HEIGHTS OF ALGEBRAIC ONE-CYCLES ON TWISTED SIEGEL THREEFOLDS

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Abstract. We compute archimedean local heights of certain compact and diagonal one-cycles on twisted Siegel threefolds realized as spin Shimura varieties. This gives evidence for the arithmetic Rallis inner product formula for Saito-Kurokawa lifts, as well as the corresponding conjecture of Beilinson-Bloch.

Contents

1. Introduction

Arithmetic heights of algebraic cycles should appear as regulators and central derivative values of automorphic L-functions, following conjectures of Birch-Swinnerton-Dyer, Beilinson-Bloch, Bloch-Kato, Deligne, and Langlands. In the setting of Shimura varieties, particularly those of orthogonal type, Kudla's programme relates the generating series of algebraic cycles to derivatives of Siegel Eisenstein series. In the special setting of Siegel threefolds realized as spin Shimura varieties, the conjectural "arithmetic Rallis inner product formula" refines this programme in the spirit of the Rallis inner product formula and higher Gross-Zagier formulae, linking inner products of these generating series to central derivative values of spin L-functions of $\mathrm{GSp}_4(\mathbf{A})$-automorphic forms, to realize the corresponding Beilinson-Bloch conjecture for rank one.

We obtain results in this direction for $\mathrm{GSp}_4(\mathbf{Q})$-automorphic forms arising as Saito-Kurokawa lifts from $\mathrm{GL}_2(\mathbf{A})$, using the theory of regularized Borcherds theta lifts developed by Kudla [33] (cf. [32], [41], [49]), Bruinier-Yang [11], and Andreatta-Goren-Howard-Madapusi Pera [1] to show higher Gross-Zagier formulae in terms of Faltings heights of arithmetic special divisors evaluated along CM-cycles of spin Shimura varieties. Here, we compute archimedean local heights of certain pairs of codimension-two cycles in terms of integrals of Green's currents realized in terms of pullback-pushforward of Green's functions for the threefold, after "passing to a divisor". Realizing the archimedean local height in this way, we compute special cases of such an "arithmetic Rallis inner product formula" corresponding to Saito-Kurokawa lifts from $\mathrm{GL}_2(\mathbf{A})$-automorphic forms, with applications to the corresponding conjecture of Beilinson-Bloch for rank one. We explain this intricate setup in more detail for Theorem 1.3, Conjecture 1.7, and Theorem 1.8 below. Roughly speaking, we pair together a one-cycle Y given by a projective curve and a diagonal one-cycle $Z(f)$ parametrized by a harmonic Maass form f of weight $-1/2$ on the twisted quaternionic Siegel threefold X determined by the spin group of a rational quadratic space (V, Q) of signature $(3, 2)$ and even lattice $L \subset V$. We prove an analogue of the Gross-Zagier formula [22] (cf. [11], [1]) in Theorem 1.3. We then explain how this fits into the conjectures of Beilinson-Bloch and Bloch-Kato for $\mathrm{GSp}_4(\mathbf{A})$-automorphic forms arising as Saito-Kurokawa lifts from $\mathrm{GL}_2(\mathbf{A})$ via Conjecture 1.7 and Theorem 1.8.

1.1. A height formula.

Fix (V, Q) a rational quadratic space of signature $(3, 2)$, with V denoting the vector space over $\mathbf{Q}$, and $Q : V \rightarrow \mathbf{Q}$ the quadratic form of signature $(3, 2)$. Let $\mathrm{GSpin}(V)$ denote the general spin group of V, so the reductive algebraic group over $\mathbf{Q}$ which sits in a short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

Each maximal even lattice $L \subset V$ determines a compact open subgroup $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$, and we consider the corresponding spin Shimura variety $X_L = X_{K_L}$ with complex points given by

$$X_L(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q}) \backslash \mathrm{GSpin}(V)(\mathbf{A}_f) \times D(V)/K_L,$$

where

$$D(V) = \{z \subset V(\mathbf{R}) : \dim(z) = 2, Q|_z < 0\}$$

denotes the Grassmannian of oriented negative hyperplanes in $V(\mathbf{R})$. This hermitian symmetric domain can be identified with the Siegel upper-half plane $\mathcal{H}_2$ of genus 2. Moreover, $\mathrm{GSpin}(V)$ is an inner form of GSp_4; there is an exceptional isomorphism of algebraic groups $\mathrm{GSpin}(V) \cong \mathrm{GSp}_4$ over $\overline{\mathbf{Q}}$, defined over any number field F over which the quadratic space (V, Q) splits. Hence if V is split over $\mathbf{Q}$, then we can identify

$$X_L \cong \mathrm{Sh}_{K_L}(\mathrm{GSp}_4, \mathcal{H}_2)$$

as a Siegel threefold representing a moduli space of abelian surfaces with extra structure. More generally, X_L is a twisted (quaternionic) version of the Siegel threefold $\mathrm{Sh}_{K_L}(\mathrm{GSp}_4, \mathcal{H}_2)$ over $\mathbf{Q}$. We describe this setup in (15) and Proposition 2.2 beflow. In summary, X_L is a quasiprojective variety defined over $\mathbf{Q}$ which is compact if and only if $V = (V, Q)$ is anisotropic. Since any indefinite quadratic form Q over $\mathbf{Q}$ in five or more variables is isotropic by Meyer's theorem [48], [51], the threefold X_L is never compact[1]. It is smooth if K_L is neat, in which case it represents a moduli problem by Kudla-Rapoport [41] (Proposition 2.2).

1.1.1. *Special cycles.* The spin Shimura variety $X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$ comes equipped with a special m-cycle $Z(U)$ of codimension $r := 3 - m$ for each rational quadratic subpace $U \subset V$ of signature $(m, 2)$ (for $0 \leq m \leq 2$), as studied systematically by Kudla [31]. To describe this briefly, each such subspace U has a corresponding spin group $\mathrm{GSpin}(U) \subset \mathrm{GSpin}(V)$, Grassmannian $D(U) \subset D(V)$, and lattice $L_U = L \cap U$, and determines a Shimura subvariety $Z(U) \subset X_L$ of codimension r, with complex points given by

$$Z(U)(\mathbf{C}) = \mathrm{GSpin}(U)(\mathbf{Q}) \backslash \mathrm{GSpin}(U)(\mathbf{A}_f) \times D(U)/K_{L_U}, \quad K_{L_U} = K_L \cap \mathrm{GSpin}(U)(\mathbf{A}_f).$$

For each $g \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we can define a g-conjugated cycle $Z(U, g) \in Z^r(X_K)$ with complex points

$$Z(U, g)(\mathbf{C}) = \mathrm{GSpin}(U)(\mathbf{Q}) \backslash \mathrm{GSpin}(U)(\mathbf{A}_f) \times D(U)/K_{L_U, g}, \quad K_{L_U, g} = g K_{L_U} g^{-1}.$$

To describe such cycles more explicitly, we fix a vector $x = (x_i)_{i=0}^{r-1} \in V^r$ for which $Q(x) > 0$, so that the corresponding span $W = \mathbf{Q}(x) = \mathbf{Q}(x_0, \ldots, x_{r-1})$ determines a positive definite subspace $W \subset V$ of signature $(r, 0)$. The orthogonal complement $U = W^{\perp} \subset V$ then determines a quadratic subspace $(U, Q_U) = (U, Q|_U)$

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[1] Strictly speaking, we should pass to the compactification of X_L in parts of this discussion, or to a number field F over which there are suitable anisotropic quadratic forms of signatures $(3, 2)$ and $(1, 2)$ if (V, Q) is split.

of signature $(m,2)=(3-r,2)$, as well as a an m-cycle $Z(U)\in Z^r(X_L)$, equivalently a subvariety $Z(U)\subset X_L$ of codimension r. We also consider the following weighted cycles. Write

$$(x,x)=((x_i,x_j))_{1\leq i,j\leq r}\in \mathrm{Sym}_r(\mathbf{Q})$$

to denote the symmetric matrix of inner products $(x_i,x_j)=Q(x_i+x_j)-Q(x_i)-Q(x_j)$. Given a positive definite symmetric matrix $\beta={}^t\beta\in\mathrm{Sym}_r(\mathbf{Q})$, we consider the corresponding hyperboloid

$$\Omega_\beta(\mathbf{Q})=\left\{x\in V^r:(x,x)=\beta\right\}.$$

Fix $x\in\Omega_\beta(\mathbf{Q})$. Given $\varphi\in\mathcal{S}(V(\mathbf{A}_f)^r)$ a K_L-invariant Schwartz function, we consider the decomposition

$$\mathrm{supp}(\varphi)\cap\Omega_\beta(\mathbf{A}_f)=\coprod_\zeta K_L\zeta^{-1}x,$$

where the product runs over a fixed set of representatives $\zeta\in\mathrm{GSpin}(V)(\mathbf{A}_f)$. We then consider the cycle $Z(\beta,\varphi)\in Z^r(X_K)$ given by the weighted sum

$$Z(\beta,\varphi)=\sum_\zeta\varphi(\zeta^{-1}x)Z(U,\zeta).$$

We note that by Witt's theorem, this weighted cycle is independent of the choice of basepoint $x\in\Omega_\beta(\mathbf{Q})$ and coset representatives $\{\zeta\}$. We shall often take $\varphi=\mathbf{1}_\mu\otimes\cdots\otimes\mathbf{1}_\mu$ for

$$\mathbf{1}_\mu=\mathrm{char}\left(\mu+L\otimes\widehat{\mathbf{Z}}\right)$$

the characteristic function of a coset μ in the discriminant group $L^\vee/L$ of L. For codimension $r=1$, this construction gives the well-studied (arithmetic) divisors

$$Z(m,\mu)=Z(m,\mathbf{1}_\mu)\in Z^1(X_L) \tag{1}$$

appearing in the theory of Borcherds regularized theta lifts (see e.g. [33], [7]). Here, we shall be primarily interested in the setting of codimension $r=2$, with diagonal cycles coming from any choice of vector $x=(x_0,x_1)\in V^r$ with $(x_0,x_0)=(x_1,x_1)=m$ for $m\in\mathbf{Q}_{>0}$,

$$Z(\beta,(\mu,\mu))=Z\left((x,x),\mathbf{1}_\mu\otimes\mathbf{1}_\mu\right)=Z\left(\begin{pmatrix}m&a\\a&m\end{pmatrix},\mathbf{1}_\mu\otimes\mathbf{1}_\mu\right)\in Z^2(X_L),\quad a=(x_0,x_1)=(x_1,x_0). \tag{2}$$

1.1.2. *Setup: projective curves against diagonal one-cycles for harmonic Maass forms.* To describe our main result, fix an anisotropic subspace $V_{1,2}\subset V$ of signature $(1,2)$, with corresponding algebraic one-cycle $Y=Z(V_{1,2})\in Z^2(X_L)$. Hence, $Y=Z(V_{1,2})$ determines a projective curve on X_L, with complex points

$$Y(\mathbf{C})=\mathrm{GSpin}(V_{1,2})(\mathbf{Q})\backslash\,\mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)\times D(V_{1,2})/K_{L_{1,2}},\quad K_{L_{1,2}}=K_L\cap\mathrm{GSpin}(V_{1,2})(\mathbf{A}_f).$$

On the other hand, let $f(\tau)=f^+(\tau)+f^-(\tau)\in H_{-1/2}(\omega_L)$ be a harmonic weak Maass form of weight $-1/2=1-3/2$ and Weil representation ω_L in $\tau=u+iv\in\mathfrak{H}$ (see Definition 4.3), with holomorphic part

$$f^+(\tau)=\sum_{m\gg-\infty}\sum_{\mu\in L^\vee/L}c_f^+(m,\mu)e(m\tau)\mathbf{1}_\mu,\quad e(z):=\exp(2\pi iz).$$

We assume that the Fourier coefficients $c_f^+(m,\mu)$ are integers for all $m<0$. We can then construct from these coefficients the diagonal one-cycle $Z(f)\in Z^2(X_L)$ defined by the corresponding linear combination

$$Z(f)=\sum_{m>0}\sum_{\mu\in L^\vee/L}c_f^+(-m,\mu)Z\left(\begin{pmatrix}m&a\\a&m\end{pmatrix},(\mu,\mu)\right)\in Z^2(X_L). \tag{3}$$

Here, we use the same conventions as in (2) above. Hence, for each coset $\mu\in L^\vee/L$, we choose for each of the finitely many $m\in\mathbf{Z}+Q(\mu)$ contributing to the sum a vector $x=(x_0,x_1)\in V^2$ with $m=(x_0,x_0)=(x_1,x_1)$ and $a=(x_0,x_1)=(x_1,x_0)$; the corresponding cycle $Z(f)\in Z^2(X_L)$ does not depend on these choices.

We compute the archimedean local height $[Y,Z(f)]_\infty=[Z(V_{1,2}),Z(f)]_\infty$ of this pair of one-cycles on X_L.

1.1.3. *Characterization of archimedean local heights.* To define the archimedean local height, the theorem of Gillet-Soulé (Theorem 4.5) associates to each m-cycle $Z \in Z^r(X_L)$ a Green's current $g_Z \in D^{r-1,r-1}(X_L)$ characterized by its inclusion in the Green's current equation

$$dd^c g_Z + \delta_Z = [\omega_Z] \in D^{r,r}(X_L),$$

where δ_Z denotes the Dirac current of integration, and $\omega_Z \in A^{r,r}(X_L)$ a smooth, Poincaré dual representative of the homological cycle $[Z]$. Such a current is unique up to the addition of an element of $\mathrm{im}(\partial) \oplus \mathrm{im}(\overline{\partial})$. The archimedean local height $[Y, Z(f)]_\infty = [Z(V_{1,2}), Z(f)]_\infty$ is characterized, essentially, by the integral

$$[Z(f), Y]_\infty = [Z(f), Z(V_{1,2})]_\infty = \int\limits_{Y(\mathbf{C})} g_{Z(f)}.$$

This integral is well-defined if the one-cycles $Y, Z(f) \in Z^2(X_L)$ are homologically equivalent to zero, or more generally, if the Green's current $g_{Z(f)} \in D^{1,1}(X_L)$ is admissible in the sense of Zhang [61]; see Definition 4.9.

The explicit realization of automorphic Green's functions $g_{Z(m,\mu)} = [G_{Z(m,\mu)}] \in D^{0,0}(X_L)$ for arithmetic divisors $Z(m,\mu) \in Z^1(X_L)$ is now well-understood via the theory of regularized Borcherds theta lifts:

$$G_{Z(m,\mu)} = \Phi(F_{m,\mu}(\tau, 5/4), \cdot) \in L^{1+\varepsilon}(X_L).$$

We refer to the discussions in [7], [11, §4], and [33], outlined below. In brief, writing $G_{Z(m,\mu)} \in A^{0,0}(X_L)$ for the function giving rise to the current $g_{Z(m,\mu)} \in D^{0,0}(X_L)$, we know that $G_{Z(m,\mu)}$ can be realized explicitly as the regularized theta lift

$$\begin{aligned} G_{Z(m,\mu)}(z,h) = \Phi(F_{m,\mu}(\tau, 5/4), z, h) &:= \int_{\mathcal{F}}^{\star} \langle\langle F_{m,\mu}(\tau, 5/4), \theta_L(\tau, z, h)\rangle\rangle d\mu(\tau) \\ &= \mathrm{CT}_{s=0}\left(\lim_{T\to\infty} \int\limits_{\mathcal{F}_T} \langle\langle F_{m,\mu}(\tau, 5/4), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right) \end{aligned}$$

of the Hejhal-Poincaré series $F_{m,\mu}(\tau, s)$ defined in (25) at the point $s = 5/4$, where it determines a Laplacian eigenfunction of eigenvalue zero. Here, we write $\mathrm{CT}_{s=0}\, F(s)$ to denote the constant term in the Laurent series expansion around $s = 0$ of a function $F(s)$ in $s \in \mathbf{C}$. We refer to Theorem 4.4 and §4.3.3 for more details. On the other hand, the explicit or canonical realization of automorphic Green's currents $g_Z \in D^{r-1,r-1}(X_L)$ for cycles $Z \in Z^r(X_L)$ of codimension $r = 2$ appears to be an open problem.

We address this problem for the special case $[Z(f), Y]_\infty$ of $Z(f) \in Z^2(X_L)$ diagonal as defined in (3) and $Y = Z(V_{1,2}) \in Z^2(X_L)$ a projective curve as follows. Starting with the abstract characterization of Gillet-Soulé of the Green's current $g_{Z(f)} \in D^{1,1}(X_L)$ for the cycle $Z(f) \in Z^2(X_L)$ defined in (3), we use a nested sequence of arithmetic divisors to realize this current in terms of certain "pushforward-pullbacks" of Green's functions given by regularized theta lifts. To describe this in more detail, consider the cycle (2) in the expansion of $Z(f)$ corresponding to (m,μ). Hence, we fix a vector $x = (x_0, x_1) \in V^2$ with $(x_i, x_i) = m > 0$ for each of $i = 0, 1$. Let us write

$$U(x) = U(x_0, x_1) := \mathbf{Q}(x_0, x_1)^\perp \subset V$$

for the rational quadratic subspace $U(x)$ is signature $(1,2)$ determined by the orthogonal complement of the span $\mathbf{Q}(x_0, x_1) \subset V$. For either of the vectors $x_i \in V$, we can also consider the subspace of signature $(2,2)$ defined by the orthogonal complement $U(x_i) := \mathbf{Q}(x_i)^\perp \subset V$ of the space $\mathbf{Q}(x_i)$. In this way, we obtain for each of $i = 0, 1$ a nested sequence of quadratic spaces

$$U(x) \subset U(x_i) \subset V$$

of respective signatures $(1,2)$, $(2,2)$, and $(3,2)$. This gives us a corresponding nested sequence of divisors

$$Z(U(x)) \subset Z(U(x_i)) \subset X_L,$$

together with a nested sequence of weighted divisors

$$Z(\beta, (\mu,\mu)) = Z\left(\begin{pmatrix} m & (x_0, x_1) \\ (x_1, x_0) & m \end{pmatrix}, (\mu,\mu) \right) \subset Z_i(m,\mu) \subset X_L.$$

Let us for simplicity write $Z = Z(\beta, (\mu, \mu))$, $Z_i = Z_i(m, \mu)$, and $X = X_L$. Viewing $Z \in Z^1(Z_i)$ as a divisor, we have a corresponding Green's function $g_{Z/Z_i} = [G_{Z/Z_i}] \in D^{0,0}(Z_i)$. Similarly, viewing $Z_i \in Z^1(X)$ as a divisor, we have a corresponding Green's function $g_{Z_i/X} = [G_{Z_i/X}] \in D^{0,0}(X)$. Pushing forward along the closed embedding $\iota : Z_i \hookrightarrow X$, we obtain from the Green's function $g_{Z/Z_i} \in D^{0,0}(Z_i)$ a pushforward Green's current $\iota_* g_{Z/Z_i} \in D^{1,1}(X)$ for the cycle $Z \in Z^2(X)$ (Lemma 36). On the other hand, we can consider the pullback along differential forms $\iota^! : A^{r,r}(X) \to A^{r,r}(Z_i)$ of the Green's function $G_{Z_i/X} \in A^{0,0}(X)$ to get $\iota^! G_{Z_i/X} \in A^{0,0}(Z_i)$ with corresponding current $[\iota^! G_{Z_i/X}] \in D^{1,1}(Z_i)$ and pushforward $\iota_*[\iota^! G_{Z_i/X}] \in D^{1,1}(X)$. Using the explicit realization of the associated Poincaré dual representatives ω_Z and $\omega_{Z_i/X}$ as Kudla-Millson theta series (Theorem 3.9), we can then deduce the following realization of the archimedean local height.

Theorem 1.1 (Theorem 4.8, Theorem 4.9). *In the setup described above, we can realize the Green's current $g_Z \in D^{1,1}(X)$ of the one-cycle $Z = Z(\beta, (\mu, \mu)) \in Z^2(X)$ as the pushforward current for the pullback $\iota^! G_{Z_i/X} \in A^{0,0}(Z_i)$ of the Green's function $G_{Z_i/X} \in A^{0,0}(X)$,*

$$g_Z = \iota_*[\iota^! G_{Z_i/X}] \in D^{1,1}(X).$$

Hence, we can characterize the archimedean local height as

$$[Z, Y]_\infty = \int_Y \iota_* \left[\iota^! G_{Z_i/X}\right].$$

Moreover, we can realize the pushforward-pullback $\iota_* \iota^! G_{Z_i/X} \in A^{0,0}(Z_i)$ of $G_{Z_i/X} = G_{Z_i(m,\mu)}$ explicitly as

$$\iota_* \iota^! \Phi(F_{m,\mu}(\tau, 5/4), z, h) = \mathrm{CT}_{s=0} \left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle F_{m,\mu}(\tau, 5/4), v^{-1/2} \theta_L(\tau, z, h) \rangle\rangle v^{-s} d\mu(\tau) \right).$$

More generally, we can deduce the following relevant characterization:

Corollary 1.2 (Corollary 4.10). *Let $f(\tau) \in H_{-1/2}(\omega_L)$ be a harmonic weak Maass form with one-cycle $Z(f) \in Z^2(X)$ as defined in (2). We can realize the Green's current $g_{Z(f)} \in D^{1,1}(X)$ as $g_{Z(f)} = \iota_*[\iota^! \Phi(f, \cdot)]$, where the pushforward-pullback $\iota_* \iota^! \Phi(f, \cdot)$ can be realized explicitly as the modified theta lift*

$$\iota_* \iota^! \Phi(f(\tau), z, h) := \mathrm{CT}_{s=0} \left(\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), v^{-1/2} \theta_L(\tau, z, h) \rangle\rangle v^{-s} d\mu(\tau) \right).$$

Consequently, we can define the archimedean local height $[Y, Z(f)]_\infty$ explicitly as

$$[Z(f), Y]_\infty = \int_{Y(\mathbf{C})} \iota_* \left[\iota^! \Phi(f, z, h)\right] = \int_{Y(\mathbf{C})} \iota_* \iota^! \Phi(f(\tau), z, h).$$

1.1.4. *Computation of the archimedean local height.* We compute the archimedean local height $[Z(f), Y]_\infty$ using the characterization of Corollary 1.2. Since we assume the subspace $V_{1,2} \subset V$ is anisotropic, the corresponding one-cycle $Y = Z(V_{1,2}) \in Z^2(X_L)$ is a projective curve. Writing $L_{1,2} = L \cap V_{1,2}$ for the corresponding sublattice of signature $(1, 2)$, with $L_{1,2}^\perp \subset V$ its orthogonal complement of signature $(2, 0)$, we first restrict the harmonic weak Maass form $f \in H_{-1/2}(\omega_L)$ to the sublattice $M = L_{1,2} \oplus L_{1,2}^\perp$, and argue (via Lemma 5.1) that we can replace the Siegel theta series $\theta_L(\tau, z, h)$ with $\theta_{L_{1,2} \oplus L_{1,2}^\perp}(\tau, z, h)$. For any point $(z_1, h) \in Y(\mathbf{C})$, we then have the decomposition

$$\theta_{L_{1,2} \oplus L_{1,2}^\perp}(\tau, z_1, h) = \theta_{L_{1,2}}(\tau, z_1, h) \otimes \theta_{L_{1,2}^\perp}(\tau, 1, 1).$$

Here, $\theta_{L_{1,2}^\perp}(\tau) = \theta_{L_{1,2}^\perp}(\tau, 1, 1)$ is a holomorphic Siegel theta series of weight 1. Hence, we compute

$$\int_{Y(\mathbf{C})} \iota_* \iota^! \Phi(f(\tau), z, h) = \sum_{(z_1, h) \in Y(\mathbf{C})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2} \theta_{L_{1,2}}(\tau, z_1, h) \otimes \theta_{L_{1,2}^\perp}(\tau) \rangle\rangle d\mu(\tau).$$

Writing $K_{1,2} = K_{L_{1,2}} = K_L \cap \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$ for the compact open subgroup determined by $L_{1,2} \subset L$, with $\mathrm{vol}(K_{1,2})$ its volume with respect to the Haar measure on $\mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$ chosen for Lemma 5.3 below, we first relate this to the integral

$$\frac{\deg(Y)}{2} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_{1,2}}(\tau, z_1, h) \otimes v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\rangle\rangle d\mu(\tau) dh,$$

where

$$\deg(Y) = \deg(Z(V_{1,2})) = \frac{2}{\mathrm{vol}(K_{1,2})}.$$

We can then use the convergent Siegel-Weil formula (Theorem 4.1, Corollary 4.2)

$$\frac{1}{2} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \theta_{L_{1,2}}(\tau, z_1, h) dh = E_{L_{1,2}}(\tau, 1/2; -1/2)$$

to relate this integral to the value at $s = s_0(V_{1,2}) = \dim(V_{1,2})/2 - 1 = 1/2$ of the Eisenstein series $E_{L_{1,2}}(\tau, s; -1/2)$ of weight $-1/2$ associated to the lattice $L_{1,2}$. Hence, we relate the height to the integral

$$\deg(Y) \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau) \otimes E_{L_{1,2}}(\tau, 1/2; -1/2)\rangle\rangle d\mu(\tau) dh.$$

Following the approach of Kudla [33] (cf. [49], [11]), we can relate this Eisenstein series $E_{L_{1,2}}(\tau, s; -1/2)$ to the image under the Maass weight-lowering operator $L_{\frac{3}{2}}$ of the corresponding Eisenstein series $E_{L_{1,2}}(\tau, s; 3/2)$ of weight $3/2$ for the same lattice $L_{1,2}$ as

$$L_{\frac{3}{2}} E_{L_{1,2}}(\tau, s; 3/2) = \frac{1}{2}(s - 1/2) \cdot E_{L_{1,2}}(\tau, s; -1/2).$$

Taking the derivative with respect to s then evaluating at $s = 1/2$ gives the relation

$$L_{\frac{3}{2}} E'_{L_{1,2}}(\tau, 1/2; 3/2) = \frac{1}{2} \cdot E_{L_{1,2}}(\tau, 1/2; -1/2).$$

We note that the higher-weight Eisenstein series $E_{L_{1,2}}(\tau, s; 3/2)$ is incoherent in the sense of Kudla [32], and hence has an analytic continuation $E^{\star}_{L_{1,2}}(\tau, s; 3/2)$ to all $s \in \mathbf{C}$ satisfying an odd functional equation

$$E^{\star}_{L_{1,2}}(\tau, s; 3/2) = -E^{\star}_{L_{1,2}}(\tau, -s; 3/2).$$

In particular, this Eisenstein series vanishes at the central point $s = 0$. The corresponding derivative $E'_{L_{1,2}}(\tau, s; 3/2) = \frac{d}{ds} E_{L_{1,2}}(\tau, s; 3/2)$ with respect to s determines a nonholomorphic Eisenstein series

$$E'_{L_{1,2}}(\tau, s; 3/2) = E'^{+}_{L_{1,2}}(\tau, s; 3/2) + E'^{-}_{L_{1,2}}(\tau, s; 3/2) \in H_{3/2}(\omega_{L_{1,2}}).$$

We write $\mathcal{E}_{L_{1,2}}(\tau) = E'^{+}_{L_{1,2}}(\tau, 0; 3/2)$ to denote its holomorphic part at $s = 0$, with Fourier series expansion

$$\mathcal{E}_{L_{1,2}}(\tau) = \sum_{\nu \in L_{1,2}^{\vee}/L_{1,2}} \sum_{m \geq 0} \kappa_{L_{1,2}}(r, \nu) e(m\tau) \mathbf{1}_{\nu} \in M_{3/2}(\omega_{L_{1,2}}).$$

We then write

$$\mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle$$

to denote the constant term in the Fourier series expansion of the scalar-valued form defined by

$$\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle \sum_{\substack{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\perp} \\ \nu \in L_{1,2}^{\vee}/L_{1,2}}} f^{+}_{\lambda+\nu}(\tau) \theta_{L_{1,2}^{\perp}, \lambda}(\tau) \mathcal{E}_{L_{1,2}, \nu}(\tau).$$

On the other hand, let ξ_l denote the Bruinier-Funke antilinear differential operator

$$\xi_l : H_l(\omega_L) \to M_{2-l}(\omega_L^{\vee}), \quad \xi_l(f)(\tau) := v^{l-2}\overline{L_l f(\tau)}, \quad L_l = -2iv^2 \frac{\partial}{\partial \overline{\tau}}.$$

We consider the holomorphic form

$$\xi_{-1/2}(f)(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \geq 0} c_{\xi_{-1/2}(f)}(m,\mu) e(m\tau) \mathbf{1}_\mu \in M_{5/2}(\omega_L^\vee)$$

of weight $5/2$ and type $\omega_L^\vee$ determined by the image of $f(\tau) \in H_{-1/2}(\omega_L)$. Writing the Fourier series expansion of the holomorphic theta series $\theta_{L_{1,2}^\perp}(\tau)$ attached to the negative definite sublattice $L_{1,2}^\perp \subset L$ as

$$\theta_{L_{1,2}^\perp}(\tau) = \sum_{\lambda \in (L_{1,2}^\perp)^\vee / L_{1,2}^\perp} \sum_{m \geq 0} r_{L_{1,2}^\perp}(m,\lambda) e(m\tau) \mathbf{1}_\lambda \in M_1(\omega_{L_{1,2}^\perp}),$$

we consider the corresponding Rankin-Selberg L-series $L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp})$. We can define this first for $\Re(s) \gg 1$ by the convergent Dirichlet series

$$L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) := (4\pi)^{-\left(\frac{s+1/2}{2}\right)} \Gamma\left(\frac{s+1/2}{2}\right) \sum_{m \geq 1} \sum_{\lambda \in (L_{1,2}^\perp)^\vee / L_{1,2}^\vee} \overline{c_{\xi_{-1/2}(f)}(m,\lambda)} r_{L_{1,2}^\perp}(m,\lambda) m^{-\left(\frac{s+1/2}{2}\right)},$$

then show by a standard unfolding argument (46) that this series equals the Petersson inner product

$$L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) = \langle \xi_{-1/2}(f), \theta_{L_{1,2}^\perp}(\tau) \otimes E_{L_{1,2}}(\cdot, s; 3/2) \rangle.$$

Hence, this L-function inherits from $E_{L_{1,2}}(\tau, s; 3/2)$ an analytic continuation

$$L^\star(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) := \langle \xi_{-1/2}(f), \theta_{L_{1,2}^\perp}(\tau) \otimes E^\star_{L_{1,2}}(\cdot, s; 3/2) \rangle$$

to all $s \in \mathbf{C}$, and satisfies the odd symmetric functional equation

$$L^\star(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) = -L^\star(-s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}).$$

It therefore vanishes at the central point $s = 0$; we consider the central derivative value $L'(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp})$.

Theorem 1.3 (Theorem 5.6, Corollary 4.10)**.** *Let $f(\tau) = f^+(\tau) + f^-(\tau) \in H_{-1/2}(\omega_L)$ be any harmonic weak Maass form whose holomorphic part*

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \gg -\infty} c_f^+(m,\mu) e(m\tau) \mathbf{1}_\mu$$

has integer Fourier coefficients $c_f^+(m,\mu)$ for all negative $m \in \mathbf{Z} + Q(\mu)$. Consider the corresponding diagonal one-cycle $Z(f) \in Z^2(X_L)$ defined in (3) *above. Let $V_{1,2} \subset V$ be any anisotropic subspace of signature* $(1,2)$ *with corresponding projective one-cycle $Y = Z(V_{1,2}) \in Z^2(X_L)$. We have the archimedean height formula*

$$[Z(f), Y]_\infty = -2\deg(Y) \cdot \left(L'(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) + \mathrm{CT}\langle\langle f^+(\tau), \theta_{L_{1,2}^\perp}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau) \rangle\rangle \right).$$

1.1.5. *Nonarchimedean local heights.* We expect that the constant term $\mathrm{CT}\langle\langle f^+(\tau), \theta_{L_{1,2}^\perp}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau) \rangle\rangle$ computes most of the nonarchimedean local height $[Z(f), Y]_{\mathrm{fin}}$, cf. [11, Conjecture 5.1, Conjecture 5.2], [1]. If $L \subset V$ is chosen so that the corresponding level $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ is neat, then $X = X_L$ has an integral model coming from the moduli description (Definition 2.1, Proposition 2.2); see Kudla-Rapoport [41]. Viewed more generally as a spin Shimura variety of Hodge type, X is known to have a flat integral model $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathbf{Z})$ by theorems of Kisin [27], Madapusi Pera [47], and Kim-Madapusi Pera [26]. As explained in [1], the special divisors $Z \in Z^1(X)$ have extensions $\mathcal{Z} \longrightarrow \mathcal{X}$ via the Kuga-Satake abelian scheme. Being spin Shimura subvarieties, it follows that special cycles $Z \in Z^r(X)$ of each codimension $0 \leq p \leq 2$ have integral models $\mathcal{Z} \in Z^p(\mathcal{X})$. We can therefore consider the corresponding Faltings arithmetic height, in the sense of Arakelov theory, after adding the nonarchimedean local component:

$$[\mathcal{Z}(f), \mathcal{Y}] = [\mathcal{Z}(f), \mathcal{Y}]_{\mathrm{fin}} + [\mathcal{Z}(f), \mathcal{Y}]_\infty .$$

Conjecture 1.4. *In the setup of Theorem 1.3, we expect for each coset $\mu \in L^\vee/L$ and negative $m \in \mathbf{Z}+Q(\mu)$ to have the nonarchimedean local height formula*

$$\left[\mathcal{Z}\left(\begin{pmatrix} m & \\ & m \end{pmatrix}, (\mu,\mu)\right), \mathcal{Y}\right]_{\mathrm{fin}} = 2\deg(\mathcal{Y}) \sum_{\substack{\mu_1 \in (L_{1,2}^\perp)^\vee/L_{1,2}^\perp \\ \mu_2 \in L_{1,2}^\vee/L_{1,2} \\ \mu_1+\mu_2 \equiv \mu \bmod L}} \sum_{\substack{m_1,m_2 \geq 0 \\ m_1+m_2=m}} r_{L_{1,2}^\perp}(m_1,\mu_1)\kappa_{L_{1,2}}(m_2,\mu_2), \tag{4}$$

so that

$$[\mathcal{Z}(f),\mathcal{Y}]_{\mathrm{fin}} = 2\deg(\mathcal{Y}) \sum_{\mu \in L^\vee/L} \sum_{m>0} c_f^+(-m,\mu) \sum_{\substack{\mu_1 \in (L_{1,2}^\perp)^\vee/L_{1,2}^\perp \\ \mu_2 \in L_{1,2}^\vee/L_{1,2} \\ \mu_1+\mu_2 \equiv \mu \bmod L}} \sum_{\substack{m_1,m_2 \geq 0 \\ m_1+m_2=m}} r_{L_{1,2}^\perp}(m_1,\mu_1)\kappa_{L_{1,2}}(m_2,\mu_2).$$

In other words, we expect that the nonarchimedean local height $[\mathcal{Z}(f),\mathcal{Y}]_{\mathrm{fin}}$ is given by $2\deg(\mathcal{Y})$ times the Fourier coefficient at (m,μ) of $\langle\langle f^+(\tau), \theta_{L_{1,2}^\perp}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle$. Consequently, the arithmetic height

$$[\mathcal{Z}(f),\mathcal{Y}] = [\mathcal{Z}(f),\mathcal{Y}]_{\mathrm{fin}} + [\mathcal{Z}(f),\mathcal{Y}]_\infty$$

should be

$$[\mathcal{Z}(f),\mathcal{Y}] = -2\deg(\mathcal{Y}) \cdot \left(c_f^+(0,0) \cdot \kappa_{L_{1,2}}(0,0) + L'(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp})\right).$$

We note that Kudla-Rapoport [41, Theorem 0.2 with $n_1 = n_2 = 2$; $r = 2$] gives evidence for this conjecture, calculating the p-adic local intersection of a pair of cycles $\mathcal{Z}(\beta_i, (\mu_{i,1}, \mu_{i,2})) \in Z^2(\mathcal{X}_{K_L^p})$ extended to a good reduction p-integral model $\mathcal{X}_{K_L^p}$ in terms of sums of Fourier coefficients of derivatives of Siegel Eisenstein series for the metaplectic group $\mathrm{Mp}_4(\mathbf{A})$. We describe these results in Theorem 3.4 and Theorem 3.5. It would be very interesting to develop these ideas to address or even deduce Conjecture 1.4 via (4) after relating $\theta_{L_{1,2}^\perp}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)$ to an Eisenstein series on $\mathrm{Mp}_4(\mathbf{A})$. In this approach, we expect Conjecture 1.4 to follow after characterizing the generalized Shimura-Waldspurger lift of Gan-Ichino [16] of the Saito-Kurokawa lift (or the Maass lift of the Eichler-Zagier lift) of $\theta_{L_{1,2}^\perp}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)$ to $\mathrm{GSp}_4(\mathbf{A})$. Since the realization of this idea would seem to imply a generalization of the explicit correspondences of Kohnen [30] and Kohnen-Zagier [29] to this higher-rank setting, we leave the details for a subsequent work.

1.1.6. *Remarks.* While we do not show that the realization $\iota_*[\iota^! \Phi(f,\cdot)] \in D^{1,1}(X_L)$ of the Green's current we use to compute $[Z(f),Y]_\infty$ is admissible in the sense of Definition 4.9, our subsequent calculation shows that we obtain the value predicted by the arithmetic Rallis inner product formula (Conjecture 1.7). We explain this connection in more detail below, noting that our special setup with $Z(f) \in Z^2(X_L)$ diagonal as in (3) and $Y = Z(V_{1,2}) \in Z^2(X_L)$ a projective curve can only recover central derivative values of $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-functions, and hence can only be viewed as evidence for this conjectural formula for the case of Saito-Kurokawa lifts. Theorem 1.3 also allows us to deduce unconditional evidence for the corresponding Beilinson-Bloch conjecture (Conjecture 1.6) for rank one; see Theorem 1.8. It is interesting to consider whether some more systematic study of "passing to divisors" in this way would allow for the calculation of heights of higher-codimensional cycles via Green's functions. In any case, the calculations we present here hint at a more general picture, where the height of a pair of codimension-two cycles $[Z(F),Y] = [Z(F), Z(V_{1,2})]$ should compute the central derivative value of the degree-eight Rankin-Selberg L-function

$$L(s, \mathrm{spin}, \varphi \times \theta_{k(L_{1,2}^\perp)}) = L(s, \mathrm{spin}, \varphi) L(s, \mathrm{spin}\, \varphi \otimes \chi_{k(L_{1,2}^\perp)})$$

of the spin L-function $L(s, \mathrm{spin}, \varphi)$ of a $\mathrm{GSp}_4(\mathbf{A})$-automorphic form $\varphi \in M_3^2$ times the Hecke theta series

$$\theta_{k(L_{1,2}^\perp)} \in M_1(\Gamma_0(|d_{k(L_{1,2}^\perp)}|), \chi_{k(L_{1,2}^\perp)})$$

attached to the imaginary quadratic field $k(L_{1,2}^\perp) = L_{1,2}^\perp \otimes \mathbf{Q}$ of discriminant $d_{k(L_{1,2}^\perp)}$ and Dirichlet character $\chi_{d_{k(L_{1,2}^\perp)}}(\cdot) = \left(\frac{d_{k(L_{1,2}^\perp)}}{\cdot}\right)$ determined by the negative-definite lattice $L_{1,2}^\perp$. Here, we write

$$F = \varphi^{\text{Sh-W}} \in M_{5/2}^2$$

to denote the generalized Shimura-Waldspurger lift [16] of φ to the metaplectic group $\mathrm{Mp}_4(\mathbf{A})$, and

$$Z(F) = \langle F, A_2 \rangle \in \mathrm{CH}^2(X_L)$$

the class of the one-cycle determined by pairing F with the generating series A_2 of special cycles $Z \in Z^2(X_L)$.

1.2. **Ambient conjectures, results.** We now explain how this fits into the landscape of cycles conjectures.

1.2.1. *The refined conjecture of Beilinson-Bloch.* Recall that for X a smooth projective scheme over $\mathbf{Q}$ of odd dimension $2r-1$, we can consider the motivic L-function

$$L(s, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r)))$$

of the middle-degree l-adic cohomology of X. We refer to the expository article [50] for definitions and background, but use normalizations of L-functions as in Li-Liu [42], [43]. Conjecturally, this L-function has an analytic continuation to a meromorphic function

$$L^{\star}(s, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r)))$$

defined on all $s \in \mathbf{C}$ which does not depend on the choice of prime l, and which satisfies a functional equation $s \mapsto -s$ making $s = 0$ the centre of symmetry. Let $\mathrm{CH}^r(X)^0 \subset \mathrm{CH}^r(X)$ denote the Chow group of algebraic cycles of codimension r which are homologically trivial on $X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$. We have the following standard conjecture, generalizing that of Birch-Swinnerton-Dyer for the special case of $X = E$ an elliptic curve:

Conjecture 1.5 (Beilinson-Bloch)**.** *Let X be any smooth projective scheme of dimension $2r-1$ over $\mathbf{Q}$.*

(i) *The Chow group $\mathrm{CH}^r(X)^0 \subset \mathrm{CH}^r(X)$ of null-homologous cycles has finite $\mathbf{Q}$-dimension $\mathrm{rk}\,\mathrm{CH}^r(X)^0$.*

(ii) *There exists a nondegenerate height pairing $\langle\ ,\ \rangle_r : \mathrm{CH}^r(X)^0 \times \mathrm{CH}^r(X)^0 \longrightarrow \mathbf{R}$.*

(iii) *We have the rank formula $\mathrm{rk}\,\mathrm{CH}^r(X)^0 = \mathrm{ord}_{s=0} L(s, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r)))$.*

(iv) *We have the refined formula*

$$L^{\star}(s, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r))) = C_{H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r)))}(0) \cdot \det\langle\ ,\ \rangle_r \mod \mathbf{Q}^{\times},$$

where

$$C_{H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r)))}(0) \in \mathbf{R}^{\times}/\mathbf{Q}^{\times}$$

denotes the Deligne period.

Taking X to the the compactification of $X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$ for (V, Q) a rational and $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ neat with codimension $r = 2$, our Theorem 1.3 gives some evidence for Conjecture 1.5 (ii). We explain this connection in more detail below. First, we remark that the works of Li-Liu [42], [43] suggest the following automorphic refinement of Conjecture 1.5. To describe this, we first recall the following equivariant version given in terms of the conjectural Chow motive $h^{2r-1}(X)(r)_{\mathbf{C}}$ of degree $2r-1$. Let us therefore assume that X admits an action of a Hecke algebra $\mathbf{T}$ via étale correspondences, so that $\mathbf{T}$ acts on both $\mathrm{CH}^r(X)^0$ and $H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}), \mathbf{Q}_l(r))$. Let π be a nonzero irreducible finite-dimensional complex representation of $\mathbf{T}$. For each prime l and each embedding $\mathbf{Q}_l \to \mathbf{C}$, we have an L-function

$$L(s, \mathrm{Hom}_{\mathbf{T}}(\pi, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r))_{\mathbf{C}})).$$

We again expect this L-function to have an analytic continuation

$$L^{\star}(s, \mathrm{Hom}_{\mathbf{T}}(\pi, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r))_{\mathbf{C}}))$$

to a meromorphic function of all $s \in \mathbf{C}$ satisfying a functional equation $s \mapsto -s$, along with the identification

$$\mathrm{ord}_{s=0} L^{\star}(s, \mathrm{Hom}_{\mathbf{T}}(\pi, H^{2r-1}(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(r))_{\mathbf{C}})) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^r(X)^0_{\mathbf{C}}).$$

Conjecture 1.6 (Beilinson-Bloch for GSp_4)**.** *Let $X = \{X_K\}_K = \{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathcal{H}_2)\}_K$ denote the GSp_4 Shimura variety of dimension $2r - 1 = 3$ ($r = 2$) defined over $\mathbf{Q}$. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{GSp}_4(\mathbf{A})$ of cohomological type. Let $L(s, \mathrm{spin}, \pi) = \prod_{v \leq \infty} L(s, \mathrm{spin}, \pi_v)$ denote the corresponding spin L-function, normalized so that its functional equation relates $s \mapsto 1 - s$, making $s = 1/2$ the central point. We expect to have the relations*

$$\mathrm{ord}_{s=1/2} L(s, \mathrm{spin}, \pi) = \mathrm{ord}_{s=0} L^\star(s, \mathrm{Hom}_{\mathbf{T}}(\pi, H^3(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(2))_{\mathbf{C}})) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^2(X)^0_{\mathbf{C}}).$$

If π is cuspidal, with archimedean component π_∞ belonging to the discrete series, and π is not endoscopic or CAP (=cuspidal automorphic associated with a parabolic), then π_∞ contributes to the cohomology of X in the middle degree. It also has a basechange to a cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ to $\mathrm{GL}_4(\mathbf{A})$ by Asgari-Shahidi [3] [4] *and Hundley-Sayag* [25], *with standard L-function $L(s, \Pi) = \prod_{v \leq \infty} L(s, \Pi_v)$. The standard L-function $L(s, \Pi)$ is normalized so that its functional equation relates $s \to 1 - s$, making $s = 1/2$ the central point. In this case, we expect to have the relations*

$$\mathrm{ord}_{s=1/2} L(s, \Pi) = \mathrm{ord}_{s=0} L^\star(s, \mathrm{Hom}_{\mathbf{T}}(\pi, H^3(X \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l(2))_{\mathbf{C}})) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^2(X)^0_{\mathbf{C}}).$$

We remark that if the automorphic representation π of $\mathrm{GSp}_4(\mathbf{A})$ is (everywhere locally) tempered, then it cannot be CAP or endoscopic. We refer to the discussions in [59] and [55, §0.6] for details. As we shall see below, we can interpret the setup we consider above as a CAP example involving the Saito-Kurokawa lift.

1.2.2. *The Bloch-Kato Main Conjecture.* We also have the following analogue of Bloch-Kato [12]; see [55] for a more extensive account. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A})$ of cohomological type. Here, we focus on the case where the archimedean component π_∞ belongs to the discrete series L-packet of parallel weight $(3, 3)$, so that π appears in the étale cohomology of the $\mathrm{GSp}_4(\mathbf{A})$ Shimura variety $X = \{X_K\}_K = \{\mathrm{Sh}_K(\mathrm{GSp}_4, \mathcal{H}_2)\}_K$ with trivial coefficients. Fix a prime number p. Taking $[E : \mathbf{Q}_p] \gg 1$ to be a sufficiently large coefficient field, we have for each prime ideal $\mathfrak{p} \mid p$ in E a p-adic Galois representation

$$\rho_{\pi,\mathfrak{p}} : G_{\mathbf{Q}} \longrightarrow \mathrm{GSp}_4(E_{\mathfrak{p}})$$

of the absolute Galois group $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Writing $V_{\pi,\mathfrak{p}}$ to denote the $E_{\mathfrak{p}}[G_{\mathbf{Q}}]$-module of dimension 4 determined by $\rho_{\pi,\mathfrak{p}}$, and $I_{\mathbf{Q}_l} \subset G_{\mathbf{Q}}$ the local inertia subgroup at a prime $l \neq p$, we consider the corresponding Bloch-Kato Selmer group

$$H^1_f(\mathbf{Q}, V_{\pi,\mathfrak{p}}) := \ker \left(H^1(\mathbf{Q}, V_{\pi,\mathfrak{p}}) \longrightarrow H^1(\mathbf{Q}_p, V_{\pi,\mathfrak{p}} \otimes B_{\mathrm{cris}}) \times \prod_{l \neq p} H^1(I_{\mathbf{Q}_l}, V_{\pi,\mathfrak{p}}) \right). \tag{5}$$

Writing $L(s, \mathrm{spin}, \pi)$ again to denote the spin L-function of π, the Bloch-Kato Main Conjecture asserts that

$$\dim_{E_{\mathfrak{p}}} H^1_f(\mathbf{Q}, V_{\pi,\mathfrak{p}}) = \mathrm{ord}_{s=1/2} L(s, \mathrm{spin}, \pi).$$

Hence, Conjecture 1.6 would also describe the rank of the Bloch-Kato Selmer group $H^1_f(\mathbf{Q}, V_{\pi,\mathfrak{p}})$ as

$$\dim_{E_{\mathfrak{p}}} H^1_f(\mathbf{Q}, V_{\pi,\mathfrak{p}}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^2(X)^0_{\mathbf{C}}). \tag{6}$$

Assuming this cuspidal representation π of parallel weight $(3, 3)$ has trivial central character, and is not CAP or endoscopic, Sweeting [55, Theorem A, Theorem C] proves results towards this Bloch-Kato Main Conjecture (5) for ranks zero and one under standard technical hypotheses. However, the case of rank one [55, Theorem C] is stated implicitly in terms of the conjectural identity (6), and requires a still-conjectural "arithmetic Rallis inner product formula" relating the codimension-2 cycles $\mathrm{CH}^2(X)^0_{\mathbf{C}}$ to the central derivative values $L'(1/2, \mathrm{spin}, \pi)$ to deduce the corresponding Bloch-Kato Main Conjecture identity (5) in this setting.

1.2.3. *The "arithmetic Rallis inner product formula".* Broadly speaking, this should realize the identity

$$\mathrm{ord}_{s=1/2} L(s, \mathrm{spin}, \pi) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^2(X)^0_{\mathbf{C}}). \tag{7}$$

of Conjecture 1.6 in the case of rank one as an arithmetic height formula for $L'(1/2, \mathrm{spin}, \pi)$ involving special codimension-two cycles $Z \in \mathrm{CH}^2(X)^0$, giving generalizations of both the Rallis inner product formula for the central value $L(1/2, \mathrm{spin}, \pi)$ and the (higher) Gross-Zagier formula [22], [11], [1] to this setting.

To give a precise statement in this setting, we assume for simplicity[2] that (V, Q) is a split rational quadratic space of signature $(3, 2)$, so that we have an identification of algebraic groups $\mathrm{GSpin}(V) \cong \mathrm{GSp}_4$ over $\mathbf{Q}$. Fix an even lattice $L \subset V$ with corresponding dual lattice $L^\vee$ and discriminant group $L^\vee/L$. Writing K_L to denote the corresponding compact open subgroup of $\mathrm{GSpin}(V)(\mathbf{A}_f) \cong \mathrm{GSp}_4(\mathbf{A}_f)$, we can then identify the corresponding spin Shimura variety as a Siegel threefold:

$$X = X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)) \cong \mathrm{Sh}_{K_L}(\mathrm{GSp}_4, \mathcal{H}_2).$$

Let us for each possible genus $1 \leq g \leq 3$ write $\mathfrak{S}_{L,g}$ to denote the vector space of functions $(L^\vee/L)^g \to \mathbf{C}$:

$$\mathfrak{S}_{L,g} \cong \mathbf{C}[(L^\vee/L)^g] \subset \mathrm{Aut}(\mathcal{S}(V(\mathbf{A}_f)^g)).$$

The metaplectic group $\mathrm{Mp}_{2g}(\mathbf{Z})$ acts on $\mathfrak{S}_{L,g}$ through the Weil representation $\omega_{L,g}$. Let $M_k^g(\omega_{L,g})$ denote the space of Siegel modular forms of parallel (scalar) weight (k, k) for $k \in \frac{1}{2}\mathbf{Z}$, genus g, and type $\omega_{L,g}$. We have for each function $\varphi \in \mathfrak{S}_{L,g}$ and positive symmetric matrix $T \in \mathrm{Sym}_g(\mathbf{Q})_{\geq 0}$ the special weighted cycle $Z(T, \varphi) \in Z^g(X_L)$. This gives a class in the Chow group $\mathrm{CH}^g(X_L)$, as well as its complexification $\mathrm{CH}^g(X_L)_{\mathbf{C}}$, which we denote by the same symbol $Z(T, \varphi)$. Let $Z(T)$ denote the element of $\mathrm{Hom}(\mathfrak{S}_{L,g}, \mathrm{CH}^g(X_L)_{\mathbf{C}})$ defined by the map $\varphi \mapsto Z(T, \varphi)$. Consider the formal generating series

$$A_g(\tau) = \sum_{T \in \mathrm{Sym}_g(\mathbf{Q})_{\geq 0}} Z(T) q^T, \quad q^T = e(\tau \operatorname{Tr}(T)) = \exp(2\pi i \tau \operatorname{Tr}(T))$$

with coefficients in $\mathfrak{S}_{L,g}^\vee \otimes_{\mathbf{C}} \mathrm{CH}^g(X_L)$. Bruinier-Raum [10, Theorem 6.2] confirms Kudla's modularity conjecture that $A_g(\tau)$ determines a $\mathrm{CH}^g(X_L)_{\mathbf{C}}$-valued Siegel modular form in $M_{5/2}^g(\omega_{L,g}^\vee)$. In particular, we have for genus $g = 2$ the modular generating series

$$A_2(\tau) = \sum_{(\mu_1, \mu_2) \in (L^\vee/L)^2} \sum_{T \in \mathrm{Sym}_2(\mathbf{Q})_{\geq 0}} Z(T, (\mu_1, \mu_2)) e(\tau \operatorname{Tr}(T)) \mathbf{1}_{\mu_1} \otimes \mathbf{1}_{\mu_2} \in \mathrm{CH}^2(X_L)_{\mathbf{C}} \otimes M_{5/2}^2(\omega_{L,2}^\vee).$$

Let $\pi' = \otimes_v \pi'_v$ denote the automorphic representation of $\mathrm{Mp}_4(\mathbf{A})$ corresponding to π under the generalized Shimura-Waldspurger correspondence of Gan-Ichino [16] (cf. [17], [52], and [57]). Fix

$$F(\tau) = \sum_{(\mu_1, \mu_2) \in (L^\vee/L)^2} \sum_{T \in \mathrm{Sym}_2(\mathbf{Q})_{\geq 0}} c_F(T, (\mu_1, \mu_2)) e(\tau \operatorname{Tr}(T)) \mathbf{1}_{\mu_1} \otimes \mathbf{1}_{\mu_2} \in M_{5/2}^2(\omega_{L,2}^\vee)$$

giving a vector-valued generator of this metaplectic representation π'. Consider the scalar product

$$\theta(F)(\tau) = \langle\langle F(\tau), \overline{A_g(\tau)} \rangle\rangle \in \mathrm{CH}^2(X_L)_{\mathbf{C}} \otimes M_{5/2}^2$$

determined by pairing the generating series $A_2(\tau)$ with this generator $F(\tau)$,

$$\theta(F)(\tau) = \sum_{(\mu_1, \mu_2) \in (L^\vee/L)^2} \sum_{T_1 \in \mathrm{Sym}_2(\mathbf{Q})_{\geq 0}} c_F(T_1, (\mu_1, \mu_2)) e(\tau \operatorname{Tr}(T_1)) \sum_{T_2 \in \mathrm{Sym}_2(\mathbf{Q})_{\geq 0}} \overline{Z(T_2, (\mu_1, \mu_2)) e(\tau \operatorname{Tr}(T_2))}.$$

Writing $\approx$ to denote equality up to a nonvanishing constant, we expect to have an arithmetic height formula

$$\text{“}[\theta(F), \theta(F)] \approx L'(1/2, \mathrm{spin}, \pi).\text{”} \tag{8}$$

Let us for each projective one-cycle $Y = Z(V_{1,2}) \in Z_c^2(X_L)$ parametrized by some anisotropic quadratic subspace $V_{1,2} \subset V$ consider the linear functional $I_{Y,\infty} : \mathrm{CH}^2(X_L) \longrightarrow \mathbf{C}$ defined by

$$I_{Y,\infty} : \mathrm{CH}^2(X_L) \longrightarrow \mathbf{C}, \quad Z \longmapsto [Z, Y]_\infty = \int_Y g_Z.$$

In fact, this functional is defined on the arithmetic Chow group $\widehat{CH}^2(X_L) = \{(Z, g_Z), Z = [Z] \in \mathrm{CH}^2(X_L)\}$. We can also define the corresponding global arithmetic linear functional in the natural way as

$$I_Y : \widehat{CH}^2(X_L) \longrightarrow \mathbf{C}, \quad Z \longmapsto [Z, Y] = [Z, Y]_{\mathrm{fin}} + [Z, Y]_\infty.$$

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[2] Strictly speaking, we would have to pass to a number field F over which the split rational quadratic space (V, Q) admits an anisotropic subspace of signature $(1, 2)$ over F. To describe the general case where the rational quadratic space (V, Q) is not split, we pass to a Jacquet-Langlands transfer π' of π to the general spin group $\mathrm{GSpin}(V)(\mathbf{A})$, with π' its generalized Shimura-Waldspurger lift to $\mathrm{Mp}_4(\mathbf{A})$.

Let $\theta(F) \in \mathrm{CH}^2(X_L)$ denote the class of the codimension-two cycle underlying $\theta(F)(\tau) \in \mathrm{CH}^2(X_L)_{\mathbf{C}} \otimes M^2_{5/2}$. Let $\pi(\theta_{L_{1,2}^{\perp}})$ denote the dihedral automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by the holomorphic theta series $\theta_{L_{1,2}^{\perp}}(\tau)$ of weight one associated to the lattice $L_{1,2}^{\perp} \subset L$ of the complement of $L_{1,2} = L \cap V_{1,2}$.

Conjecture 1.7. *Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $\mathrm{GSp}_4(\mathbf{A}) \cong \mathrm{GSpin}(V)(\mathbf{A})$ whose archimedean local component π_∞ belongs to the discrete series L-packet of parallel weight $(3,3)$, with level $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ determined by some even lattice $L \subset V$. Write $L(s, \mathrm{spin}, \pi) = \prod_{v \leq \infty} L(s, \mathrm{spin}, \pi_v)$ to denote its corresponding spin L-function with*

$$L(s, \mathrm{spin}, \pi \times \pi(\theta)) = \prod_{v \leq \infty} L(s, \mathrm{spin}, \pi_v \times \pi(\theta)_v)$$

its degree-eight Rankin-Selberg product with the standard L-function of the dihedral $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(\theta) = \otimes_v \pi(\theta)_v$ induced from a theta series θ. Let π' denote the lifting of π to an automorphic representation of $\mathrm{Mp}_4(\mathbf{A})$ by the generalized Shimura-Waldspurger theta correspondence of [16]*. Fix a Siegel modular form $F(\tau) \in M^2_{5/2}(\omega_{L,2})$ generating π'. Let $\theta(F) \in \mathrm{CH}^2(X_L)_{\mathbf{C}}$ denote the class of the cycle determined by the pairing of $F(\tau)$ with the modular generating series $A_2(\tau) \in \mathrm{CH}^2(X_L)_{\mathbf{C}} \otimes M^2_{5/2}(\omega_{L,2^\vee})$.*

(i) *We have for each projective curve $Y = Z(V_{1,2}) \in Z^2(X_L)$ parametrized by an anisotropic subspace $V_{1,2} \subset V$ and sublattice $L_{1,2} = L \cap V_{1,2}$ an archimedean local height formula*

$$I_{Y,\infty}\theta(F) = [Y, \theta(F)]_\infty \approx L'(1/2, \mathrm{spin}, \pi \times \pi(\theta_{L_{1,2}^{\perp}})).$$

(ii) *If π is cuspidal, then we have for each $Y = Z(V_{1,2}) \in Z^2(X_L)$ an arithmetic height formula*

$$I_Y\theta(F) = [Y, \theta(F)] \approx L'(1/2, \mathrm{spin}, \pi \times \pi(\theta_{L_{1,2}^{\perp}})).$$

If this π is everywhere locally tempered, or not CAP/endoscopic, we can express this equivalently as

$$I_Y\theta(F) = [Y, \theta(F)] \approx L'(1/2, \mathrm{spin}, \Pi \times \pi(\theta_{L_{1,2}^{\perp}})),$$

where Π denotes the basechange of π to $\mathrm{GL}_4(\mathbf{A})$.

Here, we remark that in the same way the formulae of Gross-Zagier [22] and higher Gross-Zagier [11], [1] require the passage to a zero/CM cycle $Z(U) \in Z^n(\mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)))$ corresponding to an imaginary quadratic field $k(U)$ determined by the negative definite subspace $U \subset V$ (of (V,Q) of signature $(n,2)$), we require the passage to a one-cycle $Y = Z(V_{1,2})$ corresponding to a projective curve determined by an anisotropic quadratic subspace $V_{1,2} \subset$ of signature $(1,2)$. The deduction of the relation (8) is then accessible after consideration of the complementary theta series $\theta_{L_{1,2}^{\perp}}(\tau)$, in the style of the arguments of Gross-Kohnen-Zagier [23] and their reinterpretation by Bruinier-Yang [11, §7, Theorem 7.7, Corollary 7.8]. To be more precise, the negative definite lattice $L_{1,2}^{\perp}$ of signature $(2,0)$ can be associated to the imaginary quadratic field $k = k(L_{1,2}^{\perp}) := L_{1,2}^{\perp} \otimes \mathbf{Q}$ of discriminant d_k and quadratic Dirichlet character $\chi(\cdot) = (\frac{d_k}{\cdot})$. After identifing the holomorphic Siegel theta series $\theta_{L_{1,2}^{\perp}}(\tau)$ as a Hecke theta series $\theta_k(\tau) \in M_1(\Gamma_0(|d_k|), \chi_k)$, we see there is a decomposition into a product of two degree-four spin L-functions

$$L(s, \mathrm{spin}, \pi \times \pi(\theta_{L_{1,2}^{\perp}})) = L(s, \mathrm{spin}, \pi) L(s, \mathrm{spin}, \pi \otimes \chi_k).$$

Granted suitable conditions on the choice of imaginary quadratic field $k = k(L_{1,2}^{\perp})$ to ensure the nonvanishing of the central value $L(1/2, \mathrm{spin}, \pi \otimes \chi_k)$, we can then derive an inner product formula for the central derivative value $L'(1/2, \mathrm{spin}, \pi)$. Moreover, we expect to find a generalization of the theorems of Waldspurger [58], Kohnen-Zagier [29, Theorem 1], and Gross-Kohnen-Zagier [23, § II. 4, Corollary 1], relating the central value $L(1/2, \mathrm{spin}, \pi \otimes \chi_k)$ to the modulus squared $|c_F(\left(\mathrm{diag}(d_k, d_k)\right)|^2$ of the "Fourier coefficient at $|d_k|$" of the metaplectic form $F(\tau) \in M^2_{5/2}(\omega_{L,2})$ generating the Gan-Ichino Shimura π' lift of π to $\mathrm{Mp}_4(\mathbf{A})$, so that

$$L'(1/2, \mathrm{spin}, \pi \times \pi(\theta_k)) \approx |c_F(\mathrm{diag}(d_k, d_k))|^2 \cdot L'(1/2, \mathrm{spin}, \pi),$$

so that a derivation of (8) would follow from a similar argument. In this way, we see that Conjecture 1.7 would give evidence of Conjecture 1.6 for rank one, in the form of the implication

$$L'(1/2, \mathrm{spin}, \pi) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \mathrm{CH}^2(X_L)_{\mathbf{C}}) > 0. \tag{9}$$

1.2.4. *Saito-Kurokawa lifts.* Finally, we explain how to derive this implication (9) in the special case where the automorphic representation π of $\mathrm{GSp}_4(\mathbf{A})$ arises as a partial Saito-Kurokawa lift of the form

$$\xi_{-1/2}(f)(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \geq 0} c_h(m, \mu) e(m\tau) \in M_{5/2}(\omega_L^\vee)$$

appearing in Theorem 1.3. Let us write $h(\tau) \in M_{5/2}(\Gamma_0(4N))$ to denote the canonical lifting of $\xi_{-1/2}(f)(\tau)$ to a scalar-valued form (see [54, Theorem 5.2 and 5.4], and §5.3 below), with $L(s,h)$ its standard L-function. We again normalize standard L-functions to have central value at $s = 1/2$, and assume that $L(s,h)$ has an odd symmetric functional equation so that $L(1/2,h) = 0$. As we explain in (46) and (48) below, we have an identification of completed Rankin-Selberg L-functions

$$L(s-1/2, h \times \theta_{k(L_{1,2}^\perp)}) = L^\star(2s-2, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) := \Lambda(2s, \chi_{k(L_{1,2}^\perp)}) L(2s-2, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}),$$

where $\theta_{k(L_{1,2}^\perp)}$ denotes the Hecke theta series associated to the imaginary quadratic field $k(L_{1,2}^\perp) = L_{1,2}^\perp \otimes \mathbf{Q}$. Via quadratic basechange equivalence, we also have decompositions into $\mathrm{GL}_2(\mathbf{A})$-automorphic L-functions

$$\begin{aligned} L(s-1/2, h \times \theta_{k(L_{1,2}^\perp)}) &= L(s-1/2, h) L(s-1/2, h \otimes \chi_{k(L_{1,2}^\perp)}) \\ &= L^\star(2s-s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^\perp}) = L^\star(2s-2, \xi_{-1/2}(f)) L^\star(2s-2, \xi_{-1/2}(f) \otimes \chi_{k(L_{1,2}^\perp)}). \end{aligned} \tag{10}$$

Recall that the Saito-Kurokawa lift can be realized explicitly as the composition

$$\sigma_l : M_{2l-2}(\Gamma_0(N)) \xrightarrow{\sigma_l^{\mathrm{Sh}}} M_{l-1/2}^+(\Gamma_0(4N)) \xrightarrow{\sigma_k^{\text{E-Z}}} J_{l,4N} \xrightarrow{\sigma_k^{\mathrm{Ma}}} M_l^2$$

of the Shimura correspondence

$$\sigma_l^{\mathrm{Sh}} : M_{2l-2}(\Gamma_0(4N)) \longrightarrow M_{l-1/2}^+(\Gamma_0(4N))$$

from the space of modular forms $M_{2l-2}(\Gamma_0(N))$ of weight $2l-2$ on $\Gamma_0(N)$ to the Kohnen plus space $M_{l-1/2}^+(\Gamma_0(4N)) \subset M_{l-1/2}(\Gamma_0(4N))$ of forms of weight $l - 1/2$ on $\Gamma_0(4N)$ (see [52], [28]), with the Eichler-Zagier correspondence

$$\sigma_l^{\text{E-Z}} : M_{l-1/2}^+(\Gamma_0(4N)) \longrightarrow J_{l,N}$$

to the space of Jacobi forms $J_{l,N}$ of weight l and index N (see [15], [23, §II]), with the lifting

$$\sigma_l^{\mathrm{Ma}} : J_{l,N} \longrightarrow M_l^2$$

of Maass [44], [45], [46], Andrianov [2], Zagier [60], and also Gritsenko [21] to the space of Siegel (paramodular) forms M_l^2 of parallel weight (l,l) and genus 2 for any integer $l \geq 1$. Assuming that $h \in M_{5/2}^+(\Gamma_0(4N))$ lies in the plus space, we can then consider the corresponding Siegel modular form

$$\varphi_h(\tau) := \sigma_3^{\mathrm{Ma}} \circ \sigma_3^{\text{E-Z}}(h)(\tau) \in M_3^2.$$

This can be thought of as a partial Saito-Kurokawa lift in that it factors through the Saito-Kurokawa lift

$$\varphi_h(\tau) = \sigma_3(h_0)(\tau), \quad \sigma_3 = \sigma_3^{\mathrm{Ma}} \circ \sigma_3^{\text{E-Z}} \circ \sigma_3^{\mathrm{Sh}}$$

of elliptic modular form $h_0 \in M_4(\Gamma_0(N))$ of weight N associated to $h(\tau)$ by the Shimura correspondence. This $\varphi_h(\tau) \in M_3^2$ generates an automorphic representation π of $\mathrm{GSp}_4(\mathbf{A})$ whose archimedean component π_∞ is a discrete series of parallel weight $(3,3)$. It has a metaplectic lift $F_h(\tau) \in M_{5/2}^2$ generating the automorphic representation π' of $\mathrm{Mp}_4(\mathbf{A})$ associated to π by the Shimura-Waldspurger correspondence of Gan-Ichino [16].

Moreover, normalizing[3] L-functions to have central value at $s = 1/2$, we have the relation of L-functions

$$L(s, h_0)\zeta(s + 7/2)\zeta(s + 5/2) = L(s, h)\zeta(s + 7/2)\zeta(s + 5/2) = L(s, \varphi_h) \tag{11}$$

if the underlying $h_0(\tau) \in M_4(\Gamma_0(N))$ is a Hecke eigenform. Twisting by $\chi_{k(L_{1,2}^{\perp})}$, we also obtain

$$L(s, h \otimes \chi_{k(L_{1,2}^{\perp})})L(s + 7/2, \chi_{k(L_{1,2}^{\perp})})L(s + 5/2, \chi_{k(L_{1,2}^{\perp})}) = L(s, \varphi_h \otimes \chi_{k(L_{1,2}^{\perp})}). \tag{12}$$

Using that $L(1/2, h) = 0$ with the Artin decomposition (10), we can then deduce from (11) and (12) that

$$\begin{aligned} L'(1/2, \varphi_h \times \theta_{k(L_{1,2}^{\perp})}) &= L'(1/2, \varphi_h)L(1/2, \varphi_h \otimes \chi_{k(L_{1,2}^{\perp})}) \\ &= L'(1/2, h)L(1/2, h \otimes \chi_{k(L_{1,2}^{\perp})}) \cdot \zeta(4)L(4, \chi_{k(L_{1,2}^{\perp})})\zeta(3)L(3, \chi_{k(L_{1,2}^{\perp})}) \\ &= L^{\star\prime}(0, \xi_{-1/2}(f))L^{\star}(0, \xi_{-1/2}(f) \otimes \chi_{k(L_{1,2}^{\perp})}) \cdot \zeta(4)L(4, \chi_{k(L_{1,2}^{\perp})})\zeta(3)L(3, \chi_{k(L_{1,2}^{\perp})}) \\ &= L^{\star\prime}(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) \cdot \zeta(4)L(4, \chi_{k(L_{1,2}^{\perp})})\zeta(3)L(3, \chi_{k(L_{1,2}^{\perp})}). \end{aligned}$$

Equivalently, writing $\pi(\varphi_h)$ to denote the automorphic representation of $\mathrm{GSp}_4(\mathbf{A})$ generated by φ_h, and $k = k(L_{1,2}^{\perp}) = L_{1,2}^{\perp} \otimes \mathbf{Q}$ for simplicity to denote the imaginary quadratic field, this gives us the relations

$$\begin{aligned} &L'(1/2, \mathrm{spin}, \pi(\varphi_h) \times \chi_k) \\ &= L'(1/2, \mathrm{spin}, \pi(\varphi_h))L(1/2, \mathrm{spin}, \pi(\varphi_h) \otimes \chi_k) \\ &= L'(1/2, h \times \theta_k) \cdot \zeta(3)L(3, \chi_k)\zeta(4)L(4, \chi_k) \\ &= L'(1/2, h)L(1/2, h \otimes \chi_k) \cdot \zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k) \\ &= L^{\star\prime}(0, \xi_{-1/2}(f))L^{\star}(0, \xi_{-1/2}(f) \otimes \chi_k) \cdot \zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k) \\ &= L^{\star\prime}(0, \xi_{-1/2}(f) \otimes \theta_{L_{1,2}^{\perp}}) \cdot \zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k). \end{aligned} \tag{13}$$

In particular, we deduce from (11), (12), and (13) that

$$L'(1/2, \mathrm{spin}, \pi(\varphi_h)) = L^{\star\prime}(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) \cdot \frac{\zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k)}{L(1/2, \mathrm{spin}, \pi(\varphi_h) \otimes \chi_k)} = \frac{L^{\star\prime}(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}})}{L(1/2, h \otimes \chi_k)},$$

where

$$L(1/2, h \otimes \chi_k) = L(1/2, h_0 \otimes \chi_k) \approx |c_h(|d_k|)|^2$$

by theorems of Waldspurger [58], Kohnen-Zagier [29], and Gross-Kohnen-Zagier [23, § II.4, Corollary 1], relating $L(1/2, h_0 \otimes \chi_k)(= L(1/2, h \otimes \chi_k))$ to the Fourier coefficient $c_h(|d_k|)$ of h as $|d_k|$. Here, we use the symbol $\approx$ to denote equality up to the explicit constants worked out in [23, § II.4, Corollary 1], which we omit from this discussion in the interests of clarity. We then obtain from Theorem 1.3 the relation

$$L'(1/2, \mathrm{spin}, \pi(\varphi_h)) \approx -\left(\frac{[Z(f), Z(V_{1,2})]_{\infty}}{\deg(Z(V_{1,2}))} + \mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle\right) \cdot \frac{\zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k)}{|c_h(|d_k|)|^2}. \tag{14}$$

We expect that $Z(f) = Z(F_h) \in \mathrm{CH}^2(X_L)$, which with Conjecture 1.4 would verify Conjecture 1.7 here:

$$L'(1/2, \mathrm{spin}, \pi(\varphi_h)) \approx -\frac{[Z(F_h), Z(V_{1,2})]}{\deg(Z(V_{1,2}))} \cdot \frac{\zeta(4)L(4, \chi_k)\zeta(3)L(3, \chi_k)}{|c_h(|d_k|)|^2}.$$

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[3] In the arithmetic normalization, given $h_0(\tau) \in M_{2l-2}(\Gamma_0(N))$ an eigenform, we have by [44], [45], [46], [2], and [60] the identification of L-functions $L(s, h_0)\zeta(s - l + 2)\zeta(s - l + 1)(= L(s, h)\zeta(s - l + 2)\zeta(s - l + 1)) = L(s, \sigma_l(h_0))$. Write the Fourier series expansion in arithmetic and normalized terms respectively as

$$h_0(\tau) = \sum_{m \geq 0} c_{h_0}(m)e(m\tau) = \sum_{m \geq 0} a_{h_0}(m)m^{\frac{w(h_0)-1}{2}}e(m\tau)$$

for $w(h_0) = 2l - 2$ the weight of h_0. Write $L(s, \pi(h_0))$ for the standard L-function associated $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(h_0)$ determined by h_0 with the automorphic normalization. Its finite part $L_f(s, \pi(h_0))$ has the expansion

$$L_f(s, \pi(h_0)) = \sum_{m \geq 1} a_{h_0}(m)m^{-s} = \sum_{m \geq 1} c_{h_0}(m)m^{-\frac{w(h_0)-1}{2}}m^{-s} =: L_f(s + (w(h_0) - 1)/2, h_0).$$

Hence, we deduce the relation $L(s, \pi(h_0)) = L(s + (2l - 3)/2, h_0)$, so that the general relation in the automorphic normalization is given by $L(s, \pi(h_0))\zeta(s + (2l + 1)/2)\zeta(s + (2l - 1)/2) = L(s, \pi(h))\zeta(s + (2l + 1)/2)\zeta(s + (2l - 1)/2) = L(s, \pi(\sigma_l(h_0)))$.

In any case, we can deduce from (14) the implication (9), in the direction of Conjecture 1.6 for rank one.

Theorem 1.8. *Let $h(\tau) \in M_{5/2}(\Gamma_0(4N))$ denote the scalar-valued lift of the vector-valued holomorphic form $\xi_{-1/2}(f) \in M_{5/2}(\omega_L^\vee)$ of Theorem 1.3. Assume that $h(\tau)$ arises as the Shimura lift of some Hecke eigenform $h_0(\tau) \in M_4(\Gamma_0(N))$, equivalently that $h(\tau) \in M_{5/2}^+(\Gamma_0(4N))$ lies in the Kohnen plus space. Assume that $h_0(\tau)$ is invariant under the Fricke involution, so that the corresponding standard L-function $L(s, h_0) = L(s, h)$ has odd symmetric functional equation and vanishing central value $L(1/2, h_0) = L(1/2, h) = 0$. Let φ_h denote the Saito-Kurokawa lift of h (via h_0) generating an automorphic representation $\pi(\varphi_h)$ of $\mathrm{GSp}_4(\mathbf{A})$. Then, by consideration of Theorem 1.3 for all imaginary quadratic fields $k = k(L_{1,2}^\perp)$ indexed by rational quadratic spaces $V_{1,2} \subset V$ of signature (1,2) giving functionals $I_{Y,\infty} = I_{Z(V_{1,2}),\infty} \in \mathrm{Hom}_{\mathbf{T}}(\pi(\varphi_h), \mathrm{CH}^2(X_L))$, we deduce*

$$L'(1/2, \mathrm{spin}, \pi(\varphi_h)) \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}\left(\pi(\varphi_h), \mathrm{CH}^2(X_L)\right) \neq 0.$$

1.2.5. *Acknowledgements.* It is a pleasure to thank Spencer Bloch for so many conversations about this project over the past several years, and to dedicate this work to him. I would also like to thank Shouwu Zhang for suggesting to me in 2024 that there should be a "pullback lemma" to reduce to Green's functions.

2. SPIN SHIMURA VARIETIES AS TWISTED SIEGEL THREEFOLDS

We describe twisted Siegel threefolds realized as spin Shimura varieties, following [31, § 1] and [41, §1, A].

2.1. **General spin groups.** Let (V, Q) be a rational quadratic space of signature $(3, 2)$, hence $\dim_{\mathbf{Q}}(V) = 5$. We write $(x, y) = Q(x + y) - Q(x) - Q(y)$ to denote the corresponding inner product on $x, y \in V$.

2.1.1. *Clifford algebras.* Let $C(V)$ denote the Clifford algebra of V, given by the quotient $C(V) = T(V)/I(V)$ of the tensor algebra $T(V) = \bigoplus_{m \geq 0} V^{\otimes m}$ of V with the ideal $I(V)$ generated by elements of the form $x \otimes x - Q(x)$ for $x \in V$. We write

$$C(V) = C^0(V) \oplus C^1(V)$$

to denote its $\mathbf{Z}/2\mathbf{Z}$-grading into the even subalgebra $C^0(V)$ generated by even products of vectors, and the odd subalgebra $C^1(V)$ generated by odd products of vectors. Hence, there is a canonical $\mathbf{Q}$-linear embedding $V \hookrightarrow C^1(V)$, which allows us to view V as a subspace. Let us write $v_1 \cdots v_r$ for simplicity to denote the tensor product $v_1 \otimes \cdots \otimes v_r$ in the Clifford algebra $C(V) = C^0(V) \oplus C^1(V)$. Let $c \mapsto c^\iota$ denote the Clifford involution of $C(V)$, the unique $\mathbf{Q}$-linear anti-involution which acts as the identity on $V \subset C^1(V)$. Hence,

$$(v_1 \cdots v_r)^\iota = v_r^\iota \cdots v_1^\iota.$$

Given $v_1, \ldots v_5 \in V$ any basis, the product $\delta = v_1 \cdots v_5$ lies in the centre $Z(C(V))$, and is invariant under the Clifford involution $\delta = \delta^\iota$. The rule $c \mapsto cc^\iota$ defines the spinor norm map

$$\nu : C(V) \longrightarrow \mathbf{Q}, \quad c \longmapsto cc^\iota.$$

2.1.2. *Spin groups.* We consider the general spin group

$$\mathrm{GSpin}(V) := \left\{ g \in C^0(V) : gVg^{-1} = V \right\}.$$

This $\mathrm{GSpin}(V)$ is a reductive algebraic group defined over $\mathbf{Q}$. It sits in a short exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1,$$

from which we can view it as an extension of the special orthogonal group $\mathrm{SO}(V)$. The restriction of the spinor norm to $\mathrm{GSpin}(V)$ gives us another short exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{GSpin}(V) \xrightarrow{\nu} \mathbf{G}_m \to 1,$$

where

$$\mathrm{Spin}(V) = \{ g \in \mathrm{GSpin}(V) : \nu(g) = 1 \} \subset \mathrm{GSpin}(V)$$

is identified as the derived subgroup. We refer to [31, §1], [41, Appendix A], and [8, §2.2] for more details.

2.2. **Identifications.** We now describe the exceptional isomorphism

$$\mathrm{GSpin}(V) \cong \mathrm{GSp}_4 \tag{15}$$

of reductive algebraic groups over $\overline{\mathbf{Q}}$, following [41, Appendix A].

2.2.1. *The spin representation.* Suppose we can fix a Witt decomposition $V = V_+ \oplus V_0 \oplus V_-$ over $\mathbf{Q}$, where the $V_\pm \subset V$ are maximal isotropic subspaces of $\dim(V_\pm) = 2$ and $V_0 \subset V$ is an anistropic line, $\dim(V_0) = 1$. Note that if we cannot fix such a decomposition over $\mathbf{Q}$, we pass to a finite extension F of $\mathbf{Q}$ where such an extension holds, then repeat the arguments given below to derive the corresponding isomorphism of reductive algebraic groups (15) over F. Starting with such a Witt decomposition, fix a basis vector $v_0 \in V_0$ with $Q(v_0) = 1$. We have identifications of representations of Clifford algebras

$$C(V)/C(V)C(V_-)_{>0} \cong C(V_+ \oplus V_-) \cong C(V_+)(1 + v_0) \oplus C(V_+)(1 - v_0),$$

where the last two components $C(V_+)(1 \pm v_0)$ are isomorphic at $C^0(V)$-modules. Either one of these modules defines the four-dimensional spin representation W of $\mathrm{GSpin}(V)$. Fixing an isomorphism $\wedge^2 V_+ \cong \mathbf{Q}$, let

$$l : W \longrightarrow \mathbf{Q}$$

denote the linear functional obtained by composing this isomorphism with the projection of $C(V_+) = \wedge(V_+)$ onto $\wedge^2 V_+$. This defines an alternative $\mathbf{Q}$-form

$$\langle x, y\rangle := l(x^\iota y).$$

As shown in [41, Lemma A.1], writing $\sigma(c)$ for the spin representation action of c on W, we have for each $c \in C(V)$ and $x, y \in W$ the relation $\langle \sigma(c)x, y\rangle = \langle x, \sigma(c')y\rangle$. In particular, for $g \in \mathrm{GSpin}(V)$, we have

$$\langle \sigma(g)x, \sigma(g)y\rangle = \nu(g)\langle x, y\rangle.$$

Here again, $\nu : \mathrm{GSpin}(V) \longrightarrow \mathbf{Q}^\times, \nu(g) = gg^\iota$ denotes the restriction to $\mathrm{GSpin}(V)$ of the spinor norm. As a consequence of this relation, we obtain from the spin representation σ the exceptional isomorphism (15), i.e.

$$\sigma : \mathrm{GSpin}(V) \cong \mathrm{GSp}(W).$$

2.2.2. *Parametrization in terms of quaternion algebras.* As explained in [41, Appendix A.3], we have the following parametrization of our rational quadratic space (V, Q) in terms of a quaternion algebra B over $\mathbf{Q}$.

More generally, if (V, Q) is any nondegenerate rational quadratic space of dimension $\dim_{\mathbf{Q}}(V) = 5$, with any choice of basis $v_1, \ldots, v_5 \in V$ and corresponding pseudoscalar/volume element $\delta = v_1 \cdots v_5 \in C(V)$, the subspace $\delta V \subset C^0(V)$ can be characterized as

$$\delta V = \left\{x \in C^0(V) : x^\iota = x,\ \mathrm{tr}(x) = 0\right\}.$$

As part of the proof of this identification ([41, Lemma A.3]), the space (V, Q) is characterized in terms of a quaternion algebra as follows. Let B be a quaternion algebra defined over $\mathbf{Q}$ with main involution $x \mapsto x^\iota$. Let $C = M_2(B)$ with involution $x \mapsto x' = {}^t x^\iota$. Let V_B denote the vector space over $\mathbf{Q}$ defined by

$$V_B = \{x \in C = M_2(B) :\ x' = x,\ \mathrm{tr}(x) = 0\} = \left\{ x = \begin{pmatrix} a & b \\ b^\iota & -a \end{pmatrix} :\ a \in \mathbf{Q},\ b \in B \right\}.$$

Let $Q_B : V_B \to \mathbf{Q}$ denote the quadratic form defined by $Q_B(x) = xx'$. Hence, (V_B, Q_B) determines a rational quadratic space. We consider its corresponding Clifford algebra $C(V_B) = C^0(V_B) \oplus C^1(V_B)$. Observe that

$$xx' = x^2 = \begin{pmatrix} a^2 + \nu(b) & \\ & a^2 + \nu(b) \end{pmatrix}.$$

The inclusion $V_B \hookrightarrow M_2(B)$ induces an isomorphism $C(V_B) \cong M_2(B)$ which is compatible with the involutions. Conversely, let (V, Q) be any nondegenerate rational quadratic space of dimension $\dim_{\mathbf{Q}}(V) = 5$. The Clifford involution $x \mapsto x'$ induces an isomorphism $C^0(V) \cong C^0(V)^{\mathrm{op}}$, from which it follows that

$$C^0(V) \cong M_2(B)$$

for some quaternion algebra B defined over $\mathbf{Q}$. We can choose this isomorphism to be compatible with the Clifford involution, so that the map carries δV into V_B. Comparing dimensions, we then obtain an isometry

$$(V, \delta^2 \cdot Q) \cong (V_B, Q_B).$$

2.3. **Shimura/Siegel threefolds.** We now describe the spin Shimura varieties associated to $\mathrm{GSpin}(V)$, which by the exceptional isomorphism (15) determine twisted (quaternionic) Siegel modular threefolds.

2.3.1. *Hermitian symmetric domains.* Let $D(V)$ denote the space of oriented, negative 2-planes in $V_{\mathbf{R}}$,

$$D(V) = \{ z \subset V_{\mathbf{R}} : \dim(z) = 2, Q|_z < 0 \} .$$

This space has two connected components $D^{\pm}(V)$, each one of which is isomorphic to a bounded symmetric domain of $\mathbf{C}^3$. We fix one of these connected components throughout. Note that we have identifications

$$D(V) \cong \mathrm{SO}(V)(\mathbf{R})/(\mathrm{SO}(3)\times\mathrm{SO}(2)) \cong \mathrm{SO}(3,2)/(\mathrm{SO}(3)\times\mathrm{SO}(2)) \tag{16}$$

(cf. [8, §2.4], [31, §1][41, Appendix A]). Here, $\mathrm{SO}(3,2)$ has two connected components $\mathrm{SO}(3,2)^{\pm}$, of which we take $\mathrm{SO}(3,2)^{+}$ to be the component of the identity. In line with (15) above, we also have an exceptional isomorphism $\mathrm{SO}(3,2)^{+} \cong \mathrm{Sp}_4(\mathbf{R})$. On the other hand, recall that we have the Siegel upper-half space

$$\mathcal{H}_2 = \{ Z = X + iY \in \mathrm{Sym}_2(\mathbf{C}) : Y > 0 \} .$$

Via the map $Z = (z_{i,j})_{i,j} \mapsto (z_{i,j})_{j \geq 1}$, we can identify $\mathcal{H}_2$ with a subset of $\mathbf{C}^3$. The symplectic group $\mathrm{Sp}_4(\mathbf{R})$ can be characterized as the group fixing the alternating form

$$\sum_{j=1}^{2} x_j y_{-j} \longmapsto \sum_{j=1}^{2} x_{-j} y_j .$$

In matrix coordinates $A, B, C, D \in M_2(\mathbf{R})$, writing $I_2 \in M_2(\mathbf{R})$ to denote the identity, it is given by

$$\mathrm{Sp}_4(\mathbf{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} A^tC = C^tA & A^tD - C^tB = I_2 \\ D^tA - B^tC = I_2 & B^tD = D^tB \end{matrix} \right\} .$$

This group $\mathrm{Sp}_4(\mathbf{R})$ acts simply transitively on $\mathcal{H}_2$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1} .$$

The matrix

$$J_2 = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$$

acts as an involution on $\mathcal{H}_2$, having only $i \cdot I_2$ as its fixed point, and so

$$\mathcal{H}_2 \cong \mathrm{Sp}_4(\mathbf{R}) \cdot iI_2 .$$

The stabilizer of $i \cdot I_2$ is

$$K := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_4(\mathbf{R}) : A^TA + B^TB = I_2, A^TB = B^TA \right\} \cong U(2) = \left\{ Z = A + iB : \overline{Z}^t Z = I_2 \right\}$$

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB .$$

In this way, we obtain group-stabilizer identifications of the Siegel upper-half space of genus 2,

$$\mathcal{H}_2 \cong \mathrm{Sp}_4(\mathbf{R})/K \cong \mathrm{Sp}_4(\mathbf{R})/U(2) .$$

On the other hand, using the central exact sequence

$$1 \longrightarrow U(1) \longrightarrow U(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1$$

with $U(1) \cong \{ e^{i\theta} \cdot I_2 \} \cong \mathrm{SO}(2)$ the central circle in $U(2)$, we obtain identifications

$$U(2)/U(1) \cong U(2)/\mathrm{SO}(2) \cong \mathrm{SO}(3)$$

and $U(2) \cong \mathrm{SO}(3) \times \mathrm{SO}(2)$ up to finite central kernel. That is, we have a surjective isogeny of finite kernel

$$U(2) \cong (\mathrm{SU}(2) \times U(1))/\{(-I, -1)\} \longrightarrow \mathrm{SO}(2) \times \mathrm{SO}(3) .$$

This implies we have a Lie algebra decomposition $\mathfrak{u}(2) = \mathfrak{so}(2) + \mathfrak{so}(3)$, from which we obtain identifications

$$\mathcal{H}_2 \cong \mathrm{Sp}_4(\mathbf{R})/(\mathrm{SO}(2)\times\mathrm{SO}(3)) \cong \mathrm{SO}(3,2)^{+}/(\mathrm{SO}(2)\times\mathrm{SO}(3)) \cong D^{\pm}(V) \tag{17}$$

of the corresponding hermitian symmetric domains. Here, we choose the connected component $D^{\pm}(V)$ of $D(V)$ to match the connected component $\mathrm{SO}(V)^{+} \cong \mathrm{SO}(3,2)^{+}$ of the identify in $\mathrm{SO}(V) \cong \mathrm{SO}(3.2)$.

2.3.2. *Shimura spin/Siegel threefolds.* Following the approach of Deligne [13] (cf. [41, §1], [31, §1]), we explain how $(\mathrm{GSpin}(V), D(V))$ determines a Shimura datum, and hence a Shimura variety

$$\mathrm{Sh}(\mathrm{GSpin}(V), D(V)) = \varprojlim_{K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)} \mathrm{Sh}_K(\mathrm{GSpin}(V), D(V))$$

with a canonical model defined over its reflex field $E(\mathrm{GSpin}(V), D(V)) = \mathbf{Q}$.

Fix a point $z \in D(V)$ with a properly oriented basis $z = [z_0, z_1]$ such that $((z_i, z_j))_{1 \leq i,j \leq 2} = -I_2$. Then,

$$j_z = z_0 \cdot z_1 \in C^0(V_{\mathbf{R}})$$

determines an element $j_z \in \mathrm{GSpin}(V)(\mathbf{R})$ such that $j_z^2 = -1$. Write $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\, \mathbf{G}_m$ to denote the Deligne torus. Let us for each element $a + b\sqrt{-1} \in \mathbf{S}_{\mathbf{R}} \cong \mathbf{C}^{\times}$ define the map[4]

$$h_z : \mathbf{S}_{\mathbf{R}} \longrightarrow \mathrm{GSpin}(V)(\mathbf{R}), \quad a + b\sqrt{-1} \longmapsto a + bj_z.$$

The domain $D(V)$ can be realized as the space of conjugacy classes of this morphism $h_z : \mathbf{S}_{\mathbf{R}} \rightarrow \mathrm{GSpin}(V)(\mathbf{R})$ under the action of $\mathrm{GSpin}(V)(\mathbf{R})$. Hence, $(\mathrm{GSpinV}, D(V)) = (\mathrm{GSpin}(V), h_z)$ determines a Shimura datum. By the theory of Shimura and Deligne [13], we then have for any compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ an algebraic variety $\mathrm{Sh}_K(\mathrm{GSpin}(V), D(V)) = \mathrm{Sh}_K(\mathrm{GSpin}(V), h_z)$ with complex points given by

$$\mathrm{Sh}_K(\mathrm{GSpin}(V), D(V))(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q}) \backslash D^{\pm}(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K,$$

and with a canonical model defined over a uniquely-determined number field $E(\mathrm{GSpin}(V), D(V))$ known as the "reflex field". In this setting, the reflex field $E(\mathrm{GSpin}(V), h_z)$ is the rational number field $\mathbf{Q}$. Since each compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ arises as the stabilizer

$$K = K_L = \mathrm{Stab}_{\mathrm{GSpin}(V)(\mathbf{A}_f)}(L \otimes \widehat{\mathbf{Z}})$$

of the adelization $L \otimes \widehat{\mathbf{Z}}$ of some uniquely-determined lattice $L \subset V$ under the action of $\mathrm{GSpin}(V)(\mathbf{A}_f)$ by conjugation, we shall also write $\mathrm{Sh}_L(\mathrm{GSpin}(V), D(V)) = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)) = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), h_z)$ to spell out that this level structure depends only on the lattice. As explained in [31] and [41], each of these Shimura varieties $\mathrm{Sh}_L(\mathrm{GSpin}(V), D(V))$ determines a quasiprojective variety of dimension 3 over $\mathbf{Q}$. Fixing any set of representatives $\{h\}$ of the finite set $\mathrm{GSpin}(V)(\mathbf{Q}) \backslash \mathrm{GSpin}(V)(\mathbf{A}_f)/K_L$, we have a decomposition into geometrically connected components

$$\mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))(\mathbf{C}) = \coprod_h \Gamma_h \backslash D^{\pm}(V), \quad \Gamma_h := \mathrm{GSpin}(V)(\mathbf{Q}) \cap hK_L h^{-1}. \tag{18}$$

Now, recall from (17) above that we can identify the hermitian symmetric domain $D(V) = D^{+}(V) \coprod D^{-}(V)$ with two copies of the Siegel upper-half space $\mathcal{H}_2 = \{Z = X + iY \in \mathrm{Sym}_2(\mathbf{C}) : Y > 0\} \subset \mathbf{C}^3$ of genus 2. The isomorphism of algebraic groups (15) allows us to identify each of the discrete subgroups Γ_h of $\mathrm{GSpin}(V)(\mathbf{R})$ as a discrete subgroup of $\mathrm{GSp}_4(\mathbf{R})$, and to view each of the threefolds on the right-hand side of (18) as a (quaternionic) Siegel threefold. In the special case where $B = M_2(\mathbf{Q})$ is the matrix quaternion algebra, $\mathrm{Sh}(\mathrm{GSpin}(V), D(V)) \cong \mathrm{Sh}(\mathrm{GL}_2(B), D(M_2(B(\mathbf{R}))))$ is the classical Siegel modular threefold of genus 2.

2.3.3. *Moduli description.* We have the following moduli description for the Siegel threefolds considered above (see [41, §1]). Fix a lattice $L \subset V$ and let $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ denote the corresponding compact open subgroup. Recall that $C = M_2(B)$, with B the indefinite quaternion algebra over $\mathbf{Q}$ parametrizing the rational quadratic space (V, Q). We write $x \rightarrow x^{\iota}$ for the main involution on B, and $x \rightarrow x' = {}^{t}x^{\iota}$ for the induced main involution on C. Let $D(B)$ denote the discriminant of B, i.e. the product of primes p for which $B_p = B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is a division algebra. Fix a maximal order $\mathcal{O}_B \subset B$ such that $\mathcal{O}_B^{\iota} = \mathcal{O}_B$. We choose an element $\tau \in B^{\times}$ such that $\tau^{\iota} = -\tau$ with $\tau^2 = -D(B)$ and $\tau \mathcal{O}_B \tau^{-1} = \mathcal{O}_B$. Consider the involution $x^{*} = \tau x^{\iota} \tau^{-1}$ on B; this preserves the maximal order $\mathcal{O}_B$.

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[4] In the context of the discussion above, we have $j_z^{\iota} = -j_z$ with $j_z^2 = z_1^2 z_2^2 = -1$ and $j_z j_z^{\iota} = 1$. There is an isomorphism of algebras $\mathbf{C} \cong C^0(z)$ over $\mathbf{R}$ given by $i \mapsto z_1 z_2$, and the composition of this isomorphism with the natural inclusion/identification $C^0(z) \subset C^0(V_{\mathbf{R}}) \cong C(\mathbf{R}) = M_2(B(\mathbf{R}))$ induces the same morphism $h_z : \mathbf{S} \longrightarrow \mathrm{GSpin}(V)$ defined over $\mathbf{R}$. Note that $h_z(i) = j_z$.

Definition 2.1 (Moduli problem)**.** *Define a functor $X_L = X_{K_L} = X_K$ by sending a $\mathbf{Q}$-scheme $S \in \mathrm{Sch}/\mathbf{Q}$ to the set $X_L(S)$ of isomorphism classes of quadruples $(A, \iota, \lambda, \overline{\eta})$, where*

(i) *A is an abelian scheme (of relative dimension 8) over S, up to isogeny,*

(ii) *$\iota : C \longrightarrow \mathrm{End}^0(A)$ is a homomorphism for which*

$$\det\left(\iota(c); \mathrm{Lie}(A)\right) = N^o(c)^2,$$

where N^o denotes the reduced norm homomorphism on $C = M_2(B)$.

(iii) *λ is a $\mathbf{Q}$-class of polarization on A which induces the involution $*$ on C:*

$$\lambda \circ \widehat{\iota(c)} \circ \lambda^{-1} = \iota(c^*) \quad \forall c \in C = M_2(B).$$

(iv) *$\overline{\eta}$ is a K_L-class of isomorphisms*

$$\eta : \widehat{V}(A) \cong U \otimes \mathbf{A}_f, \quad \widehat{V}(A) := \prod_l T_l(\mathbf{A}) \otimes \mathbf{Q}$$

which are C-linear for the left-module structure on U, and respect symplectic structures on both sides up to multiplication by a constant in $\mathbf{A}_f^\times$ (see Kottwitz [30, p. 390] *and* [41, §1] *for details).*

Proposition 2.2. *Let $L \subset V$ be any lattice whose associated compact open subgroup $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ is neat. Then, the moduli problem $X_L = X_{K_L} = X_K$ is represented by a smooth quasiprojective scheme, again denoted by $X_L = X_{K_L} = X_K$ over $\mathbf{Q}$. Moreover,*

$$X_L(\mathbf{C}) \cong \mathrm{Sh}_L(\mathrm{GSpin}(V), D(V))(\mathbf{C}).$$

Proof. See [30] for representability (under neatness), and [41, Proposition 1.1] for the isomorphism. □

3. ALGEBRAIC CYCLES

We now summarize the construction and relevant properties of special cycles on each Shimura/Siegel threefold $X_L = X_{K_L} = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$ following [31]. To describe this very briefly: Each subspace $(U, Q_U) = (U, Q|_U)$ of signature $(m, 2)$ of (V, Q) with $m = 0, 1, 2$ determines a Shimura subvariety

$$Z(U) = \mathrm{Sh}_{K_L \cap \mathrm{GSpin}(U)(\mathbf{A}_f)}(\mathrm{GSpin}(U), D(U)) \subset X_L$$

of dimension m defined over $\mathbf{Q}$. That is, each such subvariety determines a cycle $Z(U) \subset X_L$ of codimension $r := 3 - m$. These so-called special cycles satisfy nice functorial properties such as compatibility in profinite limits and under Hecke correspondences. We also have explicit constructions of the Poincaré dual representatives as Kudla-Millson theta series according to [31, §8], [35], [36], and [37]. We later describe the construction of arithmetic special divisors associated to special divisors ($r = 1$) via regularized Borcherds theta lifts as well as “admissible Green's currents" associated to special cycles of higher codimension $r \geq 2$.

3.1. Special cycles.

More formally, let $W \subset V$ be any subspace[5] of dimension $\dim(W) = r \leq 3$ such that $(\cdot, \cdot)|_W$ is positive definite, equivalently such that $Q|_W$ is positive definite. Its orthogonal complement $U = W^\perp \subset V$ determines a rational quadratic subspace $(U, Q_U) = (U, Q|_U)$ of signature $(m, 2) = (3 - r, 2)$ of (V, Q). We write $\mathrm{GSpin}(U) = \left\{ g \in C^0(U) : gUg^{-1} = U \right\}$ to denote the corresponding general spin group,

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(U) \longrightarrow \mathrm{SO}(U) \to 1,$$

with

$$D(U) = \{ z \subset U_{\mathbf{R}} : \dim(z) = 2, Q_U|_z < 0 \}$$

the Grassmannian of oriented negative 2-planes in $U_{\mathbf{R}}$. Again, this has a complex structure, with two connected components $D^\pm(U)$. As explained in [31, §2], we have a natural morphism of groups

$$\mathrm{GSpin}(U) \longrightarrow \mathrm{GSpin}(V),$$

and can identify $\mathrm{GSpin}(U)$ with the pointwise stabilizer of W in $\mathrm{GSpin}(V)$. Writing $L \subset V$ again to denote the lattice giving rise to the compact open subgroup $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$, the corresponding sublattice

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[5] We remark that Kudla [31] uses U to denote the positive definite space $Q(x)$ whose orthogonal complement $U^\perp$ gives the desired subspace of signature $(\dim(X) - \dim(U), 2)$, with $r = \dim(X)$ and $n = \dim(X) - \dim(U)$ the codimension.

$L_U = L \cap U$ gives rise to the compact open subgroup $K_{L_U} = K_L \cap \mathrm{GSpin}(U)(\mathbf{A}_f)$. Putting this together, we find there is a natural morphism of the Shimura varieties defined over $\mathbf{Q}$,

$$\mathrm{Sh}_{K_{L_U}}(\mathrm{GSpin}(U), D(U)) \longrightarrow \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)).$$

To be more precise, we write $Z(U) = Z(U, K_L)$ to denote the image of the Shimura subvariety $\mathrm{Sh}_{K_{L_U}}(\mathrm{GSpin}(U), D(U))$ under this morphism. Hence, $Z(U)$ has complex points given by

$$Z(U)(\mathbf{C}) = \mathrm{GSpin}(U)(\mathbf{Q}) \backslash D^{\pm}(U) \times \mathrm{GSpin}(U)(\mathbf{A}_f)/K_{L_U}.$$

We can define for each $g \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ a conjugated level structure $K_{L_U,g} = \mathrm{GSpin}(U)(\mathbf{A}_f) \cap g K_{L_U} g^{-1}$ and corresponding special cycle $Z(U,g) = Z(U, g, K_L)$ with complex points given by

$$Z(U,g)(\mathbf{C}) = \mathrm{GSpin}(U)(\mathbf{Q}) \backslash D^{\pm}(U) \times \mathrm{GSpin}(U)(\mathbf{A}_f)/K_{L_U,g}.$$

As explained in [31, Lemma 2.2], these latter cycles satisfy the following functorial properties:

(i) For all $k \in K$, we have $Z(U, gk, K_L) = Z(U, g, K_L)$.
(ii) For all $h \in \mathrm{GSpin}(U)(\mathbf{A}_f)$, we have $Z(U, hg, K_L) = Z(U, g, K_L)$.
(iii) For all $\gamma \in \mathrm{GSpin}(V)(\mathbf{Q})$, we have $Z(\gamma U, \gamma g, K_L) = Z(U, g, K_L)$.
(iv) For all $g_0, g_1 \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we have $Z(U, g_1 g_0^{-1}, g_0 K_L g_0^{-1}) \cdot g_0 = Z(U, g, K_L)$.

We refer to [31, §§2-4] for more on these cycles, and how they behave with respect to the decomposition (18).

3.1.1. *Weighted cycles.* We consider the following weighted cycles corresponding to K_L-invariant Schwartz functions $\varphi \in \mathcal{S}(V(\mathbf{A}_f)^r)$ (see [31, §5]). Fix a codimension $0 \leq r \leq 3$. Fix a positive definite symmetric matrix $\beta = {}^t\beta \in M_r(\mathbf{Q})$. Given a vector $x = (x_0, \ldots, x_{r-1}) \in V^r$, let $(x, x) = ((x_i, x_j))_{0 \leq i,j \leq r-1} \in \mathrm{Sym}_r(\mathbf{Q})$ denote the matrix of inner products $(x_i, x_j) = Q(x_i + x_j) - Q(x_i) - Q(x_j)$ of the components of x. Let Ω_β denote the corresponding hyperboloid

$$\Omega_\beta = \Omega_\beta^r = \left\{ x = (x_0, \ldots, x_{r-1}) \in V^r : (x,x) = \beta \right\}.$$

As explained in [31, §5], this hyperboloid determines a Zariski closed subset of V^r. The map

$$\mathrm{GSpin}(V)(\mathbf{A}_f) \longrightarrow \Omega_\beta(\mathbf{A}_f), \quad h \longmapsto h \cdot x,$$

is open, so that for $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ any compact open subgroup, the corresponding $K \cdot x$ is an open neighbourhod of x in $\Omega_\beta(\mathbf{A}_f)$. Moreover, we have

- $\Omega_\beta(\mathbf{Q}) = \emptyset \iff \Omega_\beta(\mathbf{A}_f) = \emptyset \iff \Omega_\beta(\mathbf{A}) = \emptyset$
- if $r < 3$, then $\Omega_\beta(\mathbf{Q}) \neq \emptyset$ implies that $K \cdot \Omega_\beta(\mathbf{Q}) = \Omega_\beta(\mathbf{A}_f)$.

Fix $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ a compact open subgroup and $\varphi \in \mathcal{S}(V(\mathbf{A}_f)^r)$ a K-invariant Schwartz function. Fix $\beta = {}^t\beta \in M_p(\mathbf{Q})$ a positive definite symmetric matrix, and $x \in \Omega_\beta(\mathbf{Q})$ a basepoint. Writing

$$\mathrm{supp}(\varphi) \cap \Omega_\beta(\mathbf{A}_f) = \coprod_{\zeta} K \cdot \zeta^{-1} \cdot x$$

for some finite set of representatives $\zeta \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we then use the same notations as above with

$$U = U(x) := \mathbf{Q}(x)^{\perp} \subset V$$

the orthogonal complement of the space $\mathbf{Q}(x)$ generated by $x \in V^r$ to define the weighted cycle

$$Z(\beta, \varphi, K) = \sum_{\zeta} \varphi(\zeta^{-1} \cdot x) Z(U(x), \zeta, K). \tag{19}$$

Note that this cycle is independent of the choice of basepoint $x \in \Omega_\beta(\mathbf{Q})$ and coset representatives $\{\zeta\}$.

Proposition 3.1 (Kudla). *The weighted cycle $Z(\beta, \varphi, K)$ defined in* (19) *satisfies the following properties.*

(i) *It can be expressed in terms of the decomposition* (18) *as*

$$Z(\beta, \varphi, K) = \sum_{h} \sum_{x \in \Omega_\beta(\mathbf{Q}) \bmod \Gamma_h} \varphi(h^{-1} \cdot x) C(U(x), g, K),$$

where $C(U(x), g, K)$ denotes the cycle on the connected component $\Gamma_h \backslash D(V)$ defined by the image of

$$(\Gamma_h \cap \mathrm{GSpin}(U(x))(\mathbf{A}_f)) \backslash D(U(x)) \longrightarrow \Gamma_h \backslash D(V).$$

(ii) *Writing* $\omega(h)$ *to denote the natural action of* $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ *on Schwartz functions* $\mathcal{S}(V(\mathbf{A}_f)^r)$ *by* $(\omega(h)\varphi)(x) = \varphi(h^{-1}x)$*, we have for any* $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ *the equivariance property*

$$Z(\beta, \omega(h)\varphi, hKh^{-1}) = Z(\beta, \varphi, K) \cdot h^{-1}.$$

(iii) *If* $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ *is a neat compact open subgroup with subgroup* $K' \subset K$ *and corresponding natural projection* $\mathrm{pr} : X_{K'} \longrightarrow X_K$ *of Shimura varieties, then we have the pullback formula*

$$\mathrm{pr}^* \left(Z(\beta, \varphi, K)\right) = Z(\beta, \varphi, K').$$

Proof. See [31, Proposition 5.4] for (i), [31, Proposition 5.9] for (ii), and [31, Proposition 5.10] for (iii). ☐

Example 3.1 (Special divisors or "Heegner cycles")**.** We have weighted cycles $Z(m, \mu) = Z(m, \mathbf{1}_\mu, K)$ of codimension $r = 1$ indexed by positive rationals $m \in \mathrm{Sym}_p(\mathbf{Q}) \cong \mathbf{Q}^\times$ and characteristic functions

$$\mathbf{1}_\mu = \mathrm{char}(\mu + L \otimes \widehat{\mathbf{Z}}) \in \mathcal{S}(V(\mathbf{A}_f))$$

of cosets $\mu \in L^\vee/L$ of the discriminant group of the lattice $L \subset V$. To describe these concretely in terms of the decomposition (18) of $X_L(\mathbf{C}) = X_{K_L}(\mathbf{C})$ into connected components, let us for each of the chosen representatives h of the finite set $\mathrm{GSpin}(V)(\mathbf{Q})\backslash \mathrm{GSpin}(V)(\mathbf{A}_f)/K_L$ write $\mu_h \in L_h^\vee/L_h$ to denote the cosets of the discriminant group of the lattice L_h determined by $L_h \otimes \widehat{\mathbf{Z}} = hL \otimes \widehat{\mathbf{Z}}$. Given a vector $x \in V$ with $Q(x) > 0$, we write

$$D(V)_x = \{z \in D(V) : (x, z) = 0\} \subset D(V)$$

to denote its complement in $D(V)$. The special divisor $Z(m, \mu) = Z(m, \mathbf{1}_\mu, K_L)$ has complex points given by

$$Z(m,\mu)(\mathbf{C}) = \coprod_h \Gamma_h \Big\backslash \left(\coprod_{\substack{x \in \mu_h + L_h \\ 2Q(x) = m}} D(V)_x \right). \tag{20}$$

Such special divisors generalize the construction of Heegner divisors on the modular curve, and are sometimes referred to in the literature as *Heegner cycles* for this reason.

3.2. **Moduli description of special cycles.** We have the following description of the weighted special cycles $Z(\beta, (\mu_1, \cdots, \mu_r)) \in Z^r(X_L)$ for the moduli description of Definition 2.1 and Proposition 2.2, according to Kudla-Rapoport [41, §2]. Suppose the quadratic space (V, Q) is parametrized as $V \subseteq C = M_2(B)$ for B an indefinite quaternion algebra defined over $\mathbf{Q}$, with Q the quadratic form defined by $x^2 = Q(x) \cdot I_2 \in M_2(B)$.

3.2.1. *Special endomorphisms and positive-definite spaces.* Given a point $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$, a *special endomorphism* of ξ is an element $j \in \mathrm{End}_S^0(A, \iota)$ which is invariant under the Rosati involution of λ and has reduced trace zero:

$$j^* = j \quad \text{and} \quad \mathrm{trd}(j) = 0.$$

Here, we note that $\mathrm{End}_S^0(A, \iota)$ is a finite dimensional semisimple $\mathbf{Q}$-algebra, there a sensible notion of reduced trace $\mathrm{trd} : \mathrm{End}_S^0(A, \iota) \to \mathbf{Q}$. Given j a special endormorphism of $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$ for S a connected scheme, its square $j^2 = Q(j) \cdot \mathrm{id}$ determines a quadratic form with $Q(j) \in \mathbf{Q}$ ([41, Lemma 2.1]). Hence, for S a connected scheme and $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$ a point, we can define

$$C_\xi^0 = \mathrm{End}_S^0(A, \iota)^{\mathrm{op}}$$

and

$$V_\xi^0 = \left\{ x \in C_\xi^0 : x^* = x, \ \mathrm{trd}(x) = 0 \right\}$$

the finite-dimensional rational vector space of special endomorphisms with quadratic form $Q_\xi : V_\xi^0 \to \mathbf{Q}$ given by $x^2 = Q_\xi(x) \cdot \mathrm{id}_A$. Hence, each point $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$ determines a rational quadratic space (V_ξ^0, Q_ξ^0). Writing $C(V_\xi^0, Q_\xi^0)$ to denote its Clifford algebra, we know by the universal property that there is a natural homomorphism $C(V_\xi^0, Q_\xi^0) \longrightarrow C_\xi^0$ which is compatible with respect to specialization.

To describe this setup in classical terms ([41, Lemma 2.3]), fix a point $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$ with corresponding parameter $(z, g) \in \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$. Let $A_z^{\mathrm{top}} = U(\mathbf{R})/U(\mathbf{Z})$ denote the real torus underlying the abelian variety $A_z = A(\mathbf{C})$. Then, we have the following identifications:

$$C(\mathbf{Q}) \cong \mathrm{End}^0(A^{\mathrm{top}}, \iota)^{\mathrm{op}}, \quad C_\xi^0 \cong \mathrm{Cent}_{C(\mathbf{Q})}(j_z), \quad V_\xi^0 \cong \{x \in V(\mathbf{Q}) : xj_z = j_z x\}, \quad \mathrm{Cent}_{C(\mathbf{R})} \cap V(\mathbf{R}) = z^\perp.$$

In particular, we can identity $V_\xi = V(\mathbf{Q}) \cap z^\perp$ and deduce that $0 \leq \dim_{\mathbf{Q}}(V_\xi) \leq 3$. If $\xi = (A, \iota, \lambda, \overline{\eta}) \in X_L(S)$ for S a connected scheme, then we also know that the space (V_ξ, Q_ξ) is positive definite ([41, Lemma 2.4]).

3.2.2. *Moduli description of weighted cycles.* We have the following moduli description of the weighted cycles

$$Z(\beta, (\mu_1, \cdots \mu_r)) = Z(\beta, \mathbf{1}_{\mu_1} \otimes \cdots \otimes \mathbf{1}_{\mu_r}) = Z(\beta, \varphi, K_L) \in Z^r(X_L)$$

introduced above, for Schwartz functions $\varphi = \mathbf{1}_{\mu_1} \otimes \cdots \otimes \mathbf{1}_{\mu_r} \in S(V(\mathbf{A}_f)^r)$ given by the characteristic functions of cosets $\mu_1, \cdots \mu_r \in L^\vee/L$ in the discriminant group of the underlying lattice $L \subset V$. In this generality, we view a cycle as being a finite unramified morphism into the ambient scheme X_L.

Definition 3.2 (Cycle moduli problem)**.** *Fix $L \subset V$ a even lattice with corresponding compact open subgroup $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$. Define a functor sending a $\mathbf{Q}$-scheme $S \in \mathrm{Sch}/\mathbf{Q}$ to the set*

$$\mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r))(S) = \mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r), K_L)(S)$$

of isomorphism classes of 5-tuples $(A, \iota, \lambda, \overline{\eta}, \mathbf{j})$, where:

(i) $(A, \iota, \lambda, \overline{\eta}) \in X_L(S)$,

(ii) $\mathbf{j} = (j_1, \cdots j_r) \in \mathrm{End}^0(A, \iota)^r$ *is an r-tuple of special endomorphisms of A such that*

- *For some (and hence for all) $\eta \in \overline{\eta}$, the element $\eta^*(\mathbf{j}) \in \mathrm{End}_C(U(\mathbf{A}_f))^r$ lies in the set*

$$\mu_1 \otimes \cdots \otimes \mu_r + L^r \otimes \widehat{\mathbf{Z}} \in V(\mathbf{A}_f)^r.$$

- $Q(\mathbf{j}) = \frac{1}{2}(\mathbf{j}, \mathbf{j}) = \frac{1}{2}((j_i, j_j))_{1 \leq i,j \leq r} = \beta \in \mathrm{Sym}_r(\mathbf{Q})$.

Proposition 3.3 (Kudla-Rapoport)**.** *The functor of Definition 3.2 has a coarse moduli scheme $\mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r))$. If the compact open subgroup $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ is neat, then $\mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r))$ is a fine moduli scheme, and the forgetful functor $\mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r)) \longrightarrow X_L$ is finite and unramified. Moreover, we have*

$$\mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r))(\mathbf{C}) = Z(\beta, (\mu_1, \cdots, \mu_r)) \in Z^r(X_L).$$

Proof. See [41, Proposition 2.5], taking $\omega_i = \mu_i + L \otimes \widehat{\mathbf{Z}}$ for each $1 \leq i \leq r$ and $\omega = \omega_1 \times \cdots \times \omega_r \subset V(\mathbf{A}_f)^r$. □

3.2.3. *Extension to the p-integral model.* Fix an odd prime $p > 2$ not dividing the discriminant of B. Fix an even lattice $L \subset V$ with corresponding compact open subgroup $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$. We decompose K as $K = K^p K_p$ and assume $K^p \subset \mathrm{GSpin}(V)(\mathbf{A}_f^p)$ is neat. As described in [41, §1, p. 704], we have a p-integral version of the moduli problem (Definition 2.1, Proposition 2.2), which we denote here by $\mathcal{X}_{K^p}$. Given a point $\xi = (A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{X}_{K^p}(S)$, we have natural analogues of the quadratic spaces and special cycles defined above, with $\mathbf{Z}_{(p)}$ algebra

$$C_\xi = \mathrm{End}_S(A, \iota)^{\mathrm{op}} \otimes \mathbf{Z}_{(p)}$$

and corresponding $\mathbf{Z}_{(p)}$-module

$$V_\xi = \{x \in C_\xi : x^* = x, \mathrm{trd}(x) = 0\},$$

as well as the analogous notion of a special endomorphism $\mathbf{j}$ of a point $\xi = (A, \iota, \lambda, \overline{\eta}^p) \in \mathcal{X}_{K^p}(S)$. Writing $\omega_i = \mu_i + L \otimes \widehat{\mathbf{Z}}$ for each $1 \leq i \leq r$ and $\omega = \omega_1 \times \cdots \times \omega_r \subset V(\mathbf{A}_f)^r$, we can decompose $\omega = \omega_p \times \omega^p$ into $\omega_p = \mu_1 \otimes \cdots \otimes \mu_r + L \otimes \mathbf{Z}_p \in V(\mathbf{Z}_p)^r$ and $\omega^p = \mu_1 \otimes \cdots \otimes \mu_r + L \otimes \widehat{\mathbf{Z}}^p \in V(\mathbf{A}_f^p)^r$. We then define

$$\mathcal{Z}(\beta, (\mu_1^p, \cdots \mu_r^p)) = \mathcal{Z}(\beta, \omega^p) \in Z^r(\mathcal{X}_{K^p})$$

for the corresponding characteristic functions

$$\varphi = \mathbf{1}_{\omega^p} = \mathbf{1}_{\mu_1^p} \otimes \cdots \otimes \mathbf{1}_{\mu_r^p} \in \mathcal{S}(V(\mathbf{A}_f^p)^r),$$

using the corresponding version of Definition 3.2. In this case, [41, Proposition 2.6] shows that this functor $\mathcal{Z}(\beta, \omega^p)$ is representable by a scheme which maps by a finite unramified morphism to $\mathcal{X}_{K^p}$, and that

$$\mathcal{Z}(\beta, (\mu_1^p, \cdots \mu_r^p)) \times_{\mathrm{Spec}\, \mathbf{Z}_{(p)}} \mathrm{Spec}\, \mathbf{Q} = \mathcal{Z}(\beta, (\mu_1, \cdots, \mu_r)). \tag{21}$$

3.2.4. *Fibre products and local intersections of codimension 2 cycles.* As mentioned above, Kudla-Rapoport [41] compute p-adic local intersection numbers of these special cycles (21). We describe their main results for pairs of codimension 2 cycles $\mathcal{Z}(\beta_1, (\mu_1^p, \mu_2^p))$ and $\mathcal{Z}(\beta_2, (\mu_3^p, \mu_4^p))$ briefly. We consider the fibre product

$$\mathcal{Z} = \mathcal{Z}(\beta_1, (\mu_1^p, \mu_2^p)) \times_{\mathcal{X}} \mathcal{Z}(\beta_2, (\mu_3^p, \mu_4^p)).$$

To each point

$$\xi = (A, \iota, \lambda, \overline{\eta}^p, \mathbf{j}) = (A, \iota, \lambda, \overline{\eta}^p, (j_{1,1}, j_{1,2}, j_{2,1}, j_{2,2})) \in \mathcal{Z},$$

we can associate a fundamental matrix

$$T_\xi = Q(\mathbf{j}) = \frac{1}{2}\left((j_{i,j}, j_{i,j})\right)_{1 \leq i,j \leq 2} \in \mathrm{Sym}_4(\mathbf{Z}_{(p)}), \quad \mathrm{diag}(T_\xi) = \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix}.$$

Note that we have the decomposition

$$\mathcal{Z} := \mathcal{Z}(\beta_1, (\mu_1^p, \mu_2^p)) \times_{\mathcal{X}} \mathcal{Z}(\beta_2, (\mu_3^p, \mu_4^p)) = \coprod_{\substack{T \in \mathrm{Sym}_4(\mathbf{Z}_{(p)})_{\geq 0} \\ \mathrm{diag}(T) = \begin{pmatrix} \beta_1 & \\ & \beta_2 \end{pmatrix}}} \mathcal{Z}(T, (\mu_1, \mu_2, \mu_3, \mu_4)).$$

Theorem 3.4 (Kudla-Rapoport)**.** *Let $\xi = (A, \iota, \lambda, \overline{\eta}^p, \mathbf{j}) \in \mathcal{Z}$ be any point with $\det(T_\xi) \neq 0$. Then, ξ is an isolated intersection point if and only if the fundamental matrix T_ξ represents 1 over $\mathbf{Z}_p$. Moreover, when this is the case, the underlying abelian variety A is a product of a supersingular elliptic curve.*

Proof. See [41, Theorem 0.1] for the special case of $n_1 = n_2 = 2$ (so $r = 2$). ☐

Kudla-Rapoport [41, Theorem 0.2] also compute the local intersection numbers

$$\langle \mathcal{Z}(\beta_1, (\mu_1^p, \mu_2^p)), \mathcal{Z}(\beta_2, (\mu_3^p, \mu_4^p)) \rangle_p^{\mathrm{proper}} = \sum_{\substack{\xi \in \mathcal{Z} \\ \mathrm{isolated}}} e(\xi),$$

where each isolated intersection point $\xi = (A, \iota, \lambda, \overline{\eta}^p, \mathbf{j})$ of $\mathcal{Z}$ appears with multiplicity

$$e(\xi) = \mathrm{length}\, \mathcal{O}_{Z,\xi}$$

given by the length of the local ring of $\mathcal{Z}$ at ξ. To describe the computation briefly for the context of Conjecture 1.4, fix rational symplectic spaces W_1 and W_2 of dimension 4, and put $W = W_1 + W_2$. Let

$$i : \mathrm{Mp}_2(\mathbf{A}) \times \mathrm{Mp}_2(\mathbf{A}) \longrightarrow \mathrm{Mp}_4(\mathbf{A})$$

denote the embedding of the corresponding metaplectic groups $\mathrm{Mp}(W_1) \times \mathrm{Mp}(W_2) \to \mathrm{Mp}(W)$. Fix a standard section $\Phi(s) = \Phi_\infty(s) \otimes \Phi_f(s) \in I(s, \chi)$ in the induced representation of $\mathrm{Mp}_4(\mathbf{A})$. More precisely, writing (V', Q') for the rational quadratic space of signature $(5, 0)$, take $\varphi_\infty(s)$ the standard Gaussian function for $V'(\mathbf{R})$, and $\Phi_\infty(s) = \lambda_\infty(\varphi_\infty)(s)$ its image under the local intertwining operator $\lambda_\infty : \mathcal{S}(V'(\mathbf{R})) \to I_\infty(s, \chi)$. For the nonarchimedean local component, we take $\varphi_f = \mathbf{1}_{\mu_1} \otimes \mathbf{1}_{\mu_2} \otimes \mathbf{1}_{\mu_3} \otimes \mathbf{1}_{\mu_4} \in \mathcal{S}(V(\mathbf{A}_f)^4)$ to be the corresponding characteristic function, with $\Phi_f(s) = \lambda_f(\varphi_f)(s)$ its image under the nonarchimedean local intertwinining operator $\lambda_f : \mathcal{S}(V(\mathbf{A}_f)^4) \to I_f(0, \chi)$. Writing $P \subset \mathrm{Mp}_4$ to denote the Siegel parabolic, the corresponding incoherent Eisenstein series

$$E(h, s, \Phi) = \sum_{\gamma \in P(\mathbf{Q}) \backslash \mathrm{Mp}_4(\mathbf{Q})} \Phi(\gamma h, s), \quad h \in \mathrm{Mp}_4(\mathbf{A})$$

converges for $\Re(s) > 5/2$, and has an analytic continuation to all $s \in \mathbf{C}$. We consider its Fourier series expansion along the unipotent radical of P, denoted by

$$E(h, s, \Phi) = \sum_{T \in \mathrm{Sym}_4(\mathbf{Q})} E_T(h, s, \Phi).$$

Since $E(h, s, \Phi)$ is incoherent, it vanishes at $s = 0$, and we can consider the values $E_T'(h, 0, \Phi) = \frac{d}{ds} E_T(h, s, \Phi)|_{s=0}$.

Theorem 3.5 (Kudla-Rapoport). *We have for any pair* $h_i \in \mathrm{Mp}(W_i)(\mathbf{A}) \cong \mathrm{Mp}_2(\mathbf{A})$ $(i = 1, 2)$ *that*

$$\sum_{T \in \mathrm{Sym}_4(\mathbf{Q})_{\geq 0}} E'_T(i(h_1, h_2), 0, \Phi) = \frac{\mathrm{vol}(V'(\mathbf{R}))}{2} \cdot W_{\beta_1}^{\frac{5}{2}}(h_1) \cdot W_{\beta_2}^{\frac{5}{2}}(h_2) \cdot \log(p) \cdot \mathrm{vol}(\mathrm{pr}(K_L))$$
$$\times \langle \mathcal{Z}(\beta_1, (\mu_1^p, \mu_2^p)), \mathcal{Z}(\beta_2, (\mu_3^p, \mu_4^p)) \rangle_p^{\mathrm{proper}}.$$

Here, the $W_{\beta_i}^k(h_i)$ *denote the generalized Whittaker functions defined as in* (22) *below.*

Proof. See [41, Theorem 0.2] for the special case of $n_1 = n_2 = 2$ (so $r = 2$). ☐

3.3. **Cycle classes and the intersection ring.** Given $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ a compact open subgroup, we again write $X_K = \mathrm{Sh}_K(\mathrm{GSpin}(V), D(V))$ to denote the corresponding Shimura variety, with $X = \varprojlim_K X_K$ the projective limit. Given an integer $0 \leq r \leq 3$, we write $Z^r(X_K)$ to denote the free abelian group on irreducible subvarieties of codimension r (so dimension $m = 3 - r$), with

$$\mathrm{CH}^r(X_K) = Z^r(X_K) / \sim_{\mathrm{rat}}$$

the Chow group of rational equivalence classes. We have a natural projection map $Z^r(X_K) \longrightarrow \mathrm{CH}^r(X_K)$, as well as a natural cycle class map $\mathrm{cl} : \mathrm{CH}^r(X_K) \longrightarrow H^{2r}(X_K, \mathbf{Q})$. Passing to limits, we consider the groups

$$Z^r(X) = \varinjlim_K Z^r(X_K)$$
$$\mathrm{CH}^r(X) = \varinjlim_K \mathrm{CH}^r(X_K)$$
$$H^{2r}(X, \mathbf{Q}) = \varinjlim_K H^{2r}(X_K, \mathbf{Q}),$$

as well as the corresponding natural maps $Z^r(X) \longrightarrow \mathrm{CH}^r(X) \longrightarrow H^{2r}(X, \mathbf{Q})$. In particular, we obtain from the construction of weighted cycles above (cf. [31, Corollary 5.11, (6.6), (6.7)]) classes

$$\mathcal{S}(V(\mathbf{A}_f)^r)_R \longrightarrow Z^r(X)_R, \qquad \varphi \longmapsto \varinjlim_K Z(\beta, \varphi, K)$$
$$\mathcal{S}(V(\mathbf{A}_f)^r)_R \longrightarrow \mathrm{CH}^r(X)_R, \qquad \varphi \longmapsto \{\beta, \varphi\} := \varinjlim_K \{Z(\beta, \varphi, K)\}_K$$
$$\mathcal{S}(V(\mathbf{A}_f)^r)_R \longrightarrow H^{2r}(X, R), \qquad \varphi \longmapsto [\beta, \varphi] := \mathrm{cl}\,(\{\beta, \varphi\})$$

for any ring R.

Remark This construction can be adapted for β singular; see [31, § (6.9)-(6.11)]. In brief, suppose V is anisotropic, and that $\beta \in \mathrm{Sym}_r(\mathbf{Q})$ is positive semidefinite, so that $\det(\beta) = 0$. Since V is anistropic, the the inner sum in the corresponding weighted cycle (19) is finite, and so we can define

$$Z(\beta, \varphi, K)^0 = \sum_h \sum_{x \in \Omega_\beta(\mathbf{Q}) \bmod \Gamma_h} \varphi(h^{-1} \cdot x) c(U(x), h, K).$$

Here, each $U(x) = \mathbf{Q}(x)^\perp \subset V$ has complement $\mathbf{Q}(x)$ of dimension equal to $b = \mathrm{rk}(\beta)$, and we have

$$Z(\beta, \varphi, K) \in Z^b(X_K), \quad \{Z(\beta, \varphi)\} =: \{\beta, \varphi\}^0 \in \mathrm{CH}^b(X_K), \quad [Z(\beta, \varphi)] =: [\beta, \varphi]^0 \in H^{2b}(X_K, \mathbf{Q}).$$

Let $k_{D^\pm(V)}(z_1, z_2)$ denote the Bergman kernel function for each connected component $D^\pm(V)$. Let

$$\Omega(z) = \Omega_{D^\pm(V)}(z) := \partial\overline{\partial} \log k_{D^\pm(V)}(z, z)$$

denote the corresponding Kähler form. For each compact open subgroup $K \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$,

$$\frac{1}{2\pi i} \cdot \Omega_{D^\pm(V)}(z) = \frac{1}{2\pi i} \cdot \partial\overline{\partial} k_{D^\pm(V)}(z, z)$$

induces a $(1, 1)$ form on X_K which is the Chern form for the canonical bundle $\mathfrak{K}_{X_K}$ on X_K. Taking the cup product with this $\Omega_{D^\pm(V)}$ induces the Lefshetz operator

$$L : H^q(X_K, \mathbf{Q}) \longrightarrow H^{q+2}(X_K, \mathbf{Q}),$$

as well as its corresponding operator on Chow groups

$$L : \mathrm{CH}^q(X_K)_{\mathbf{Q}} \longrightarrow \mathrm{CH}^{q+1}(X_K)_{\mathbf{Q}}.$$

We obtain from this a commutative diagram

$$\begin{tikzcd}
L : \mathrm{CH}^r(X_K)_{\mathbf{Q}} \arrow[r] \arrow[d, swap, "\mathrm{cl}"] & \mathrm{CH}^{r+1}(X_K)_{\mathbf{Q}} \arrow[d, swap, "\mathrm{cl}"] \\
L : H^{2r}(X_K, \mathbf{Q}) \arrow[r] & H^{2r}(X_K, \mathbf{Q}),
\end{tikzcd}$$

which after taking direct limits becomes

$$\begin{tikzcd}
L : \mathrm{CH}^r(X)_{\mathbf{Q}} \arrow[r] \arrow[d, swap, "\mathrm{cl}"] & \mathrm{CH}^{r+1}(X)_{\mathbf{Q}} \arrow[d, swap, "\mathrm{cl}"] \\
L : H^{2r}(X, \mathbf{Q}) \arrow[r] & H^{2r}(X, \mathbf{Q}).
\end{tikzcd}$$

As explained in [31, Proposition 6.1], the Lefshetz operator L on $\mathrm{CH}^r(X)$ and on $H^r(X, \mathbf{Q})$ commutes with the action of $\mathrm{GSpin}(V)(\mathbf{A}_f)$. Consequently, for any positive semidefinite matrix $\beta \in \mathrm{Sym}_r(\mathbf{Q})$, we have

$$\{\beta, \varphi\} = L^{r-\mathrm{rk}(\beta)} \cdot \{\beta, \gamma\}^0 \in \mathrm{CH}^r(X)_{\mathbf{Q}}$$

and

$$[\beta, \varphi] = L^{r-\mathrm{rk}(\beta)} \cdot [\beta, \gamma]^0 \in H^{2r}(X, \mathbf{Q}).$$

We also have the following cup product formula for classes coming from special cycles.

Theorem 3.6 (Kudla)**.** *Assume the quadratic space V is anisotropic. For each $0 \leq r \leq 3 = \dim(X)$, let $C^r(X)$ denote the subspace of $H^{2r}(X, \mathbf{Q})$ spanned by the classes $[\beta, \varphi] = \mathrm{cl}\left(\{\beta, \varphi\}\right)$, with $\beta \in \mathrm{Sym}_r(\mathbf{Q})$ ranging over positive semidefinite matrices, and $\varphi \in \mathrm{S}(V(\mathbf{A}_f)^r)_{\mathbf{Q}}$ ranging over Schwartz functions.*

(i) *The space $C(X) = \bigoplus_{r=0}^{3} C^p(X)$ is a subring of $H(X, \mathbf{Q})$.*

(ii) *For $\beta_i \in \mathrm{Sym}_{r_i}(\mathbf{Q})$ and $\varphi_i \in \mathcal{S}(V(\mathbf{A}_f)^{r_i})$ $(i = 1, 2)$ the cup product of $[\beta_1, \varphi_1]$ and $[\beta_2, \varphi_2]$ is given by the formula*

$$[\beta_1, \varphi_1] \cdot [\beta_2, \varphi_2] = \sum_{\substack{\beta \in \mathrm{Sym}_{r_1+r_2}(\mathbf{Q}) \\ \beta = \begin{pmatrix} \beta_1 & \alpha \\ {}^t\alpha & \beta_2 \end{pmatrix}}} [\beta, \varphi_1 \otimes \varphi_1],$$

where the sum runs over matrices $\alpha \in M_{r_1, r_2}(\mathbf{Q})$ for which $\begin{pmatrix} \beta_1 & \alpha \\ {}^t\alpha & \beta_2 \end{pmatrix}$ is positive semidefinite.

Proof. See [31, Theorem 6.2]. ☐

3.4. **Poincaré dual representatives.** We now describe the theory of Kudla-Millson [35], [36], [37] in this setting, particularly the construction of Poincaré dual representatives for the classes $[\beta, \varphi] \in H^{2r}(X_K, \mathbf{C})$, following [31, §7-8]. Here, we fix an integer $1 \leq r \leq 3$ corresponding to the chosen codimension of the special cycles on the Siegel threefold $X = \{X_K\}_K = \{\mathrm{Sh}_K(\mathrm{GSpin}(V), D(V))\}_K$.

3.4.1. *r-th Schwartz forms.* We work over the ring $R = \mathbf{C}$, with $H^r(X_K) = H^r(X_K, \mathbf{C})$. Fixing a basepoint $z_0 \in D(V)$, we now write K to denote the corresponding stabilizer in $\mathrm{GSpin}(V)(\mathbf{R})$, so that the identification (16) of the Grassmannian is equivalent to

$$D(V) \cong \mathrm{GSpin}(V)(\mathbf{R})/K \cong \mathrm{SO}(n, 2)/\left(\mathrm{SO}(n) \times \mathrm{SO}(2)\right).$$

Write $\mathfrak{g}_0 = \mathrm{Lie}(\mathrm{GSpin}(V)(\mathbf{R}))$ and $\mathfrak{k}_0 = \mathrm{Lie}(K)$ to denote the corresponding Lie algebras, with $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}$ and $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$ their complexifications. We have a Harish-Chandra decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-$$

and a space of smooth differential forms on $D(V)$ of type (a, b),

$$A^{a,b}(D(V)) \cong \left[C^{\infty}(\mathrm{GSpin}(V)(\mathbf{R})) \otimes \wedge^{a,b}(\mathfrak{p}^*)\right]^K.$$

Let $\mathrm{Mp}_{2r}(\mathbf{R})$ denote the metaplectic cover of $\mathrm{Sp}_{2r}(\mathbf{R})$. Recall this group acts on the space $\mathcal{S}(V(\mathbf{R})^p)$ of Schwartz functions on $V(\mathbf{R})^r$ through the Weil representation $\omega = \omega_{\psi_\infty}$ associated to the standard additive character $\psi_\infty(x) = e(x) = \exp(2\pi i x)$. Let K' denote the inverse image of the maximal compact subgroup

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + ib \in U(p) \right\} \subset \mathrm{Sp}_{2r}(\mathbf{R}).$$

Given a point $z \in D(V)$, we again write $(\cdot,\cdot)_z$ to denote the majorant determined by z, so

$$(x,x)_z = \begin{cases} (x,x) & \text{if } x \in z^\perp \\ -(x,x) & \text{if } x \in z. \end{cases}$$

We consider the corresponding Gaussian Φ^0 defined by

$$\Phi^0(x,z) := \exp\left(-\pi \operatorname{tr}(x,x)_z\right).$$

Note that $\Phi^0(hx,hz) = \Phi^0(x,z)$ for all $h \in \mathrm{GSpin}(V)(\mathbf{R})$, and hence

$$\Phi^0(\cdot,\cdot) \in \left[\mathcal{S}(V(\mathbf{R})^r) \otimes C^\infty(D(V))\right]^{\mathrm{GSpin}(V)(\mathbf{R})}.$$

It is also an eigenfunction for K' such that $\omega(k')\Phi^0 = \det(k')^{\frac{1}{2}}\Phi^0$.

Theorem 3.7 (Kudla-Millson). *There exists for each integer $1 \leq r \leq 3$ a nonzero Schwartz form*

$$\Phi^{(r)}(\cdot,\cdot) \in \left[\mathcal{S}(V(\mathbf{R})^r) \otimes A^{r,r}(D(V))\right]^{\mathrm{GSpin}(V)(\mathbf{R})} \cong \left[\mathcal{S}(V(\mathbf{R})^r) \otimes \wedge^{r,r}(\mathfrak{p}^*)\right]^K$$

for which the following properties hold:

(i) $d\Phi^{(r)} = 0$, *so $\Phi^{(r)}(x,\cdot)$ is a closed $\mathrm{GSpin}(V)(\mathbf{R})_x$-invariant (r,r)-form on $D(V)$ for each $x \in V(\mathbf{R})^r$.*

(ii) *Under the action of $K^{'}$ on $\mathcal{S}(V(\mathbf{R})^r)$ through the Weil representation ω, $\omega(k')\Phi^{(r)} = \det(k')^{\frac{r+2}{2}}\Phi^{(r)}$.*

(iii) *The Schwartz forms combine under wedge products as $\Phi^{(r_1)} \wedge \Phi^{(r_2)} = \Phi^{(r_1+r_2)}$, with $\Phi^{(r)} = 0$ if $r > 3$.*

(iv) $\Phi^{(r)}(0) = c(r)\Omega^r$ *for some constant $c(r) > 0$.*

Proof. See [31, Theorem 7.1]. ☐

Suppose that $W, (\cdot,\cdot)_W$ is a positive definite inner product space over $\mathbf{R}$, with Gaussian $\Phi_W^0 \in \mathcal{S}(W(\mathbf{R})^r)$ defined by $\Phi_W^0(x) = \exp(-\pi \operatorname{tr}(x,x)_W)$ and associated Weil representation ω_W of $\mathrm{Mp}_{2r}(\mathbf{R})$ on $\mathcal{S}(W(\mathbf{R})^r)$ acting via $\omega_W(k')\Phi_W^0 = \det(k')^{\frac{5}{2}}\Phi_W^0$. Given a vector $x \in W(\mathbf{R})^r$ with $(x,x)_W = \beta \in \mathrm{Sym}_r(\mathbf{R})$ and $g \in \mathrm{Mp}_{2r}(\mathbf{R})$, we consider the generalized Whittaker function defined by

$$W_\beta(g) = \omega_W(g)\Phi_W^0(x). \tag{22}$$

Theorem 3.8 (Kudla-Millson). *The following assertions are true.*

(i) *For any $g \in \mathrm{Mp}_{2r}(\mathbf{R})$, $\omega(g)\Phi^{(r)}(x)$ is a closed $\mathrm{GSpin}(U)(\mathbf{R})$-invariant (r,r)-form on $D(V)$.*

(ii) *Let $\Gamma_U \subset \mathrm{GSpin}(U)(\mathbf{R})^+$ be any discrete subgroup such that $\Gamma_U \backslash D^+(U)$ is compact. Then, for any closed and bounded $(3-r,3-r)$-form η on $\Gamma_U \backslash D^+(U)$, we have*

$$\int_{\Gamma_U \backslash D^+(V)} (\omega(g)\Phi^{(r)})(x) \wedge \eta = W_\beta(g) \int_{\Gamma_U \backslash D^+(U)} -\frac{1}{2\pi} x^*(1,2)^{r-q} \wedge \eta,$$

where $\beta = (x,x)$ and $q = \mathrm{rk}(\beta) = \dim \mathbf{Q}(x)$.

(iii) *Fix* $W = \mathbf{Q}(x)$ *a positive* q*-plane determined by the span of a vector* $x \in V(\mathbf{R})^q$ *for* $0 \leq q < r$. *Let* $U = U(x) = \mathbf{Q}(x)^{\perp} \subset V$ *be its orthogonal complement, a subspace of signature* $(3-q, 2)$, *with corresponding Grassmannian* $D(U)$ *of connected components* $D^{\pm}(U)$. *Let* $\Phi_W^0 \in \mathcal{S}(W(\mathbf{R})^r)$ *be the corresponding Gaussian, as described above. Under pullback on differential forms*

$$\iota_U^! : A^{r,r}(D(V)) \rightarrow A^{r,r}(D(U)),$$

we have the identification

$$\iota_U^! \Phi^{(r)} = \Phi_W^0 \otimes \Phi_U^{(r)},$$

where $\Phi_U^{(r)}$ *denotes the* r*-th Schwartz form for* U *and* $\mathrm{GSpin}(U)(\mathbf{R})$,

$$\Phi_U^{(r)} \in \left[\mathcal{S}(U(\mathbf{R})^r) \otimes A^{r,r}(D(U))\right]^{\mathrm{GSpin}(U)(\mathbf{R})}.$$

Proof. See [31, p. 67] for (i), [31, Theorem 7.2 (Thom lemma)] for (ii), and [31, Lemma 7.3] for (iii). ☐

3.4.2. *Dual forms.* We have the following construction of Poincaré dual representatives for the classes $[\beta, \varphi]$ introduced above. Here, we first introduce the Kudla-Millson theta series $\Theta_L^{(r)}$ associated to the p-th Schwartz form $\Phi^{(r)}$. Hence, fixing the integer codimension $1 \leq r \leq 3$, we chose a decomposable Schwartz function

$$\Phi = \Phi_\infty \otimes \Phi_f \in \left[\mathcal{S}(V(\mathbf{A})^r) \otimes A^{r,r}(D(V))\right]^{\mathrm{GSpin}(V)(\mathbf{R})}$$

with archimedean part given by the p-Schwartz form

$$\Phi_\infty = \Phi^{(r)} \in \left[\mathcal{S}(V(\mathbf{R})^r) \otimes A^{r,r}(D(V))\right]^{\mathrm{GSpin}(V)(\mathbf{R})},$$

and nonarchimedean part matching the chosen function parametrizing our weighted special cycles $Z(\beta, \varphi)$,

$$\Phi_f = \varphi \in \mathcal{S}(V(\mathbf{A}_f)^r).$$

Fix $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$ a compact open subgroup with underlying lattice $L \subset V$. Let $\psi = \otimes_v \psi_v$ denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, with archimedean component $\psi_\infty(x) = e(x) = \exp(2\pi i x)$. Recall that the metaplectic group $\mathrm{Mp}_{2r}(\mathbf{A})$ acts on the space of Schwartz-Bruhat functions $\mathcal{S}(V(\mathbf{A}_f)^r)$ via the Weil representation $\omega_L = \omega_{L,\psi}$. For any K-invariant Schwartz function $\Phi_f \in \mathcal{S}(V(\mathbf{A}_f)^r)$, we define the corresponding Kudla-Millson theta series on $g \in \mathrm{Mp}_{2r}(\mathbf{A})$ and $h \in \mathrm{GSpin}(V)(\mathbf{A})$ by

$$\Theta_L^{(r)}(h, g, \varphi) = \sum_{x \in V(\mathbf{Q})^r} \omega_L(g)\Phi(h^{-1}x) = \sum_{x \in V(\mathbf{Q})^r} \omega_L(g)\Phi^{(r)} \otimes \varphi\left(h^{-1}x\right).$$

Note that this theta series defines a closed (r,r)-form on the threefold $X_L = X_{K_L}$. Its cohomology class is related to the classes $[\beta, \varphi] \in H^{2r}(X_L, \mathbf{C})$ described above as follows. We define the generalized Whittaker coefficients $W_\beta(g) = \omega_L(g)\Phi^0(x)$ for $g \in \mathrm{Mp}_{2r}(\mathbf{R}) \subset \mathrm{Mp}_{2r}(\mathbf{A})$ as in (22). Let us write $\Theta_L^{(r)}(g, \varphi) = \Theta_L^{(r)}(1, g, \varphi)$.

Theorem 3.9 (Kudla-Millson). *Fix* $1 \leq r \leq 3$, *and a* $K = K_L$*-invariant Schwartz function* $\varphi \in \mathcal{S}(V(\mathbf{A}_f)^r)$.

(i) *We have for each* $g \in \mathrm{Mp}_{2r}(\mathbf{R}) \subset \mathrm{Mp}_{2r}(\mathbf{A})$ *an identification of cohomology classes*

$$[\Theta_L^{(r)}(g, \varphi)] = \sum_{\substack{\beta \in \mathrm{Sym}_r(\mathbf{Q}) \\ \beta \geq 0}} W_\beta(g)[\beta, \varphi] \in H^{2r}(X_L, \mathbf{C}).$$

(ii) *Given* $\beta \in Sym_r(\mathbf{Q})$ *with* $\beta \geq 0$ *(positive semidefinite),* $g \in \mathrm{Mp}_{2r}(\mathbf{R})$, *and* $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, *let*

$$\Theta_{L,\beta}^{(r)}(h, g, \varphi) = \sum_{x \in \Omega_\beta(\mathbf{Q})} \omega_L(g)\Phi(h^{-1}x) = \sum_{x \in \Omega_\beta(\mathbf{Q})} \omega_L(g)\Phi^{(r)} \otimes \varphi(h^{-1}x).$$

This is a (r,r)*-form on* X_K, *and a smooth Poincaré dual form for the weighted cycle* $[\beta, \varphi]$:

$$[\Theta_{L,\beta}^{(r)}(g, \varphi)] = [\beta, \varphi] \cdot W_\beta(g) \in H^{2r}(X_L, \mathbf{C}).$$

Proof. See [31, Theorem 8.1] for (i), and [31, Corollary 8.2] for (ii). ☐

4. AUTOMORPHIC GREEN'S FUNCTIONS AND CURRENTS

We now describe automorphic Green's functions and currents for spin Shimura varieties $X = X_L$, leading to a characterization of the archimedean local heights for the setup with twisted Siegel threefolds introduced above. Here we begin with Siegel theta series related to Siegel Eisenstein series via the (convergent) Siegel-Weil formula. We then introduce regularized theta lifts following [33], [31], and [11], including their characterization as automorphic Green's functions for divisors. We note that this theory is now well-known, with various detailed expositions (e.g. [1], [5], [7], [11], [33]). Here, we describe only what we require for our later calculations. We then give a general account of Green's currents, including the notation of an admissible Green's current (due to Zhang [61]) to define the archimedean local height of two algebraic cycles $Y, Z \subset X$ for which $\dim(Y) + \dim(Z) = \dim(X) - 1$. Finally, returning to our setting of twisted Siegel threefolds

$$X_L \cong \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)),$$

we consider how to compute the archimedean local height $[Y, Z]_\infty$ of a pair of one-cycles $Y, Z \in Z^2(X)$. Taking the diagonal one-cycle $Z = Z(\beta, (\mu, \mu))$ as in (2) above, we show how passing to a divisor $Z_i \subset X$ containing $Z \subset Z_i$ as a divisor allows us to realize the Green's current $g_Z \in D^{1,1}(X)$ as the pushfoward $\iota_*[\iota^! G_{Z_i/X}]$ of the current $[\iota^! G_{Z_i/X}] \in D^{0,0}(Z_i)$ determined by the pullback $\iota^! G_{Z_i/X}$ along differential forms $A^{0,0}(X) \to A^{0,0}(Z_i)$ of the Green's function $G_{Z_i/X}$ of any surface $\iota : Z_i \hookrightarrow X$ containing $Z \subset Z_i$ as a divisor (Theorem 4.8). This allows us to give a more precise characterization of the archimedean local height $[Y, Z]_\infty$ of two one-cycles $Y, Z \in Z^2(X)$ in Corollary 4.9 and Corollary 4.10.

4.1. Siegel theta series.

Let us first suppose more generally that (V, Q) is any rational quadratic space of signature $(n, 2)$, with corresponding inner product $(\cdot, \cdot)$ and Grassmannian $D(V) = D^\pm(V)$. We also fix an integral lattice $L \subset V$, with dual lattice $L^\vee$, and write $L^\vee/L$ to denote the discriminant group.

4.1.1. *Construction.* Let $\psi = \otimes_v \psi_v$ denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, with archimedean component $\psi_\infty(x) = e(x) := \exp(2\pi i x)$. We write $\omega_L = \omega_{L,\psi}$ to denote the corresponding Weil representation

$$\omega_L : G(\mathbf{A}) \times H(\mathbf{A}) \longrightarrow \mathrm{Aut}\left(\mathcal{S}(V \otimes \mathbf{A})\right)$$

of the product of the two-fold metaplectic cover $G(\mathbf{A}) = \mathrm{Mp}_2(\mathbf{A})$ of $\mathrm{Sp}_2(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A})$ and the general spin group $H(\mathbf{A}) = \mathrm{GSpin}(V)(\mathbf{A})$ on the space of Bruhat-Schwartz functions $\mathcal{S}(V(\mathbf{A}))$, as described in [33, §1], [11], or [1] (for instance). Given $\Phi \in \mathcal{S}(V \otimes \mathbf{A})$, $g \in G(\mathbf{A})$, and $h \in H(\mathbf{A})$, we consider the theta function

$$\vartheta_L(g, h; \Phi) = \sum_{x \in V(\mathbf{Q})} \left(\omega_L(g, h)\Phi\right)(x) = \sum_{x \in V(\mathbf{Q})} \omega_L(g)\Phi(h^{-1}x).$$

This theta function is both left $G(\mathbf{Q})$-invariant and left $H(\mathbf{Q})$-invariant.

Given $z \in D(V)$, we consider the corresponding majorant defined on $x \in V(\mathbf{R})$ by

$$(x, x)_z := (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z) = \begin{cases} (x, x) & \text{if } x \in z^\perp \\ -(x, x) & \text{if } x \in z. \end{cases}$$

Note that this determines a positive definite quadratic form on $V(\mathbf{R})$. The corresponding Gaussian

$$\Phi_\infty(x, z) = \exp(-\pi (x, x)_z)$$

determines an archimedean local Schwartz function $\Phi_\infty(x, \cdot) \in \mathcal{S}(V(\mathbf{R}))$, and is left $H(\mathbf{R})$-invariant. To be more precise, $\Phi_\infty(hx, hz) = \Phi_\infty(x, z)$ for each $h \in \mathbf{R}$. This Gaussian function also has weight $1 - n/2$ under the action of the maximal compact subgroup of $G(\mathbf{R})$. Taking any finite local Schwartz function $\Phi_f \in \mathcal{S}(V \otimes \mathbf{A}_f)$, and fixing a basepoint $z_0 \in D(V)$, we write the theta function corresponding to this choice of Schwartz function $\Phi = \Phi_\infty(\cdot, z_0) \otimes \Phi_f$ as

$$\theta_L(g, h; \Phi_f) = \vartheta_L(g, h; \Phi_\infty(\cdot, z_0) \otimes \Phi_f(\cdot)).$$

Descending via the Iwasawa decomposition for $\mathrm{Mp}_2(\mathbf{R})$, we can write this as a function of $\mathfrak{H} \times D(V)$ to recover the classical Siegel theta function $\theta_L(\tau, z, h) : \mathfrak{H} \times D(V) \longrightarrow \mathfrak{S}_L = \mathbf{C}[L^\vee/L]$ as follows. Given $z \in D(V)$, choose an element $h_z \in H(\mathbf{R})$ for which $h \cdot z_0 = z$. Observe that we then have $\omega_L(h_z)\Phi_\infty(\cdot, z_0) = \Phi_\infty(\cdot, z)$. Writing $\tau = u + iv \in \mathfrak{H}$, let $g_\tau' = (g_\tau, 1) \in G(\mathbf{R})$ denote the image of the mirabolic matrix

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & 0 \\ 0 & v^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}).$$

Given $h_f \in H(\mathbf{A}_f)$, we then consider the theta function defined by

$$\begin{aligned}\theta_L(\tau, z, h_f; \Phi_f) &= v^{-\frac{n}{4}+\frac{1}{2}} \vartheta_L(g_\tau', h_z h_f, \Phi_\infty(\cdot, z_0) \otimes \Phi_f(\cdot)) \\ &= v^{-\frac{n}{4}+\frac{1}{2}} \sum_{x \in V(\mathbf{Q})} \omega_L(g_\tau')\left(\Phi_\infty(\cdot, z) \otimes \omega_L(h_f)\Phi_f\right)(x),\end{aligned}$$

which after using explicit formulae for the Weil representation ω_L to deduce that

$$v^{-\frac{n}{4}+\frac{1}{2}} \omega_L(g_\tau')\left(\Phi_\infty(\cdot, z)\right)(x) = v \cdot e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right)$$

takes the more explicit form

$$\theta_L(\tau, z, h_f; \Phi_f) = v \sum_{x \in V(\mathbf{Q})} e\left(Q(x_{z^\perp})\tau + Q(x_z)\overline{\tau}\right) \otimes \Phi_f(h_f^{-1}x).$$

We shall often write $h = h_f \in H(\mathbf{A}_f)$ for simplicity in what follows. Taking $\varphi_f = \mathbf{1}_\mu = \mathrm{char}(\mu + L \otimes \widehat{\mathbf{Z}})$ to be the characteristic function of a coset $\mu \in L^\vee/L$, we consider the corresponding theta function

$$\theta_{L,\mu}(\tau, z, h) = \theta_L(\tau, z, h_f; \mathbf{1}_\mu).$$

Note that $\mathbf{1}_\mu$ belongs to the subset $\mathfrak{S}_L \subset \mathrm{Aut}(\mathcal{S}(V \otimes \mathbf{A}_f))$ of functions supported on $L^\vee \otimes \widehat{\mathbf{Z}}$ which are constant on cosets of $L \otimes \widehat{\mathbf{Z}}$, and that we have the identification

$$\mathfrak{S}_L = \bigoplus_{\mu \in L^\vee/L} \mathbf{C}\mathbf{1}_\mu \cong \mathbf{C}[L^\vee/L] \subset \mathrm{Aut}(\mathcal{S}(V \otimes \mathbf{A}_f)).$$

Note as well that $\mathfrak{S}_L$ is invariant the under action of the inverse image of $\mathrm{SL}_2(\widehat{\mathbf{Z}})$ in $G(\mathbf{A}_f)$ by ω_L. Taking the sum over cosets $\mu \in L^\vee/L$, we obtain the classical Siegel theta series

$$\theta_L(\tau, z, h) = \sum_{\mu \in L^\vee/L} \theta_{L,\mu}(\tau, z, h)\mathbf{1}_\mu : \mathfrak{H} \times D(V) \longrightarrow \mathfrak{S}_L. \tag{23}$$

As a function in Grassmannian variable $z \in D(V)$, this theta series is invariant under the arithmetic subgroup $\Gamma_h = H(\mathbf{Q}) \cap hK_Lh^{-1}$. As a function in $\tau \in \mathfrak{H}$, it determines a nonholomorphic vector-valued modular form of weight $1 - n/2$ and representation $\omega_L^\vee$.

4.2. **Eisenstein series and the Siegel-Weil formula.**

We can describe averages of theta Siegel theta series (23) as special values of certain Langlands Eisenstein series via the Siegel-Weil formula. Here, we first describe the general setup for any rational quadratic space (U, Q_U) of signature $(n, 2)$ following [33, §4.1]. We then describe the corresponding convergent Siegel-Weil formula. Later, we shall apply this to the special setting of $U = V_{1,2} \subset V$ an anisotropic quadratic subspace of signature $(1, 2)$.

4.2.1. *Langlands Eisenstein series.* Let $\psi = \otimes_v \psi_v$ again denote the standard additive character on $\mathbf{A}/\mathbf{Q}$. Let (U, Q_U) be any rational quadratic space of signature $(p(U), q(U)) = (n, 2)$, for $n \geq 0$ an integer. Fix an integral lattice $L_U \subset U$ with dual lattice $L_U^\vee$ and discriminant group $L_U^\vee/L_U$. We write

$$\omega_{L_U} = \omega_{L_U,\psi} : \mathrm{Mp}_2(\mathbf{A}) \times \mathrm{GSpin}(U)(\mathbf{A}) \longrightarrow \mathfrak{S}_{L_U}$$

for the corresponding Weil representation, as introduced above. We write $G = \mathrm{Mp}_2$ and $H_U = \mathrm{GSpin}(U)$ again for simplicity to denote the algebraic groups over $\mathbf{Q}$. We write $P = MN$ denote the standard parabolic subgroup of upper triangular matrices in SL_2, with Levi subgroup M and unipotent subgroup N, so

$$M = \left\{ m(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix},\ a \in \mathbf{G}_m \right\}, \quad N = \left\{ n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},\ b \in \mathbf{G}_a \right\}.$$

Let $K_\infty = \mathrm{SO}_2(\mathbf{R})$ denote the maximal compact subgroup of $\mathrm{SL}_2(\mathbf{R})$, with $K = \mathrm{SL}_2(\widehat{\mathbf{Z}})$ the maximal compact subgroup of $\mathrm{SL}_2(\mathbf{A}_f)$. Hence, we have the Iwasawa decomposition $\mathrm{SL}_2(\mathbf{A}) = N(\mathbf{A})M(\mathbf{A})K_\infty K$.

Writing M', N', K', and K'_∞ to denote the respective images in the metaplectic group $G = \mathrm{Mp}_2$, we also have Iwasawa decomposition $G(\mathbf{A}) = N(\mathbf{A})'M(\mathbf{A})'K'_\infty K'$. We shall consider the metaplectic group G as a semidirect product $G = \mathrm{Mp}_2(\mathbf{A}) = \mathrm{SL}_2(\mathbf{A}) \times \{\pm 1\}$, with multiplication on the right given by $[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)]$, with c the cocycle defined in [19] and [57].

Given an idele class character $\chi : \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$, we define a character $\chi^\psi : M'(\mathbf{A}) \to \mathbf{C}^\times$ by the rule

$$\chi^\psi([m(a), \epsilon]) = \epsilon \chi(a) \gamma(a, \psi)^{-1},$$

where $\gamma(\cdot, \psi)$ denotes the global Weil index. Let $(\cdot, \cdot)_\mathbf{A}$ denote the Hilbert symbol on $\mathbf{A}^\times$, and $\det(U)$ the Gram determinant of U. We shall consider the idele class character $\chi = \chi_U$ defined on $x \in \mathbf{A}^\times$ by

$$\chi_U(x) = \left(x, (-1)^{\frac{\dim(U)}{2}} \det(U)\right)_\mathbf{A}.$$

Given $s \in \mathbf{C}$, let $I(s, \chi_U)$ denote the principal series representation of $G(\mathbf{A})$ induced from the quasicharacter $\chi_U(\cdot)|\cdot|^s$. Explicitly, $I(s, \chi_U)$ is the space of all smooth functions $\phi(g', s)$ on $g' \in G(\mathbf{A})$ such that

$$\phi([n(b), 1][m(a), \epsilon]g', s) = \begin{cases} \chi_U^\psi([m(a), \epsilon])|a|^{s+1}\phi(g', s) & \text{if } p(U) \equiv 1 \bmod 2 \\ \chi_U(a)|a|^{s+1}\phi(g, s) & \text{if } p(U) \equiv 0 \bmod 2 \end{cases}$$

for all $a \in \mathbf{A}^\times$ and $b \in \mathbf{A}$, with $\mathrm{Mp}_2(\mathbf{A})$ acting by right translation. We have a $G(\mathbf{A})$-intertwining map

$$\lambda : \mathcal{S}(U(\mathbf{A})) \longrightarrow I(s_0(U), \chi_U), \quad \lambda(\Phi)(g') := (\omega_L(g')\Phi)(0) \quad \text{for } s_0(U) := \frac{\dim(U)}{2} - 1.$$

This gives rise to a unique standard section in the following sense. A section $\phi(s) \in I(s, \chi_U)$ is said to be *standard* if its restriction to the maximal compact subgroup $K'_\infty K \subset \mathrm{Mp}_2(\mathbf{A})$ does not depend on the complex variable s. We deduce from the Iwasawa decomposition for $G(\mathbf{A})$ that each $\lambda(\Phi) \in I(s_0(U), \chi_U)$ has a unique extension to a standard section $\lambda(\Phi)(s) \in I(s, \chi_U)$ such that $\lambda(\Phi)(s_0(U)) = \lambda(\Phi)$.

Given a standard section $\phi(s) \in I(s, \chi_U)$, we define the Eisenstein series on $g' \in G(\mathbf{A})$

$$E_L(g', s; \phi) = \sum_{\gamma' \in P'(\mathbf{Q}) \backslash G(\mathbf{Q})} \phi(\gamma' g', s).$$

Here, the sum converges absolutely if $\Re(s) \gg 1$, and determines an automorphic form on $g' \in G(\mathbf{A})$. This Eisenstein series $E(g', s)$ has a well-known analytic continuation to a meromorphic function of $s \in \mathbf{C}$ via the Langlands functional equation, which relates $E_L(g', s; \phi)$ to $E_L(g', -s; M\phi)$, for M the intertwining operator.

We shall consider Eisenstein series constructed from standard sections related to the Siegel theta series (23). To describe these in general, let χ_l for any $l \in \frac{1}{2}\mathbf{Z}$ denote the character of $K_\infty = \mathrm{SO}_2(\mathbf{R})$ defined by

$$\chi_l(k_\theta) = e^{il\theta} = \exp(il\theta), \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

Let $\phi_\infty^l(s) \in I(s, \chi_U)$ denote the unique standard section of weight l, i.e. for which $\phi_\infty^l(k_\theta, s) = \chi_l(k_\theta) = e^{il\theta}$. Writing $D(U) = \{z \subset U(\mathbf{R}) : \dim(u) = p(U), Q|_z < 0\}$ for the Grassmannian of U, we define for a given basepoint $z \in D(U)$ the majorant $(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z)$ on $x \in U(\mathbf{R})$ and standard Gaussian

$$\Phi_\infty(x, z) = \exp\left(-(x, x)_z\right).$$

This Gaussian satisfies the familiar property $\Phi_\infty(hx, hz) = \Phi_\infty(x, z)$ for all $h \in \mathrm{GSpin}(U)(\mathbf{R})$. As a function of $x \in U(\mathbf{R})$, it determines an archimedean local Schwartz function $\Phi_\infty \in \mathcal{S}(U(\mathbf{R}))$. The maximal compact subgroup $K'_\infty \subset G(\mathbf{R})$ acts on $\Phi_\infty(x, z)$ via the Weil representation ω_L with weight $\frac{p(U)-q(U)}{2}$. Hence, it is natural to consider the Eisenstein series $E_L(g', s; \lambda(\Phi))$ with archimedean local section given by applying the intertwining operator to this Gaussian,

$$\lambda_\infty(\Phi_\infty(\cdot, z)) = \phi_\infty^{\frac{p(U)-q(U)}{2}}(s_0(U)), \quad s_0(U) := \frac{\dim(U)}{2} - 1.$$

More generally, we shall consider the corresponding Siegel Eisenstein series of weight l defined by

$$E_L(\tau, s; l) := \sum_{\mu \in L^\vee/L} E(g'_\tau, s; \phi_\infty^l \otimes \lambda(\mathbf{1}_\mu))\mathbf{1}_\mu. \tag{24}$$

4.2.2. *The Siegel-Weil formula.* In the special case where the quadratic space (U, Q_U) is anisotropic or more generally, has sufficiently small Witt index $r = r(U)$ relative to $m = \dim_{\mathbf{Q}}(U)$ so that the "Weil convergence criterion" ($\iff m - r > 2$) ensures the absolute convergence of the theta integral

$$I(g, \varphi) = \int\limits_{\mathrm{O}(V)(\mathbf{Q})\backslash \mathrm{O}(V)(\mathbf{A})} \vartheta_L(g, h; \Phi) dh \qquad \forall \Phi \in \mathcal{S}(V(\mathbf{A})),$$

we know by classical theorems of Siegel, Weil, and Kudla-Rallis [38] [39] that the values $E_L(g', s_0(U), \lambda(\Phi))$ are holomorphic, and can be realized as multiples of the corresponding convergence theta integral $I(g, \lambda\Phi)$.

Theorem 4.1 ("Siegel-Weil", Kudla-Rallis). *Let (U, Q) be a rational quadratic space of signature $(p(U), q(U))$. Assume that (U, Q) is anisotropic, or more generally that $\dim_{\mathbf{Q}}(U) - r(U) > 2$, where $r(U)$ denotes the Witt index of U (the dimension of the largest isotropic subspace). Let $L \subset U$ be any integral lattice, and $\Phi \in \mathcal{S}(U(\mathbf{A}))$ any Schwartz function. The corresponding Eisenstein series $E_L(g, s; \Phi)$ is holomorphic at*

$$s_0(U) := \frac{\dim(U)}{2} - 1 = \frac{(p(U) + q(U))}{2} - 1,$$

where it is given by the formula

$$\frac{\kappa}{2} \int\limits_{\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A})} \vartheta_L(g, h; \Phi) dh = E_L(g, s_0(U); \lambda(\Phi)), \quad \kappa := \begin{cases} 2 & \text{if } \dim_{\mathbf{Q}}(U) \leq 2 \\ 1 & \text{otherwise} \end{cases}$$

Here, dh denotes the Tamagawa measure on $\mathrm{SO}(U)(\mathbf{A})$.

Proof. See [33, Theorem 4.1 (ii)]. The anisotropic case is shown in [38], and the isotropic convergent case (with the Weil convergence criterion $\dim_{\mathbf{Q}}(U) - r(U) > 2$) is shown in [39]. ☐

Corollary 4.2. *We have the following special version of Theorem 4.1 for the Siegel theta series* (23) *related to the Eisenstein series* (24) *of weight $l = l(U) := (p(U) - q(U))/2$:*

$$\frac{\kappa}{2} \int\limits_{\mathrm{SO}(U)(\mathbf{Q})\backslash \mathrm{SO}(U)(\mathbf{A})} \theta_L(\tau, h) dh = E_L(\tau, s_0(U); l).$$

Proof. This follows from Theorem 4.1, using the matching special choices of theta series (23) and Eisenstein series (24) for the weight l determined by the signature of the space (U, Q_U). ☐

In our main calculations below, we shall take $U = V_{1,2}$ to be an anisotropic subspace of signature $(1, 2)$ (hence $m = \dim_{\mathbf{Q}}(U) = 3$) with corresponding projective one-cycle $Z(V_{1,2}) \subset X_L = X_{K_L}$ so that we can use the convergent Siegel-Weil formula of Theorem 4.1 and Corollary 4.2. We remark that in the more general setting where such a signature $(1, 2)$ subspace $V_{1,2} \subset V$ is not anisotropic, and hence for which the Weil convergence criterion fails, there should be a regularized first-term identity in the spirit of [40] and [34], as well as a second-term identity in the spirit of [18].

4.3. **Regularized theta lifts and automorphic Green's functions.** We now describe regularized theta lifts giving automorphic Green's functions. Here, we retain the more general setup introduced above, with (V, Q) a rational quadratic space of signature $(n, 2)$ for any integer $n \geq 1$. We write $\mathrm{GSpin}(V)$ to denote the corresponding spin group, with $D(V)$ the Grassmannian of oriented negative 2-planes in $V(\mathbf{R})$. Fixing an integral lattice $L \subset V$ with corresponding compact open subgroup $K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$, we write $X_L = X_{K_L} = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$ to denote the corresponding Shimura variety with complex points

$$X_L(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q})\backslash D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K_L.$$

In this generality, X_L is a quasiprojective variety of dimension n defined over $\mathbf{Q}$, compact if V is anisotropic, and smooth if K_L is neat. We write $\omega_L = \omega_{L,\psi}$ to denote the corresponding Weil representation, with $\psi = \otimes_v \psi_v$ the standard additive character of $\mathbf{A}/\mathbf{Q}$, and $\omega_L^\vee$ the dual representation.

4.3.1. *Harmonic weak Maass forms.* Fix a weight $l \in \frac{1}{2}\mathbf{Z}$. Write $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, and Γ' for its inverse image in the metaplectic group $\mathrm{Mp}_2(\mathbf{R})$.

Definition 4.3. *A twice continuously differentiable function $f : \mathfrak{H} \longrightarrow \mathfrak{S}_L$ is a harmonic weak Maass form of weight l and type ω_L for Γ' if*

(i) *$f|_{l,\omega_L}\gamma' = f$ for all $\gamma' \in \Gamma'$. Here, $|_{l,\omega_L}$ denote the Petersson slash operator of weight l and type ω_L, defined for $\gamma' = (\gamma, \epsilon) \in \mathrm{Mp}_2(\mathbf{Z})$ by*

$$f(\tau)|_{l,\omega_L}\gamma' = \epsilon(\tau)^{-2l}\omega_L(\gamma')^{-1}f(\gamma\tau).$$

(ii) *There exists an $\mathfrak{S}_L$-valued polynomial*

$$P_f(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \gg -\infty} c_f^+(m,\mu)e(m\tau)\mathbf{1}_\mu,$$

known as the principal part of f, such that $f(\tau) - P_f(\tau) = O(e^{-v\varepsilon})$ for some $\varepsilon > 0$ as $v = \Im(\tau) \to \infty$.

(iii) *f is harmonic, i.e. annihilated $\Delta_l f = 0$ by the weight l hyperbolic Laplacian operator*

$$\Delta_l = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ilv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

We write $H_l(\omega_L)$ to denote the vector space over $\mathbf{C}$ of harmonic weak Maass forms of weight l and type ω_L for Γ'. As shown in [9], and such form $f(\tau) \in H_l(\tau)$ has a unique decomposition $f(\tau) = f^+(\tau) + f^-(\tau)$ given on the level of Fourier series expansions by

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \gg -\infty} c_f^+(m,\mu)e(m\tau)\mathbf{1}_\mu$$

and

$$f^-(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m<0} c_f^-(m,\mu)e(m\tau)W_l(2\pi m v)\mathbf{1}_\mu.$$

Here, W_l denotes the Whittaker function defined by $W_l(a) = \int_{-2a}^{\infty} e^{-t}t^{-l}dt = \Gamma(1-l, 2a|)$ for $a < 0$. The series $f^+(\tau)$ is known as the *holomorphic part of f*, and $f^-(\tau)$ is *non-holomorphic part of f*. Bruinier-Funke [9, Proposition 3.2] study the antilinear differential operator ξ_l on $H_l(\omega_L)$ defined by

$$\xi_l(f)(\tau) = v^{l-2}\overline{L_l f(\tau)}, \quad L_l := -2iv^2\frac{\partial}{\partial\overline{\tau}}.$$

Writing $M_l^!(\omega_L)H_l(\omega_L)$ to denote the subspace of weakly holomorphic forms (with poles at cusps), with $M_l(\omega_L) \subset M_l^!(\omega_L)$ the subspace of holomorphic forms, and $S_l(\omega_L) \subset M_l(\omega_l) \subset M_l^!(\omega_L) \subset H_l(\omega_L)$ the subspace of cuspidal holomorphic forms, the operator ξ_l determines a short exact sequence

$$0 \longrightarrow M_l^!(\omega_L) \longrightarrow H_l(\omega_L) \xrightarrow{\xi_l} M_{2-l}^!(\omega_L^\vee) \longrightarrow 0$$

of vector spaces over $\mathbf{C}$, with the identification $\ker(\xi_l) = M_l^!(\omega_L)$. Given harmonic weak Maass forms

$$f(\tau) = \sum_{\mu \in L^\vee/L} f_\mu(\tau)\mathbf{1}_\mu \in H_l(\omega_L)$$

and

$$g(\tau) = \sum_{\mu \in L^\vee/L} g_\mu(\tau)\mathbf{1}_\mu \in H_{-l}(\omega_L^\vee),$$

we define their corresponding inner product

$$\langle\langle f(\tau), g(\tau)\rangle\rangle = \sum_{\mu \in L^\vee/L} f_\mu(\tau)g_\mu(\tau)$$

and Petersson inner product

$$\langle f, g\rangle = \int_{\mathcal{F}} \langle\langle f(\tau), g(\tau)\rangle\rangle v^l d\mu(\tau), \quad d\mu(\tau) = \frac{dudv}{v^2}.$$

Here,

$$\mathcal{F} = \{\tau \in \mathfrak{H} : -1/2 \leq \Re(v) \leq 1/2, \tau\overline{\tau} \geq 1\}$$

denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$.

4.3.2. *Regularized theta lifts as Green's functions of special divisors.* Let $\theta_L(\tau, z, h)$ denote the Siegel theta series defined in (23) above. Note that as a modular form in the variable $\tau = u + iv \in \mathfrak{H}$, this theta series has weight $n/2 - 1$ and representation $\omega_L^\vee$. Let $f(\tau) = f^+(\tau) + f^-(\tau) \in H_l(\omega_L)$ be any harmonic weak Maass form of weight $l = 1 - n/2$ and type ω_L. Given a function $F(s)$ of a complex variable $s \in \mathbf{C}$, let us write $\mathrm{CT}_{s=0} F(s)$ to denote the constant term in its Laurent series expansion around $s = 0$. We consider the Borcherds regularized theta lift ([5], [33]) defined on $\tau = u + iv \in \mathfrak{H}$, $z \in D^\pm(V)$, and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ by

$$\Phi(f, z, h) = \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right), \quad d\mu(\tau) = \frac{dudv}{v^2}.$$

Here, each $\mathcal{F}_T = \{\tau = u + iv \in \mathcal{F} : v \leq T\}$ denotes the truncated fundamental domain of height T. We also consider for each $m \in \mathbf{Q}_{\geq 0}$ and $\mu \in L^\vee/L$ the divisor $Z(m, \mu) = Z(m, \mathbf{1}_\mu, K) \subset X_L$ defined in (20) above,i.e. where the definition holds more generally for spin Shimura varieties $X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$ attached to any rational quadratic space (V, Q) of signature $(n, 2)$.

Theorem 4.4 (Borcherds/Bruinier)**.** *Let $f(\tau) = f^+(\tau) = f^-(\tau) \in H_{1-n/2}(\omega_L)$ be any harmonic weak Maass form whose holomorphic part $f^+(\tau)$ has integer Fourier coefficients*

$$f^+(\tau) = \sum_{\mu\in L^\vee/L}\sum_{m\gg-\infty} c_f^+(m, \mu)e(m\tau)\mathbf{1}_\mu, \quad c_f^+(\mu, m) \in \mathbf{Z}.$$

(i) *If $f(\tau) = f^+(\tau) \in \ker(\xi_{1-n/2}) \cong M^!_{1-n/2}(\omega_L)$ is weakly holomorphic, then there exists a $\overline{\mathbf{Q}}$-valued meromorphic modular form $\Psi(f, z, h) : X_L(\mathbf{C}) \longrightarrow \overline{\mathbf{Q}}$ of weight $c_f(0, 0)/2$ and divisor*

$$\mathrm{Div}(\Psi(f)^2) = Z(f) := \sum_{\mu\in L^\vee/L}\sum_{m\in\mathbf{Q}_{>0}} c_f(-m, \mu)Z(m, \mu)$$

for which

$$\Psi(f, z, h) = -2\log||\Psi(f, z, h)||^2 - c_f(0, 0)\left(2\log||y|| + \Gamma'(1)\right).$$

Here, $||\cdot||$ denotes the Petersson norm on $X_L(\mathbf{C})$.

(ii) *In general, the regularized theta lift $\Phi(f, z, h)$ is an automorphic Green's function for the divisor*

$$Z(f) := \sum_{\mu\in L^\vee/L}\sum_{m\in\mathbf{Q}_{>0}} c_f^+(-m, \mu)Z(m, \mu).$$

That is, $\Psi(f, z, h)$ is a smooth function on $X_L\backslash Z(f)$ with a logarithmic singularity along $Z(f)$. Its corresponding $(1,1)$-form $dd^c\Phi(f, z, h)$ satisfies the Green's current equation

$$dd^c\Phi(f, z, h) = \delta_{Z(f)} + [\omega_{Z(f)}],$$

where $\delta_{Z(f)}$ denotes the Dirac current for $Z(f)$, and $\omega_{Z(f)}$ a smooth closed form representing the homological cycle $[Z(f)]$, so that the cohomology class $[\omega_{Z(f)}] \in H^2(X_L, \mathbf{C})$ is the Poincaré dual of $[Z(f)]$. The regularized theta lift $\Phi(f, z, h)$ also an eigenfunction for the generalized Laplacian Δ_z,

$$\Delta_z\Phi(f, z, h) = \frac{n}{4}\cdot c_f^+(0, 0)\cdot\Phi(f, z, h).$$

Moreover, $\Phi(f, z, h) \in L^{1+\varepsilon}(X_L, d\mu(z))$ for some $\varepsilon > 0$, where $d\mu(z)$ denotes the measure on X_L induced from the Haar measure on $\mathrm{GSpin}(V)(\mathbf{A})$.

Proof. The first claim (i) follows from Borcherds [5, Theorem 13.3] (see also [33, Theorem 1.3]), with the algebraicity shown by Howard-Madapusi Pera [24, Theorem 9.1.1]. The second claim (ii) is shown by Bruinier [7]; see also [11, Theorem 4.3]. ☐

4.3.3. *Maass Poincaré series and the special divisors $Z(m,\mu)$.* We now describe the Hejhal-Poincaré series $F_{m,\mu}(\tau) \in H_l(\omega_L)$ of weight $l = 1 - \frac{n}{2}$ whose regularized theta lifts $\Phi_{m,\mu}(\cdot) = \Phi(F_{m,\mu}, \cdot)$ are the Green's functions of the special divisors $Z(m,\mu) = Z(m, \mathbf{1}_\mu, K_L)$ defined in (20) above, following [7, Definition 1.8].

To define these, let us for complex numbers $\alpha, \beta \in \mathbf{C}$ write $W_{\alpha,\beta}(z)$ and $M_{\alpha,\beta}(z)$ to denote the standard Whittaker functions, giving linearly independent solutions of the differential equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{2} + \frac{\alpha}{z} - \frac{\beta^2 - 1/4}{z^2} \right) w = 0.$$

Hence, these functions are related by

$$W_{\alpha,\beta}(z) = \frac{\Gamma(-2\beta)}{\Gamma\left(\frac{1}{2} - \beta - \alpha\right)} \cdot M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma\left(\frac{1}{2} + \beta - \alpha\right)} \cdot M_{\alpha,-\beta}(z),$$

whence $M_{\alpha,\beta}(z) = M_{\alpha,-\beta}(z)$. These functions behave as

$$\begin{aligned} M_{\alpha,\beta}(z) &\sim x^{\beta + \frac{1}{2}} && \text{for } \beta \notin -\tfrac{1}{2}\mathbf{Z}_{>0}, \\ W_{\alpha,\beta}(z) &\sim \frac{\Gamma(2\beta)}{\Gamma\big(\beta - \alpha + \frac{1}{2}\big)}\, z^{-\beta + \frac{1}{2}} && \text{for } \beta \geq \tfrac{1}{2} \end{aligned}$$

as $z \to 0$, and as

$$\begin{aligned} M_{\alpha,\beta}(z) &= \frac{\Gamma\left(1 + 2\beta\right)}{\Gamma\left(\beta - \alpha + \frac{1}{2}\right)} e^{\frac{y}{2}} y^{-\alpha} \left(1 + O(y^{-1})\right) \\ W_{\alpha,\beta}(z) &= e^{-\frac{y}{2}} y^{\alpha} \left(1 + O(y^{-1})\right) \end{aligned}$$

as $\Im(z) = y \to \infty$. Fixing a weight $l \in \frac{1}{2}\mathbf{Z}$, we define for $s \in \mathbf{C}$ and $y \in \mathbf{R}_{>0}$ the normalized functions

$$\mathcal{M}_s(y) = y^{-\frac{l}{2}} M_{-\frac{l}{2}, s - \frac{1}{2}}(y), \quad \mathcal{W}_s\left(|y|\right) = |y|^{-\frac{l}{2}} W_{\frac{l}{2}(y), s - \frac{1}{2}}\left(|y|\right).$$

These normalized functions are holomorphic in $s \in \mathbf{C}$, and related the standard Whittaker functions via

$$\mathcal{M}_{\frac{l}{2}}(y) = y^{-\frac{1}{2}} M_{-\frac{l}{2}, \frac{l}{2} - \frac{1}{2}}(y) = e^{\frac{y}{2}}, \quad \mathcal{W}_{1 - \frac{l}{2}}(y) = y^{-\frac{l}{2}} W_{\frac{l}{2}, \frac{1}{2} - \frac{l}{2}}(y) = e^{-\frac{y}{2}}.$$

Let $l = 1 - n/2$. Recall we write Γ' to denote the inverse image of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ in the metapletic group $\mathrm{Mp}_2(\mathbf{R})$. Let us also write Γ'_∞ to denote the inverse image of

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} : r \in \mathbf{Z} \right\} \subset \mathrm{SL}_2(\mathbf{Z})$$

in $\mathrm{Mp}_2(\mathbf{Z})$. Given a complex number $s \in \mathbf{C}$ (first with $\Re(s) \gg 1$), a coset $\mu \in L^\vee/L$, and an integer $m \in \mathbf{Z} + Q(\mu)$, let $F_{m,\mu}(\tau)$ denote the Poincaré series defined on $\tau = u + iv \in \mathfrak{H}$ by the summation

$$F_{m,\mu}(\tau, s) = F^L_{m,\mu}(\tau, s) = \frac{1}{\Gamma(2s)} \sum_{\gamma' \in \Gamma'_\infty \backslash \mathrm{Mp}_2(\mathbf{Z})} \left[\mathcal{M}_s(4\pi|m|v) e(m\tau) \mathbf{1}_\mu\right] |_{l,\omega_L} \gamma'. \tag{25}$$

Here again, $|_{l,\omega_L}$ denotes the Petersson slash operator defined for $\gamma' = (\gamma, \epsilon) \in \mathrm{Mp}_2(\mathbf{Z})$ by

$$f|_{l,\omega_L}(\tau)\gamma' = \epsilon(\tau)^{-2l} \omega_L(\gamma')^{-1} f(\gamma\tau).$$

The series (25) converges normally for $\tau = u + iv \in \mathfrak{H}$ and $s = \sigma + it \in \mathbf{C}$ with $\sigma > 1$. Moreover, it is an eigenfunction for the hyperbolic Laplacian Δ_l of weight l, with

$$\Delta_l F_{m,}(\tau, s) = \big(s(1-s) + (l^2 - 2l)/4\big) F_{m,\mu}(\tau, s).$$

In particular, for $s = 1 - l/2$, this Poincaré series $F_{m,\mu}(\tau, 1 - l/2)$ is annihilated by Δ_l, and hence we have

$$F_{m,\mu}(\tau, 1 - l/2) \in H_{1 - l/2}(\omega_L).$$

The Fourier series expansions of these Poincaré series are computed in [7, Proposition 1.10]; in crude form,

$$F_{m,\mu}(\tau, 1 - l/2) = \mathbf{1}_\mu e(m\tau) + \mathbf{1}_{-\mu} e(m\tau) + O(1). \tag{26}$$

Remark In the setting we describe above with $n = 3$, so $l = 1 - 3/2 = -1/2$, the regularized theta lift

$$\Phi_{m,\mu}(z,h) = \Phi(F_{m,\mu}(1 - l/2), z, h) = \Phi(F_{m,\mu}(\tau, 5/4), z, h) \in L^{1+\varepsilon}(X_L)$$

of each of these Poincaré series is the regularized theta lift in the sense of Theorem 4.4 (ii) for the weighted special divisor $Z(m,\mu) = Z(m, \mathbf{1}_\mu) \subset X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V))$. Equivalently, in the notations of Theorem 4.4, we have an identification of (arithmetic) divisors $Z(m,\mu) = Z(F_{m,\mu}) \in Z^1(X_L)$.

If (V,Q) has signature $(2,2)$ so that the corresponding variety X_L is a surface and $l = 1 - 2/2 = 0$, the special divisors $Z(m,\mu) = Z(m, \mathbf{1}) \in Z^1(X_L)$ have Green's functions given by the regularized theta lift

$$\Phi_{m,\mu}(z,h) = \Phi(F_{m,\mu}(1 - l/2), z, h) = \Phi(F_{m,\mu}(\tau, 1), z, h) \in L^{1+\varepsilon}(X_L).$$

4.4. **Green's currents and archimedean local heights.** We now characterize Green's currents and archimedean local heights. We refer to [53], [6], [20], and [61] for general Arakelov theory background.

4.4.1. *Green's currents.* Given $X = (X, \Omega)$ any Kähler variety of complex dimension n, we write $A^{p,q}(X)$ to denote the vector space of smooth $\mathbf{C}$-valued differential forms of type (p,q). The space $A^i(X)$ of smooth differential forms of degree $i \leq 2n$ then has the decomposition

$$A^i(X) = \bigoplus_{p+q=i} A^{p,q}(X). \tag{27}$$

We have the Dolbeault differentials ∂, $\overline{\partial}$, and $d = \partial + \overline{\partial}$ on these spaces,

$$\partial : A^{p,q}(X) \longrightarrow A^{p+1,q}(X), \quad \overline{\partial} : A^{p,q}(X) \longrightarrow A^{p,q+1}(X), \quad d : A^i(X) \longrightarrow A^{i+1}(X).$$

We write $D_i(X) = A^i(X)^* = \mathrm{Hom}_{\mathrm{cont}}(A^i(X), \mathbf{C})$ to denote the dual space of Schwartz-continuous linear functionals on $A^i(X)$. We obtain from (27) the corresponding decomposition on this dual space

$$D_i(X) = \bigoplus_{p+q=i} D_{p,q}(X), \quad D_{p,q}(X) := A^{p,q}(X)^*.$$

We also have the corresponding maps induced by ∂, $\overline{\partial}$, and $d = \partial + \overline{\partial}$:

$$\partial' : D^{p,q}(X) \longrightarrow D^{p+1,q}(X), \quad \overline{\partial}' : D^{p,q}(X) \longrightarrow D^{p,q+1}(X), \quad d' : D^{p,q}(X) \longrightarrow D^{p+1,q+1}(X).$$

For each of these spaces, there is a natural inclusion map

$$A^{p,q}(X) \hookrightarrow D^{p,q}(X), \quad \omega \longmapsto [\omega],$$

with $[\omega]$ the current defined by

$$[\omega](\alpha) := \int_X \omega \wedge \alpha \quad \forall \alpha \in A^{n-p,n-q}(X).$$

Let us choose an orientation on X by declaring that $dz_1 d\overline{z}_1 \cdots dx_n d\overline{z}_n$ has positive orientation on $\mathbf{C}^n$. If $p + q = n$, then Stokes' theorem gives us the relation

$$\begin{aligned}[d\omega](\alpha) &= \int_X d\omega \wedge \alpha = \int_X d(\omega \wedge \alpha) - \int_X (-1)^n \omega \wedge d\alpha \\ &= (-1)^{n+1} \int_X \omega \wedge d\alpha = (-1)^{n+1}[\omega](d\alpha) = (-1)^{n+1}(d'[\omega])(\alpha).\end{aligned}$$

Writing $\partial = (-1)^{n+1}\partial'$, $\overline{\partial} = (-1)^{n+1}\overline{\partial}'$, and $d = (-1)^{n+1}d'$, we then have the commutative diagrams

$$\begin{tikzcd}
A^{p,q} \arrow[r] \arrow[d, "\partial"'] & D^{p,q}(X) \arrow[d, "\partial"'] & A^{p,q} \arrow[r] \arrow[d, "\overline{\partial}"'] & D^{p,q}(X) \arrow[d, "\overline{\partial}"'] & A^{p,q} \arrow[r] \arrow[d, "d"'] & D^{p,q}(X) \arrow[d, "d"'] \\
A^{p+1,q}(X) \arrow[r] & D^{p+1,q}(X), & A^{p,q+1}(X) \arrow[r] & D^{p,q+1}(X), & A^{p+1,q+1}(X) \arrow[r] & D^{p+1,q+1}(X).
\end{tikzcd}$$

For each irreducible subvariety $\iota : Y \hookrightarrow X$ of codimension r, we define the *Dirac current* δ_Y by the rule

$$\delta_Y(\alpha) := \int_{Y^{\mathrm{ns}}} \iota^* \alpha \qquad \forall \alpha \in A^{n-r,n-r}(X),$$

where Y^{ns} denotes the nonsingular locus of Y. This definition extends by linearity to any analytic subvariety of X. A well-known result of Lelong shows that δ_Y is well-defined, giving a current $\delta_Y \in D^{r,r}(X)$. As explained in [53], this result can be deduced in a direct way from Hironaka's resolution of singularities. We consider this construction, together with the relations

$$d^c := \frac{1}{4\pi i}\left(\partial - \overline{\partial}\right), \quad dd^c := -\frac{1}{2\pi i}\partial\overline{\partial}.$$

Given $Y \subset X$ any analytic subvariety of codimension r, a *Green's current for* Y is a current $g_Y \in D^{r-1,r-1}$ such that for some form $\omega \in A^{r,r}(X)$, we have the identity of currents

$$dd^c g_Y + \delta_Y = [\omega] \in D^{r,r}(X).$$

Theorem 4.5 (Gillet-Soulé)**.** *Let $Y \subset X$ be any subvariety of codimension r. Let $\omega_Y \in A^{r,r}(X)$ be a closed form representing the homological cycle $[Y]$, so that the cohomology class of ω_Y is the Poincaré dual of $[Y]$. Writing δ_Z to denote the Dirac current given by integration along Z, there exists a current $g_Y \in D^{r-1,r-1}(X)$ characterized by its inclusion in the Green's current equation*

$$dd^c g_Y + \delta_Y = [\omega_Y] \in D^{r,r}(X).$$

If g_Y and g_Y' are two Green's currents for Y, then

$$g_Y - g_Y' = [\eta] + \partial S_1 + \overline{\partial} S_2$$

for some $\eta \in A^{r-1,r-1}(X)$, $S_1 \in D^{r-2,r-1}(X)$, and $S_2 \in D^{r-1,r-2}(X)$. In other words, the Green's current $g_Y \in D^{r-1,r-1}(X)$ is unique up to addition of an element of $\mathrm{im}(\partial) \oplus \mathrm{im}(\overline{\partial})$.

Proof. See [53, §1, Theorem 1], [20], or [61, Theorem 4.1]. ☐

4.4.2. *Archimedean local heights.* We use Green's currents to define the archimedean local height of algebraic cycles $Y, Z \subset X$ with $\dim(Y) + \dim(Z) = \dim(X) - 1$. We shall later focus on the example of a twisted quaternionic Siegel threefold $X = X_K = \mathrm{Sh}(\mathrm{GSpin}(V, D(V))$ with $Y, Z \subset X$ one-cycles of codimension $r = 2$.

Broadly speaking, the archimedean local height $[Z, Y]_\infty$ should be given naturally by the integration

$$[Z, Y]_\infty = \int_Y g_Z$$

along Y of a Green's current $g_Z \in D^{r,r}(X)$ for $Z \in Z^r(X)$. Since this integration rule depends on the choice of smooth Poincaré dual representative $[\omega_Z] = dd^c g_Z + \delta_Z$, we introduce the notion of an admissible Green's current following [61] to get a well-defined characterization of the archimedean local height $[Z, Y]_\infty$.

To describe the origin of this definition of admissible Green's currents, we first recall Hodge decompositions, the curvature map, and the Hodge-Lefschetz theorem. The Kähler structure on $X = (X, \Omega)$ gives adjoint maps ∂^*, $\overline{\partial}^*$, and d^* for the L^2-inner product

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha(x), \beta(x) \rangle \Omega^n, \quad \forall \alpha, \beta \in A^{p,q}(X)$$

on forms in $A^n(X)$. We define the corresponding Laplacian

$$\Delta_d = \partial\left(\partial^*\partial + \partial\partial^*\right) = \partial\left(\overline{\partial}^*\partial + \overline{\partial}\partial^*\right) = d^*d + dd^*$$

with $\mathcal{H}^{p,q}(X) = \ker(\Delta_d)$ the corresponding space of harmonic forms. The Hodge decomposition for $A^{p,q}(X)$ is the orthogonal decomposition

$$A^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(\overline{\partial}) \oplus \mathrm{im}(\overline{\partial}^*) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*).$$

By duality, the space of currents $D^{p,q}(X)$ has the corresponding Hodge decomposition

$$D^{p,q}(X) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(\partial) \oplus \mathrm{im}(\partial^*) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(\overline{\partial}) \oplus \mathrm{im}(\overline{\partial}^*) = \mathcal{H}^{p,q}(X) \oplus \mathrm{im}(d) \oplus \mathrm{im}(d^*).$$

Note that we also have identifications $H^i(X, \mathbf{C}) = \ker(\Delta_d|_{A^i(X)})$ and $H^{p,q}(X, \mathbf{C}) = \ker(\Delta_d|_{A^{p,q}(C)})$, so that

$$H^i(X, \mathbf{C}) = \bigoplus_{p+q=i} H^{p,q}(X, \mathbf{C}).$$

We have the following $\partial\overline{\partial}$ lemma (see [53, § 1, Lemma]): If $T \in D^{r,r}(X)$ is given by $T = dS$ for some other current S, then $T = dd^cU$ for some $U \in D^{r-1,r-1}(X)$. We also have the following Hodge-Lefschetz theorem. Recall that taking the wedge with the Kähler form Ω gives the Lefshetz operator

$$L : A^{p,q}(X) \rightarrow A^{p+1,q+1}(X).$$

The Hodge index theorem applied to the complex vector spaces $H^i(X, \mathbf{C})$ gives us the Hodge-Lefschetz theorem, which implies the following: For indices $p+q \leq n+1$, the operator $L^{n+1-p-q}$ induces an isomorphism

$$\mathrm{im}(\partial\overline{\partial})^{p,q} \cong \mathrm{im}(\partial\overline{\partial})^{n+1-q,n+1-p},$$

and there is a canonical splitting of the corresponding short exact sequence

$$0 \longrightarrow \mathrm{im}(\partial\overline{\partial}) \longrightarrow \ker(d) \longrightarrow H^{p,q}(X) \longrightarrow 0.$$

Recall we write $Z^r(X)$ to denote the free abelian group of algebraic cycles of codimension r of X. Let

$$\widehat{Z}^r(X) = \{(Y, g_Y) : Y \in Z^r(X), g_Y \in D^{r-1,r-1}(X) \text{ Green's current for } Y\}$$

denote the corresponding group of arithmetic cycles of codimension r. Writing $C^r(X) \subset H^{r,r}(X, \mathbf{C})$ to denote the group of cohomology classes of cycles in $Z^r(X)$, we have surjections

$$\widehat{Z}^r(X) \longrightarrow Z^r(X) \longrightarrow C^r(X), \quad (Y, g_Y) \longmapsto Y \longmapsto [\omega_Y]$$

Consider the corresponding kernels defined by

$$\begin{aligned}\widehat{Z}^r_0(X) &= \ker\left(\widehat{Z}^r(X) \longrightarrow C^r(X)\right) \\ &= \left\{(Y, g_Y) \in \widehat{Z}^r(X) : \ Y \text{ is cohomologically trivial}\right\}\end{aligned}$$

and

$$\begin{aligned}\widehat{Z}^r_1(X) &= \ker\left(\widehat{Z}^r(X) \longrightarrow Z^r(X)\right) \\ &= \left\{(Y, g_Y) \in \widehat{Z}^r(X) : \ Y = \emptyset \text{ is the empty cycle, with } dd^c g_Y = dd^c g_\emptyset \text{ smooth and exact}\right\}.\end{aligned}$$

Since $\mathrm{im}(d) \cap A^{r,r}(X) = \mathrm{im}(\partial\overline{\partial}) \cap A^{r,r}(X)$, we see that the map that sends $(\emptyset, g_\emptyset) \mapsto g_\emptyset$ defines an isomorphism

$$\widehat{Z}^r_1(X) \cong A^{r-1,r-1}(X).$$

Making this identification $\widehat{Z}^r_1(X) \cong A^{r-1,r-1}(X)$, we consider the pairing defined by

$$\widehat{Z}^r_1(X) \times \widehat{Z}^{n+1-r}(X) \longrightarrow \mathbf{R}, \quad (\phi, (Y, g_Y)) \longmapsto \int\limits_X \phi \wedge \omega_Y,$$

and write $N^r(X) \subset A^{r-1,r-1}(X) \cong \widehat{Z}^r_1(X)$ to denote its left kernel. Note that we have the inclusions

$$\mathrm{im}(\partial + \overline{\partial}) \cap A^{r-1,r-1}(X) \subset N^r(X) \subset \ker(\partial\overline{\partial}) \cap A^{r-1,r-1}(X).$$

These inclusions imply there is a surjection

$$\frac{\ker(\partial\overline{\partial}) \cap A^{r-1,r-1}(X)}{\mathrm{im}(\partial + \overline{\partial})} \longrightarrow \frac{\ker(\partial\overline{\partial}) \cap A^{r-1,r-1}(X)}{N^p(X)}$$

corresponding to

$$H^{r-1,r-1}(X, \mathbf{C}) \cong H^{n+1-r,n+1-r}(X, \mathbf{C})^* \longrightarrow C^{n+1-r}(X)^*.$$

Note that Grothendieck's standard conjectures would imply the identification $C^{n+1-r}(X)^* \cong C^{r-1}(X)$.

We now consider the numerical equivalence classes defined by

$$\overline{Z}^r(X) = \frac{\widehat{Z}^r(X)}{N^r(X)}, \quad \overline{Z}^r_0(X) = \frac{\widehat{Z}^r_0(X)}{N^r(X)}, \quad \overline{Z}^r_1(X) = \frac{\widehat{Z}^r_1(X)}{N^r(X)},$$

$$\overline{Z}^r_2(X) = \frac{\ker(\partial\overline{\partial}) \cap A^{r-1,r-1}(X)}{N^r(X)} \cong C^{n+1-r}(X)^*,$$

and

$$B^r(X) = \frac{\overline{Z}_1^r(X)}{\overline{Z}_2(X)} \cong \partial\overline{\partial}(A^{r-1,r-1}(X)).$$

These groups fit into short exact sequences

$$0 \longrightarrow \overline{Z}_1^r(X) \longrightarrow \overline{Z}^r(X) \longrightarrow Z^r(X) \longrightarrow 0$$

and

$$0 \longrightarrow \overline{Z}_2^r(X) \longrightarrow \overline{Z}_1^r(X) \longrightarrow B^r(X) \longrightarrow 0. \tag{28}$$

We consider the curvature map defined by sending an arithmetic cycle to its Poincaré dual representative:

$$\omega : \overline{Z}^r(X) \longrightarrow A^{r,r}(X), \quad (Y, g_Y) \longmapsto \omega_Y \quad \text{with} \quad [\omega_Y] = dd^c g_Y + \delta_Y.$$

Observe that since $N^r(X) \subset \ker(\partial\overline{\partial})$, this map is well-defined. We consider its image

$$\overline{C}^r(X) := \omega(\overline{Z}^r(X)).$$

Note that the cycle class map induces a surjective map

$$\overline{C}^r(X) \longrightarrow C^r(X) \subset H^{2r}(X, \mathbf{C}).$$

We then consider its kernel

$$\overline{C}_1^r(X) = \ker(\overline{C}^r(X) \longrightarrow C^r(X)).$$

Hence, we have a short exact sequence

$$0 \longrightarrow \overline{C}_1^r(X) \longrightarrow \overline{C}^r(X) \longrightarrow C^r(X) \longrightarrow 0$$

and fundamental diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{Z}_1^r(X) \arrow[r] \arrow[d, "\overline{\omega}"'] & \overline{Z}^r(X) \arrow[r] \arrow[d, "\omega"'] & Z^r(X) \arrow[r] \arrow[d, "\mathrm{cl}"'] & 0 \\
0 \arrow[r] & \overline{C}_1^r(X) \arrow[r] & \overline{C}^r(X) \arrow[r] & C^r(X) \arrow[r] & 0.
\end{tikzcd}$$

Note that by applying the snake lemma to this diagram, we obtain the short exact sequence

$$0 \longrightarrow C^{n+1-r}(X)^* \longrightarrow \ker(\omega) \longrightarrow \ker(\mathrm{cl}) \longrightarrow 0.$$

We now apply Hodge-Lefschetz to the short exact sequence (28) to get a canonical splitting

$$\overline{Z}_1^r(X) = \overline{Z}_2^r(X) \oplus B_L^r(X),$$

where

$$B_L^r(X) = \mathrm{im}\left(\partial^*\overline{\partial}^* : A^{r,r}(X) \longrightarrow \overline{Z}_1^r(X)\right).$$

Writing

$$C_L^r(X) = H^{r,r}(X, \mathbf{C}) \cap \overline{C}^r(X)$$

for the (r,r)-harmonic forms in the image of the curvature map, we also have the corresponding splitting

$$\overline{C}^r(X) = \overline{C}_1^r(X) \oplus \overline{C}_L^r(X).$$

Let $Z_L^r(X) \subset Z^r(X)$ denote the cycles with harmonic curvatures, equivalently, the orthogonal complement of $B_L^r(X)$. We have a short exact sequence

$$0 \longrightarrow C^{n+1-r}(X)^* \longrightarrow Z_L^r(X) \longrightarrow Z^r(X) \longrightarrow 0$$

which is split by the lifting $[Y] \mapsto (Y, g_Y)$, where the Green's curren g_Y is chosen so that

$$\int_X g_Y h = 0 \quad \text{for all } h \in C_L^{n+1-r}(X).$$

We call such a Green's current $g_Y \in D^{r-1,r-1}(X)$ *admissible.*

Definition 4.6 (Archimedean local heights)**.** *Let $X = (X, \Omega)$ be any Kähler variety of complex dimension n. Let $Z \subset X$ be an algebraic cycle of codimension r with corresponding Green's current $g_Z \in D^{r-1,r-1}(X)$. Given $Y \subset X$ any algebraic cycle such that $\dim(Y) + \dim(Z) = n - 1$, we define the archimedean local height*

$$[Z, Y]_\infty = \int\limits_Y g_Z.$$

This is well-defined if the Green's current $g_Z \in D^{r-1,r-1}(X)$ is admissible, i.e. chosen such that $\int_X g_Z h = 0$ for all $h \in C_L^{n+1-r}(X) := H^{n+1-r,n+1-r}(X, \mathbf{C}) \cap \overline{C}^{n+1-r}(X)$.

4.4.3. *Pushforward and pullback.* We now return to our setting with a twisted quaternionic Siegel threefold. Hence, we fix (V, Q) a rational quadratic space of signature $(3, 2)$ and an integral lattice $L \subset V$ with stabilizer $K = K_L \subset \mathrm{GSpin}(V)(\mathbf{A}_f)$, and consider the corresponding spin Shimura variety

$$X = X_L = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)).$$

We now prepare abstractly to compute examples of heights of algebraic cycles $Y, Z \subset X_L$ of codimension $r = 2$ in the subsequent section.

Let $Z = Z(U(x)) = Z(U(x_0, x_1)) \subset X_L$ be the special cycle of codimension 2 associated to the rational quadratic subspace of signature $(1, 2)$ arising as the complement $U(x) = U(x_0, x_1) := \mathbf{Q}(x_0, x_1)^\perp \subset V$ of the signature $(2, 0)$ space $\mathbf{Q}(x_0, x_1)$ generated by $x = (x_0, x_1) \in V^2$ with $Q(x_i) > 0$:

$$U(x) = U(x_0, x_1) := \mathbf{Q}(x_0, x_1)^\perp \subset V.$$

For each vector $x_i = x_0, x_1$, we can also consider the complement $U(x_i) := \mathbf{Q}(x_i)^\perp \subset V$ of the line spanned by x_i, which determines a subspace $U(x_i) \subset V$ of signature $(2, 2)$. Hence, we obtain a nested sequence

$$U(x) \subset U(x_i) := \mathbf{Q}(x_i)^\perp \subset V$$

of rational quadratic spaces of respective signatures $(1, 2)$, $(2, 2)$, and $(3, 2)$ for each index $i = 0, 1$. We write $L_{U(x)} = L \cap U(x)$ and $L_{U(x_i)} = L \cap U(x_i)$ for the corresponding lattices, with $K_{L_{U(x)}} = K \cap \mathrm{GSpin}(U(x))(\mathbf{A}_f)$ and $K_{L_{U(x_i)}} = K \cap \mathrm{GSpin}(U(x))(\mathbf{A}_f)$ the corresponding compact open subgroups. Writing $Z(U(x_i)) \subset X_L$ for the divisor corresponding to $U(x_i) \subset V$, we obtain a nested sequence of arithmetic divisors

$$Z = Z(U(x)) \hookrightarrow Z_i = Z(U(x_i)) \hookrightarrow X_L$$

with complex points given by

$$\begin{tikzcd}
Z(\mathbf{C}) = Z(U(x))(\mathbf{C}) = \mathrm{GSpin}(U(x))(\mathbf{Q})\backslash D^\pm(U(x)) \times \mathrm{GSpin}(U(x))(\mathbf{A}_f)/K_{L_{U(x)}} \cong \coprod_h \Gamma_h \backslash D^\pm(V)_{(x_0,x_1)} \arrow[d] \\
Z_i(\mathbf{C}) = Z(U(x_i))(\mathbf{C}) = \mathrm{GSpin}(U(x_i))(\mathbf{Q})\backslash D^\pm(U(x_i)) \times \mathrm{GSpin}(U(x_i))(\mathbf{A}_f)/K_{L_{U(x_i)}} \cong \coprod_h \Gamma_h \backslash D^\pm(V)_{x_i} \arrow[d] \\
X_L(\mathbf{C}) = \mathrm{GSpin}(V)(\mathbf{Q})\backslash D^\pm(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f)/K_L \cong \coprod_h \Gamma_h \backslash D^\pm(V).
\end{tikzcd}$$

Here, we use the decomposition into connected components (18), writing $D(V)_{x_i} = \{z \in D(V) : (x_i, z) = 0\}$ and $D(V)_{(x_0,x_1)} = \{z \in D(V) : (x_0, z) = (x_1, z) = 0\}$ for the orthogonal complements. We can also consider nested sequences of weighted divisors. To fix ideas, suppose we choose vectors $x_i \in V$ with $Q(x_i) = m_i \in \mathbf{Q}_{\geq 0}$, and such that $m_i \in \mathbf{Z} + Q(\mu_i)$ for given cosets $\mu_i \in L^\vee/L$. Consider the corresponding symmetric matrix

$$\beta = \begin{pmatrix} (x_0, x_0) & (x_0, x_1) \\ (x_1, x_0) & (x_1, x_1) \end{pmatrix} = \begin{pmatrix} m_0 & a \\ a & m_1 \end{pmatrix} \in \mathrm{Sym}_2(\mathbf{Q}).$$

We then have for each $i = 0, 1$ the corresponding nested sequence of divisors

$$Z = Z(\beta, (\mu_0, \mu_1)) = Z\left(\beta, \mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1}, K\right) \hookrightarrow Z_i = Z(m_i, \mu_i) = Z(m_i, \mathbf{1}_{\mu_i}, K) \hookrightarrow X_L \tag{29}$$

with complex points

$$\begin{tikzcd}
Z(\mathbf{C}) = Z(\beta, (\mu_0, \mu_1))(\mathbf{C}) = Z(\beta, \mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1}, K)(\mathbf{C}) \cong \coprod_h \Gamma_h \Bigg\backslash \left(\coprod_{\substack{x_i \in \mu_{i,h} + L_h \\ 2Q(x_i) = m_i \\ i=0,1}} D^{\pm}(V)_{(x_0, x_1)} \right) \arrow[d] \\
Z_i(\mathbf{C}) = Z(m_i, \mu_i)(\mathbf{C}) = Z(m_i, \mathbf{1}_{\mu_i}, K)(\mathbf{C}) \cong \coprod_h \Gamma_h \Bigg\backslash \left(\coprod_{\substack{x_i \in \mu_{i,h} + L_h \\ 2Q(x_i) = m_i}} D^{\pm}(V)_{x_i} \right) \arrow[d] \\
X_L(\mathbf{C}) = \coprod_h \Gamma_h \backslash D^{\pm}(V).
\end{tikzcd}$$

Let us now write $X = X_L$ for simplicity to denote the twisted (quaternionic) Siegel threefold we consider. We consider the nested sequence of weighted divisors $Z \subset Z_i \subset X$ defined in (29). Note that by the discussion of Green's functions above (Theorem 4.4 and §4.3.3), each $Z_i \subset X$ is an arithmetic divisor

$$\widehat{Z}_i = (Z_i, [G_{Z_i}]) = (Z_i, [\Phi(F_{m_i,\mu_i}(\cdot, 5/4))])$$

with Green's function

$$G_{Z_i}(z,h) = \Phi(F_{m_i,\mu_i}(\cdot, 5/4), z, h) \in A^{0,0}(X)$$

given by the regularized theta lift of the Hejhal-Poincaré series

$$F_{m_i,\mu_i}(\tau, 5/4) = F_{m_i,\mu_i}(\tau, s)|_{s=5/4} \in H_{-1/2}(\omega_L) = H_{1-3/2}(\omega_L).$$

Similarly, viewing $Z_i \in Z^1(X)$ as a surface containing $Z \in Z^2(X)$, $Z \subset Z_i$ determines an arithmetic divisor

$$\widehat{Z} = (Z, [G_{Z/Z_i}]) = (Z, [\Phi(F_{m_j,\overline{\mu_j}}(\cdot, 1))]), \quad \overline{\mu}_j \in L^{\vee}_{U(x_i)}/L_{U(x_i)}, \quad j \neq i \in \{0,1\}$$

with Green's function

$$G_{Z/Z_i}(z,h) = \Phi(F_{m_j,\overline{\mu_j}}(\tau, 1), z, h) \in A^{0,0}(X)$$

given by the regularized theta lift of the Hejhal-Poincaré series

$$F_{m_j,\overline{\mu_j}}(\tau, 1) = F_{m_j,\overline{\mu_j}}(\tau, s)|_{s=1} \in H_0(\omega_{L_{U(x_i)}}) = H_{1-2/2}(\omega_{L_{U(x_i)}}).$$

In particular, we have for each of $i = 0, 1$ a nested sequence of weighted arithmetic divisors

$$\widehat{Z} \subset \widehat{Z}_i \subset X.$$

We first consider the Green's function G_{Z/Z_i} of $Z \subset Z_i$ for either $i = 0, 1$. Here, the choice of surface $Z \subset Z_i \subset X$ does not matter; we could replace Z_i by any subsurface $S \subset X$ containing Z as a divisor, $Z \subset S \subset X$. We view this as a Green's current $g_{Z/Z_i} = [G_{Z/Z_i}] \in D^{0,0}(Z_i)$ which satisfies the Green's current equation

$$dd^c g_{Z/Z_i} + \delta_{Z/Z_i} = [\omega_{Z/Z_i}] \in D^{1,1}(Z_i). \tag{30}$$

Pushing the Green's current $g_{Z/Z_i} \in D^{0,0}(Z_i)$ forward along the closed embedding $\iota : Z_i \hookrightarrow X$, we obtain a $(1,1)$-form $\iota_* g_{Z/Z_i}$ acting on test $(3,3)$-forms η on X by $\iota_* g_{Z/Z_i}(\eta) = g_{Z/Z_i}(\iota^* \eta)$. The corresponding Green's current equation (30) then pushes forward along $\iota : Z_i \hookrightarrow X$ as

$$dd^c \iota_* g_{Z/Z_i} + \iota_* \delta_{Z/Z_i} = \iota_* [\omega_{Z/Z_i}] \in D^{2,2}(X).$$

Here, δ_{Z/Z_i} denotes the current of integration over $Z \subset Z_i$, and $\omega_{Z/Z_i} \in A^{1,1}(Z_i)$ denotes the smooth Poincaré dual associated to the homology class $[Z]$ of Z relative to Z_i. To interpret this identification of $(2,2)$-forms, we first argue that the pushforward $\iota_* \delta_{Z/Z_i}$ coincides with the current of integration $\delta_Z = \delta_{Z/X}$

over Z inside of X. This is because integrating a test form on X over Z is equivalent to integrating its restriction to Z_i along the same cycle Z. Hence, we obtain the pushforward Green's current equation

$$dd^c \iota_* g_{Z/Z_i} + \delta_Z = \iota_*[\omega_{Z/Z_i}] \in D^{2,2}(X). \tag{31}$$

We now compare this to the Green's current equation of the codimension $r = 2$ special cycle $Z \subset X$. By Theorem 4.5, we know there exists a $(1,1)$-form $g_Z = g_{Z/X} \in D^{1,1}(X)$ satisfying the current equation

$$dd^c g_Z + \delta_Z = [\omega_Z] \in D^{2,2}(X).$$

Here, $\delta_Z = \delta_{Z/X}$ is the same current of integration described above, and $\omega_Z = \omega_{Z/X} \in A^{2,2}(X)$ denotes the smooth Poincaré dual of the homology class $[Z]$ of Z relative to X. Now, let H be the $(1,1)$-form for which

$$dd^c H = \iota_*[\omega_{Z/Z_i}] - [\omega_Z] \in D^{2,2}(X).$$

Solving for H (via Theorem 3.9), we can then describe the Green's current $g_Z \in D^{1,1}(X)$ in terms of the pushfoward $\iota_* g_{Z/Z_i} = \iota_*[G_{Z/Z_i}]$ of the current $g_{Z/Z_i} = [G_{Z/Z_i}]$ associated to the Green's function G_{Z/Z_i} as

$$g_Z = \iota_* g_{Z/Z_i} - H \in D^{1,1}(X). \tag{32}$$

On the other hand, since $\iota_*[\omega_{Z/Z_i}]$ and $[\omega_Z]$ represent the same class $[\iota(Z)] = [Z]$ in $H^4(X, \mathbf{Q})$, their difference $\iota_*[\omega_{Z/Z_i}] - [\omega_Z]$ represents the trivial class. Hence, we may take $\iota_*[\omega_{Z/Z_i}] \in D^{2,2}(X)$ explicitly in the Green's current equation for $Z \subset X$, so that $g_Y = \iota_* g_{Z/Z_i}$. In this way, we show the following

Lemma 4.7 (Pushforward of Green's currents)**.** *Consider the nested sequence of weighted arithmetic divisors* (29) *above. The pushforward Green's current $\iota_* g_{Z/Z_i} \in D^{2,2}(X)$ associated to the arithmetic divisor $Z \subset Z_i$ (on the surface Z_i) defines a Green's current for the codimension $r = 2$ special cycle $Z \subset X$.*

Note that we can use Theorem 3.9 to compute the Poincaré dual representatives $\omega_Z = \Theta_Z^{(2)} \in A^{2,2}(X)$ and $\omega_{Z/Z_i} = \Theta_{Z/Z_i}^{(1)} \in A^{1,1}(Z_i)$ as Kudla-Millson theta series, and hence to solve explicitly for H in (36). To be more precise, we refer to the nested sequence of weighted divisors (29) above, with Green's function

$$G_{Z_i} = \Phi(F_{m_i,\mu_i}(\cdot, 5/4), \star) \in L^{1+\varepsilon}(X)$$

of the divisor $Z_i = Z_i(m_i, \mu_i)$ on X and Green's function

$$G_{Z/Z_i} = \Phi(F_{m_j,\overline{\mu}_j}(\cdot, 1), \star) \in L^{1+\varepsilon}(Z_i)$$

of the divisor $Z = Z(\beta, (\mu_0, \mu_1))$ on the surface $Z_i = Z_i(m_i, \mu_i)$. Recall that for $y = (y_0, y_1) \in V^2$ we write

$$(y, y) = \begin{pmatrix} (y_0, y_0) & (y_0, y_1) \\ (y_0, y_1) & (y_1, y_1) \end{pmatrix}$$

for the matrix of inner products, and that we have

$$\beta := \begin{pmatrix} (x_0, x_0) & (x_0, x_1) \\ (x_1, x_0) & (x_1, x_1) \end{pmatrix} = \begin{pmatrix} m_1 & a \\ a & m_1 \end{pmatrix} \in \mathrm{Sym}_2(\mathbf{Q}).$$

We consider the hyperboloids

$$\Omega_\beta(\mathbf{Q}) = \left\{ y = (y_0, y_1) \in V^2 : (y, y) = \beta \right\}$$

$$\Omega_{m_i}(\mathbf{Q}) = \left\{ y \in V : Q(y) = m_i \right\},$$

and

$$\overline{\Omega}_{m_j} = \left\{ y \in U(x_i) = Q|_{U(x_i)}(y) = m_j \right\}, \quad j \neq i \in \{0, 1\}.$$

The codimension $r = 2$ cycle $Z \subset X$ and the codimension $r = 1$ cycle $Z \subset Z_i$ on the surface defined by $Z_i = Z_i(m_i, \mu_i) \subset X$ and $Z = Z(\beta, (\mu_0, mu_1)) \subset Z_i$ have the corresponding Kudla-Millson theta series

$$\omega_Z = \Theta_{L,\beta}^{(2)}(\mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1}) = \sum_{y = (y_0, y_1) \in \Omega_\beta(\mathbf{Q})} \Phi^{(2)} \otimes (\mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1})(y) = \sum_{\substack{y = (y_0, y_1) \in \Omega_\beta(\mathbf{Q}) \\ y_i \in L + \mu_i \\ (m_i \in \mathbf{Z} + Q(\mu_i),\ i = 0,1)}} \Phi^{(2)}(y) \in A^{2,2}(X)$$

and

$$\omega_{Z/Z_i} = \Theta^{(1)}_{L_{U(x_i)},m_j}(\mathbf{1}_{\overline{\mu}_j}) = \sum_{y \in \overline{\Omega}_{m_j}(\mathbf{Q})} \Phi^{(1)} \otimes \mathbf{1}_{\overline{\mu}_j}(y) = \sum_{\substack{y \in \overline{\Omega}_{m_j}(\mathbf{Q}) \\ y \in \overline{\mu}_j + L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y) \in A^{1,1}(Z_i).$$

Hence, we consider the corresponding currents $[\omega_Z] \in D^{2,2}(X)$ and $[\omega_{Z/Z_i}] \in D^{1,1}(Z_i)$ characterized by

$$[\omega_Z](\alpha) = \int_X \omega_Z \wedge \alpha = \int_X \Theta^{(2)}_{L,\beta}(\mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1}) \wedge \alpha \quad \forall \alpha \in A^{1,1}(X)$$

and

$$[\omega_{Z/Z_i}](\alpha) = \int_{Z_i} \omega_Z \wedge \alpha = \int_{Z(m_i,\mu_i)} \Theta^{(1)}_{L_{U(x_i)},m_j}(\mathbf{1}_{\overline{\mu}_j}) \wedge \alpha \quad \forall \alpha \in A^{1,1}(Z_i).$$

We also consider the pushforward $\iota_*[\omega_{Z/Z_i}] \in D^{2,2}(X)$ characterized by

$$\iota_*[\omega_{Z/Z_i}](\alpha) = \int_{Z_i} \omega_{Z/Z_i} \wedge \iota^! \alpha = \int_{Z_i} \Theta^{(1)}_{L_{U(x_i)},m_j}(\mathbf{1}_{\overline{\mu}_j}) \wedge \iota^! \alpha \quad \forall \alpha \in A^{1,1}(X),$$

where $\iota^! : A^{1,1}(X) \to A^{1,1}(Z_i)$ is the pullback on differential forms. We can then compute the difference as

$$\begin{aligned}
&\iota_*[\omega_{Z/Z_i}](\alpha) - [\omega_Z](\alpha) = \int_{Z_i} \omega_{Z/Z_i} \wedge \iota^! \alpha - \int_X \omega_Z \wedge \alpha \\
&= \int_{Z_i} \Theta^{(1)}_{L_{U(x),m_j}}(\mathbf{1}_{\overline{\mu}_j}) \wedge \iota^! \alpha - \int_X \Theta^{(2)}_{L,\beta}(\mathbf{1}_{\mu_0} \otimes \mathbf{1}_{\mu_1}) \wedge \alpha \\
&= \int_{Z_i} \sum_{\substack{y_j \in \overline{\Omega}_{m_j}(\mathbf{Q}) \\ y_j \in \overline{\mu}_j + L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y_j) \wedge \iota^! \alpha - \int_X \sum_{\substack{(y_0,y_1) \in \Omega_\beta(\mathbf{Q}) \\ y_i \in L + \mu_i \\ (m_i \in \mathbf{Z} + Q(\mu_i),\ i=0,1)}} \Phi^{(2)}((y_0,y_1)) \wedge \alpha \\
&= \int_{Z_i} \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)}(y_j) = m_j \\ y_j \in \overline{\mu}_j + L_{U(x_i)},\ \overline{\mu}_j := \mu_j \bmod L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y_j) \wedge \iota^! \alpha - \int_X \sum_{\substack{y = (y_0,y_1) \in V^2 : (y,y) = \beta = \begin{pmatrix} m_0 & a \\ a & m_1 \end{pmatrix} \\ y_i \in L + \mu_i \\ (m_i \in \mathbf{Z} + Q(\mu_i),\ i=0,1)}} \Phi^{(1)} \wedge \Phi^{(1)}((y_0,y_1)) \wedge \alpha \\
&= \int_{Z_i} \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)}(y_j) = m_j \\ y_j \in \overline{\mu}_j + L_{U(x_i)},\ \overline{\mu}_j := \mu_j \bmod L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y_j) \wedge \iota^! \alpha - \int_X \sum_{\substack{y = (y_0,y_1) \in V^2 : (y,y) = \beta = \begin{pmatrix} m_0 & a \\ a & m_1 \end{pmatrix} \\ y_i \in L + \mu_i \\ (m_i \in \mathbf{Z} + Q(\mu_i),\ i=0,1)}} \Phi^{(1)}(y_0) \wedge \Phi^{(1)}(y_1) \wedge \alpha \\
&= \int_X \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)}(y_j) = m_j \\ y_j \in \overline{\mu}_j + L_{U(x_i)},\ \overline{\mu}_j := \mu_j \bmod L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y_j) \wedge \iota^! \alpha - \sum_{\substack{y = (y_0,y_1) \in V^2 : (y,y) = \beta = \begin{pmatrix} m_0 & a \\ a & m_1 \end{pmatrix} \\ y_i \in L + \mu_i \\ (m_i \in \mathbf{Z} + Q(\mu_i),\ i=0,1)}} \Phi^{(1)}(y_0) \wedge \Phi^{(1)}(y_1) \wedge \alpha
\end{aligned}$$

for any $\alpha \in D^{1,1}(X)$. We deduce from Lemma 36 that this represents Poincaré dual of the trivial class in $H^{2,2}(X, \mathbf{C}) \subset H^4(X, \mathbf{C})$. Note that if we restrict to the diagonal setting with $m = m_0 = m_1$ and $\mu = \mu_0 = \mu_1$, this difference (which represents the trivial class) is given by the simpler expression

$$\begin{aligned}&\iota_*[\omega_{Z/Z_i}](\alpha) - [\omega_Z](\alpha)\\ &= \int_X \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)}(y_j) = m \\ y_j \in \overline{\mu}_j + L_{U(x_i)},\ \overline{\mu}_j := \mu \bmod L_{U(x_i)}}} \Phi^{(1)}(y_j) \wedge \iota^! \alpha - \sum_{\substack{y = (y_0, y_1) \in V^2 : (y,y) = \beta = \begin{pmatrix} m & a \\ a & m \end{pmatrix} \\ y_i \in L + \mu}} \Phi^{(1)}(y_0) \wedge \Phi^{(1)}(y_1) \wedge \alpha\end{aligned}$$

Let us now consider pullbacks along differential forms $\iota^! : A^{r,r}(X) \longrightarrow A^{r,r}(Z_i)$ of the Green's function $G_{Z_i} = G_{Z_i/X} \in A^{0,0}(X)$ and its corresponding $dd^c G_{Z_i/X} \in A^{1,1}(X)$. Abstractly, we can characterize these pullbacks via the L^2-inner product on differential forms by

$$\langle \iota^! G_{Z_i/X}, \alpha \rangle = \langle G_{Z_i/X}, \delta_{Z_i} \wedge \alpha \rangle \quad \forall \alpha \in A^{2,2}(Z_i)$$

and

$$\langle \iota^! dd^c G_{Z_i/X}, \alpha \rangle = \langle dd^c G_{Z_i/X}, \delta_{Z_i} \wedge \alpha \rangle = \langle dd^c \iota^! G_{Z_i/X}, \alpha \rangle \quad \forall \alpha \in A^{1,1}(Z_i)$$

For the current of integration $\delta_{Z_i} = [Z_i]$, we have $[\iota^! Z_i] = 0$. We deduce from the Green's current equation

$$dd^c g_{Z_i/X} + \delta_{Z_i/X} = [\omega_{Z_i/X}] \in D^{1,1}(X)$$

for $g_{Z_i/X} = [G_{Z_i/X}] \in D^{1,1}(X)$ that we have the current identity

$$dd^c [\iota^! G_{Z_i/X}] = [\iota^! \omega_{Z_i/X}] \in D^{1,1}(Z_i). \tag{33}$$

Now, recall that we have

$$\omega_{Z_i/X} = \Theta^{(1)}_{L,m_i}(\mathbf{1}_{\mu_i}) = \sum_{\substack{y \in V : 2Q(y) = m_i \\ m_i \in \mathbf{Z} + Q(\mu_i), y \in L + \mu_i}} \Phi^{(1)}(y) \in A^{1,1}(Z_i)$$

and

$$\iota^! \omega_{Z_i/X} = \iota^! \Theta^{(1)}_{L,m_i}(\mathbf{1}_{\mu_i}) = \sum_{\substack{y = y^+ \oplus y^- \in W \oplus U(x_i) \\ 2Q|_{U(x_i)}(y) = m_i \\ m_i \in \mathbf{Z} + Q(\mu_i), y \in L + \mu_i}} \varphi^0_W(y^+) \otimes \Phi^{(1)}(y^-) \in A^{1,1}(Z_i)$$

by Kudla-Millson [37] (Theorem 3.9 and Theorem 3.8 (iii)). Let us now consider the difference of currents

$$[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}] = [\Theta^{(1)}_{L,m_i}(\mathbf{1}_{\mu_i})] - [\Theta^{(1)}_{L_{U(x_i)},m_j}(\mathbf{1}_{\overline{\mu}_j})] \in D^{1,1}(Z_i).$$

Expanding out the Kudla-Millson theta series, this is the current defined for any form $\alpha \in A^{1,1}(Z_i)$ by

$$\begin{aligned}&[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}] = [\Theta^{(1)}_{L,m_i}(\mathbf{1}_{\mu_i})] - [\Theta^{(1)}_{L_{U(x_i)},m_j}(\mathbf{1}_{\overline{\mu}_j})]\\ &= \int_{Z_i} \left(\sum_{\substack{y = y^+ \oplus y^- \in W \oplus U(x_i) \\ 2Q(y) = m_i \\ m_i \in \mathbf{Z} + Q(\mu_i), y \in L + \mu_i}} \varphi^0_W(y^+) \otimes \Phi^{(1)}(y^-) - \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)} = m_j \\ y_j \in \overline{\mu}_j + L_{U(x_i)},\ \overline{\mu}_j := \mu_j \bmod L_{U(x_i)} \\ (m_j \in \mathbf{Z} + Q|_{U(x_i)}(\overline{\mu}_j),\ j \neq i \in \{0,1\})}} \Phi^{(1)}(y_j) \right) \wedge \alpha.\end{aligned}$$

Now, if we take $m = m_0 = m_1$ and $\mu = \mu_0 = \mu_1$, then the contributions of the 1-Schwartz forms $\Phi^{(1)}$ cancel out, leaving only a contribution of the Gaussian φ_W^0 for the positive-definite line $W = \mathbf{Q}(x_i)$:

$$\begin{aligned}
&[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}] = [\Theta^{(1)}_{L,m}(\mathbf{1}_\mu)] - [\Theta^{(1)}_{L_{U(x_i)},m}(\mathbf{1}_{\overline{\mu}})] \\
&= \int\limits_{Z_i} \left(\sum_{\substack{y = y^+ \oplus y^- \in W \oplus U(x_i) \\ 2Q(y) = m}} \varphi_W^0(y^+) \otimes \Phi^{(1)}(y^-) - \sum_{\substack{y_j \in U(x_i) : 2Q|_{U(x_i)} = m \\ y_j \in \overline{\mu} + L_{U(x_i)},\ \overline{\mu} := \mu \bmod L_{U(x_i)}}} \Phi^{(1)}(y_j) \right) \wedge \alpha \qquad (34) \\
&= \int\limits_{Z_i} \left(\sum_{\substack{y^+ \in W := \mathbf{Q}(x_i) \\ 2Q(y^+) = m}} \varphi_W^0(y^+) \right) \wedge \alpha = \int\limits_{Z_i} \left(\sum_{\substack{y^+ \in W := \mathbf{Q}(x_i) \\ 2Q(y^+) = m}} \exp\left(-\pi(y^+, y^+)\right) \right) \wedge \alpha \in D^{1,1}(Z_i).
\end{aligned}$$

Now, observe that since the special cycle $Z_i = Z(U(x_i))$ is parametrized by the orthogonal complement $U(x_i) = \mathbf{Q}(x_i)^\perp = W^\perp$ of the positive space $W = Q(x_i)$, we deduce from the right-hand-side of (34) that

$$[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}] = \int\limits_{Z_i} C(W, \mu, m) \wedge \alpha$$

for $C(W, \mu, m)$ a constant. Equivalently, we can view the difference $[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}]$ as the constant $C(W, \mu, m)$ times the identity $(1,1)$-form in $A^{1,1}(Z_i)$. Equivalently, the difference $[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}]$ represents the trivial class in $H^{1,1}(Z_i, \mathbf{C}) \subset H^2(Z_i, \mathbf{C})$. Hence, we derive the following result.

Theorem 4.8 (Pullback-pushforward of Green's forms)**.** *If $m = m_0 = m_1$ and $\mu = \mu_0 = \mu_1$, then*

$$g_Z = \iota_*[\iota^! G_{Z_i/X}] \in D^{1,1}(X)$$

is a Green's current for the one-cycle $Z = Z(\beta, (\mu, \mu)) \in Z^2(X)$.

Proof. If $m = m_0 = m_1$ and $\mu = \mu_0 = \mu_1$, then we have

$$[\iota^! \omega_{Z_i/X}] - [\omega_{Z/Z_i}] = 0.$$

by the discussion above. Consequently, we have via (33) the Green's current equation

$$dd^c g_{Z/Z_i} + \delta_{Z/Z_i} = [\iota^! \omega_{Z_i/X}] = dd^c[\iota^! G_{Z_i/X}] \in D^{1,1}(Z_i).$$

Pushing currents forward along $\iota : Z_i \hookrightarrow X$ according to Lemma 36, we obtain

$$\iota_* dd^c g_{Z/Z_i} + \delta_Z = \iota_*[\iota^! \omega_{Z_i/X}] = \iota_* dd^c[\iota^! G_{Z_i/X}] \in D^{2,2}(X).$$

Hence, we can take the pushforward $\iota_*[\iota^! G_{Z_i/X}]$ of the Green's current $[\iota^! G_{Z_i/X}]$ of the pullback $\iota^! G_{Z_i/X}$ of the Green's function $G_{Z_i/X}$ to be the Green's current g_Z for the codimension $r = 2$ cycle $Z \subset X$. ☐

Theorem 4.9. *In the special setup described above of nested arithmetic divisors* (29) *with $m = m_0 = m_1$ and $\mu = \mu_0 = \mu_1$, we have the following characterization of the archimedean local height of $Z \in Z^2(X)$ against another special cycle $Y \in Z^2(X)$ of codimension $r = 2$:*

$$[Z, Y]_\infty := \int_Y g_Z = \int_Y \iota_*[\iota^! G_{Z_i/X}] = \int_Y \iota_*[\iota^! \Phi(F_{m,\mu}(\tau, 5/4), \star)].$$

Here, we argue that we can realize the pullback $\iota^! G_{Z_i/X}$ explicitly as the modified regularized theta lift

$$\iota^! \Phi(F_{m,\mu}(\tau, 5/4), z, h) := \mathrm{CT}_{s=0} \left(\lim_{T \to \infty} \int\limits_{\mathcal{F}_T} \langle\langle v^{1/4} F_{m,\mu}(\tau, 5/4), \overline{\theta}_L(\tau, z, h) \rangle\rangle v^{-s} d\mu(\tau) \right), \qquad (35)$$

where $\overline{\theta}_L(\tau, z, h)$ denotes the weight-zero restriction of the Siegel theta series $\theta_L(\tau, z, h)$ to the subspace $U(x_i)$. We then argue that we can realize the subsequent pushforward to X_L explicitly as

$$\begin{aligned}\iota_* \iota^! \Phi(F_{m,\mu}(\tau, 5/4), z, h) &:= \mathrm{CT}_{s=0}\left(\lim_{T \to \infty} \int\limits_{\mathcal{F}_T} \langle\langle v^{1/4} F_{m,\mu}(\tau, 5/4), v^{-1/4}\theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right) \\ &= \mathrm{CT}_{s=0}\left(\lim_{T \to \infty} \int\limits_{\mathcal{F}_T} \langle\langle F_{m,\mu}(\tau, 5/4), v^{-1/2}\theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau) \right). \end{aligned} \tag{36}$$

Proof. The first identification follows from Theorem 4.8. The second identification can be deduced from the characterization of the Green's function $G_{Z_i/X}$ of the arithmetic divisor $Z_i \in \widehat{Z}^1(X)$ as a regularized theta lift $\Phi(F_{m,\mu}(\tau, 5/4), \cdot)$ of some $F_{m,\mu}(\tau, 5/4) \in H_{-1/2}(\omega_L)$. Here, we argue as follows that the pullback

$$\iota^! G_{Z_i/X} = \iota^! \Phi(F_{m,\mu}(\tau, 5/4), \star) \in A^{0,0}(Z_i)$$

is realized as in (35). We first note that the regularized theta lift

$$\Phi(F_{m,\mu}(\tau, 5/4), \cdot) \in L^{1+\varepsilon}(X_L)$$

can be characterized abstractly following Kudla [33, §1, (1.25), cf. (1.28), (1.29)]: It is the theta lift in the metaplectic variable $g'_\tau = (g_\tau, 1) \in \mathrm{Mp}_2(\mathbf{R})$ of the automorphic form $\phi_{m,\mu}(g_\tau, 5/4)$ on $g_\tau \in \mathrm{SL}_2(\mathbf{A})$ lifting $F_{m,\mu}(\tau, 5/4) \in H_{1-3/2}(\omega_L)$ against the functional

$$\theta_L(g'_\tau, z, h) : \mathrm{Mp}_2(\mathbf{Q})\backslash \mathrm{Mp}_2(\mathbf{A}) \times X_L(\mathbf{C}) = \mathrm{Sh}_{K_L}(\mathrm{GSpin}(V), D(V)) \longrightarrow \left(\mathcal{S}(V(\mathbf{A}))^{K_L}\right)^\vee$$

determined by the Siegel theta series $\theta_L(g'_\tau, z, h) \in A_{1/2}(\omega_L^\vee)$ of weight $1/2 = (3-2)/2$. Recall that the underlying Siegel theta series $\theta_L(g', z, h)$ on $g' \in \mathrm{Mp}_2(\mathbf{A})$ is defined in terms of the theta kernel

$$\theta_L(g', h; \Phi_f) = \vartheta_L(g', h; \Phi_\infty(\cdot, z_0) \otimes \Phi_f(\cdot)). \tag{37}$$

Here again, fixing any basepoint $z_0 \in D(V)$, we write $\Phi_\infty(\cdot, z_0)$ to denote the standard Gaussian in $\mathcal{S}(V(\mathbf{R}))$, whose weight under the action of the maximal compact subgroup $K'_\infty \subset \mathrm{Mp}_2(\mathbf{R})$ equals $(3-2)/2 = 1/2$. The pullback $\iota'^! \Phi_\infty(\cdot, z_0)$ is then determined by restriction to the corresponding subspace of Schwartz functions

$$\iota^! : \mathcal{S}(V(\mathbf{A}_f)) \longrightarrow \mathcal{S}(U(x_i)(\mathbf{A})).$$

In this way, we deduce that $\iota^! \Phi_\infty(\cdot, z_0) \in \mathcal{S}(U(x_i)(\mathbf{R}))$ must transform under the action of $K'_\infty \subset \mathrm{Mp}_2(\mathbf{R})$ on this space[6] with the same weight $0 = (2-2)/2$ as the corresponding Gaussian in $\mathcal{S}(U(x_i)(\mathbf{R}))$. In other words, we require that $K'_\infty \subset \mathrm{Mp}_2(\mathbf{R})$ acts on $\iota^! \Phi_\infty(\cdot, z_0)$ with weight zero. Taking sums to define theta kernels as in (37), we deduce that the pullback $\iota^! \Phi(F_{m,\mu}(\tau, 5/4), \cdot)$ of $\Phi(F_{m,\mu}(\tau, 5/4), \cdot)$ along $A^{0,0}(X) \to A^{0,0}(Z_i)$ should determine a lift in $g_\tau \in \mathrm{Sp}_2(\mathbf{R}) \cong \mathrm{SL}_2(\mathbf{R})$ against the corresponding weight-zero theta functional

$$\iota^! \theta_L(g', z, h) : \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}) \times Z_i(\mathbf{C}) = \mathrm{Sh}_{K_{L \cap U(x_i)}}(\mathrm{GSpin}(U(x_i)), D(U(x_i))) \longrightarrow \left(\mathcal{S}(U(x_i)(\mathbf{A}))^{K_{L\cap U(x_i)}}\right)^\vee.$$

Explicitly, we can realize the pullback $\iota^! \theta_L(g', z, h)$ via the restriction $\overline{\theta}_L(g, z, h)$ of $\theta_L(g', z, h)$ to the subspace $U(x_i) \subset V$, with Weil representation corresponding to the sublattice $L_{U(x_i)} = L \cap U(x_i)$, and evaluating at the corresponding restrictions to $g \in \mathrm{Sp}_2(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A})$, $z \in D(U(x_i))$, and $h \in \mathrm{GSpin}(U(x_i))(\mathbf{A}_f)$. This $\overline{\theta}_L(g', z, h)$ determines a Siegel theta series of weight $0 = (2-2)/2$. When we subsequently consider the pullback to the original space (V, Q) of signature $(3, 2)$ with $z \in D(V)$ and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we take

$$\iota_* \iota^! \theta_L(\tau, z, h) = \iota_* \overline{\theta}_L(\tau, z, h) = v^{-\frac{1}{4}} \theta_L(\tau, z, h)$$

to compensate for this shift in the weight of the theta series from $1/2 = (3-2)/2$ to $0 = (2-2)/2$.

To define a pairing $\langle\langle \phi(g'), \iota^! \theta_L(g', z, h)\rangle\rangle$ against this theta kernel $\iota^! \theta_L(g', z, h) = \overline{\theta}_L(g, z, h)$ according to [33, (1.31), (1.32)], we require a vector-valued automorphic function ϕ of weight zero,

$$\phi : \mathrm{Mp}_2(\mathbf{Q})\backslash \mathrm{Mp}_2(\mathbf{A}) \longrightarrow \mathcal{S}(U(x_i)(\mathbf{A})_f)^{K \cap \mathrm{GSpin}(U(x_i))(\mathbf{A}_f)}.$$

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[6] which factors through the action of the maximal compact subgroup $K_\infty \subset \mathrm{Sp}_2(\mathbf{R}) \cong \mathrm{SL}_2(\mathbf{R})$ as $\dim(U(x_i))$ is even

That is, the two forms $\phi(g')$ and $\iota^!\theta_L(g', z, h)$ must have opposite weights summing to zero. Hence, we let ϕ be the non-genuine[7] lifting to $g \in \mathrm{Mp}_2(\mathbf{A})$ of the weight-zero Maass form $v^{1/4}F_{m,\mu}(\tau, 5/4) \in H_0(\omega_L)$. Descending back down to $\tau \in \mathfrak{H}$ via the Iwasawa decomposition to express the corresponding pairing

$$\langle\langle \phi(g'_\tau), \iota^!\theta_L(g'_\tau, z, h)\rangle\rangle = \langle\langle \phi(g'_\tau), \overline{\theta}_L(g'_\tau, z, h)\rangle\rangle$$

in classical terms leads us to the realization

$$\iota^!\Phi(F_{m,\mu}(\tau, 5/4), z, h) = \int_{\mathcal{F}}^{\star} \langle\langle v^{1/4}F_{m,\mu}(\tau, 5/4), \overline{\theta}_L(\tau, z, h)\rangle\rangle d\mu(\tau) \in A^{0,0}(Z_i(\mathbf{C}))$$

of the pullback described in (35) above, as well as to the subsequent realization

$$\iota_*\iota^!\Phi(F_{m,\mu}(\tau, 5/4), z, h) = \int_{\mathcal{F}}^{\star} \langle\langle v^{1/4}F_{m,\mu}(\tau, 5/4), v^{-1/4}\theta_L(\tau, z, h)\rangle\rangle d\mu(\tau) \in A^{0,0}(X_L(\mathbf{C})).$$

Here, we may shift the factor of $v^{1/4}$ to the dual side to express this pushforward equivalently as

$$\iota_*\iota^!\Phi(F_{m,\mu}(\tau, 5/4), z, h) = \int_{\mathcal{F}}^{\star} \langle\langle F_{m,\mu}(\tau, 5/4), v^{-1/2}\theta_L(\tau, z, h)\rangle\rangle d\mu(\tau) \in A^{0,0}(X_L(\mathbf{C})).$$

In this way, we justify the realization (36) of the pushforward-pullback Green's function $\iota_*\iota^!\Phi(F_{m,\mu}(\tau, 5/4), z, h)$, giving the Green's current $g_Z = \iota_*[\iota^!\Phi(F_{m,\mu}\tau, z, h)] = [\iota_*\iota^!\Phi(F_{m,\mu}(\tau, 5/4), z, h)] \in D^{1,1}(X_L(\mathbf{C}))$. ☐

Corollary 4.10. *Given $f(\tau) = f^+(\tau) + f^-(\tau) \in H_{-1/2}(\omega_L)$ a harmonic weak Maass form with corresponding diagonal one-cycle $Z(f) \in Z^2(X)$ as defined in (2) above, we can realize the Green's current $g_{Z(f)} \in D^{1,1}(X)$ as $g_{Z(f)} = \iota_*[\iota^!\Phi(f, \cdot)] = [\iota_*[\iota^!\Phi(f, \cdot)]]$, where the pullback $\iota^!\Phi(f, \cdot)$ can be realized explicitly as*

$$\iota^!\Phi(f(\tau), z, h) := \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int\limits_{\mathcal{F}_T} \langle\langle v^{1/4}f(\tau), \overline{\theta}_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right),$$

and the subsequent pushforward as

$$\begin{aligned}\iota_*\iota^!\Phi(f(\tau), z, h) &:= \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int\limits_{\mathcal{F}_T} \langle\langle v^{1/4}f(\tau), v^{-1/4}\theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right), \\ &= \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int\limits_{\mathcal{F}_T} \langle\langle f(\tau), v^{-1/2}\theta_L(\tau, z, h)\rangle\rangle v^{-s} d\mu(\tau)\right).\end{aligned}$$

Consequently, we can define the archimedean local height $[Y, Z(f)]_\infty$ for $Y \in Z^2(X_L)$ explicitly as

$$[Z(f), Y]_\infty = \int\limits_Y \iota_*\left[\iota^!\Phi(f, z, h)\right] = \int\limits_{Y(\mathbf{C})} \iota_*\iota^!\Phi(f, z, h).$$

Proof. Formally, we can apply the arguments of Theorem 4.8 and Corollary 4.9 in the same way to

$$Z := Z(f) = \sum_{m>0}\sum_{\mu\in L^\vee/L} c_f^+(-m, \mu) Z\left(\begin{pmatrix} m & a \\ a & m\end{pmatrix}, (\mu, \mu)\right) \in Z^2(X_L),$$

with the corresponding divisors

$$Z_i := \sum_{m>0}\sum_{\mu\in L^\vee/L} c_f^+(-m, \mu) Z(m, \mu) \in Z^1(X_L)$$

determined uniquely for each pair (m, μ) in the expansion of $Z(f)$. To be sure, we choose for each such pair (m, μ) with $m \in \mathbf{Z} + Q(\mu)$ a vector $x = (x_0, x_1) \in V^2$ such that

$$(x, x) = \begin{pmatrix} (x_0, x_0) & (x_0, x_1) \\ (x_1, x_0) & (x_1, x_1)\end{pmatrix} = \begin{pmatrix} m & a \\ a & m\end{pmatrix} \in \mathrm{Sym}_2(\mathbf{Q}).$$

By Witt's theorem (see [31, §5]), the special weighted cycles Z and Z_i do not depend on these choices. ☐

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[7] That is, we take the lift to $\mathrm{Sp}_2(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A})$, then consider this as a form on $\mathrm{Mp}_2(\mathbf{A})$ with trivial metaplectic variable.

5. Inner product calculation

We now compute $[Z(f), Y]_\infty$ as described in Corollary 4.10 above for $Y \in Z^2(X_L)$ a projective curve.

5.1. **Setup.** Let $V_{1,2} \subset V$ be an anisotropic subspace of signature $(1,2)$, so that we can use the convergent Siegel-Weil formula (Theorem 4.1). This means that the corresponding 1-cycle $Y := Z(V_{1,2})$ on the threefold X_L is a projective curve. Fix $f(\tau) = f^+(\tau) + f^-(\tau) \in H_{-1/2}(\omega_L) = H_{1-3/2}(\omega_L)$ with holomorphic part

$$f^+(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \gg -\infty} c_f^+(m,\mu) e(m\tau) \mathbf{1}_\mu.$$

Assume that the Fourier coefficients $c_f^+(m,\mu) \in \mathbf{Z}$ are integers for all negative $m \in \mathbf{Z} + Q(\mu)$. Let

$$Z(f) := \sum_{\mu \in L^\vee/L} \sum_{m>0} c_f^+(-m,\mu) Z\left(\begin{pmatrix} m & a \\ a & m \end{pmatrix}, \mathbf{1}_\mu \otimes \mathbf{1}_\mu\right) \in Z^2(X_L)$$

be the corresponding special cycle of codimension $r = 2$. Here, we take $a \in \mathbf{Q}^\times$ to be the rational number $a = (x_0, x_1) = (x_1, x_0)$ determined by the choice of vector $x = (x_0, x_1) \in V^2$ in the discussion above. We now compute the corresponding archimedean local height

$$[Z(f), Y]_\infty = [Z(f), Z(V_{1,2})]_\infty = \int_{Z(V_{1,2})} \iota_*[\iota^!\Phi(f(\tau), \star)] = \int_{Z(V_{1,2})(\mathbf{C})} \iota_*\iota^!\Phi(f(\tau), z, h). \tag{38}$$

5.1.1. *Rational quadratic spaces.* Recall that we fix (V, Q) any rational quadratic space of signature $(3,2)$, with L any maximal even lattice. We then fix $V_{1,2} \subset V$ any rational quadratic subspace of signature $(1,2)$, with $L_{1,2} = V_{1,2} \cap L$ the corresponding sublattice. As explained above following [31], we can construct such a subspace $V_{1,2}$ explicitly as the complement of a positive definite hyperplane $U = U(x)$ of V of signature $(2,0)$ spanned by some positive definite vector $x = (x_1, x_2) \in V^2$, so that $V_{1,2} = U^\perp = U(x)^\perp \subset V$. Writing $V_{1,2}^\perp \subset V$ to denote the orthogonal complement of $V_{1,2}$, we have a rational splitting

$$V = V_{1,2} \oplus V_{1,2}^\perp.$$

We also consider the complement $L_{1,2}^\perp \subset L$, and the corresponding sublattice $M = M_{1,2} := L_{1,2} \oplus L_{1,2}^\perp \subset L$.

5.1.2. *Sublattices and inner product relations.* Let $\theta_L(\tau, z, h) : \mathfrak{H} \times D(V) \times \mathrm{GSpin}(V)(\mathbf{A}_f) \longrightarrow \mathfrak{S}_L \cong \mathbf{C}[L^\vee/L]$ denote the Siegel theta series associated to the lattice L, as constucted above from the Weil representation

$$\omega_L : \mathrm{Mp}_2(\mathbf{A}_f) \longrightarrow \mathfrak{S}_L \cong \mathbf{C}[L^\vee/L]$$

following [5], cf. [33, §1] and [11, §2]. Writing $M = L_{1,2} \oplus L_{1,2}^\perp \subset L$ again, we now argue that we can replace the Siegel theta series $\theta_L(\tau, z, h)$ with that of the sublattice

$$\theta_M(\tau, z, h) = \theta_{M_{1,2}}(\tau, z, h) = \theta_{L_{1,2} \oplus L_{1,2}^\perp}(\tau, z, h).$$

When $z = z_1 \in D(V_{1,2})$ and $h = h_1 \in \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$, this latter theta series decomposes as a product

$$\theta_{L_{1,2} \oplus L_{1,2}^\perp}(\tau, z, h) = \theta_{L_{1,2}}(\tau, z, h) \otimes \theta_{L_{1,2}^\perp}(\tau) = \theta_{L_{1,2}}(\tau, z, h) \otimes \theta_{L_{1,2}^\perp}(\tau, 1, 1).$$

Here, we write $\theta_{L_{1,2}^\perp}(\tau) = \theta_{L_{1,2}^\perp}(\tau, 1, 1)$ to simplify notations.

Lemma 5.1. *Let $M \subset L$ be any finite-index sublattice. For any $l \in \frac{1}{2}\mathbf{Z}$, we have a natural restriction map*

$$r_{L/M} : H_l(\omega_L) \to H_l(\omega_M), \quad f(\tau) \longmapsto f_M(\tau)$$

and a natural trace map

$$t_{L/M} : H_l(\omega_M) \to H_l(\omega_L), \quad g(\tau) \longmapsto g^L(\tau)$$

for which there is a relation of inner products

$$\langle\langle f(\tau), g(\tau) \rangle\rangle = \langle\langle f_M(\tau), g^L(\tau) \rangle\rangle$$

for any $f \in H_l(\omega_L)$ and $g \in H_l(\omega_M)$. To describe these maps explicitly, consider the natural map

$$L^\vee/L \longrightarrow L^\vee/L, \quad \mu \longmapsto \overline{\mu}.$$

For $\mu \in M^\vee/M$ and $f \in H_l(\omega_L)$, we have

$$f_{M,\mu} = \begin{cases} f_{\overline{\mu}} & \text{if } \mu \in L^\vee/M \\ 0 & \text{otherwise} \end{cases}.$$

For $\overline{\mu} \in L^\vee/L$ with μ a fixed preimage of $\overline{\mu}$ in $L^\vee/M$ and $g \in H_l(\omega_M)$, we have

$$g^L_{\overline{\mu}} = \sum_{\nu \in L/M} g_{\nu+\mu}.$$

Proof. See [11, Lemma 3.1]. □

We deduce from this that we have for any $\tau \in \mathfrak{H}$, $z \in D(V)$, and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$ the relation

$$\theta_L(\tau, z, h) = (\theta_M(\tau, z, h))^L,$$

and hence the identification of inner products

$$\langle\langle f(\tau), \theta_L(\tau, z, h)\rangle\rangle = \langle\langle f(\tau), (\theta_{L_{1,2}\oplus L_{1,2}^\perp}(\tau, z, h))^L\rangle\rangle = \langle\langle f_{L_{1,2}\oplus L_{1,2}^\perp}(\tau), \theta_{L_{1,2}\oplus L_{1,2}^\perp}(\tau, z, h)\rangle\rangle. \tag{39}$$

for any harmonic weak Maass form $f(\tau) \in H_{1-\frac{3}{2}}(\omega_L) = H_{-\frac{1}{2}}(\omega_L)$. We shall take this relation (39) for granted in what follows, dropping the restriction map notations $f_M(\tau) = f_{L_{1,2}\oplus L_{1,2}^\perp}(\tau)$ to simplify the writing.

5.2. **Functional identities for Eisenstein series.**

We now review some properties of the Langlands Siegel Eisenstein series $E_{L_{1,2}}(\tau, s; l)$ as defined in (24) for our later calculations.

5.2.1. *Langlands Eisenstein series for the subspace $V_{1,2}$.* We consider Langlands Eisenstein series for the rational quadratic space $U = V_{1,2} = (V_{1,2}, Q|_{V_{1,2}})$ from the discussion above. Since $\dim(V_{1,2}) = 3$, these will be Eisenstein series on the metaplectic group $\mathrm{Mp}_2(\mathbf{A})$. Fixing an even lattice $L_{1,2} \subset V_{1,2}$, we write $\omega_{L_{1,2}} = \omega_{L_{1,2},\psi}$ to denote the corresponding Weil representation of $\mathrm{Mp}_2(\mathbf{A})$ acting on the space of Schwartz functions $\mathcal{S}(V_{1,2}(\mathbf{A}))$. In this case, the principal series representation $I(s, \chi_{V_{1,2}})$ of $\mathrm{Mp}_2(\mathbf{A})$ induced from $\chi_{V_{1,2}}(\cdot)|\cdot|^s$ consists of smooth functions $\phi(g', s)$ on $g' = [g, \epsilon] \in \mathrm{Mp}_2(\mathbf{A})$ such that

$$\phi([n(b), 1][m(a), \epsilon]g', s) = \chi^\psi_{V_{1,2}}([m(a), \epsilon])|a|^{s+1}\phi(g', s)$$

for all $b \in \mathbf{A}$ and $a \in \mathbf{A}^\times$, with the notations described above. In particular, we have for any $v \in \mathbf{R}_{>0}$ that

$$\phi([n(b), 1][m(v), \epsilon]g', s) = \chi^\psi_{V_{1,2}}([m(v), \epsilon])v^{s+1}\phi(g', s), \tag{40}$$

so that

$$v^{-1/2} \cdot \phi([n(b), 1][m(v), \epsilon]g', s) = \chi^\psi_{V_{1,2}}([m(v), \epsilon])v^{s+1/2}\phi(g', s).$$

Note that if $\phi(g', s) \in I(s, \chi_{V_{1,2}})$ is standard, then we can identify this latter operation with $\phi(g', s - 1/2)$. Indeed, we see from the definition of $\phi(g', s - 1/2)$ that we have the corresponding transformation property

$$\phi([n(b), 1][m(v), \epsilon]g', s - 1/2) = \chi^\psi_{V_{1,2}}([m(v), \epsilon])v^{s+1/2}\phi(g', s - 1/2) \tag{41}$$

for all $b \in \mathbf{A}$ and $v \in \mathbf{R}_{>0}$. We argue that this is equivalent to the standard section $\phi(g', s) \in I(s, \chi_{V_{1,2}})$, multiplied by $v^{-1/2}$. To see this, use Iwasawa decomposition to write $g' = [n(b), 1][m(v), \epsilon]k'$ for k' in the maximal compact subgroup of $\mathrm{Mp}_2(\mathbf{A})$. Using the identities (40) and (41) at $k' \in \mathrm{Mp}_2(\mathbf{A})'$, we compute

$$\frac{\phi(g', s)}{\phi(g', s - 1/2)} = \frac{\phi([n(b), 1][m(v), \epsilon])k', s)}{\phi([n(b), 1][m(v), \epsilon]k', s - 1/2)} = v^{\frac{1}{2}} \cdot \frac{\phi(k', s)}{\phi(k', s - 1/2)} = v^{1/2}.$$

Here, the first equality follows from the Iwasawa decomposition for g', where we could more generally consider the Iwasawa decomposition of any element of $\mathrm{Mp}_2(\mathbf{A})$ in this way. The second equality follows from a comparison of the corresponding transformations (40) and (41). The third equality then follows from the fact that $\phi(g', s)$ is standard, meaning that its restriction to the maximal compact subgroup is independent of the choice of shift s. In particular, we find that we have the identification (of standard sections)

$$v^{-1/2}\phi(g'_\tau, s) = \phi(g'_\tau, s - 1/2) \in I(s, \chi_{V_{1,2}}), \quad \tau = u + iv \in \mathfrak{H} \tag{42}$$

Here, we take $g_\tau = n(u)m(v)$ (rather than $n(u)m(v^{1/2})$) to be consistent with the discussion of [33, §4.1]. Of course, the same argument works for any power of v; we present the example of $v^{1/2}$ to use later as follows.

Recall that for any standard section $\phi(g', s) \in I(s, \chi_{V_{1,2}})$, we consider the corresponding Eisenstein series

$$E(g', s; \phi) = \sum_{\gamma' \in P'(\mathbf{Q}) \backslash \mathrm{Mp}_2(\mathbf{Q})} \phi(\gamma' g', s),$$

whose analytic properties we describe above. Writing $g'_\tau = (g_\tau, 1) \in \mathrm{Mp}_2(\mathbf{R}) \subset \mathrm{Mp}_2(\mathbf{A})$ and $\phi_\infty^l \in I(s, \chi_{V_{1,2}})$ again for the standard section of weight $l \in \frac{1}{2}\mathbf{Z}$, we consider the corresponding Eisenstein series

$$E_{L_{1,2}}(\tau, s; l) = \sum_{\mu \in L_{1,2}^\vee / L_{1,2}} E(g'_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) \mathbf{1}_\mu.$$

Observe that by (42), we obtain for any $v \in \mathbf{R}_{>0}$ the identification

$$v^{-1/2} E_{L_{1,2}}(\tau, s + 1/2; l) = E_{L_{1,2}}(\tau, s; l),$$

so that after taking derivatives in s to get

$$v^{-1/2} E'_{L_{1,2}}(\tau, s + 1/2; l) = E'_{L_{1,2}}(\tau, s; l),$$

we can evaluate at $s = 0$ to obtain the corresponding identity

$$v^{-1/2} E'_{L_{1,2}}(\tau, 1/2; l) = E'_{L_{1,2}}(\tau, 0; l), \tag{43}$$

Note that we can describe this Eisenstein series

$$E_{L_{1,2}}(\tau, s; l) = \sum_{\mu \in L_{1,2}^\vee / L_{1,2}} E(g'_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) \mathbf{1}_\mu$$

in classical terms following [34, IV.2], [11, (2.17)]. Writing $\Gamma'_\infty = P'(\mathbf{Q}) \cap \Gamma'$ in the notations above, we have

$$E(g'_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) = \sum_{\substack{\gamma' = [\gamma, \epsilon] \in \Gamma'_\infty \backslash \Gamma' \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \phi_\infty^l(\gamma'[g_\tau, 1], s) \lambda_f(\mathbf{1}_\mu)(\gamma').$$

We consider the Iwasawa decomposition

$$\gamma' g'_\tau = [\gamma g_\tau, \epsilon] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} g_\tau, \epsilon \right] = [n(\beta), 1][m(\alpha), \epsilon][k_\theta, 1]$$

for $\beta \in \mathbf{R}$, $\alpha \in \mathbf{R}_{>0}$, and $\theta \in \mathbf{R}$. Using that

$$\phi_\infty^l([n(\beta), 1][m(\alpha), \epsilon][k_\theta, 1], s) = \chi_{V_{1,2}}^\psi([m(\alpha), \epsilon]) \alpha^{s+1} e^{il\theta} = \epsilon \gamma(\alpha, \psi)^{-1} \alpha^{s+1} e^{il\theta} = \epsilon e^{-\frac{\pi i}{4}} \alpha^{s+1} e^{il\theta},$$

we find that after direct calculation expanding γ and g_τ that

$$\alpha = v^{1/2} |c\tau + d|^{-1}, \quad e^{i\theta} = \frac{c + \overline{\tau} d}{|c\tau + d|} = \frac{|c\tau + d|}{(c\tau + d)},$$

and hence

$$\begin{aligned} E(g'_\tau, s; \phi_\infty^l \otimes \lambda_f(\mathbf{1}_\mu)) &= \sum_{\gamma' = [\gamma, \epsilon] \in \Gamma'_\infty \backslash \Gamma'} \epsilon e^{-\frac{\pi i}{4}} (c\tau + d)^{-l} \frac{v^{s/2 + 1/2}}{|c\tau + d|^{s+1-l}} \lambda_f(\mathbf{1}_\mu)(\gamma') \\ &= \sum_{\gamma' = [\gamma, \epsilon] \in \Gamma'_\infty \backslash \Gamma'} \epsilon e^{-\frac{\pi i}{4}} (c\tau + d)^{-l} \frac{v^{s/2 + 1/2}}{|c\tau + d|^{s+1-l}} \langle \mathbf{1}_\mu, (\omega_{L_{1,2}}^{-1})(\gamma') \mathbf{1}_0 \rangle \end{aligned}$$

so that

$$E_{L_{1,2}}(\tau, s; l) = \sum_{\gamma' = [\gamma, \epsilon] \in \Gamma_\infty \backslash \Gamma'} \epsilon e^{-\frac{\pi i}{4}} \left[\Im(\tau)^{\frac{s+1-l}{2}} \mathbf{1}_0 \right] \Big|_{l, \omega_{L_{1,2}}} \gamma', \tag{44}$$

for $|_{l, \omega_{L_{1,2}}}$ the Petersson slash operator of weight l for the Weil representation $\omega_{L_{1,2}}$.

5.2.2. *Functional identities.* Let $E_{L_{1,2}}(\tau, s; l)$ denote the Langlands Eisenstein series of weight $l \in \frac{1}{2}\mathbf{Z}/\mathbf{Z}$ associated to the lattice $L_{1,2} \subset V_{1,2}$, as defined above. We consider the case of weight $l = (1-2)/2 = -1/2$. Recall that by the Siegel-Weil formula (Theorem 4.1, Corollary 4.2), we have

$$\frac{1}{2} \int\limits_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \theta_{L_{1,2}}(\tau, z_1, h) dh = E_{L_{1,2}}(\tau, s_0(V_{1,2}); -1/2) = E_{L_{1,2}}(\tau, 1/2; -1/2).$$

Observe that we have the following relation under the Maass weight lowering operator $L_{3/2}$,

$$L_{\frac{3}{2}} E_{L_{1,2}}(\tau, s; 3/2) = \frac{1}{2}(s - 1/2) \cdot E_{L_{1,2}}(\tau, s; -1/2).$$

Taking the derivative with respect to s on each side, we obtain from this the relation

$$L_{\frac{3}{2}} E'_{L_{1,2}}(\tau, s; 3/2) = \frac{1}{2}(s - 1/2) \cdot E'_{L_{1,2}}(\tau, s; -1/2) + \frac{1}{2} \cdot E_{L_{1,2}}(\tau, s; -1/2).$$

Evaluating at $s = 1/2$ then gives us the functional identity

$$L_{\frac{3}{2}} E'_{L_{1,2}}(\tau, 1/2; 3/2) = \frac{1}{2} \cdot E_{L_{1,2}}(\tau, 1/2; -1/2). \tag{45}$$

5.2.3. *Functional equations.* We note that each of the Eisenstein series $E_{L_{1,2}}(\tau, s; l)$ with $l \neq (1-2)/2$ is incoherent in the sense of Kudla [32]. In particular, each of these Eisenstein series has an analytic continuation $E^{\star}_{L_{1,2}}(\tau, s; l)$ to all $s \in \mathbf{C}$ which satisfies the odd functional equation

$$E^{\star}_{L_{1,2}}(\tau, s; l) = -E^{\star}_{L_{1,2}}(\tau, -s; l),$$

with vanishing central value $E^{\star}_{L_{1,2}}(\tau, 0; l) = E_{L_{1,2}}(\tau, 0; l) = 0$.

5.3. **Rankin-Selberg L-functions.** We now consider the Rankin-Selberg L-function

$$L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}})) = \langle \xi_{-1/2}(f), \theta_{L_{1,2}^{\perp}} \otimes E_{L_{1,2}}(\cdot, s; 3/2) \rangle$$

given by the Petersson inner product

$$\langle \xi_{-1/2}(f), \theta_{L_{1,2}^{\perp}} \otimes E_{L_{1,2}}(\cdot, s; 3/2) \rangle = \int\limits_{\mathcal{F}} \langle \langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes E_{L_{1,2}}(\tau, s; 3/2) \rangle \rangle v^{5/2} d\mu(\tau).$$

We write the Fourier series expansion of the holomorphic form $\xi_{-1/2}(f)(\tau) \in M_{5/2}(\omega_L^{\vee})$ as

$$\xi_{-1/2}(f)(\tau) = \sum_{\mu \in L^{\vee}/L} \sum_{m \geq 0} c_{\xi_{-1/2}(f)}(m, \mu) e(m\tau) \mathbf{1}_{\mu},$$

and that of the holomorphic theta series $\theta_{L_{1,2}^{\perp}}(\tau) \in M_1(\omega^{\vee}_{L_{1,2}^{\perp}})$ for the negative-definite sublattice $L_{1,2}^{\perp}$ as

$$\theta_{L_{1,2}^{\perp}}(\tau) = \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\perp}} \sum_{m \geq 0} r_{L_{1,2}^{\perp}}(m, \lambda) e(m\tau) \mathbf{1}_{\lambda}.$$

Here, each $r_{L_{1,2}^{\perp}}(m, \lambda)$ denotes the counting function defined by

$$r_{L_{1,2}^{\perp}}(m, \lambda) := \frac{1}{w_{k(L_{1,2}^{\perp})}} \cdot \# \left\{ x \in \lambda + L_{1,2}^{\perp} : Q(x) = m \right\},$$

where $w_{k(L_{1,2}^{\perp})}$ denotes the number of roots of unity in the imaginary quadratic field $k(L_{1,2}^{\perp})$ determined by $L_{1,2}^{\perp} \otimes \mathbf{Q}$. That is, each rational quadratic space $(V_{1,2}^{\perp}, Q_{L_{1,2}^{\perp}})$ of signature $(2,0)$ is equivalent to

$$(V_{1,2}^{\perp}, Q_{L_{1,2}^{\perp}}) = (k(L_{1,2}^{\perp}), \alpha \cdot \mathbf{N}_{k(L_{1,2}^{\perp})/\mathbf{Q}})$$

for vector space given by the imaginary quadratic field $k(L_{1,2}^{\perp})$ of discriminant $d_{k(L_{1,2}^{\perp})} = (L_{1,2}^{\perp})$, with quadratic form given by some positive scalar $\alpha \in \mathbf{Q}_{>0}$ times the corresponding norm form[8]

$$\mathbf{N}_{k(L_{1,2}^{\perp})/\mathbf{Q}}(z) := z z^{\tau}, \quad \tau \neq \mathbf{1} \in \mathrm{Gal}(k(L_{1,2}^{\perp})/\mathbf{Q}).$$

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[8] i.e. expanding z in terms of an integral basis of $\mathcal{O}_k$, this determines a positive definite binary quadratic form

We refer to [11, §2.2] and [1] for more details.

5.3.1. *Analytic continuation via Rankin-Selberg convolution.* Let us first observe that we can compute

$$\begin{aligned}
&\mathrm{CT}\langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}(\tau)} \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle := \int_0^1 \langle\langle \overline{\xi_{-1/2}(f)(u+iv)}, \theta_{L_{1,2}^{\perp}(u+iv)} \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle du \\
&= \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\vee}} \sum_{m \geq 0} \overline{c_{\xi_{-1/2}(f)}(m,\lambda)} e(miv) \sum_{n \geq 0} r_{L_{1,2}^{\perp}}(n,\lambda) e(niv) \int_0^1 e(nu - mu) du \\
&= \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\vee}} \sum_{m \geq 0} \overline{c_{\xi_{-1/2}(f)}(m,\lambda)} r_{L_{1,2}^{\perp}}(m,\lambda) e^{-4\pi m v}.
\end{aligned}$$

Here, we restrict the holomorphic form $\xi_{-1/2}(f) \in M_{5/2}(\omega_L^{\vee})$ to the sublattice $L_{1,2} \oplus L_{1,2}^{\perp} \subset L$. We can then open up gamma factors in the following Dirichlet series (defined first for $\Re(s) \gg 1$) to compute

$$\begin{aligned}
L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) &:= (4\pi)^{-\left(\frac{s+1/2}{2}\right)} \Gamma\left(\frac{s+1/2}{2}\right) \sum_{m \geq 1} \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\vee}} \overline{c_{\xi_{-1/2}(f)}(m,\mu)} r_{L_{1,2}^{\perp}}(m,\lambda) m^{-\left(\frac{s+1/2}{2}\right)} \\
&= \int_0^{\infty} \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\vee}} \sum_{m \geq 1} \overline{c_{\xi_{-1/2}(f)}(m,\mu)} r_{L_{1,2}^{\perp}}(m,\lambda) e^{-4\pi m v} v^{\frac{s+1/2}{2}} dv \\
&= \int_0^{\infty} \int_0^{1} \langle\langle \overline{\xi_{-1/2}(f)(u+iv)}, \theta_{L_{1,2}^{\perp}(u+iv)} \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle v^{\frac{s+1/2}{2}} du dv \\
&= \iint_{\Gamma_{\infty}\backslash\mathfrak{H}} \langle\langle \overline{\xi_{-1/2}(f)(u+iv)}, \theta_{L_{1,2}^{\perp}(u+iv)} \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle v^{\frac{s+1/2}{2}+2} \frac{du dv}{v^2} \\
&= \iint_{\Gamma_{\infty}\backslash\mathfrak{H}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle v^{\frac{s+1/2+4}{2}} d\mu(\tau).
\end{aligned}$$

Here, the unipotent matrices $\Gamma_{\infty} = P(\mathbf{Q}) \cap \Gamma$ act on $\mathfrak{H}$ by translation. Fixing a set of coset representatives $\{\gamma\}$ for $\Gamma_{\infty}\backslash\Gamma$ and writing $\mathcal{F}$ as usual to denote the standard fundamental domain for $\Gamma\backslash\mathfrak{H}$, we obtain a fundamental domain $\{\gamma\mathcal{F}\}$ for this translation action. Hence, we unfold to get the convolution relation

$$\begin{aligned}
&L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) = \sum_{\gamma \in \Gamma_{\infty}\backslash\Gamma} \int_{\gamma\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle v^{\frac{s+9/2}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_{\infty}\backslash\Gamma} \int_{\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\gamma\tau)}, \theta_{L_{1,2}^{\perp}}(\gamma\tau) \otimes \mathbf{1}_{L_{1,2}+0}\rangle\rangle \Im(\gamma\tau)^{\frac{s+9/2}{2}} d\mu(\tau) \\
&= \int_{\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes \left(\sum_{\gamma \in \Gamma_{\infty}\backslash\Gamma} \Im(\gamma\tau)^{\frac{s+9/2}{2}} \omega_{L_{1,2}}^{-1}(\gamma) \mathbf{1}_{L_{1,2}+0} \right) \rangle\rangle d\mu(\tau) \\
&= \int_{\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes \left(\sum_{\gamma \in \Gamma_{\infty}\backslash\Gamma} \left[\Im(\tau)^{\frac{s-1/2}{2}+5/2} \mathbf{1}_0 \right] \Big|_{3/2, \omega_{L_{1,2}}} \gamma \right) \rangle\rangle d\mu(\tau) \\
&= \int_{\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes \left(\sum_{\gamma \in \Gamma_{\infty}\backslash\Gamma} \left[\Im(\tau)^{\frac{s-1/2}{2}} \mathbf{1}_0 \right] \Big|_{3/2, \omega_{L_{1,2}}} \gamma \right) \rangle\rangle v^{5/2} d\mu(\tau) \\
&= \int_{\mathcal{F}} \langle\langle \overline{\xi_{-1/2}(f)(\tau)}, \theta_{L_{1,2}^{\perp}}(\tau) \otimes E_{L_{1,2}}(\tau, s; 3/2)\rangle\rangle v^{5/2} d\mu(\tau) = \langle \xi_{-1/2}(f), \theta_{L_{1,2}^{\perp}} \otimes E_{L_{1,2}}(\cdot, s; 3/2)\rangle.
\end{aligned} \tag{46}$$

Here, we use the expansion (44). We also treat the shift by $-5/2$ by the same principles used to establish the relations (42) and (43) for shifts by $-1/2$. Since this incoherent Eisenstein series $E_{L_{1,2}}(\tau, s; 3/2)$ has an analytic continuation $E_{L_{1,2}}^{\star}(\tau, s; 3/2)$ to all $s \in \mathbf{C}$ satisfying the odd symmetric functional equation

$E^\star_{L_{1,2}}(\tau,s;3/2) = -E^\star_{L_{1,2}}(\tau,-s;3/2)$, we deduce from (46) that the Dirichlet series $L(s,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}})$ (defined first for $\Re(s)\gg 1$) has an analytic continuation

$$L^\star(s,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}}) := \langle \xi_{-1/2}(f), \theta_{L^\perp_{1,2}}\otimes E^\star_{L_{1,2}}(\cdot,s;3/2)\rangle$$

to all $s\in\mathbf{C}$ satisfying the odd functional equation

$$L^\star(s,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}}) = -L^\star(-s,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}}). \tag{47}$$

In particular, the central value vanishes: $L^\star(0,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}})=0$. It therefore makes sense to look at the central derivative value $L^{\star\prime}(0,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}})$, which we often simply denote by $L'(0,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}})$.

5.3.2. *Relation to standard Rankin-Selberg L-functions.* We now explain how to relate $L^\star(s,\xi_{-1/2}(f)\times\theta_{L^\perp_{1,2}})$ to standard $\mathrm{GL}_2(\mathbf{A})\times\mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-functions, as outlined in the introduction for Theorem 1.8.

Let $h(\tau)=h_{\xi_{-1/2}(f)}(\tau)$ denote the lifting of $\xi_{-1/2}(f)(\tau)\in M_{5/2}(\omega_L^\vee)$ to a scalar-valued holomorphic form. To describe this according to [54, Theorems 5.2 and 5.4], let $d(L)$ denote the discriminant of the lattice $L\subset V$, given by the determinant of the Gram matrix, with $\chi_L(\cdot)=(\frac{d(L)}{\cdot})$ the corresponding character. Let $M(L)$ denote the level of the lattice $L\subset V$. Viewing $\xi_{-1/2}(f)\in M_{5/2}(\omega_L^\vee)$ as the Shimura lift of a holomorphic form of weight 4, we deduce that the scalar-valued lifting has the expansion

$$h(\tau)=h_{\xi_{-1/2}(f)}(\tau)=\sum_{\gamma'=[\gamma,\epsilon]\in\Gamma_0(M(L))\backslash\mathrm{Mp}_2(\mathbf{Z})}\left[\xi_{-1/2}(f)(\tau)\mathbf{1}_0\right]\Big|_{5/2,\omega_L}\gamma'\in M_{5/2}(\Gamma_0(M(L)),\chi_L).$$

We write the Fourier expansion of $h(\tau)$ as

$$h(\tau)=\sum_{m\geq 0}c_h(m)e(m\tau)=\sum_{m\geq 0}a_h(m)m^{\frac{5/2-1}{2}}e(m\tau)=\sum_{m\geq 0}a_h(m)m^{3/4}e(m\tau).$$

Here, the $a_h(m)$ denote the normalized Fourier coefficients, scaled so that the finite part of the standard L-function of h is given by the Dirichlet series $L(s,h)=\sum_{m\geq 1}a_h(m)m^{-s}=\sum_{m\geq 1}c_h(m)m^{-(s+3/4)}$. Let

$$\theta_{k(L^\perp_{1,2})}(\tau)=\sum_{\gamma\in\Gamma_0(|d_{k(L^\perp_{1,2})}|)\backslash\mathrm{SL}_2(\mathbf{Z})}\left[\theta_{L^\perp_{1,2}}(\tau)\mathbf{1}_0\right]\Big|_{1,\omega_{L^\perp_{1,2}}}\gamma\in M_1(\Gamma_0(|d_{k(L^\perp_{1,2})}|),\chi_{k(L^\perp_{1,2})})$$

denote the lifting of $\theta_{L^\perp_{1,2}}(\tau)\in M_1(\omega^\vee_{L^\perp_{1,2}})$ to the scalar-valued Hecke theta series associated to the imaginary quadratic field $k(L^\perp_{1,2})$ determined by $L^\perp_{1,2}$. We refer to the discussion in [56, §6.1] for more details of this correspondence, which presumably is known classically. Here, we can write

$$\theta_{k(L^\perp_{1,2})}(\tau)=\sum_{m\geq 1}r_{L^\perp_{1,2}}(m)e(m\tau)$$

to denote the Fourier expansion, with (normalized) coefficients given in terms of the counting functions

$$r_{L^\perp_{1,2}}(m)=\sum_{\lambda\in(L^\perp_{1,2})^\vee/L^\perp_{1,2}}r_{L^\perp_{1,2}}(m,\lambda)$$

introduced above. The choice of lattice $L^\perp_{1,2}$ determines an integral ideal in the imaginary quadratic field $k(L^\perp_{1,2})=L^\perp_{1,2}\otimes\mathbf{Q}$, and $r_{L^\perp_{1,2}}(m)$ counts the number of ideals of norm m in the ideal class it represents.

Let $L(s,h\times\theta_{k(L^\perp_{1,2})})$ denote the Rankin-Selberg L-function of h times $\theta_{k(L^\perp_{1,2})}$, defined for $\Re(s)\gg 1$ by

$$\begin{aligned}L(s,h\times\theta_{k(L^\perp_{1,2})})&=L(2s,\chi_{k(L^\perp_{1,2})})\sum_{m\geq 1}a_h(m)r_{L^\perp_{1,2}}(m)m^{-(s-1)}\\&=L(2s,\chi_{k(L^\perp_{1,2})})\sum_{m\geq 1}c_h(m)r_{L^\perp_{1,2}}(m)m^{-(s-1/4)}.\end{aligned}$$

This Dirichlet series has a well-known analytic continuation to an entire function

$$\Lambda(s,h\times\theta_{k(L^\perp_{1,2})}):=L_\infty(s,h\times\theta_{k(L^\perp_{1,2})})L(s,h\times\theta_{k(L^\perp_{1,2})})$$

of $s \in \mathbf{C}$ by the method of Rankin-Selberg convolution, and satisfies a symmetric functional equation

$$\Lambda(s, h \times \theta_{k(L_{1,2}^{\perp})}) = \epsilon(s, h \times \theta_{k(L_{1,2}^{\perp})})\Lambda(1-s, h \times \theta_{k(L_{1,2}^{\perp})})$$

relating values at s to $1-s$. More precisely, a variation of the unfolding calculation of (46) above shows that

$$\begin{aligned}
&(4\pi)^{-(\frac{s+1/2}{2})}\Gamma\left(\frac{s+1/2}{2}\right)\sum_{m \geq 1} c_h(m) r_{L_{1,2}^{\perp}}(m) m^{-(\frac{s+1/2}{2})} = \int_0^\infty \sum_{m \geq 1} c_h(m) r_{L_{1,2}^{\perp}}(m) e^{-4\pi m v} v^{\frac{s+1/2}{2}} dv \\
&= \int_0^\infty \int_0^1 h(\tau)\overline{\theta_{k(L_{1,2}^{\perp})}(\tau)} v^{\frac{s+1/2}{2}+2} \frac{du dv}{v^2} = \iint\limits_{\Gamma_\infty \backslash \mathfrak{H}} h(\tau)\overline{\theta_{k(L_{1,2}^{\perp})}(\tau)} v^{\frac{s+9/2}{2}} d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma\mathcal{F}} h(\tau)\overline{\theta_{k(L_{1,2}^{\perp})}(\tau)} v^{\frac{s+9/2}{2}} d\mu(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}} h(\gamma\tau)\overline{\theta_{k(L_{1,2}^{\perp})}(\gamma\tau)} \Im(\gamma\tau)^{\frac{s+9/2}{2}} d\mu(\tau) \\
&= \int_{\mathcal{F}} h(\tau)\overline{\theta_{k(L_{1,2}^{\perp})}(\tau)} \left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\gamma\tau)^{\frac{s-1/2}{2}} \right] \Big|_{3/2, \chi_{k(L_{1,2}^{\perp})}} \right) v^{5/2} d\mu(\tau) = \langle h, \theta_{k(L_{1,2}^{\perp})} E_{3/2}(\cdot, s) \rangle.
\end{aligned}$$

Here, $E_{3/2}(\tau, s) \in M_{3/2}(\Gamma_0(M(L)), \chi_{k(L_{1,2}^{\perp})})$ corresponds the scalar-valued lift of the incoherent Eisenstein series $E_{L_{1,2}}(\tau, s; 3/2)$. It inherits the twist $\chi_{k(L_{1,2}^{\perp})}$ from $\theta_{k(L_{1,2}^{\perp})}$ via the corresponding transformation law. Now, observe that we have an identification of Petersson inner products

$$\langle h, \theta_{k(L_{1,2}^{\perp})} E_{3/2}(\cdot, s) \rangle = \langle \xi_{-1/2}(f), \theta_{L_{1,2}^{\perp}} \otimes E_{L_{1,2}}(\cdot, s; 3/2) \rangle =: L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}).$$

From the vector-valued case, we deduce that $E_{3/2}(\tau, s)$ has an analytic continuation $E_{3,2}^{\star}(\tau, s)$ satisfying

$$E_{3/2}^{\star}(\tau, s) = -E_{3/2}^{\star}(\tau, -s),$$

and hence that

$$L^{\star}(s, h \times \theta_{k(L_{1,2}^{\perp})}) := \langle h, \theta_{k(L_{1,2}^{\perp})} E_{3/2}^{\star}(\cdot, s) \rangle = -L^{\star}(-s, h \times \theta_{k(L_{1,2}^{\perp})}).$$

Moreover, we deduce (cf. [11, Proposition 2.5]) that

$$E_{3/2}^{\star}(\tau, s) := \Lambda(s+1, \chi_{k(L_{1,2}^{\perp})}) E_{3/2}(\tau, s), \quad \Lambda(s, \chi_{k(L_{1,2}^{\perp})}) := |d_{k(L_{1,2}^{\perp})}|^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s, \chi_{k(L_{1,2}^{\perp})}).$$

Indeed, this gives us the relation of Dirichlet series

$$\begin{aligned}
&L^{\star}(2s-2, h \times \theta_{k(L_{1,2}^{\perp})}) = (4\pi)^{-(s-\frac{3}{4})}\Gamma\left(s - \frac{3}{4}\right) \Lambda(2s, \chi_{k(L_{1,2}^{\perp})}) \sum_{m \geq 1} c_h(m) r_{L_{1,2}^{\perp}}(m) m^{-(s-\frac{3}{4})} \\
&= (4\pi)^{-(s-\frac{3}{4})}\Gamma\left(s - \frac{3}{4}\right) \pi^{-2}\Gamma(s) |d_{k(L_{1,2}^{\perp})}|^s L(2s, \chi_{k(L_{1,2}^{\perp})}) \sum_{m \geq 1} a_h(m) r_{L_{1,2}^{\perp}}(m) m^{-(s-\frac{3}{2})} \qquad (48) \\
&= (4\pi)^{-(s-\frac{3}{4})}\Gamma\left(s - \frac{3}{4}\right) \pi^{-2}\Gamma(s) |d_{k(L_{1,2}^{\perp})}|^s L(s-1/2, h \times \theta_{k(L_{1,2}^{\perp})}).
\end{aligned}$$

We deduce that $L^{\star}(s, h \times \theta_{k(L_{1,2}^{\perp})}) = L^{\star}(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}})$ must be a shift

$$\Lambda(s-1/2, h \times \theta_{k(L_{1,2}^{\perp})}) = L^{\star}(2s-2, h \times \theta_{k(L_{1,2}^{\perp})})$$

of the standard Rankin-Selberg L-function $\Lambda(s, h \times \theta_{k(L_{1,2}^{\perp})})$, which has root number $\epsilon(1/2, h \times \theta_{k(L_{1,2}^{\perp})})) = -1$. We can also deduce from the quadratic basechange equivalence (see e.g. [56, (6.1.3)]) and the Hecke theta series that we have decompositions into products of $\mathrm{GL}_2(\mathbf{A})$-automorphic L-functions, namely

$$L(s, h \times \theta_{k(L_{1,2}^{\perp})}) = L(s, h) L(s, h \otimes \chi_{k(L_{1,2}^{\perp})}).$$

5.4. **Summation along 1-cycles.** We now compute the archimedean local height (38) in the style of the arguments of Kudla [33], Schofer [49], and Bruinier-Yang [11, Theorem 4.7].

5.4.1. *Relations of differential forms.*

Lemma 5.2. *We have for any weight $l \in \frac{1}{2}\mathbf{Z}$ and Maass form $f \in H_l(\omega_L)$ the relation of differential forms*

$$\overline{\partial}(f d\tau) = -v^{2-l}\xi_l(f) d\mu(\tau) = -L_l f d\mu(\tau).$$

In particular, we have the relation

$$\overline{\partial}(f d\tau) = -v^{\frac{5}{2}}\xi_{-\frac{1}{2}}(f) d\mu(\tau) = -L_{-\frac{1}{2}} f d\mu(\tau).$$

Proof. See e.g. [14, Lemma 2.5]. □

5.4.2. *Normalization of measures.* Recall we have a short exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathrm{GSpin}(V_{1,2}) \longrightarrow \mathrm{SO}(V_{1,2}) \longrightarrow 1. \tag{49}$$

Fix the Haar measure $dh = dh_\infty \times dh_f$ on $\mathrm{SO}(V_{1,2})(\mathbf{A})$ so that

$$\mathrm{vol}\left(\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A})\right) = 2.$$

We further normalize the archimedean component dh_∞ so that

$$\mathrm{vol}\left(\mathrm{SO}(V_{1,2})(\mathbf{R})\right) = 1.$$

Fix the Haar measure on $\mathbf{A}_f^\times$ to give $\mathrm{vol}(\mathbf{Z}_p) = 1$ for each prime p, so that $\mathrm{vol}(\widehat{\mathbf{Z}}) = 1$, and $\mathrm{vol}(\mathbf{Q}^\times\backslash\mathbf{A}_f^\times) = 1/2$. This determines the normalization of the measure dh_f on $\mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$ via (49), with

$$\mathrm{vol}(\mathrm{GSpin}(V_{1,2})(\mathbf{Q})\backslash \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)) = \mathrm{vol}(\mathbf{Q}^\times\backslash\mathbf{A}_f^\times) \cdot \mathrm{vol}\left(\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A})\right) = 1.$$

Lemma 5.3. *Let B be any function of $h \in \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$ which depends only on the image of h in $\mathrm{SO}(V_{1,2})(\mathbf{A}_f)$. Assume that B is left $\mathrm{GSpin}(V_{1,2})(\mathbf{Q})$-invariant, and right invariant under the compact open subgroup $K_{1,2} = K_{L_{1,2}} = K_L \cap \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$. We have the relation of finite sums*

$$\begin{aligned}\iota_*\iota^!\Phi(f, Z(V_{1,2})) &:= \sum_{(z_1,h)\in Z(V_{1,2})(\mathbf{C})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^\perp}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) \\ &\frac{1}{\mathrm{vol}(K_{1,2})} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^\perp}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) dh \\ &= \frac{\deg(Z(V_{1,2}))}{2} \cdot \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^\perp}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) dh,\end{aligned}$$

where

$$\deg(Z(V_{1,2})) = \frac{2}{\mathrm{vol}(K_{1,2})}.$$

Proof. See [49, Lemma 2.13] and [11, Lemma 4.4]. Put $H_{1,2} = \mathrm{GSpin}(V_{1,2})$, with $Z_{1,2}$ its centre, so that

$$\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f) = H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/Z_{1,2}(\mathbf{A}_f).$$

Since $V_{1,2}$ is anisotropic, we argue as in [49, Lemma 2.13] this spaces is compact, with finite quotient

$$\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)/\left(K_{1,2}/Z_{1,2}\right) = H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/K_{1,2}.$$

Since the function B is invariant under $K_{1,2}$, it follows that we have the identification of finite sums

$$\int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} B(h)dh = \mathrm{vol}(K_{1,2}/Z_{1,2}) \sum_{H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/(K_{1,2}/Z_{1,2})} B(h).$$

Our choice of measure normalizations implies that $\mathrm{vol}(K_{1,2}/Z_{1,2}) = \mathrm{vol}(K_{1,2})$, so that we have

$$\frac{1}{\mathrm{vol}(K_{1,2})} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} B(h)dh = \sum_{H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/(K_{1,2})} B(h).$$

Taking $B(h) = \iota_* \iota^! \Phi(f, z, h)$ (with $z \in D(V)$ fixed) as a function of $h \in H_{1,2}(\mathbf{A}_f)$, we obtain

$$\sum_{H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/K_{1,2}} \iota_* \iota^! \Phi(f, z, h) = \frac{1}{\mathrm{vol}(K_{1,2})} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} \iota_* \iota^! \Phi(f, z, h) dh.$$

On the other hand, taking the constant function $B(h) = 1$, we obtain the relation

$$\begin{aligned}\deg(Z(V_{1,2})) &= \sum_{H_{1,2}(\mathbf{Q})\backslash H_{1,2}(\mathbf{A}_f)/K_{1,2}} 1 \\ &= \frac{1}{\mathrm{vol}(K_{1,2})} \int_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f)} dh = \frac{\mathrm{vol}(\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash \mathrm{SO}(V_{1,2})(\mathbf{A}_f))}{\mathrm{vol}(K_{1,2})} = \frac{2}{\mathrm{vol}(K_{1,2})}.\end{aligned}$$

□

5.4.3. *Description of the regularized integral.* We first derive the following convenient expression for the regularized theta integral following Kudla [33, Proposition 2.5]. Hence, given $f(\tau) \in H_{-1/2}(\omega_L)$, $z_1 \in D(V_{1,2})$, and $h \in \mathrm{GSpin}(V_{1,2})(\mathbf{A}_f)$, we consider the regularized integral defined by

$$\begin{aligned}\iota_* \iota^! \Phi(f, z_1, h) &= \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2} \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) \\ &:= \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes v^{-1/2}\theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle v^{-s} d\mu(\tau) \right).\end{aligned}$$

Again, $\mathrm{CT}_{s=0}\, F(s)$ is the constant term in the Laurent series expansion around $s = 0$ of a function $F(s)$. We first consider the less convergent shift by $s = -1/2$,

$$\begin{aligned}\iota_* \iota^! \Phi^{(1/2)}(f, z_1, h) &:= \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) \\ &:= \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle v^{-s} d\mu(\tau) \right).\end{aligned}$$

Proposition 5.4. *We have the regularized integral formula*

$$\iota_* \iota^! \Phi^{(1/2)}(f, z_1, h) = \lim_{T\to\infty} \left[\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h_1)\rangle\rangle d\mu(\tau) - A_0 \log(T) \right],$$

where

$$A_0 = \mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathbf{1}_{0+L_{1,2}}\rangle\rangle.$$

denotes the constant term in the Fourier series expansion of the modular form $\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathbf{1}_{0+L_{1,2}}\rangle\rangle$.

Proof. See [33, Proposition 2.5], a variation of the same calculation works here. To be clear, we start with

$$\iota_* \iota^! \Phi^{(1/2)}(f, z_1, h) = \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle v^{-s} d\mu(\tau) \right).$$

Since

$$\int_{\mathcal{F}_1} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle v^{-s} d\mu(\tau)$$

is a holomorphic function, we can separate it from the limit to obtain the expression

$$\begin{aligned}
&\iota_*\iota^!\Phi^{(1/2)}(f, z_1, h) \\
&= \mathrm{CT}_{s=0}\left(\lim_{T\to\infty}\int\limits_1^T C(v,h)v^{-s}\frac{dv}{v}\right) + \int\limits_{\mathcal{F}_1}\langle\langle f(\tau), \theta_{L_{1,2}^\perp}(\tau)\otimes\theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau),
\end{aligned} \tag{50}$$

where each $C(v,y)$ denotes the constant coefficient defined by

$$C(v,h) = \int\limits_{-1/2}^{1/2} v^{-1}\langle\langle f(u+iv), \theta_{L_{1,2}^\perp}(u+iv)\otimes\theta_{L_{1,2}}(u+iv, z_1, h)\rangle\rangle du.$$

To evaluate this coefficient, let us first split it into parts according to the holomorphic/nonholomorphic decompostion $f(\tau) = f^+(\tau) + f^-(\tau)$. Hence, we consider $C(v,h) = C^+(v,h) + C^-(v,h)$, with

$$C^\pm(v,h) = \int\limits_{-1/2}^{1/2} v^{-1}\langle\langle f^\pm(u+iv), \theta_{L_{1,2}^\perp}(u+iv)\otimes\theta_{L_{1,2}}(u+iv, z_1, h)\rangle\rangle du.$$

Writing $M = L_{1,2}\oplus L_{1,2}^\perp$ for simplicity so that we have the Fourier expansions

$$f^+(\tau) = \sum_{\mu\in M^\vee/M} f_\mu^+(\tau)\mathbf{1}_\mu = \sum_{\mu\in M^\vee/M}\sum_{\substack{m\in\mathbf{Q}\\ m\gg-\infty}} c_f^+(\mu, m)e(m\tau)\mathbf{1}_\mu,$$

$$f^-(\tau) = \sum_{\mu\in M^\vee/M} f_\mu^-(\tau)\mathbf{1}_\mu = \sum_{\mu\in M^\vee/M}\sum_{\substack{m\in\mathbf{Q}\\ m<0}} c_f^-(\mu, m)W_{-1/2}(2\pi my)e(m\tau)\mathbf{1}_\mu,$$

and

$$\theta_M(\tau, z_1, h) = v\sum_{\mu\in M^\vee/M}\theta_{M,\mu}(z_1,h)\mathbf{1}_\mu = v\sum_{\mu\in M^\vee/M}\sum_{\substack{x\in V(\mathbf{Q})\\ x\in h\mu}} e\left(\tau Q(x_{z_1^\perp}) + \overline{\tau}Q(x_{z_1})\right)\mathbf{1}_\mu,$$

we then compute

$$\begin{aligned}
C^+(v,h) &= v^{-1}\int_{-1/2}^{1/2}\sum_{\mu\in M^\vee/M} f_{M,\mu}^+(u+iv)\theta_{M,\mu}(u+iv, z_1, h)du \\
&= \sum_{\mu\in M^\vee/M}\sum_{\substack{m\in\mathbf{Q}\\ m\gg-\infty}} c_f^+(\mu,m)e(miv)\sum_{\substack{x\in V(\mathbf{Q})\\ x\in h\mu}} e\left(ivQ(x_{z_1^\perp}) - ivQ(x_{z_1})\right)\int_{-1/2}^{1/2} e\left(mu + Q(x_{z_1^\perp})u + Q(x_{z_1})u\right)du \\
&= \sum_{\mu\in M^\vee/M}\sum_{\substack{m\in\mathbf{Q}\\ m\gg-\infty}}\sum_{\substack{x\in V(\mathbf{Q})\\ x\in h\mu\\ m=-Q(x_{z_1^\perp})-Q(x_{z_1})}} c_f^+(\mu,m)e(miv)e\left(ivQ(x_{z_1^\perp}) - ivQ(x_{z_1})\right) \\
&= \sum_{\mu\in M^\vee/M}\sum_{\substack{x\in V(\mathbf{Q})\\ x\in h\mu}} c_f^+(\mu, -Q(x_{z_1^\perp}) - Q(x_{z_1}))e^{4\pi vQ(x_{z_1^\perp})}
\end{aligned}$$

and similarly

$$C^-(v,h) = \sum_{\mu\in M^\vee/M}\sum_{\substack{x\in V(\mathbf{Q})\\ x\in h\mu}} c_f^-(\mu, -Q(x_{z_1^\perp}) - Q(x_{z_1}))W_{-1/2}\left(2\pi v\left(-Q(x_{z_1^\perp}) - Q(x_{z_1})\right)\right)e^{4\pi vQ(x_{z_1^\perp})}.$$

Now, observe that the sum

$$\lim_{T\to\infty}\int_1^T C(v,h)v^{-s}\frac{dv}{v} = \lim_{T\to\infty}\int_1^T\left(C^+(v,h) + C^-(v,h)\right)v^{-s}\frac{dv}{v}$$

converges absolutely, as the restricted quadratic form $Q|_{z_1} < 0$ for $z_1 \in D(V_{1,2})$, and the Fourier coefficients are sufficiently bounded. Let us now consider the contributions from x orthogonal to z_1, so $(x, z_1) = 0$, equivalently $x_{z_1} = 0$ so that $Q(x_{z_1}) = 0$ and $x \in V_{1,2}^{\perp}$. It is easy to see that these contributions are given by

$$C^{+}_{V_{1,2}^{\perp}}(v,h) = \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\perp}} \sum_{\substack{x \in V_{1,2}^{\perp}(\mathbf{Q}) \\ x \in \lambda}} c_f^{+}(\lambda, -Q(x_{z_1^{\perp}})) = C^{+}_{V_{1,2}^{\perp}}$$

and

$$C^{-}_{V_{1,2}^{\perp}}(v,h) = \sum_{\lambda \in (L_{1,2}^{\perp})^{\vee}/L_{1,2}^{\perp}} \sum_{\substack{x \in V_{1,2}^{\perp}(\mathbf{Q}) \\ x \in \lambda}} W_{-1/2}\left(-2\pi v Q(x_{z_1^{\perp}})\right) c_f^{-}(\lambda, -Q(x_{z_1^{\perp}})) = C^{-}_{V_{1,2}^{\perp}}(v).$$

Notice that neither of the coefficients depends on h, and the first coefficient $C^{+}_{V_{1,2}^{\perp}}$ does not even depend on v. We have for $\Re(s) > 0$ that

$$\lim_{T\to\infty} \int_1^T C^{+}_{V_{1,2}^{\perp}}(v,h) v^{-s} \frac{dv}{v} = C^{+}_{V_{1,2}^{\perp}} \cdot \lim_{T\to\infty} \int_1^T v^{-s} \frac{dv}{v} = C^{+}_{V_{1,2}^{\perp}} \cdot \lim_{T\to\infty} \frac{(1 - T^{-s})}{s},$$

and in the limit to $s = 0$ that

$$\lim_{s\to 0} \left(\lim_{T\to\infty} \int_1^T C^{+}_{V_{1,2}^{\perp}}(v,h) v^{-s} \frac{dv}{v} \right) = C_{V_{1,2}^{\perp}} \cdot \lim_{T\to\infty} \log(T).$$

For the constributions of the x not orthogonal to z_1, we find that

$$C^{+}_{V_{1,2}}(v,h) = v^{-1/2} \sum_{\lambda \in L_{1,2}^{\vee}/L_{1,2}} \sum_{\substack{x \in V_{1,2}(\mathbf{Q}) \\ x \in h\lambda}} c_f^{+}(\lambda, -Q(x_{z_1}) - Q(x_{z_1^{\perp}})) e^{4\pi v Q(x_{z_1})}$$

and

$$C^{-}_{V_{1,2}}(v,h) = v^{-1/2} \sum_{\lambda \in L_{1,2}^{\vee}/L_{1,2}} \sum_{\substack{x \in V_{1,2}(\mathbf{Q}) \\ x \in h\lambda}} c_f^{-}(\lambda, -Q(x_{z_1}) - Q(x_{z_1^{\perp}})) W_{-1/2}\left(-2\pi v \left(Q(x_{z_1}) + Q(x_{z_1^{\perp}})\right)\right) e^{4\pi v Q(x_{z_1})}.$$

As explained in the proof of [33, Proposition 2.5], the integrals defined for $t > 0$ by

$$\beta_{s+1}(t) = \int_1^{\infty} e^{-tv} v^{-s} \frac{dv}{v}$$

are convergent for all $s \in \mathbf{C}$, and determine holomorphic functions of s. In this way, we deduce that

$$\begin{aligned} \mathrm{CT}_{s=0} \left(\lim_{T\to\infty} \int_1^T C(v,h) v^{-s} \frac{dv}{v} \right) &= \lim_{T\to\infty} \left(\int_1^T C(v,h) v^{-s} \frac{dv}{v} - C^{+}_{V_{1,2}^{\perp}} \cdot \log(T) \right) \\ &= \lim_{T\to\infty} \left(\int_1^T C(v,h) v^{-s} \frac{dv}{v} - A_0 \cdot \log(T) \right). \end{aligned}$$

Substituting this back into the initial expression, we obtain the desired limit

$$\lim_{T\to\infty} \left(\int_{\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h) \rangle\rangle v^{-s} d\mu(\tau) - A_0 \log(T) \right).$$

□

Corollary 5.5. *The regularized theta integral $\iota_* \iota^! \Phi(f, z_1, h)$ is absolutely convergent, and given by the limit*

$$\iota_* \iota^! \Phi(f, z_1, h) = \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f(\tau), v^{-1/2} \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h_1) \rangle\rangle d\mu(\tau).$$

Proof. This follows from an easy variation of the proof of Proposition 5.4, scaling each $C^{\pm}_{V_{1,2}^{\perp}}(v,h)$ by $v^{-1/2}$.

□

5.4.4. *Summing up.* Applying Lemma 5.3, we obtain the expression

$$\begin{aligned}&\sum_{(z_1,h)\in Z(V_{1,2})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\otimes\theta_{L_{1,2}}(\tau,z_1,h)\rangle\rangle d\mu(\tau)\\ &= \frac{1}{\mathrm{vol}(K_{1,2})} \int\limits_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash\,\mathrm{SO}(V_{1,2})(\mathbf{A}_f)/K_{1,2}} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\otimes\theta_{L_{1,2}}(\tau,z_1,h)\rangle\rangle d\mu(\tau).\end{aligned}$$

Applying Corollary 5.5, this is the same as

$$\frac{1}{\mathrm{vol}(K_{1,2})} \int\limits_{\mathrm{SO}(V_{1,2})(\mathbf{Q})\backslash\,\mathrm{SO}(V_{1,2})(\mathbf{A}_f)/K_{1,2}} \lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\otimes\theta_{L_{1,2}}(\tau,z_1,h)\rangle\rangle d\mu(\tau).$$

Switching the order of summation with the finite sum over h in the limiting integral, then applying the Siegel-Weil formula (Theorem 4.1, Corollary 4.2), we obtain the expression

$$\begin{aligned}&\sum_{(z_1,h)\in Z(V_{1,2})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\otimes\theta_{L_{1,2}}(\tau,z_1,h)\rangle\rangle d\mu(\tau)\\ &= \lim_{T\to\infty}\frac{2}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} \langle\langle f(\tau), v^{-1/2}\theta_{L_{1,2}^{\perp}}(\tau)\otimes E_{L_{1,2}}(\tau,1/2;-1/2)\otimes|\tau|^{-1/2}\rangle\rangle d\mu(\tau).\end{aligned}\tag{51}$$

Applying Lemma 5.2 to the functional identity (45) to obtain the relation of differential forms

$$-2\overline{\partial}E'_{L_{1,2}}(\tau,1/2;3/2)d\tau = E_{L_{1,2}}(\tau,1/2;-1/2)d\mu(\tau),$$

we then use the relation $d=\partial+\overline{\partial}$ to express (51) equivalently as

$$\begin{aligned}&-\lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2)\rangle d\tau\\ &+\lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} \langle\langle \overline{\partial}f(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2)\rangle\rangle d\tau.\end{aligned}\tag{52}$$

Using Lemma 5.2 again to evaluate

$$\overline{\partial}(fd\tau) = -v^{5/2}\xi_{-1/2}(f)d\mu(\tau) = -L_{-1/2}fd\mu(\tau)$$

for each of the truncated integrals in the second term of (52), we then obtain the equivalent expression

$$\begin{aligned}&\sum_{(z_1,h)\in Z(V_{1,2})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}\theta_{L_{1,2}}(\tau,z_1,h)\rangle\rangle d\mu(\tau)\\ &= -\lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2)\rangle\rangle d\tau\\ &-\lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} \langle\langle \xi_{-1/2}(f)(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2)\rangle\rangle v^{\frac{5}{2}}d\mu(\tau).\end{aligned}\tag{53}$$

To evaluate this latter expression (53), we first note by (43) that we can identify the shift in each integral as a shift in the s-variable of the Eisenstein series:

$$v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2) = E'_{L_{1,2}}(\tau,0;3/2).$$

We can then identify the second expression on the right-hand side of (53) as the Rankin-Selberg integral

$$\begin{aligned}&\lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} \langle\langle \xi_{-1/2}(f)(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes v^{-1/2}E'_{L_{1,2}}(\tau,1/2;3/2)\rangle\rangle v^{\frac{5}{2}}d\mu(\tau)\\ &= \lim_{T\to\infty}\frac{4}{\mathrm{vol}(K_{1,2})}\int_{\mathcal{F}_T} \langle\langle \xi_{-1/2}(f)(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes E'_{L_{1,2}}(\tau,0;3/2)\rangle\rangle v^{\frac{5}{2}}d\mu(\tau)\\ &= \frac{4}{\mathrm{vol}(K_{1,2})}\cdot\langle \xi_{-1/2}(f)(\tau), \theta_{L_{1,2}^{\perp}}(\tau)\otimes E'_{L_{1,2}}(\tau,0;3/2)\rangle\\ &= \frac{4}{\mathrm{vol}(K_{1,2})}\cdot L'(0,\xi_{-1/2}(f)\times\theta_{L_{1,2}^{\perp}}),\end{aligned}$$

where we consider the Rankin-Selberg L-function defined by the Petersson inner product

$$L(s, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) = \langle \xi_{-1/2}(f)(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes E_{L_{1,2}}(\tau, s; 3/2) \rangle.$$

We then evaluate the first integral in the expression (53) via Stokes' theorem to find that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes v^{-1/2} E'_{L_{1,2}}(\tau, 1/2; 3/2)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes E'_{L_{1,2}}(\tau, 0; 3/2)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{\partial\mathcal{F}_T} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes E'_{L_{1,2}}(\tau, 0; 3/2)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{\tau=iT}^{1+iT} \langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes E'_{L_{1,2}}(\tau, 0; 3/2)\rangle\rangle d\tau \\
&= \lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{u=0}^{1} \langle\langle f(u+iT), \theta_{L_{1,2}^{\perp}}(u+iT) \otimes E'_{L_{1,2}}(u+iT, 0; 3/2)\rangle\rangle du \\
&= \frac{4}{\mathrm{vol}(K_{1,2})} \cdot \mathrm{CT}\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes E'_{L_{1,2}}(\tau, 0; 3/2)\rangle\rangle.
\end{aligned}$$

To further evaluate this constant coefficient term, we split the derivative Eisenstein series into holomorphic and nonholomorphic parts $E'_{L_{1,2}}(\tau, 0; 3/2) = E^{+\prime}_{L_{1,2}}(\tau, 0; 3/2) + E^{-\prime}_{L_{1,2}}(\tau, 0; 3/2)$ then consider the corresponding integral following the arguments of [11, Theorem 4.7] and [14, Theorem 3.2], using the rapid decay of $E^{-\prime}_{L_{1,2}}(u+iT, 0; 3/2)$ with $T \to \infty$ to see that its contribution to the integral is negligible in the limit. Writing

$$\mathcal{E}_{L_{1,2}}(\tau) = E^{+\prime}_{L_{1,2}}(\tau, 0; 3/2)$$

for the holomorphic part, we find in this way that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{4}{\mathrm{vol}(K_{1,2})} \int_{\mathcal{F}_T} d\langle\langle f(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes v^{-1/2} E'_{L_{1,2}}(\tau, 1/2; 3/2)\rangle\rangle d\tau \\
&= \frac{4}{\mathrm{vol}(K_{1,2})} \cdot \mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle.
\end{aligned}$$

Hence, we have now shown the following

Theorem 5.6. *We have that*

$$\begin{aligned}
\iota_* \iota^! \Phi(f, Z(V_{1,2})) &:= \sum_{(z_1, h) \in Z(V_{1,2})} \int_{\mathcal{F}}^{\star} \langle\langle f(\tau), v^{-1/2} \theta_{L_{1,2}^{\perp}}(\tau) \otimes \theta_{L_{1,2}}(\tau, z_1, h)\rangle\rangle d\mu(\tau) \\
&= -\frac{4}{\mathrm{vol}(K_{1,2})} \cdot \left(L'(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) + \mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle \right).
\end{aligned}$$

Hence by Corollary 4.10 and Lemma 5.3, we compute the archimedean local height (38) *as*

$$[Z(f), Y]_{\infty} = -2\deg(Y) \cdot \left(L'(0, \xi_{-1/2}(f) \times \theta_{L_{1,2}^{\perp}}) + \mathrm{CT}\langle\langle f^{+}(\tau), \theta_{L_{1,2}^{\perp}}(\tau) \otimes \mathcal{E}_{L_{1,2}}(\tau)\rangle\rangle \right).$$

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