A NOTE ON LOCATING "TAMAGAWA MOTIVES" IN THE AUTOMORPHIC COHOMOLOGY OF GL_n

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Abstract. We consider analogues of Bloch's Tamagawa number formulation of the Birch-Swinnerton-Dyer conjecture in automorphic cohomology, with some new results and perspectives for the case of Mordell-Weil rank one.

Contents

1. Overview

Let E be an elliptic curve defined over the rational number field $\mathbf{Q}$, with $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ its corresponding cuspidal eigenform, and $\pi(f)$ the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$. Hence, writing the standard L-function of $\pi(f)$ as $\Lambda(s, \pi(f)) = L(s, \pi_\infty(f))L(s, \pi(f)) = \Lambda(s, f) = L_\infty(s, f)L(s, f)$, and the Hasse-Weil L-function of E as $L(E, s)$ we have by the modularity theorem of Wiles [48], Taylor-Wiles [47], and Breuil-Conrad-Diamond-Taylor [9] an identification of completed L-functions

$$\Lambda(E, s) = L_\infty(E, s)L(E, s) := \Lambda(s - 1/2, \pi(f)) = \Lambda(s - 1/2, f). \tag{1}$$

More generally, for f any cuspidal eigenform on $\Gamma_0(N)$ with corresponding $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f)$ and standard L-function $\Lambda(s, \pi(f)) = L(s, \pi_\infty(f))L(s, \pi(f)) = L_\infty(s, f)L(s, f)$, we write

$$r = r(f) = r(\pi) := \mathrm{ord}_{s=1/2} L(s, \pi(f)) = \mathrm{ord}_{s=1/2} L(s, f)$$

to denote the analytic rank, so the order of vanishing at the central point for the corresponding functional equation. Writing $\times$ to denote the normalized unitary induction, and inspired by the volume computations of Bloch in [5, (1.13)], we consider automorphic representations $\Pi^{(r)}(f)$ of $\mathrm{GL}_{2+r}(\mathbf{A})$ defined by

$$\Pi^{(r)}(f) := \pi(f) \times \underbrace{|\cdot|^{\frac{1}{2}} \times \cdots \times |\cdot|^{\frac{1}{2}}}_{r\text{-fold unitary induction}} . \tag{2}$$

Note that the corresponding completed L-function $\Lambda(s, \Pi(f)) = \Lambda(s, \Pi_\infty(f))L(s, \Pi(f))$ of this automorphic representation $\Pi(f)$ of $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A})^r \subset \mathrm{GL}_{2+r}(\mathbf{A})$ is given by the product

$$\Lambda(s, \Pi(f)) = \Lambda(s, \pi(f)) \cdot \Lambda(s + 1/2)^r = \Lambda(s, f) \cdot \Lambda(s + 1/2)^r$$

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of the standard L-function $\Lambda(s,\pi(f)) = \Lambda(s,f)$ of $\pi(f)$ times r-fold product of the completed Riemann zeta function $\Lambda(s) = \Gamma_{\mathbf{R}}(s)\zeta(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ shifted by 1/2. Hence, the finite part $L(s,\Pi(f))$ of this L-function given by the product

$$L(s,\Pi(f)) = L(s,\pi(f)) \cdot \zeta(s+1/2)^r = L(s,f)\zeta(s+1/2)^r.$$

Observe that we could also define the analytic rank equivalently to be the least integer $r \geq 0$ for which the corresponding value $\Lambda(1/2,\pi(f)) \cdot \Lambda(1)^r$ does not vanish. Either way, this analytic rank $r = r(f)$ is determined uniquely by the $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f)$, or equivalently by the eigenform f. The analytic rank $r = r(f)$ in then predicted by the conjecture of Birch and Swinnerton-Dyer to equal the algebraic rank of the elliptic curve, so that $E(\mathbf{Q}) \cong \mathbf{Z}^r \oplus E(\mathbf{Q})_{\mathrm{tors}}$ as finitely generated abelian groups.

1.1. **Locating twists of $\Pi(f_k)$ in the $\mathrm{GL}(2+r)$ automorphic cohomology.** Let us suppose now that $f_k \in S_k(\Gamma_0(N))$ is any cuspidal eigenform of even weight $k \geq 2$ and trivial nebentype character. We shall later consider a Hida family $\{f_k\}_k$ interpolating the base eigenform $f = f_2 \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizing an elliptic curve E of conductor N defined over $\mathbf{Q}$ with Mordell-Weil group $E(\mathbf{Q}) \cong \mathbf{Z}^r \oplus E(\mathbf{Q})_{\mathrm{tors}}$. We can then consider the family of automorphic representations

$$\Pi^{(r)}(f_k) = \pi(f_k) \times \underbrace{|\cdot|^{\frac{1}{2}} \times \cdots \times |\cdot|^{\frac{1}{2}}}_{r\text{-fold unitary induction}}$$

of the Levi subgroup $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A})^r$ of $\mathrm{GL}_{2+r}(\mathbf{A})$. We see from the definitions that each $\Pi^{(r)}(f_k)$ determines an automorphic representation of $\mathrm{GL}_{2+r}(\mathbf{A})$ which is algebraic but not cohomological (see [15, pp. 84-89], [16]). That is, replacing $\Pi(f_k)$ by its unique isobaric quotient if needed, we see following the definition given in [15, § 1.2.3] that the associated Langlands parameter is a representation of the Weil group $W_{\mathbf{R}}$ whose restriction to $W_{\mathbf{C}} \cong \mathbf{C}^\times$ is given by

$$z \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}}, \left(\frac{z}{\overline{z}}\right)^{-\left(\frac{k-1}{2}\right)}, \underbrace{(z\cdot\overline{z})^{\frac{1}{2}}, \ldots, (z\cdot\overline{z})^{\frac{1}{2}}}_{r \text{ times}} \right).$$

Thus, each $\Pi^{(r)}(f_k)$ determines an algebraic representation representation $\Pi^{(r)}(f_k) \in \mathrm{Alg}(n) = \mathrm{Alg}(2+r)$, with corresponding weights

$$p = (p_j)_{j=1}^{r+2} = \left(\frac{k-1}{2}, \frac{1-k}{2}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{r \text{ times}} \right) \quad \text{and} \quad q = (q_j)_{j=1}^{r+2} = \left(\frac{1-k}{2}, \frac{k-1}{2}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{r \text{ times}} \right).$$

While each of the weights p_j and q_j of each $\Pi(f_k)$ is half-integral, and hence not regular according to the definition given in [15, p. 84], each of the corresponding differences $p_j - q_j$ is integral. Hence, $\Pi(f_k)$ is algebraic according to the definition given in [15, pp. 89-91].

Let us now refine this setup. Let $f_k \in S_k^{\mathrm{new}}(\Gamma_0(N))$ be any newform of even weight $k \geq 2$ on $\Gamma_0(N)$. Later, we shall also assume that f_k belongs to the Hida family $\{f_k\}_k$ associated to a cuspidal newform $f = f_2 \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizing an elliptic curve E over $\mathbf{Q}$ of conductor N and Mordell-Weil rank $r = r(f) = 1$, so that $E(\mathbf{Q}) \cong \mathbf{Z} \oplus E(\mathbf{Q})_{\mathrm{tors}}$. In any case, we consider the algebraic representations

$$\Pi(f_k) := \pi(f_k) \times |\cdot|^{\frac{1}{2}} \in \mathrm{Alg}(3)$$

of the Levi subgroup $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A})$ of $\mathrm{GL}_3(\mathbf{A})$ henceforth. We can then consider cohomological twists of these representations by taking products with so-called cohomological character $|\cdot|^{1/2}$ (see [11, §5]). That is, we can consider the "cohomological twists" defined by the products

$$\Pi'(f_k) := |\cdot|^{\frac{1}{2}} \cdot \Pi(f_k) \in \mathrm{Alg}(3).$$

Since the corresponding weights are seen by inspection to be integral for $k \geq 4$ even, and hence regular for $k \geq 4$ even, each such $\Pi'(f_k)$ with $k \geq 4$ even is seen to be a cohomological automorphic representation of $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$. Note that the case of $k = 2$ is an outlier, with the algebraic representation

$\Pi'(f_2) \in \mathrm{Alg}(3)$ having non-regular weights $p = (1, 0, 1)$ and $q = (0, 1, 1)$. Hence, writing $\mathrm{Coh}(3) \subset \mathrm{Alg}(3)$ to denote the set of all cohomological automorphic representations of $\mathrm{GL}_3(\mathbf{A})$, we have that

$$\Pi'(f_k) \in \begin{cases} \mathrm{Alg}(3) \backslash \mathrm{Coh}(3) & \text{if } k = 2 \\ \mathrm{Coh}(3) & \text{if } k \geq 4 \end{cases}.$$

Here, we view both $\Pi(f_k)$ and $\Pi'(f_k)$ as algebraic representations of the Levi subgroup

$$M(\mathbf{A}) = M_{(2,1)}(\mathbf{A}) \cong \mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$$

coming from the parabolic subgroup $P = MN$ associated to the partition $2 + 1 = 3$.

Is it simple to see from definitions that the standard L-function $\Lambda(s, \Pi'(f_k))$ of each of the algebraic representations $\Pi'(f_k) \in \mathrm{Aut}(3)$ equals a shift by $s = 1/2$ of that of $\Pi(f_k) \in \mathrm{Aut}(3)$, so that

$$\Lambda(s, \Pi'(f_k)) = \Lambda(s + 1/2, \Pi(f_k)) = \Lambda(s + 1/2, \pi(f_k))\Lambda(s + 1).$$

Motivated by the main theorem of Bloch [5], and especially the volume computations given in [5, (1.13)], we first seek to locate these cohomological automorphic representations $\Pi'(f_k) \in \mathrm{Coh}(3)$ for $k \geq 4$ even in the $\mathrm{GL}_3(\mathbf{A})$ automorphic cohomology. Taking for granted the setup with symmetric spaces used by Clozel [16] and Franke-Schwermer [20] (for instance), and writing $\mathcal{L}_k$ to denote the uniquely-determined coefficient sheaf associated to each eigenform $f_k \in S_k(\Gamma_0(N))$, we seek to address the following motivating questions.

Question 1.1. *(i) Let $f = f_2 \in S_2^{\mathrm{new}}(\Gamma_0(N))$ denote the cuspidal newform of weight $k = 2$ parametrizing an elliptic curve E over $\mathbf{Q}$ of conductor N and Mordell-Weil rank $r = 1$, so that $E(\mathbf{Q}) \cong \mathbf{Z} \oplus E(\mathbf{Q})_{\mathrm{tors}}$. Let $\{f_k\}_k$ denote the Hida family of eigenforms $f_k \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of even weights $k \geq 2$ corresponding to $f = f_2$ and any ordinary prime p. To which degree in the $\mathrm{GL}_3(\mathbf{A})$-automorphic cohomology does each of the representations $\Pi'(f_k) \in \mathrm{Coh}(3)$ with $k \geq 4$ contribute, and what are the properties of the corresponding limit $\mathfrak{K}(\{f_k\}_k) = \varinjlim_{k \geq 4} \Pi'(f_k) \in \mathrm{Coh}(3)$?*

(ii) In the more general setting where $f = f_2 \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizes an elliptic curve E defined over $\mathbf{Q}$ having any Mordell-Weil rank $r \geq 1$, can we find some class in the $\mathrm{GL}_{2+r}(\mathbf{A})$-automorphic cohomology whose L-function coincides with that of the algebraic representation $\Pi^{(r)}(f) = \Pi^{(r)}(f_2)$ of $\mathrm{GL}_{2+r}(\mathbf{A})$?

Let us remark that Question 1.1 is linked to the open problem of finding a motive $M(X)$ attached to the torus extension class X – or more generally a motive $M(\mathcal{T})$ attached to the torus bundle $\mathcal{T}$ – described in the works of Bloch, [5] and [6]. To be more precise, recall that given G an algebraic group defined over $\mathbf{Q}$, we can choose a lifting to a group scheme over the S-integers $\mathbf{Z}_S$ for a finite set of places S, which allows us to define $G(\mathbf{Z}_p)$ for each finite place $p \notin S$, and in this way the group of adelic points $G(\mathbf{A})$. If $G(\mathbf{Q})$ embeds discretely into $G(\mathbf{A})$, then we can consider the classical Tamagawa number conjecture for G: Writing $L(G, s) = \prod_{v \notin S} L_v(G, s)$ to denote the corresponding L-function, let $r(G) \leq 0$ be the integer for which

$$\lim_{s \to 1} \frac{L(G, s)}{(s-1)^{r(G)}} \neq 0, \infty. \tag{3}$$

Fixing the Haar measure on $G(\mathbf{A})$ by insisting that $G(\mathbf{Z}_v)$ for each $v \notin S$ gets measure one, we then define the Tamagawa number $\tau(G)$ of G to be the corresponding volume times the limiting quantity (3),

$$\tau(G) := \mathrm{Vol}\left(G(\mathbf{A})/G(\mathbf{Q})\right) \cdot \lim_{s \to 1} \frac{L(G, s)}{(s-1)^{r(G)}}.$$

Taking for granted the folklore conjecture that the corresponding Tate-Shafarevich group

$$Ш(G) := \ker\left(H^1(\mathbf{Q}, G(\overline{\mathbf{Q}})) \longrightarrow \prod_v H^1(\mathbf{Q}_v, G(\overline{\mathbf{Q}}))\right)$$

is finite, and writing $\mathrm{Pic}(G)_{\mathrm{tors}}$ to denote the torsion subgroup of the Picard group $\mathrm{Pic}(G)$, the Tamagawa number conjecture for G asserts that the Tamagawa number $\tau(G)$ can be computed as the quotient

$$\tau(G) = \frac{\#\,\mathrm{Pic}(G)_{\mathrm{tors}}}{\#Ш(G)}. \tag{4}$$

On the other hand, recall that for E an elliptic curve of conductor N defined over $\mathbf{Q}$ as above, the conjecture of Birch and Swinnerton-Dyer predicts that $r = r(E) = r(f) = \mathrm{rk}_{\mathbf{Z}}\, E(\mathbf{Q})$. The refined conjecture of Birch and Swinnerton-Dyer predicts in addition to this that we have the class-number-like formula

$$\lim_{s \rightarrow 1} \frac{\Lambda(E, s)}{(s-1)^r} = \frac{\#\text{Ш}(E) \cdot \det\langle\ ,\rangle \cdot V_{\infty} \cdot V_{\mathrm{bad}}}{\# E(\mathbf{Q})_{\mathrm{tors}} \cdot \#\operatorname{Pic}(E)_{\mathrm{tors}}} = \frac{\#\text{Ш}(E) \cdot \det\langle\ ,\rangle \cdot V_{\infty} \cdot V_{\mathrm{bad}}}{\# E(\mathbf{Q})^2_{\mathrm{tors}}}.$$

Here, writing $\langle\ ,\ \rangle : E(\mathbf{Q}) \times E(\mathbf{Q}) \longrightarrow \mathbf{R}$ to denote the Néron-Tate height pairing (as reconstructed in [5]),

$$\text{Ш}(E) := \ker \left(H^1(\mathbf{Q}, E(\overline{\mathbf{Q}})) \longrightarrow \prod_v H^1(\mathbf{Q}_v, E(\overline{\mathbf{Q}})) \right)$$

denotes the Tate-Shafarevich group (which again is conjectured to be finite),

$$\det\langle\ ,\rangle := \det\left(\langle P_i, P_j \rangle\right)_{i,j}, \quad \{P_i\}_i \text{ any basis of } E(\mathbf{Q})/E(\mathbf{Q})_{\mathrm{tors}}$$

the regulator,

$$V_{\infty} = \operatorname{Vol}\left(E \otimes_{\mathbf{Q}} \mathbf{R}\right)$$

denotes the real period, and

$$V_{\mathrm{bad}} := \operatorname{Vol}\left(\prod_{\substack{v \in S \\ v < \infty}} E(\mathbf{F}_v) \right)$$

denotes the product of local Tamagawa factors at primes of bad reduction for E. Bloch shows in [5] that each $\alpha \in \operatorname{Pic}(E)$ corresponds to a $\mathbf{G}_m$-torsor $X_{\alpha} \longrightarrow E$, which for $\alpha \in \operatorname{Pic}^0(E) = E(\mathbf{Q})$ can be thought of as a group extension of E by $\mathbf{G}_m$, and in this way builds an extension

$$0 \longrightarrow T \longrightarrow X \longrightarrow E \longrightarrow 0, \tag{5}$$

where T denotes the $\mathbf{Q}$-split torus with character group $X^*(T) = E(\mathbf{Q})/E(\mathbf{Q})_{\mathrm{tors}}$. We have the following important theorem, which is shown in [5] more generally for abelian varieties over number fields.[1]

Theorem 1.2 (Bloch)**.** *The extension X defined via the exact sequence* (5) *determines an algebraic group over $\mathbf{Q}$, with rational points $X(\mathbf{Q})$ embedding discretely into the adelic points $X(\mathbf{A})$, with Tate-Shafarevich group $\text{Ш}(X) = \text{Ш}(E)$, and with $\operatorname{Pic}(X)_{\mathrm{tors}} = E(\mathbf{Q})_{\mathrm{tors}}$. Moreover, the refined conjecture of Birch and Swinnerton-Dyer for E is true if and only if the Tamagawa number conjecture for X is true, equivalently if and only if the Tamagawa number of X is given by $\tau(X) = E(\mathbf{Q})_{\mathrm{tors}}/\text{Ш}(E) \in \mathbf{Q}\backslash\{0\}$.*

It would be interesting to give a completely automorphic description of the extension class X, leading to the underlying automorphic motive $M(X)$ (e.g. through the corresponding Galois representation). In particular, the identification of such an automorphic Tamagawa motive $M(X)$ would reduce the remaining open cases of the conjecture of Birch and Swinnerton-Dyer for $E(\mathbf{Q})$ *for any rank* $r \geq 0$ to the corresponding Bloch-Kato main conjecture [7] for $M(X)$. This would offer some framework for approaching the remaining open cases of Birch-Swinnerton-Dyer, namely via the Bloch-Kato (or Iwasawa-Greenberg) main conjectures for $M(X)$. In particular, if $M(X)$ could be viewed as the Galois representation $\rho_{M(X)}$ associated to some automorphic form, then it would be reasonable to expect progress on the corresponding Bloch-Kato main conjecture for $\rho_{M(X)}$ though some Euler system construction. The Bloch-Kato main conjecture for this $M(X)$ (a variant of the Tamagawa number conjecture for X) would then imply the conjecture of Birch and Swinnerton-Dyer for E. In this direction, we pose another question.

Question 1.3. *Let E be an elliptic curve of conductor N defined over $\mathbf{Q}$ and Mordell-Weil rank r, with corresponding eigenform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ and $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f)$. Let $\{f_k\}_k$ denote a Hida family of eigenforms $f_k \in S_k(\Gamma_0(N))$ of higher even weights $k \geq 2$ specializing to $f = f_2$ for any ordinary prime p. Can the corresponding family of algebraic representations $\Pi^{(r)}(f_k) \in \mathrm{Alg}(2{+}r)$ be used with deformation theoretic methods to construct a suitable extension class $M(f)$ associated to the corresponding*

[1] In fact, Bloch shows the analogue of the theorem stated here for any abelian variety A defined over a global field k of characteristic zero.

Galois representations $\rho_{\pi(f)} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_l)$ *or* $\rho_{\Pi'(f)} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_{2+r}(\overline{\mathbf{Q}}_l)$*, or perhaps even the "Tamagawa motive"* $M(X)$ *itself?*

The phrase "deformation theoretic methods" refers to something like the techniques of Skinner-Urban [44] and [43]. These works attach to a newform $f(z) = \sum_{n \geq 1} a_f(n)e(nz) \in S_2(\Gamma_0(N))$ of odd analytic rank – equivalently to any newform $f \in S_2(\Gamma_0(N))$ which is invariant under the Fricke involution w_N – and any prime p for which f is ordinary (i.e. for which $|a_f(p)|_p = 1$) an extension of the corresponding p-adic Galois representation V_f using Galois representations $\rho_{\mathcal{V}}$ associated to Siegel modular forms which are congruent modulo high powers of p to the Saito-Kurakawa lift $\mathrm{SK}(f)$ of f. The ideas underlying this construction are rooted in the approach of Harder [27], who used a geometric approach with Eisenstein cohomology to give a (conditional) construction of similar extension classes. They can also be extended to the case of even analytic rank via unitary groups, as outlined in Skinner-Urban [46] and Bellaïche-Chenevier [4]. While these works broadly construct extension classes related to the Bloch-Kato Selmer group of some twist of the p-adic Galois representation V_f constructed from f, we seek to address the more basic question of the existence or provenance of an automorphic motive $X = X(f)$ corresponding to Bloch's extension class. Latently, the idea is to relate difficult open problems to finding suitable integral presentations for the derivative central values $L^{(r)}(1/2, \pi) = L^{(r)}(E, s)$ for any Mordell-Weil rank r (particularly $r \geq 2$) to the better-understood study of central values $L(1/2, \Pi) = L(1, X)$ of some automorphic form on a higher-rank group such as $\mathrm{GL}_{2+r}(\mathbf{A})$.

1.2. **Summary of results.** We establish the following for Question 1.1 (i). Here, we use the constructions of Galois representations due to to Scholze [39] and Clozel [16] to determine the degree to which the cohomological twists $\Pi'(f_k) \in \mathrm{Coh}(3)$ (with $k \geq 4$ even) contributes. We then consider such classes in a Hida family corresponding to f to specialize to weight $k = 2$, and in this way address Question 1.1 (i). Let us first remark that we are not able to realize this cohomological twist $\Pi'(f_k) \in \mathrm{Coh}(3)$ via unnormalized induction from the algebraic representation $\Pi'(f_k)$ of the Levi subgroup $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A})$ of $\mathrm{GL}_3(\mathbf{A})$ in these constructions. Rather, we use the constructions of [39] and [16] to view the $\mathrm{GL}_3(\mathbf{A})$ locally symmetric space as a Levi component in an $\mathrm{Sp}_6(\mathbf{A})$ Shimura variety; we then use Poincaré duality with the vanishing theorems of Lan-Suh [34] to determine the degrees to which the cohomological representations $\Pi'(f_k) \in \mathrm{Coh}(3)$ with $k \geq 4$ even contribute. The framework of [39] and [16] with the boundary of the Borel-Serre compactification of an ambient Shimura variety applies to any cohomological representation $\Pi \in \mathrm{Coh}(3)$ and more generally $\mathrm{Coh}(n)$ (for instance). However, somewhat awkwardly, we must first realize our cohomological representations $\Pi'(f_k) \in \mathrm{Coh}(3)$ (for $k \geq 4$ even) as the unnormalized inductions of certain non-algebraic representations of the Levi subgroup $M_{2,1} \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \subset \mathrm{GL}_3$ to describe precisely how our questions fit into this framework.

To fix ideas, let us start with the algebraic representation $\Pi(f_k)$ of $M_{2,1}(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$ defined for any $f_k \in S_k(\Gamma_0(N))$ ($k \geq 2$ even) by the normalized parabolic induction

$$\Pi(f_k) := \pi(f_k) \times |\cdot|^{1/2}.$$

Each such representation $\Pi(f_k) \in \mathrm{Alg}(3)$ has Langlands parameter

$$z \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}}, \left(\frac{z}{\overline{z}}\right)^{-\left(\frac{k-1}{2}\right)}, (z\overline{z})^{\frac{1}{2}} \right)$$

and algebraic weights

$$p(\Pi(f_k)) = \left(\frac{k-1}{2}, \frac{1-k}{2}, \frac{1}{2} \right) \quad \text{and} \quad q(\Pi(f_k)) = \left(\frac{1-k}{2}, \frac{k-1}{2}, \frac{1}{2} \right).$$

Again, we consider the twist $\Pi'(f_k) := |\cdot|^{\frac{1}{2}} \cdot (\pi(f_k) \times |\cdot|^{\frac{1}{2}})$, which has Langlands parameter

$$z \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}} (z\overline{z})^{\frac{1}{2}}, \left(\frac{z}{\overline{z}}\right)^{-\left(\frac{k-1}{2}\right)} (z\overline{z})^{\frac{1}{2}}, (z\overline{z}) \right)$$

and weights

$$p(\Pi'(f_k)) = \left(\frac{k}{2}, \frac{2-k}{2}, 1 \right) \quad \text{and} \quad q(\Pi'(f_k)) = \left(\frac{2-k}{2}, \frac{k}{2}, 1 \right).$$

Since the weights are integral for each $k \equiv 0 \bmod 2$ with distinct entries for each $k \geq 4$, each corresponding representation $\Pi'(f_k)$ of $M_{(2,1)}(\mathbf{A}) \cong \mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$ is cohomological for $k \geq 2$ even. In all cases, $\Pi'(f_k)$ has the corresponding standard L-function $\Lambda(s, \Pi'(f_k)) = \Lambda(s+1/2, \pi(f_k))\Lambda(s+1)$.

Unfortunately, the normalized parabolic induction $\Pi(f_k)$ and its cohomological twist $\Pi'(f_k)$ do not fit well into the constructions of Galois representations given in [16] and [39] (via unnormalized induction). As we explain in Proposition 2.6 below, we can realize the cohomological twist $\Pi'(f_k) := \Pi(f_k) \times |\cdot|^{\frac{1}{2}}$ equivalently as the unnormalized parabolic induction of a non-algebraic representation $\Pi_0''(f_k) := |\cdot| \left(\pi(f_k) \times |\cdot|^{-1}\right)$. Here, starting with a cuspidal eigenform $f_k \in S_2(\Gamma_0(N))$ with even weight $k \geq 2$, we first consider the representation $\Pi_0(f_k)$ of the Levi subgroup $M_{2,1}(\mathbf{A}) \cong \mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$ defined by

$$\Pi_0(f_k) := \pi(f_k) \times |\cdot|^{-1}.$$

Hence, this $\Pi_0(f_k)$ determines a non-algebraic representation of $M_{2,1}(\mathbf{A})$, with Langlands parameter

$$z \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}}, \left(\frac{\overline{z}}{z}\right)^{\frac{k-1}{2}}, (z\overline{z})^{-1} \right)$$

and weights

$$p(\Pi_0(f_k)) = \left(\frac{k-1}{2}, \frac{1-k}{2}, -1\right) \quad \text{and} \quad q(\Pi_0(f_k)) = \left(\frac{1-k}{2}, \frac{k-1}{2}, -1\right).$$

Similarly, the twist $\Pi_0''(f_k) := |\cdot| \cdot \Pi_0(f_k) = |\cdot| \cdot (\pi(f_k) \times |\cdot|^{-1})$ determines a non-algebraic representation of $M_{2,1}(\mathbf{A})$, with Langlands parameter

$$z \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}} \cdot (z\overline{z}), \left(\frac{\overline{z}}{z}\right)^{\frac{k-1}{2}} \cdot (z\overline{z}), (z\overline{z})^0 \right)$$

and weights

$$p(\Pi_0''(f_k)) = \left(\frac{k+1}{2}, \frac{3-k}{2}, 0\right) \quad \text{and} \quad q(\Pi_0''(f_k)) = \left(\frac{3-k}{2}, \frac{k+1}{2}, 0\right).$$

However, the unnormalized parabolic induction $\Pi^\star(f_k) = \chi_{\Pi_0''(f_k)}$ of $\Pi_0''(f_k)$ recovers the cohomological twist $\Pi'(f_k)$, and has the desired L-function $\Lambda(s, \Pi^\star(f_k)) = \Lambda(s, \chi_{\Pi_0''(f_k)}) = \Lambda(s, \Pi'(f_k)) = \Lambda(s, \pi(f_k))\Lambda(s+1/2)$.

When the weight $k \geq 4$ of $f \in S_k(\Gamma_0(N))$ is even, the construction of [39] (and more generally [16]) allows us to realize the representation $\Pi^\star(f_k) = \Pi'(f_k) \in \mathrm{Coh}(3)$ in the cohomology of the boundary of the Borel-Serre compactification of an ambient $\mathrm{Sp}_6(\mathbf{A})$ Shimura variety[2]. We use this construction to deduce that each $\Pi^\star(f_k) = \Pi'(f_k) \in \mathrm{Coh}(3)$ contributes nontrivially to either $H^1(S_K, \mathcal{L}_k)$ or $H_c^2(S_K, \mathcal{L}_k)$, where S_K denotes the corresponding locally symmetric space for $\mathrm{GL}_3(\mathbf{A})$. The corresponding symmetric space for the $\mathrm{Sp}_6(\mathbf{A})$ Shimura variety has a larger dimension than that of $\mathrm{GL}_3(\mathbf{A})$. This in particular allows us to use vanishing theorems such as those of Lan-Suh [34] (cf. [13], [14]) with Poincaré duality to deduce the vanishing of the $H_c^2(S_K, \mathcal{L}_k)$, and hence the following result.

Theorem 1.4 (Theorem 2.19). *Let $f_k \in S_k(\Gamma_0(N))$ be a cuspidal eigenform of even weight $k \geq 4$, with corresponding $\mathrm{GL}_2(\mathbf{A})$-automorphic representation denoted by $\pi(f_k)$. Obtained via unnormalized induction of the non-algebraic representation $\Pi_0''(f_k)$ defined above, the cohomological representation $\Pi^\star(f_k) = \Pi'(f_k)$ of $\mathrm{GL}_3(\mathbf{A})$ factors through the automorphic cohomology in the first degree $H^1(S_K, \mathcal{L}_k)$, giving rise to a class in $\mathfrak{K}(f_k) \in H^1(S_K, \mathcal{L}_k)$.*

Corollary 1.5. *Let E be an elliptic curve of conductor N defined over $\mathbf{Q}$, parametrized by a cuspidal newform $f(z) = \sum_{n \geq 1} a_f(n)e(nz) \in S_2^{\mathrm{new}}(\Gamma_0(N))$. Fix a prime p for which which f is ordinary, so $|a_f(p)|_p = 1$, and*

[2]More generally, if we pass to a totally real or CM number fields as we may with these arguments, these constructions lead to ambient Shimura varieties attached to the symplectic group $\mathrm{Sp}_{2(2+r)}$ over a totally real field, or to the unitary group $U(2+r, 2+r)$ over a CM field.

let $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{Z}_p)$ denote the corresponding p-adic Galois representation. Let $\{f_k\}_k$ denote the Hida family associated to ρ_f, with weight-two specialization $f_2 = f$. Taking the direct limit

$$\mathfrak{K}(\{f_k\}_k) = \varinjlim_{k \geq 4} \mathfrak{K}(f_k)$$

of the corresponding cohomology classes $\mathfrak{K}(f_k)$ in $\varinjlim_k H^1(S_K, \mathcal{L}_k)$, we specialize to weight $k=2$ to obtain a class $\mathfrak{K}(f) = \mathfrak{K}(f_2)$ in the $\mathrm{GL}_3(\mathbf{A})$ completed cohomology $\varprojlim_K \varinjlim_k H^1(S_K, \mathcal{L}_k)$ corresponding to the algebraic representation $\Pi'(f) = \Pi'(f_2) \in \mathrm{Alg}(3)$ with standard L-functions $\Lambda(s, \Pi'(f)) = \Lambda(s+1/2, \Pi(f))$, where $\Lambda(s, \Pi(f)) = \Lambda(s, \pi(f))\Lambda(s+1/2)$.

So, while we cannot realize the algebraic representation $\Pi(f) = \Pi(f_2) \in \mathrm{Alg}(3)$ or its twist $\Pi'(f) = \Pi'(f_2)$ in the $\mathrm{GL}_3(\mathbf{A})$-automorphic cohomology $H^\star(S_K, \mathbf{C})$, this does give rise via the limit $\mathfrak{K}(\{f_k\}_k) = \varinjlim_{k\geq 4} \mathfrak{K}(f_k)$ to a class in the completed cohomology for $\mathrm{GL}_3(\mathbf{A})$, in the sense of Calegari-Emerton (see e.g. [12], [18])[3]. It would be interesting to investigate these classes in completed cohomology further, as well the more general setting of classes arising from the boundaries of the Borel-Serre compactifications of Sp_{2n} Shimura varieties, with the aim of constructing p-adic realizations $M_p(X)$ of the desired extension class $M(X)$.

We conclude with some more remarks about motivation, and the connection to periods. Suppose the underlying elliptic curve E has Mordell-Weil rank $r = 1$, and hence that f has analytic rank $r(f) = 1$ by various theorems on cyclotomic main conjectures towards the conjecture of Birch and Swinnerton-Dyer – namely those of Kato [31], Rohrlich [38], Skinner-Urban [46], and Kolyvagin [32]. Hence equivalently, the central value of the corresponding induced L-function $\Lambda(s, \Pi(f)) = \Lambda(s, \pi(f))\Lambda(s+1/2)$ does not vanish, i.e. as $\Lambda(1/2, \pi(f)) = (E/\mathbf{Q}, 1) = 0$. In this setting, we know by the theorem of Gross-Zagier [23, (7.3)] (cf. [24]) that for some rational point $P \in E(\mathbf{Q})$, we have

$$\Lambda'(E/\mathbf{Q}, 1) = \Lambda'(1/2, \pi(f)) = \Lambda(1/2, \Pi(f)) = A \cdot \Omega \cdot \langle P, P \rangle, \tag{6}$$

where $\Omega = V_\infty$ denotes the real period of a regular differential operator on $E/\mathbf{Q}$, and A is some nonzero rational number. To be more precise, since we know that $\Lambda'(E/\mathbf{Q}, 1) = \Lambda'(1/2, \pi(f)) = \Lambda(1/2, \Pi(f)) \neq 0$, we deduce that from [23, Theorem (7.3)] that $P \in E(\mathbf{Q})/E(\mathbf{Q})_{\mathrm{tors}}$ is a generator. Using various Euler characteristic calculations (see [30] and [49] with [10, Theorem 3.10]), we can also deduce that, up to powers of 2 and 3, this rational number A is given by the Birch-Swinnerton-Dyer constant

$$\frac{\#\text{Ш}(E) \cdot V_{\mathrm{bad}}}{\#E(\mathbf{Q})_{\mathrm{tors}}^2} \in \mathbf{Q}.$$

Now, we know by the argument given in Kontsevich-Zagier [33, §3.5] that the product $A \cdot \Omega \cdot \langle P, P \rangle$ is a period. Hence from (6), we deduce that the central value $\Lambda(1/2, \Pi(f))$ is a period. It would be interesting if we could use the mysterious completed completed class $\mathfrak{K}(\{f_k\}_k)$ to give a different calculation of this period $A \cdot \Omega \cdot \langle P, P \rangle$, in the style of the arguments of [3] and Kontsevich-Zagier [33, §3.5] applied to the central value $\Lambda(1/2, \Pi(f)) = \Lambda(1/2, \Pi^{(1)}(f))$, perhaps after some suitable interpretation of the class $\mathfrak{K}(\{f_k\}_k)$ in a p-adic completion of the Eisenstein cohomology (cf. [27]). We can ask the same question more generally. That is, if E is an elliptic curve defined over $\mathbf{Q}$ with any Mordell-Weil rank $r = r_E(\mathbf{Q}) \geq 0$, we know from our initial setup that we have an identification of central (derivative) values

$$\Lambda^{(r)}(E/\mathbf{Q}, 1) = \Lambda^{(r)}(1/2, \pi(f)) = \Lambda(1/2, \Pi^{(r)}(f)), \tag{7}$$

and that the refined conjecture of Birch and Swinnerton-Dyer predicts an identification

$$\Lambda^{(r)}(E/\mathbf{Q}, 1) = \Lambda^{(r)}(1/2, \pi(f)) = \Lambda(1/2, \Pi^{(r)}(f)) \stackrel{?}{=} \frac{\#\text{Ш}(E) \cdot V_{\mathrm{bad}}}{\#E(\mathbf{Q})_{\mathrm{tors}}^2} \cdot \Omega \cdot \det\langle , \rangle. \tag{8}$$

[3]Note that Theorem 1.4 and Corollary 1.5 can be extended naturally to totally real and CM number fields. That is if E is a modular elliptic curve over a totally real field F, then arguments for Theorem 1.4 remain the same, with the ambient Shimura variety attached to $\mathrm{Sp}_6(\mathbf{A}_F)$. If E is a modular elliptic curve over a CM field F, then the arguments for Theorem 1.4 remain almost the same after replacing the ambient Shimura variety attached to the unitary group $U(3,3)(\mathbf{A}_F)$. The version of Corollary 1.5 for the corresponding $\mathrm{GL}_2(\mathbf{A}_F)$-automorphic representation is then given similarly. If F is a totally real field, then this class is realized in the cohomology of the Borel-Serre boundary of a $\mathrm{Sp}_6(\mathbf{A}_F)$ Shimura variety $S_{\underline{K}}(\mathrm{Sp}_6(\mathbf{A}_F))$. If F is a CM field, then it is realized in the cohomology of the Borel-Serre boundary of a $U(3,3)(\mathbf{A}_F)$ Shimura variety $S_{\underline{K}}(U(3,3)(\mathbf{A}_F))$.

The value on the right-hand side of (8) is shown to be a period in [33, §4]. On the other hand, the central derivative values $\Lambda^{(r)}(E/\mathbf{Q}, 1) = \Lambda^{(r)}(1/2, \pi(f))$ are conjectured to be periods (e.g. [33, Question 4]). It would be interesting to develop the ideas of Beilinson [3] and Kontsevich-Zagier [33, § 3.5] to show that the central values $\Lambda(1/2, \Pi^{(r)}(f))$ of the $\mathrm{GL}_{2+r}(\mathbf{A})$-automorphic L-function $\Lambda(1/2, \Pi^{(r)}(f))$ are periods to this end, perhaps using p-adic analogues of the constructions from Eisenstein cohomology as in the approach of Harder [27], and p-adic realizations in the completed cohomology as suggested by our discussion above. However, we save these tasks for some future work.

Outline. We first review the construction of Clozel [16] generalizing Scholze [39] in § 2, then explain the relation to ambient Shimura varieties as developed in [39] and [1] in §3. This leads to some discussion of Poincaré duality and vanishing theorems (and conjectures) for the cohomology of the locally symmetric spaces we consider. The main argument then appears Theorem 2.19.

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2. Symmetric spaces, Shimura varieties, and Borel-Serre compactifications

Here, we first describe the relevant construction of Eisenstein cohomology classes given in [16], [39], and [1]. We then describe some more general results including various vanishing theorems (and conjectures) before giving our main arguments.

2.1. **Symmetric spaces.** Let G be any reductive group over $\mathbf{Q}$, and $P = MN$ any parabolic $\mathbf{Q}$-subgroup. Following [16], we let $A = A_G$ denote the neutral component in the group of real points of some split maximal central torus of G, and $K_\infty \subset G(\mathbf{R})$ the maximal compact subgroup. We shall also write $K \subset G(\mathbf{A}_f)$ to denote a fixed compact open subgroup. We consider the cohomology of the symmetric spaces

$$X_G = G(\mathbf{R})/A_G K_\infty,$$

as well as their quotients $\Gamma\backslash X_G$ by congruence subgroups $\Gamma \subset G(\mathbf{Q})$. In fact, we consider the adelic analogues of these symmetric spaces given by the double coset spaces

$$S_K = G(\mathbf{Q})\backslash G(\mathbf{A})/A_G K_\infty K,$$

each of which can be written as a finite union of "classical" spaces of the form $\Gamma\backslash X_G$.

2.1.1. *Cohomology of symmetric spaces.* We consider the cohomology $H^*(S_K, \mathbf{C})$ of the symmetric spaces S_K, noting that these can be thought of naturally as components the profinite limit

$$H^*(S, \mathbf{C}) = \varprojlim_{K \subset G(\mathbf{A}_f)} H^*(S_K, \mathbf{C})$$

over all compact open subgroups $K \subset G(\mathbf{A}_f)$ (see e.g. [20]). Theorems of Borel and Franke [19] show that these spaces can be computed in terms of automorphic forms via the following relation to relative Lie cohomology. Writing $\mathfrak{g}$ to denote the Lie algebra of G, and $\mathcal{A}(G_K)$ the space of automorphic forms on

$$G_K := G(\mathbf{Q})\backslash G(\mathbf{A})/A_G K,$$

these theorems (e.g. [19, Theorem 18]) give an identification of cohomology groups

$$H^*(S_K, \mathbf{C}) \cong H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G_K)). \tag{9}$$

Here, the $(\mathfrak{g}, K_\infty)$-cohomology on the right hand side is defined via the action of $G(\mathbf{R})$ by right multiplication. This identification (9) can be further stratified thanks to the Langlands direct sum decomposition

$$\mathcal{A}(G_K) = \bigoplus_{P \in \mathcal{C}} \mathcal{A}_P(G_K) \tag{10}$$

into the classes $\mathcal{C} = \{P\}$ of associate $\mathbf{Q}$-parabolic subgroups $P \subset G$, where each $\mathcal{A}_P(G_K) \subset \mathcal{A}(G_K)$ denotes the subspace of forms obtained via induction from $P \subset G$. That is, we obtain from (10) a decomposition

$$H^*(S_K, \mathbf{C}) \cong \bigoplus_{P \in \mathcal{C}} H^*(\mathfrak{g}, K_\infty, \mathcal{A}_P(G_K)). \tag{11}$$

We refer to the discussions in [20], [22], and [16] for more details about this latter decomposition (11) in general, i.e. which we present here only in a simplified form for illustration. In brief, when the subspace of cuspidal forms of $\mathcal{A}(G_K)$ is sufficiently well-understood, we also have the decomposition of the part relative to G,

$$H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G_K)) \cong H^*(\mathfrak{g}, K_\infty, \mathcal{A}_{\mathrm{cusp}}(G_K)) \cong \bigoplus_{\pi \subset L^2_{\mathrm{cusp}}(G_K)} H^*(\mathfrak{g}, K_\infty, \pi).$$

Here, $\mathcal{A}_{\mathrm{cusp}}(G_K) \subset \mathcal{A}(G_K)$ denotes the subspace of cuspidal forms, and the sum runs over cuspidal summands $\pi \subset L^2(G_K)$ in the space of L^2-automorphic forms $L^2(G_K)$ on the double quotient space G_K.

2.1.2. *Approaches to constructions of classes.* Note that Harder and Schwermer have proposed a programme for constructing cohomology classes associated to each $\mathcal{A}_P(G_{K_f})$ for $P = MN \neq G$ in terms of differential-form-valued Eisenstein series induced from cuspidal forms on the corresponding Levi subgroup $M \subset P$. Harder obtained important results in this direction for the case of $G = \mathrm{GL}_2$ ([25], [26], [28], [29]), and more general constructions for $G = \mathrm{GL}_n$ are given by Schwermer [42], [41] and Franke-Schwermer [20]; see also the discussion in Grbac [22]. The approach of Scholze [39] and its generalization to general reductive groups given by Clozel [16] differs from these works in that it is purely topological, using properties of the boundary of the Borel-Serre compactifications of the symmetric spaces S_K to given novel constructions of such classes. However, without further innovation, these topological constructions appear to be limited to the special case where $P \subset G$ is a maximal parabolic subgroup. In particular, the discussion in Clozel [16, §5] shows why the construction is a priori degenerate (vanishing) when the parabolic subgroup $P \subset G$ is not maximal.

2.1.3. *Parabolics associated to the partitions* $2 + r = 2 + 1 + \ldots 1$. Let us for future reference spell out the choices of groups which appear in our motivating examples. As indicated above, we consider the linear reductive group $G = \mathrm{GL}_n$ with $n = 2 + r \geq 2$ (and $r \geq 0$ fixed). The corresponding parabolic subgroup $P = MN \subset G$ we consider above is associated to the partition $(2, 1, \ldots, 1)$ of $n = 2 + r$. Hence, we have the more explicit Levi decomposition

$$P_{(2,1,\ldots,1)} = M_{(2,1,\ldots,1)} N_{(2,1,\ldots 1)} \subset \mathrm{GL}_{2+r},$$

where the Levi subgroup $M = M_{(2,1,\ldots,1)}$ can be identified in a natural way with the product $M \cong \mathrm{GL}_2 \times \mathrm{GL}_1^r$, and the unipotent radical $N = N_{(2,1,\ldots,1)}$ with the subgroup of the standard unipotent subgroup $N_n = N_{2+r}$ of upper triangular matrices given on adelic points by

$$N(\mathbf{A}) = N_{(2,1,\ldots,1)}(\mathbf{A}) = \left\{ y = \begin{pmatrix} 1 & 0 & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & u_{2,4} & \cdots & u_{2,n} \\ & & 1 & u_{3,4} & \cdots & u_{3,n} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & u_{n-1,n} \\ & & & & & 1 \end{pmatrix} : u_{i,j} \in \mathbf{A} \right\} \subset N_n(\mathbf{A}).$$

Hence, we have the semi-direct product decomposition

$$N_n \cong N_2 \ltimes N_{(2,1,\cdots 1)}. \tag{12}$$

Let us remark that the unipotent radical $N = N_{(2,1,\ldots,1)}$ implicit in this example is the same unipotent radical $N = N_{(2,1,\ldots,1)} = Y_{n,1}$ appearing in the classical works of Ginzburg, Jacquet, Piatetski-Shapiro, and Shalika on Rankin-Selberg L-functions for $\mathrm{GL}_n \times \mathrm{GL}_m$ (our setup corresponding to the case with $m = 1$), and also crucially in the proofs of converse theorems for GL_n-automorphic L-functions shown by Cogdell. We refer to the exposition in Cogdell [17] (for instance) for more background on this latter topic, and note that the integrals over the quotients of these unipotent subgroups

$$N(\mathbf{Q}) \backslash N(\mathbf{A}) = N_{(2,1,\ldots,1)}(\mathbf{Q}) \backslash N_{(2,1,\ldots,1)}(\mathbf{A}) = Y_{n,1}(\mathbf{Q}) \backslash Y_{n,1}(\mathbf{A})$$

correspond up to a factor of $|\cdot|^{\frac{n-2}{2}}$ to the classical projection operator $\mathbb{P}_1^n$. These integrals satisfy various remarkable properties, determining L^2-automorphic forms on the mirabolic subgroup $P_2(\mathbf{A}) \subset \mathrm{GL}_2(\mathbf{A})$, and moreover preserve the L-function coefficient of cuspidal forms on $\mathrm{GL}_n(\mathbf{A})$ through their Fourier-Whittaker expansions, i.e. as L^2-automorphic forms on the mirabolic subgroup $P_2(\mathbf{A}) \subset \mathrm{GL}_2(\mathbf{A})$.

2.2. **Cohomology of Borel-Serre compactifications of symmetric spaces.** Let us now describe the constructions of [39] and [16] for any reductive group G over $\mathbf{Q}$.

2.2.1. *Borel-Serre compactifications of symmetric spaces.* Given a $\mathbf{Q}$-parabolic subgroup $P \subset G$, we write $P = MN$ to denote the Levi decomposition, with $M \subset P$ the Levi subgroup, and $N \subset P$ the unipotent radical. Let us fix a compact open subgroup $K \subset G(\mathbf{A}_f)$. We shall assume this compact open subgroup decomposes as a direct product $K = \prod_{p<\infty} K_p$, with each $K_p \subset G(\mathbf{Q}_p)$ a subgroup. We shall also assume that K is neat, in the strict sense[4]. That is, recall that an element $g = (g_p)_p \in G(\mathbf{A}_f)$ is said to be *neat* if for any faithful representation ρ of G over $\mathbf{Q}$, fixing an embedding of $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$ for each prime p, and writing Γ_p to denote the torsion subgroup $\overline{\mathbf{Q}}_p$ generated by the eigenvalues of g_p, we have that

$$\bigcap_{p<\infty} \Gamma_p = \{1\}\,.$$

We then say that K is *neat (in the strict sense)* if each element $g = (g_p)_p \in K$ is neat. Fixing such a neat compact open subgroup $K \subset G(\mathbf{A}_f)$, we shall then write K_H for any connected linear algebraic subgroup or quotient H of G to denote the corresponding intersection $K_H := K \cap H(\mathbf{A}_f)$.

Let us now return to the symmetric space defined by

$$S_K = G(\mathbf{Q})\backslash G(\mathbf{A})/A_G K_\infty K = G(\mathbf{Q})\backslash \left(X_G \times G(\mathbf{A})\right)/K,$$

which recall can be written in classical terms as a finite disjoint union

$$S_K = \coprod_i \Gamma_i \backslash X_G \tag{13}$$

of quotients of X_G by congruence subgroups $\Gamma_i \subset G(\mathbf{Q})$. Note that our choice of $\mathbf{Q}$-parabolic $P = MN \subset G$ gives rise to the corresponding symmetric spaces

$$S_{K_P} = P(\mathbf{Q})\backslash P(\mathbf{A})/A_M K_{\infty,M} K_P$$

and

$$S_{K_M} = M(\mathbf{Q})\backslash M(\mathbf{A})/A_M K_{\infty,M} K_M.$$

Here, viewing M as a reductive group, we write A_M to denote the neutral component of the set of $\mathbf{R}$-points of the maximal split $\mathbf{Q}$-torus in the centre of M, and $K_{\infty,M} = M(\mathbf{R})$ the maximal compact subgroup. These spaces have equivalent archimedean descriptions analogous to (13). We also have a natural projection map

$$\pi_P : S_{K_P} \longrightarrow S_{K_M}$$

for which the quotient S_K of S_K can be realized as a fibre bundle with compact fibre $N(\mathbf{Q})\backslash N(\mathbf{A})/K_N$, where $K_N = K \cap N(\mathbf{A}_f)$ is the corresponding (neat) compact open subgroup.

To describe the Borel-Serre bordification X_G^{BS} of X_G (in the terminology of [8]), let $\mathcal{P}$ to denote the set of $\mathbf{Q}$-parabolic subgroups $P \subset G$, and $\mathcal{P}' \subset \mathcal{P}$ the subset of proper parabolic subgroups. We then define

$$X_G^{\mathrm{BS}} = \coprod_{P \in \mathcal{P}} e(P).$$

Here, the product runs over all $\mathbf{Q}$-parabolic subgroups $P \subset G$, and $e(G) = X_G$. The boundary $\partial X_G^{\mathrm{BS}}$ of this manifold with corners X_G^{BS} is given by the subproduct over proper $\mathbf{Q}$-parabolic subgroups

$$\partial X_G^{\mathrm{BS}} = \coprod_{P \in \mathcal{P}'} e(P).$$

[4]In the looser, classical sense: $K \subset G(\mathbf{A}_f)$ is neat if the intersection $K \cap G(\mathbf{Q})$ has no nontrivial torsion element. Note that the strict neat compact open subgroup form a basis of all compact open subgroups of $G(\mathbf{A}_f)$.

Here, for each proper parabolic subgroup $P \in \mathcal{P}'$, we write $e(P)$ to denote a copy of X_G/A_P with $A_P = A_M$, with the "geodesic flow" of $A_P = A_M$ commuting with the left action by $P(\mathbf{R})$.

Let us now consider similar notions for the symmetric space S_K. Here, we define

$$S_K^{\mathrm{BS}} = G(\mathbf{Q})\backslash \left(X_G^{\mathrm{BS}} \times G(\mathbf{A}_f)\right)/K.$$

As explained in [16, §2], since the quotient $G(\mathbf{A})/K$ is discrete, and since $G(\mathbf{Q})$ acts with no fixed points, we deduce that this space S_K^{BS} is also a manifold without corners. Writing

$$G(\mathbf{A}_f) = \coprod_g G(\mathbf{Q})gK$$

to denote the finite decomposition underling the decomposition (13), we can write S_K^{BS} equivalently as

$$S_K^{\mathrm{BS}} = \coprod_g \Gamma_g \backslash X_G^{\mathrm{BS}}, \quad \Gamma_g := G(\mathbf{Q}) \cap gKg^{-1}.$$

We have the following decompositions of the bordification S_K^{BS}. Note that the set of $\mathbf{Q}$-parabolic subgroups of G modulo $G(\mathbf{Q})$ is finite. Let us fix a set of representatives $\mathcal{Q}$ for this finite set. We then argue that we can write S_K^{BS} equivalently as the finite product

$$S_K^{\mathrm{BS}} = \coprod_{P \in \mathcal{Q}} S_K^{\mathrm{BS}}(P), \quad S_K^{\mathrm{BS}}(P) := G(\mathbf{Q})\backslash \left(e(P) \times G(\mathbf{A}_f)\right)/K. \tag{14}$$

Here, each of the components $S_K^{\mathrm{BS}}(P)$ can be thought of as the contribution of the parabolic P to S_K^{BS}. Moreover, in passing to the limit $K_f \to 1$, the corresponding limits of cohomology groups $H^*(S_K^{\mathrm{BS}}(P))$ can be seen as the realization of an induced representation of $G(\mathbf{A}_f)$ from $P(\mathbf{A}_f)$.

Now, we can decompose each of the contributions $S_K^{\mathrm{BS}}(P)$ in (14) as follows. Notice that the double coset space $P(\mathbf{Q})\backslash G(\mathbf{A}_f)/K$ is finite, as $P(\mathbf{A}_f)\backslash G(\mathbf{A}_f)$ can be viewed as the set of $\mathbf{A}_f$-points of a projective variety. We may thus fix a set of representatives $\mathcal{R}(P)$ for this space to derive the corresponding decomposition

$$S_K^{\mathrm{BS}}(P) = \coprod_{h \in \mathcal{R}(P)} P(\mathbf{Q})\backslash \left(e(P) \times P(\mathbf{A}_f)\right)/K_P(h), \quad K_P(h) := P(\mathbf{A}_f) \cap hKh^{-1}. \tag{15}$$

On the other hand, using properties of the "geodesic action" (according to the description in [16, (2.1)]), we can identify each $e(P) = X_G/A_G$ in this latter expression (15) in terms of the parabolic subgroup as

$$e(P) = X_G/A_G = \left(P(\mathbf{R})K_\infty/K_\infty\right)/A_P = P(\mathbf{R})/A_M K_{\infty,M}.$$

In this way, we can rewrite the decomposition (15) as

$$S_K^{\mathrm{BS}}(P) = \coprod_{h \in \mathcal{R}(h)} S_{K_P(h)}, \quad S_{K_P(h)} := P(\mathbf{Q})\backslash P(\mathbf{A})/A_M K_{\infty,M} K_P(h). \tag{16}$$

Here, each component $S_{K_P(h)}$ can be viewed as the symmetric space corresponding to $K_P(h) \subset P(\mathbf{A}_f)$. Hence in summary, we get the decomposition of S_K^{BS} into a finite disjoint of symmetric spaces

$$S_K^{\mathrm{BS}} = \coprod_{P \in \mathcal{Q}} S_K^{\mathrm{BS}}(P) = \coprod_{P \in \mathcal{Q}} \coprod_{h \in \mathcal{R}(P)} S_{K_P(h)}. \tag{17}$$

Writing $\mathcal{Q}' \subset \mathcal{Q}$ to denote the representatives corresponding to maximal $\mathbf{Q}$-parabolic subgroups $P \subset G$, the boundary $\partial S_K^{\mathrm{BS}}$ can then be described as

$$\partial S_K^{\mathrm{BS}} = \coprod_{P \in \mathcal{Q}'} S_K^{\mathrm{BS}}(P) = \coprod_{P \in \mathcal{Q}'} \coprod_{h \in \mathcal{R}(P)} S_{K_P(h)}. \tag{18}$$

Finally, let us note that

$$\dim \partial S_K^{\mathrm{BS}} = \dim X_G - 1.$$

If P is a $\mathbf{Q}$-parabolic with split component $A_P = A_M$, then then

$$\dim e(P) = \dim X_G - \left(\dim A_P - \dim A_G\right).$$

As well, for $P \in \mathcal{Q}'$ maximal, the quotients $S_K^{\mathrm{BS}}(P)$ correspond to the open cells of the boundary $\partial S_K^{\mathrm{BS}}$.

2.2.2. *Actions of unramified Hecke algebras.* Let us retain the setup described above, and write S to denote the finite set of places for which $K_p \subset G(\mathbf{Q}_p)$ is not hyperspecial. Let $K^S = \prod_{p \notin S, p < \infty} K_p$, and similarly $\mathbf{A}_f^S = \prod_{p \notin S, p < \infty} \mathbf{Q}_p$. We then consider the corresponding unramified Hecke algebras of bi-invariant functions

$$\begin{aligned} \mathcal{H}^S(G) &= \mathcal{C}_c\left(K^S \backslash G(\mathbf{A}_f^S)/K^S, \mathbf{Z}\right) = \bigotimes_{p \notin S} \mathcal{H}_p(G), \quad \mathcal{H}_p(G) := \mathcal{C}_c\left(K_p \backslash G(\mathbf{Q}_p)/K_p, \mathbf{Z}\right), \\ \mathcal{H}^S(P) &= \mathcal{C}_c\left(K_P^S \backslash P(\mathbf{A}_f^S)/K_P^S, \mathbf{Z}\right) = \bigotimes_{p \notin S} \mathcal{H}_p(P), \quad \mathcal{H}_p(G) := \mathcal{C}_c\left(K_{p,P} \backslash G(\mathbf{Q}_p)/K_{p,P}, \mathbf{Z}\right) \\ \mathcal{H}^S(M) &= \mathcal{C}_c\left(K_M^S \backslash M(\mathbf{A}_f^S)/K_M^S, \mathbf{Z}\right) = \bigotimes_{p \notin S} \mathcal{H}_p(M), \quad \mathcal{H}_p(G) := \mathcal{C}_c\left(K_{p,M} \backslash G(\mathbf{Q}_p)/K_{p,M}, \mathbf{Z}\right), \end{aligned} \tag{19}$$

where the decompositions on the right hand sides denote the usual restricted tensor products in each case. Here, for a given prime p, we also write $K_{p,P} = K_p \cap P(\mathbf{Q}_p)$ and $K_{p,M} = K_p \cap M(\mathbf{Q}_p)$.

Let us assume that the set of representatives $\mathcal{Q}$ of $\mathbf{Q}$-parabolic subgroups modulo $G(\mathbf{Q})$ coincides with the set of standard parabolic subgroups, i.e. those parabolic subgroups which contain a fixed minimal parabolic subgroup P_0. This assumption allows us for each prime p and for each $\mathbf{Q}$-parabolic representative $P \in \mathcal{Q}$ to find a maximal subgroup $K_p^0 \subset G_p(\mathbf{Q}_p)$ for which

$$G(\mathbf{Q}_p) = K_p^0 P(\mathbf{Q}_p) = P(\mathbf{Q}_p) K_p^0.$$

We shall then assume that our fixed compact open $K \subset G(\mathbf{A}_f)$ can be decomposed into local components $K = \prod_{p < \infty} K_p$ with $K_p = K_p^0$ for all $p \notin S$. This latter assumption means that each of the subgroups and subquotients $P = MN$ we consider comes equipped with a natural $\mathbf{Z}_p$-structure for each $p \notin S$. We can also choose local Haar measures on each of these subgroups and subquotients so that the volumes of $\mathbf{Z}_p$-points equal one for all $p \notin S$.

We have the following morphisms between these algebras. On the one hand, we have the natural map

$$\rho : \mathcal{H}^S(G) \longrightarrow \mathcal{H}^S(P)$$

given by restriction of functions. On the other hand, we have the unnormalized constant term map

$$\lambda : \mathcal{H}^S(P) \longrightarrow \mathcal{H}^S(M), \quad \varphi \longmapsto \lambda\varphi(m) := \int_{N(\mathbf{A}_f)} \varphi(mn) dn \quad (m \in M(\mathbf{A}_f^S)).$$

Let us now consider cohomology, taking the coefficients over any local Artin ring κ[5]. Each of the Hecke algebras (19) acts in a natural way on the cohomology groups $H^*(S_K^{\mathrm{BS}}, \kappa)$. For instance, if for a given prime p we consider the double coset operator $K_p g K_p \in \mathcal{H}_p(G)$ for some $g \in G(\mathbf{Q}_p)$, then the action can be given explicitly in terms of correspondences for the bordifications as follows: Writing R_g to denote the action by right translation of g on $S_K^{\mathrm{BS}} = G(\mathbf{Q}) \backslash \left(X_G^{\mathrm{BS}} \times G(\mathbf{A}_f)\right)/K$, we have the commutative diagram

$$\begin{array}{ccc} S^{\mathrm{BS}}_{K \cap gKg^{-1}} & \xrightarrow{R_g} & S^{\mathrm{BS}}_{g^{-1}Kg \cap K} \\ \downarrow & & \downarrow \\ S_K^{\mathrm{BS}} & & S_K^{\mathrm{BS}}. \end{array} \tag{20}$$

Let us also note that this action of $\mathcal{H}^S(G)$ respects each of the decompositions above leading to (16). In particular, the action of $\mathcal{H}^S(G)$ on $H^*(\partial S_K^{\mathrm{BS}}, \kappa)$ is compatible with the map $H^*(S_K^{\mathrm{BS}}, \kappa) \longrightarrow H^*(\partial S_K^{\mathrm{BS}}, \kappa)$.

2.2.3. *Main construction for maximal parabolic subgroups.* Let us assume now that $P \in \mathcal{Q}'$ is a maximal $\mathbf{Q}$-parabolic subgroup of G, with Levi decomposition $P = MN$ and fixed set of representatives $\mathcal{R}(P) = \{h\}$ for the finite set $P(\mathbf{Q}) \backslash G(\mathbf{A}_f)/K$. As explained in [16], the components $S_K^{\mathrm{BS}}(P)$ and $S_{K_P(h)}$ are then open in $\partial S_K^{\mathrm{BS}}$, and we obtain in this way a map

$$j_* : H_c^i(S_{K_P(h)}, \kappa) \longrightarrow H^i(\partial S_K^{\mathrm{BS}}, \kappa).$$

[5]The arguments described here work for any coefficient ring, and although we are ultimately most interested in the case where κ is a number field, this more general setup allows us to consider coefficients $\kappa = \mathbf{Z}/l^n\mathbf{Z}$ as in Scholze [39].

We also have a natural projection map

$$\pi : S_{K_P} \longrightarrow S_{K_M},$$

whose pullback has compact fibres. In this way, we deduce that the pullback π^* gives the map on cohomology

$$\pi^* : H^i_c(S_{K_M}, \kappa) \longrightarrow H^i_c(S_{K_P}, \kappa).$$

Lemma 2.1. *The maps j_* and π^* defined above satisfy the following properties.*

(i) *The map $j_* : H^i(S_{K_{P(h)}}, \kappa) \longrightarrow H^i(\partial S_K^{\mathrm{BS}}, \kappa)$ is equivariant under the action of $\mathcal{H}^S(G)$, with $\mathcal{H}^S(G)$ acting naturally on $H^i(\partial S_K^{\mathrm{BS}}, \kappa)$, and via composition with the restriction map ρ on $H^i(S_{K_{P(h)}}, \kappa)$.*

(ii) *The map $\pi^* : H^i_c(S_{K_M}, \kappa) \longrightarrow H^i(S_{K_P}, \kappa)$ is equivariant under the action of $\mathcal{H}^S(P)$, with $\mathcal{H}^S(P)$ acting naturally on $H^i(S_{K_P}, \kappa)$, and via composition with the constant term map λ on $H^i(S_{K_M}, \kappa)$.*

Proof. See [16, Lemma 2.1] or [39, Lemma V.2.3]. □

Let us now consider the "diagram"

$$H^i_c(S_{K_P}, \kappa) \xrightarrow{\ j_*\ } H^i(\partial S_K^{\mathrm{BS}}, \kappa) \xrightarrow{\ j^*\ } H^i(S_{K_P}, \kappa),$$

writing the composition of these maps as

$$j_c := j^* j_* : H^i_c(S_{K_P}, \kappa) \longrightarrow H^i(S_{K_P}, \kappa).$$

Observe that the image of this map lies in the compactly supported part of the cohomology, i.e. in the inner cohomology $H^i_!(S_{K_P}, \kappa)$. In this way, we see that the image of j_* gives us a surjective map

$$\mathrm{Im}(j_*) \longrightarrow H^i_!(S_{K_P}, \kappa). \tag{21}$$

We can now consider the "full diagram"

$$\begin{array}{ccc} H^i_c(S_{K_P}, \kappa) & \xrightarrow{\ j_c\ } & H^i(S_{K_P}, \kappa) \\ \uparrow \pi^* & & \downarrow \pi_* \\ H^i_c(S_{K_M}, \kappa) & \xrightarrow{\ j_c^M\ } & H^i(S_{K_M}, \kappa). \end{array} \tag{22}$$

Here, the vertical map π_* on the right-hand side is described more precisely as follows. A priori, the pushforward π_* determines an identification $H^*(S_{K_P}, \kappa) = H^*(S_{K_M}, R\pi_*\kappa)$. However, as explained in [16, (2.2)], this cohomology group is associated to the local system $H^*(\mathcal{F})$, where $\mathcal{F} = \Gamma_N \backslash N(\mathbf{R})$ is the fibre of the projection $\pi : S_{K_P} \to S_{K_M}$, which can be covered by the contractible space $\widetilde{\mathcal{F}} \cong N(\mathbf{R})$ to obtain a map of local systems $H^*(\mathcal{F}) \to H^*(\widetilde{\mathcal{F}}) = \kappa$ on S_{K_M}. In this way, we see that π_* induces a map $\pi_* : H^i(S_{K_P}, \kappa) \longrightarrow H^*(S_{K_M}, \kappa)$.

Lemma 2.2. *The map $\pi_* : H^i(S_{K_P}, k) \longrightarrow H^i(S_{K_M}, \kappa)$ is equivariant under the action of $\mathcal{H}^S(P)$, with $\mathcal{H}^S(P)$ acting naturally on $H^i(S_{K_P}, \kappa)$, and via composition with the constant term map λ on $H^i(S_{K_M}, \kappa)$.*

Proof. See [16, Lemma 2.2]. □

Now, observe that the by the commutativity of (22), the map $\pi_* : H^*(S_{K_P}, \kappa) \longrightarrow H^*(S_{K_M}, \kappa)$ must be surjective. We can deduce the following more general result via the fundamental diagram (22).

Corollary 2.3. *There exists a surjective map*

$$\mathrm{Im}(j_*) \longrightarrow H^i_!(S_{K_M}, \kappa),$$

where again j_* denotes the map $j_* : H^i(S_{K_P}, \kappa) \longrightarrow H^i(\partial S_K^{\mathrm{BS}}, \kappa)$. Moreover, this surjective map is equivariant for the action of $\mathcal{H}^S(G)$, where $\mathcal{H}^S(G)$ acts naturally on $\mathrm{Im}(j_*)$, and via composition with $\lambda \circ \rho$ on $H^i_!(S_{K_M}, \kappa)$.

Let us at last describe the crux of the argument, which involves the long exact sequence for the cohomology of the manifold with boundary $\partial S_K^{\mathrm{BS}}$,

$$\cdots \longrightarrow H_c^i(S_K,\kappa) \longrightarrow H^i(S_K^{\mathrm{BS}},\kappa) \longrightarrow H^i(\partial S_K^{\mathrm{BS}},\kappa) \longrightarrow H_c^{i+1}(S_K,\kappa) \longrightarrow \cdots . \tag{23}$$

Note that the submodule

$$H := j_* H_c^i(S_{K_P},\kappa) \subset H^i(\partial S_K^{\mathrm{BS}},\kappa)$$

admits a filtration

$$0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0,$$

with

$$H' \subset H^i(S_K^{\mathrm{BS}},\kappa) = H^i(S_K,\kappa) \quad \text{and} \quad H'' \subset H_c^{i+1}(S_K,\kappa).$$

Let us also note that all the cohomology spaces we consider here are finite over κ. Writing $\mathfrak{m} \subset \kappa$ to denote the maximal ideal, each module H admits a (finite) filtration

$$0 = \mathfrak{m}^r H \subset \mathfrak{m}^{r-1} H \subset \ldots \subset \mathfrak{m} H \subset H,$$

with each successive quotient being stable under the action of $\mathcal{H}^S(G)$. That is, writing $\mathbf{F}_\kappa = \kappa/\mathfrak{m}$ to denote the residue field, $\mathcal{H}^S(G)$ acts on each quotient via $\mathcal{H}^S(G) \otimes \mathbf{F}_\kappa$. Let us write $\mathfrak{m}_{\mathcal{H}} \subset \mathcal{H}^S(G) \otimes \mathbf{F}_\kappa$ to denote a maximal ideal. We can now deduce the following more substantial result via Corollary 2.3.

Theorem 2.4 (Clozel, d'après Scholze)**.** *The following assertions are true.*

(i) *Each irreducible subquotient of the $\mathcal{H}^S(G)$-submodule $H_!^i(S_{K_M},\kappa)$ is a subquotient of either $H^i(S_K,\kappa)$ or $H_c^{i+1}(S_K,\kappa)$.*

(ii) *Given any maximal ideal $\mathfrak{m}_{\mathcal{H}} \subset \mathcal{H}^S(G) \otimes \mathbf{F}_\kappa$ (not necessarily non-Eisenstein) for which the localization $H_!^i(S_{K_M},\kappa)_{\mathfrak{m}_{\mathcal{H}}}$ does not vanish, either $H^i(S_K,\kappa)_{\mathfrak{m}_{\mathcal{H}}} \neq 0$ or $H_c^{i+1}(S_K,\kappa)_{\mathfrak{m}_{\mathcal{H}}} \neq 0$.*

Proof. See [16, Theorem 2.4 and Theorem 2.5], and also [39, Corollary 5.2.4]. □

Hence, we have summarized the main construction of classes of [39] and [16]. We refer to [16, § 3, Theorem 3.4 and Proposition 3.5] for a detailed account of the setup with complex coefficients (summarized below), as well as to [16, §4] for a description of the compatibility of this construction with Eisenstein cohomology, and to [16, §5] for a discussion of the degenerate case where the $\mathbf{Q}$-parabolic subgroup P is not maximal, and in particular a proof that the corresponding composition map $j^* j_*$ in that case vanishes.

2.2.4. *Complex coefficients.* Since it is relevant to Question 1.1, let us give a brief description of the general construction given above for complex coefficients. Note however we can also deduce a relevant version of Theorem 2.4 to this end after rationalization of each of the cohomology groups, using that the ambient groups can be computed in terms of automorphic forms on G. In any case, we give some more details for the abstract classical theorems in this setting, again following Clozel [16, §3].

Again, we fix a (neat) compact open subgroup $K \subset G(\mathbf{A}_f)$, together with a maximal $\mathbf{Q}$-parabolic subgroup $P = MN$. Let us assume we are also given a complex algebraic representation L of G. This gives rise in a natural way to a local system $\mathcal{L}$ on S_K. Since the bordification S_K^{BS} defines a homotopy equivalence, it is easy to see that $\mathcal{L}$ extends to a local system on S_K^{BS}. Moreover, it is not hard to justify that most of the discussion above carries over the with Artinian local ring k replaced by the local system $\mathcal{L}$. Here, we have a natural map

$$H^i(S_K^{\mathrm{BS}},\mathcal{L}) \longrightarrow H^i(\partial S_K^{\mathrm{BS}},\mathcal{L}).$$

The total space of the vector bundle $\mathcal{L}$ on S_K is given by

$$G(\mathbf{Q}) \backslash (X \times G(\mathbf{A}_f)/K) \times L, \tag{24}$$

with $G(\mathbf{Q})$ acting diagonally on each component. The action R_g of an element $g \in G(\mathbf{A}_f)$ via right translation can be described accordingly, in the usual way [6]. We obtain the same equivariant actions of the unramified Hecke algebra $\mathcal{H}^S(G)$ on the cohomology groups $H^*(S_K^{\mathrm{BS}}, \mathcal{L})$ and $H^*(\partial S_K^{\partial}, \mathcal{L})$ as described above, with corresponding equivariant map $j_* : H^i(S_{K_P}, \mathcal{L}) \longrightarrow H^i(\partial S_K^{\mathrm{BS}}, \mathcal{L})$.

In this generality, Harder and Schwermer describe $H^*(S_{K_P}, \mathcal{L})$ as a cohomology space on S_{K_M} via the degeneracy of the Leray spectral sequence for the fibration $S_{K_P} \longrightarrow S_{K_M}$ by compact nilmanifolds. Let us again write the Levi decomposition as $P = MN$, and $\mathfrak{n}$ for the Lie algebra of the unipotent radical N. For each integer $j \geq 0$, the cohomology group $H^j(\mathfrak{n}, L)$ determines an $M(\mathbf{R})$-module, and hence a local system $\mathcal{H}^j(\mathfrak{n}, L)$ on S_{K_M}. Theorems of Harder [25] and Schwermer [42] show that we have an $\mathcal{H}^S(M)$-equivariant decomposition

$$H^i(S_{K_P}, \mathcal{L}) \cong \bigoplus_{j+k=i} H^k(S_{K_M}, \mathcal{H}^j(\mathfrak{n}, L)) \tag{25}$$

on compactly supported cohomology, which is also equivariant under the action of $\mathcal{H}^S(P)$ ([16, Lemma 3.2]).

The construction of classes in this setting works in the same way as described above, incorporating the decomposition (26). That is, we obtain the from the "diagram"

$$H_c^i(S_{K_P}, \mathcal{L}) \xrightarrow{\ j_*\ } H^i(\partial S_{K_P}^{\mathrm{BS}}, \mathcal{L}) \xrightarrow{\ j^*\ } H^i(S_{K_P}, \mathcal{L})$$

a surjective map of $\mathcal{H}^S(G)$-modules $\mathrm{Im}(j_*) \longrightarrow H_!^i(S_{K_P}, \mathcal{L})$, which can be described more precisely in terms of the decomposition (25) as a $\mathcal{H}^S(G)$-equivarient surjective map

$$\mathrm{Im}(j_*) \longrightarrow H_!^i(S_{K_P}, \mathcal{L}) \cong \bigoplus_{i=j+k} H^k(S_{K_M}, \mathcal{H}^j(\mathfrak{n}, L)). \tag{26}$$

We then consider the corresponding long exact cohomology sequence

$$\cdots \longrightarrow H_c^i(S_K, \mathcal{L}) \longrightarrow H^i(S_K^{\mathrm{BS}}, \mathcal{L}) \longrightarrow H^i(\partial S_K^{\mathrm{BS}}, \mathcal{L}) \longrightarrow H_c^{i+1}(S_K, \mathcal{L}) \longrightarrow \cdots$$

for this decomposition to deduce that the subquotients of (26) on which $\mathcal{H}^S(G)$ acts via the composition $\lambda \circ \rho$ occur in either $H^i(S_K, \mathcal{L})$ or $H_c^{i+1}(S_K, \mathcal{L})$. Here, we can give the following abstract version of Theorem 2.4, described in terms of characters of the Hecke algebra $\mathcal{H}^S(G)$ rather than maximal ideals. Let us write

$$G_K = G(\mathbf{Q}) \backslash G(\mathbf{A}) / A_G K,$$

so that $S_K = G_K / K_\infty$. We then consider the usual inclusion of $G(\mathbf{R})$-representations given by the inclusion of the space of L^2-cuspidal automorphic forms on G_K inside the discrete spectrum

$$L^2_{\mathrm{cusp}}(G_K) \subset L^2_{\mathrm{dis}}(G_K),$$

with corresponding inclusion of spaces of automorphic forms denoted by

$$\mathcal{A}_{\mathrm{cusp}}(G_K) \subset \mathcal{A}_{\mathrm{dis}}(G_K).$$

Taking $(\mathfrak{g}, K_\infty)$-cohomology here gives the corresponding cuspidal and L^2-cohomology groups respectively. Here, the so-called L^2-cohomology $H^*_{(2)}(S_K)$ corresponding to the discrete spectrum can be represented by L^2-harmonic forms of S_K, and we have the inclusions $H^*_{\mathrm{cusp}}(S_K) \subset H^i_!(S_K) \subset S^*_{(2)}(S_K)$. Taking coefficients $\mathcal{L}$, we can then describe the cuspidal cohomology on S_K and S_M in terms of relative Lie cohomology,

$$H^*_{\mathrm{cusp}}(S_K, \mathcal{L}) = H^*(\mathfrak{g}, K, \mathcal{A}_{\mathrm{cusp}}(G_K) \otimes L), \quad H^*_{\mathrm{cusp}}(S_{K_M}, \mathcal{L}) \quad = H^*(\mathfrak{m}, \mathcal{A}_{\mathrm{cusp}}(G_{K_M}) \otimes L).$$

[6] In brief, put $K' = K \cap gKg^{-1}$ and $K'' = g^{-1}Kg \cap K$. Given a point $y \in S_{K'}$, we can construct a map $L_y \longrightarrow L_{R_g y}$ as follows. Write a representative of L_y in (24) as $(x, h) \times L$ for $(x, h) \in X \times G(\mathbf{A}_f)$ a representative of y. The map R_g is then given by the rule $(x, h) \longmapsto (x, hg)$. In a similar way, $L_{R_g y}$ can be represented by $(x, hg) \times L$. The map $(x, h, l) \longmapsto (x, hg, l)$ for $l \in L$ is then seen to descend to the quotient (24), hence giving the the desired map. In this way, we can study of the corresponding commutative diagram to extend most of the construction leading to Theorem 2.4 to this setting, i.e. by a similar study of

$$\begin{array}{ccc} S_{K'}^{\mathrm{BS}} & \xrightarrow{\ R_g\ } & S_{K''}^{\mathrm{BS}} \\ \downarrow & & \downarrow \\ S_K^{\mathrm{BS}} & & S_K^{\mathrm{BS}}. \end{array}$$

Here, we write $\mathfrak{m}$ to denote the Lie algebra of M, and

$$G_{K_M} = M(\mathbf{Q})\backslash M(\mathbf{A})/A_M K_M.$$

We can now state the following version of Theorem 2.4, according to [16, Theorem 3.4 and Proposition 3.5].

Theorem 2.5. *Let G be any reductive group over $\mathbf{Q}$, with $P = MN$ any maximal $\mathbf{Q}$-parabolic subgroup, and L any complex algebraic representation of G. The following assertions hold for the setup described above.*

(i) *Let χ be a character of $\mathcal{H}^S(M)$ which occurs nontrivially in $H^k_!(S_{K_M}, \mathcal{H}^j(\mathfrak{n}, L))$ for some integers $j, k \geq 0$. Putting $i = j + k$, the character of $\mathcal{H}^S(G)$ defined by $\chi' = \chi \circ (\lambda \circ \rho)$ occurs nontrivially in either $H^i(S_K, \mathcal{L})$ or $H^{i+1}_c(S_K, \mathcal{L})$.*

(ii) *If χ occurs in the cuspidal cohomology $H^k_{\mathrm{cusp}}(S_{K_M}, \mathcal{H}^j(\mathfrak{n}, L)) = H^k(\mathfrak{m}, K_M, \mathcal{A}_{\mathrm{cusp}}(G_M) \otimes H^j(\mathfrak{n}, L))$, then the character $\chi' = \chi \circ (\lambda \circ \rho)$ of $\mathcal{H}^S(G)$ occurs in either $H^i(S_K, \mathcal{L})$ or $H^{i+1}_c(S_K, \mathcal{L})$.*

(iii) *If χ again is a character of $\mathcal{H}^S(M)$ with corresponding character $\chi' = \chi \circ (\lambda \circ \rho)$ of $\mathcal{H}^S(G)$, then we have the following dimension bound for isotypic components (for any degree $i \geq 0$):*

$$\dim H^i(S_K, \mathcal{L})_{\chi'} + \dim H^{i+1}_c(S_K, \mathcal{L})_{\chi'} \geq \sum_{i=j+k} \dim H^k_!(S_{K_M}, \mathcal{H}^j(\mathfrak{n}, L))_{\chi}.$$

2.2.5. *Langlands parameters and weights.* Let us now consider the corresponding automorphic character χ of Theorems 2.4 and 2.5 for the setup we consider. The definition here carries some subtlety in that these constructions are given in terms of unnormalized[7] induction. Let L_k denote the complex algebraic representation of $\mathrm{GL}_3(\mathbf{Q})$ determined by the archimedean component $\Pi(f_k)'_\infty$ of the cohomological twist $\Pi(f_k)'$. Note that this is determined by the weight of the underlying eigenform $f_k \in S_k(\Gamma_0(N))$. We write $\mathcal{L}_k$ to denote the corresponding coefficient sheaf. Let us consider the map

$$j_* : H^i_c(S_{K_P}, \mathcal{L}_k) \longrightarrow H(\partial S_K^{\mathrm{BS}}, \mathcal{L}_k),$$

as well as the surjection

$$\mathrm{Im}(j_*) \longrightarrow H^i_!(S_{K_P}, \mathcal{L}_k).$$

Note that we have the decomposition

$$H^i_!(S_{K_P}, \mathcal{L}_k) = \bigoplus_{i=j_1+j_2} H^{j_1}_!(S_{K_M}, \mathcal{H}^{j_2}(\mathfrak{n}, L_k)).$$

We write $\alpha \in H^i(S_{K_P}, \mathcal{L}_k)$ to denote the generalized eigenclass associated to the character $\chi = \lambda \circ \rho$, as in [16, Theorem 3.7]. We decompose the symmetric space S_{K_M} for the Levi subgroup $M = M_{2,1} \subset \mathrm{GL}_3$ into the corresponding product

$$S_{K_M} = S_{M,1} \times S_{M,2}; \quad S_{M,j} := \mathbf{R}^\times_{>0} \backslash \mathrm{GL}_j(\mathbf{A})/K_{j,\alpha}K \quad \text{for each of } j = 1, 2.$$

Since the first factor $S_{M,1}$ is finite, we consider the cohomology in degree zero, using the Künneth formula for the product. Let us now consider the $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f_k)$ determined by the cuspidal eigenform $f_k \in S_k(\Gamma_0(N))$, with $\pi(f_k) \times |\cdot|^{\frac{1}{2}}$ the representation of $\mathrm{GL}_3(\mathbf{A})$ defined by unitary induction on the second factor $S_{M,2}$, which we deduce constrains us to looking at $j_1 = 1$ and $H^{j_2}(\mathfrak{n}, L_k)$ as a representation of M. That is, we consider the corresponding decomposition

$$H^1_!(S_{K_P}, \mathcal{L}_k) = H^0(S_{M,1}, \mathcal{L}_k) \oplus H^1_!(S_{M,2}, \mathcal{L}_k).$$

We shall see later that the cohomological twist $\Pi(f_k)'$ contributes nontrivially to $H^1_!(S_{K_P}, \mathcal{L}_k)$, and hence to $\mathrm{Im}(j_*)$. The associated character of the Hecke algebra $\mathcal{H}_{\mathrm{GL}_3}$ is obtained from that of $\mathcal{H}_M$ via the composition $\chi_{\mathrm{GL}_3} = \chi_M \circ \lambda \circ \rho$. Note that the constant term map $\lambda : \mathcal{H}_P \to \mathcal{H}_M$ is unnormalized, and hence that the character of $\mathcal{H}_{\mathrm{GL}_3}$ occurring in $\mathrm{Im}(j_*)$ will be given correspondingly by unnormalized induction, in that it

[7] I am grateful to Laurent Clozel for pointing this out to me.

will give a representation of $\mathrm{GL}_3(\mathbf{A})$ of the form $\mathrm{Ind}_P^{\mathrm{GL}_3}(\pi(f_2) \otimes \mathbf{1})$. Now, writing δ_P to denote the module of P, we know that the unitary induction $\mathrm{ind}(*)$ is related to this unnormalized induction by the relation

$$\mathrm{ind}(*) = \mathrm{Ind}_P^{\mathrm{GL}_3}(* \otimes \delta_P^{\frac{1}{2}}). \tag{27}$$

To work out the relation (27) to this unnormalized induction out explicitly, we first recall that δ_P is given locally for any $m \in M$ by the rule $\delta_P(m) := |\det(\mathrm{Ad}\, m|\mathfrak{n})|$. Here, we write

$$M = \left\{ \begin{pmatrix} g & \\ & z \end{pmatrix} : g \in \mathrm{GL}_2, z \in \mathrm{GL}_1 \right\} \quad \text{and} \quad \mathfrak{n} = \left\{ Y := \begin{pmatrix} 0 & X \\ & 0 \end{pmatrix}, X \in F^2 \right\},$$

so that via the relation

$$mYm^{-1} = \begin{pmatrix} 0 & gXz^{-1} \\ & 0 \end{pmatrix}$$

we have

$$\delta_P(m) = |\det(g)| \cdot |z|^{-2} \quad \Longrightarrow \quad \delta_P(m)^{\frac{1}{2}} = |\det(g)|^{\frac{1}{2}} \cdot |z|^{-1}.$$

In this way, we deduce that

$$\mathrm{Ind}_P^{\mathrm{GL}_3}(\Pi(f_k) \otimes \mathbf{1}) = \mathrm{ind}\left(\Pi(f_k) \otimes \mathbf{1} \otimes |\det(*)|^{-\frac{1}{2}} \otimes |\cdot|\right).$$

Hence, the Langlands parameter of the character $\chi_{\pi(f_k) \times |\cdot|^{\frac{1}{2}}}$ associated to the (normalized) unitary induction $\pi(f_k) \times |\cdot|^{\frac{1}{2}}$ is in fact given by the character

$$z \in W_{\mathbf{C}} \cong \mathbf{C}^\times \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}} \cdot (z\overline{z})^{-\frac{1}{2}}, \left(\frac{\overline{z}}{z}\right)^{\frac{k-1}{2}} \cdot (z\overline{z})^{-\frac{1}{2}}, (z\overline{z}) \right)$$

with weights (in z and $\overline{z}$ respectively) given by

$$p(\chi_{\pi(f_k) \times |\cdot|^{\frac{1}{2}}}) = \left(\frac{k-2}{2}, -\frac{k}{2}, 1\right) \quad \text{and} \quad q(\chi_{\pi(f_k) \times |\cdot|^{\frac{1}{2}}}) = \left(-\frac{k}{2}, \frac{k-2}{2}, 1\right).$$

Hence, this character $\chi_{\pi(f_k) \times |\cdot|^{\frac{1}{2}}}$ of the Hecke algebra $\mathcal{H}_{\mathrm{GL}_3}$ arising in the constructions of Theorems 2.4 and 2.5 determines a cohomological automorphic representation $\sigma(f_k)$ of $\mathrm{GL}_3(\mathbf{A})$, with corresponding L-function

$$\Lambda(s, \sigma(f_k)) = \Lambda(s - 1/2, \pi(f_k)) \cdot \Lambda(s+1).$$

This is *not* the L-function $\Lambda(s, \Pi(f_2)) = \Lambda(s, \pi(f_k))\Lambda(s+1/2)$ we want to consider for arithmetic applications.

On the other hand, we can consider the character $\chi_{\Pi_0''(f_k)}$ of $\mathcal{H}_{\mathrm{GL}_3}$ associated with the non-algebraic representation $\Pi_0''(f_k) := |\cdot| \cdot (\pi(f_k) \times |\cdot|^{-1})$. This representation $\Pi_0''(f_k)$ has Hodge-Tate weights

$$p(\Pi_0''(f_k)) = \left(\frac{k+1}{2}, \frac{3-k}{2}, 0\right) \quad \text{and} \quad q(\Pi_0''(f_k)) = \left(\frac{3-k}{2}, \frac{k+1}{2}, 0\right).$$

Its unnormalized induction gives this cohomological representation $\Pi^\star(f_k) = \sigma_0''(f_k) = \chi_{\Pi_0''(f_k)}$ of $M_{(2,1)}(\mathbf{A})$ in $\mathrm{GL}_3(\mathbf{A})$, with Langlands parameter

$$z \in W_{\mathbf{C}} \cong \mathbf{C}^\times \longmapsto \left(\left(\frac{z}{\overline{z}}\right)^{\frac{k-1}{2}} \cdot (z\overline{z})^{\frac{1}{2}}, \left(\frac{\overline{z}}{z}\right)^{\frac{k-1}{2}} \cdot (z\overline{z})^{\frac{1}{2}}, (z\overline{z}) \right),$$

and hence with Hodge-Tate weights (in z and $\overline{z}$ respectively) given by the desired vectors

$$p(\Pi^\star(f_k)) = \left(\frac{k}{2}, \frac{2-k}{2}, 1\right) \quad \text{and} \quad q(\Pi^\star(f_k)) = \left(\frac{2-k}{2}, \frac{k}{2}, 1\right)$$

Observe that while $\Pi^\star(f_k) = \sigma_0''(f_k)$ is not regular for $k = 2$, i.e. as the Hodge-Tate weights $p_0'' = (1, 0, 1)$ and $q_0'' = (0, 1, 1)$ when $k = 2$, it determines a cohomological representation for all even weights $k \geq 2$ which is regular for $k \geq 4$. Moreover, for any even weight $k \geq 2$, the corresponding standard L-function $\Lambda(s, \Pi^\star(f_k), s)$ is given by a shift by $1/2$ of the one we wish to consider. That is, we have that

$$\Lambda(s, \Pi^\star(f_k)) = \Lambda(s + 1/2, \pi(f_k)) \cdot \Lambda(s+1).$$

Let us summarize this discussion as follows.

Proposition 2.6. *Let $f_k \in S_2(\Gamma_0(N))$ be any cuspidal eigenform of even weight $k \geq 2$, with corresponding $\mathrm{GL}_2(\mathbf{A})$-automorphic representation $\pi(f_k)$. The unnormalized induction $\Pi^\star(f_k)$ of the non-algebraic automorphic representation $\Pi_0''(f_k) := |\cdot| \cdot \Pi_0(f_k)$ of the Levi subgroup $M_{(2,1)}(\mathbf{A}) \cong \mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_1(\mathbf{A}) \subset \mathrm{GL}_3(\mathbf{A})$ defined via unitary induction $\Pi_0(f_k) := \pi(f_k) \times |\cdot|^{-1}$ is a cohomological automorphic representation with corresponding standard L-function $\Lambda(s, \Pi^\star(f_k), s) = \Lambda(s+1/2, \pi(f_k)) \cdot \Lambda(s+1)$. It has Hodge-Tate weights $p^\star = p_0'' = (\frac{k}{2}, \frac{2-k}{2}, 1)$ in z and $q^\star = q_0'' = (\frac{2-k}{2}, \frac{k}{2}, 1)$ in $\overline{z}$. Moreover, for even weights $k \geq 4$, this representation $\Pi^\star(f_k) = \Pi_0''(f_k)$ is regular.*

2.3. **Relations to Shimura varieties.** We now explain the following novel variations of the long exact sequence (23) which appear in the arguments of [1], [13], and [14] (see also [40, §5]). For the special case of $G = \mathrm{GL}_n$ we consider, the locally symmetric spaces X_K and S_K are not generally hermitian. However, they can be realized as components in the Borel-Serre compactifications of certain ambient hermitian symmetric spaces $X_{\underline{K}}$ and corresponding Shimura varieties $S_{\underline{K}}$, from which we can extract a variation of the exact sequence (23). Essentially, this allows us to use Poincaré duality ([1, Proposition 2.1.12 and Corollary 2.1.13]) and vanishing theorems in the style of Caraiani-Scholze [13] and [14] (cf. [1, Theorem 1.1.1]) and Lan-Suh [34, Theorem 10.1] to deduce the vanishing the higher degree cohomology group $H_c^{i+1}(S_K)$ in the setting we consider above, at least in the case of rank $n = 2 + r = 3$ corresponding to Mordell-Weil rank $r = 1$.

2.3.1. *Poincaré duality for cohomology of locally symmetric spaces.* Let us first state the following relevant version of Poincaré duality satisfied for the cohomology groups we consider. Here, we can suppose more generally that G is any connected linear reductive group defined over any number field F, and consider the corresponding symmetric space

$$X^G = G(F_\infty)/K_\infty \mathbf{R}^\times$$

for G, where $K_\infty \subset G(F_\infty)$ denotes the maximal compact subgroup. Let us write $d = d(F) = \dim_{\mathbf{R}} X^G$ to denote the dimension. Given a compact open subgroup $K \subset G(\mathbf{A}_F)$, we then write X_K^G to denote the corresponding symmetric space defined by

$$X_K^G = G(F) \backslash \left(X^G \times G(\mathbf{A}_{F,f})/K \right).$$

Fixing a finite set of prime S of F as above, we shall also write $\mathcal{H}^S(G)$ to denote the corresponding unramified algebra of Hecke operators. We shall also assume that K decomposes as product $K = \prod_{v<\infty} K_v$ with each $K_v \subset G(F_v)$, and use the standard upstairs/downstairs notations $K^S = \prod_{v \notin S} K_v$ and $K_S = \prod_{v \in S} K_v$. Fix a prime l, and let E be a finite extension of $\mathbf{Q}_l$ contained in $\overline{\mathbf{Q}}_l$, taken to be large enough to contain the image of each embedding $F \hookrightarrow \overline{\mathbf{Q}}_l$. We then take $\mathcal{O} = \mathcal{O}_E$ to be the ring of integers of E, with $\varpi = \varpi_E \in \mathcal{O}$ a fixed uniformizer.

Proposition 2.7. *Let κ denote either $\kappa = \mathcal{O}$ or $\kappa = \mathcal{O}/\varpi^m$ for some integer $m \geq 1$. Let $K \subset G(\mathbf{A}_{F,f})$ be a neat (in the strict sense[8]) compact open subgroup which decomposes as $K = \prod_{v<\infty} K_v$, with each $K_v \subset G(F_v)$. Let $\mathcal{V}$ be any $\kappa[K_S]$-module which is finite and free as a κ-module, with $\mathcal{V}^\vee = \mathrm{Hom}(\mathcal{V}, \kappa)$ its Pontrjagin dual. Then, the duality involution*

$$\iota : \mathcal{H}^S(G) \longrightarrow \mathcal{H}^S(G), \quad \left[K^S g K^S\right] \longmapsto \left[K^S g^{-1} K^S\right] \tag{28}$$

descends to an isomorphism of κ-algebras

$$\mathcal{H}^S(G)\left(H_c^j(X_K^G, \mathcal{V})\right) \cong \mathcal{H}^S(G)\left(H^{d-j}(X_K^G, \mathcal{V}^\vee)\right)$$

for any integer $j \geq 0$. In particular, if $\mathfrak{m} \subset \mathcal{H}^S(G)$ is any maximal ideal in the support of $H^*(X_K^G, \mathcal{V}^\vee)$, then the corresponding dual $\mathfrak{m}^\vee$ is in the support of $H_c^*(X_K^G, \mathcal{V})$.

Proof. See [37, Proposition 3.7], as well as [1, Proposition 2.2.12 and Corollary 2.2.13]. □

Let us now return to the setup described above, leading up to the statements of Theorems 2.4 and 2.5. We can now deduce the following result.

---

[8]Defined in the same way as above, after extending scalars to $\mathbf{A}_{F,f}$

Corollary 2.8. *We have for each integer $j \geq 0$ an identification of $\mathcal{H}^S(G)$-modules $H_c^j(S_K, \mathcal{V}) \cong H^{d-j}(S_K, \mathcal{V}^\vee)$, as well as $H_c^j(S_K, \kappa) \cong H^{d-j}(S_K, \kappa)$, via the duality involution* (28)*. In particular, we obtain from* (23) *the respective induced long exact sequences*

(29)
$$\cdots \longrightarrow H_c^j(S_K, \mathcal{V}) \longrightarrow H^j(S_K^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H^j(\partial S_K^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H^{d-(j+1)}(S_K, \mathcal{V}^\vee) \longrightarrow \cdots$$

and

(30)
$$\cdots \longrightarrow H_c^j(S_K, \kappa) \longrightarrow H^j(S_K^{\mathrm{BS}}, \kappa) \longrightarrow H^j(\partial S_K^{\mathrm{BS}}, \kappa) \longrightarrow H^{d-(j+1)}(S_K, \kappa) \longrightarrow \cdots.$$

2.3.2. *Ambient Shimura varieties.* We now explain how the cohomology groups of the GL_n-symmetric spaces $H^*(X_G, \kappa)$ and $H^*(S_K, \kappa)$ we consider above above (particularly in the induced exact sequence (30)) can be realized in the cohomology of Borel-Serre compactifications of certain ambient auxiliary Shimura varieties. Here, we follow the general setup described in [39, § V], as well as [40, § 5], [37, §5], [1, § 2], [13], and [14].

Let F be either a totally real or CM number field, writing F^+ to denote the maximal totally real subfield. Hence, in the event that F is totally real, we have $F = F^+$. In the event that F is CM, by which we mean F is a totally imaginary quadratic extension of its maximal totally real subfield F^+, we have $[F : F^+] = 2$. Let us then take G to be the reductive algebraic group over $\mathbf{Q}$ defined by

$$G = \begin{cases} \mathrm{Res}_{F/\mathbf{Q}} \mathrm{Sp}_{2n} & \text{if } F = F^+ \text{ is totally real} \\ \mathrm{Res}_{F/\mathbf{Q}} U(n,n)_{/F^+} & \text{if } F \neq F^+ \text{ is CM.} \end{cases}$$

In either case, the linear algebraic group

$$M = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_{n/F}$$

appears as a maximal Levi subgroup of G. Let us in each case take D to be the standard conjugacy class of $u : U(1) \longrightarrow G_{\mathbf{R}}^{\mathrm{ad}}$, so that (G, D) determines a Shimura datum, and hence a corresponding Shimura variety $\mathrm{Sh}(G, D) = \{S_{\overline{K}}\}_{\overline{K}}$, with $\underline{K} \subset G(\mathbf{A}_f)$ ranging over all compact open subgroups. In particular, we can use this setup to view the cohomology of the GL_n-locally symmetric spaces $S_K = S_{\underline{K}_M}$ we consider above in terms of the cohomology of the boundary components of the Borel-Serre compactifications of the corresponding Shimura variety $S_{\underline{K}}$. To describe this in more detail, let us fix a parabolic subgroup $P \subset G$, with Levi decomposition $P = MN$. Given $\underline{K}_M \subset M(\mathbf{A}_f)$ a compact open subgroup, and writing $K_{\infty,M} \subset M(\mathbf{R})$ again to denote the maximal compact subgroup, we consider the corresponding locally symmetric space

$$S_{\underline{K}_M} = M(\mathbf{Q}) \backslash [M(\mathbf{R})/\mathbf{R}_{>0} K_{M,\infty}] \times M(\mathbf{A}_f)/K_M.$$

Given a compact open subgroup $\underline{K}_P \subset P(\mathbf{A}_f)$, we consider the corresponding locally symmetric space

$$S_{\underline{K}_P} = P(\mathbf{Q}) \backslash [P(\mathbf{R})/\mathbf{R}_{>0} K_{M,\infty}] \times P(\mathbf{A}_f)/K_P.$$

As explained in [39, Lemma V.2.2], given $\underline{K}_P \subset P(\mathbf{A}_f)$ a compact open subgroup with image $\underline{K}_M \subset M(\mathbf{A}_f)$, we have a natural projection $S_{\underline{K}_P} \longrightarrow S_{\underline{K}_M}$, which can be viewed as a bundle with fibres $(\mathbf{S}^1)^m$, where m denotes the dimension of the unipotent radical N of $P = MN$. Here again, given $\underline{K} \subset G(\mathbf{A}_f)$ a compact open subgroup, we write $S_{\underline{K}}^{\mathrm{BS}}$ to denote the Borel-Serre compactification of the corresponding Shimura variety $S_{\underline{K}}$. Hence, $S_{\underline{K}}^{\mathrm{BS}}$ is a compactifaction of the manifold with corners $S_{\underline{K}}$, and the inclusion $S_{\underline{K}} \hookrightarrow S_{\underline{K}}^{\mathrm{BS}}$ a homotopy equivalence. Writing $\partial S_{\underline{K}}^{\mathrm{BS}} = S_{\underline{K}}^{\mathrm{BS}} \backslash S_{\underline{K}}$ again to denote the Borel-Serre boundary, we have the exact sequences

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow \cdots.$$

Writing $\underline{K}_P = \underline{K} \cap P(\mathbf{A}_f)$ to denote the image of $\underline{K}$ in $P(\mathbf{A}_f)$, we also have an open embedding

$$S_{\underline{K}_P} \longrightarrow \partial S_{\underline{K}}^{\mathrm{BS}},$$

and this gives rise to natural maps

$$H_c^i(S_{\underline{K}_P}, \kappa) \longrightarrow H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H^i(S_{\underline{K}_P}, \kappa).$$

Here again, we consider the actions of Hecke algebras in the same way as described above for the more general case, and note that we have a commutative diagram of k-modules

$$\begin{array}{ccccc} & & H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \kappa) & & \\ & \nearrow & & \searrow & \\ H_c^i(S_{\underline{K}_P}, \kappa) & & \longrightarrow & & H^i(S_{\underline{K}_P}, \kappa) \\ \uparrow & & & & \downarrow \\ H_c^i(S_{\underline{K}_M}, \kappa) & & \longrightarrow & & H^i(S_{\underline{K}_M}, \kappa) \end{array}$$

This key property underlies the constructions of Galois representations given in [39], [37], and [1]; cf. [40, § 5]. Here, in either case described above, we can realize the GL_n-symmetric spaces $X_{\mathrm{GL_n}}$ and S_K as components of $\partial S_{\underline{K}}^{\mathrm{BS}}$ of $S_{\underline{K}}$ in the corresponding long exact sequence

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H_c^{i+1}(S_{\underline{K}}, \kappa) \longrightarrow \cdots .$$

To be more precise, we have the following analogues of the long exact sequences appearing in Theorem 2.4 and 2.5. We refer to [37, §5.1] (cf. [1, Proposition 2.2.8]) and [1, §2.2] for more explicit descriptions of rational and integral coefficient systems that can be specified for related constructions of Galois representations. These works allow us to deduce that we have long exact sequences of $\mathcal{H}^S(\underline{G})$-modules

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H_c^{i+1}(S_{\underline{K}}, \kappa) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \mathcal{V}) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H^i(\partial S_{\underline{K}}^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H_c^{i+1}(S_{\underline{K}}, \mathcal{V}) \longrightarrow \cdots ,$$

where the boundary $\partial S_{\underline{K}}^{\mathrm{BS}}$ in each case can be viewed as a torus bundle $\mathcal{T}(S_{\underline{K}_M}) = \mathcal{T}(S_K)$ of the GL_n locally symmetric space $S_{\underline{K}_M}$. Note that this $S_{\underline{K}_M}$ was denoted in our previous discussion above by S_K, so that we now identify the GL_n locally symmetric space as $S_K = S_{\underline{K}_M}$. We have the following results.

Proposition 2.9. *We have a long exact sequences of $\mathcal{H}^S(\underline{G})$-modules*

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H^i(\mathcal{T}(S_{\underline{K}_M}), \kappa) \longrightarrow H_c^{i+1}(S_{\underline{K}}, \kappa) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \mathcal{V}) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H^i(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}) \longrightarrow H_c^{i+1}(S_{\underline{K}}, \mathcal{V}) \longrightarrow \cdots .$$

Applying Poincaré duality (Proposition 2.7), we can then derive the following variation for later use.

Corollary 2.10. *Writing $\underline{d} = \dim_{\mathbf{R}} X_{\underline{G}} = [F^+ : \mathbf{Q}] \cdot n^2$ to denote the dimension of the symmetric space attached to G, we have induced long exact sequences of $\mathcal{H}^S(\underline{G})$-modules*

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \kappa) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \kappa) \longrightarrow H^i(\mathcal{T}(S_{\underline{K}_M}), \kappa) \longrightarrow H^{\underline{d}-(i+1)}(S_{\underline{K}}, \kappa) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H_c^i(S_{\underline{K}}, \mathcal{V}) \longrightarrow H^i(S_{\underline{K}}^{\mathrm{BS}}, \mathcal{V}) \longrightarrow H^i(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}) \longrightarrow H^{\underline{d}-(i+1)}(S_{\underline{K}}, \mathcal{V}^\vee) \longrightarrow \cdots ,$$

Relations to unitary Shimura varieties We now describe the CM setting with the unitary group $U(n, n)$. Let us suppose that F is any CM field, so a totally imaginary quadratic extension of its maximal totally real subfield F^+. We write $c \in \mathrm{Gal}(F/F^+)$ to denote the nontrivial automorphism (complex conjugation), with $\mathcal{O}_F$ as usual to denote the ring of integers of F, and $\mathcal{O}_{F^+}$ that of the maximal totally real subfield F^+. Let us also fix an integer $n \geq 1$, and write Ψ_n denote the $n \times n$ matrix with ones along the antidiagonal,

$$\Psi_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

We then write J_n to denote the $2n \times 2n$ matrix defined by

$$J_n = \begin{pmatrix} & \Psi_n \\ -\Psi_n & \end{pmatrix}.$$

Note that this matrix J_n gives rise to a perfect hermitian pairing,

$$\langle \cdot, \cdot \rangle : \mathcal{O}_F^{2n} \times \mathcal{O}_F^{2n} \longrightarrow \mathcal{O}_F, \quad (x, y) \longmapsto \langle x, y \rangle := {}^t x \; J_n \; y^c.$$

Let us now consider the group $\underline{G}$ defined over $\mathcal{O}_{F^+}$ by the following rule: For any $\mathcal{O}_{F^+}$-algebra R,

$$\underline{G}(R) = \left\{ g \in \mathrm{GL}_{2n}(\mathcal{O}_F \otimes_{\mathcal{O}_{F^+}} R) : {}^t g \; J_n \; g^c = J_n \right\}.$$

Let $\underline{P} \subset \underline{G}$ denote the subgroup which stabilizes the subspace $\mathcal{O}_F^n \oplus 0^n \subset \mathcal{O}_F^{2n}$, and $\underline{M} \subset \underline{P}$ the closed subgroup which stabilizes the factors $\mathcal{O}_F^n \oplus 0^n$ and $0^n \oplus \mathcal{O}_F^n$. We then let $\underline{T} \subset \underline{G}$ denote the standard diagonal torus, with $\underline{B} \subset \underline{G}$ the standard Borel subgroup of upper triangular matrices, and $\underline{S} \subset \underline{T}$ the subtorus of elements having determinant contained in $\mathcal{O}_{F^+}$. In the style of [1, § 2] and [37], we drop the underline notations to denote the corresponding basechange to F-fibres, so that

$$G = \underline{G} \times_{\mathcal{O}_{F^+}} F, \quad P = \underline{P} \times_{\mathcal{O}_{F^+}} F, \quad M = \underline{M} \times_{\mathcal{O}_{F^+}} F$$

and

$$T = \underline{T} \times_{\mathcal{O}_{F^+}} F, \quad B = \underline{B} \times_{\mathcal{O}_{F^+}} F, \quad S = \underline{S} \times_{\mathcal{O}_{F^+}} F.$$

Thus, after basechange to the CM field F, we can view $P \subset G$ as a parabolic subgroup with $M \subset P$ the unique Levi subgroup containing T, and with $S \subset T$ a maximal F^+-split torus of G having the property that $T = Z_G(S)$. Moreover, we have the identification $G = U(n, n)$ with the quasi-split unitary group $U(n, n)$ over F. Note that all of these groups are all reductive. Moreover, we have the following result.

Lemma 2.11. *The following assertions are true.*

- (i) *If v is a finite place of F^+ which is unramified (split or inert) in F, then $\underline{G}_{\mathcal{O}_{F_v^+}}$ is reductive, and hence $G_{F_v^+}$ is unramified.*
- (ii) *Let $\underline{N} \subset \underline{P}$ denote the closed subgroup which acts trivially on the subspaces $\mathcal{O}_F^n \oplus 0^n$ and $0^n \oplus \mathcal{O}_F^n$. Then, we have the semi-direct product decomposition $\underline{P} \cong \underline{M} \ltimes \underline{N}$, as well as the identification*

$$\underline{M} \cong \mathrm{Res}_{\mathcal{O}_F / \mathcal{O}_{F^+}} \mathrm{GL}_n.$$

Proof. See [37, Lemma 5.1]. □

2.3.3. *Vanishing theorems.* Let us also record the following general results about vanishing of cohomology groups for the setting described above, noting that many (but not all) of the results described in this paragraph apply to the unitary setting with F a CM field.

We first describe the general theorem of Lan-Suh [34, Theorem 10.1]. Although this applies in general to both cases we consider, we illustrate the "sufficient regularity" condition in terms of a specific coefficient system $\mathcal{V}$. Here, we follow [37] (cf. [1, §2.2]) for the setting of unitary Shimura varieties described above, but note that the analogous theorem holds for any PEL-type Shimura variety. Let us thus fix a prime l, together with any finite extension E of $\mathbf{Q}_l$ which contains the image of each embedding $\tau : F \longrightarrow \overline{\mathbf{Q}}_l$. Write $\mathcal{O} = \mathcal{O}_E$ to denote its ring of integers. Fix a uniformizer $\varpi \in \mathcal{O}$, and let $k = \mathcal{O}/\varpi$ to denote the residue field. Let us also introduce the following notations, writing

$$\mathbf{Z}_+^n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n : \lambda_1 \geq \ldots \geq \lambda_n\}$$

to denote the set of dominant weights, and

$$\mathbf{Z}_{++}^n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n : \lambda_1 > \ldots > \lambda_n\}$$

the set of strictly dominant weights. To any tuple of dominant weights $\lambda = (\lambda_\tau) \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, we can associate a local system $\mathcal{V}_\lambda$ of finite free $\mathcal{O}$-modules on the GL_n-symmetric space S_K we consider above (defined in [37, §2.2] and [1, §2.2]). The corresponding cohomology groups $H^*(S_K, \mathcal{V}_\lambda)$ are finite $\mathcal{O}$-modules, and we can associate to place $v \notin S$ a family of local Hecke operators $T_v^1, \ldots, T_v^n \in \mathcal{H}_v(\mathrm{GL}_n)$ in terms of double coset operators[9]. On the other hand, in the setup linking to the unitary group $U(n,n)$ outlined above, a similar choice of local system on $S_{\underline{K}}$ leads to a compatible choice of local system $\mathcal{V}_{\widetilde{\lambda}}$, as defined in [37, §5.1] and [1, §2.2]. To describe this briefly, let us retain the general setup, with F a CM field having maximal totally real subfield F^+, and $c \in \mathrm{Gal}(F/F^+)$ the nontrivial automorphism corresponding to complex conjugation. Write I_l to denote the set of embeddings $\tau : F^+ \rightarrow E$. Let us for each such embedding $\tau \in I_l$ fix a lifting of $\widetilde{\tau} : F \rightarrow E$ of τ to the totally imaginary quadratic extension F/F^+, and write $\widetilde{I}_l = \{\widetilde{\tau} : \tau \in I_l\}$ to denote the set of these liftings. Let $\underline{T}_n \subset \underline{M}$ denote the standard diagonal maximal torus, so that our fixed embedding $\underline{M} \rightarrow \underline{G}$ induces an identification $\underline{T}_n \cong \underline{T}$. This latter identification can be used to relate local systems on $S_{\underline{K}}$ and S_K in the setup above as follows. Fix a place $v \in S$ above the fixed rational prime l, and let $\tau \in I_l$ be the embedding inducing v. The choice of lifting $\widetilde{\tau} : F \rightarrow E$ then determines canonical identifications $M \otimes_{F^+,\tau} E \cong \mathrm{GL}_n \times \mathrm{GL}_n$, $T_n \otimes_{F^+,\tau} E \cong \mathrm{GL}_1^n \times \mathrm{GL}_1^n$, and hence $X^*(T_{n,E,\tau}) \cong \mathbf{Z}^n \times \mathbf{Z}^n$. An element $(\lambda_{\widetilde{\tau}}, \lambda_{\widetilde{\tau}^c}) \in X^*(T_{n,E,\tau}) \cong \mathbf{Z}^n \times \mathbf{Z}^n$ lies in the dominant subset of $X^*(T_{n,E,\tau})^+ \subset X^*(T_{n,E,\tau})$ if any only if it factors through $\mathbf{Z}_+^n \times \mathbf{Z}_+^n$. Given a dominant weight $(\lambda_{\widetilde{\tau}}, \lambda_{\widetilde{\tau}^c}) \in \mathbf{Z}_+^n \times \mathbf{Z}_+^n$, we consider the $\mathcal{O}[\underline{M}(\mathcal{O}_{F_v^+})]$-module $\mathcal{V}_{\lambda_{\widetilde{\tau}}, \lambda_{\widetilde{\tau}}^c}$ as defined in [37, §2.2], which determines a finite $\mathcal{O}$-module. Given a tuple of dominant weights $\lambda = (\lambda_{\widetilde{\tau}}) \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)} = (\mathbf{Z}_+^n, \mathbf{Z}_+^n)^{\mathrm{Hom}(F^+,E)}$, we consider the tensor product over $\mathcal{O}$,

$$\mathcal{V}_\lambda := \bigotimes_{\tau \in I_l} \mathcal{V}_{\lambda_{\widetilde{\tau}}, \lambda_{\widetilde{\tau}^c}}.$$

This tensor product $\mathcal{V}_\lambda$ determines an $\mathcal{O}[\underline{M}(\mathcal{O}_{F^+} \otimes_{\mathbf{Z}} \mathbf{Z}_l)]$-module which is finite and free as an $\mathcal{O}$-module. It is also to be viewed as the local system corresponding to our symmetric space $S_{\underline{M}} = S_K$. To describe the local system associated to the unitary Shimura variety $S_{\underline{K}}$, as well as its relation to this V_λ, we consider the following setup. Let us again choose a place $v \in S$ above l, with $\tau \in I_l$ the embedding $\tau : F^+ \rightarrow E$ inducing v. The choice of lifting $\widetilde{\tau} \in \widetilde{I}_l$ then determines canonical isomorphisms $G \otimes_{F^+,\tau} E \cong \mathrm{GL}_{2n}$, $T \otimes_{F^+,\tau} E \cong \mathrm{GL}_1^{2n}$, and hence $X^*(T_{E,\tau}) \cong \mathbf{Z}^{2n}$. An element $\mu_\tau \in X^*(T_{E,\tau}) \cong \mathbf{Z}^{2n}$ lies in the dominant subspace $X^*(T_{E,\tau})^+ \subset X^*(T_{E,\tau})$ is and only if it factors through $\mathbf{Z}_+^{2n}$. Now, note that under the induced isomorphism of character groups $X^*(T_{n,E,\tau}) \cong X^*(T_{E,\tau})$, we have the following correspondence of weights:

$$\begin{gathered} X^*(T_{n,E,\tau}) \cong X^*(T_{E,\tau}), \\ \lambda_\tau := (\lambda_{\widetilde{\tau},1}, \ldots, \lambda_{\widetilde{\tau},n}, \lambda_{\widetilde{\tau}^c,1}, \ldots, \lambda_{\widetilde{\tau}^c,n}) \leftrightarrow \widetilde{\lambda}_\tau := (-\lambda_{\widetilde{\tau}^c,n}, \ldots, -\lambda_{\widetilde{\tau}^c,1}, \lambda_{\widetilde{\tau},1}, \ldots, \lambda_{\widetilde{\tau},n}). \end{gathered} \tag{31}$$

In this way, the subset of dominant weights $X^*(T_{E,\tau})^+ \subset X^*(T_{n,E,\tau})^+$ can also be described by the simpler condition $-\lambda_{\widetilde{\tau}^c,1} \geq \lambda_{\widetilde{\tau},1}$. Given a dominant weight $\widetilde{\lambda}_\tau \in X^*(T_{E,\tau})^+$, we consider the the $\mathcal{O}[\underline{G}(\mathcal{O}_{F_v^+})]$-module $\mathcal{V}_{\widetilde{\lambda}_\tau}$ defined in [37, §2.2], which again determines a finite $\mathcal{O}$-module. Given a tuple of dominant weights $\widetilde{\lambda} = (\widetilde{\lambda}_\tau) \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, we can again consider the tensor product over $\mathcal{O}$,

$$\mathcal{V}_{\widetilde{\lambda}} := \bigotimes_{\tau \in I_l} \mathcal{V}_{\widetilde{\lambda}_\tau}.$$

This tensor product $\mathcal{V}_{\widetilde{\lambda}}$ determines an $\mathcal{O}[\underline{G}(\mathcal{O}_{F^+} \otimes_{\mathbf{Z}} \mathbf{Z}_l)]$-module which is finite and free as an $\mathcal{O}$-module. The modules $\mathcal{V}_\lambda$ and $\mathcal{V}_{\widetilde{\lambda}}$ can be related more explicitly according to [37, Lemma 5.4], which states roughly that for matching dominant weights $\lambda \leftrightarrow \widetilde{\lambda}$ as in (31), there is a direct sum decomposition of $\mathcal{O}[K_{M,l}]$-modules

$$\mathrm{Res}_{K_{M,l}}^{K_l} \mathcal{V}_{\widetilde{\lambda}} = \mathcal{V}_\lambda \oplus \kappa,$$

with $\mathcal{V}_\lambda \subset \mathcal{V}_{\widetilde{\lambda}}^{K_{N,l}}$. These modules also satisfy convenient properties after twisting; see [37, Lemma 5.5] and [1, Proposition 2.2.14]. Now, we have general vanishing theorem for the cohomology groups $H^*(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}})$, provided at the dominant weight $\lambda \leftrightarrow \widetilde{\lambda}$ as described in (31) is "sufficiently regular" in the sense of [34].

Theorem 2.12 (Lan-Suh). *Let us suppose first that we are in the setting with $F \neq F^+$ a CM field, with $G = \underline{G}$ as described above. Suppose the chosen prime l is unramified in F. Fix $\underline{K} \subset \underline{G}(\mathbf{A}_{F,f})$ a decomposable*

[9]Note that [37] write $\mathbb{T}^S(H^*(S_K, \mathcal{V}_\lambda))$ (in our notations) to denote the commutative $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(H^*(S_K, \mathcal{V}_\lambda))$ generated by these operators.

compact open subgroup which is neat in the strict sense, and whose component at l is maximal. Let us choose a strictly dominant weight $\widetilde{\lambda} = (\widetilde{\lambda}_\tau)_\tau = ((\widetilde{\lambda}_{\tau,j})_j)_\tau \in (\mathbf{Z}^{2n}_{++})^{\mathrm{Hom}(F^+,E)}$ subject to the constraint

$$[F:\mathbf{Q}]\cdot n(n+1) + \sum_{\tau \in I_p}\sum_{j=1}^{2n}\left(\widetilde{\lambda}_{\tau,j} - 2\left\lfloor \frac{\widetilde{\lambda}_{\tau,2n}}{2}\right\rfloor\right) < l.$$

let $\underline{d} := [F^+:\mathbf{Q}]\cdot n^2$ denote the dimension of the symmetric space attached to G, and let us take for granted the correspondence of strictly dominant weights $\lambda \leftrightarrow \widetilde{\lambda}$ described in (31) above. Then for each $0 \leq i < \underline{d}$, we have the vanishing of the corresponding cohomology

$$H^i(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) = H^i(S_{\underline{K}}, \mathcal{V}_\lambda \otimes_{\mathcal{O}} \kappa) = 0. \tag{32}$$

Similarly, for each $i > \underline{d}$, we have the vanishing of the compactly supported cohomology with dual coefficients:

$$H^i_c(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) = H^i_c(S_{\underline{K}}, \mathcal{V}^\vee_\lambda \otimes_{\mathcal{O}} \kappa) = 0. \tag{33}$$

In the setting where $F = F^+$ is totally real and $S_{\underline{K}}$ is a symplectic Shimura variety according to our discussion above, the same vanishing identifications (32) and (33) hold for $\mathcal{V}$ a local system of "sufficiently regular" weight, with $\underline{d} = \frac{1}{2}\cdot n \cdot (n+1)$ the dimension of the corresponding locally symmetric space $S_{\underline{K}}$.

Proof. See [34, Theorem 10.1], as well as the relevant summary for this case given in [37, Theorem 5.10]. □

Observe that via Proposition 2.9, we then derive the long exact sequence

(34)

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(\mathcal{T}(S_{\underline{K_M}}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow 0,$$

or equivalently

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(\partial S^{\mathrm{BS}}_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow 0.$$

Corollary 2.13. *We have the short exact sequence of cohomology groups*

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(\mathcal{T}(S_{\underline{K_M}}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow 0$$

Equivalently,

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}}(\partial S^{\mathrm{BS}}_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow 0.$$

Proof. Consider the last term $H^{\underline{d}+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}})$ in the exact sequence (34). We argue we can assume without loss of generality that $\mathcal{V}_{\widetilde{\lambda}}$ is also sufficiently regular in the sense of [34, Theorem 10.1] (Theorem 2.12), twisting by a character according to the description given in [37, Lemma 5.5] if necessary. We can then invoke Theorem 2.12 again to deduce that $H^{\underline{d}+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) = 0$, from which the claimed short exact sequence follows immediately via (34). □

We also have the following related vanishing theorems, which we state to give additional context. Let us start by recounting the general expectation for the cohomology of GL_n locally symmetric spaces.

Conjecture 2.14 (Folklore)**.** *Let $G = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n(F)$ for any integer $n \geq 1$ and any number field F. Fix $K \subset G(\mathbf{A}_f)$ any compact open subgroup, and let X_G and S_K denote the corresponding symmetric spaces, as introduced above. Writing $K_\infty \subset G(\mathbf{R})$ again to denote the maximal compact subgroup, with $A_\infty = A_G$ the identity component of $\mathbf{R}$-points of the maximal $\mathbf{Q}$-split torus of the centre Z_G, and $d = \dim_{\mathbf{R}} X^G$ the dimension of the symmetric space for G, let $l_0 = \mathrm{rk}\,G(\mathbf{R}) - \mathrm{rk}(K_\infty) - \mathrm{rk}(A_\infty)$ and $q_0 = \frac{1}{2}(d - l_0)$. Then, for any rational prime l, we have that $H^j(S_K, \mathbf{Z}_l) = 0$ unless $j \in [q_0, q_0 + l_0]$. That is, the cohomology of the locally symmetric space S_K with coefficients in $\mathbf{Z}_l$ is supported only in degrees $j \in [q_0, q_0 + l_0]$.*

Although this conjecture remains largely open (see e.g. [14]), we have the following results. First, we have the following theorem due to Li and Schwermer [36].

Theorem 2.15 (Li-Schwermer)**.** *Let G be a connected, semisimple algebraic group defined over $\mathbf{Q}$, and $\Gamma \subset G(\mathbf{Q})$ a torsionfree congruence subgroup. Let (ν, E) be any finite-dimensional representation of $G(\mathbf{R})$ on a complex vector space E for which the highest weight is regular. Writing $K_\infty \subset G(\mathbf{R})$ again to denote the maximal compact subgroup, with $X_G = G(\mathbf{R})/K_\infty$ the symmetric space, we consider the cohomology groups $H^*(\Gamma\backslash X_G, E)$. In particular, for all $j < \frac{1}{2} \cdot [\dim X_G - (\mathrm{rk}(G(\mathbf{R}) - \mathrm{rk}(K_\infty)))] = \frac{1}{2} \cdot (2q_0 + l_0 - l_0) = q_0$,*

$$H^j(\Gamma\backslash X_G, E) = 0.$$

We also have the following results for the CM setup described above, with the split unitary group $U(n, n)$. To describe these, let us again take F to be a CM field with maximal totally real subfield F^+. Fix $\underline{K} \subset \underline{G}(\mathbf{A}_{F^+,f})$ a decomposable compact open subgroup, which is neat in the strict sense. Fix a finite set of places S of F^+ including all places v for which $\underline{K}_v \neq \underline{G}(F_v^+)$. Fix a rational prime l, together with a maximal ideal $\underline{\mathfrak{m}} \subset \mathcal{H}^S(\underline{G})$ appearing in the support of the cohomology $H^*(S_{\underline{K}}, \mathbf{F}_l)$. Let us enlarge the set S to include l, and fix an embedding $\mathcal{H}^S(\underline{G}) \longrightarrow \mathbf{F}_l$. The constructions of [39] and [1] then show the existence of a $2n$-dimensional Galois representation

$$\overline{\rho}_{\underline{\mathfrak{m}}} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_{2n}(\overline{\mathbf{F}}_l)$$

whose characteristic polynomials of Frobenius $\overline{\rho}_{\underline{\mathfrak{m}}}(\mathrm{Frob}_v)$ for all $v \notin S$ are given by the usual Hecke polynomial (described in terms of local Hecke operators at v) modulo $\underline{\mathfrak{m}}$.

Theorem 2.16 (Caraiani-Scholze)**.** *Let us assume, in the setup described above, that:*

(i) *The maximal totally real subfield F^+ is nontrivial, $[F^+ : \mathbf{Q}] > 1$.*

(ii) *The residual Galois representation $\overline{\rho}_{\underline{m}}$ has length at most 2.*

(iii) *There exists a rational prime $p \neq l$ which splits completely in the CM field F, and such that for each $v \mid p$ in F, the residual Galois representation $\overline{\rho}_{\underline{\mathfrak{m}}}$ is unramified at v, and the set of eigenvalues $\{\alpha_{1,v}, \ldots, \alpha_{2n,v}\}$ of $\overline{\rho}_{\underline{\mathfrak{m}}}(\mathrm{Frob}_v)$ satisfies the condition that $\alpha_{i,v} \neq p \cdot \alpha_{j,v}$ for all $i \neq j$.*

Then, writing $\underline{d} = [F^+ : \mathbf{Q}] \cdot n^2$ again to denote the dimension of the symmetric space attached to $\underline{G}$,

$$\begin{cases} H^i(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} = 0 & \text{if } i < \underline{d} \\ H^i_c(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} = 0 & \text{if } i > \underline{d}. \end{cases}$$

Proof. See [14, Theorem 1.1], and also the descriptions given in [1, Theorem 1.1.1] and [40, Theorem 5.2]. ☐

Corollary 2.17. *We have the short exact sequence of cohomology groups*

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow H^{\underline{d}}(\mathcal{T}(S_{\underline{K_M}}), \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow 0.$$

Equivalently, we have

$$0 \longrightarrow H^{\underline{d}}_c(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow H^{\underline{d}}(S_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow H^{\underline{d}}(\partial S^{\mathrm{BS}}_{\underline{K}}, \mathbf{F}_l)_{\underline{\mathfrak{m}}} \longrightarrow 0.$$

Proof. The claim follows from the long exact sequence appearing in Proposition 2.9; cf. also [40, §5]. ☐

We also have the following result for rational coefficients in the setup outlined above.

Theorem 2.18 (Allen-Calegari-Caraiani-Gee-Helm-Le Hung-Newton-Scholze-Taylor-Thorne)**.** *Let F/F^+ be a CM field, and $K \subset \mathrm{GL}_n(\mathbf{A}_{F,f})$ a decomposable and neat (in the strict sense) compact open subgroup. Put $q_0 = [F^+ : \mathbf{Q}]n(n-1)/2$ and $l_0 = [F^+ : \mathbf{Q}]n - 1$. Let $\mathfrak{m} \subset \mathcal{H}^S(G)$ be a maximal ideal whose corresponding residual Galois representation $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, equivalently "non-Eisenstein". Then,*

$$H^j(S_K, \mathcal{V}_\lambda)_{\mathfrak{m}} \left[\frac{1}{l}\right] = 0 \text{ if } j \notin [q_0, l_0 + q_0].$$

Moreover, if $H^j(S_K, \mathcal{V}_\lambda)_{\mathfrak{m}} \left[\frac{1}{l}\right] \neq 0$ for some $j \in [q_0, l_0+q_0]$, then $H^j(S_K, \mathcal{V}_\lambda)_{\mathfrak{m}} \left[\frac{1}{l}\right] \neq 0$ for all $j \in [q_0, l_0+q_0]$.

Proof. See [1, Theorem 2.4.9 (2)]. ☐

2.4. **Return to the motivating question for Mordell-Weil rank $r = 1$.** Let us now return to our motivating Question 1.1 with $f = f_2 \in S_2^{\mathrm{new}}(\Gamma_0(N))$ an eigenform parametrizing an elliptic curve E of conductor N over $\mathbf{Q}$ whose corresponding standard L-function $\Lambda(E,s) = \Lambda(s-1/2,\pi(f))$ has analytic rank $r = r(f) = 1$. We first pass to higher-weight forms, taking $k \geq 4$ to be any even integer, taking $f_k \in S_k(\Gamma_0(N))$ any cuspidal eigenform of even weight $k \geq 4$. We shall later consider a Hida family $\{f_k\}_k$ of such forms with weight-two specialization $f_2 = f \in S_2^{\mathrm{new}}(\Gamma_0(N))$.

Theorem 2.19. *Let $f_k \in S_k(\Gamma_0(N))$ be any cuspidal eigenform of even weight $k \geq 4$ and corresponding cuspidal automorphic representation $\pi(f_k)$ of $\mathrm{GL}_2(\mathbf{A})$. The cohomological representation $\Pi^\star(f_k) = \Pi'(f_k)$ as described in Proposition 2.6 contributes to the cohomology in the first degree $H^1(S_K,\mathcal{L}_k)$, as opposed to $H_c^2(S_K,\mathcal{L}_k)$. Here, S_K denotes the symmetric space defined from the corresponding compact open subgroup $K \subset G(\mathbf{A}_f) = \mathrm{GL}_3(\mathbf{A}_f)$ determined by $\Pi^\star(f_k) = \Pi'(f_k) \in \mathrm{Coh}(3)$, and $\mathcal{L}_k$ the coefficient system determined uniquely by the eigenform $f_k \in S_k(\Gamma_0(N))$.*

Proof. Again, we view $\Pi^\star(f_k)$ as a representation of the Levi subgroup $M = M_{2,1} \cong \mathrm{GL}_2 \times \mathrm{GL}_1 \subset G = \mathrm{GL}_3$, and use the constructions of Theorem 2.4 and 2.5 to realize this unnormalized parabolic induction explicitly through the cohomology of the Borel-Serre compactification of a Shimura variety for Sp_6. Writing $P = MN$ again to denote the corresponding Levi decomposition $P_{(2,1)} = M_{(2,1)}N_{(2,1)}$, and using the Harder-Schwermer decomposition (25), we see that the cohomological representation $\Pi^\star(f_k) = \Pi'(f_k) \in \mathrm{Coh}(3)$ is realized through the image $\mathrm{Im}(j_*)$ with its induced decomposition (26),

$$\mathrm{Im}(j_*) \longrightarrow H_!^i(S_{K_P},\mathcal{L}_k) \cong \bigoplus_{i=j_1+j_2} H^{j_2}(S_{K_M},\mathcal{H}^{j_1}(\mathfrak{n},L_k)).$$

In this way, we deduce that $\Pi^\star(f_k)$ factors through one of the summands $H^*(S_{K_M},\mathcal{L}_k) \subset H^*(S_{K_M},\mathcal{H}^*(\mathfrak{n},L_k))$. We then deduce by Theorem 2.5 or Theorem 2.4 that $\Pi^\star(f_k)$ contributes to either $H^1(S_K,\mathcal{L}_k)$ or $H_c^2(S_K,\mathcal{L}_k)$. To be clear, we know that the symmetric space X^G corresponding to $G = \mathrm{GL}_3$ has dimension 5. We deduce from this that the cohomology $H^i(S_K,\mathcal{L}_k)$ of corresponding quotient S_K of the symmetric space X^G should sit in degrees $i \in [0, \dim X^G - \mathrm{rk}\, G] = [0, 5-3] = [0,2]$, and by Poincaré duality that the compactly supported cohomology $H_c^i(S_K,\mathcal{L}_k^\vee)$ should sit in degrees $i \in [3,5]$. On the other hand[10], we argue that the cohomological dimension of the symmetric space S_{K_M} corresponding to $M = M_{2,1}$ is 1 as opposed to 2, as the corresponding boundary component is essentially an arithmetic quotient of $\mathrm{GL}_2(\mathbf{R})/O_2(\mathbf{R})$ as opposed to $\mathrm{GL}_2(\mathbf{R})/\mathbf{R}^\times$, and so that the class corresponding to $\Pi^\star(f_k)$ appears in either $H^1(S_{K_M},\mathcal{L}_k)$ or $H_c^2(S_{K_M},\mathcal{L}_k)$. Hence, we can restrict our attention to $H^1(S_K,\mathcal{L}_k)$ and $H_c^2(S_K,\mathcal{L}_k)$.

Writing $\mathcal{L} = \mathcal{L}_k$, we consider the inclusion of each $H^i(S_K,\mathcal{L})$ in the long exact sequence (23),

$$(35) \quad \cdots \longrightarrow H_c^i(S_K,\mathcal{L}) \longrightarrow H^i(S_K,\mathcal{L}) \longrightarrow H^i(\partial S_K^{\mathrm{BS}},\mathcal{L}) \longrightarrow H_c^{i+1}(S_K,\mathcal{L}) \longrightarrow \ldots,$$

which after applying Poincaré duality (Proposition 2.7) to the last term gives the induced exact sequence

(36)

$$\cdots \longrightarrow H_c^i(S_K,\mathcal{L}) \longrightarrow H^i(S_K,\mathcal{L}) \longrightarrow H^i(\partial S_K^{\mathrm{BS}},\mathcal{L}) \longrightarrow H^{d-(i+1)}(S_K,\mathcal{L}^\vee) \longrightarrow \ldots.$$

Here again, we note that $d = 5$.

On the other hand, fixing an isomorphism $\overline{\mathbf{Q}}_l \cong \mathbf{C}$, and realizing the GL_3 symmetric space S_K as the Levi component $S_{\underline{K}_M}$ corresponding to the Sp_6 Shimura variety $S_{\underline{K}}$ over the totally real field $F^+ = \mathbf{Q}$, we can also realize each $H^i(S_K,\mathcal{L})$ inside the corresponding exact sequence

(37)

$$\cdots \longrightarrow H_c^i(S_{\underline{K}},\mathcal{V}) \longrightarrow H^i(S_{\underline{K}},\mathcal{V}) \longrightarrow H^i(\mathcal{T}(S_{\underline{K}_M}),\mathcal{V}) \longrightarrow H_c^{i+1}(S_{\underline{K}},\mathcal{V}) \longrightarrow \ldots,$$

which after applying Poincaré duality to the last term gives the induced exact sequence

(38)

$$\cdots \longrightarrow H_c^i(S_{\underline{K}},\mathcal{V}) \longrightarrow H^i(S_{\underline{K}},\mathcal{V}) \longrightarrow H^i(\mathcal{T}(S_{\underline{K_M}}),\mathcal{V}) \longrightarrow H^{\underline{d}-(i+1)}(S_{\underline{K}},\mathcal{V}^\vee) \longrightarrow \ldots.$$

Here, letting R denote either $\mathcal{O}$ or $\mathcal{O}/\varpi^m$ for some integer $m \geq 1$, we can take $\mathcal{V}$ to be any $R[K_S]$-module which is finite and flat as an R-module, as in the statement of Proposition 2.7. We also have the corresponding dimension formula $\underline{d} = \frac{1}{2}n(n+1) = \frac{1}{2}3(4) = 6$, and can assume from the discussion above that $i = 1$.

[10] I am grateful to Laurent Clozel for pointing this out to me.

At this point we can argue in several ways. To describe the first, we consider $H^2_c(S_K, \mathcal{L})$, which by Poincaré duality can be identified as $H^2_c(S_K, \mathcal{L}) \cong H^{d-2}(S_K, \mathcal{L}^\vee) = H^3(S_K, \mathcal{L}^\vee)$. Via the fixed isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_l$, we can then realize this latter group $H^3(S_K, \mathcal{L}^\vee)$ as $H^3(S_{\underline{K}_M}, \mathcal{V})$ for a corresponding Levi component $S_{\underline{K}_M}$ for the hermitian symmetric space $S_{\underline{K}}$ attached to the Sp_6 Shimura variety, as described above. Here, we argue that we can take the local system $\mathcal{V} = \mathcal{V}_{\widetilde{\lambda}}$ associated to a sufficiently regular weight $\widetilde{\lambda}$ for the vanishing theorem of Lan-Suh [34, Theorem 10.1] (Theorem 2.12), namely by embedding the cohomology groups we consider into such ambient cohomology groups. We can then consider the corresponding exact sequence (37),

$$\cdots \longrightarrow H^3_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^4_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow \ldots,$$

where the second term $H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}})$ is seen immediately to vanish by Theorem 2.12. On the other hand, we argue that we can use Poincaré duality to identify the last term in this sequence as

$$H^4_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^{\underline{d}-4}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) = H^2(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}).$$

We now argue that we can assume without loss of generality that $\mathcal{V}^\vee_{\widetilde{\lambda}}$, is sufficiently regular, twisting by a character as in [37, Lemma 5.5] if necessary to reduce to this case. We may then apply Theorem 2.12 again to deduce that $H^4_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^2(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) = 0$, from which we deduce that $H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}})$. Using Theorem 2.12 again, we see that $H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}}) = 0$. We then deduce via Poincaré duality that $H^2_c(S_K, \mathcal{L}) = 0$, so that that our representation $\Pi^\star(f_k)$ of $M \subset G$ must contribute to $H^1(S_K, \mathbf{C})$.

In a similar way, we could also consider the final term $H^{i+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}})$ in (37), which by Poincaré duality (Proposition 2.7) can be identified as $H^{i+1}_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^{\underline{d}-(1+i)}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) = H^{6-(1+i)}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}})$. Taking $i = 1$ and $n = 3$ for our example, we are looking at $H^2_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^4(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}})$. Taking $\mathcal{V}^\vee_{\widetilde{\lambda}}$ to be sufficiently regular in the sense of [34, Theorem 10.1] (Theorem 2.12) then we can deduce that this term must vanish. Thus via (37), we derive the corresponding exact sequence

$$\ldots \longrightarrow H^1_c(S_{\underline{K}}, \mathbf{C}) \longrightarrow H^1(S_{\underline{K}}, \mathbf{C}) \longrightarrow H^1(\mathcal{T}(S_{\underline{K}_M}), \mathbf{C}) \longrightarrow 0,$$

and via (35) the exact sequence

$$\cdots \longrightarrow H^1_c(S_K, \mathbf{C}) \longrightarrow H^1(S_K, \mathbf{C}) \longrightarrow H^1(\partial S^{\mathrm{BS}}_K, \mathbf{C}) \longrightarrow 0.$$

In particular, this would also prove the claim.

As a variation of the latter argument, we could consider $H^{d-(i+1)}(S_K, \mathbf{C}) = H^{d-2}(S_K, \mathbf{C}) = H^3(S_K, \mathbf{C})$ in the exact sequence (36). That is, we could then consider the exact sequences

$$\cdots \longrightarrow H^3_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^4_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow \ldots,$$

and

$$\cdots \longrightarrow H^3_c(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}-4}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) \longrightarrow \ldots.$$

corresponding respectively to (37) and (38). Let us look at the latter sequence, which after applying Poincaré duality to the first term can be put into the more convenient form

$$\cdots \longrightarrow H^{\underline{d}-3}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) \longrightarrow H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}}) \longrightarrow H^{\underline{d}-4}(S_{\underline{K}}, \mathcal{V}^\vee_{\widetilde{\lambda}}) \longrightarrow \ldots.$$

Arguing again that we can take $\mathcal{V}^\vee_{\widetilde{\lambda}}$ to be sufficiently regular in the sense of Theorem 2.12, we then see that the outer terms vanish, so that we get an induced isomorphism $H^3(S_{\underline{K}}, \mathcal{V}_{\widetilde{\lambda}}) \cong H^3(\mathcal{T}(S_{\underline{K}_M}), \mathcal{V}_{\widetilde{\lambda}})$. On the other hand, arguing again that we can assume without loss of generality that $\mathcal{V}_{\widetilde{\lambda}}$ is sufficiently regular in the same sense of Theorem 2.12, possibly after twisting by a character as described in [37, Lemma 5.5], we can again use Theorem 2.12 to deduce the vanishing of these terms, i.e. as $3 < \underline{d} = 6$. ☐

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