}, then we can find coprime F-integers p and q for which $f_{A_{0,\alpha}}(p,q) = a_{A_0}$. We can then find F-integers r and s with $ps - rs = 1$ and consider

$$f_{A_0,\alpha}(px+ry, qx+sy) = f_{A_0,\alpha}(p,q)x^2 + \left(2a_{A_{0,\alpha}}pr + b_{A_{0,\alpha}}ps + b_{A_{0,\alpha}}rq + 2c_{A_{0,\alpha}}qs\right)xy + f_{A_0,\alpha}(r,s)y^2.$$

This gives us a non-reduced representative

$$f_{0,\alpha}(x,y) := f_{A_0,\alpha}(px+ry, qx+sy) = a_{A_0}x^2 + B_{A_0,\alpha}xy + C_{A_0,\alpha}y^2,$$

with coefficients

$$B_{A_0,\alpha} = B_{A_0,\alpha}(p,q,r,s) = 2a_{A_{0,\alpha}}pr + b_{A_{0,\alpha}}ps + b_{A_{0,\alpha}}rq + 2c_{A_{0,\alpha}}qs$$

and

$$C_{A_0,\alpha} = C_{A_0,\alpha}(p,q,r,s) = f_{A_0,\alpha}(r,s)$$

constrained by

$$B_{A_0,\alpha}^2 - 4a_{A_0}C_{A_0,\alpha} = \mathfrak{D}\mathfrak{p}^{2\alpha} = \mathfrak{D}\mathfrak{p}^{2\gamma_{A_0}+2\alpha'_{A_0}} = \mathfrak{p}^{2\alpha'_{A_0}} \cdot (b_{A_0}^2 - 4a_{A_0}c_{A_0}).$$

Let us choose this non-reduced representative so that the middle coefficient $B_{A_0,\alpha}$ is minimal, equivalently so that the last coefficient $C_{A_0,\alpha}$ is maximal. We argue that we can also work with this non-reduced representative $f_{0,\alpha}(x,y)$ in place of $f_{A_{0,\alpha}}(\alpha)$ in all of our arguments[9]. Taking the limit $\alpha \to \infty$, we then always have the required property $a_{A_0} \ll \mathbf{N}C_{A_{0,\alpha}}^{\frac{1}{2}}$ to justify that the error terms tend to zero. Using these representatives $f_{0,\alpha}(x,y)$ instead of the reduced quadratic form representatives $f_{A_{0,\alpha}}(x,y)$ as we may to parametrize the counting functions in our sums, the claimed estimate is simple to deduce from the calculation and bounds derived above. Here, we can also use the unconditional approximation $\delta_0 = 103/512$ of Blomer-Harcos [6] (via the approximation $\theta_0 = 7/64$ of Blomer-Brumley [5]). In this way, we deduce that the average converges with $\alpha \to \infty$ to the constant term (34). To derive the claimed nonvanishing of this term, let us first write rewrite each residual term as

$$R(\pi,\chi,a_{A_0}) = \sum_{q \mid a_{A_0}} \mathcal{L}_{\chi,q}(1), \quad \mathcal{L}_{\chi,q}(1) = \left(\frac{\mu(q)\lambda^{(q)}(a_{A_0})\omega(q)\chi(\mathbf{N}a_{A_0})}{\mathbf{N}a_{A_0}^{\frac{1}{2}}}\right) \cdot \frac{L_{\mathbf{1},q}^{\star}(1, \mathrm{Sym}^2\,\pi^{(q)} \otimes \chi \circ \mathbf{N})}{L_{\mathbf{1}}^{\star}(2, \omega\chi^2 \circ \mathbf{N})}.$$

We argue by inspection of the Dirichlet series defining each $L_{\mathbf{1},q}^{\star}(1, \mathrm{Sym}^2\,\pi^{(q)} \otimes \chi \circ \mathbf{N})$ from the full series $L_{\mathbf{1},1}^{\star}(1, \mathrm{Sym}^2\,\pi^{(q)} \otimes \chi \circ \mathbf{N}) = L_{\mathbf{1}}^{\star}(1, \mathrm{Sym}^2\,\pi^{(q)} \otimes \chi \circ \mathbf{N})$ that each of the summands $\mathcal{L}_{\chi,q}(1)$ is nonvanishing. A more intrinsic way to see this is that the standard contour arguments used to derive the corresponding statement for $L(1, \mathrm{Sym}^2\,\pi \otimes \chi)$, as given in [12, Lemma 4.2, cf. Lemma 4.1] (for instance), can be applied

[9]Note that choice of reduced representative was somewhat arbitrary here. We have the freedom to take any choice of representative for each class $A \in C(\alpha)$.

directly to each of these partial Dirichlet series. We can also deduce by the argument given in Theorem 2.9 (i) that for each divisor $q \mid a_{A_0}$ and class $A_{0,\alpha} \in C_0(\alpha)$, the corresponding sum of residual terms

$$\mathcal{L}_{\chi,q}(1) + \epsilon(\chi) \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \cdot \mathcal{L}_{\overline{\chi},q}(1)$$

does not vanish. Similarly, we can see the sum over divisors $q \mid a_{A_0}$ of these terms

$$\sum_{q \mid a_{A_0}} \mathcal{L}_{\chi,q}(1) + \epsilon(\chi) \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \cdot \sum_{q \mid a_{A_0}} \mathcal{L}_{\overline{\chi},q}(1)$$

does not vanish. For instance, if we assume otherwise that it vanishes, it would have to follow that

$$\sum_{q \mid a_{A_0}} \mathfrak{L}_{\chi,q}(1) = -\epsilon(\chi) \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \sum_{q \mid a_{A_0}} \mathfrak{L}_{\overline{\chi},q}(1),$$

equivalently that

$$\frac{L_\infty(\frac{1}{2})}{\widetilde{L}_\infty(\frac{1}{2})} \cdot \frac{\sum_{q \mid a_{A_0}} \mathfrak{L}_{\chi,q}(1)}{\sum_{q \mid a_{A_0}} \mathfrak{L}_{\overline{\chi},q}(1)} = -\epsilon(\chi). \tag{35}$$

Using that the root number $\epsilon(\chi) \in \overline{\mathbf{Q}}$ is an algebraic number, we can again take a product over all of the Galois conjugates to deduce that (35) is equivalent to the impossible identity $L_\infty(\frac{1}{2}) = -\widetilde{L}_\infty(\frac{1}{2})$. It is then easy to check that the sum (34) is nonvanishing by a variation of the same argument. To be clear, if we assume otherwise that it vanishes, it would have to follow that

$$\sum_{A_{0,\alpha} \in C_0(\alpha) \cong C_0} \rho_0(A_{0,\alpha}) \sum_{q \mid a_{A_0}} \mathfrak{L}_{\chi,q}(1) = -\epsilon(\chi) \cdot \frac{\widetilde{L}_\infty(\frac{1}{2})}{L_\infty(\frac{1}{2})} \cdot \sum_{A_{0,\alpha} \in C_0(\alpha) \cong C_0} \rho_0(A_{0,\alpha}) \sum_{q \mid a_{A_0}} \mathfrak{L}_{\overline{\chi},q}(1),$$

equivalently that

$$\frac{L_\infty(\frac{1}{2})}{\widetilde{L}_\infty(\frac{1}{2})} \cdot \frac{\sum_{A_{0,\alpha} \in C_0(\alpha) \cong C_0} \rho_0(A_{0,\alpha}) \sum_{q \mid a_{A_0}} \mathfrak{L}_{\chi,q}(1)}{\sum_{A_{0,\alpha} \in C_0(\alpha) \cong C_0} \rho_0(A_{0,\alpha}) \sum_{q \mid a_{A_0}} \mathfrak{L}_{\overline{\chi},q}(1)} = -\epsilon(\chi). \tag{36}$$

Again using that the root number $\epsilon(\chi) \in \overline{\mathbf{Q}}$ is an algebraic number, we take a product over all Galois conjugates to deduce that (36) is equivalent to the impossible identity $L_\infty(\frac{1}{2}) = -\widetilde{L}_\infty(\frac{1}{2})$.

The third claim (iii) is deduced similarly as for (ii) from the average formula. $\square$

Let us conclude with a few remarks on the limitations of the method we use here to derive bounds, which is drawn out in Appendix A below. In short, to derive bounds for the off-diagonal sums $D_{A,j}^{\dagger}(\pi, \chi; Z)$ we consider via spectral decompositions of shifted convolution sums, we need to derive integral presentations in terms of Fourier coefficients of some distinct automorphic form. This imposes some constraints on the coefficients of the binary quadratic form class representative $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$ that can be used here, i.e. to parametrize the counting functions $r_A(\mathfrak{n})$. In particular, the first coefficient a_A must be small relative to the last coefficient c_A. It is for this reason that we do not derive an unconditional nonvanishing estimate for the Galois subaverage $G_\alpha(\pi, \chi, x)$ directly – we do not know the relative sizes of the coefficients a_A and c_A for the classes that contribute, and this remains and interesting open problem to consider. In general, without constraints on the coefficients of $f_A(x,y)$, we can prove the following result, a special adaptation of the proof of Theorem 2.4 for the abovementioned sums. However, since the choice of archimedean local vector in the Kirillov model does not seem to be admissible without introducing a smooth partition of unity and dyadic decomposition, it cannot be used in a direct way to bound the sums we consider here suitably[10] – a fact that is consistent with the cutoff functions V_j having poles near zero.

[10]This is because we need to consider each length R as in the proof of Theorem 2.9 (i), and so have to include the contributions of small lengths R of size close to one in the sum. At the same time, although we do not describe it here (see rather [58]), it

Proposition 2.11. *Let π as above be any cuspidal automophic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ with unitary central character ω. Let χ be any primitive Dirichlet character of conductor p^β with $\xi = \chi \circ \mathbf{N}$ the corresponding idele class character of F, and $A \in C(\alpha)$ and ring class of conductor $\mathfrak{p}^\alpha$ with associated reduced quadratic form class group representative $f_A(x,y) = a_A x^2 + b_A xy + c_A y^2$. Again, we write $C = \mathbf{N}(\mathfrak{D}^2 c(\pi_K))p^{4d\max(\alpha,\beta)}$ for simplicity to denote the conductor of the L-functions in the average. Let us take $V \in \mathcal{C}^\infty(\mathbf{R}_{>0})$ to be any smooth function of rapid decay at infinity with bounded derivatives $V^{(i)} \ll 1$ for all $i \geq 0$. More specifically, we assume that the function $\exp(2\pi y)V(y)$ of $y \in \mathbf{R}_{>0}$ is square integrable, which will always be the case e.g. if V is compactly supported. Then for any choice of real number $Y > 1$, the sum*

$$\frac{1}{w_K} \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{\overline{\omega}\eta(\mathfrak{m})\overline{\chi}^2(\mathbf{N}\mathfrak{m})}{\mathbf{N}\mathfrak{m}^2} \sum_{\substack{a,b \in \mathcal{O}_F \\ f_A(a,b) \neq 0}} \frac{\overline{\lambda_{\pi\otimes\xi}(f_A(a,b))}}{\mathbf{N}f_A(a,b)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}f_A(a,b)\mathbf{N}\mathfrak{m}^2}{Y}\right)$$

as well as the corresponding contragredient sum can be bounded above in modulus by the quantity

$$\ll_{\pi,\chi,\varepsilon} C^{\frac{1}{2}+\varepsilon} \cdot (D_K p^{d\beta}) \cdot Y^{-\frac{1}{2}}.$$

Proof. Ignoring $\mathfrak{m}$-sums for simplicity, we argue as follows, writing $V = V_j$ for $j = 1, 2$ to denote the relevant cutoff function. We have by orthogonality of characters $\rho \in C(\alpha)^\vee$ that

$$\sum_{a,b \in \mathcal{O}_F} \frac{\lambda_{\pi\otimes\xi}(f_A(a,b))}{\mathbf{N}f_A(a,b)^{\frac{1}{2}}} V\left(\frac{\mathbf{N}f_A(a,b)}{Y}\right) = \frac{1}{\#C(\alpha)} \sum_{\rho \in C(\alpha)^\vee} \sum_{A \in C(\alpha)} \rho(A) \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_{\pi\otimes\xi}(\mathfrak{n}) r_A(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{n}}{Y}\right).$$

Estimating the ρ-sum trivially, this latter sum is bounded above in modulus by

$$\sum_{A \in C(\alpha)} \rho(A) \sum_{\mathfrak{n} \subset \mathcal{O}_F} \frac{\lambda_{\pi\otimes\xi}(\mathfrak{n}) r_A(\mathfrak{n})}{\mathbf{N}\mathfrak{n}^{\frac{1}{2}}} V\left(\frac{\mathbf{N}\mathfrak{n}}{Y}\right),$$

which after Mellin inversion is the same as

$$\int_{\Re(s)=2} L(s+1/2, \pi \otimes \xi \times \rho) Y^s \widehat{V}(s) \frac{ds}{2\pi i}.$$

Shifting the contour leftward to $\Re(s) = -1/2$ and estimating the contribution of $L(s+1/2, \pi\otimes\xi\times\rho)$ trivially by $C^{\frac{1}{2}}$ on the line $\Re(s) = 0$ recovers essentially the same bound. Let us also remark that a similar bound can be obtained by replacing the metaplectic theta series θ_Q in the proof of Theorem 2.4 below with the binary theta series θ_{f_A} associated to the fixed quadratic form representative $f_A(x,y)$, i.e. then decomposing the constant coefficient in the Fourier-Whittaker expansion of $\Phi_A = \phi\overline{\theta}_{f_A}$ for some suitable choice of pure tensor $\phi = \otimes_v \phi_v \in V_\pi$ spectrally in terms of the constant coefficient of $\mathrm{GL}_2(\mathbf{A}_F)$ Eisenstein series[11]. ☐

2.5. **Galois conjugate values.** Assume now that π_∞ is a holomorphic discrete series of weight $k = (k_j)_{j=1}^d$ with each $k_j \geq 2$, so that π arises from a holomorphic cuspidal Hilbert modular eigenform of "arithmetic weight $k \geq 2$". The Hecke eigenvalues $\lambda(\mathfrak{n}) = \lambda_\pi(\mathfrak{n})$ are then known by a theorem of Shimura [49] to be algebraic numbers. Writing $\langle \pi, \pi \rangle$ to denote the Petersson norm of π, another theorem of Shimura shows [49] shows essentially that the values

$$\mathcal{L}(1/2, \pi \times \mathcal{W}) = \frac{L(1/2, \pi \times \mathcal{W})}{8\pi^2 \langle \pi, \pi \rangle}$$

are algebraic numbers, and moreover acted upon in a natural way by automorphisms $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. More precisely, writing $\mathbf{Q}(\pi)$ to denote the finite extension of $\mathbf{Q}$ obtained by adjoining the eigenvalues of π, and $\mathbf{Q}(\pi, \mathcal{W})$ the finite extension of $\mathbf{Q}(\pi)$ obtained by adjoining the values of $\mathcal{W}$, the values $\mathcal{L}(1/2, \pi \times \mathcal{W})$ lie in $\mathbf{Q}(\pi, \mathcal{W})$. These algebraic values are Galois conjugate in the sense that $\sigma \in \mathrm{Aut}(\mathbf{C})$ acts on them via the rule $\sigma(\mathcal{L}(1/2, \pi \times \mathcal{W})) = \mathcal{L}(1/2, \pi^\sigma \times \mathcal{W}^\sigma)$. Here, π^σ denotes the representation of $\mathrm{GL}_2(\mathbf{A}_F)$ obtained from π by applying σ to its eigenvalues, and $\mathcal{W}^\sigma$ the character defined on nonzero ideals $\mathfrak{a} \subset \mathcal{O}_K$ by the rule $\mathfrak{a} \mapsto \mathcal{W}(\mathfrak{a})^\sigma$. Restricting to embeddings σ of $\mathbf{Q}(\pi, \mathcal{W})$ into $\mathbf{C}$ which fix $\mathbf{Q}(\pi)$, we obtain a Galois conjugate family of values $\mathcal{L}(1/2, \pi \times \mathcal{W})$, where the action fixes π but varies over Galois conjugate characters $\mathcal{W}$.

seems a slightly better bound can be derived in the classical setup over $F = \mathbf{Q}$, essentially as we can detect more cancellation thanks to a more explicit knowledge of the constant coefficients of the Eisenstein series appearing in the spectral decomposition.

[11] We omit the details of this alternative argument for brevity.

When $\mathbf{Q}(\pi)$ is linearly disjoint over $\mathbf{Q}$ to the cyclotomic extension $\mathbf{Q}(\mathcal{W})$ obtained by adjoining the values of $\mathcal{W}$, then the (well-defined) weighted average

$$G_{[\mathcal{W}]}(\pi) := \frac{1}{[\mathbf{Q}(\pi, \mathcal{W}) : \mathbf{Q}(\pi)]} \sum_{\substack{\sigma : \mathbf{Q}(\pi, \mathcal{W}) \rightarrow \mathbf{C} \\ \sigma(\mathbf{Q}(\pi)) = \mathbf{Q}(\pi)}} L(1/2, \pi \times \mathcal{W}^{\sigma})$$

over all complex embeddings $\sigma : \mathbf{Q}(\pi, \mathcal{W}) \rightarrow \mathbf{C}$ which fix $\mathbf{Q}(\pi)$ consists of Galois conjugate values. It is easy to see from this that the sum defining $G_{[\mathcal{W}]}(\pi)$ vanishes if and only if each of the summands vanishes. This allows us to deduce the following direct consequences of Corollary 2.10 above.

Theorem 2.12. *Assume π is a holomorphic discrete series of weight $(k_j)_{j=1}^d$ with each $k_j \geq 2$ and conductor $c(\pi) \subset \mathcal{O}_F$. Fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ with underlying rational prime p, together with a totally imaginary quadratic extension K/F of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$. Assume that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$, and that $\mathbf{Q}(\pi)$ is linearly disjoint over $\mathbf{Q}$ to the cyclotomic tower obtained by adjoining all p-power roots of unity $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbf{Q}(\zeta_{p^n})$. Fix a primitive even Dirichlet character $\chi \bmod p^\beta$ for some integer $\beta \geq 0$. In the event that $\beta = 0$ (and hence χ is trivial), let us also assume that the root number $\epsilon(1/2, \pi \times \rho)$ for ρ ranging over primitive ring class characters characters of conductor $\mathfrak{p}^\alpha$ for each $\alpha \gg 1$ sufficiently large is not equal to -1. Then for each sufficiently large integer $\beta \geq 1$, there exists a primitive ring class character ρ of conductor $\mathfrak{p}^\alpha$ for which the Galois average $G_{[\rho\chi \circ \mathbf{N}]}(\pi)$ does not vanish. Hence, the central value $L(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \rho\chi \circ \mathbf{N})$ does not vanish for $\mathcal{W} = \rho\chi \circ \mathbf{N}$ ranging over such Hecke characters taking values in roots of unity of exact order $\mathrm{lcm}(p^\beta, \mathrm{ord}(\rho))$, i.e. where $\mathrm{ord}(\rho) \mid (\#C(\alpha) - \#C(\alpha-1))$ denotes the exact order of the character ρ.*

Proof. The claim follows from Corollary 2.10, after using Shimura's algebraicity theorem. $\square$

3. p-adic L-functions

Let us assume from now on that π is a holomorphic discrete series at each of weight $(k_j)_{j=1}^d$ with $k_j \geq 2$. We explain in this setting how to derive stronger nonvanishing results from the existence of a suitable p-adic L-function. This can be viewed as a more efficient way of using the algebraicity theorem of Shimura [49] (as done in Theorem 2.12 above) together with congruences to derive stronger results from the nonvanishing of a single character twist (as supplied by Corollary 2.10 above). To ensure the existence of such a p-adic L-function, we shall assume for simplicity that π is $\mathfrak{p}$-ordinary at our fixed prime $\mathfrak{p} \subset F$, i.e. that the image of the Hecke eigenvalue $\lambda(\mathfrak{p}) = \lambda_\pi(\mathfrak{p})$ under our fixed embedding $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ is a p-adic unit.

3.1. **Some background.** Let us first establish some background notions.

3.1.1. *Iwasawa algebras.* Let $\mathcal{O}$ be a finite extension of $\mathbf{Z}_p$, $\mathcal{G}$ a profinite group, and $\mathcal{O}[[\mathcal{G}]] = \varprojlim_{\mathcal{U} \subset \mathcal{G}} \mathcal{O}[\mathcal{G}/\mathcal{U}]$ its $\mathcal{O}$-Iwasawa algebra. The limit here runs over open normal subgroups $\mathcal{U} \subset \mathcal{G}$. Note that if $\mathcal{G}$ is abelian and finitely generated, then the elements $\mathcal{L}$ of $\mathcal{O}[[\mathcal{G}]]$ can be viewed as $\mathcal{O}$-valued measures $d\mathcal{L}$ on $\mathcal{G}$.

3.1.2. *Choice of profinite group $\mathcal{G}$.* Let us henceforth consider the profinite group

$$\mathcal{G} = \varprojlim_{\alpha, \beta} C(\mathcal{O}_{\mathfrak{p}^\alpha}) \times (\mathbf{Z}/p^\beta \mathbf{Z})^\times.$$

Composing with the Artin reciprocity map rec_K gives us for each integer $\alpha \geq 0$ an identification

$$\mathrm{rec}_K : C(\mathcal{O}_{\mathfrak{p}^\alpha}) \cong \mathit{\Omega}_\alpha := \mathrm{Gal}(K[\mathfrak{p}^\alpha]/K),$$

where $K[\mathfrak{p}^\alpha]$ denotes the ring class extension of conductor $\mathfrak{p}^\alpha$ of K. The torsion subgroup $\Omega_0 = \mathit{\Omega}_{\mathrm{tors}}$ of $\mathit{\Omega} := \varprojlim_\alpha \mathit{\Omega}_\alpha$ is a finite group (see e.g. [15, §2]), and the quotient Ω of $\mathit{\Omega}$ by Ω_0 is isomorphic as a topological group to $\mathbf{Z}_p^\delta$, where $\delta = \delta_\mathfrak{p} = [F_\mathfrak{p} : \mathbf{Q}_p]$ is the residue degree of $\mathfrak{p}$. On the other hand, let us for each integer $\beta \geq 0$ write $\mathit{\Gamma}_\beta$ to denote the Galois group $\mathrm{Gal}(K(\zeta_{p^\beta})/K)$, where $K(\zeta_{p^\beta})$ is the extension obtained from K by adjoining a primitive p^β-th root of unity ζ_{p^β}. The corresponding limit $\mathit{\Gamma} = \varprojlim_\beta \mathit{\Gamma}_\beta$ is isomorphic as a topological group to $\mathbf{Z}_p^\times$, and hence its torsion subgroup $\Gamma_0 = \mathit{\Gamma}_{\mathrm{tors}}$ is also finite. Let us write Γ to denote the quotient of $\mathit{\Gamma}$ by Γ_0, so that Γ is isomorphic as topological group to $\mathbf{Z}_p$. Writing $K_\infty^{(\mathfrak{p})} = \bigcup_{\alpha,\beta\geq 0} K[\mathfrak{p}^\alpha]K(\zeta_{p^\beta})$ to denote the compositum of the extensions $K[\mathfrak{p}^\alpha]$ and $K(\zeta_{p^\beta})$, the reciprocity map gives an identification

$$\mathrm{rec}_K : \mathcal{G} \longrightarrow \mathrm{Gal}(K_\infty^{(\mathfrak{p})}/K) = \mathit{\Omega} \times \mathit{\Gamma} = \varprojlim_{\alpha,\beta\geq 0} \mathit{\Omega}_\alpha \times \mathit{\Gamma}_\beta.$$

Let us now write $G \approx \mathbf{Z}_p^{\delta+1}$ to denote the quotient of $\mathcal{G}$ modulo its finite torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$. We can describe this $\mathcal{G}$ in terms of the following tower of Galois extensions:

$$\begin{tikzcd}
 & & K_\infty^{(\mathfrak{p})} \arrow[dll, dash, "\mathrm{Gal}(K_\infty^{(\mathfrak{p})}/K[\mathfrak{p}^\infty]) \approx \mathit{\Gamma} \approx \Gamma \times \Gamma_0"'] \arrow[d, dash, "G_0"'] \arrow[drr, dash, "\mathrm{Gal}(K_\infty^{(\mathfrak{p})}/K(\zeta_{p^\infty})) \approx \mathit{\Omega} \approx \Omega \times \Omega_0"] & & \\
 & & K_\infty \arrow[dl, dash] \arrow[dd, dash, "G \approx \mathbf{Z}_p^{1+\delta}"'] \arrow[dr, dash] & & \\
K[\mathfrak{p}^\infty] = \bigcup_{\alpha\geq 0} K[\mathfrak{p}^\alpha] \arrow[r, dash, "\Omega_0"'] & D_\infty \arrow[dr, dash, "\Omega \approx \mathbf{Z}_p^\delta"'] & & K^{\mathrm{cyc}} \arrow[dl, dash, "\Gamma \approx \mathbf{Z}_p"] & K(\zeta_{p^\infty}) = \bigcup_{\beta\geq 0} K(\zeta_{p^\beta}) \arrow[l, dash, "\Gamma_0"] \\
 & & K \arrow[d, dash] & & \\
 & & F & &
\end{tikzcd}$$

Consider the corresponding $\mathcal{O}$-Iwasawa algebra $\mathcal{O}[[\mathcal{G}]] \approx \mathcal{O}[G_0][[G]]$. Here, we have an injection

$$\mathcal{O}[[\mathcal{G}]] \longrightarrow \bigoplus_{\mathcal{W}_0 \in G_0^\vee} \mathcal{O}[[G]], \ \lambda \longmapsto (\mathcal{W}_0(\lambda))_{\mathcal{W}_0 \in G_0^\vee}, \tag{37}$$

where the sum runs over characters $\mathcal{W}_0 = \rho_0\psi_0 = \rho_0(\chi_0 \circ \mathbf{N})$ of the torsion subgroup G_0, and each $\mathcal{W}_0(\lambda)$ denotes the specialization of the given element $\lambda \in \mathcal{O}[\mathcal{G}]]$ to the character $\mathcal{W}_0$ of G_0, but not to any character of $G \approx \Omega \times \Gamma \approx \mathbf{Z}_p^{\delta+1}$. Thus, each $\mathcal{W}_0(\lambda)$ denotes a genuine element of the Iwasawa algebra $\mathcal{O}[[G]]$ rather than simply a value in $\mathcal{O}$. Note that when G_0 has order prime to p, this injection (37) is also a surjection.

3.1.3. *Relation to formal power series.* Taking $G = \mathrm{Gal}(K_\infty/K) \approx \Omega \times \Gamma \cong \mathbf{Z}_p^{\delta+1}$ as above, we view the corresponding $\mathcal{O}$-Iwasawa algebra $\mathcal{O}[[G]]$ as a multivariable power series ring in the following standard way. Let $r \geq 2$ denote the integer defined by $r = \delta + 1 = [F_\mathfrak{p} : \mathbf{Q}_p] + 1$. Fixing a system of topological generators $\gamma_1, \ldots \gamma_\delta$ of Ω and γ_r of Γ, we have an isomorphism to the formal power series ring $\mathcal{O}[[T_1, \ldots, T_r]]$ in r indeterminates $T_1, \ldots, T_r$ given by

$$\mathcal{O}[[G]] \approx \mathcal{O}[[\Omega \times \Gamma]] \longrightarrow \mathcal{O}[[T_1, \ldots, T_r]], \ (\gamma_1, \ldots, \gamma_r) \longmapsto (T_1 + 1, \ldots, T_r + 1). \tag{38}$$

3.1.4. *Weierstrass preparation theorem.* Let R be any complete local ring (e.g. $R = \mathcal{O}[[T]]$) with maximal ideal $\mathfrak{m}_R$, and fix a uniformizer ϖ_R of R. Consider the formal power series ring $R[[T]]$ in the indeterminate T. Recall that a polynomial $g(T)$ in $R[T]$ is said to be *distinguished (or Weierstrass)* if it takes the form

$$g(T) = T^n + b_{n-1}T^{n-1} + \ldots + b_0$$

for some integer $n \geq 1$, with each coefficient b_i lying in the maximal ideal $\mathfrak{m}_R$.

Proposition 3.1 (Weierstrass preparation theorem). *Let $h(T) = \sum_{j\geq 0} a_j T^j$ be an element of the formal power series ring $R[[T]]$. If $h(T)$ is not identically zero, then it can be expressed uniquely as*

$$h(T) = u(T)g(T)\varpi_R^{\mu_R}$$

of some unit $u(T)$ in $R[[T]]$ times some distinguished polynomial $g(T)$ in $R[T]$ times some integer power $\mu_R \geq 0$ of the fixed uniformizer ϖ_R of R.

Proof. The result is standard, see e.g. [35, Ch. IV, Theorem 9.2]. ☐

Given a nonzero element $h(T) = u(T)g(T)\varpi_R^{\mu_R}$ of a formal power series ring $R[[T]]$ as above, the degree of the distinguished polynomial $g(T)$ is known as the *Weierstrass degree of* $h(T)$, and the positive integer $\mu = \mu_R$ as the *μ-invariant*. Note that this Weierstrass degree can also be characterized as the least integer $j \geq 0$ for which the coefficient a_j in the power series expansion $h(T) = \sum_{j\geq 0} a_j T^j \in R[[T]]$ is a unit in R.

3.1.5. *Multivariable p-adic L-functions.* Fix $\mathcal{O}$ a finite extension of $\mathbf{Z}_p$ containing the Hecke eigenvalues of π. Recall that for $\mathcal{W}$ a Hecke character of K, a well-known theorem of Shimura [49] shows that the values

$$\mathcal{L}(1/2, \pi \times \mathcal{W}) = \frac{L(1/2, \pi \times \mathcal{W})}{8\pi^2 \langle \pi, \pi \rangle} \tag{39}$$

are algebraic, and moreover that they lie in the finite extension of $\mathbf{Q}$ defined by the compositum of the Hecke field $\mathbf{Q}(\pi)$ with the cyclotomic extension $\mathbf{Q}(\mathcal{W})$ of $\mathbf{Q}$ obtained by adjoining the values of the character $\mathcal{W}$. Via our fixed embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$, we view the values (39) as element in $\overline{\mathbf{Q}}_p$. We have the following construction, stated here in a simplified form[12] that will suffice for our subsequent arguments.

Theorem 3.2. *Suppose π is associated to a holomorphic $\mathfrak{p}$-ordinary Hilbert modular form of arithmetic weight $k \geq 2$. There exists a measure $\mathcal{L}_\mathfrak{p} = \mathcal{L}_\mathfrak{p}(\pi) \in \mathcal{O}[[\mathcal{G}]]$ such that for any nontrivial finite-order character $\mathcal{W}$ of $\mathcal{G}$, we have the interpolation formula*

$$\mathcal{W}(\mathcal{L}_\mathfrak{p}) = \eta(\pi, \mathcal{W}) \cdot \mathcal{L}(1/2, \pi \times \overline{\mathcal{W}}) \in \overline{\mathbf{Q}}_p. \tag{40}$$

Here, $\eta(\pi, \mathcal{W})$ is some algebraic number which does not vanish if $\mathfrak{p}$ does not divide the conductor of π, and $\mathcal{W}(\mathcal{L}_\mathfrak{p}) = \int_\mathcal{G} \mathcal{W}(\sigma) d\mathcal{L}_\mathfrak{p}(\sigma)$ denotes the specialization of the measure $\mathcal{L}_\mathfrak{p}$ to the character $\mathcal{W}$.

Proof. See e.g. [26], or the constructions of [45, §2], [27], and [46, §5.1]. Each of these constructions shows the existence of such an element; again we suppress the exact form of the $\eta(\pi, \mathcal{W})$. ☐

3.2. **Power series expansions.** Let $\mathcal{L}_\mathfrak{p} \in \mathcal{O}[[\mathcal{G}]]$ be the p-adic L-function of Theorem 3.2 above. Fix a character $\mathcal{W}_0$ of the finite torsion subgroup G_0, and consider the corresponding partially-specialized p-adic L-function $\mathcal{W}_0(\mathcal{L}_\mathfrak{p}) \in \mathcal{O}[[G]]$. Let us then write $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \cdots, T_r) \in \mathcal{O}[[T_1, \ldots, T_r]]$ to denote the image of $\mathcal{W}_0(\mathcal{L}_\mathfrak{p})$ under the non-canonical isomorphism (38). Recall that we label the indeterminates here so that $T_1, \ldots, T_\delta$ denote the anticyclotomic variables corresponding to a fixed system $\gamma_1, \ldots, \gamma_\delta$ of generators of the anticyclotomic Galois group $\Omega \approx \mathbf{Z}_p^\delta$, and $T_r = T_\delta$ denotes the cyclotomic variable corresponding to a fixed generator $\gamma_r = \gamma_{\delta+1}$ of the cyclotomic Galois group $\Gamma \approx \mathbf{Z}_p$. We now consider expansions of the p-adic L-function $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \ldots, T_r)$ is each T_j. Let $\mathfrak{P}$ denote the maximal ideal of the (complete local) ring $\mathcal{O}$.

3.2.1. *Expansion in the cyclotomic variable.* Let us now expand the p-adic L-function $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \ldots, T_r)$ in the cyclotomic variable T_r. Hence, writing $k \in \{0, 1\}$ to denote the integer for which the anticyclotomic root number $\epsilon(1/2, \pi \times \rho) = (-1)^k$ for all almost all characters ρ of Ω (cf [15, §1]), we have

$$\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \cdots, T_r) = \sum_{j \geq k} a_j(T_1, \ldots, T_\delta) T_r^j \in \mathcal{O}[[T_1, \cdots, T_\delta]][[T_r]]. \tag{41}$$

Proposition 3.3 (Least nonvanishing criterion via the cyclotomic variable). *Let $\mathcal{W}_0$ be any character of the finite torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$. Assume that $\mathrm{ord}_\mathfrak{P}(\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; 0, \ldots, 0, T_r)) = 0$. Assume that for some character $\mathcal{W} = \mathcal{W}_0 \mathcal{W}_w = \mathcal{W}_0 \rho_w \psi_w$ factoring through $\mathcal{G} \approx G_0 \times \Omega \times \Gamma$, we know that $L(1/2, \pi \times \mathcal{W}) \neq 0$, or equivalently that $\mathcal{W}(\mathcal{L}_\mathfrak{p}) = \mathcal{L}_\mathfrak{p}(\mathcal{W}_0; \rho_w(\gamma_1) - 1, \ldots, \rho_w(\gamma_\delta) - 1, \psi_w(\gamma_r) - 1) \neq 0$. Then, there exists a minimal exponent $\beta_0 \geq 0$ such that for all characters ψ_w of the cyclotomic Galois group Γ of exact order p^β with $\beta \geq \beta_0$, the central value $L(1/2, \pi \times \mathcal{W}_0 \rho_w \psi_w)$ does not vanish for any character ρ_w of the anticyclotomic Galois group Ω.*

Proof. See [58, Part II, Proposition 3.1 (i)]. Using Proposition 3.1, we deduce from the hypothesis that $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0, T_1, \ldots, T_r)$ has a finite Weierstrass degree $w(T_r)$ in T_r. As $\mathrm{ord}_\mathfrak{P}(\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; \cdots, 0, T_r)) = 0$, this latter fact has the following consequence for the power series expansion (41): There exists a least integer $j_0 \geq k$ such that $a_{j_0}(T_1, \cdots T_\delta)$ is a unit in $\mathcal{O}[[T_1, \cdots, T_\delta]]$. Since units never specialize to zero, we deduce that there exists a least integer $\beta_0 = \beta_0(w(T_r))$ (depending on the Weierstrass degree $w(T_r)$) such that for all

[12] i.e. without spelling out the interpolation formula

characters ψ_w of Γ of exact order p^β with $\beta \geq \beta_0$, we have $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; \rho_w(\gamma_1)-1, \cdots, \rho_w(\gamma_\delta)-1, \psi_w(\gamma_r)-1) \neq 0$ for all characters ρ_w of Ω. This latter assertion is equivalent to the stated claim. ☐

Corollary 3.4. *Assume π is $\mathfrak{p}$-ordinary with $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Fix a character $\mathcal{W}_0$ of the torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$, and assume that the cyclotomic analytic μ-invariant of the corresponding p-adic L-function $\mathcal{W}_0(\mathcal{L}_\mathfrak{p}) \in \mathcal{O}[[G]]$ vanishes. There exists a minimal exponent $\beta_0 \geq 0$ such that for all characters ψ_w of the cyclotomic Galois group Γ of exact order p^β with $\beta \geq \beta_0$, the central value $L(1/2, \pi \times \mathcal{W}_0\rho_w\psi_w)$ does not vanish for any character ρ_w of the anticyclotomic Galois group Ω.*

Proof. Taking both $\alpha \gg \beta$ to be sufficiently large, we can use the results of Corollary 2.10 (ii) as input for the argument Proposition 3.3 described above. To be clear, we can assume without loss of generality that the $\mathcal{W}_0 = \rho_0\psi_0 = \rho_0\chi_0 \circ \mathbf{N}$ is purely cyclotomic $\mathcal{W}_0 = \rho_0$, i.e. after replacing the χ in our main theorem with $\chi\chi_0$ as we may. The desired input then follows from Corollary 2.10 (ii) for the weighted average over primitive characters $P(\alpha, \rho_0)$. ☐

3.2.2. *Specialization in the anticyclotomic variables.* Let us now fix an index $1 \leq l \leq \delta$, and consider the expansion of $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \ldots, T_r)$ in each anticyclotomic variable T_l:

$$\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \ldots, T_r) = \sum_{i \geq 0} b_i(T_1, \ldots, T_{l-1}, T_{l+1}, \cdots, T_\delta) T_l^i \in \mathcal{O}[[T_1, \ldots, T_{l-1}, T_{l+1}, \cdots, T_\delta]][[T_r]]. \tag{42}$$

Proposition 3.5 (Least nonvanishing criterion via the anticyclotomic variable)**.** *Let $\mathcal{W}_0$ be any character of the torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$. Assume that* $\mathrm{ord}_\mathfrak{P}(\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; 0, \ldots, 0, T_l, 0, \ldots, 0)) = 0$ *for each index* $1 \leq l \leq \delta$*. Assume that for some character $\mathcal{W} = \mathcal{W}_0\mathcal{W}_w = \mathcal{W}_0\rho_w\psi_w$ factoring through $\mathcal{G}$, we have $L(1/2, \pi \times \mathcal{W}) \neq 0$, equivalently $\mathcal{W}(\mathcal{L}_\mathfrak{p}) = \mathcal{L}_\mathfrak{p}(\mathcal{W}_0; \rho_w(\gamma_1) - 1, \ldots, \rho_w(\gamma_\delta - 1), \psi_w(\gamma_r) - 1) \neq 0$. Then, there exists a minimal exponent $\alpha_0 \geq 0$ such that for all characters ρ_w of the anticyclotomic Galois group Ω of exact order p^α with $\alpha \geq \alpha_0$, the central value $L(1/2, \pi \times \mathcal{W}_0\rho_w\psi_w)$ does not vanish for any character ψ_w of the cyclotomic Galois group Γ.*

Proof. We apply the same argument as given for Proposition 3.3 in each anticylotomic indeterminate T_l. In this way, we deduce that for each index $1 \leq l \leq \delta$, there exists a minimal exponent $\alpha_0(l)$ such that $\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; \rho_w(\gamma_1) - 1, \cdots, \rho_w(\gamma_\delta) - 1, \psi_w(\gamma_r) - 1) \neq 0$ for all characters ρ_w of Ω of exact order p^α for $\alpha \geq \alpha_0(l)$ and all characters ψ_w of Γ. Taking $\alpha_0 = \max_{1 \leq l \leq \delta} \alpha_0(l)$ then proves the claim. ☐

Corollary 3.6. *Assume π is $\mathfrak{p}$-ordinary with $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Fix a character $\mathcal{W}_0$ of the torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$, and assume that the anticyclotomic analytic μ-invariant of the corresponding p-adic L-function $\mathcal{W}_0(\mathcal{L}_\mathfrak{p}) \in \mathcal{O}[[G]]$ vanishes. Then, there exists a minimal exponent $\alpha_0 \geq 0$ such that for all characters ρ_w of the anticyclotomic Galois group Ω of exact order p^α with $\alpha \geq \alpha_0$, the central value $L(1/2, \pi \times \mathcal{W}_0\rho_w\psi_w)$ does not vanish for any character ψ_w of the Galois group Γ.*

Proof. Again, we can use the result of Corollary 2.10 as input to deduce the claim. ☐

Finally, we can deduce the following unconditional result in this direction thanks to [10].

Theorem 3.7. *Assume that the cuspidal automorphic representation π as described above is $\mathfrak{p}$-ordinary, that $(c(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$, and also that the residual Galois representation associated to π by constructions of Carayol* [9]*, Taylor* [54]*, and Wiles* [62] *is absolutely irreducible. Fix a character $\mathcal{W}_0$ of the torsion subgroup $G_0 = \mathcal{G}_{\mathrm{tors}}$. There exists a minimal exponent $\alpha_0 \geq 0$ such that for all characters ρ_w of the anticyclotomic Galois group $\Omega \approx \mathbf{Z}_p^\delta$ of exact order p^α with $\alpha \geq \alpha_0$, the central value $L(1/2, \pi \times \mathcal{W}_0\rho_w\psi_w)$ does not vanish for any character ψ_w of the cyclotomic Galois group $\Gamma \approx \mathbf{Z}_p$.*

Proof. We use (as input for Corollary 3.6) the result of Chida-Hsieh [10, Theorem C], which implies that

$$\mathrm{ord}_\mathfrak{P}\left(\mathcal{L}_\mathfrak{p}(\mathcal{W}_0; T_1, \ldots, T_\delta, 0)\right) = 0$$

under the stated hypotheses on the conductor $c(\pi)$ and the Galois representation associated to π. ☐

Appendices

Appendix A. Shifted convolution sums over totally real fields

We now prove Theorem 2.4, followed by a variation with decompositions into Poincaré series to prove Theorem 2.5. The idea for Theorem 2.4 is to use the surjectivity of the archimedean Kirillov map for π to realize the shifted convolution sum on the left hand side as the Fourier-Whittaker coefficient at q of a certain genuine automorphic form Φ on the two-fold metaplectic cover $\overline{G}(\mathbf{A}_F)$ of $\mathrm{GL}_2(\mathbf{A}_F)$ (see [21]). Decomposing Φ spectrally, and using its convergence in the Sobolev norm topology, the stated bound can be derived from existing bounds for the Fourier-Whittaker coefficients of each form appearing in the decomposition. Let us now now explain this idea in more detail as follows, noting that the special case of $F = \mathbf{Q}$ is worked out in [55, Theorem 1] (cf. also [6]).

Fourier-Whittaker expansions. Let $\psi = \otimes \psi_v$ denote the standard additive character on $\mathbf{A}_F/F$. Hence, ψ is trivial on F, agrees with the function $x = (x_j)_{j=1}^d \mapsto \exp(2\pi i(x_1 + \cdots x_d))$ on the archimedean component $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$ of $\mathbf{A}_F$, and each finite place v is trivial on the local inverse different $\mathfrak{d}_{F,v}^{-1}$ but nontrivial on $v^{-1}\mathfrak{d}_{F,v}^{-1}$. Let $\phi \in V_\pi$ be any vector in the representation space of π. We have for $x \in \mathbf{A}_F$ any generic adele and $y \in \mathbf{A}_F^\times$ any generic idele the Fourier-Whittaker expansion

$$\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in F^\times} W_\phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)\psi(-\gamma x). \tag{43}$$

Here, for any $g \in \mathrm{GL}_2(\mathbf{A}_F)$, we write

$$W_\phi(g) := \int_{\mathbf{A}_F/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-x)dx$$

to denote the Whittaker function of ϕ. Note that if we decompose the idele $y \in \mathbf{A}_F^\times$ into its corresponding nonarchimedean and archimedean components as $y = y_f y_\infty$, with $y_f \in \mathbf{A}_{F,f}^\times$ and $y_\infty \in F_\infty^\times \cong (\mathbf{R}^\times)^d$, then we can also decompose the (specialized) Whittaker function W_ϕ its corresponding nonarchimedean component

$$\rho_\phi(y_f) = \rho_\phi\left(\begin{pmatrix} y_f & \\ & 1 \end{pmatrix}\right) := W_\phi\left(\begin{pmatrix} y_f & \\ & 1 \end{pmatrix}\right)$$

and archimedean component

$$W_\phi(y_\infty) := W_\phi\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right),$$

so that (43) is the same as

$$\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in F^\times} \rho_\phi(\gamma y_f) W_\phi(\gamma y_\infty)\psi(-\gamma x). \tag{44}$$

Note as well that if $\phi \in V_\pi$ is a new vector, and we write $|\cdot|$ to denote the idele norm, then the coefficients $\rho_\phi(\gamma y_f)$ are related to the L-function coefficients $\lambda(\gamma y_f) = \lambda_\pi(\gamma y_f)$ in the sense that the Fourier-Whittaker expansion (43) (or (44)) is equivalent to

$$\phi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = \sum_{\gamma \in F^\times} \frac{\lambda(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} W_\phi(\gamma y_\infty)\psi(\gamma x). \tag{45}$$

In what follows, we shall always take $\phi \in V_\pi$ to be a pure tensor $\phi = \otimes_v \phi_v$ whose nonarchimedean components are each essential Whittaker vectors. The corresponding Whittaker coefficients ρ_ϕ are then related to the L-function coefficients λ as in (45). That is, the local vectors ϕ_v are then related directly via Mellin transformation to the corresponding local Euler factors of $L(s,\pi)$. We shall also make a precise choice of the archimedean local vectors $\phi_\infty = \otimes_{v|\infty}\phi_v$ as follows. Namely, we shall use the surjectivity of the archimedean Kirillov map $\phi \mapsto W_\phi$, which as explained in [6, § 2.5 (37)] (for instance) induces an isometry between the representation space V_π and the Whittaker model $\mathcal{W}(\pi)$ of π. In particular:

Proposition A.1. *Let $W \in L^2(F_\infty^\times) \cong L^2((\mathbf{R}^\times)^d)$ be any smooth*[13] *function on $F_\infty^\times \cong (\mathbf{R}^\times)^d$. Let π be any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Then, there exists a vector $\phi \in V_\phi$ whose corresponding (archimedean) Whittaker function W_ϕ satisfies $W_\phi(y_\infty) = W(y_\infty)$ as function(s) of $y_\infty \in F_\infty^\times$.*

Proof. The result is standard; see [6, (37), Lemma 3] and [55, Proposition 2.1] (for instance). □

We use this result to choose our archimedean vector $\phi_\infty = \otimes_{v|\infty}\phi_v$ in such a way that the corresponding archimedean Whittaker coefficient $W_\phi(y_\infty)$ in the expansion (45) matches the chosen test function that appears in the statement of Theorem 2.4. That is, we use Proposition A.1 to derive an integral presentation for the shifted convolution sum appearing in Theorem 2.4 as follows. Let us now consider the F-rational quadratic form defined on $\gamma \in F$ by $Q(\gamma) = \gamma^2$. Let θ_Q denote the corresponding half-integral weight theta series, viewed as an automorphic form on the metaplectic cover $\overline{G}(\mathbf{A}_F)$ of $\mathrm{GL}_2(\mathbf{A}_F)$. This theta series has the following expansion (at archimedean components): For $x_\infty \in F_\infty \cong \mathbf{R}^d$ and $y_\infty \in F_\infty^\times \cong (\mathbf{R}^d)^\times$,

$$\theta_Q\left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix}\right) = |y_\infty|^{\frac{1}{4}} \sum_{\gamma \in F} \psi(Q(\gamma)(x_\infty + iy_\infty)) = |y_\infty|^{\frac{1}{4}} \sum_{q \in \mathcal{O}_F} \psi(Q(q)(x_\infty + iy_\infty)).$$

Let us also write $\overline{\theta}_Q = T_{-1}\theta_Q$ to denote the image of θ_Q under the Hecke operator T_{-1} corresponding to the classical Hecke operator sending $z \in \mathfrak{H}$ to $-\overline{z}$. The corresponding form $\overline{\theta}_Q$ then has the expansion

$$\overline{\theta}_Q\left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix}\right) = |y_\infty|^{\frac{1}{4}} \sum_{\gamma \in F} \psi(-Q(\gamma)(x_\infty - iy_\infty)) = |y_\infty|^{\frac{1}{4}} \sum_{\gamma \in \mathcal{O}_F} \psi(-Q(\gamma)(x_\infty - iy_\infty)). \tag{46}$$

Fixing $\phi = \otimes_v \phi_v \in V_\pi$ a pure tensor as described above, we consider the product $\Phi = \phi\overline{\theta}_Q$. Note that this Φ is a genuine automorphic form on the metaplectic group $\overline{G}(\mathbf{A}_F)$. It has the Fourier expansion

$$\begin{aligned}\Phi\left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix}\right) &= \sum_{\gamma \in F} W_\Phi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right)\psi(\gamma x_\infty) \\ &= \sum_{q \in \mathcal{O}_F} W_\Phi\left(\begin{pmatrix} q & \\ & 1 \end{pmatrix}\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right)\psi(q x_\infty),\end{aligned}$$

where for each F-integer $\mathfrak{q}$,

$$W_\Phi\left(\begin{pmatrix} q & \\ & 1 \end{pmatrix}\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = \int_{I \cong [0,1]^d \subset F_\infty} \Phi\left(\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix}\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right)\psi(-q x_\infty) dx_\infty.$$

Proposition A.2. *Let $W \in L^2(F_\infty^\times) \cong L^2((\mathbf{R}^\times)^d)$ be any smooth function on $F_\infty^\times \cong (\mathbf{R}^\times)^d$. Fix a nonzero F-integer $q \in \mathcal{O}_F$, as well as an archimedean idele $Y_\infty \in F_\infty^\times$ with idele norm $|Y_\infty| \gg |q|$. Let π be any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Fix $\phi = \otimes_v \phi_v \in V_\pi$ a pure tensor whose nonarchimedean local components ϕ_v are each essential Whittaker vectors, and whose archimedean local component $\phi_\infty = \otimes_{v|\infty}\phi_v$ is chosen in such a way that*

$$W_\phi(y_\infty) := W_\phi\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix}\right) = \psi(-iy_\infty)\psi\left(\frac{iq}{Y_\infty}\right)W(y_\infty)$$

as a function of $y_\infty \in F_\infty^\times \cong (\mathbf{R}^\times)^d$. Then, the coefficient at q in the expansion of

$$\Phi\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) = \phi\overline{\theta}_Q\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right)$$

is given by

$$\int_{I \cong [0,1]^d \subset F_\infty} \Phi\left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix}\right)\psi(-q x_\infty) dx_\infty = |Y_\infty|^{-\frac{1}{4}} \sum_{\gamma \in F^\times} \frac{\lambda(Q(\gamma) + q)}{|Q(\gamma) + q|^{\frac{1}{2}}} W\left(\frac{Q(\gamma) + q}{Y_\infty}\right).$$

[13] Strictly speaking, we should impose the condition that W be compactly supported to match the statements of results in the literature. In practice however, this condition is really only imposed to ensure the square summability of the function.

Equivalently, we have the integral presentation

$$|Y_\infty|^{\frac{1}{4}} \int_{I \cong [0,1]^d \subset F_\infty} \phi \overline{\theta}_Q \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-q x_\infty) dx_\infty = \sum_{\gamma \in F^\times} \frac{\lambda(\gamma^2 + q)}{|\gamma^2 + q|^{\frac{1}{2}}} W \left(\frac{\gamma^2 + q}{Y_\infty} \right). \tag{47}$$

Here, each of the sums is supported only on nonzero F-integers.

Proof. Cf. [55, §6.1]. We use the expansions (45) and (46) to compute

$$\begin{aligned} &\int_{I \cong [0,1]^d \subset F_\infty} \phi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \overline{\theta}_Q \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-q x_\infty) dx_\infty \\ &= \int_{I \cong [0,1]^d \subset F_\infty} \sum_{\gamma_1 \in F^\times} \frac{\lambda(\gamma_1)}{|\gamma_1|^{\frac{1}{2}}} W_\phi \left(\frac{\gamma_1}{Y_\infty} \right) \psi(\gamma_1 x_\infty) \cdot |Y_\infty|^{-\frac{1}{4}} \sum_{\gamma_2 \in F} \psi \left(\frac{i Q(\gamma_2)}{Y_\infty} \right) \psi(-Q(\gamma_2) x_\infty) \psi(-q x_\infty) dx_\infty \\ &= |Y_\infty|^{-\frac{1}{4}} \sum_{\gamma_1 \in F^\times} \frac{\lambda(\gamma_1)}{|\gamma_1|^{\frac{1}{2}}} W_\phi \left(\frac{\gamma_1}{Y_\infty} \right) \sum_{\gamma_2 \in F} \psi \left(\frac{i Q(\gamma_2)}{Y_\infty} \right) \int_{I \cong [0,1]^d \subset F_\infty} \psi(\gamma_1 x_\infty - Q(\gamma_2) x_\infty - q x_\infty) dx_\infty. \end{aligned}$$

To compute the integral, recall that the characters on the compact abelian group $I \cong [0,1]^d \cong (\mathbf{R}/\mathbf{Z})^d$ can be parametrized by $\psi(\gamma x_\infty)$ for any fixed $x_\infty \in I$ with $\gamma \in F$ varying. We can then use the orthogonality of these characters to deduce that the integral vanishes unless $\gamma_1 - Q(\gamma_2) - q = 0$, so

$$\int_{I \cong [0,1]^d \subset F_\infty} \Phi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-q x_\infty) dx_\infty = |Y_\infty|^{-\frac{1}{4}} \sum_{\gamma \in F^\times} \frac{\lambda(Q(\gamma) + q)}{|Q(\gamma) + q|^{\frac{1}{2}}} W_\phi \left(\frac{Q(\gamma) + q}{Y_\infty} \right) \psi \left(\frac{i Q(\gamma)}{Y_\infty} \right).$$

Now, by our choice of archimedean local vector $\phi_\infty = \otimes_{v|\infty} \phi_v$, this latter identity is the same as

$$\int_{I \cong [0,1]^d \subset F_\infty} \Phi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-q x_\infty) dx_\infty = |Y_\infty|^{-\frac{1}{4}} \sum_{\gamma \in F^\times} \frac{\lambda(Q(\gamma) + q)}{|Q(\gamma) + q|^{\frac{1}{2}}} W \left(\frac{Q(\gamma) + q}{Y_\infty} \right)$$

or equivalently

$$|Y_\infty|^{\frac{1}{4}} \int_{I \cong [0,1]^d \subset F_\infty} \phi \overline{\theta}_Q \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-q x_\infty) dx_\infty = \sum_{\gamma \in F^\times} \frac{\lambda(\gamma^2 + q)}{\mathbf{N}(\gamma^2 + q)^{\frac{1}{2}}} W \left(\frac{\gamma^2 + q}{Y_\infty} \right).$$

Note that these expansions are supported only on nonzero F-integers. The claim (47) follows. □

Upper bounds for Whittaker functions. We shall use the following general bounds for classical Whittaker functions, as derived via contour integral arguments in [55, § 7] for the case of $F = \mathbf{Q}$. These bounds can be applied componentwise to derive bounds in the more general setting we consider here. Let us for arbitrary complex numbers $\kappa, \nu \in \mathbf{C}$ consider the classical Whittaker function $W_{\kappa,\nu}(y)$ defined on a positive real variable $y \in \mathbf{R}_{>0}$ as in [55, § 7.1]. To be clear about the choice of normalization of $W_{\kappa,\nu}$, we note that the Mellin transform of $e^{\frac{y}{2}} W_{\kappa,\nu}(y)$ at $s \in \mathbf{C}$ with $\Re(s) > \frac{1}{2} \pm \nu$ is known via direct calculation to equal

$$\int_0^\infty e^{\frac{y}{2}} W_{\kappa,\nu}(y) y^s \frac{dy}{y} = \frac{\Gamma\left(\frac{1}{2} + s + \nu\right) \Gamma\left(\frac{1}{2} + s - \nu\right)}{\Gamma\left(1 + s - \kappa\right)}.$$

Proposition A.3. *There exists a constant $A > 0$ for which we have the following uniform bounds in $y \in \mathbf{R}_{>0}$ with $y \to 0$, i.e. for any choice of real parameter $y > 0$ in the interval $0 < y < 1$:*

(i) *If $\kappa, \nu \in \mathbf{R}$, then we have for any choice of $\varepsilon > 0$ the upper bound*

$$\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa + i\nu\right)} \ll_\varepsilon (|\kappa| + |\nu| + 1)^A y^{\frac{1}{2} - \varepsilon}.$$

(ii) *If $\kappa, \nu \in \mathbf{R}$ with $0 < \nu < \frac{1}{2}$, then we have for any choice of $\varepsilon > 0$ the upper bound*

$$\frac{W_{\kappa,\nu}(y)}{\Gamma\left(\frac{1}{2} + \kappa\right)} \ll_\varepsilon (|\kappa| + 1)^A y^{\frac{1}{2} - \nu - \varepsilon}.$$

(iii) *If $\kappa, \nu \in \mathbf{R}$ with $\kappa - \nu - \frac{1}{2} \in \mathbf{Z}_{\geq 0}$ and $\nu > -\frac{1}{2} + \varepsilon$, then we have the upper bound*

$$\frac{W_{\kappa,\nu}(y)}{\left|\Gamma\left(\frac{1}{2}+\kappa-\nu\right)\Gamma\left(\frac{1}{2}+\kappa+\nu\right)\right|^{\frac{1}{2}}} \ll_{\varepsilon} (|\kappa|+|\nu|+1)^{A}\, y^{\frac{1}{2}-\varepsilon}.$$

Here, $|\cdot|$ denotes the complex absolute value. Each of these bounds is uniform in the choice of weight κ and spectral parameter ν, with the implied constant depending only on the corresponding choice of $\varepsilon > 0$.

Proof. See [55, Proposition 3.1]; the bounds are derived via contour presentations in [55, §7]. □

Let us now return to the setting we consider throughout this work, with F a totally real number field of degree $d = [F : \mathbf{Q}]$. Given d-tuples $\kappa = (\kappa_j)_{j=1}^d \in F_\infty \cong \mathbf{R}^d$ and $\nu = (\nu_j)_{j=1}^d \in F_\infty \cong \mathbf{R}^d$, we consider the Whittaker function $W_{\kappa,\nu}(y_\infty)$ defined on $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times \cong (\mathbf{R}^\times)^d$ in the natural way via the product

$$W_{\kappa,\nu}(y_\infty) = \prod_{j=1}^{d} W_{\kappa_j,\nu_j}(|y_{\infty,j}|). \tag{48}$$

We can then derive the following immediate consequence from Proposition A.3 for the setting of totally real fields we consider here, i.e. where the notations are now altered accordingly to reflect this.

Corollary A.4. *There exists a constant $A > 0$ for which we have the following uniform bounds in the archimedean idele variable $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times \cong (\mathbf{R}^\times)^d$ with $0 < |y_{\infty,j}| < 1$ for each index $1 \leq j \leq d$.*

(i) *For all $\kappa = (\kappa_j)_{j=1}^d, \nu = (\nu_j)_{j=1}^d \in F_\infty \cong \mathbf{R}^d$, we have for any $\varepsilon > 0$ the bound*

$$\frac{W_{\kappa,\nu}(y_\infty)}{\prod_{j=1}^d \Gamma\left(\frac{1}{2}+\kappa_j+i\nu_j\right)} \ll_{\varepsilon} |y_\infty|^{\frac{1}{2}-\varepsilon} \prod_{j=1}^{d} (|\kappa_j|+|\nu_j|+1)^{A},$$

where $|y_\infty| = \prod_{j=1}^d |y_{\infty,j}|$ denotes the idele norm of $y_\infty = (y_{\infty,j})_{j=1}^d$.

(ii) *For all $\kappa = (\kappa_j)_{j=1}^d, \nu = (\nu_j)_{j=1}^d \in F_\infty \cong \mathbf{R}^d$ with $0 < \nu_j < \frac{1}{2}$ for each index $1 \leq j \leq d$, we have for any choice $\varepsilon > 0$ the bound*

$$\frac{W_{\kappa,\nu}(y)}{\prod_{j=1}^d \Gamma\left(\frac{1}{2}+\kappa_j\right)} \ll_{\varepsilon} \prod_{j=1}^{d} (|\kappa_j|+1)^{A} |y_j|^{\frac{1}{2}-\nu_j-\varepsilon}.$$

(iii) *If $\kappa = (\kappa_j)_{j=1}^d, \nu = (\nu_j)_{j=1}^d \in F_\infty \cong \mathbf{R}^d$ with $\kappa_j - \nu_j - \frac{1}{2} \in \mathbf{Z}_{\geq 0}$ and $\nu_j > -\frac{1}{2} + \varepsilon$ for each index $1 \leq j \leq d$, we have the bound*

$$\frac{W_{\kappa,\nu}(y)}{\prod_{j=1}^d \left|\Gamma\left(\frac{1}{2}+\kappa_j-\nu_j\right)\Gamma\left(\frac{1}{2}+\kappa_j+\nu_j\right)\right|^{\frac{1}{2}}} \ll_{\varepsilon} |y_\infty|^{\frac{1}{2}-\varepsilon} \prod_{j=1}^{d} (|\kappa_j|+|\nu_j|+1)^{A}.$$

Here again, $|y_\infty| = \prod_{j=1}^d |y_{\infty,j}|$ denotes the idele norm of $y_\infty = (y_{\infty,j})_{j=1}^d$.

These bounds are uniform in the weight $\kappa = (\kappa_j)_{j=1}^d$ and spectral parameter $\nu = (\nu_j)_{j=1}^d$.

Spectral decomposition of genuine metaplectic forms. We now consider the genuine automorphic form $\Phi = \phi\overline{\theta}_Q$ on $\overline{G}(\mathbf{A}_F)$ appearing in (47). We shall decompose Φ spectrally to prove Theorem 2.4. Viewing GL_2 as an algebraic group, we let $\overline{G}$ denote its two-fold metaplectic cover, as constructed via cocycles in [21]. The adelic points $\overline{G}(\mathbf{A}_F)$ fit into the exact sequence

$$1 \longrightarrow C_2 \longrightarrow \overline{G}(\mathbf{A}_F) \longrightarrow \mathrm{GL}_2(\mathbf{A}_F) \longrightarrow 1,$$

where $C_2 = \{\pm 1\}$ denotes the group of square roots of unity. We recall that an automorphic form on $\overline{G}(\mathbf{A}_F)$ which transforms nontrivially under C_2 is said to be genuine, in which case it corresponds to a classical Hilbert modular form of half-integral weight. We write $L^2(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F), \omega)$ to denote the space of such genuine automorphic forms (of central character ω), although the notation is perhaps ambiguous, i.e. as this space does not include non-genuine forms arising as liftings of $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms. This space of genuine automorphic forms on $\overline{G}(\mathbf{A}_F)$ decomposes into a Hilbert direct sum

$$L^2(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega) = L^2_{\mathrm{disc}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right) \oplus L^2_{\mathrm{cont}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$$

of a discrete spectrum $L^2_{\mathrm{disc}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ plus a continuous spectrum $L^2_{\mathrm{cont}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ spanned by analytic continuations of Eisenstein series. The discrete spectrum decomposes further into a direct sum

$$L^2_{\mathrm{disc}}(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega) = L^2_{\mathrm{cusp}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right) \oplus L^2_{\mathrm{res}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$$

of cuspidal forms $L^2_{\mathrm{cusp}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ defined by the usual vanishing condition over unipotent integrals plus residual forms $L^2_{\mathrm{res}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ which arise as residues of Eisenstein series. We note that the latter space is spanned by theta series, and more specifically translates of the metaplectic theta series θ_Q introduced above (see [21, § 6]). It is then apparent (cf. [55, § 6]) that we can find a basis $\mathcal{B}$ of $L^2(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega)$ consisting of:

- An orthonormal basis $\{f_i\}_i$ consisting of cuspidal forms $f_i \in L^2_{\mathrm{cusp}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ of respective weights $\kappa_i = (\kappa_{i,j})_{j=1}^d$ and spectral parameters $\nu_i = (\nu_{i,j})_{j=1}^d$.
- An orthonormal basis $\{\vartheta_\xi\}_\xi$ consisting of residual forms $\vartheta_\xi \in L^2_{\mathrm{res}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$ of respective weights $\kappa_\xi = (\kappa_{\xi,j})_{j=1}^d$ and spectral parameters $\nu_\xi = (\nu_{\xi,j})_{j=1}^d$.
- An orthonormal basis $\{\mathcal{E}_\varpi\}_\varpi$ consisting of (contour integrals of) Eisenstein series ϖ of respective weights $\kappa_\varpi = (\kappa_{\varpi,j})_{j=1}^d$ and spectral parameters $\nu_{s,\varpi} = (\nu_{s,\varpi,j})_{j=1}^d$ for $L^2_{\mathrm{cont}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$,

$$L^2_{\mathrm{cont}}\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right) = \int_{\Re(s)=1/2} \bigoplus_\varpi \mathcal{E}_\varpi(*,s)\frac{ds}{2\pi i}.$$

Decomposing $\Phi = \phi\overline{\theta}_Q$ in terms of such a basis $\mathcal{B}$, we obtain for any $\overline{g} = (g,\zeta) \in \overline{G}(\mathbf{A}_F)$ the decomposition

$$\Phi(\overline{g}) = \sum_i \langle \Phi, f_i\rangle \cdot f_i(\overline{g}) + \sum_\xi \langle \Phi, \vartheta_\xi\rangle \cdot \vartheta_\xi(\overline{g}) + \sum_\varpi \int_{\Re(s)=1/2} \langle \Phi, \mathcal{E}_\varpi(*,s)\rangle \cdot \mathcal{E}_\varpi(\overline{g},s)\frac{ds}{2\pi i}. \tag{49}$$

Here, $\langle \cdot,\cdot\rangle$ denotes the inner product on $L^2\left(\mathrm{GL}_2(F)\backslash\overline{G}(\mathbf{A}_F),\omega\right)$. To see that the sums over coefficients $\mathfrak{K}_i = \langle \Phi, f_i\rangle$, $\mathfrak{K}_\xi = \langle \Phi, \vartheta_\xi\rangle$, and $\mathfrak{K}_\varpi = \langle \Phi, \mathcal{E}_\varpi(*,s)\rangle$ appearing in (49) are bounded in a suitable way, we shall use the fact that Φ has convergent Sobolev norm. To be more precise, let us first recall the definition of the Sobolev norm on any sufficiently smooth L^2-automorphic form ϕ on $\mathrm{GL}_2(\mathbf{A}_F)$, i.e. on any sufficiently smooth automorphic form $\phi \in L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F),\omega)$. We refer to [6, §2.10] (cf. [39], [55, § 6]) for more background. In short, the action of of $\mathrm{GL}_2(F_\infty)$ on $L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F),\omega)$ induces an action of its Lie algebra $\mathfrak{gl}_2(F_\infty)$ on this space, and hence an action of its Lie subalgebra $\mathfrak{g} = \mathfrak{sl}_2(F_\infty)$ on this space. Writing $e_j = (0,\cdots,0,1,0,\cdots,0)$ with 1 at position j for each index $1 \leq j \leq d$, this latter action is generated by the linearly independent vectors

$$H_j = \begin{pmatrix} e_j & 0 \\ 0 & e_j \end{pmatrix} \qquad R_j = \begin{pmatrix} 0 & e_j \\ 0 & 0 \end{pmatrix} \qquad L_j = \begin{pmatrix} 0 & 0 \\ e_j & 0 \end{pmatrix}.$$

We also have the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ on $L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F),\omega)$ via differential operators. Writing a generic element $k(\vartheta) \in \mathrm{SO}_2(F_\infty)$ for $\vartheta = (\vartheta_j)_{j=1}^d \in (\mathbf{R}/\mathbf{Z})^d$ as

$$k(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in \mathrm{SO}_2(F_\infty),$$

the operators corresponding to the basis elements are given explicitly by

$$\begin{aligned} dH_j &= -2y_j \sin(2\vartheta_j)\partial_{x_j} + 2y_j\cos(2\vartheta_j)\partial_{y_j} + \sin(2\vartheta_j)\partial_{\vartheta_j} \\ dR_j &= y_j\cos(2\vartheta_j)\partial_{x_j} + y_j \sin(2\vartheta_j)\partial_{y_j} + \sin^2(\vartheta_j)\partial_{\vartheta_j} \\ dL_j &= y_j\cos(2\vartheta_j)\partial_{x_j} + y_j \sin(2\vartheta_j)\partial_{y_j} - \cos^2(\vartheta_j)\partial_{\vartheta_j}. \end{aligned}$$

As explained [6, §2.10], we know that the action of any $\mathcal{D} \in \mathcal{U}(\mathfrak{g})$ commutes with the corresponding spectral decomposition of ϕ. One can extend this discussion in a natural way to the space of genuine metaplectic forms $L^2(\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F), \omega)$ (see [55, §6.2] and more generally [39]). In particular, there is a corresponding action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ on $L^2(\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F), \omega)$. This action commutes with the spectral decomposition (49) of $\Phi \in L^2(\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F), \omega)$ in the sense that for any $\mathcal{D} \in \mathcal{U}(\mathfrak{g})$, we have the relation

$$(50) \qquad ||\mathcal{D}\Phi||^2 = \sum_i \langle \Phi, f_i\rangle^2 \cdot ||\mathcal{D}\Phi||^2 + \sum_\xi \langle \Phi, \vartheta_\xi\rangle^2 \cdot ||\mathcal{D}\vartheta_\xi||^2 + \int_{\Re(s)=1/2} \sum_\varpi \langle \Phi, \mathcal{E}_\varpi(*,s)\rangle^2 \cdot ||\mathcal{D}\mathcal{E}_\varpi||^2 \frac{ds}{2\pi i}.$$

Such a relation of course holds for any $\Phi \in L^2(\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F), \omega)$, and it is then natural for any choice of integer $B \geq 0$ to define the corresponding Sobolev norm

$$||\Phi||_{\mathcal{S}^B} = \sum_{\mathrm{ord}(\mathcal{D}) \leq B} ||\mathcal{D}\Phi||^2,$$

where the sum runs over all monomials in H_{j_1}, R_{j_2}, and L_{j_3} of order at most B. It is well-known that any smooth $\Phi \in L^2(\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F), \omega)$ is convergent in this Sobolev norm in the following sense.

Lemma A.5. *Given ϕ any smooth automorphic form in $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F)$ or more generally $\mathrm{GL}_2(F)\backslash \overline{G}(\mathbf{A}_F)$ of a given central character (not necessarily $\mathcal{K}$-finite) and $B \geq 0$ any integer, we have the Sobolev norm bound $||\phi||_{\mathcal{S}^B} \ll 1$. Here, the implied constant depends only on the integer $B \geq 1$.*

Proof. See [55, Lemma 6.1] or more generally [39, § 2.4]. □

Using this, we can deduce from (50) that the sums of coefficients in the spectral expansion (49) are bounded suitably in terms of the corresponding spectral parameters. We refer to the discussions in [55, § 5-6] and [6, § 2.10] for more details. This convergence allows us to proceed with the proof of Theorem 2.4 via the spectral decomposition (49) of the genuine metaplectic form $\Phi = \phi\overline{\theta}_Q$ as follows.

Proof of Theorem 2.4. Recall that via the integral presentation (47) above, it will suffice to bound the Fourier-Whittaker coefficient at a give nonzero F-integer q of the genuine metaplectic form $\Phi = \phi\overline{\theta}_Q$. To be more precise, it will do to consider only the spectral decomposition of the coefficient at q of Φ, which we can expand out via the spectral decomposition (49) of Φ at the elements

$$\overline{g} = (g, \zeta) = \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix}, 1 \right) \in \overline{G}(F_\infty).$$

Suppressing the metaplectic variable $\zeta \in C_2 = \{\pm 1\}$ to lighten notation, we derive the presentation

$$\begin{aligned} &\sum_{\gamma \in F^\times} \frac{\lambda(\gamma^2+q)}{|\gamma^2+q|^{\frac{1}{2}}} W\left(\frac{\gamma^2+q}{Y_\infty}\right) \\ &= |Y_\infty|^{\frac{1}{4}} \int_{I \cong [0,1]^d \subset F_\infty} \Phi\left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty \\ &= |Y_\infty|^{\frac{1}{4}} \sum_i \langle \Phi, f_i \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} f_i \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty \\ &+ |Y_\infty|^{\frac{1}{4}} \sum_\xi \langle \Phi, \vartheta_\xi \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} \vartheta_\xi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty \\ &+ |Y_\infty|^{\frac{1}{4}} \sum_\varpi \int_{\Re(s)=1/2} \langle \Phi, \mathcal{E}_\varpi(*,s) \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} \mathcal{E}_\varpi \left(\left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right), s \right) \psi(-qx_\infty) dx_\infty \frac{ds}{2\pi i}. \end{aligned}$$

Note that the contributions of the spectral coefficients in this expression are bounded via the convergence of Φ in the Sobolev norm (cf. (50) with Lemma A.5). We can now follow essentially the same argument as given in [55, § 6.6-6.8] to derive the stated estimate of Theorem 2.4 for the shifted convolution sum on the left hand side of this expression. To be more precise, let us first consider the second integral over residual terms $\{\vartheta_\xi\}_\xi$ in this spectral expansion, which after writing c_{ϑ_ξ} to denote the coefficients in the Dirichlet series of each Mellin transform (L-function) $L(s, \vartheta_\xi)$ can be expressed equivalently as

$$\begin{aligned}&|Y_\infty|^{\frac{1}{4}} \sum_\xi \langle \Phi, \vartheta_\xi \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} \vartheta_\xi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty \\ &= |Y_\infty|^{\frac{1}{4}} \sum_\xi \langle \phi \overline{\theta}_Q, \vartheta_\xi \rangle \cdot \frac{c_{\vartheta_\chi}(q)}{|q|^{\frac{1}{2}}} W_{\vartheta_\xi} \left(\frac{q}{Y_\infty} \right).\end{aligned}$$

Hence (cf. [55, §6.8]), we can derive the crude estimate

$$|Y_\infty|^{\frac{1}{4}} \sum_\xi \langle \Phi, \vartheta_\xi \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} \vartheta_\xi \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty = I(W) M_{\pi,q}$$

for some linear functional $I(W)$ in the chosen weight function W, where $M_{\pi,q} \geq 0$ is a constant depending only on our initial cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$ and the chosen F-integer q. More precisely, we see from inspection of the inner products $\langle \phi \overline{\theta}_Q, \vartheta_\xi \rangle$ that $M_{\pi,q}$ vanishes unless π is dihedral and q totally positive, i.e. since the theory of the Shimura integral (see e.g. [55, § 4.7]) implies that the inner product vanishes unless the completed symmetric square L-function $\Lambda(s, \mathrm{Sym}^2 \pi)$ has a pole at $s = 1$, and since the coefficients in the Fourier-Whittaker expansion of ϑ_ξ are only supported on totally positive F-integers. Since we assume throughout that the representation π is non-dihedral, we deduce that $M_{\pi,q} = 0$, and hence that this term does not contribute to the estimate. To bound the remaining contributions, we first argue that it will suffice to consider the cuspidal spectrum, as the contribution of the continuous spectrum can be estimated via a minor variation of the same argument (cf. [6], [55, § 6]). To estimate the cuspidal contribution, we again write the L-function coefficients of each cusp form f_i as c_{f_i}, so that

$$\begin{aligned}&|Y_\infty|^{\frac{1}{4}} \sum_i \langle \Phi, f_i \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} f_i \left(\begin{pmatrix} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{pmatrix} \right) \psi(-qx_\infty) dx_\infty \\ &= |Y_\infty|^{\frac{1}{4}} \sum_i \langle \phi \overline{\theta}_Q, f_i \rangle \cdot \frac{c_{f_i}(q)}{|q|^{\frac{1}{2}}} W_{f_i} \left(\frac{q}{Y_\infty} \right).\end{aligned} \tag{51}$$

Let us now write $0 < \delta_0 \leq 1/2$ to denote the best exponent approximation for the Fourier coefficients of genuine metaplectic forms, i.e. so that $c_{f_i}(q) \ll_\varepsilon |q|^{\delta_0 + \varepsilon}$ for any $\varepsilon > 0$ and nonzero F-integer q. Note that by the theorem of Kohnen-Zagier [33] and more generally Baruch-Mao [2], this exponent δ_0 is seen to be equivalent to the best exponent approximation the generalized Lindelöf hypothesis for $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic forms, as mentioned above. Using Corollary A.4 to bound the contribution of each archimedean Whittaker function $W_{f_i}(y_\infty) = W_{\kappa_i, \nu_i}(y_\infty)$ of $y_\infty \in F_\infty^\times$, we deduce that (51) can be bounded above by the quantity

$$\begin{aligned}&\ll_\varepsilon |Y_\infty|^{\frac{1}{2}} \cdot \sum_i \langle \phi \overline{\theta}_Q, f_i \rangle \cdot |q|^{\delta_0 + \varepsilon - \frac{1}{2}} \cdot \left| \frac{q}{Y_\infty} \right|^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} \cdot \prod_{j=1}^d \left(|\kappa_{i,j}| + |\nu_{i,j}| + 1 \right)^A \cdot ||f_i|| \\ &= |Y_\infty|^{\frac{1}{4}} \cdot \sum_i \langle \phi \overline{\theta}_Q, f_i \rangle \cdot |q|^{\delta_0 + \varepsilon - \frac{1}{2}} \cdot \left| \frac{q}{Y_\infty} \right|^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon} \cdot \prod_{j=1}^d \left(|\kappa_{i,j}| + |\nu_{i,j}| + 1 \right)^A\end{aligned}$$

for any choice of $\varepsilon > 0$. Here, we use that $||f_i|| = 1$ for each i (by our choice of orthonormal basis), and that the spectral parameter ν can be approximated by $\theta_0/2$ (cf. [55, Proposition 3.1 and Lemma 6.2]). We also choose the archimedean idele representative $Y_\infty \in F_\infty^\times$ in a suitable way so that the bounds of Corollary A.4 can be applied. Now, using the Sobolev norm convergence (Lemma A.5) and more precisely the Plancherel formula with itererated applications of the generalized Laplacian operator to deduce that

$$\sum_i \langle \phi \overline{\theta}_Q, f_i \rangle^2 \cdot \prod_{j=1}^d (|\kappa_{i,j}| + |\nu_{i,j}| + 1)^{2A} \ll ||\phi \overline{\theta}_Q||^2 \ll_{2A} 1,$$

we can then apply the Cauchy-Schwarz inequality to the quantity in the previous bound to deduce that

$$|Y_\infty|^{\frac{1}{4}} \sum_i \langle \Phi, f_i \rangle \cdot \int_{I \cong [0,1]^d \subset F_\infty} f_i \left(\left(\begin{array}{cc} \frac{1}{Y_\infty} & x_\infty \\ & 1 \end{array} \right) \right) \psi(-qx_\infty) dx_\infty \ll_{\varepsilon, 2A} |Y_\infty|^{\frac{1}{4}} \cdot |q|^{\delta_0 - \frac{1}{2}} \cdot \left| \frac{q}{Y_\infty} \right|^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

The contributions from the continuous spectrum are bounded in the same way, using a variation of the argument given above to bound spectral coefficients (via adjoint properties of the inner product). The claimed estimate of Theorem 2.4 follows. □

Proof of Theorem 2.5. We now give the following variation of the proof of Theorem 2.4 to show Theorem 2.5. Let us take for granted the setup described for the statement of Theorem 2.5. To begin, fix $W \in L^2(F_\infty^\times)$ any smooth weight function as in the statement, i.e. after composing with the norm $|\cdot| = \mathbf{N} : F_\infty^\times \to \mathbf{R}_{>0}$. Let $\phi = \otimes_v \phi_v \in V_\pi$ be any pure tensor whose nonarchimedean local components are each essential Whittaker vectors, and whose archimedean local components are chosen so that as functions of $y_\infty \in F_\infty^\times \cong (\mathbf{R}^\times)^d$,

$$W_\phi(y_\infty) := W_\phi \left(\left(\begin{array}{cc} y_\infty & \\ & 1 \end{array} \right) \right) = W(y_\infty).$$

It is then easy to deduce the properties of the Fourier-Whittaker expansion of ϕ outlined above that

$$\sum_{\gamma \in F^\times} W_\phi \left(\left(\begin{array}{cc} \frac{q(\gamma)}{Y_\infty} & \\ & 1 \end{array} \right) \right) = \sum_{\gamma \in F^\times} \frac{\lambda_\pi(q(\gamma))}{|\gamma|^{\frac{1}{2}}} W \left(\frac{q(\gamma)}{Y_\infty} \right). \tag{52}$$

Here, we use all of the same conventions as above, taking $Y_\infty \in F_\infty^\times$ to be a fixed, totally positive archimedean idele representative of norm $|Y_\infty| = Y$. Note that the sum (52) here is supported only on F-integers $\gamma = n \in \mathcal{O}_F$. We shall henceforth write (n) to denote the principal ideal generated by such an F-integer to simplify notations.

Let us also say a few words about how to choose this vector $\phi \in V_\pi$ with $W_\varphi(y_\infty) = W(y_\infty)$ explicitly. First, let us recall that for φ a cuspidal automorphic form on $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $k = (k_j)_{j=1}^d$ and spectral parameter $\nu = (\nu_j)_{j=1}^d$, we can describe the Fourier-Whittaker expansion of φ more explicitly in terms of the Whittaker functions defined in (48) above as follows. Following the discussion in [6, §2.4] (cf. [7]), let us define the normalized Whittaker functions

$$\tilde{W}_{\frac{k}{2}, \nu}(y_\infty) = \prod_{j=1}^d \tilde{W}_{\frac{k_j}{2}, \nu_j}(y_{\infty, j})$$

on $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times \cong (\mathbf{R}^\times)$ by taking each component to be

$$\widetilde{W}_{\frac{k_j}{2}, \nu_j}(y_{\infty, j}) = \frac{i^{\mathrm{sgn}(y_{\infty,j}) \frac{k_j}{2}} \cdot W_{\mathrm{sgn}(y_{\infty,j}) \frac{k_j}{2}, \nu_j}(4\pi |y_{\infty,j}|)}{\left\{ \Gamma \left(\frac{1}{2} - \nu_j + \mathrm{sgn}(y_{\infty,j}) \frac{k_j}{2} \right) \Gamma \left(\frac{1}{2} + \nu_j + \mathrm{sgn}(y_{\infty,j}) \frac{k_j}{2} \right) \right\}^{\frac{1}{2}}}.$$

As explained in [6, § 2.4] and [7, § 4], fixing a parity $k_j \equiv \varepsilon \bmod 2$, these functions $\widetilde{W}_{\frac{k_j}{2}, \nu_j}$ form an orthonormal basis of the Hilbert space $L^2(\mathbf{R}^\times, dy^\times)$ in the sense that

$$L^2(\mathbf{R}^\times, dy^\times) = \bigoplus_{\substack{k_j \in \mathbf{Z} \\ k_j \equiv \varepsilon \bmod 2}} \mathbf{C} \tilde{W}_{\frac{k_j}{2}, \nu_j}, \quad \left\langle \tilde{W}_{\frac{k_j}{2}, \nu_j}, \tilde{W}_{\frac{k_j'}{2}, \nu_j} \right\rangle = \delta_{k_j, k_j'}.$$

In this way, we deduce that the normalized Whittaker functions $\tilde{W}_{\frac{k}{2}, \nu}(y_\infty)$ defined on $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times$ furnish an orthonormal basis for the corresponding Hilbert space $L^2(F_\infty^\times, dy_\infty^\times)$. Let us now consider a pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ whose nonarchimedean local components are each essential Whittaker vectors, and whose archimedean component has weight $k = (k_j)_{j=1}^d$ and spectral parameter $\nu = (\nu_j)_{j=1}^d$. As explained in

[6, § 2.5] (using our notations defined above), we then know that for some character $\varepsilon_\pi : \{\pm 1\}^d \longrightarrow \{\pm 1\}$ depending only on π, this φ has for $y = y_f \cdot y_\infty \in \mathbf{A}_F^\times$ and $x \in \mathbf{A}_F$ the more explicit Fourier expansion

$$\varphi\left(\left(\begin{array}{cc} y & x \\ & 1 \end{array}\right)\right) = \sum_{\gamma \in F^\times} \frac{\rho_\varphi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} W_\varphi(\gamma y_\infty)\psi(\gamma x) = \sum_{\gamma \in F^\times} \frac{\lambda_\pi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \varepsilon_\pi(\mathrm{sgn}(\gamma y_\infty)) \tilde{W}_{\frac{k}{2},\nu}(\gamma y_\infty)\psi(\gamma x). \tag{53}$$

Now, we can apply Maass weight-raising operators as follows. Let us for a given weight $k = (k_j)_{j=1}^d$ consider the corresponding operator $R_k = (R_{k_j})_{j=1}^d$ defined componentwise on $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times = (\mathbf{R}^\times)^d$ by

$$R_{k_j} := i \cdot y_{\infty,j} \cdot \frac{\partial}{\partial x_{\infty,j}} + y_{\infty,j} \cdot \frac{\partial}{\partial y_{\infty,j}} + \frac{k_j}{2}.$$

This R_{k_j} is none other than the classical Maass weight-raising operator of weight k_j, which transforms a form of weight k_j to a form of weight $k_j + 2$. It is also well-known that for any integer n,

$$R_{k_j}\left(W_{\frac{k_j}{2},\nu_j}(4\pi|y_{\infty,j}|) \cdot e(nx_{\infty,j})\right) = c_{\frac{k_j}{2},\nu_j} \cdot W_{\frac{k_j+2}{2},\nu_j}(4\pi|y_{\infty,j}|) \cdot e(nx_{\infty,j}),$$

where

$$c_{\frac{k_j}{2},\nu_j} = \begin{cases} -1 & \text{if } n > 0 \\ -\left(\nu_j^2 - \left(\frac{k_j+1}{2}\right)^2\right) & \text{if } n < 0. \end{cases}$$

Putting $c_{\frac{k}{2},\nu} = \prod_{j=1}^d c_{\frac{k_j}{2},\nu_j}$, and using the fact that these operators do not affect the nonarchimedean Whittaker coefficients, we deduce that

$$R_k\varphi\left(\left(\begin{array}{cc} y & x \\ & 1 \end{array}\right)\right) = \sum_{\gamma \in F^\times} \frac{\lambda_\pi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot \varepsilon_\pi(\mathrm{sgn}(\gamma y_\infty)) \cdot c_{\frac{k}{2},\nu} \cdot \tilde{W}_{\frac{k+2}{2},\nu}(\gamma y_\infty) \cdot \psi(\gamma x). \tag{54}$$

Here, we write $k+2$ to denote the vector $(k_j+2)_{j=1}^d$. Let us also note that if φ arises as the lift to $\mathrm{GL}_2(\mathbf{A}_F)$ of a holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$, then the normalized Whittaker functions that appear in the expansion (53) are derived from the classical Whittaker functions of the simpler form

$$W_{\frac{k}{2},\frac{k+1}{2}}(y_\infty) = \prod_{j=1}^d W_{\frac{k_j}{2},\frac{k_j-1}{2}}(y_{\infty,j}), \quad W_{\frac{k_j}{2},\frac{k_j-1}{2}}(y_{\infty,j}) := y_{\infty,j}^{\frac{k_j}{2}} \cdot e^{-\frac{y_{\infty,j}}{2}}.$$

Now finally, for a given sequence of d-tuples of complex numbers

$$\{\mathfrak{K}_{k+2i}\}_{i\geq 0} = \left\{(\mathfrak{K}_{k_j+2i})_{j=1}^d\right\}_{i\geq 0}, \quad \mathfrak{K}_{k_j+2i} \in \mathbf{C}\ \forall\ 1 \leq j \leq d, i \geq 0,$$

let us define the corresponding d-tuple of linear combinations of Maass weight-raising operators

$$\mathcal{R}_k = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}_{i\geq 0}\right) = \sum_{i\geq 0} \mathfrak{K}_{k+2i} \cdot R_{k+2i} = \left(\sum_{i\geq 0} \mathfrak{K}_{k_j+2i} R_{k_j+2i}\right)_{j=1}^d.$$

Applying such an operator $\mathcal{R}_k = \mathcal{R}_k(\{\mathfrak{K}_{k+2i}\}_{i\geq 0})$ to φ via the expansion (53) as in (54), we obtain

$$\mathcal{R}_k\varphi\left(\left(\begin{array}{cc} y & x \\ & 1 \end{array}\right)\right) = \sum_{\gamma \in F^\times} \frac{\lambda_\pi(\gamma y_f)}{|\gamma y_f|^{\frac{1}{2}}} \cdot \varepsilon_\pi(\mathrm{sgn}(\gamma y_\infty)) \cdot \mathcal{R}_k \tilde{W}_{\frac{k}{2},\nu}(\gamma y_\infty)\psi(\gamma x). \tag{55}$$

Our choice of pure tensor ϕ above with archimedean Whittaker coefficient $W_\phi(y_\infty) = W(y_\infty)$ given by the fixed weight function W then corresponds to taking $\phi = \mathcal{R}_k\varphi = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}_{i\geq 0}\right)\varphi$ for some φ of weight k as above, and insisting that the sequence of vectors of complex coefficients $\{\mathfrak{K}_{k+2i}\}_{i\geq 0}$ is chosen so that

$$\mathcal{R}_k \tilde{W}_{\frac{k}{2},\nu}(y_\infty) = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}_{i\geq 0}\right)\left(\tilde{W}_{\frac{k_j}{2},\nu_j}(y_\infty)\right) = W(y_\infty) \tag{56}$$

as functions of $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times \cong (\mathbf{R}^\times)^d$. Note that this procedure is equivalent to decomposing the chosen weight function $W \in L^2(F_\infty, dy_\infty^\times)$ into an orthonormal basis of normalized Whittaker functions $\tilde{W}_{\frac{k}{2},\nu}$ according to the discussion above, then giving a name to the corresponding vector in V_π. In particular, observe that by our standing assumptions that the chosen weight function W decays rapidly as $|y_\infty| \to \infty$

and moderately as $|y_\infty| \to 0$, or even that W is compactly supported, we can assume that the expansions (55) and (56) into infinite linear combinations of Maass weight-raising operators are absolutely convergent. We shall henceforth take for granted that this vector $\phi = \mathcal{R}_k \varphi \in V_\pi$ is constructed via such a convergent infinite linear combination from a form φ of weight $k = (k_j)_{j=1}^d$, and later take this φ to be the lift to $\mathrm{GL}_2(\mathbf{A}_F)$ of a holomorphic Hilbert modular form.

Our strategy is to decompose this chosen pure tensor ϕ into a linear combination of Poincaré series on $\mathrm{GL}_2(\mathbf{A}_F)$ in the style of Blomer [4, §3], then apply Poisson summation to identify the coefficients in the sum with the Fourier-Whittaker coefficients of certain genuine Poincaré series on the metaplectic cover $\overline{G}(\mathbf{A}_F)$. Making such an identification, we can then decompose these genuine forms spectrally according to the discussion given above, passing to unipotent integrals describing the coefficients rather than invoking any Kuznetsov trace formula directly. In fact, this procedure allows us to give a "soft" generalization of the argument of [4], choosing suitable vectors in the Kirillov model and Schwartz functions parametrizing Poincaré series to reduce to a variation of the proof given for Theorem 2.4 above.

To make this argument rigorous, we first need to recall some more background about Poincaré series in this generality. Let us again write $N_2 \subset \mathrm{GL}_2$ to denote the subgroup of upper-triangular unipotent matrices, and consider the space $\mathcal{S}(N_2(\mathbf{A}_F)\backslash\overline{G}(\mathbf{A}_F);\psi)$ of smooth functions f on $\overline{G}(\mathbf{A})$ whose action by $N_2(\mathbf{A}_F)$ is given by a chosen additive character ψ of $\mathbf{A}_F/F$ (extended in the natural way to $N_2(F)\backslash N_2(\mathbf{A})$), i.e. so that $f(n\overline{g}) = \psi(n) f(\overline{g})$ for all $n \in N_2(\mathbf{A}_F)$ and $\overline{g} = (g,\zeta) \in \overline{G}(\mathbf{A}_F)$ with $g \in \mathrm{GL}_2(\mathbf{A}_F)$ and $\zeta \in Z_2$. Fix $\Gamma \subset \mathrm{GL}_2(\mathbf{A}_F)$ a discrete, finite co-volume subgroup. Let $\Gamma_\infty \cong N_2(F)$ denote the subgroup of Γ stabilized by the cusp at infinity. Let us also fix an idele class character ω of $\mathbf{A}_F$, viewed as a character of $\gamma \in \Gamma$ in the usual way via evaluation at the lower left entry. We also fix a multiplier system $\vartheta : \mathrm{GL}_2 \to \mathbf{C}$, as described in [47] and [21, §2] for each of the real places. We then consider the Poincaré series $P_{f,\omega,\vartheta}$ defined on $\overline{g} = (g,\zeta) \in \overline{G}(\mathbf{A}_F)$ by

$$P_{f,\omega,\vartheta}(\overline{g}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega(\gamma) \cdot \overline{\vartheta(\gamma)} \cdot f(\gamma \overline{g}). \tag{57}$$

This Poincaré series $P_{f,\omega,\vartheta}$ is absolutely convergent, uniformly on compact subsets, and determines a smooth L^2-automorphic on $\overline{G}(\mathbf{A}_F)$ (see e.g. [13]). Note that by restricting to the component $\mathrm{GL}_2(\mathbf{A}_F)$ of $\overline{G}(\mathbf{A}_F)$ we obtain a natural subspace inclusion $\mathcal{S}(N_2(\mathbf{A}_F)\backslash \mathrm{GL}_2(\mathbf{A}_F);\psi) \subseteq \mathcal{S}(N_2(\mathbf{A}_F)\backslash \mathrm{GL}_2(\mathbf{A}_F);\psi)$, and hence recover the corresponding construction of Poincaré series on $\mathrm{GL}_2(\mathbf{A}_F)$ (taking the multiplier term to be trivial). We can also compute Fourier-Whittaker expansions as follows, noting that our setup here is simpler than the case of general number fields, as we consider only sums over principal ideals in (52). Following [13], let us consider the set of F-rational numbers determined by the set

$$\Omega(\Gamma) = \left\{ c \in F_\infty^\times : N_2(F_\infty) \cdot w \cdot \underline{c} \cdot N_2(F_\infty) \cap \Gamma \neq \emptyset \right\},$$

where we use the shorthand notations

$$\underline{c} := \begin{pmatrix} c & \\ & 1 \end{pmatrix} \quad \text{and} \quad w := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Given $c \in \Omega(\Gamma)$, we then consider the subgroup $\Gamma_c \subset \Gamma$ defined by

$$\Gamma_c := N_2(F_\infty) \cdot w \cdot \underline{c} \cdot N_2(F_\infty) \cap \Gamma,$$

which is left and right invariant by Γ_∞. We can also decompose each $\gamma \in \Gamma_c$ according to the Bruhat decomposition as $\gamma = n_1(\gamma) \cdot w \cdot \underline{c} \cdot n_2(\gamma)$ with $n_j(\gamma) \in N_2(F_\infty)$ for each of $j = 1, 2$. More explicitly, we have

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_c \quad \Longrightarrow \quad n_1(\gamma) = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix} \quad \text{and} \quad n_2(\gamma) = \begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix},$$

which can be deduced from the elementary matrix decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} c & \\ & c^{-1}(ad-bc) \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ & 1 \end{pmatrix}.$$

Proposition A.6. *Given nontrivial additive characters ψ_1, ψ_2 of $\mathbf{A}_F/F$, the Fourier-Whittaker coefficient*

$$W_{P_{f,\omega,\vartheta}}(\overline{g}) = W_{P_{f,\omega,\vartheta},\psi_2}(\overline{g}) := \int_{N_2(F)\backslash N_2(\mathbf{A}_F)} P_{f,\omega,\vartheta}\left(n\overline{g}\right) \psi_2^{-1}(n) dn$$

with respect to ψ_2 of the Poincaré series $P_{f,\omega,\vartheta}$ constructed from some $f \in \mathcal{S}(N_2(\mathbf{A}_F)\backslash\overline{G}(\mathbf{A}_F);\psi_1)$ is given by the following formula: For any $\overline{g} = (g,\zeta) \in \overline{G}(\mathbf{A}_F)$, we have that

$$W_{P_{f,\omega,\vartheta}}(\overline{g}) = \int_{\mathbf{A}_F/F} f(\overline{g})\psi_1(x)\psi_2(-x)dx$$
$$+ \sum_{c\in\Omega(\Gamma)} \left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot\psi_1(n_1(\gamma))\cdot\psi_2(n_2(\gamma)) \right) \int_{N_2(\mathbf{A}_F)} f(w\cdot\underline{c}\cdot n\cdot\overline{g})\psi_2^{-1}(n)dn.$$

In particular, fixing ψ to be the standard additive character, and then taking $\psi_1(x) = \psi_\infty(mx)$ and $\psi_2(x) = \psi_\infty(rx)$ for nonzero F-integers $m,r\in\mathcal{O}_F$, with $\vartheta(\gamma) = \left(\frac{c}{d}\right)^*$ as described (for each component) in [21, §2] *and* [47]*, we arrive at the simpler expression*

$$W_{P_{f,\omega,\vartheta}}(\overline{g}) = \int_{\mathbf{A}_F/F} f(\overline{g})\psi_\infty(mx-rx)dx + \sum_{c\in\Omega(\Gamma)} \mathrm{Kl}_{\Gamma,\omega,\vartheta}(m,r;c)\cdot\mathcal{F}_{f,r,c}(\overline{g})$$

where $\mathrm{Kl}_{\Gamma,\omega,\vartheta}(m,r;c)$ denotes the Kloosterman sum defined by

$$\mathrm{Kl}_{\Gamma,\omega,\vartheta}(m,r;c) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot\psi_\infty(mn_1(\gamma)+rn_2(\gamma))$$
$$= \sum_{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \omega(d)\left(\frac{c}{d}\right)^*\psi_\infty\left(\frac{ma+rd}{c}\right),$$

and $\mathcal{F}_{f,,r,c}$ the intertwining operator defined by

$$\mathcal{F}_{f,r,c}(\overline{g}) = \int_{N_2(\mathbf{A}_F)} f(w\cdot\underline{c}\cdot n\cdot\overline{g})\psi_2^{-1}(n)dn = \int_{\mathbf{A}_F} f\left(w\underline{c}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\overline{g}\right)\psi_\infty(-rx)dx.$$

That is, the Fourier-Whittaker coefficient of $P_{f,\omega,\vartheta}(\overline{g})$ at r (with respect to the standard additive character) is described in this way.

Proof. Cf. [13, Proposition 2.5]; the same calculation works here. Given $\overline{g} = (g,\zeta)\in\overline{G}(\mathbf{A}_F)$, ω, u, and $f\in S(N_2(\mathbf{A}_F)\backslash\overline{G}(\mathbf{A}_F);\psi_1)$ as above, we compute

$$W_{P_{f,\omega,\vartheta},\psi_2}(\overline{g}) = \int_{N_2(F)\backslash N_2(\mathbf{A}_F)} P_{f,\omega,\vartheta}(n\overline{g})\psi_2^{-1}(n)dn = \int_{\mathbf{A}_F/F} P_{f,\omega,\vartheta}\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\overline{g}\right)\psi_2(-x)dx$$
$$= \int_{\mathbf{A}_F/F}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot f\left(\gamma\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\overline{g}\right)\cdot\psi_2(-x)dx,$$

which after using the Bruhat decomposition expansion $\Gamma = \Gamma_\infty\cup\bigcup_{c\in\Omega(\Gamma)}\Gamma_c$ (see [13, § 2]) equals

$$\int_{\mathbf{A}_F/F} f\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\overline{g}\right)\psi_2(-x)dx + \int_{\mathbf{A}_F/F}\sum_{c\in\Omega(\Gamma)}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c}\omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot f\left(\gamma\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\overline{g}\right)\cdot\psi_2(-x)dx.$$

Here, it is easy to see from the fact that $f\in\mathcal{S}(N_2(\mathbf{A}_F)\backslash\overline{G}(\mathbf{A});\psi_1)$ that the first integral equals

$$\int_{\mathbf{A}_F/F} f(\overline{g})\,\psi_1(x)\psi_2(-x)dx.$$

To compute the second integral, we use the identifications $\mathbf{A}_F/F \cong N_2(F)\backslash N_2(\mathbf{A}_F) \cong \Gamma_\infty\backslash N_2(\mathbf{A}_F)$ to find

$$\int_{\mathbf{A}_F/F} \sum_{c\in\Omega(\Gamma)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c} \omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot f\left(\gamma\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\overline{g}\right)\cdot\psi_2(-x)dx$$
$$= \sum_{c\in\Omega(\Gamma)} \int_{\Gamma_\infty\backslash N_2(\mathbf{A}_F)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c} \omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot f(\gamma n\overline{g})\cdot\psi_2^{-1}(n)dn$$
$$= \sum_{c\in\Omega(\Gamma)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \int_{N_2(\mathbf{A}_F)} f(\gamma n\overline{g})\psi_2^{-1}(n)dn,$$

which after decomposing each $\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty$ in Bruhat form as $\gamma = n_1(\gamma)\cdot w\cdot\underline{c}\cdot n_2(\gamma)$ and making a change of variables $n\to n_2(\gamma)^{-1}\cdot n$ is the same as the stated integral

$$\sum_{c\in\Omega(\Gamma)} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty} \omega(\gamma)\cdot\overline{\vartheta(\gamma)}\cdot\psi_1(n_1(\gamma))\cdot\psi_2(n_2(\gamma))\int_{N_2(\mathbf{A}_F)} f(w\cdot\underline{c}\cdot n\cdot\overline{g})\psi_2^{-1}(n)dn.$$

□

Let us now consider the latter Poincaré series above, i.e. for a given nonzero F-integer $m\in\mathcal{O}_F$ fix $\psi_1(x) = \psi_\infty(mx)$, and fix a suitable rapidly decaying Schwartz function $f\in\mathcal{S}(N_2(\mathbf{A}_F)\backslash\operatorname{GL}_2(\mathbf{A}_F);\psi_1)$. We then consider the corresponding smooth Poincaré series P_m on $g\in\operatorname{GL}_2(\mathbf{A}_F)$ defined by

$$P_m(g) := P_f(g) = \sum_{\Gamma_\infty\backslash\Gamma} f(\gamma g).$$

Note that this standard notation keeps track of the shift of the standard archimedean additive character ψ_∞ by the nonzero F-integer m (cf. e.g. [29, § 14.2]). However, it suppresses the way in which this dependence is carried through the choice of Schwartz function $f\in\mathcal{S}(N_2(\mathbf{A}_F)\backslash\operatorname{GL}_2(\mathbf{A}_F);\psi_1)$, with ψ_1 the nontrivial additive character defined on $x = x_f\cdot x_\infty\in\mathbf{A}_F$ by

$$\psi_1(x) := \psi_\infty(mx) = \exp\left(2\pi i m\left(x_{\infty,1}+\ldots+x_{\infty,d}\right)\right).$$

We ask the reader to please keep in mind the meaning of this standard but opaque notation for the Poincaré series $P_m = P_{f,\mathbf{1},\mathbf{1}}$ constructed from a given f which transforms under such a shift by $m\in\mathcal{O}_F\backslash\{0\}$ of the standard additive character ψ_∞. To be clear, we also take the central character $\omega = \mathbf{1}$ and the multiplier $\vartheta = \mathbf{1}$ in the definition above to be trivial, then drop the corresponding subscripts from the notations. We shall later write $\operatorname{Kl}_\Gamma(m,r;c) = \operatorname{Kl}_{\Gamma,\mathbf{1},\mathbf{1}}(m,r;c)$ to denote the corresponding Kloosterman sums. We argue that we can decompose our cuspidal pure tensor $\phi\in V_\pi$ into a linear combination of such Poincaré series:

Lemma A.7. *Given $\phi\in V_\pi$ any cuspidal pure tensor, there exists a decomposition of ϕ into a convergent (usually infinite) linear combination of the Poincaré series P_m introduced above. To be more precise, we have as functions of $g\in\operatorname{GL}_2(\mathbf{A}_F)$ the decomposition*

$$\phi(g) = \sum_m c_m(\phi)\cdot P_m(g), \quad c_m(\phi)\in\mathbf{C}^\times. \tag{58}$$

Moreover, in the special case where $\phi\in V_\pi$ corresponds to the lift of a holomorphic Hilbert modular form, the sum (58) *is finite.*

Proof. To justify (58) in the general case, we argue by a minor variation of the argument given for the classical case of holomorphic cuspidal modular forms following [29, Lemma 14.3 and Corollary 14.4]. Roughly, the same application of orthogonality of additive characters used to derive the standard identity [29, Lemma 14.3] for the Fourier-Whittaker coefficients of the cuspidal form ϕ as Petersson inner products $\langle\phi, P_m\rangle$ applies here as well. To be more precise, consider the Poincaré series $P_m = P_{f,\mathbf{1},\mathbf{1}}$ associated to a nonzero F-integer m, say with $f\in\mathcal{S}(N_2(\mathbf{A}_F)\backslash\operatorname{GL}_2(\mathbf{A}_F),\psi_1)$ a Schwartz function which transforms under unipotents via the additive character ψ_1 defined on $x\in\mathbf{A}_F$ by $\psi_1(x) = \psi_\infty(mx_\infty)$. Let us write $\overline{P}_m = T_{-1}P_m$ to denote the image of P_m under the Hecke operator at infinity, sending

$$\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -x_\infty \\ & 1 \end{pmatrix} \in N_2(F_\infty).$$

We consider the Petersson inner product $\langle P_m, \phi \rangle$, which after expanding out definitions is the same as

$$\begin{aligned}\langle \phi, P_m \rangle &= \int_{Z_2(\mathbf{A}_F)\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F)} \phi(g)\overline{P_m(g)} dg \\ &= \int_{Z_2(\mathbf{A}_F)\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F)} \phi(g) \left(\overline{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g)} \right) dg \\ &= \int_{Z_2(\mathbf{A}_F)\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F)} \phi(g)\overline{f(g)} dg.\end{aligned}$$

Here, we use the automorphy of ϕ, note that $\Gamma_\infty \cong N_2(F)$, and collapse the summation in the usual way for the last step. Let us now write $F_{\infty,+}^\times \subset F_\infty^\times$ for the totally positive archimedean ideles of $\mathbf{A}_F^\times$, so $F_{\infty,+}^\times \cong \mathbf{R}_{>0}^d$. Passing to a standard fundamental domain, we can then compute the latter integral as

$$\begin{aligned}&\int_{Z_2(\mathbf{A}_F)\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F)} \phi(g)\overline{f(g)} dg \\ &= \int_{F_{\infty,+}^\times} \int_{I \cong [0,1]^d \subset F_\infty} \phi\left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \right) f\left(\begin{pmatrix} y_\infty & -x_\infty \\ & 1 \end{pmatrix} \right) \frac{dx_\infty dy_\infty}{y_\infty^2} \\ &= \int_{F_{\infty,+}^\times} f\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix} \right) \sum_{\gamma \in F^\times} W_\phi\left(\begin{pmatrix} \gamma y_\infty & \\ & 1 \end{pmatrix} \right) \left(\int_{I \cong [0,1]^d \subset F_\infty} \psi_\infty(\gamma x_\infty - m x_\infty) dx_\infty \right) \frac{dy_\infty}{y_\infty^2} \\ &= \int_{F_{\infty,+}^\times} f\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix} \right) W_\phi\left(\begin{pmatrix} m y_\infty & \\ & 1 \end{pmatrix} \right) \frac{dy_\infty}{y_\infty^2} \\ &= \frac{\rho_\phi(m)}{|m|^{\frac{1}{2}}} \int_{F_{\infty,+}^\times} f\left(\begin{pmatrix} y_\infty & \\ & 1 \end{pmatrix} \right) W_\phi(m y_\infty) \frac{dy_\infty}{y_\infty^2}.\end{aligned}$$

Here, we open up the Fourier-Whittaker expansion of ϕ in the usual way as

$$\phi\left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \right) = \sum_{\gamma \in F^\times} W_\phi\left(\begin{pmatrix} \gamma y_\infty & \\ & 1 \end{pmatrix} \right) \psi(\gamma x_\infty),$$

using for $y = y_f y_\infty \in \mathbf{A}_F^\times$ the shorthand notations

$$\frac{\rho_\phi(y_f m)}{|y_f m|^{\frac{1}{2}}} := W_\phi\left(\begin{pmatrix} m y_f & \\ & 1 \end{pmatrix} \right), \quad W_\phi(m y_\infty) := W_\phi\left(\begin{pmatrix} m y_\infty & \\ & 1 \end{pmatrix} \right)$$

to denote the nonarchimedean and archimedean local components of the Fourier-Whittaker coefficients (where the nonarchimean idele y_f is taken to be the identity in $\mathbf{A}_{F,f}^\times$). We also use the transformation property of the Schwartz function $f \in \mathcal{S}(N_2(\mathbf{A})\backslash \operatorname{GL}_2(\mathbf{A}_F), \psi_1)$ to expand out the kernel function before evaluating via orthogonality of additive characters. In this way, we show the analogue of [29, Lemma 14.3]. Although it is not necessarily finite dimensional, we can deduce from this inner product calculation that the span $\langle P_m \rangle_m$ of such Poincaré series P_m, i.e. with m running over nonzero F-integers, has closure equal to the cuspidal spectrum[14] $L^2_{\text{cusp}}(\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F), \mathbf{1})$. The claim is then simple to justify from the inner product calculations given above. In the special case where ϕ arises from the lift to $\operatorname{GL}_2(\mathbf{A}_F)$ of a holomorphic Hilbert modular form, we restrict to the spaces of such forms as in the classical argument (e.g. [29, Lemma 14.3]). Hence, the corresponding space of cuspidal forms describing the closure is finite dimensional, from which we deduce that (58) must be a finite sum. $\square$

[14] Let us also remark that each of the Poincaré series P_m we consider above belongs to the continuous spectrum $L^2_{\text{cont}}(\operatorname{GL}_2(F)\backslash \operatorname{GL}_2(\mathbf{A}_F), \mathbf{1})$ spanned by analytic continuations of Eisenstein series, e.g. after considering the Mellin inversion formula. This can be deduced for instance from the classical discussion in Kubota [34, §7.4], which shows that the continuous spectrum $L^2_{\text{cont}}(\operatorname{GL}_2(\mathbf{Q})\backslash \operatorname{GL}_2(\mathbf{A}), \omega)$ is in fact spanned by Poincaré series. More precisely, the Mellin inversion formula described in [34, 7.4.4] relates these so-called "incomplete theta series" with analytic continuations of Eisenstein series, which are known to furnish the continuous spectrum.

Let us now return to our chosen pure tensor $\phi \in V_\pi$ described above, with each nonarchimedean local component an essential Whittaker vector, and archimedean component chosen so that $W_\phi(y_\infty) = W(y_\infty)$ as functions of $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times$. Henceforth, we assume that ϕ is constructed explicitly as the image $\phi = \mathcal{R}_k\varphi = \mathcal{R}_k(\{\mathfrak{K}_{k+2i}\}_{i\geq 0})\varphi$ of the lift φ to $\mathrm{GL}_2(\mathbf{A}_F)$ of some holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$ under the operator $\mathcal{R}_k = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}\right)_{i\geq 0})$ associated to the coefficient of d-tuples $\{\mathfrak{K}_{k+2i}\}_{i\geq 0}$ for which the relation (56) holds. In this special setting, we have the following relevant decomposition.

Corollary A.8. *Let $\phi = \otimes_v \phi_v \in V_\pi$ be a pure tensor as described above, with each finite component being an essential Whittaker vector, and archimedean component chosen so that $W_\phi(y_\infty) = W(y_\infty)$ as functions of $y_\infty = (y_{\infty,j})_{j=1}^d \in F_\infty^\times$. Equivalently, let us assume that $\phi = \mathcal{R}_k\varphi = \mathcal{R}_k(\{\mathfrak{K}_{k+2i}\}_{i\geq 0})\varphi$ for φ the lift of some holomorphic Hilbert modular form of weight $k = (k_j)_{j=1}^d$ under the operator $\mathcal{R}_k = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}\right)_{i\geq 0})$ associated to the coefficient of d-tuples $\{\mathfrak{K}_{k+2i}\}_{i\geq 0}$ for which the relation* (56) *holds. In this case, ϕ decomposes into a finite linear combination of (non-K-finite) Poincaré series,*

$$\phi = \sum_m c_m(\phi) \cdot P_m.$$

Proof. Given that φ is the lift to $\mathrm{GL}_2(\mathbf{A}_F)$ of a holomorphic Hilbert modular form, we know from Lemma A.7 that we can decompose φ into a finite linear combination of Poincaré series,

$$\varphi = \sum_m c_m(\varphi) \cdot P_m, \quad c_m(\varphi) \in \mathbf{C}^\times.$$

Applying the operator $\mathcal{R}_k = \mathcal{R}_k\left(\{\mathfrak{K}_{k+2i}\}_{i\geq 0}\right)$ described above to this chosen decomposition of the lifted holomorphic form φ, we get the corresponding expansion

$$\mathcal{R}_k\varphi = \sum_m c_m(\varphi) \cdot \mathcal{R}_k P_m = \sum_m c_m(\varphi) \cdot \sum_{i\geq 0} \mathfrak{K}_{k+2i} R_{k+2i} P_m,$$

which after relabelling is none other than our desired expansion into Poincaré series

$$\sum_m c_m(\phi) \cdot P_m', \quad P_m' := \mathcal{R}_k P_m = \sum_{i\geq 0} \mathfrak{K}_{k+2i} R_{k+2i} P_m.$$

That is, these non-K-finite Poincaré series P_m' are defined from those appearing in the fixed expansion of the lifted holomorphic form φ. We also have $c_m(\phi) = c_m(\varphi)$ for each of the finitely many indices m, these complex coefficients depending only on the underlying representation π and form $\varphi \in V_\pi$. $\square$

In particular, we can ignore the contributions of these coefficients $c_m(\phi)$ in our subsequent calculations, as in the classical version of the argument due to Blomer [4]. It follows from (58) that our shifted convolution sum (52) can be decomposed accordingly as

$$\sum_{(n)} W_\phi\left(\begin{pmatrix} \frac{q(n)}{Y_\infty} & \\ & 1\end{pmatrix}\right) = \sum_m c_m(\phi) \sum_{(n)} W_{P_m}\left(\begin{pmatrix} \frac{q(n)}{Y_\infty} & \\ & 1\end{pmatrix}\right). \tag{59}$$

Here, each coefficient

$$\begin{aligned} W_{P_m}\left(\begin{pmatrix} \frac{q(n)}{Y_\infty} & \\ & 1\end{pmatrix}\right) &:= \int_{\mathbf{A}_F/F} P_m\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)\psi(-q(n)x)dx \\ &= \int_{I\cong[0,1]^d\subset F_\infty} P_m\left(\begin{pmatrix} 1 & x_\infty \\ & 1\end{pmatrix}\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)\psi(-q(n)x_\infty)dx_\infty \end{aligned}$$

on the right hand side of (59) is the Fourier-Whittaker coefficient at $q(n)$ of

$$P_m\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)$$

so that we take $r = q(n)$, we use the second formula of Proposition A.7 to express (59) equivalently as

$$\sum_m c_m(\phi) \sum_{(n)} \left(\int_{\mathbf{A}_F/F} f\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)\psi(mx - q(n)x)dx + \sum_{c\in\Omega(\Gamma)} \mathrm{Kl}_\Gamma(m, q(n); c)\mathcal{F}_{f,q(n),c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)\right). \tag{60}$$

Again, as we assume that $\phi = \mathcal{R}_k\varphi$ is constructed from the lift φ of holomorphic Hilbert modular form, we know by Corollary A.8 that the decomposition into Poincaré series (58) is a finite sum. This in particular justifies that we can switch the order of summation in these expressions for the decomposition of ϕ. We now argue that the integral term in this latter expression (60) (which typically vanishes by orthogonality) is negligible relative to the second term, so that it is enough to estimate the simpler sum

$$\sum_m c_m(\phi) \sum_{(n)} \sum_{c \in \Omega(c)} \mathrm{Kl}_\Gamma(m, q(n); c) \mathcal{F}_{f,q(n),c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right). \tag{61}$$

Opening the Kloosterman sums $\mathrm{Kl}_\Gamma(m, q(n); c)$ and switching the order of summation, (61) is the same as

$$\begin{aligned} &\sum_m c_m(\phi) \sum_{(n)} \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \omega(d) \psi_\infty\left(\frac{ma + q(n)d}{c}\right) \mathcal{F}_{f,q(n),c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \\ &= \sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \omega(d) \psi_\infty\left(\frac{ma}{c}\right) \sum_{(n)} \psi_\infty\left(\frac{q(n)d}{c}\right) \mathcal{F}_{f,q(n),c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right), \end{aligned}$$

which after partitioning the (n)-sum into congruence classes $u \bmod c$ is the same as

$$\sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \omega(d) \psi_\infty\left(\frac{ma}{c}\right) \sum_{u \bmod c} \sum_{\substack{(n) \\ n \equiv u \bmod c}} \psi_\infty\left(\frac{q(u)d}{c}\right) \mathcal{F}_{f,q(n),c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right). \tag{62}$$

Here, the sum over $u \bmod c$ denotes the sum over a full set of representatives for the classes $(\mathcal{O}_F / c\mathcal{O}_F)$. We now use Poisson summation (cf. [4, Lemma 1]) to evaluate the inner sum. That is, given $\mathcal{F}$ a sufficiently well-behaved Schwartz class function on $x_\infty \in F_\infty \cong \mathbf{R}^d$, we can use the Poisson summation formula

$$\sum_{\substack{(n) \\ n \equiv u \bmod c}} \mathcal{F}(n) = \frac{1}{|c|} \sum_{(h)} \widehat{\mathcal{F}}\left(\frac{h}{c}\right) \psi_\infty\left(\frac{hu}{c}\right), \tag{63}$$

where

$$\widehat{\mathcal{F}}(x_\infty) = \int_{F_\infty \cong \mathbf{R}^d} \mathcal{F}(z_\infty) \psi_\infty(-z_\infty x_\infty) dz_\infty$$

denotes the Fourier transform of $\mathcal{F}$. Applying (63) to the inner sum in (62) (as in [4, (3.2)]) then gives us

$$\sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \omega(d) \psi_\infty\left(\frac{ma}{c}\right) \frac{1}{|c|} \sum_{u \bmod c} \sum_{(h)} \psi_\infty\left(\frac{q(u)d + hu}{c}\right) \widehat{\mathcal{F}}_{f, \frac{q(h)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right), \tag{64}$$

where

$$\begin{aligned} \widehat{\mathcal{F}}_{f, \frac{q(h)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) &= \int_{F_\infty \cong \mathbf{R}^d} \mathcal{F}_{f, \frac{q(z_\infty)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \psi_\infty(-hz_\infty) dz_\infty \\ &= \int_{F_\infty \cong \mathbf{R}^d} \left(\int_{\mathbf{A}_F} f\left(w \cdot \underline{c} \cdot \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \psi_\infty\left(-\frac{q(z_\infty)}{c} \cdot x\right) dx\right) \psi_\infty(-hz_\infty) dz_\infty \end{aligned}$$

is absolutely convergent and rapidly decreasing for Schwartz functions $f \in \mathcal{S}(N_2(\mathbf{A}_F) \backslash \mathrm{GL}_2(\mathbf{A}_F); \psi_1)$.

Let us now consider each quadratic Gauss sum in the latter expression (64), i.e. opening up the quadratic polynomials $q(u) = ru^2 + su + t$ in the latter expression to find the sums

$$\begin{aligned}\sum_{u \bmod c} \psi_\infty\left(\frac{q(u)d + hu}{c}\right) &= \sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u + dt}{c}\right) \\ &= \psi_\infty\left(\frac{dt}{c}\right)\sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right).\end{aligned} \tag{65}$$

To evaluate the inner Gauss sums on the right hand side of (65) explicitly, we start with the following observations. First, since ψ_∞ is the archimedean component of the standard additive character $\psi = \otimes_v \psi_v$ on $\mathbf{A}_F/F$, it is defined on any $z_\infty = (z_{\infty,j})_{j=1}^d \in F_\infty \cong \mathbf{R}^d$ by the more explicit formula

$$\psi_\infty(z_\infty) = e(\mathrm{Tr}(z_\infty)) = e\left(\sum_{j=1}^{[F:\mathbf{Q}]} z_{\infty,j}\right) = \exp\left(2\pi i \cdot \sum_{j=1}^{[F:\mathbf{Q}]} z_{\infty,j}\right).$$

Here, Tr denotes sum over real embeddings $\sigma_j : F \to \mathbf{R}$. We then see that the inner sum in (65) equals

$$\sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right) = \sum_{u \bmod c} e\left(\mathrm{Tr}\left(\frac{dru^2 + (ds+h)u}{c}\right)\right). \tag{66}$$

Writing $\alpha_j = \sigma_j(\alpha)$ to denote the image under a given embedding $\sigma_j : F \to \mathbf{R}$ of $\alpha \in F$, we can then use distributivity to justify interchanging the sum and the product on the right of (66), which in particular gives us the following relation to classical generalized quadratic Gauss sums:

$$\sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)}{c}\right) = \prod_{j=1}^{[F:\mathbf{Q}]} \sum_{u_j \bmod c_j} e\left(\frac{d_j r_j u_j^2 + (d_j s_j + h_j)u_j}{c_j}\right). \tag{67}$$

Proposition A.9. *Given F-integers a, b, and c with a and c totally positive and $\mathbf{N}(ac) \equiv 0 \bmod 2$, consider the quadratic Gauss sum defined by*

$$T(a,b,c) := \sum_{u \bmod c} \psi_\infty\left(\frac{au^2 + 2bu}{2c}\right).$$

We have the formula

$$T(a,b,c) = \left(\frac{\mathbf{N}c}{\mathbf{N}a}\right)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{b^2}{2ac}\right) \cdot \overline{T(c,b,a)}. \tag{68}$$

In particular, taking $b = 0$, we recover the formula

$$T(a,0,c) = \left(\frac{\mathbf{N}c}{\mathbf{N}a}\right)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8}\right) \cdot \overline{T(c,0,a)}. \tag{69}$$

Proof. Decomposing via the relation (67), we have that

$$T(a,b,c) = \prod_{j=1}^{[F:\mathbf{Q}]} T_j(a,b,c), \quad T_j(a,b,c) := \sum_{u_j \bmod c_j} e\left(\frac{a_j u_j^2 + 2b_j u_j}{2c_j}\right).$$

Hence, it will do to prove the claim for each of the classical Gauss sums in the product, namely that

$$T_j(a,b,c) = e\left(\frac{1}{8} - \frac{b_j^2}{2a_j c_j}\right) \cdot \overline{T_j(c,b,a)} \tag{70}$$

for each embedding $\sigma_j : F \to \mathbf{R}$. Note that this latter formula is well-known classically for the special case of $b_j = 0$. For instance, it can be derived via Poisson summation formula as in [40, Theorem 9.15]. A minor generalization gives the stated identity (70). In any case, the more general identity (68) follows via the decomposition (67). For the convenience of the reader, we present a distinct proof of (70) via the residue theorem due to G. Harcos (see [25]). Let us for each embedding $\sigma_j : F \to \mathbf{R}$ consider the entire function $g_j : \mathbf{C} \to \mathbf{C}$ defined on $z \in \mathbf{C}$ by

$$g_j(z) := e\left(\frac{a_j z^2 + 2b_j z}{2c_j}\right).$$

Consider the line $\mathcal{L}$ parametrized by $t \longmapsto -\frac{1}{2} + e\left(\frac{1}{8}\right) t$ for $t \in \mathbf{R}$. By the residue theorem, we have that

$$T_j(a,b,c) = \sum_{n=0}^{c_j-1} e\left(\frac{a_j n^2 + 2b_j n}{2c_j}\right) = \int\limits_{c_j+\mathcal{L}} \frac{g_j(z)}{e(z)-1} dz - \int\limits_{\mathcal{L}} \frac{g_j(z)}{e(z)-1} dz = \int\limits_{\mathcal{L}} \frac{g_j(z+c_j) - g_j(z)}{e(z)-1} dz.$$

On the other hand, since $a_j c_j \equiv 0 \bmod 2$, we have that

$$g_j\left(z + c_j\right) = e\left(\frac{a_j z^2 + 2a_j c_j z + a_j c_j^2 + 2b_j z + 2b_j c_j}{2c_j}\right) = e\left(\frac{a_j z^2 + 2a_j c_j z + 2b_j z}{2c_j}\right) = g_j(z) e(a_j z),$$

from which it follows that

$$\begin{aligned}\frac{g_j(z+c_j) - g_j(z)}{e(z)-1} = g_j(z) \cdot \left(\frac{e(a_j z) - 1}{e(z)-1}\right) &= \sum_{m=0}^{a_j-1} e\left(\frac{a_j z^2 + 2b_j z + 2c_j m z}{2c_j}\right) \\ &= \sum_{m=0}^{a_j-1} e\left(\frac{(a_j z + b_j + c_j m)^2 - (b_j + c_j m)^2}{2a_j c_j}\right)\end{aligned}$$

and hence

$$T_j(a,b,c) = \int\limits_{\mathcal{L}} \frac{g_j(z+c_j) - g_j(z)}{e(z)-1} dz = \sum_{m=0}^{a_j-1} e\left(-\frac{(b_j + c_j m)^2}{2a_j c_j}\right) \int\limits_{\mathcal{L}} e\left(\frac{(a_j z + b_j + c_j m)^2}{2a_j c_j}\right) dz.$$

Applying the residue theorem again to this latter expression, we observe that each inner contour integral is independent of the choices of m and b_j, and equal to

$$\int\limits_{e\left(\frac{1}{8}\right)\mathbf{R}} e\left(\frac{a_j z^2}{2c_j}\right) = e\left(\frac{1}{8}\right) \cdot \sqrt{\frac{c_j}{a_j}} \cdot \int\limits_{\mathbf{R}} e^{-\pi i^2} dt = e\left(\frac{1}{8}\right) \cdot \sqrt{\frac{c_j}{a_j}}.$$

In this way, we derive

$$T_j(a,b,c) = \int\limits_{\mathcal{L}} \frac{g_j(z+c_j) - g_j(z)}{e(z)-1} dz = e\left(\frac{1}{8}\right) \cdot \sqrt{\frac{c_j}{a_j}} \sum_{m=0}^{a_j-1} e\left(-\frac{(b_j + c_j m)^2}{2a_j c_j}\right) = e\left(\frac{1}{8}\right) \cdot \sqrt{\frac{c_j}{a_j}} \cdot \overline{T_j(c_j, b_j, a_j)}$$

to justify (70), and hence (68) via (67). ☐

We can now use this to evaluate the inner quadratic Gauss sum appearing in (66) more explicitly via quadratic reciprocity. Let us for a given rational integer c_j write

$$\epsilon_{c_j} = \begin{cases} 1 & \text{if } c_j > 0 \\ i & \text{if } c_j > 0. \end{cases}$$

Given an F-integer c with $c_j = \sigma_j(c)$ for each real embedding $\sigma_j : F \to \mathbf{R}$ as above, we can then write

$$\epsilon_c = \prod_{j=1}^{[F:\mathbf{Q}]} \epsilon_{c_j}.$$

Similarly, given rational integers a_j and q_j, we write $\left(\frac{a_j}{p_j}\right)$ to denote the Jacobi symbol. Given F integers a and q with respective images $a_j = \sigma_j(a)$ and $q_j = \sigma_j(q)$ under a given embedding $\sigma_j : F \to \mathbf{R}$, we then consider the corresponding quadratic character symbol defined by the product

$$\left(\frac{a}{q}\right) = \prod_{j=1}^{[F:\mathbf{Q}]} \left(\frac{a_j}{q_j}\right).$$

Here, for simplicity, we use the same notation for the composition with the norm $(\cdot) \circ \mathbf{N}$, taking for granted that the context will make the meaning clear.

Corollary A.10. *We have the following explicit identity for the inner quadratic Gauss sum*

$$T(2dr, ds+h, c) = \sum_{u \bmod c} \psi_\infty\left(\frac{2dru^2 + 2(ds+h)u}{2c}\right) = \sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right)$$

appearing in (66)*. That is, we have the exact formula*

$$T(2dr, ds+h, c) = \left(\frac{\mathbf{N}(c)}{\mathbf{N}(2dr)}\right)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{(ds+h)^2}{4drc}\right) \cdot \overline{T(c, 0, 2dr)}, \tag{71}$$

which after evaluating the Gauss sum $\overline{T(c,0,2dr)}$ via quadratic reciprocity takes the more explicit form

$$\begin{aligned} &T(2dr, ds+h, c) \\ &= \mathbf{N}(c)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{(ds+h)^2}{4drc}\right) \cdot (1-i)^{[F:\mathbf{Q}]} \cdot \left(\frac{c/2}{2dr}\right) \cdot \prod_{j=1}^{[F:\mathbf{Q}]} \times \begin{cases} 0 & \text{if } 2d_jr_j \equiv 2 \bmod 4 \\ \overline{\epsilon}_{c_j/2}^{-1} & \text{if } 2d_jr_j \equiv 0 \bmod 4 \end{cases} \\ &= \begin{cases} \mathbf{N}(c)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{(ds+h)^2}{4drc}\right) \cdot (1-i)^{[F:\mathbf{Q}]} \cdot \left(\frac{c/2}{2dr}\right) \cdot \epsilon_{c/2} & \text{if } 2d_jr_j \equiv 0 \bmod 4 \ \forall\ 1 \leq j \leq [F:\mathbf{Q}] \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{72}$$

In particular, we have that

$$\begin{aligned} &\sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right) \\ &= \begin{cases} \mathbf{N}(c)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{(ds+h)^2}{4drc}\right) \cdot (1-i)^{[F:\mathbf{Q}]} \cdot \left(\frac{c/2}{2dr}\right) \cdot \epsilon_{c/2} & \text{if } 2d_jr_j \equiv 0 \bmod 4 \ \forall\ 1 \leq j \leq [F:\mathbf{Q}] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and hence

$$\begin{aligned} &\psi_\infty\left(\frac{dt}{c}\right) \sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right) \\ &= \psi_\infty\left(\frac{dt}{c}\right) \cdot \begin{cases} \mathbf{N}(c)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8} - \frac{(ds+h)^2}{4drc}\right) \cdot (1-i)^{[F:\mathbf{Q}]} \cdot \left(\frac{c/2}{2dr}\right) \cdot \epsilon_{c/2} & \text{if } 2d_jr_j \equiv 0 \bmod 4 \ \forall\ 1 \leq j \leq [F:\mathbf{Q}] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{73}$$

Proof. We can deduce (71) from the simpler identity (69), which in this case takes the form

$$T(2dr, 0, c) = \left(\frac{\mathbf{N}c}{\mathbf{N}(2dr)}\right)^{\frac{1}{2}} \cdot \psi_\infty\left(\frac{1}{8}\right) \cdot \overline{T(c, 0, 2dr)}. \tag{74}$$

Consider our initial Gauss sum

$$T(2dr, ds+h, c) = \sum_{u \bmod c} \psi_\infty\left(\frac{2dru^2 + 2(ds+h)u}{2c}\right) = \sum_{u \bmod c} \psi_\infty\left(\frac{dru^2 + (ds+h)u}{c}\right)$$

Completing the square via the elementary identity

$$\left(u + \frac{(ds+h)}{2dr}\right)^2 = u^2 + \frac{2(ds+h)}{2dr}u + \frac{(ds+h)^2}{4(dr)^2} \qquad \implies$$

$$\frac{2dr}{2c}\left(u^2 + \frac{2(ds+h)}{2dr}u\right) = \frac{2dr}{2c}\left(u + \frac{(ds+h)}{2dr}\right)^2 - \frac{2dr}{2c}\frac{(ds+h)^2}{4(dr)^2}$$

we deduce that

$$\psi_\infty\left(\frac{2dru^2+2(ds+h)u}{2c}\right)=\psi_\infty\left(-\frac{(ds+h)^2}{4dcr}\right)\psi_\infty\left(\frac{2dr}{2c}\left(u+\frac{(ds+h)}{2dr}\right)^2\right)$$

and hence

$$\sum_{u \bmod c}\psi_\infty\left(\frac{2dru^2+2(ds+h)u}{2c}\right)=\psi_\infty\left(-\frac{(ds+h)^2}{4dcr}\right)\sum_{u \bmod c}\psi_\infty\left(\frac{2dru^2}{2c}\right)$$

so that

$$T(2dr,ds+h,c)=\psi_\infty\left(-\frac{(ds+h)^2}{4dcr}\right)T(2dr,0,c). \tag{75}$$

Applying (74) to the quadratic Gauss sum $T(2dr,0,c)$ in (75) then gives the desired identity

$$T(2dr,ds+h,c)=\psi_\infty\left(\frac{1}{8}-\frac{(ds+h)^2}{4dcr}\right)\cdot\left(\frac{\mathbf{N}(c)}{\mathbf{N}(2dr)}\right)^{\frac{1}{2}}\cdot\overline{T(c,0,2dr)}.$$

Let us note that we can also consider the identity (68) here, which gives

$$T(2dr,ds+h,c)=\left(\frac{\mathbf{N}(c)}{\mathbf{N}(2dr)}\right)^{\frac{1}{2}}\cdot\psi_\infty\left(\frac{1}{8}-\frac{(ds+h)^2}{4drc}\right)\cdot\overline{T(c,ds+h,2dr)}. \tag{76}$$

Completing the square via the elementary identity

$$\left(u+\frac{(ds+h)}{c}\right)^2=u^2+\frac{2(ds+h)}{c}u+\frac{(ds+h)^2}{c^2} \implies$$

$$\frac{c}{4dr}\left(u^2+\frac{2(ds+h)}{c}u\right)=\frac{c}{4dr}\left(u+\frac{(ds+h)}{c}\right)^2-\frac{(ds+h)^2}{4cdr},$$

we deduce that

$$\psi_\infty\left(\frac{cu^2+2(ds+h)u}{4dr}\right)=\psi_\infty\left(-\frac{(ds+h)^2}{4cdr}\right)\psi_\infty\left(\frac{c}{4dr}\left(u+\frac{(ds+h)}{c}\right)^2\right)$$

and hence

$$\sum_{u \bmod c}\psi_\infty\left(\frac{cu^2+2(ds+h)u}{4dr}\right)=\psi_\infty\left(-\frac{(ds+h)^2}{4cdr}\right)\sum_{u \bmod c}\psi_\infty\left(\frac{cu^2}{4dr}\right),$$

so that

$$T(c,ds+h,dr)=\psi_\infty\left(-\frac{(ds+h)^2}{4cdr}\right)\cdot T(c,0,2dr).$$

Applying this latter identity to the right hand side of (76) (after complex conjugation) then gives us

$$T(2dr,ds+h,c)=\left(\frac{\mathbf{N}(c)}{\mathbf{N}(2dr)}\right)^{\frac{1}{2}}\cdot\psi_\infty\left(\frac{1}{8}-\frac{(ds+h)^2}{4drc}\right)\cdot\psi_\infty\left(\frac{(ds+h)^2}{4drc}\right)\cdot\overline{T(c,0,2dr)}.$$

In any case, we can evaluate the remaining Gauss sum $\overline{T(c,0,2dr)}$ via quadratic reciprocity as follows. Let us for simplicity of notation derive the corresponding formula for $T(c,0,2dr)$, then apply complex conjugation at the last step. Recall that the relation (67) allows us to pass to classical Gauss sums via the decomposition

$$T(c,0,2dr)=\sum_{u \bmod 2dr}\psi_\infty\left(\frac{cu^2}{4dr}\right)=\prod_{j=1}^{[F:\mathbf{Q}]}\sum_{u_j \bmod 2d_jr_j}e\left(\frac{c_ju_j^2}{4d_jr_j}\right).$$

We know by quadratic reciprocity that for each index $1\leq j\leq[F:\mathbf{Q}]$, we have

$$\sum_{u_j \bmod 2d_jr_j}e\left(\frac{c_j/2\cdot u_j^2}{2d_jr_j}\right)=\begin{cases}0 & \text{if } 2d_jr_j\equiv 2 \bmod 4\\ (1+i)\cdot\sqrt{2d_jr_j}\cdot\epsilon_{c_j/2}^{-1}\cdot\left(\frac{c_j/2}{2d_jr_j}\right) & \text{if } 2d_jr_k\equiv 0 \bmod 4,\end{cases}$$

where again

$$\epsilon_{c_j/2} = \begin{cases} 1 & \text{if } c_j > 0 \\ i & \text{if } c_j < 0. \end{cases}$$

We can then evaluate

$$T(c,0,2dr) = (1+i)^{[F:\mathbf{Q}]} \cdot \mathbf{N}(2dr)^{\frac{1}{2}} \cdot \left(\frac{c/2}{2dr}\right) \cdot \prod_{j=1}^{[F:\mathbf{Q}]} \times \begin{cases} 0 & \text{if } 2d_jr_j \equiv 2 \bmod 4 \\ \epsilon_{c_j/2}^{-1} & \text{if } 2d_jr_j \equiv 0 \bmod 4, \end{cases}$$

and hence after complex conjugation

$$\overline{T(c,0,2r)} = (1-i)^{[F:\mathbf{Q}]} \cdot \mathbf{N}(2dr)^{\frac{1}{2}} \cdot \left(\frac{c/2}{2dr}\right) \cdot \prod_{j=1}^{[F:\mathbf{Q}]} \times \begin{cases} 0 & \text{if } 2d_jr_j \equiv 2 \bmod 4 \\ \overline{\epsilon}_{c_j/2}^{-1} & \text{if } 2d_jr_j \equiv 0 \bmod 4. \end{cases}$$

Applying this latter formula to (71), factoring out terms that cancel, we then arrive at (72). ☐

It follows (cf. [4, (3.5)]) after switching the order of summation that (64) can be evaluated via (73) as

(77)

$$\sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty} \omega(d) \psi_\infty\left(\frac{ma}{c}\right) \frac{1}{|c|} \sum_{(h)} \widehat{\mathcal{F}}_{f, \frac{q(h)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \sum_{u \bmod c} \psi_\infty\left(\frac{q(u)d + hu}{c}\right)$$

$$= \psi_\infty\left(\frac{1}{8}\right) \sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \frac{(1-i)^{[F:\mathbf{Q}]}}{|c|^{\frac{1}{2}}} \sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty \\ 2d_jr_j \equiv 2 \bmod 4 \ \forall \ 1 \leq j \leq [F:\mathbf{Q}]}} \omega(d) \cdot \overline{\epsilon}_{c/2}^{-1} \cdot \left(\frac{c/2}{2dr}\right) \cdot \psi_\infty\left(\frac{ma}{c}\right)$$

$$\times \sum_{(h)} \widehat{\mathcal{F}}_{f, \frac{q(h)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \psi_\infty\left(\frac{dt}{c} - \frac{(ds+h)^2/4}{dcr}\right),$$

which after using the elementary calculation

$$\psi_\infty\left(\frac{dt}{c} - \frac{(ds+h)^2/4}{dcr}\right) = \psi_\infty\left(\frac{dt}{c} - \frac{[d^2s^2 + 2dsh + h^2]/4}{cdr}\right)$$
$$= \psi_\infty\left(\frac{dt}{c} - \frac{ds^2/4}{dcr}\right) \psi_\infty\left(\frac{-sh/2}{cr} - \frac{h^2/4}{dcr}\right) = \psi_\infty\left(\frac{-d(s^2-4rt)}{4cr}\right) \psi_\infty\left(\frac{-sh/2}{cr} - \frac{h^2/4}{cdr}\right)$$

and grouping together like terms gives us the neater expression

(78)

$$\psi_\infty\left(\frac{1}{8}\right) \sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \frac{(1-i)^{[F:\mathbf{Q}]}}{|c|^{\frac{1}{2}}}$$
$$\sum_{\substack{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_c / \Gamma_\infty \\ 2d_jr_j \equiv 2 \bmod 4 \ \forall \ 1 \leq j \leq [F:\mathbf{Q}]}} \omega(d) \cdot \overline{\epsilon}_{c/2}^{-1} \cdot \left(\frac{c/2}{2dr}\right) \cdot \psi_\infty\left(\frac{am}{c}\right) \cdot \psi_\infty\left(\frac{-d(s^2-4rt)}{4cr}\right)$$
$$\times \sum_{(h)} \widehat{\mathcal{F}}_{f, \frac{q(h)}{c}, c}\left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}\right) \psi_\infty\left(\frac{-sh/2}{cr} - \frac{h^2/4}{cdr}\right).$$

Opening up the Fourier transform and switching the order of summation, this sum (78) is equivalent to

$$\begin{aligned}
&\psi_\infty\left(\frac{1}{8}\right)\sum_m c_m(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)^{[F:\mathbf{Q}]}}{|c|^{\frac{1}{2}}} \\
&\sum_{\substack{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty \\ 2d_jr_j\equiv 2 \bmod 4 \ \forall \ 1\leq j\leq[F:\mathbf{Q}]}} \omega(d)\cdot\overline{\epsilon}_{c/2}^{-1}\cdot\left(\frac{c/2}{2dr}\right)\cdot\psi_\infty\left(\frac{am}{c}\right)\cdot\psi_\infty\left(\frac{-d(s^2-4rt)}{4cr}\right) \\
&\times\sum_{(h)}\psi_\infty\left(\frac{-dsh/2-h^4/4}{cdr}\right)\int_{\mathbf{A}_F} f\left(w\cdot\underline{c}\cdot\begin{pmatrix}\frac{1}{Y_\infty} & x \\ & 1\end{pmatrix}\right)\int_{F_\infty}\psi_\infty\left(\frac{-q(z_\infty)x_\infty-hcz_\infty}{c}\right)dz_\infty dx.
\end{aligned}\tag{79}$$

Now for each $x_\infty=(x_{\infty,j})_{j=1}^d\in F_\infty$ with norm $|x_\infty|\neq 0$ (hence with each component $x_{\infty,j}\neq 0$), we can evaluate the inner integral

$$\begin{aligned}
&\int_{F_\infty}\psi_\infty\left(\frac{-q(z_\infty)x_\infty-hcz_\infty}{c}\right)dz_\infty \\
&=\int_{F_\infty}\psi_\infty\left(-\left(\frac{rx_\infty}{c}\right)z_\infty^2-\left(\frac{sx_\infty+hc}{c}\right)z_\infty-\left(\frac{tx_\infty}{c}\right)\right)dz_\infty \\
&=\left|\frac{c}{2irx_\infty}\right|^{\frac{1}{2}}\cdot\psi_\infty\left(\frac{(s^2-4rt)x_\infty}{4rc}\right)\cdot\psi_\infty\left(\frac{2sx_\infty hc+h^2c^2}{4crx_\infty}\right)
\end{aligned}$$

in this latter expression of (79) as by the integral formula

$$\int_{-\infty}^{\infty}e^{-(A(x_{\infty,j})z_{\infty,j}^2+B(x_{\infty,j})z_{\infty,j}+C(x_{\infty,j}))}dz_{\infty,j}=\sqrt{\frac{\pi}{A(x_{\infty,j})}}\cdot e^{\frac{B(x_{\infty,j})^2-4A(x_{\infty,j})C(x_{\infty,j})}{4A(x_{\infty,j})}}$$

applied to each component of $z_\infty=(z_{\infty,j})_{j=1}^d\in F_\infty\in\mathbf{R}^d$ (in the notations described above) with

$$A(x_{\infty,j})=2\pi i\left(\frac{r_jx_{\infty,j}}{c_j}\right),\quad B(x_{\infty,j})=2\pi i\left(\frac{s_jx_{\infty,j}+h_jc_j}{c_j}\right),\quad C(x_{\infty,j})=2\pi i\left(\frac{t_jx_{\infty,j}}{c_j}\right).$$

In this way, we argue that for some suitable Schwartz function $f'\in\mathcal{S}(N_2(\mathbf{A}_F)\backslash\overline{G}(\mathbf{A}_F);\psi_1)$ modulo $N_2(F_\infty)$, we can reduce to bounding the simpler expression

$$\begin{aligned}
&\psi_\infty\left(\frac{1}{8}\right)\sum_m c_m(\phi)\sum_{c\in\Omega(\Gamma)}\frac{(1-i)^{[F:\mathbf{Q}]}}{|c|^{\frac{1}{2}}} \\
&\times\sum_{\substack{\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma_c/\Gamma_\infty \\ 2d_jr_j\equiv 2 \bmod 4 \ \forall \ 1\leq j\leq[F:\mathbf{Q}]}} \omega(d)\cdot\overline{\epsilon}_{c/2}^{-1}\cdot\left(\frac{c/2}{2dr}\right)\cdot\psi_\infty\left(\frac{am}{c}\right)\cdot\psi_\infty\left(\frac{-d(s^2-4rt)}{4cr}\right) \\
&\times\sum_{(h)}\psi_\infty\left(\frac{-dsh/2-h^2/4}{cdr}\right)\int_{\mathbf{A}_F} f'\left(w\cdot\underline{c}\cdot\begin{pmatrix}1 & x \\ & 1\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{Y_\infty} & \\ & 1\end{pmatrix}\right)\psi_\infty\left(\frac{\Delta}{4rc}\cdot x\right)dx.
\end{aligned}\tag{80}$$

Indeed, after switching the order of summation, each inner (h)-sum in (79) is equivalent to

$$\int_{\mathbf{A}_F} f\left(w\underline{c}\begin{pmatrix}\frac{1}{Y_\infty} & x \\ & 1\end{pmatrix}\right)\left\{\sum_{(h)}\psi_\infty\left(\frac{-dsh/2-h^2/4}{cdr}\right)\int_{F_\infty}\psi_\infty\left(\frac{-q(z_\infty)x_\infty-hcz_\infty}{c}\right)dz_\infty\right\}dx,$$

which we argue can be approximated in terms of the $|x_\infty| \neq 0$ contributions and evaluated as

$$\int_{\substack{x \in \mathbf{A}_F \\ |x_\infty| \neq 0}} f\left(w \underline{c} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix} \right)$$
$$\times \left\{ \left| \frac{c}{2irx_\infty} \right|^{\frac{1}{2}} \sum_{(h)} \psi_\infty \left(\frac{-dsh/2 - h^2/4}{cdr} \right) \psi_\infty \left(\frac{shc/2}{cr} \right) \psi_\infty \left(\frac{h^2 c^2}{4crx_\infty} \right) \right\} \psi_\infty \left(\frac{\Delta}{4rc} \right) dx.$$

Hence we argue that each inner (h)-sum in this latter expression can be approximated as a Gaussian integral and treated as a constant. Now, to interpret (80) in the style of [4, § 3], we make two observations. First, we make a changes of variables $c \to c'' = 4rc$ to each inner sum to relate (80) to the simpler sum

(81)
$$\psi_\infty \left(\frac{1}{8} \right) \sum_m c_m(\phi) \sum_{c \in \Omega(\Gamma)} \frac{(1-i)^{[F:\mathbf{Q}]}}{|4rc|^{\frac{1}{2}}}$$
$$\times \sum_{\gamma = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_{c''} / \Gamma_\infty} \omega(d'') \cdot \left(\frac{c''}{d''} \right) \cdot \overline{\epsilon}_{c''/2}^{-1} \cdot \psi_\infty \left(\frac{a'' m}{c''} \right) \cdot \psi_\infty \left(- \frac{d'' \Delta}{c''} \right) \mathcal{F}_{f', -\Delta, c''} \left(\begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix} \right),$$

again for some suitable function $f' \in \mathcal{S}(N_2(\mathbf{A}_F) \backslash \overline{G}(\mathbf{A}_F); \psi_1)$. We then see that the inner sum can be related to the generalized Kloosterman sum of modulus $4rc$, central character ω, and half-integral weight theta multiplier ϑ given by the expansion

$$\mathrm{Kl}_{\Gamma, \omega, \vartheta}(m, -\Delta; 4rc) := \sum_{\gamma = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_{4rc} / \Gamma_\infty} \omega(d'') \cdot \left(\frac{c''}{d''} \right) \cdot \epsilon_{c''/2}^{-1} \cdot \psi_\infty \left(\frac{a'' m}{c''} \right) \psi_\infty \left(\frac{-d'' \Delta}{c''} \right).$$

Here, we take for granted the description given in Gelbart [21, Proposition 2.16] (for instance), or the explicit classical realization described in Proskurin [47] (which carries over component-wise) of ϑ. Writing $\mathcal{P}_m$ to denote the corresponding Poincaré series defined on $\overline{g} \in \overline{G}(\mathbf{A}_F)$ by

$$\mathcal{P}_m(\overline{g}) = P_{f', \omega, \vartheta}(\overline{g}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \omega(\gamma) \cdot \overline{\vartheta(\gamma)} \cdot f'(\gamma \overline{g})$$

for each m, it is then simple to deduce from (77) and (78) that it is enough to bound the sum

$$\sum_m c_m(\phi) \cdot W_{\mathcal{P}_m} \left(\begin{pmatrix} -\frac{\Delta}{Y_\infty} & \\ & 1 \end{pmatrix} \right)$$

to derive a bound for the initial sum (52), where for each m we write $W_{\mathcal{P}_m}$ to denote the unipotent integral

$$\begin{aligned} W_{\mathcal{P}_m} \left(\begin{pmatrix} -\frac{\Delta}{Y_\infty} & \\ & 1 \end{pmatrix} \right) &= W_{\mathcal{P}_m} \left(\begin{pmatrix} -\frac{\Delta}{Y_\infty} & \\ & 1 \end{pmatrix}, 1 \right) \\ &= \int_{\mathbf{A}_F / F} \mathcal{P}_m \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}, 1 \right) \psi(-\Delta x) dx \\ &= \int_{I \cong [0,1]^d \subset F_\infty} \mathcal{P}_m \left(\begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{Y_\infty} & \\ & 1 \end{pmatrix}, 1 \right) \psi(-\Delta x_\infty) dx_\infty. \end{aligned}$$

But this reduces us to a special case of the proof of Theorem 2.4, with the totally positive nonzero F-integer $-\Delta$ replacing q. That is, we decompose each of the Poincaré series P_m spectrally, then pass to unipotent integrals to bypass the Kuznetsov trace formula. Since the level is divisible by the leading coefficient r, we use Weyl's law to deduce that there will be roughly $\ll_\pi \mathbf{N}r$ contributions. Applying the previous argument to the spectral decomposition(s) of the Poincaré series then gives us the bound

$$\sum_{(n)} W_\phi \left(\begin{pmatrix} \frac{q(n)}{Y_\infty} & \\ & 1 \end{pmatrix} \right) \ll \sum_m c_m(\phi) \cdot W_{\mathcal{P}_m} \left(\begin{pmatrix} -\frac{\Delta}{Y_\infty} & \\ & 1 \end{pmatrix} \right) \ll_{\pi,\varepsilon} Y^{\frac{1}{4}} \cdot \mathbf{N}r \cdot \mathbf{N}\Delta^{\delta_0 - \frac{1}{2}} \left(\frac{\mathbf{N}\Delta}{Y} \right)^{\frac{1}{2} - \frac{\theta_0}{2} - \varepsilon}.$$

Here, we have to multiply in the factor of $Y^{\frac{1}{4}}$ at the last step to compensate for the fact that the Fourier expansions (and hence Fourier coefficients) of genuine parallel half-integral weight forms evaluated at the matrix $\mathrm{diag}(1/Y_\infty, 1)$ are proportional to $Y^{-\frac{1}{4}} = |Y_\infty|^{-\frac{1}{4}}$. Again, the m-sum here is finite thanks to Corollary A.8, with the coefficients $c_m(\phi)$ depending only on π and the underlying holomorphic Hilbert modular form, and so this finishes the proof.

Appendix B. Iwasawa main conjectures for GL_2 over CM fields

Let F be a totally real number field of degree d, integers $\mathcal{O}_F$, and adeles $\mathbf{A}_F$. Let π be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation of conductor $c(\pi) \subset \mathcal{O}_F$ and unitary central character $\omega = \omega_\pi$. Let K be a totally imaginary quadratic extension of F of relative discriminant $\mathfrak{D} = \mathfrak{D}_{K/F} \subset \mathcal{O}_F$. Fix a prime $\mathfrak{p} \mid p$ in F of residue degree $\delta = \delta_\mathfrak{p} = [F_\mathfrak{p} : \mathbf{Q}_p]$. We explain in this self-contained section how to deduce many cases of the Iwasawa-Greenberg main conjectures for π in the maximal pro-p abelian extension of K unramified outside of $\mathfrak{p}$ when π corresponds to a holomorphic Hilbert modular form of parallel weight two. We then explain how to use such results together with the nontriviality of the corresponding p-adic L-functions to derive bounds for Mordell-Weil ranks of abelian varieties in this $\mathbf{Z}_p^r$-extension of K, where $r = \delta + 1$. In particular, we obtain results for the cyclotomic $\mathbf{Z}_p$-extension of K without using a new Euler system construction, and hence without generalizing the construction of Kato [32] to number fields.

Iwasawa main conjectures. We first describe some relevant divisibilities for the so-called r-variable Iwasawa-Greenberg main conjecture (see [61, Conjecture 1]) that can be deduced from various works in the "basechange anticyclotomic variables". We assume $\pi = \otimes_v \pi_v$ is associated to a p-ordinary cuspidal Hilbert eigenform of parallel weight 2 and trivial character.[15] We then have by constructions of Carayol [9], Taylor [54], and Wiles [62] a Galois representation $\rho_\pi : G_F := \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow \mathrm{GL}_2(\mathcal{O})$ subject to the usual conditions so that $L(s, \rho_\pi) = L(s, \pi)$. Let us use the setup of [61, §1], and hence assume the following

Hypothesis B.1. *The following conditions are met:*

(1) *The fixed prime p is not ramified in F, and each prime $\mathfrak{p} \mid p \subset \mathcal{O}_F$ splits in K.*

(2) *The totally imaginary quadratic extension K/F is not contained in the narrow class field of F.*

(3) *The residual representation $\overline{\rho}_\pi$ of ρ_π is irreducible in the sense of* [61, (irred)].

(4) *The $\mathcal{O}_L^\times$-valued characters giving the actions of $G_{F_\mathfrak{p}}$ on $V_\mathfrak{p}$ and $V_\mathfrak{p}/V_\mathfrak{p}^+$ are distinct in the sense of* [61, (dist)]. *Hence according to* [61, §1.1], *for each prime $\mathfrak{p} \mid p \subset \mathcal{O}_F$, the restriction of ρ_π to the decomposition group $G_{F_\mathfrak{p}} \cong \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\mathfrak{p})$ is isomorphic to some upper triangular representation $V_\mathfrak{p}$ admitting a distinguished one-dimensional subspace $V_\mathfrak{p}^+$ with proscribed Galois action. Fixing L a finite extension of $\mathbf{Q}_p$ which is sufficiently large to contain the Hecke field $\mathbf{Q}(\pi)$, we assume this condition on the $\mathcal{O}_L^\times$-valued characters giving the actions of $G_{F_\mathfrak{p}}$.*

Let K_∞ denote the $\mathbf{Z}_p^r$-extension of K, this being the compositum of the anticyclotomic $\mathbf{Z}_p^\delta$-extension of K with the cyclotomic $\mathbf{Z}_p$-extension of K. In general, for each prime $\mathfrak{p}$ dividing p in $\mathcal{O}_F$, we can associate to the Galois representation $\rho_\pi = V$ a Selmer group $\mathrm{Sel}(\pi, K_\infty)$. In the setting where π is associated to an abelian variety A/F (as described below), this coincides with the classical $\mathfrak{p}$-primary Selmer group of A over K_∞. The first part of the Iwasawa-Greenberg main conjecture for GL_2 over K asserts that the Pontryagin dual $X(\pi, K_\infty)$ of $\mathrm{Sel}(\pi, K_\infty)$ has the structure of a finitely generated torsion $\mathcal{O}[[G]]$-module, in which case it has a characteristic power series. Let us write $\mathrm{char}(X(\pi, K_\infty)) = \mathrm{char}_{\mathcal{O}[[G]]}(X(\pi, K_\infty)) \in \mathcal{O}[[G]]$

[15] Note that we also could consider the more general setting described in [61, Conjecture] here, but at the expense of clarity.

to denote this characteristic power series. We know thanks to various constructions (among them the one given in [61], see also [17], [19], [26], [28], and [45]) that there exists for each $\mathfrak{p} \mid p$ a p-adic L-function $\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty) \in \mathcal{O}[[G]]$. The main conjecture for GL_2 over K then posits that we have an equality of principal ideals $(\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty)) = (\mathrm{char}(X(\pi, K_\infty)))$ in $\mathcal{O}[[G]]$.

Theorem B.2. *Let $p \geq 5$ be a rational prime, and let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ associated to a Hilbert modular eigenform of parallel weight 2, level $\mathfrak{c}(\pi)$, and trivial nebentype character satisfying Hypothesis B.1 above. Assume that $(\mathfrak{c}(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Writing $\mathfrak{c}(\pi)^+\mathfrak{c}(\pi)^-$ to denote the decomposition of $\mathfrak{c}(\pi)\mathcal{O}_K$ into a product of split primes $\mathfrak{c}(\pi)^+$ and inert primes $\mathfrak{c}(\pi)^{-1}$, let us also assume (5) that $\mathfrak{c}(\pi)^-$ is the squarefree product of a number of primes $v \subset \mathcal{O}_F$ congruent to $d \bmod 2$, hence the corresponding root number $\epsilon \in \{\pm 1\}$ of $L(s, \pi_K)$ is $+1$. As well, let us assume (6) that the residual representation $\overline{\rho}_\pi$ is ramified at each prime $v \mid \mathfrak{c}(\pi)^- \subset \mathcal{O}_F$, and (7) that the following technical conditions are met:*

(A) *The restriction $\overline{\rho}_\pi|_{F(\zeta_p)}$ of $\overline{\rho}_\pi$ to the field $F(\zeta_p)$ obtained by adjoining a primitive p-th root of unity ζ_p to F is absolutely irreducible.*

(B) *The following case is excluded when $p = 5$: the projective image J of $\overline{\rho}_\pi$ is isomorphic to $\mathrm{PGL}_2(\mathbf{F}_p)$, and the mod p cyclotomic character factors through $G_F \longrightarrow J^{\mathrm{ab}} \approx \mathbf{Z}/2\mathbf{Z}$.*

(C) *There is a minimal modular lifting of $\overline{\rho}_\pi$.*

(D) *Ihara's lemma for Shimura curves over totally real number fields is true; see* [57, Hypothesis 11.5].

(E) *For each finite place v of F, if $\overline{\rho}_\pi|_{I_{F_v}}$ is absolutely irreducible, then the cardinality of the residue field at v is not congruent to $-1 \bmod p$.*

(F) *The completed character group associated to the residual representation $\overline{\rho}_\pi$ satisfies the multiplicity one condition of* [57, Hypothesis 11.5].

Then, the r-variable main conjecture is true: We have $(\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty)) = (\mathrm{char}(X(\pi, K_\infty)))$ in $\mathcal{O}[[G]]$.

Proof. The result can be derived from [59, Proposition 4.3], which carries over with minor changes to deal with the more general setting of $\delta > 1$, along with [61, Theorem 3], [36, Theorem 1.5], and [57, Theorem 1.2]. That is, putting together [61, Theorem 3] with [36, Theorem 1.5] shows that we have an equality of principal ideals $(\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty)|_\Omega) = (\mathrm{char}(X(\pi, K_\infty)|_\Omega))$ in $\mathcal{O}[[\Omega]]$. Writing K_n for an integer $n \geq 1$ to denote the degree-p^n extension of K contained in K^{cyc}, with $\Omega^{(n)} = \mathrm{Gal}(K_n D_\infty / K_n) \approx \mathbf{Z}_p^\delta$, we obtain from [57, Theorem 1.2] (b) that for each $n \geq 1$, there is an inclusion of ideals $(\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty)|_{\Omega^{(n)}}) \subseteq (\mathrm{char}(X(\pi, K_\infty)|_{\Omega^{(n)}}))$ in $\mathcal{O}[[\Omega^{(n)}]]$. Using (a) and (b) as input for [59, Proposition 4.3], we deduce there is an inclusion of ideals $(\mathcal{L}_{\mathfrak{p}}(\pi, K_\infty)) \subseteq (\mathrm{char}(X(\pi, K_\infty)))$ in $\mathcal{O}[[G]]$. The other inclusion is the proven in [61, Theorem 3]. □

Applications to Mordell-Weil ranks. Let us again write $\mathbf{Q}(\pi)$ to denote the Hecke field of π. It is conjectured and known in many cases that one can associate to π an abelian variety $A = A_\pi$ over F (defined uniquely up to isogeny) for which the following properties hold:

(i) The dimension of A equals the degree of the Hecke field of π, i.e. $\dim(A) = [\mathbf{Q}(\pi) : \mathbf{Q}]$.

(ii) The ring of endomorphisms of A is given by the integers of the Hecke field of π, i.e. $\mathrm{End}_F(A) = \mathcal{O}_{\mathbf{Q}(\pi)}$.

(iii) The Hasse-Weil L-function $L(s, A/F)$ of A over F is described (Euler factor for Euler factor) in terms of the finite part $L(s, \pi)$ of the standard L-function of π by the relation
$$L(s - 1/2, \pi) = \Gamma_{\mathbf{C}}(s) L(s, A/F), \text{ where } \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-1}\Gamma(s).$$

The first two conditions (i) and (ii) imply that A/F is of GL_2-*type*, and also of *strict* GL_2-*type* in the sense of [65, §3.2]. Let $Ш(A/K_\infty)$ denote the Tate-Shafarevich group of A over K_∞, with $Ш(A/K_\infty)[p^\infty]$ its

p-primary subgroup. Using the Kummer exact sequence

$$0 \longrightarrow A(K_\infty) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathrm{Sel}(\pi, K_\infty) \longrightarrow Ш(A/K_\infty)[p^\infty] \longrightarrow 0$$

with the interpolation property satisfied by $\mathcal{L}_\mathfrak{p}(\pi, K_\infty)$ (described in Theorem 3.2 above) we can derive the following result from Theorem B.2. Given a finite-order character $\mathcal{W}$ of G, let us write $\pi(\mathcal{W})$ to denote the corresponding induced representation of $\mathrm{GL}_2(\mathbf{A}_F)$, and also $L(s, \pi \times \mathcal{W}) = L(s, \pi \times \pi(\mathcal{W}))$ to the denote the corresponding $\mathrm{GL}_2(\mathbf{A}_F) \times \mathrm{GL}_2(\mathbf{A}_F)$ Rankin-Selberg L-function of π times $\pi(\mathcal{W})$.

Corollary B.3. *Assume the conditions of Theorem B.2. If for some finite-order character $\mathcal{W}$ of G the central value $L(1/2, \pi \times \mathcal{W}) = L(1/2, \pi \times \pi(\mathcal{W}))$ does not vanish, then the $\mathcal{W}$-isotypical components of both $A(K_\infty)$ and $Ш(A/K_\infty)[p^\infty]$ are finite.*

Hence, we obtain from Theorem 2.12 the following result.

Theorem B.4. *Let $p \geq 5$ be a prime, and let $\pi \cong \widetilde{\pi}$ be a cuspidal $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation attached to a Hilbert modular eigenform of parallel weight 2, level $\mathfrak{c}(\pi)$, and trivial character satisfying Hypothesis B.1. Assume the hypotheses of Theorem B.2. Assume as well that that the Hecke field $\mathbf{Q}(\pi)$ is linearly disjoint over $\mathbf{Q}$ to the cyclotomic tower $\mathbf{Q}(\zeta_{p^\infty}) = \bigcup_{n \geq 1} \mathbf{Q}(\zeta_{p^n})$. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$. and absolute discriminant D_K. Let $\rho = \rho_w$ be any ring class character factoring through $G = \mathrm{Gal}(K_\infty/K)$. There exists an integer $\beta_0(\rho)$ such that for all characters $\psi = \psi_w$ of the cyclotomic Galois group $\Gamma = \mathrm{Gal}(K^{\mathrm{cyc}}/K)$ of exact order p^β with $\beta \geq \beta_0(\rho)$, the central value $L(1/2, \pi \times \rho\psi)$ does not vanish, and hence the corresponding $\rho\psi$-isotypical components of both $A(K_\infty)$ and $Ш(A/K_\infty)[p^\infty]$ are finite.*

Proof. The result follows from Theorem 2.12 (via Corollary 2.10), using the Weierstrass preparation theorem in the cyclotomic variable of the corresponding $(\delta_\mathfrak{p} + 1)$-variable p-adic L-function for each choice of ρ. $\square$

We can also deduce the following rank formula. Let $\epsilon \in \{\pm 1\}$ denote the sign of the Hasse-Weil L-function $L(s, A/K)$ of A over K, equivalently the root number $\epsilon(1/2, \pi_K)$ of the L-function $L(s, \pi_K)$ of the basechange $\pi_K = \mathrm{BC}_{K/F}(\pi)$ of π to K. Let K^{cyc} denote the cyclotomic $\mathbf{Z}_p$-extension of K, with Galois group $\Gamma = \mathrm{Gal}(K^{\mathrm{cyc}}/K) \cong \mathbf{Z}_p$. Let D_∞ denote the anticyclotomic $\mathbf{Z}_p^\delta$-extension of K, with Galois group $\Omega = \mathrm{Gal}(D_\infty/K) \cong \mathbf{Z}_p^\delta$. Let K_∞ denote the compositum $D_\infty K^{\mathrm{cyc}}$ of these extensions, with Galois group $G = \mathrm{Gal}(K_\infty/K) \approx \mathbf{Z}_p^r$.

Theorem B.5. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ associated to a Hilbert modular eigenform of parallel weight 2, conductor $\mathfrak{c}(\pi)$, and trivial character. Let K/F be a totally imaginary quadratic extension of relative discriminant $\mathfrak{D} \subset \mathcal{O}_F$ and absolute discriminant D_K. Suppose that π has associated to it an abelian variety A/F satisfying conditions (i), (ii), and (iii), and that $(\mathfrak{c}(\pi), \mathfrak{D}\mathfrak{p}) = (\mathfrak{p}, \mathfrak{D}) = 1$. Assume additionally that the following conditions hold: (iv) if A acquires CM after basechange to some quadratic extension K_π/F, then this extension K_π is not contained in K_∞ when $\epsilon = +1$, and (v) A has good ordinary reduction at all primes above p in F. Finally, if the residue degree $\delta = [F_\mathfrak{p} : \mathbf{Q}_p]$ is greater than one, then we also assume the conditions of Theorems B.2, 3.7, and 2.12 above (including the vanishing of the anticyclotomic μ-invariant), and that the absolute discriminant D_K is sufficiently large. Then, $A(K_\infty)$ is finitely generated if $\epsilon = +1$, and otherwise $A(K_\infty)/A(D_\infty)$ is finitely generated if $\epsilon = -1$.*

Proof. Note that if $\delta = 1$, then we can use the argument of [59, Proposition 3.14] (cf. [58, Part II, §1]) to deduce the result. In brief, this argument shows that we have the rank formula

$$\mathrm{corank}_{\mathcal{O}[[H]]}\left(A(K_\infty) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right) = \begin{cases} 0 & \text{if } \epsilon = +1 \\ 1 & \text{if } \epsilon = -1, \end{cases} \tag{82}$$

where $\mathcal{O}$ is a finite extension of $\mathbf{Z}_p$ large enough to contain the integers of $\mathbf{Q}(\pi)$, $H = \mathrm{Gal}(K_\infty/K^{\mathrm{cyc}}) \cong \mathbf{Z}_p^\delta$, and $\mathcal{O}[[H]]$ is the $\mathcal{O}$-Iwasawa algebra of H, in other words the completed group ring coming from $\mathcal{O}$-valued measures on H. We deduce from such a formula when $\delta = 1$ that $A(K_\infty)$ is finitely generated when $\epsilon = +1$, and (using the nontriviality theorem of [15, Theorem 1.15] with the formula of [65]) that $A(K_\infty)/A(K_{\mathfrak{p}^\infty})$ is finitely generated when $\epsilon = -1$.

Suppose now that $\delta \geq 2$. If $\epsilon = -1$, then the stated formula can be deduced again from the argument of [59, Proposition 3.14], using the nontriviality theorem of [15, Theorem 1.15] with [65, Theorem 1.2] to derive a suitable growth-of-rank formula for each of the anticyclotomic lines. To describe this, let us for each integer $n \geq 0$ write K_n to denote the extension of degree p^n contained in the cyclotomic extension K^{cyc}. Consider the anticyclotomic extension defined by the compositum $K_n D_\infty$, which is contained in the anticyclotomic $\mathbf{Z}_p^{\delta p^n}$-extension of K_n. Let $\Omega^{(n)} = \mathrm{Gal}(K_n D_\infty / K_n) \cong \mathbf{Z}_p^\delta$ denote the corresponding Galois group. Using the argument with results mentioned above, we derive the rank formula

$$\mathrm{corank}_{O[[\Omega^{(n)}]]}\left(A(K_n D_\infty) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right) = 1$$

for each integer $n \geq 0$, where $\mathcal{O}[[\Omega^{(n)}]]$ denotes the corresponding Iwasawa algebra. Passing to the limit implies the corresponding $\mathcal{O}[[H]]$-module formula (82). Suppose now that $\epsilon = +1$. Fix an integer $n \geq 0$. We can use [42, Theorem (A$'$)], fixing an auxiliary prime l and $\mathfrak{l} \mid l$ in $\mathbf{Q}(\pi)$ satisfying conditions [42, (A1$'$), (A2$'$), (A3$'$)], to deduce the following implication: If for any finite-order character $\rho^{(n)}$ of $\Omega^{(n)}$ (arising via composition with the norm $\rho^{(n)} = \rho \circ \mathbf{N}$ from a character ρ of $\Omega^{(0)}$) the central value

$$L(1/2, \pi^{(n)} \times \rho^{(n)}) = \prod_{\psi \in \mathrm{Gal}(K_n/K)^\vee} L(1/2, \pi \times \rho\psi) = L(1, A/K_n, \rho^{(n)})$$

does not vanish, then the $\rho^{(n)}$-isotypical component of the Mordell-Weil group $A(K_n[c(\rho^{(n)})])$ is finite. Here, $\pi^{(n)}$ denotes the basechange of π to the degree-p^n extension of F contained in its cyclotomic $\mathbf{Z}_p$-extension, with $L(s, \pi^{(n)} \times \rho^{(n)})$ the corresponding Rankin-Selberg L-function, and the first equality denotes the Artin decomposition into Rankin-Selberg L-functions of $\mathrm{GL}_2(\mathbf{A}_F)$-representations. As well, $K_n[c(\rho^{(n)})]$ denotes the ring class extension of K_n of conductor $c(\rho^{(n)})$, where $c(\rho^{(n)})$ denotes the conductor of $\rho^{(n)}$. This allows us to deduce from Theorem 2.12 that

$$\mathrm{corank}_{O[[\Omega^{(n)}]]}\left(A(K_n D_\infty) \otimes \mathbf{Q}_p/\mathbf{Z}_p\right) = 0$$

for each integer $n \geq 0$. Passing to the limit again implies the corresponding $\mathcal{O}[[H]]$-module formula (82).

To deduce the stated formula for Mordell-Weil ranks in either case on the sign ε when $\delta \geq 2$, we argue as follows these $\mathcal{O}[[H]]$-module formulae will suffice. Let us now impose the conditions of Theorem B.2. We are then reduced (via Theorem B.4) to considering specializations of the underlying p-adic L-function $\mathcal{L}_\mathfrak{p}(\mathbf{1}; T_1, \cdots, T_\delta, T_{\delta+1})$ to check the claim directly. Assuming the conditions of Theorem 3.7, we deduce that there exists a minimal anticyclotomic exponent α_0 such that each character ρ_w of Ω of exact order p^α with $\alpha \geq \alpha_0$, the corresponding specialization $\mathcal{L}_\mathfrak{p}(\mathbf{1}; \rho_w(\gamma_1) - 1, \cdots, \rho_w(\gamma_\delta - 1, \psi_w(\gamma_{\delta+1}) - 1)$ does not vanish for any character ψ_w factoring through the cyclotomic Galois group $\Gamma \cong \mathbf{Z}_p$. To deal with the remaining anticyclotomic exponents $\leq \alpha_0$, we apply the result of Corollary 2.10 for the setting associated to each character ρ_w of $\Omega \cong \mathbf{Z}_p^\delta$ of exact order p^α with $0 \leq \alpha \leq \alpha_0$ and $\beta = 1$ to deduce the claim, e.g. after applying the Weierstrass preparation theorem to the corresponding basechange cyclotomic variable. $\square$

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