L-FUNCTIONS OF ELLIPTIC CURVES IN RING CLASS EXTENSIONS OF REAL QUADRATIC FIELDS VIA REGULARIZED THETA LIFTINGS

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ABSTRACT. We derive integral presentations for central derivative values of L-functions of elliptic curves defined over the rationals, basechanged to a real quadratic field K, and twisted by ring class characters of K. In particular, we derive an explicit formula for the central derivative value in terms of special values of automorphic Green's functions for certain linear combinations of Hirzebruch-Zagier divisors on certain Hilbert modular surfaces. In special cases, we can also describe these central derivative values as periods via a reinterpretation of the corresponding Birch-Swinnerton-Dyer constant in terms of special values of these automorphic Green's functions.

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1. INTRODUCTION

Let E be an elliptic curve of conductor N defined over the rational number field \mathbf{Q} , with corresponding Hasse-Weil L-function denoted by L(E, s). The modularity theorem of Wiles, Taylor-Wiles, and Breuil-Conrad-Diamond-Taylor implies that L(E, s) has an analytic continuation $\Lambda(E, s)$ via the Mellin transform

(1)
$$\Lambda(E,s) = \Lambda(s,f) := \int_0^\infty f\left(\frac{iy}{\sqrt{N}}\right) y^s \frac{dy}{y} = N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(s,f)$$

of some weight-two newform

$$f(\tau) = f_E(\tau) = \sum_{n \ge 1} c_f(n) e(n\tau) = \sum_{n \ge 1} a_f(n) n^{\frac{1}{2}} e(n\tau) \in S_2^{\text{new}}(\Gamma_0(N))$$

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with L-function corresponding to the Mellin transform (first for $\Re(s) > 1$)

$$L(s,f) := \sum_{n \ge 1} a_f(n) n^{-s} = \sum_{n \ge 1} c_f(n) n^{-(s+1/2)}.$$

That is, writing $\pi = \bigotimes_v \pi_v$ to denote the cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ associated to f, with $\Lambda(s,\pi) = \prod_{v \leq \infty} L(s,\pi_v)$ its standard *L*-function¹ we have equivalences of *L*-functions

$$\Lambda(E,s) = \Lambda(s - 1/2, f) = \Lambda(s - 1/2, \pi).$$

Let k be any number field. The Mordell-Weil theorem implies that the group of k-rational points E(k)has the structure of a finitely generated abelian group $E(k) \cong \mathbf{Z}^{r_E(k)} \oplus E(k)_{\text{tors}}$. It is a fundamental open problem to characterize the rank $r_E(k) = \operatorname{rk}_{\mathbf{Z}} E(k)$. Writing L(E/k, s) to denote the Hasse-Weil L-function of E/k, Birch and Swinnerton-Dver conjectured that this generating series L(E/k, s), defined a priori only for $\Re(s) > 3/2$, has an analytic continuation $\Lambda(E/k, s)$ to all $s \in \mathbf{C}$, with $\Lambda(E/k, s)$ satisfying a functional equation relating values at s to 2-s (so that s=1 is the central point). Taking for granted this preliminary hypothesis², the conjecture of Birch and Swinnerton-Dyer predicts that the rank $r_E(k)$ is given by the order of vanishing $\operatorname{ord}_{s=1} \Lambda(E/k, s)$ at this central point. Although this conjecture has been verified over the past several decades for $r_E(k) \leq 1$ with $k = \mathbf{Q}$ or k an imaginary quadratic field, it remains open at large, without a single known example for $r_E(k) \ge 2$. The most stunning progress to date has come through the Iwasawa theory of elliptic curves, using as a starting point special value formulae for the values $\Lambda^{(r_E(k))}(E/k, 1)$. In particular, the celebrated theorem of Gross-Zagier [24] (with generalizations such as [48] and [8]) for the central derivative value $\Lambda'(E/k, \chi, 1)$, with χ a class group character of an imaginary quadratic field k, has played a major role underlying most of this progress for rank one. This tour de force makes use of all that is known about the theory of complex multiplication and explicit class field theory for imaginary quadratic fields, and especially a construction of points $e_H \in E(k[1])$ dating back to Heegner to relate the central derivative values $\Lambda'(E/k, \chi, 1)$ for χ a character of the class group $\operatorname{Pic}(\mathcal{O}_k) \cong \operatorname{Gal}(k[1]/k)$ (with k[1]/k the Hilbert class field) to the regulator term $R_E(k) = [e_H, e_H]$ (with $[\cdot, \cdot]$ the Néron-Tate height pairing).

Here, we return to the more mysterious setting of k = K a real quadratic field $K = \mathbf{Q}(\sqrt{d})$ of discriminant

$$d_K = \begin{cases} d & \text{if } d \equiv 1 \mod 4\\ 4d & \text{if } d \equiv 2, 3 \mod 4 \end{cases}$$

prime to N, and corresponding even Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$. Let χ be any ring class character of K of conductor $c \in \mathbf{Z}_{\geq 1}$ prime to $d_K N$. Hence, we view χ a character of the corresponding ring class group $\operatorname{Pic}(\mathcal{O}_c) \cong \operatorname{Gal}(K[c]/K)$ of the **Z**-order $\mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K,

$$\chi: \operatorname{Pic}(\mathcal{O}_c) := \mathbf{A}_K^{\times} / \mathbf{A}^{\times} K_{\infty}^{\times} K^{\times} \widehat{\mathcal{O}}_c^{\times} \longrightarrow \mathbf{S}^1, \quad \widehat{\mathcal{O}}_c^{\times} := \prod_{v < \infty} \mathcal{O}_{c,v}^{\times}.$$

Via (1), the theories of Rankin-Selberg convolution and quadratic basechange imply that the Hasse-Weil *L*-function $L(E/K, \chi, s)$ has an analytic continuation $\Lambda(E/K, \chi, s)$ to all $s \in \mathbb{C}$ via a functional equation relating values at s to 2-s. Writing $\pi(\chi)$ to denote the automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ of level $d_K c^2$ and character η induced from the ring class character χ , this completed *L*-function $\Lambda(E/K, \chi, s)$ is equivalent to the corresponding shifted $\operatorname{GL}_2(\mathbf{A}) \times \operatorname{GL}_2(\mathbf{A})$ Rankin-Selberg *L*-function $\Lambda(s - 1/2, \pi \times \pi(\chi))$. Writing $\Pi = \operatorname{BC}_{K/\mathbf{Q}}(\pi)$ to denote the quadratic basechange lifting of π to a cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A}_K)$, the *L*-function $\Lambda(E/K, \chi, s)$ is also equivalent to the shifted $\operatorname{GL}_2(\mathbf{A}_K) \times \operatorname{GL}_1(\mathbf{A}_K)$ automorphic *L*-function $\Lambda(s - 1/2, \Pi \otimes \chi)$. Hence, we see the analytic continuation through the equivalent presentations

$$\Lambda(E/K,\chi,s) = \Lambda(s-1/2,\pi\times\pi(\chi)) = \Lambda(s-1/2,\Pi\otimes\chi).$$

As explained in (6) below, each $\Lambda(E/K, \chi, s)$ satisfies a symmetric functional equation. This gives the following immediate consequence, whose proof we explain in the discussion leading to Hypothesis 2.1 below:

Lemma 1.1. Let *E* be an elliptic curve of conductor *N* defined over \mathbf{Q} , and $\pi = \pi(f)$ the cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ associated to the eigenform $f \in S_2^{\operatorname{new}}(\Gamma_0(N))$ parametrizing *E*. Let *K* be a real quadratic field of discriminant d_K prime to *N*, with $\eta(\cdot) = \eta_K(\cdot) = \left(\frac{d_K}{d_K}\right)$ the corresponding Dirichlet

¹using the unitary normalization so that s = 1/2 is the central value

²which remains open in general

character. Hence, we can write $N = N^+N^-$ for N^+ the product of prime divisors $q \mid N$ which split in K, and N^- the product of prime divisors $q \mid N$ which remain inert in K, and $\eta(-N) = \eta(N) = \eta(N^-)$. If N^- is the squarefree product of an odd number of primes, then we have the vanishing of the central value

$$\Lambda(E/K,\chi,1) = \Lambda(1/2,\pi \times \pi(\chi)) = \Lambda(1/2,\Pi \otimes \chi) = 0$$

for any ring class character χ of K of conductor c prime to $d_K N$.

In the setup of forced vanishing described for Lemma 1.1, we study the central derivative values

$$\Lambda'(E/K,\chi,1) = \Lambda'(1/2,\pi \times \pi(\chi)) = \Lambda'(1/2,\Pi \otimes \chi).$$

We derive integral presentations for these derivative values as twisted linear combinations of special values of automorphic Green's functions for certain Hirzebruch-Zagier-like divisors on Hilbert modular surfaces. To do this, we adapt and develop calculation of Bruinier-Yang [8, Theorem 4.7], related to their distinct proof of the Gross-Zagier formula [8, §7], cf. [24] and [48]. This allows us to show some preliminary analogue of the Gross-Zagier formula for the mysterious setting of real quadratic fields. While there is no known global analogue of the Heegner point construction in this setting, we present some depiction of the provenance of such points $e_{??} \in E(K[c])$ in "geodesic" sets $Z(V_{A,2})$ associated to embeddings of the modular curve $Y_0(N)$ as a Hirzebruch-Zagier divisor into a quaternionic Hilbert modular surface.

Fix a primitive ring class character χ of K of conductor c prime to $d_K N$ (which we shall assume exists). For each class $A \in \operatorname{Pic}(\mathcal{O}_c)$, we fix an integral representative $\mathfrak{a} \subset \mathcal{O}_K$ so that $A = [\mathfrak{a}] \in \operatorname{Pic}(\mathcal{O}_c)$, and write $Q_{\mathfrak{a}}(z) := \mathbf{N}_{K/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ to denote the corresponding norm form of signature (1, 1). Here, we write $\mathbf{N}_{K/\mathbf{Q}}(z) = zz^{\tau}$ to denote the corresponding norm homomorphism, where $\tau \in \operatorname{Gal}(K/\mathbf{Q})$ denotes the nontrivial automorpoms. We also fix a **Z**-basis $\mathfrak{a} = [1, z_{\mathfrak{a}}]$ and write $\mathfrak{a}_{\mathbf{Q}} := \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{Q} = [1, z_{\mathfrak{a}}]\mathbf{Q}$ to denote the corresponding fractional ideal. We consider the quadratic space (V_A, q_A) of type (2, 2) defined by

$$V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}, \quad Q_A(z) = Q_A((z_1, z_2)) := Q_\mathfrak{a}(z_1) - Q_\mathfrak{a}(z_2).$$

We consider the corresponding spin group $\operatorname{GSpin}(V_A)$. As we explain in Proposition 3.3 below, we have an exceptinal isomorphism $\operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2$ of algebraic groups over **Q**. Consider the Grassmannian

$$D(V_A) = \{ z \subset V_A(\mathbf{R}) : \dim(z) = 2, Q_A|_z < 0 \}$$

of oriented negative definite³ hyperplanes in $V_A(\mathbf{R})$. Note that $D(V_A)$ has two connected components $D^{\pm}(V_A)$ corresponding to the choice of orientation. We shall fix one of these $D^{\pm}(V_A) \cong \mathfrak{H}^2$ consistently throughout. For any compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$, we can then consider the corresponding spin Shimura variety $\operatorname{Sh}(D(V_A), \operatorname{GSpin}(V_A))$ with complex points

$$\operatorname{Sh}_{U_A}(D(V_A)^{\pm}, \operatorname{GSpin}(V_A))(\mathbf{C}) = \operatorname{GSpin}(V_A)(\mathbf{Q}) \setminus (D(V_A)^{\pm} \times \operatorname{GSpin}(V_A)(\mathbf{A}_f)/U_A).$$

This gives a quasiprojective surface over \mathbf{Q} , which can be identified with a (quaternionic) Hilbert modular surface. Via the identification $\operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2$, we can take U_A to be the compact open subgroup of $\operatorname{GSpin}(V_A)(\mathbf{A}_f)$ corresponding to the two-fold product of congruence subgroup $\Gamma_0(N)$ (see (9)). We then have the more precise identification

$$\operatorname{Sh}_{U_A}(D(V_A)^{\pm}, \operatorname{GSpin}(V_A))(\mathbf{C}) \cong \operatorname{GL}_2(\mathbf{Q})^2 \setminus \left(\mathfrak{H}^2 \times \operatorname{GL}_2(\mathbf{A}_f)^2 / U_A\right) \cong Y_0(N) \times Y_0(N).$$

These surfaces come equipped with special Hirzebruch-Zagier divisors. To describe them, define for each $m \in \mathbf{Q}_{>0}$

$$\Omega_{m,A}(\mathbf{Q}) = \{ x \in V_A : Q_A(x) = m \}.$$

Consider the natural projection pr : $D(V_A)^{\pm} \times \operatorname{GSpin}(V_A)(\mathbf{A}_f) \longrightarrow \operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$. Given a vector $x \in V_A(\mathbf{R})$, consider the orthogonal projection $D(V_A)_x^{\pm} = \{z \in D(V_A)^{\pm} : z \perp x\}$. Let $\Lambda_A \subset V_A$ denote the maximal lattice associated with the compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)$, with $\Lambda_A^{\#}$ its dual lattice, and $\Lambda_A^{\#}/\Lambda_A$ the corresponding discriminant group. We define for each $\mu \in \Lambda_A^{\#}/\Lambda_A$ the divisor

$$Z_A(\mu, m) = \sum_{x \in (\mathrm{GSpin}(V_A)(\mathbf{Q}) \cap U_A) \setminus \Omega_{A,m}(\mathbf{Q})} \mathbf{1}_{\mu}(x) \operatorname{pr}(D(V_A)_x^{\pm}).$$

³We could just as well consider positive definite hyperplanes, the choice makes no difference.

Sums over cosets of these special divisors can be related to classical Hirzebruch-Zagier divisors. We consider these divisors in relation to the anisotropic subspaces $V_{A,2} \subset V_A$ of signature (1, 1) cut out by the integer ideal representatives \mathfrak{a} :

$$(V_{A,2}, Q_{A,2}), \quad V_{A,2} := \mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes \mathbf{Q}, \quad Q_{A,2}(\lambda) = Q_{\mathfrak{a}}(\lambda) = \frac{\mathbf{N}(\lambda)}{\mathbf{N}\mathfrak{a}} = \frac{\lambda\lambda^{\tau}}{\mathbf{N}\mathfrak{a}} \quad (\tau \neq \mathbf{1} \in \operatorname{Gal}(K/\mathbf{Q})).$$

Each such subspace $(V_{A,2}, Q_{A,2})$ gives rise to a "geodesic" set

 $z_{V_{A,2}} \in D(V_{A,2}) = \{ z \in V_{A,2}(\mathbf{R}) : \dim(z) = 1, \ Q_{A,2}|_z < 0 \}$

in the corresponding subdomain $D(V_{A,2})$ for $V_{A,2} \subset V_A$. Again, we have two connected components $D^{\pm}(V_{A,2})$ in $D(V_A)$ corresponding to the orientation of a hyperbolic line z in $V_{A,2}(\mathbf{R}) = \mathfrak{a}_{\mathbf{Q}} \otimes \mathbf{R} = [1, z_{\mathfrak{a}}]\mathbf{R}$. Each $D^{\pm}(V_{A,2})$ determines an open subset of real projective space of dimension one with a fixed orientation,

$$D^{\pm}(V_{A,2}) = \left\{ z^{\pm} = [x:y] \in \mathbf{P}^{1}(\mathbf{R}), \text{ orientation } \pm : Q_{A,2}(x,y) < 0 \right\}.$$

Each oriented hyperbolic line $z^{\pm} \in D^{\pm}(V_{A,2})$ determines a real curve of dimension one – equivalent to a real geodesic in the upper-half plane embedded into the Shimura surface $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$. Via the identifications $\operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2$ and $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \cong Y_0(N) \times Y_0(N)$ described above, each hyperbolic line $z^{\pm} \in D^{\pm}(V_{A,2})$ determines a real geodesic on $Y_0(N)$ embedded into $Y_0(N) \times Y_0(N)$. We consider for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ the corresponding "geodesic" set $Z(V_{A,2})$ associated to $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$:

$$Z(V_{A,2}) = \operatorname{GSpin}(V_{A,2})(\mathbf{Q}) \setminus \left(D^{\pm}(V_{A,2}) \times \operatorname{GSpin}(V_{A,2})(\mathbf{A}_f) / (U_A \cap \operatorname{GSpin}(V_{A,2})(\mathbf{A}_f)) \right) \subset Y_0(N).$$

To describe what we prove, we again write $\Lambda_A \subset V_A$ denote the maximal lattice corresponding to the chosen compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$, with $\Lambda_A^{\#}$ its dual lattice, and $\Lambda_A^{\#}/\Lambda_A$ its discriminant group. Let $\theta_{\Lambda_A}(\tau, z, h_f)$ denote the corresponding Siegel theta series defined on $\tau \in \mathfrak{H}$, $z \in D(V_A)$, and $h_f \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$. Let H_{0,Λ_A} denote the space of harmonic weak Maass forms of weight 0 and Weil representation r_{ψ,Λ_A} (defined below), with $M_{0,\Lambda_A}^! \subset H_{0,\Lambda_A}$ the subspace of weakly holomorphic forms, $M_{0,\Lambda_A} \subset M_{0,\Lambda_A}^!$ the subspace of holomorphic forms, and $S_{0,\Lambda_A} \subset M_{k,\Lambda_A}$ the subspace of cuspidal forms. Bruinier-Funke [7] define an antilinear differential operator

$$\xi_0: H_{0,\Lambda_A} \longrightarrow S_{2,\Lambda_A}, \quad \xi_0(\phi) := 2i \overline{\left(\frac{\partial \phi}{\partial \overline{\tau}}\right)}.$$

Note that this is related to the classical weight-lowering operator $L_0 = -2iv^2 \frac{\partial}{\partial \overline{\tau}}$ by $\xi_0(\phi)(\tau) = v^{-2} \overline{L_0 \phi(\tau)}$. In particular, this operator determines a short exact sequence of spaces of vector-valued modular forms

$$0 \longrightarrow M^!_{0,\Lambda_A} \longrightarrow H_{0,\Lambda_A} \xrightarrow{\xi_0} S_{2,-\Lambda_A} \longrightarrow 0,$$

where the subspace of weakly holomorphic forms

$$\phi(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \phi_{\mu}(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \gg -\infty \atop m \gg -\infty} c_{\phi}(\mu, m) e(m\tau) \mathbf{1}_{\mu} \in M^!_{0,\Lambda_A} \subset H_{0,\Lambda_A}$$

can be identified with ker(ξ_0). Writing A_{0,Λ_A} for the space of all smooth modular forms of weight 0 and representation r_{ψ,Λ_A} , so that $M_{0,\Lambda_A}^! \subset H_{0,\Lambda_A} \subset A_{0,\Lambda_A}$, we define a scalar product $\langle\langle f,g \rangle\rangle$ on forms

$$f(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} f_{\mu}(\tau) \mathbf{1}_{\mu} \in A_{0,\Lambda_A} \quad \text{and} \quad g(\tau) \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} g_{\mu}(\tau) \mathbf{1}_{\mu} \in A_{0,-\Lambda_A}$$

by the rule

$$\langle\langle f,g\rangle\rangle = \sum_{\mu\in\Lambda_A^{\#}/\Lambda_A} f_{\mu}(\tau)g_{\mu}(\tau).$$

Writing $\mathcal{F} = \{\tau = u + iv \in \mathfrak{H} : |u| \leq 1/2, u^2 + v^2 \geq 1\}$ to denote the standard fundamental domain for $SL_2(\mathbf{Z})$ acting on \mathfrak{H} , we define the Petersson inner product (when it converges) by the integral

$$\langle f,g \rangle = \int_{\mathcal{F}} \langle \langle f(\tau), \overline{g(\tau)} \rangle \rangle \frac{dudv}{v^2}.$$

Given a harmonic weak harmonic weak Maass form $f_0 \in H_{0,\Lambda_A}$, we then consider the regularized theta lift

$$\vartheta_{f_0}^{\star}(z,h_f) = \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \langle \langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle \rangle \frac{dudv}{v^2} = \mathrm{CT}_{s=0} \left\{ \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle \rangle v^{-s} \frac{dudv}{v^2} \right\}$$

given by the constant term in the Laurent series expansion around s = 0 of the function

$$\lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle \rangle v^{-s} \frac{dudv}{v^2},$$

where the limit is taken truncated fundamental domains $\mathcal{F}_T = \{\tau = u + iv \in \mathfrak{H} : |u| \leq 1/2, \tau \overline{\tau} \geq 1, \text{ and } v \leq T\}$ for the action of $SL_2(\mathbf{Z})$ on \mathfrak{H} .

A theorem of Bruinier [5], refining work of Borcherds [4], allows us to view these regularized theta lifts $\vartheta_{f_0}^{\star}$ as automorphic Green's functions in the sense of Arakelov theory. That is, if the Fourier coefficients $c_{f_0}^+(\mu, m)$ of the holomorphic part of f_0 are integers, then we define the corresponding divisor

$$Z_A(f_0) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_0}^+(\mu, -m) Z_A(\mu, m).$$

This theorem allows us to view the regularized theta lift $\vartheta_{f_0}^{\star}$ as the automorphic Green's function $G_{Z_A(f_0)}$ for this divisor $Z_A(f_0)$. We refer to Theorem 4.5 below for a more precise description, which gives us an arithmetic divisor $\widehat{Z}_A(f_0) = (Z_A(f_0), G_{Z_A(f_0)})$. To be more precise, for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ we take $f_{0,\eta,A} \in H_{0,-\Lambda_A}$ to be the harmonic weak Maass form of weight 0 and Weil representation $\overline{r}_{\psi,\Lambda_A} = r_{\psi,-\Lambda_A}$ whose image $g_{\eta,A} = \xi_0(f_{0,\eta,A}) \in S_{2,-\Lambda_A}$ under the differential operator $\xi_0 : H_{0,\Lambda_A} \to S_{2,-\Lambda_A}$ has a canonical lift as described in Theorem 4.6 to the twisted scalar-valued eigenform $f \otimes \eta \in S_2^{\operatorname{new}}(\Gamma_0(d_K^2N),\eta)$. Here, we write $-\Lambda_A$ to denote the quadratic space determined by $(\Lambda_A, -q_A)$. Each of the vector-valued cusp forms $g_{A,\eta}$ has Fourier series expansion given explicitly in terms of the Fourier coefficients of the eigenform $f \in S_2^{\operatorname{new}}(\Gamma_0(N))$ parametrizing the elliptic curve E. To be more precise, we have for each class $A = [\mathfrak{a}] \in \operatorname{Pic}(\mathcal{O}_c)$ the relation

$$g_{\eta,A}(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} g_{\eta,A,\mu}(\tau) \mathbf{1}_{\mu} = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \left(\sum_{\substack{m \in \mathbf{Q}_{>0} \\ m \equiv d_K^2 N Q_A(\mu) \bmod d_K^2 N}} c_f(m) \eta(m) s(m) e\left(\frac{m\tau}{d_K^2 N}\right) \right) \mathbf{1}_{\mu}.$$

Here, using the standard notations $\tau = u + iv \in \mathfrak{H}$ and $e(\tau) = \exp(2\pi i\tau)$, we write s to denote the function defined on classes $m \mod d_K N$ by $s(m) = 2^{\Omega(m, d_K^2 N)}$, where $\Omega(m, d_K^2 N)$ is the number of divisors of the greatest common divisor $(m, d_K^2 N)$ of m and $d_K N$. This Maass form $f_{0,\eta,A} \in H_{0,\Lambda_A}$ determined by $\xi_0 f_{0,\eta,A}(\tau) = g_{\eta,A}(\tau)$ has a decomposition $f_{0,\eta,A}(\tau) = f_{0,\eta,A}^+(\tau) + f_{0,\eta,A}^-(\tau)$ into a holomorphic part $f_{0,\eta,A}^+(\tau)$ and an antiholomorphic part $f_{0,\eta,A}^-(\tau)$. We write the respective Fourier series expansions as

$$f_{0,\eta,A}^{+}(\tau) = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} f_{0,\eta,A,\mu}^{+}(\tau) \mathbf{1}_{\mu} = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} \left(\sum_{m \in \mathbf{Q} \atop m \gg -\infty} c_{f_{0,\eta,A}}^{+}(\mu,m) e(m\tau) \right) \mathbf{1}_{\mu}$$

and

$$f_{0,\eta,A}^{-}(\tau) = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} f_{0,\eta,A,\mu}^{-}(\tau) \mathbf{1}_{\mu} = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} \left(\sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_{f_{0,\eta,A}}^{-}(\mu,m) W_{0}(2\pi m v) e(m\tau) \right) \mathbf{1}_{\mu}$$

with Whittaker function $W_0(m) = \int_{-2m}^{\infty} e^{-t} dt = \Gamma(1, 2|m|)$ defined for m < 0.

Our main results, Theorem 4.19 and Corollary 4.20, allow us to express the central derivative value $\Lambda'(1/2, \Pi \otimes \chi)$ in terms of sums of these Green's functions $G_{Z(f_{0,\eta,A})}$ along the "geodesic" spaces $Z(V_{A,2})$. To give the relation more precisely, we first describe how we decompose the theta series $\theta_{\Lambda_A}(\tau, z, h_f)$ for our main calculation. We consider the anisotropic subspaces $V_{A,1} := \mathfrak{a}_{\mathbf{Q}}$ with $Q_{A,1}(z) = -Q_{\mathfrak{a}}(z)$ and $V_{A,2} = \mathfrak{a}_{\mathbf{Q}}$ with $Q_{A,2}(\lambda) = Q_{\mathfrak{a}}(z)$ of type (1, 1). We consider for each j = 1, 2 the sublattice $\Lambda_{A,j} := \Lambda_A \cap V_{A,j}$, and the corresponding (nonholomorphic) Siegel theta series $\theta_{\Lambda_{A,j}}(\tau, z, h_f) : \mathfrak{H} \times D(V_{A,j}) \longrightarrow \mathcal{S}_{\Lambda_{A,j}}$ of weight zero and representation $r_{\psi,\Lambda_{A,j}}$, where $D(V_{A,j})$ denotes the corresponding subdomain of $D(V_A)$. As explained below,

since we evaluate at elements $z_{V_{A,2}} \in D(V_{A,2})$ and $h_f \in \operatorname{GSpin}(V_{A,2})(\mathbf{A}_f)$, we can replace the Siegel theta series $\theta_{\Lambda_A}(\tau, z_{V_{A,2}}, h_f)$ with the corresponding product of specializations $\theta_{\Lambda_{A,1}}(\tau, 1, 1) \otimes \theta_{\Lambda_{A,2}}(\tau, z_{V_{A,2}}, h_f)$. We use the Siegel-Weil theorem (Theorem 4.8 and Corollary 4.9) to interpret the sum

$$2\int_{\mathrm{SO}(V_{A,2})(\mathbf{Q})\backslash \operatorname{SO}(V_{A,2})(\mathbf{A})} \theta_{\Lambda_{A,2}}(\tau, z_{V_{A,2}}, h) dh$$

as the value at s = 0 of an $S_{\Lambda_{A,2}}$ -valued Eisenstein series $E_{\Lambda_{A,2}}(\tau, s; 0)$ of weight 0, which is holomorphic at s = 0. Following the approach of Kudla [34], we interpret this Eisenstein series as the image under the antilinear differential weight-lowering operator ξ_2 of a derivative Eisenstein series $E'_{\Lambda_{A,2}}(\tau, 0; 2)$ of weight two. We remark that this is not an "incoherent" Eisenstein series, but rather a classical Siegel Eisenstein series of weight zero associated to the lattice $\Lambda_{A,2}$. We describe it in more detail below, together with the Langlands functional equation; see Propositions 4.10 and 4.12. Let $\mathcal{E}_{\Lambda_{A,2}}(\tau)$ denote the holomorphic part of $E'_{\Lambda_{A,2}}(\tau, 0; 2)$. Writing $\theta^+_{\Lambda_{A,1}}(\tau)$ to denote the holomorphic part of the theta series $\theta_{\Lambda_{A,1}}(\tau)$, we consider the constant coefficient

(2)
$$\operatorname{CT}\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}^{+}(\tau) \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle$$

in the Fourier series expansion of $\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+(\tau) \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle$. Observe that this constant coefficient (2) is an algebraic number. Let h_K denote the class number of K, and ϵ_K the fundamental unit, so that $\epsilon_K = \frac{1}{2}(t + u\sqrt{d_K})$ is the least integral solution (with u minimal) to Pell's equation $t^2 - d_K u^2 = 4$.

Theorem 1.2 (Theorem 4.19, Corollary 4.5). In the setup described above, we have the integral presentation $\Lambda'(1/2, \Pi \otimes \chi) = \Lambda'(E/K, \chi, 1)$

$$= -\frac{\sqrt{d_K}}{\log \epsilon_K \cdot h_K} \cdot \frac{1}{2} \sum_{\substack{A \in \operatorname{Pic}(\mathcal{O}_C) \\ A = [\mathfrak{a}]}} \chi(A) \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \rangle + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \vartheta_{f_{0,\eta,A}}^{\star}(z^{\pm},h) \right) \cdot \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \vartheta_{f_{0,\eta,A}}^{\star}(z^{\pm},h) \right) \cdot \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{CT} \langle \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,2}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{1,\eta,A}^+(\tau), \theta_{1,\eta,A}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{1,\eta,A}^+(\tau), \theta_{1,\eta,A}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{1,\eta,A}^+(\tau), \theta_{1,\eta,A}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{1,\eta,A}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \right) \right| + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z^{\pm},h) \in Z(V_{A,2})} \left| \left(\operatorname{V} \langle f_{1,\eta,A}^+ \otimes$$

Equivalently, writing $G_{Z(f_{0,\eta,A})}$ for each class A to denote the automorphic Green's function for the divisor

$$Z(f_{0,\eta,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \in \mathbf{Q}_{>0}} c^+_{f_{0,\eta,A}}(\mu, -m) \cdot Z_A(\mu, m)$$

given by linear combination of special (Hirzebruch-Zagier) divisors $Z_A(\mu, m)$ on $Sh_{U_A}(D_{V_A}, GSpin(V_A))$, let

$$G_{Z(f_{0,\eta,A})}(V_{A,2}) = \sum_{(z^{\pm},h)\in Z(V_{A,2})} \vartheta_{f_{0,\eta,A}}^{\star}(z^{\pm},h)$$

denote the sum along the geodesic $Z(V_{A,2})$ in $\operatorname{Sh}_{U_A}(D(V_A), \operatorname{GSpin}(V_A))$. We obtain the integral presentation

$$\Lambda'(1/2,\Pi\otimes\chi) = \Lambda'(E/K,\chi,1)$$

$$= -\frac{\sqrt{d_K}}{\log\epsilon_K \cdot h_K} \cdot \frac{1}{2} \sum_{\substack{A \in \operatorname{Pic}(\mathcal{O}_C)\\A = [\mathfrak{a}]}} \chi(A) \left(\operatorname{CT}\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+ \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle + \frac{\operatorname{vol}(U_{A,2})}{4} G_{Z(f_{0,\eta,A})}(V_{A,2})\right).$$

If we assume the ersatz Heegner hypothesis (Lemma 1.1, Hypothesis 2.1) that the inert level N^- is given by the squarefree product of an odd number of primes, then $L(1/2, \Pi \otimes \chi) = 0$ by symmetric functional equation (6), and so the central derivative value $\Lambda'(1/2, \Pi \otimes \chi)$ described by our formula is not forced to vanish. The analogous formula for central values $\Lambda(1/2, \Pi \otimes \chi)$ in the setting where $\eta(-N) = \eta(N) = +1$ is given by Popa [39, § 1, Theorem 6.3.1]. This develops Waldspurger's theorem [45] to give an exact toric period formula for these central values, and generalizes the formula of Gross [22] for the analogous setup with K an imaginary quadratic field. Roughly speaking, Waldspurger's theorem [45] equates the nonvanishing of the central value $\Lambda(1/2, \pi \times \pi(\chi))$ with that of the period integral

$$\int_{\mathbf{A}_{K}^{\times}/K^{\times}}\varphi(t)\chi(t)dt$$

for $\varphi \in \pi^{JL}$ a vector in the Jacquet-Langlands lift π^{JL} of π to an *indefinite* quaternion algebra B over \mathbf{Q} with ramification given by the inert level: Ram $(B) = \{q \mid N^-\}$. Popa [39] gives an exact and even classical formula

for $L(1/2, \pi \times \pi(\chi))$ as such as toric integral, which according to the discussion in [39, § 6] can be viewed as a twisted sum over geodesic on the modular curve $X_0(N)$ parametrizing E. Our Theorem 4.19 can be viewed as an analogue of Popa's theorem for the central derivative values $\Lambda'(1/2, \Pi \otimes \chi) = \Lambda'(1/2, \pi \times \pi(\chi))$ when the generic root number is $\eta(-N) = \eta(N) = -1$ (i.e. when Hypothesis 2.1 holds).

1.0.1. A geometric interpretation. Let us consider the geodesic sets $Z(V_{A,2})$ associated to the subspaces $(V_{A,2}, q_{A,2})$ of signature (1, 1). We describe these in more detail in §4.3.5 below.

We can identify the Grassmannian $D(V_{A,2}) \cong \{z = [x : y] \in \mathbf{P}^1(\mathbf{R}) : q_{A,2}(x,y) < 0\}$ of hyperbolic lines with the symmetric space $D(\operatorname{GSpin}(1,1))$ of $\operatorname{GSpin}(1,1) \cong \mathbf{G}_m \times \operatorname{SO}(1,1)$. On the other hand, we can consider the symplectic group $\operatorname{GSp}_4(W)$ acting on a four-dimensional symplectic space W. The Siegel parabolic $P = \{g \in \operatorname{GSp}_4(W) : gL = L\}$ of $\operatorname{GSp}_4(W)$ stabilizing a (maximal isotropic) two-dimensional Lagrangian subspace $L \subset W$ has Levi subgroup $M_P \cong \mathbf{G}_m \times \operatorname{GL}_2$. Viewing GL_2 as an extension of $\operatorname{SO}(1,1)$ via the inclusion

$$SO(1,1) \subset GSpin(1,1) \cong \mathbf{G}_m \times \mathbf{G}_m \longrightarrow GL_2, \quad (t_1,t_2) \longmapsto \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},$$

we obtain an embedding of $D(V_{A,2})$ into the corresponding symmetric space $D(M_P)$ for M_P . In this way, we can realize each geodesic set $Z(V_{A,2})$ inside a component of the boundary of the Borel-Serre compactification of a $GSp_4(W)$ Shimura variety.

To state this more formally, let $(\tilde{V}_{A,2}, \tilde{q}_{A,2})$ be any rational quadratic space of signature (3, 2) into which $(V_{A,2}, q_{A,2})$ embeds. Consider the corresponding spin group $\operatorname{GSpin}(\tilde{V}_{A,2})$ and Grassmannian of negative definite hyperplanes $D(\tilde{V}_{A,2})$. Let $\tilde{\Lambda}_{A,2} \subset \tilde{V}_{A,2}$ be any lattice for which $\tilde{\Lambda}_{A,2} \cap V_{2,A} = \Lambda_{A,2} = \mathfrak{a}$, and let $\tilde{U}_{A,2}$ denote the corresponding compact open subgroup of $\operatorname{GSpin}(\tilde{V}_{A,2})(\mathbf{A}_f)$. The spin Shimura variety $\operatorname{Sh}_{\tilde{U}_{A,2}}(\operatorname{GSpin}(\tilde{V}_{A,2}), D(\tilde{V}_{A,2}))$ with complex points

$$\operatorname{Sh}_{\widetilde{U}_{A,2}}(\operatorname{GSpin}(\widetilde{V}_{A,2}), D(\widetilde{V}_{A,2}))(\mathbf{C}) = \operatorname{GSpin}(\widetilde{V}_{A,2})(\mathbf{Q}) \setminus D(\widetilde{V}_{A,2}) \times \operatorname{GSpin}(\widetilde{V}_{A,2})(\mathbf{A}_f)/\widetilde{U}_{A,2}$$

defines a quasiprojective variety of dimension 3 over \mathbf{Q} . Via the accidental isomorphisms

$$\operatorname{Spin}(3,2) \cong \operatorname{Sp}_4(W), \quad \operatorname{GSpin}(3,2) \cong \operatorname{GSp}_4(W)$$

it can be identified as a Siegel threefold $X_{A,2} \cong \operatorname{Sh}_{\widetilde{U}_{A,2}}(\operatorname{GSpin}(\widetilde{V}_{A,2}), D(\widetilde{V}_{A,2}))$. Hence, the symmetric space $D(V_{A,2})$ can be realized as a component in the boundary $\partial X_{A,2}^{\operatorname{BS}}$ of the Borel-Serre compactification $X_{A,2}^{\operatorname{BS}}$ of X_A . Via Theorem 1.2, this suggests that the study of the boundaries of Borel-Serre compactifications of Siegel threefolds $X_{A,2}$ of this type – realized as spin Shimura varieties associated to rational quadratic spaces of signature (3, 2) – might shed light on the provenance of "Stark-Heegner" points in $X_0(N)(K[c]) \longrightarrow E(K[c])$. This observation also allows us to interpret our main formula in terms of $\partial X_{A,2}^{\operatorname{BS}}$ for any such Siegel threefold $X_{A,2}$. We hope to return to this idea in a subsequent work. Let us note that the strategy of realizing locally symmetric spaces for GL_n in the boundaries of Borel-Serre compactifications of ambient symplectic or unitary Shimura varieties, which seems to go back to Clozel (cf. [13]), is used crucially in the constructions by Scholze [42], Harris-Lan-Taylor-Thorne [26], and Allen-Calegari-Caraiani-Gee-Helm-Le Hung-Newton-Scholze-Taylor-Thorne [1] of Galois representations associated to cuspidal GL_n-automorphic representations.

1.0.2. Other remarks. (i). The regularized theta lifts $\vartheta_{f_{0,\eta,A}}^{\star} = G_{Z(f_{0,\eta,A})}$ can be related to the theta lifts constructed by Kudla-Millson in [37] by the arguments of Bruinier-Funke [7, Theorems 1.4 and 1.5]. Such relations, which hold for any signature (p, q), suggest another potential geometric development of this formula.

(ii). The role played by the holomorphic projection in [24] is replaced here by the holomorphic part $\mathcal{E}_{\Lambda_{A,2}}(s,\tau)$ of the derivative Eisenstein series $E'_{\Lambda_{A,2}}(s,\tau;2)$. More precisely, applying the Siegel-Weil formula to $\theta_{\Lambda_{A,2}}$ gives the value at $s_0 = 0$ of a weight zero Eisenstein series $E_{L_{A,2}}(s,\tau;0)$. We can realize this $E_{L_{A,2}}(s,\tau;0)$ as the image under the weight-lowering operator L_2 of the derivative at s = 0 of a weight two Eisenstein series $E_{\Lambda_{A,2}}(s,\tau;2)$ (see Proposition 4.12). This derivative value $E'_{L_{A,2}}(s,\tau;2)|_{s=0}$ appears in the Rankin-Selberg integral presentation of $L'(0,\xi_0(f_{0,\eta,A}) \times \theta_{\Lambda_{A,1}})$. (iii). Recall that a complex number is a *period* if its real and imaginary parts can be expressed as integrals of rational functions with rational coefficients, over domains in \mathbb{R}^n given by polynomials inequalities with rational coefficients. We expect the values $\Lambda'(E/K, \chi, 1)$ are always periods (cf. [32, Question 4]), as this would be implied refined conjecture of Birch and Swinnerton-Dyer. We note that this can be deduced in the special cases described in Corollary 5.1 via the argument given in [32, §4] for the Birch-Swinnerton-Dyer constant. We expect that the values taken by the regularized theta lifts $\vartheta_{f_0}^*$ here are periods. The following heuristic suggests that the values of the regularized theta lift $\vartheta_{f_0}^*$ at special divisors should always be periods: We can decompose any cuspidal harmonic weak Maass form f_0 into a linear combination of Poincaré series $F_{\mu,m}$ as in [5, Theorem 2.14]. Ignoring issues of convergence, we obtain a decomposition for the regularized theta lift $\vartheta_{f_0}^*$ into a linear combination of the regularized theta lift $\vartheta_{f_0}^*$ at rational series at the series $\vartheta_{F_{\mu,m}}^*$. Evaluated at the "points" we consider, these constituents $\vartheta_{F_{\mu,m}}^*$ can be computed as a rational linear combination of the Gaussian hypergeometric function $_2F_1$ at rationals – which are known to be periods.

In this direction, we expect the values $\Lambda'(E/K, \chi, 1)$ on the right-hand side of Theorem 1.2 can be expressed as some algebraic number times the arithmetic height of some algebraic cycle, and in this way seen to be a period – in the same way that the Birch-Swinnerton-Dyer constant⁴ is shown to be a period in Kontsevich-Zagier [32, § 3.5]. Note that such a relation to arithmetic heights can be established for the more general setting of Green's functions evaluated along CM cycles of spin Shimura varieties for (n, 2) by the combined works of Bruinier-Yang [8, Theorem 1.2] and Andreatta-Goren-Howard-Madapusi Pera [2, Theorem A].

1.0.3. Applications towards Birch-Swinnerton-Dyer. Theorem 4.19 also suggests a possible origin of points in the K[c]-rational Mordell-Weil groups E(K[c]) in via embeddings of Hirzebruch-Zagier divisors into spin Shimura varieties. In this spirit, we also describe how the refined Birch and Swinnerton-Dyer conjecture suggests new characterizations of the Tate-Shafarevich group $III_E(K[c])$ and regulator term $R_E(K[c])$. We refer to (62), (63), and below for more details of what can be deduced here. One consequence is the following.

Corollary 1.3 (Theorem 5.1). Assume the ersatz Heegner hypothesis (Lemma 1.1, Hypothesis 2.1) that the inert level N^- is given by the squarefree product of an odd number of primes, then $L(1/2, \Pi \otimes \chi) = 0$ by symmetric functional equation (6). Writing E again to denote the underlying elliptic curve over \mathbf{Q} , we write $E^{(d_K)}$ to denote its quadratic twist. Let us also assume that E has semistable reduction so that its conductor N is squarefree, with N coprime to the discriminant d_K of K, and for each prime $p \geq 5$:

- The residual Galois representations E[p] and $E^{(d_K)}[p]$ attached to E and $E^{(d_K)}$ are irreducible.
- There exists a prime divisor $l \parallel N$ distinct from p where the residual representation E[p] is ramified, and a prime divisor $q \parallel Nd_K$ distinct from p where the residual representation $E^{(d_K)}[p]$ is ramified.

For either elliptic curve $A = E, E^{(d_K)}$, let us write $III_A(\mathbf{Q})$ to denote the Tate-Shafarevich group, with $T_A(\mathbf{Q})$ the product over local Tamagawa factors, and ω_A a fixed invariant differential for A/\mathbf{Q} . Suppose that $ord_{s=1} \Lambda(E/K, 1) = 1$, so that either $\Lambda(E, 1) = \Lambda(1/2, \pi)$ or the quadratic twist $\Lambda(E^{(d_K)}, 1) = \Lambda(1/2, \pi \otimes \eta)$ vanishes. Writing [e, e] to denote the regulator of either E or $E^{(d_K)}$ according to which factor vanishes, we have the following unconditional identity, up to powers of 2 and 3:

⁴We remark that the idea of the deduction, not given explicitly in [32, §3.5], is to use the formulae of Gross-Zagier [24] and Gross-Kohnen-Zagier [23] to verify that $L'(E, 1) = \alpha \cdot R \cdot \Omega$, where α denotes some nonzero rational number, $R = R_E(\mathbf{Q}) = \langle e, e \rangle$ the regulator (given by the arithmetic height of a Heegner divisor on the modular curve $X_0(N)$), and $\Omega = \Omega_E(\mathbf{Q})$ the real period. Assuming the finiteness of the Tate-Shafarevich group $III_E(\mathbf{Q})$ (implicitly), the argument of Kontsevich-Zagier [32, § 3.5] shows that the Birch-Swinnerton-Dyer constant $\kappa_E(\mathbf{Q}) := (R_E(\mathbf{Q}) \cdot T_E(\mathbf{Q}) \cdot III_E(\mathbf{Q}) \cdot \Omega_E(\mathbf{Q}))/\#E(\mathbf{Q})^2$ is a period. In other words, their deduction consists of first relating L'(E, 1) to $\kappa_E(\mathbf{Q})$ via the Gross-Zagier formula, then using the fact that $\kappa_E(\mathbf{Q})$ is known to be a period to deduce that L'(E, 1) must be a period. There does not seem to be any direct proof in the literature that the central derivative value L'(E, 1) is a period.

$$\frac{\#\operatorname{III}_{E}(\mathbf{Q}) \cdot \#\operatorname{III}_{E^{(d_{K})}}(\mathbf{Q}) \cdot [e, e] \cdot T_{E}(\mathbf{Q}) \cdot T_{E^{(d_{K})}}(\mathbf{Q})}{\#E(\mathbf{Q})^{2}_{\operatorname{tors}} \cdot \#E^{(d_{k})}(\mathbf{Q})^{2}_{\operatorname{tors}}} \cdot \int_{E(\mathbf{R})} |\omega_{E}| \cdot \int_{E^{(d_{K})}(\mathbf{R})} |\omega_{E^{(d_{k})}}| \\
= -\frac{\sqrt{d_{K}}}{\log \epsilon_{K}} \cdot \frac{1}{2} \sum_{A \in \operatorname{Pic}(\mathcal{O}_{K})} \left(\operatorname{CT}\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}^{+} \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z_{V_{A,2}},h) \in Z(V_{A,2})} \vartheta_{f_{0,\eta,A}}^{*}(z_{V_{A,2}},h) \right) \\
= -\frac{\sqrt{d_{K}}}{\log \epsilon_{K}} \cdot \frac{1}{2} \sum_{A \in \operatorname{Pic}(\mathcal{O}_{K})} \left(\operatorname{CT}\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}^{+} \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle + \frac{\operatorname{vol}(U_{A,2})}{4} G_{Z(f_{0,\eta,A})}(Z_{A,2}) \right).$$

Note that the value on the left-hand side is known to be a period via the argument of $[32, \S4]$.

It would be interesting to develop these relations in connection to the real quadratic Borcherds products studied by [15], perhaps leading to a global analogue of Darmon's conjecture [14, Conjecture 5.6] via the Borel-Serre compactifications of Siegel threefolds arising as spin Shimura varieties associated to rational quadratic subspaces $(\tilde{V}_{A,2}, \tilde{q}_{A,2}) \supset (V_{A,2}, q_{A,2})$ of signature (3,2). It would also be interesting to use the same setup with K replaced by an imaginary quadratic field of discriminant d_k prime to N to develop a new proof of the Gross-Zagier formula, developing the ideas of [8, §7-8] in this setup to derive a unified description for quadratic fields, and perhaps in this way realizing the geodesics sets $Z(V_{A,2})$ we consider here as boundary components in compactifications of higher-dimensionam Shimura varieties, e.g. for GSp₄.

Outline. We first describe the setup with L-functions and their functional equations in §2, then spin Shimura varieties in §3. We describe regularized theta lifts in §4.4, leading to the main Theorem 4.19 and Corollary 4.5. Our main results are derived in Theorem 4.16 (using Proposition 4.12), Theorem 4.19, and Corollary 4.20. Finally, we describe relations to the Birch and Swinnerton-Dyer conjecture in §5.

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2. Background on L-functions

2.1. Equivalences of *L*-functions and symmetric functional equations. Let *E* be an elliptic curve of conductor *N* defined over **Q**, parametrized via modularity by a cuspidal newform $f \in S_2(\Gamma_0(N))$. Let $\pi = \bigotimes_v \pi_v$ denote the cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ generated by *f*. Hence we have identifications of completed *L*-functions

(3)
$$\Lambda(E,s) = \Lambda(s-1/2,f) = \Lambda(s-1/2,\pi) = \prod_{v \le \infty} L(s-1/2,\pi_v).$$

Again, we fix K a real quadratic field of discriminant d_K prime to the conductor N, and write $\eta = \eta_{K/\mathbf{Q}}$ to denote the corresponding Dirichlet character. As well, we fix a ring class character χ of K of some conductor $c \in \mathbf{Z}_{\geq 1}$ coprime to $d_K N$. Let K[c] denote the ring class extension of K of conductor c. Inspired by the conjecture of Darmon [14, Conjecture 5.6] and the theorem of Gross-Zagier [24], we seek to detect Heegner-like points in the Mordell-Weil group E(K[c]) of K[c]-rational points through the study of integral presentations of the central derivative value $\Lambda'(E/K, \chi, 1)$ of the completed Hasse-Weil L-function $\Lambda(E/K, \chi, s)$ of E basechanged to K and twisted by χ . By the theory of Rankin-Selberg convolution (cf. e.g. [24]), we deduce from (3) that the Hasse-Weil L-function $L(E/K, \chi, s)$ has an analytic continuation $\Lambda(E/K, \chi, s)$ to all $s \in \mathbf{C}$ via its identification with the Rankin-Selberg L-function $\Lambda(s, \pi \times \pi(\chi))$ of π times the representation $\pi(\chi) = \bigotimes_v \pi(\chi)_v$ of $\operatorname{GL}_2(\mathbf{A})$ induced by π :

(4)
$$\Lambda(E/K,\chi,s) = \Lambda(s-1/2,\pi\times\pi(\chi)) = \prod_{v\leq\infty} L(s-1/2,\pi_v\times\pi(\chi)_v).$$

On the other hand, recall that by the theory of cyclic basechange (in the sense of [38], [3]), we can attach to π a cuspidal automorphic representation $\Pi = BC_{K/\mathbf{Q}}$ of $GL_2(\mathbf{A}_K)$. It is then well-known that the Rankin-Selberg

L-function $\Lambda(s, \pi \times \pi(\chi))$ for $\operatorname{GL}_2(\mathbf{A}) \times \operatorname{GL}_2(\mathbf{A})$ is equivalent to the twisted standard or Godement-Jacquet L-function $\Lambda(s, \Pi \otimes \chi)$ on $\operatorname{GL}_2(\mathbf{A}_K) \times \operatorname{GL}_1(\mathbf{A}_K)$. This gives us another equivalence of L functions

(5)
$$\Lambda(E/K,\chi,s) = \Lambda(s-1/2,\Pi\otimes\chi) = \prod_{w\leq\infty} L(s-1/2,\Pi_w\otimes\chi_w),$$

where we view χ as an idele class character $\chi = \bigotimes_w \chi_w$ of K having trivial archimedean component $\chi_\infty \equiv 1$.

In each of these presentations (4) and (5), the *L*-function $L(s, \pi \times \pi(\chi)) = L(s, \Pi \otimes \chi)$ has a well-known analytic continuation to all $s \in \mathbf{C}$, and satisfies a functional equation relating values at s to 1 - s. Moreover, since $\pi \cong \tilde{\pi}$ is self-dual, and ring class characters equivariant under complex conjugation, the Rankin-Selberg *L*-function $\Lambda(s, \pi \times \pi(\chi))$ satisfies a symmetric functional equation

(6)
$$\Lambda(s, \pi \times \pi(\chi)) = \epsilon(s, \pi \times \pi(\chi))\Lambda(1 - s, \pi \times \pi(\chi))$$

with epsilon factor

$$\epsilon(s, \pi \times \pi(\chi)) = c(\pi \times \pi(\chi))^{\frac{1}{2}-s} \cdot \epsilon(1/2, \pi \times \pi(\chi)) = (d_K^2 N^2 c^4)^{\frac{1}{2}-s} \cdot \epsilon(1/2, \pi \times \pi(\chi))$$

and root number $\epsilon(1/2, \pi \times \pi(\chi)) \in \{\pm 1\} \subset \mathbf{S}^1$ given by the simple formula

(7)
$$\epsilon(1/2, \pi \times \pi(\chi)) = \eta(-N) = \eta(N).$$

Here, we write $c(\pi \times \pi(\chi)) = d_K^2 N^2 c^4$ to denote the conductor of the *L*-function $\Lambda(s, \pi \times \pi(\chi))$, and use that the quadratic Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$ is even (as *K* is a real quadratic field). Note that this formula (7) holds for any choice of ring class character χ of *K* of conductor *c* coprime to the product $d_K N$, and that this functional equation does not depend on the choice of ring class character χ . Since the functional equation (6) is symmetric, we deduce that must be forced vanishing of the central value $\Lambda(1/2, \pi \times \pi(\chi)) = \Lambda(1/2, \Pi \otimes \chi) = 0$ when $\eta(N) = -1$. We can therefore impose the following condition on the level *N* of π , equivalently the conductor *N* of *f* and *E*, to ensure this forced vanishing. Here, since we assume that *N* is coprime to the disciminant d_K , we can assume that the conductor *N* factorizes as $N = N^+N^-$, where for each prime $q \mid N$,

$$\begin{array}{cccc} q \mid N^+ & \Longleftrightarrow & \eta(q) = 1 & \Longleftrightarrow & q \text{ splits in } K \\ q \mid N^- & \Longleftrightarrow & \eta(q) = -1 & \Longleftrightarrow & q \text{ is inert } K. \end{array}$$

Hypothesis 2.1 (Ersatz Heegner hypothesis). Let us assume that the inert level N^- is the squarefree product of an odd number of primes, and hence that the root number of $\Lambda(s, \pi \times \pi(\chi))$ for χ any ring class character of K of conductor c prime to $d_K N$ is given by $\epsilon(1/2, \pi \times \pi(\chi)) = \eta(-N) = \eta(N) = \eta(N^-) = -1$.

If the condition of Hypothesis 2.1 is met, then the corresponding central value $\Lambda(1/2, \pi \times \pi(\chi))$ is forced by the functional equation (6) to vanish: $\Lambda(1/2, \pi \times \pi(\chi)) = \Lambda(1/2, \Pi \otimes \chi) = 0$. It then makes sense to derive integral presentations for the central derivative values in this case,

$$\Lambda'(1/2, \pi \times \pi(\chi)) = \Lambda'(1/2, \pi_K \otimes \chi) = ?$$

The conjectures of Birch-Swinnerton-Dyer, Darmon [14, Conjecture 5.6], Kudla, and even Bruinier-Yang [8, Conjecture 1.1] (for instance) suggest that this central derivative value should be related to the height of a CM-type point on some Shimura variety associated to the modular curve $X_0(N)$.

2.2. The basechange representation. Let us now consider the quadratic basechange lifting $\Pi = BC_{K/\mathbf{Q}}(\pi)$ of π to $GL_2(\mathbf{A}_K)$, which exists by the theory of Langlands [38] and more generally Arthur-Clozel [3]. Note that this basechange representation Π of $GL_2(\mathbf{A}_K)$ has trivial central character. We refer to the article of Gérardin-Labesse [20] for more background on the general properties of cyclic basechange representations. Let us first record that this quadratic representation is known to be cuspidal.

Proposition 2.2. Let $\pi = \pi(f)$ be a cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ of trivial central character corresponding to a newform $f \in S_2^{\operatorname{new}}(\Gamma_0(N))$ parametrizing an elliptic curve E/\mathbf{Q} of conductor N. Let K be any real quadratic field. Let $\Pi = \operatorname{BC}_{K/\mathbf{Q}}(\pi)$ denote the quadratic basechange lifting of π to an automorphic representation of $\operatorname{GL}_2(\mathbf{A}_K)$. Then, Π must be cuspidal.

Proof. We know by Langlands [38, Ch. 2, (B), p. 19] that the quadratic basechange representation Π is cuspidal if and only if $\Pi \cong \Pi^{\tau}$ for $\tau \in \operatorname{Gal}(K/\mathbf{Q})$ the nontrivial automorphism. On the other hand, by the characterization given in [38, Ch. 2, (i), (ii)], we see that this condition must always hold here. Roughly speaking, this characterization amounts to the condition $L(s,\Pi) = L(s,\pi \circ \mathbf{N}_{K/\mathbf{Q}})$. Since π is defined over **Q** and hence invariant under the action of $\tau \in \operatorname{Gal}(K/\mathbf{Q})$, so too is the composition of π with the norm homomorphism $N_{K/\mathbf{Q}}$. In this way, we see that $L(s,\Pi^{\tau}) = L(s,\pi \circ \mathbf{N}_{K/\mathbf{Q}}) = L(s,\Pi) = L(s,\pi)L(s,\pi \otimes \eta)$ and hence $\Pi \cong \Pi^{\tau}$, so that Π must be cuspidal.

We can also consider the basechange of the elliptic curve E/\mathbf{Q} to the quadratic field K, with E(K) its Mordell-Weil group. The theorem of Freitas-Le Hung-Siksek [17, Theorem 1] shows that E(K) is modular. Hence, its completed L-function $\Lambda(E/K, s)$ is equivalent to the shift by 1/2 of the corresponding L-function $L(s,\sigma)$, with $\sigma = \bigotimes_w \sigma_w$ a cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A}_K)$ determined uniquely by E(K). On the other hand, using the modularity of $E(\mathbf{Q})$ with the Artin basechange decomposition described above (which implies that $L(s,\Pi) = L(s,\pi)L(s,\pi\otimes\eta)$), it follows that

$$\Lambda(E/K,s) = \Lambda(s-1/2,\pi)\Lambda(s-1/2,\pi\otimes\eta) = \Lambda(s-1/2,\Pi).$$

Hence, we deduce that $\sigma = \Pi$, which gives us another proof that Π must be cuspidal.

Corollary 2.3. Let E/\mathbf{Q} be an elliptic curve of conductor N parametrized via modularity by a cuspidal newform $f \in S_2^{\text{new}}(\Gamma_0(N))$ of weight 2, trivial character, and level N. Let $\pi = \pi(f)$ denote the corresponding cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$ of level $c(\pi) = N$ and trivial central character whose archimedean component is a holomorphic discrete series of weight 2. Using the unitary normalization for the automorphic L-functions (so that s = 1/2 is the central value), we have the equivalences of L-functions

$$\Lambda(E,s) = \Lambda(s - 1/2, f) = \Lambda(s - 1/2, \pi).$$

Let K be any real quadratic field. The basechanged elliptic curve E(K) can be associated to a cuspidal Hilbert newform **f** of parallel weight two, trivial central character, and level $\mathfrak{N} \subset \mathcal{O}_K$ equal to the conductor of E/K, with $\Pi = BC_{K/\mathbf{Q}}(\pi)$ the corresponding cuspidal automorphic representation of $GL_2(\mathbf{A}_K)$ of level $c(\Pi) = \mathfrak{N} \subset \mathcal{O}_K$ and trivial central character whose archimedean component is a holomorphic discrete series of parallel weight two. We then have the corresponding equivalences of L-functions

$$\Lambda(E/K,s) = \Lambda(s-1/2,\mathbf{f}) = \Lambda(s-1/2,\Pi)$$

= $\Lambda(s-1/2,\pi)\Lambda(s-1/2,\pi\otimes\eta) = \Lambda(s-1/2,f)\Lambda(s-1/2,f\otimes\eta).$

3. Spin groups and orthogonal groups

We now describe spin groups associated to rational quadratic spaces of type (2, 2). Here, we follow $[6, \S]$ [2.3-2.7] and $[8, \S, 2-4]$, but adapt for the special setting we consider in Proposition 3.3 below.

3.1. Rational quadratic spaces of type (2,2). Let (V,Q) be any rational quadratic space (V,Q) of type (2,2) bilinear form $(v_1, v_2) = Q(v_1+v_2) - Q(v_1) - Q(v_2)$. We shall later focus on the special example described above. That is, we consider the real quadratic field $K = \mathbf{Q}(\sqrt{d})$ with d > 0. Recall that for an integer $c \geq 1$, we consider the ring class group $\operatorname{Pic}(\mathcal{O}_c)$ of the **Z**-order $\mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K through which our fixed ring class character χ factors. We shall only consider this group when it exists. Note that this will always be so for c = 1, in which case $\operatorname{Pic}(\mathcal{O}_c) = \operatorname{Pic}(\mathcal{O}_K)$ can be identified with the ideal class group of \mathcal{O}_K . We fix for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ of the class $A = [\mathfrak{a}] \in \operatorname{Pic}(\mathcal{O}_c)$ with a **Z**-basis $[1, z_{\mathfrak{a}}]$.

Definition 3.1. Writing $Q_{\mathfrak{a}}(z) = \mathbf{N}_{K/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ to denote the corresponding norm form of signature (1,1), we consider the quadratic space defined by $V_A = \mathbf{Q} \oplus z_{\mathfrak{a}} \mathbf{Q} \oplus \mathfrak{a}_{\mathbf{Q}} \cong \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$ for $\mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes \mathbf{Q}$ with of the two following (essentially equivalent) quadratic forms q_A and Q_A :

- (i) $V_A = \mathbf{Q} \oplus z_{\mathfrak{a}} \mathbf{Q} \oplus \mathfrak{a}_{\mathbf{Q}}$ with $q_A(x, y, \lambda) := Q_{\mathfrak{a}}(\lambda) xy = \mathbf{N}\mathfrak{a}^{-1} \cdot \mathbf{N}_{K/\mathbf{Q}}(\lambda) xy$, (ii) $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$ with $Q_A(z) = Q_A(z_1, z_2) := Q_{\mathfrak{a}}(z_1) Q_{\mathfrak{a}}(z_2)$.

We see by inspection that (V_A, q_A) is a rational quadratic space of type (2, 2) as d > 0 is positive ⁵. We also see by inspection that (V_A, Q_A) has type (2, 2) if $d \neq 0$ is positive or negative⁶. For either choice of quadratic form, we write $(\cdot, \cdot)_A : V_A \times V_A \to \mathbf{Q}$ to denote the corresponding hermitian bilinear form.

3.2. Spin groups and exceptional isomorphisms. Let (V, Q) be any rational quadratic space of type (2, 2). Let C_V denote the corresponding Clifford algebra over **Q**. That is, consider the tensor algebra

$$T_V = \bigoplus_{m \ge 0} V^{\otimes m} = \mathbf{Q} \oplus V \oplus (V \otimes_{\mathbf{Q}} V) \oplus \cdots,$$

with $I_V \subset T_V$ the two-sided ideal generated by $v \otimes v - Q(v)$ for $v \in V$. We define $C_V = T_V/I_V$. So, C_V is a **Q**-module of rank 4, there are canonical embeddings of **Q** and V into C_V . By definition, we have that $Q(v) = v^2$ and uv + vu = (u, v) := Q(u + v) - Q(u) - (v) for any $u, v \in C_V$. We shall denote an element of the form $v_1 \otimes \cdots \otimes v_m$ in C_V for $v_i \in V$ by $v_1 \cdots v_m$ for simplicity.

Let $C_V^0 \subset C_V$ denote the **Q**-subalgebra generated by products of even numbers of basis vectors of V. Writing $C_V^1 \subset C_V$ to denote the **Q**-subalgebra generated by products of odd numbers of basis vectors of V, we have the decomposition $C_V \cong C_V^0 \oplus C_V^1$. Multiplication by -1 defines an isometry of V and gives rise to an algebra homomorphism $J: C_V \longrightarrow C_V$ known as the canonical automorphism. It is known that we can characterize the even Clifford algebra equivalently as

$$C_V^0 = \{ v \in C_V : J(v) = v \}.$$

We have the canonical anti-involution on C_V , defined by ${}^tC_V \longrightarrow C_V, (x_1 \otimes \cdots \otimes x_m)^t := x_m \otimes \cdots \otimes x_1$, from which we can define the Clifford norm

$$N_{C_V}: C_V \longrightarrow C_V, \quad N_{C_V}(x) := x^t x.$$

Note that for $x \in V$, we have $N_{C_V}(x) = Q(x)$. Hence, we see that the Clifford norm N_{C_V} is an extension of the quadratic form Q. It is not generally multiplicative.

Theorem 3.2. Let (V,Q) be any rational quadratic space of type (2,2), with Clifford algebra C_V and even subalgebra $C_V^0 \subset C_V$. We again write (x,y) = Q(x+y) - Q(x) - Q(y) to denote the associated bilinear form.

- (i) Fix any orthogonal basis v_1, v_2, v_3, v_4 of V, and put $\delta = v_1 v_2 v_3 v_4$. We can identify the centre $Z(C_V)$ of the Clifford algebra C_V with \mathbf{Q} , and the centre $Z(C_V^0)$ of its even part C_V^0 with $\mathbf{Q} + \mathbf{Q}\delta$.
- (ii) Fix any basis $v_1, v_2, v_3, v_4 \in V$ and let $S = ((v_i, v_j))_{i,j}$ denote the corresponding Gram matrix. The determinant $d(V) = \det(S)$ does not depend on the chosen basis and defines the discriminant of V. Moreover, we have the relation $\delta^2 = 2^{-4} d(V) \in \mathbf{Q}^{\times}/(\mathbf{Q}^{\times})^2$ for the volume form δ defined in (i).

Proof. See [6, § 2.2, Theorem 2.6 and Remark 2.5], these results are standard.

Let us now for the general case (V, Q) consider the corresponding Clifford group CG_V defined by

$$CG_V = \left\{ x \in C_V : x \text{ invertible }, xVJ(x)^{-1} = V \right\}.$$

This definition allows us to associate to each $x \in C_V$ an automorphism α_x of V defined by $\alpha_x(v) = xvJ(x)^{-1}$ (for any $v \in V$). We obtain from this a linear representation $\alpha : \operatorname{CG}_V \longrightarrow \operatorname{Aut}_{\mathbf{Q}}(V), \ x \mapsto \alpha_x$ known as the vector representation. Note that the involution $x \mapsto x^t$ sends CG_V to itself, and so $N_{C_V}(x) \in \operatorname{CG}_V$ for any $x \in C_V$. We also know (see [6, Lemma 2.11]) that the kernel of the vector representation $\alpha : \operatorname{CG}_V \to \operatorname{Aut}_{\mathbf{Q}}(V)$ equals \mathbf{Q}^{\times} , that the Clifford norm N_{C_V} induces a homomorphism $\operatorname{CG}_V \to \mathbf{Q}^{\times}$, and that N_{C_V} in this setting is multiplicative.

We now consider the general spin group $\operatorname{GSpin}_V = \operatorname{CG}_V \cap C_V^0$ and underlying spin group

$$\operatorname{Spin}(V) = \left\{ x \in \operatorname{GSpin}_V = \operatorname{CG}_V \cap C_V^0 : N_{C_V}(x) = 1 \right\}$$

⁵That the space has signature (2, 2) when d > 0 can be seen directly after putting the quadratic form into diagonal form. That is, we can introduce coordinates u = x + y and v = x - y corresponding to a change of basis to $\{(1, z_{\mathfrak{a}}), (1, -z_{\mathfrak{a}})\}$ for the subspace $\mathbf{Q} + z_{\mathfrak{a}}\mathbf{Q}$. Checking that $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, we find $q_A(x, y, \lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda)/\mathbf{N}\mathfrak{a} - \frac{1}{4}(u^2 - v^2)$ in this new basis.

⁶Here, the norm form $Q_{\mathfrak{a}}(z)$ has type (1,1) if d > 0 and type (2,0) of d < 0, so that Q_A has type (2,2) in either case.

As the vector representation α here is surjective with kernel \mathbf{Q}^{\times} , we see that the Clifford group GC_V is a central extension of the orthogonal group $\mathrm{O}(V)$, and that the general spin group GSpin_V is a central extension of the special orthogonal group $\mathrm{SO}(V)$. That is, we have exact sequences

$$1 \longrightarrow \mathbf{Q}^{\times} \longrightarrow \mathrm{CG}_V \longrightarrow \mathrm{O}(V) \longrightarrow 1,$$

$$1 \longrightarrow \mathbf{Q}^{\times} \longrightarrow \operatorname{GSpin}(V) \longrightarrow \operatorname{SO}(V) \longrightarrow 1.$$

As explained in [6, Lemma 2.14], we also have the simpler characterizations of spin groups

$$\operatorname{GSpin}(V) = \left\{ x \in C_V^0 : N_{C_V}(x) \in \mathbf{Q}^{\times} \right\}, \quad \operatorname{Spin}(V) = \left\{ x \in C_V^0 : N_{C_V}(x) = 1 \right\}.$$

We can now deduce via Theorem 3.2 that we have the following identifications of algebraic groups.

Proposition 3.3. We have the following identifications of spin groups for the rational quadratic spaces (V_A, q_A) and (V_A, Q_A) described in Definition 3.1. Fix any class $A \in \text{Pic}(\mathcal{O}_c)$ with integer ideal representative $\mathfrak{a} \subset \mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ and \mathbf{Z} -basis $\mathfrak{a} = [1, z_{\mathfrak{a}}]$. We again write $Q_{\mathfrak{a}}(z) = \mathbf{N}_{K/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ to denote the norm form, as well as $\mathbf{N}_{K/\mathbf{Q}}(z) = zz^{\tau}$ and $\text{Tr}_{K/\mathbf{Q}}(z) = z + z^{\tau}$ for the nontrivial automorphism $\tau \in \text{Gal}(K/\mathbf{Q})$ to denote the norm and trace homomorphisms.

- (i) Consider the quadratic space (V_A, q_A) given by $V_A = \mathbf{Q} \oplus z_a \mathbf{Q} \oplus \mathfrak{a}_{\mathbf{Q}} \cong \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$ and quadratic form $q_A(x, y, \lambda) := Q_{\mathfrak{a}}(\lambda) xy$. Then, the centre $Z(C_{V_A}^0)$ of the even Clifford algebra $C_{V_A}^0$ is given by K, and we have an exceptional isomorphism $\operatorname{Spin}(V_A) \cong \operatorname{Res}_{K/\mathbf{Q}} \operatorname{SL}_2(K)$ of algebraic groups over \mathbf{Q} .
- (ii) Consider the quadratic space (V_A, Q_A) of type (2,2) given by V_A = a_Q ⊕ a_Q ⊕ a_Q with the altered quadratic form Q_A(z) = Q_A((z₁, z₂)) := Q_a(z₁) Q_a(z₂). Then, the centre Z(C⁰_{VA}) of the even Clifford algebra C⁰_{VA} is given by Q, and we have exceptional isomorphisms Spin(V_A) ≅ SL²₂ and GSpin(V_A) ≅ GL²₂ of algebraic groups over Q.

Proof. Cf. the discussion in [6, §2.7] for the similar but distinct quadratic space $V_0 := \mathbf{Q} \oplus \mathbf{Q} \oplus K$ with quadratic form $q_0(x, y, \lambda) := \mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$, where it is shown that we can identify the centre⁷ of the even Clifford algebra as $Z(C_{V_0}^0) = K$, and that we have the accidental isomorphism $\operatorname{Spin}(V_0) \cong \operatorname{Res}_{K/\mathbf{Q}} \operatorname{SL}_2(K)$ of algebraic groups over \mathbf{Q} . We note that the spaces (V_A, q_A) and (V_A, Q_A) we consider here are distinct, as we shall show through direct calculations of the determinants and volume forms.

Let us start with (i). Hence, for the quadratic space (V_A, q_A) , we fix the basis

$$v_1 = (1, z_{\mathfrak{a}}, 0), \quad v_2 = (1, -z_{\mathfrak{a}}, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (0, 0, z_{\mathfrak{a}}).$$

⁷Note however that [6, Example 2.10, p. 133] cannot be true for the special case where the centre $Z = Z(C_V^0)$ of the even Clifford algebra C_V^0 is $k + k\delta = \mathbf{Q}$, i.e. where $k = \mathbf{Q}$ and the volume form $\delta = v_1 v_2 v_3 v_4$ is rational. That is, the Clifford algebra C_V has dim $\mathbf{Q} C_V = 2^{\dim(V)} = 2^4$, and so dim $\mathbf{Q} C_V^0 = 8$. Since this latter dimension is not a square, C_V^0 cannot be a quaternion algebra over \mathbf{Q} . Nor can it be a central simple algebra of dimension $4 = 2^2$ over K, as its centre is \mathbf{Q} . Rather, we deduce from the standard classification of Clifford algebras over \mathbf{R} that $C_{V \otimes \mathbf{R}} \cong M_4(\mathbf{R})$ and $C_{V \otimes \mathbf{R}}^0 \cong M_2(\mathbf{R}) \oplus M_2(\mathbf{R})$ that we must have $C_V^0 \cong B \oplus B$ for B an indefinite quaternion algebra over \mathbf{Q} of discriminant d(V). In particular, if d(V) = 1 (is a square), then $C_V^0 \cong M_2(\mathbf{Q}) \oplus M_2(\mathbf{Q})$.

We first compute the inner products

$$\begin{split} (v_1, v_1)_A &= -2 \cdot 2z_{\mathfrak{a}} + 2 \cdot (1 \cdot z_{\mathfrak{a}}) = -2z_{\mathfrak{a}} \\ (v_1, v_2)_A &= (v_2, v_1)_A = -2 \cdot 0 + 1 \cdot z_{\mathfrak{a}} + 1(-z_{\mathfrak{a}}) = 0 \\ (v_1, v_3)_A &= (v_3, v_1)_A = -1 \cdot z_{\mathfrak{a}} + Q_{\mathfrak{a}}(1) + 1 \cdot z_{\mathfrak{a}} - Q_{\mathfrak{a}}(1) = 0 \\ (v_1, v_4)_A &= (v_4, v_1)_A = -1 \cdot z_{\mathfrak{a}} + Q_{\mathfrak{a}}(z) + 1 \cdot z_{\mathfrak{a}} - Q_{\mathfrak{a}}(z) = 0 \\ (v_2, v_2)_A &= 2 \cdot 2z_{\mathfrak{a}} - 1 \cdot z_{\mathfrak{a}} - 1 \cdot z_{\mathfrak{a}} = 2z_{\mathfrak{a}} \\ (v_2, v_3)_A &= (v_3, v_2)_A = 1 \cdot z_{\mathfrak{a}} + Q_{\mathfrak{a}}(1) - 1 \cdot z_{\mathfrak{a}} - Q_{\mathfrak{a}}(1) = 0 \\ (v_2, v_4)_A &= (v_4, v_2)_A = 1 \cdot z_{\mathfrak{a}} + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - 1 \cdot z_{\mathfrak{a}} - Q_{\mathfrak{a}}(1) = 0 \\ (v_3, v_3)_A &= Q_{\mathfrak{a}}(2) - 2 \cdot Q_{\mathfrak{a}}(1) = \mathbf{N}\mathfrak{a}^{-1}2 \\ (v_3, v_4)_A &= (v_4, v_3)_A = Q_{\mathfrak{a}}(1 + z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1}\operatorname{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) \\ (v_4, v_4)_A &= Q_{\mathfrak{a}}(2z_{\mathfrak{a}}) - 2Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1}2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}). \end{split}$$

We then compute the determinant $d(V_A) = \det((v_i, v_j)_A)$ of the corresponding Gram matrix

$$d(V_{A}) = \det \begin{pmatrix} -2z_{\mathfrak{a}} & 0 & 0 & 0\\ 0 & 2z_{\mathfrak{a}} & 0 & 0\\ 0 & 0 & \frac{2}{\mathbf{N}\mathfrak{a}} & \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \\ 0 & 0 & \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & \frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \end{pmatrix} = -2z_{\mathfrak{a}} \begin{vmatrix} 2z_{\mathfrak{a}} & 0 & 0\\ 0 & \frac{2}{\mathbf{N}\mathfrak{a}} & \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \\ 0 & \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & \frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \end{vmatrix} \\ = -\frac{4z_{\mathfrak{a}}^{2}}{\mathbf{N}\mathfrak{a}^{2}} \cdot \left(4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) - \mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2}\right) = \frac{4z_{\mathfrak{a}}^{2}}{\mathbf{N}\mathfrak{a}^{2}} \cdot \left(\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2} - 4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})\right) \in \mathbf{Q}^{\times}/(\mathbf{Q}^{\times})^{2}$$

Hence, we find that $d(V_A) = \operatorname{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^2 - 4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) \in \mathbf{Q}^{\times}/(\mathbf{Q}^{\times})^2$. Writing $z_{\mathfrak{a}} = \alpha + \beta\sqrt{d}$ again, we find

$$d(V_A) = \operatorname{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^2 - 4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) = \left(\alpha + \beta\sqrt{d} + \alpha - \beta\sqrt{d}\right)^2 - 4(\alpha + \beta\sqrt{d})(\alpha - \beta\sqrt{d})$$
$$= 4\alpha^2 - 4(\alpha^2 - \beta^2 d) = 4(\alpha^2 - \alpha^2 + \beta^2 d) = 4\beta^2 d \equiv d \mod (\mathbf{Q}^{\times})^2.$$

Hence, we find that $\delta^2 = 2^{-4}d(V_A)$ so that $\delta = 2^{-2}\sqrt{d}$ and $Z(C_{V_A}^0) = \mathbf{Q} + \delta \mathbf{Q} = K$. It is then easy to deduce that we have an isomorphism $\operatorname{Spin}(V_A) \cong \operatorname{Res}_{K/\mathbf{Q}} \operatorname{SL}_2(K)$ of algebraic groups over \mathbf{Q} . Let us now consider (ii). In this case, we start with the same underlying vector space $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$, but consider the slightly altered quadratic form $Q_A(z) = Q_A((z_1, z_2)) := Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2)$. Fix the basis

$$w_1 = (1,0), \quad w_2 = (z_{\mathfrak{a}},0), \quad w_3 = (0,1), \quad w_4 = (0,z_{\mathfrak{a}}).$$

Writing $(w_i, w_j)_A = Q_A(w_i + w_j) - Q_A(w_i) - Q_A(w_j)$ again to denote the inner product, we compute

$$\begin{split} (w_1, w_1)_A &= Q_{\mathfrak{a}}(2) - Q_{\mathfrak{a}}(1) = \mathbf{N}\mathfrak{a}^{-1}2 \\ (w_1, w_2)_A &= (w_2, w_1)_A = Q_{\mathfrak{a}}(1 + z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1}\operatorname{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) \\ (w_1, w_3)_A &= (w_3, w_1)_A = Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(1) + Q_{\mathfrak{a}}(1) = 0 \\ (w_1, w_4)_A &= (w_4, w_1)_A = Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(1) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = 0 \\ (w_2, w_2)_A &= Q_{\mathfrak{a}}(2z_{\mathfrak{a}}) - 2Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = \mathbf{N}\mathfrak{a}^{-1}2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) \\ (w_2, w_3)_A &= (w_3, w_2)_A = Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(1) + Q_{\mathfrak{a}}(1) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = 0 \\ (w_2, w_4)_A &= (w_4, w_2)_A = Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) - Q_{\mathfrak{a}}(z_{\mathfrak{a}}) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = 0 \\ (w_3, w_3)_A &= -Q_{\mathfrak{a}}(2) + 2Q_{\mathfrak{a}}(1) = -\mathbf{N}\mathfrak{a}^{-1}2 \\ (w_3, w_4)_A &= (w_4, w_3)_A = -Q_{\mathfrak{a}}(1 + z_{\mathfrak{a}}) + Q_{\mathfrak{a}}(1) + Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = -\mathbf{N}\mathfrak{a}^{-1}\operatorname{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}}) \\ (w_4, w_4)_A &= -Q_{\mathfrak{a}}(2z_{\mathfrak{a}}) + 2Q_{\mathfrak{a}}(z_{\mathfrak{a}}) = -\mathbf{N}\mathfrak{a}^{-1}2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}}). \end{split}$$

We then compute the determinant $d(v_A) = \det((w_i, w_j))_{i,j}$ of the corresponding Gram matrix

$$d(V_A) = \det \begin{pmatrix} \frac{2z_{\mathfrak{a}}}{\mathrm{N}\mathfrak{a}} & \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} & 0 & 0\\ \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} & \frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} & 0 & 0\\ 0 & 0 & -\frac{2}{\mathrm{N}\mathfrak{a}} & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} \\ 0 & 0 & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} & -\frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathrm{N}\mathfrak{a}} \end{pmatrix} \in \mathbf{Q}^{\times}/(\mathbf{Q}^{\times})^2$$

via the Lagrange cofactor method as

$$\begin{split} d(V_A) \\ &= \frac{2}{\mathbf{N}\mathfrak{a}} \left| \begin{array}{c} \frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & 0 & 0\\ 0 & -\frac{2}{\mathbf{N}\mathfrak{a}} & -\frac{2}{\mathbf{N}\mathfrak{a}} \\ 0 & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \\ -\frac{2\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \\ \end{array} \right| - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \left| \begin{array}{c} \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} & 0 & 0\\ 0 & -\frac{2}{\mathbf{N}_{K}} \\ -\frac{2}{\mathbf{N}_{K}/\mathbf{Q}}(z_{\mathfrak{a}}) \\ 0 & -\frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}} \\ \end{array} \right| \\ = \frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^{2}} \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^{2}} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2}}{\mathbf{N}\mathfrak{a}^{2}} \right) - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2}}{\mathbf{N}\mathfrak{a}^{2}} \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^{2}} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2}}{\mathbf{N}\mathfrak{a}^{2}} \right)^{2} \\ = \left(\frac{4\mathbf{N}_{K/\mathbf{Q}}(z_{\mathfrak{a}})}{\mathbf{N}\mathfrak{a}^{2}} - \frac{\mathrm{Tr}_{K/\mathbf{Q}}(z_{\mathfrak{a}})^{2}}{\mathbf{N}\mathfrak{a}^{2}} \right)^{2} \equiv 1 \in \mathbf{Q}^{\times}/(\mathbf{Q}^{\times})^{2}. \end{split}$$

That is, we compute the discriminant $d(V_A)$ to be trivial, whence the volume form $\delta = 2^{-4} \in \mathbf{Q}$ is rational. Hence, we know by Theorem 3.2 that the centre $Z(C_{V_A}^0) = \mathbf{Q} + \delta \mathbf{Q}$ is simply \mathbf{Q} . In this setting, since $\dim_{\mathbf{Q}} C_{V_A}^0 = 8$ and $C_{V_A \otimes \mathbf{R}} \cong C_{2,2}(\mathbf{R}) \cong M_4(\mathbf{R})$, we deduce that $C_{V_A}^0 \cong B \oplus B$ for B an indefinite quaternion algebra over \mathbf{Q} . Morover, since the discriminant $d(V_A) = 1$, we deduce that this must be the matrix algebra $B \cong M_2(\mathbf{Q})$. It is then easy to deduce from the discussion above that we obtain the exceptional isomorphisms $\operatorname{Spin}(V_A) \cong \operatorname{SL}_2^2$ and $\operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2$ of algebraic groups over \mathbf{Q} .

Relation to quadratic basechange liftings. Consider the quadratic space $V_0 = \mathbf{Q} \oplus \mathbf{Q} \oplus K$ with quadratic form $q_0(x, y, \lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$. Although we do not use this quadratic space (V_0, q_0) for our main calculations, we note that the accidental isomorphism $\operatorname{Spin}(V_0) \cong \operatorname{Res}_{K/\mathbf{Q}}(\operatorname{SL}_2(K))$ can be used to realize the quadratic basechange lifting $\Pi = \operatorname{BC}_{K/\mathbf{Q}}(\pi)$ of the cuspidal automorphic representation $\pi = \pi(f)$ to $\operatorname{GL}_2(\mathbf{A}_K)$ as a theta lift from $\operatorname{SL}_2(\mathbf{A})$ to $\operatorname{Spin}(V_0)(\mathbf{A})$, which after extending to similitudes can be viewed as a theta lift from $\operatorname{GL}_2(\mathbf{A})$ to $\operatorname{GSpin}(V_0)(\mathbf{A})$. We refer to [6, §2-3] for a classical description of this setup.

4. Regularized theta lifts and automorphic Green's functions

We now introduce regularized theta lifts associated with the quadratic spaces (V_A, Q_A) described in Proposition 3.3 above following Borcherds [4], Kudla [34], Bruinier [5], Bruinier-Funke [7], and Bruinier-Yang [8]. We compute these theta lifts along the anisotropic subspace $(V_{A,2}, Q_{A,2}) = (V_{A,2}, Q_A|_{V_{A,2}})$ of type (1, 1) defined by the ideal representative $V_{A,2} := \mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes \mathbf{Q}$ and the restricted quadratic form $Q_{A,2}(\lambda) = Q_{\mathfrak{a}}(\lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda)/\mathbf{N}\mathfrak{a}$. In this way, we derive new integral presentations for the central derivative values $\Lambda'(E/K, \chi, 1) = \Lambda'(1/2, \Pi \otimes \chi) = \Lambda'(1/2, f \times \theta(\chi))$.

4.1. Setup. Fix a primitive ring class character χ of K of some conductor $c \in \mathbb{Z}_{\geq 1}$ coprime to Nd_K , which we assume exists. (This is always the case for conductor c = 1, whence χ is a class group character). Thus, χ factors through the ring class group $\operatorname{Pic}(\mathcal{O}_c)$. Let us for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ fix an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ of $A = [\mathfrak{a}] \in \operatorname{Pic}(\mathcal{O}_c)$ with \mathbb{Z} -basis $\mathfrak{a} = [1, z_{\mathfrak{a}}]$. We consider the rational quadratic space (V_A, Q_A) of type (2, 2) defined in Definition 3.1 (ii), hence with vector space $V_A = \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{a}_{\mathbb{Q}}$ and quadratic form $Q_A(z) = Q_A((z_1, z_2)) = Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2)$.

4.1.1. Exceptional isomorphisms. By Proposition 3.3, we have an isomorphism of algebraic groups over
$$\mathbf{Q}$$

(8)
$$\zeta : \operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2.$$

We then take $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ to be the compact open subgroup $U_A = \prod_{p < \infty} U_{A,p}$, where each $U_{A,p} \subset \operatorname{GSpin}(V_A)(\mathbf{Q}_p)$ is determined by the condition that each component $\zeta(U_{A,p}) \cong K_{0,p}(N) \times K_{0,p}(N)$

is given by the Cartesian product of congruence subgroup

(9)
$$K_{0,p}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\mathbf{Z}_p) : c \in N\mathbf{Z}_p \right\} \subset \operatorname{GL}_2(\mathbf{Z}_p).$$

Via (8), we can identify the cuspidal automorphic representation $\pi = \pi(f)$ of $\operatorname{GL}_2(\mathbf{A})$ determined by the eigenform $f \in S_2^{\operatorname{new}}(\Gamma_0(N))$ parametrizing E/\mathbf{Q} as a cuspidal automorphic representation of $\operatorname{GSpin}(V_A)(\mathbf{A})$ which is right-invariant under the action of $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$. We shall also take $\Lambda_A \subset V_A$ to be the maximal lattice corresponding to this subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$. Given any lattice $\Lambda_A \subset V_A$, we write $\Lambda_A^{\#}$ to denote the corresponding dual lattice, and $\Lambda_A^{\#}/\Lambda_A$ the corresponding discriminant group.

4.1.2. Weil representations. Let $\psi = \bigotimes_v \psi_v$ denote the standard additive character of \mathbf{A}/\mathbf{Q} . We write

$$r_{\psi} = r_{\psi,\Lambda_A} : \mathrm{SO}(V_A)(\mathbf{A}) \times \mathrm{SL}_2(V)(\mathbf{A}) \longrightarrow \mathcal{S}(V_A(\mathbf{A}))$$

for the corresponding Weil representation of $SO(V_A)(\mathbf{A}) \times SL_2(\mathbf{A})$ on the space $\mathcal{S}(V_A(\mathbf{A}))$ of Schwartz-Bruhat functions on $V_A(\mathbf{A})$, as well as its extension to the similitude group

$$R(\mathbf{A}) := \{(h,g) \in \mathrm{GO}(V_A)(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A}) : \nu(h) = \det(g)\} \subset \mathrm{GO}(V_A)(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A}).$$

Note that since $\dim_{\mathbf{Q}}(V_A) = 4$ is even, $r_{\psi} = r_{\psi,\Lambda_A}$ factors through $\mathrm{SL}_2(\mathbf{A})$ rather than its two-fold metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{A})$. The action of $\mathrm{SL}_2(\mathbf{A})$ on $\mathcal{S}(V_A(\mathbf{A}))$ commutes with that of $\mathrm{SO}(V_A)(\mathbf{A})$. We write $r_{\psi}(h)\Phi(x) = \Phi(h^{-1}x)$ for $h \in \mathrm{SO}(V_A)(\mathbf{A})$ and $\Phi \in \mathcal{S}(V_A(\mathbf{A}))$ to denote the latter action.

4.1.3. Subspaces of Schwartz functions. Let $\mathcal{S}_{\Lambda_A} \subset \mathcal{S}(V_A(\mathbf{A}_f))$ denote the subspace of Schwartz functions with support on $\widehat{\Lambda}_A^{\#} := \Lambda_A^{\#} \otimes \widehat{\mathbf{Z}}$ which are constant on cosets of $\widehat{\Lambda}_A := \Lambda_A \otimes \widehat{\mathbf{Z}}$. Note that \mathcal{S}_{Λ_A} admits a basis of characteristic functions $\mathbf{1}_{\mu} = \operatorname{char} \left(\mu + \widehat{\Lambda}_A \right)$,

(10)
$$\mathcal{S}_{\Lambda_A} = \bigoplus_{\mu \in \Lambda_A^{\#}/\Lambda_A} \mathbf{C} \cdot \mathbf{1}_{\mu} \subset \mathcal{S}(V_A(\mathbf{A}_f)).$$

This space S_{Λ_A} is stable under the action of $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ through the Weil representation r_{ψ} . The space of Schwartz-Bruhat functions $S(V_A(\mathbf{A}_f))$ can be expressed as the direct limit $\varinjlim_{\Lambda_A} S_{\Lambda_A}$ of these subsapces.

4.1.4. Anisotropic subspaces. For each of the quadratic spaces (V_A, Q_A) described in Definition 3.1 (ii) above, we consider the anisotropic subspace $(V_{A,2}, Q_{A,2}) = (V_{A,2}, Q_A|_{V_{A,2}})$ of type (1,1) defined by the fractional ideal $V_{A,2} := \mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes \mathbf{Q}$ and restricted quadratic form $Q_{A,2}(\lambda) = Q_{\mathfrak{a}} = \mathbf{N}_{K/\mathbf{Q}}(\lambda)/\mathbf{N}\mathfrak{a}$. We also consider the anisotropic subspace $(V_{A,1}, Q_{A,1}) = (V_{A,1}, Q_A|_{V_{A,1}})$ of type (1,1) defined by $V_{A,1} := \mathfrak{a}_{\mathbf{Q}}$ and restricted quadratic form $Q_{A,1}(x, y) = -Q_{\mathfrak{a}}$. We write $(V_{A,j}, Q_{A,j})$ for j = 1, 2 to denote either of these spaces.

Writing $K^1 \subset K^{\times}$ to denote the elements of norm one, it is easy to see that $\text{Spin}(V_{A,j}) \cong \text{SO}(V_{A,j}) \cong K^1$ for each of j = 1, 2. Writing $K^1_{\mathbf{A}}$ to denote the adelic points, we have the Hilbert exact sequence

$$1 \longrightarrow \mathbf{A}^{\times} \longrightarrow \mathbf{A}_{K}^{\times} \longrightarrow K_{\mathbf{A}}^{1} \longrightarrow 1.$$

In particular, we obtain natural identifications for the corresponding adelic quotient spaces

$$\operatorname{Spin}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{Spin}(V_{A,j})(\mathbf{A}) \cong \operatorname{SO}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{SO}(V_{A,j})(\mathbf{A}) \cong \mathbf{A}_K^{\times}/\mathbf{A}^{\times}K^{\times}$$

Hence, we can view the ring class character $\chi : \mathbf{A}_{K}^{\times}/\mathbf{A}^{\times}K^{\times} \to \mathbf{C}^{\times}$ as an automorphic representation of $\mathrm{SO}(V_{A,j})(\mathbf{A})$. In a similar way, we have natural identifications

$$\operatorname{GSpin}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{GSpin}(V_{A,j})(\mathbf{A}) \cong \operatorname{GO}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{GO}(V_{A,j})(\mathbf{A}) \cong \mathbf{A}_K^{\times}/K^{\times}.$$

Here, strictly speaking, we fix one of the two connected components $\mathrm{GO}^{\pm}(V_{A,j})$ of $\mathrm{GO}(V_{A,j})$ so that

$$\operatorname{GSpin}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{GSpin}(V_{A,j})(\mathbf{A}) \cong \operatorname{GO}^{\pm}(V_{A,j})(\mathbf{Q}) \setminus \operatorname{GO}^{\pm}(V_{A,j})(\mathbf{A}) \cong \mathbf{A}_{K}^{\times}/K^{\times}.$$

We refer to the discussion in [39, Theorem 2.3.3] for more background leading to this identification.

4.2. Hermitian symmetric domains. The symmetric spaces associated to each quadratic space (V_A, Q_A) are hermitian symmetric domains, i.e. have a complex structure. We have the following equivalent realizations.

4.2.1. The Grassmannian model. Let

$$D(V_A) = \{ z \subset V_A(\mathbf{R}) : \dim(z) = 2, Q_A|_z < 0 \}$$

denote the Grassmannian of oriented hyperplanes of $V_A(\mathbf{R})$ on which Q_A is negative definite. As explained above, we fix a choice of orientation $D^{\pm}(V_A)$. We also write $D^{\pm}(V_{A,j})$ to denote the sub-Grassmannian for the signature (1, 1) subspace $V_{A,j} \subset V_A$ for each index j = 1, 2, each consisting of oriented hyperbolic lines:

$$D^{\pm}(V_{A,j}) = \{ z \subset V_{A,j}(\mathbf{R}) : \dim(z) = 1, Q_{A,j}|_{z} < 0 \} = \{ [x : y] \in \mathbf{P}^{1}(\mathbf{R}) : \text{orientation } \pm 1, Q_{A,j}(x, y) < 0 \}.$$

4.2.2. The projective model. Note that $D^{\pm}(V_A)$ can be identified with the complex surface defined by

$$Q(V_A) = \left\{ w \in V_A(\mathbf{C}) : (w, w)_A = 0, (w, \overline{w})_A < 0 \right\} / \mathbf{C}^{\times} \subset \mathbf{P}(V_A(\mathbf{C}))$$

via the map

(11)
$$D^{\pm}(V_A) \longrightarrow Q(V_A), \quad z \longmapsto v_1 - iv_2 = w,$$

for v_1, v_2 a properly-oriented standard basis of $D^{\pm}(V_A)$ with $(v_1, v_1)_A = (v_2, v_2)_A = -1$ and $(v_1, v_2)_A = 0$. We refer to this identifications $D^{\pm}(V_A) \cong Q(V_A)$ as the projective model.

4.2.3. The tube domain model. Fix a Witt decomposition $V_A(\mathbf{R}) = V_{A,0} + \mathbf{R} \cdot e + \mathbf{R} \cdot f$, with e and f chosen so that $(e, e)_A = (f, f)_A = 0$ and $(e, f)_A = 1$, and $C(V_A) = \{y \in V_{A,0} : (y, y)_A < 0\}$ its negative cone. We can then identify $D^{\pm}(V_A) \cong Q(V_A)$ with the corresponding tube domain

$$\mathcal{H}(V_A) := \{ z \in V_{A,0}(\mathbf{C}) : \Im(z_0) \in C(V_A) \} \cong \mathfrak{H}^2$$

via the map $\mathcal{H}(V_A) \longrightarrow V_A(\mathbf{C})$ sending $z \longmapsto w(z) := z + e - q_A(z)f$ composed with the projection to $Q(V_A)$. We call $\mathcal{H}(V_A) \subset V_{0,A}(\mathbf{C}) \cong \mathbf{C}^2$ the tube domain model.

4.3. Spin Shimura varieties. We now describe the Shimura varieties associated with each group $\operatorname{GSpin}(V_A)$. Here, we can take $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ any compact open subgroup. Later, we shall choose U_A to correspond to the level of the basechange representation $\Pi = \operatorname{BC}_{K/\mathbf{Q}}(\pi(f))$ of $\operatorname{GL}_2(\mathbf{A}_K)$, i.e. which determines a compact open subgroup of $\operatorname{SL}_2(\mathbf{A}_{K,f})$ corresponding to a congruence subgroup of $\operatorname{SL}_2(\mathcal{O}_K)$. In the special case we consider here, we can also use the exceptional isomorphism $\operatorname{GSpin}(V_A) \cong \operatorname{GL}_2^2$, to choose this level structure more explicitly as in (9) above.

4.3.1. Orbifolds. Consider the Shimura varieties $Sh_{U_A}(D^{\pm}(V_A), GSpin(V_A))$ with complex points given by

$$Sh_{U_A}(D^{\pm}(V_A), GSpin(V_A))(\mathbf{C}) = GSpin(V_A)(\mathbf{Q}) \setminus (D^{\pm}(V_A) \times GSpin(V_A)(\mathbf{A}_f)/U_A)$$
$$\cong GSpin(V_A)(\mathbf{Q}) \setminus (\mathfrak{H}^2 \times GSpin(V_A)(\mathbf{A}_f)/U_A).$$

Note that this is a quasiprojective surface defined over \mathbf{Q} . Via the exceptional isomorphism (8) with choice of level (9), we obtain the identification $\operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A))(\mathbf{C}) \cong \operatorname{GL}_2(\mathbf{Q})^2 \setminus (\mathfrak{H}^2 \times \operatorname{GL}_2(\mathbf{A}_f)^2 / \zeta(U_A))$ with the two-fold product $Y_0(N) \times Y_0(N)$ of the noncompactified modular curve $Y_0(N) = \Gamma_0(N) \setminus \mathfrak{H}$.

4.3.2. Decompositions. Fixing a (finite) set of representatives $h_j \in \operatorname{GSpin}(V_A)(\mathbf{Q}) \setminus \operatorname{GSpin}(V_A)(\mathbf{A}_f)/U_A$, we get the decomposition

(12)
$$\operatorname{GSpin}(V_A)(\mathbf{A}) = \coprod_j \operatorname{GSpin}(V_A)(\mathbf{Q}) \operatorname{GSpin}(V_A)(\mathbf{R})^0 h_j U_A$$

where $\operatorname{GSpin}(V_A)(\mathbf{R})^0$ denotes the identity component of $\operatorname{GSpin}(V_A)(\mathbf{R}) \cong \operatorname{GSpin}(2,2)$. This gives us the corresponding decomposition of the Shimura variety as

(13)
$$\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) = \coprod_j X_{A,j}, \quad \text{where } X_{A,j} = \Gamma_j \setminus D^{\pm}(V_A)$$

for the arithmetic subgroup $\Gamma_{A,j} = \operatorname{GSpin}(V_A)(\mathbf{Q}) \cap (\operatorname{GSpin}(V_A)(\mathbf{R})^0 h_j U h_j^{-1})$. Chosing U_A according to (9) via (8), this simply recovers the decomposition $\operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A)) \cong Y_0(N) \times Y_0(N)$.

4.3.3. Special divisors. We now introduce special (arithmetic) divisors on $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$). Here, we follow the discussion of Kudla [34] (cf. [33]), which we note applies to cycles of any codimension. Given a vector $x \in V_A(\mathbf{Q})$ with $Q_A(x) < 0$, let $V_{A,x} := x^{\perp} \subset V_A$ denote the orthogonal complement, and

$$D^{\pm}(V_A)_x = \{z \in D^{\pm}(V_A) : x \perp z\}$$

the corresponding Grassmannian. Let $\operatorname{GSpin}(V_{A,x})(\mathbf{A}_f)$ denote the stabilizer in $\operatorname{GSpin}(V_A)(\mathbf{A}_f)$ of x. We have a natural map defined on $h \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ by (14)

$$\operatorname{GSpin}(V_{A,x})(\mathbf{Q}) \setminus D^{\pm}(V_A)_x \times \operatorname{GSpin}(V_{A,x})(\mathbf{A}_f) / \left(\operatorname{GSpin}(V_{A,x})(\mathbf{A}_f) \cap hU_A h^{-1}\right) \longrightarrow \operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A))$$
$$[z, h_1] \longmapsto [z, h_1 h].$$

Definition 4.1. Given $x \in V_A(\mathbf{Q})$ with $Q_A(x) < 0$ and $h \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$, let $Z_A(x,h) = Z_A(x,h,U_A)$ denote the image of the map (14). Here, we drop the compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ from the notation when the context is clear.

This image $Z_A(x, h) = Z_A(x, h, U_A)$ determines a special cycle $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$ which is defined over \mathbf{Q} . As explained in [34, §1] and [33], these cycles satisfy many nice properties, including compatibility with Hecke operators. To illustrate a couple of the relevant properties we shall use here, let is for a given positive rational number $m \in \mathbf{Q}_{>0}$ write $\Omega_{A,m}(\mathbf{Q})$ to denote the corresponding quadric

$$\Omega_{A,m}(\mathbf{Q}) = \{ x \in V_A : Q_A(x) = m \}$$

If $\Omega_{A,m}(\mathbf{Q})$ is not the empty set, in which case we fix a point $x_0 \in \Omega_{A,m}(\mathbf{Q})$, the corresponding finite adelic points $\Omega_{A,m}(\mathbf{A}_f)$ determine a closed subgroup of $V_A(\mathbf{A}_f)$. Given $\Phi \in \mathcal{S}(V_A(\mathbf{A}_f))^{U_A}$, we then write

(15)
$$\operatorname{supp}(\Phi) \cap \Omega_{A,m}(\mathbf{A}_f) = \coprod_r U_A \cdot \zeta_r^{-1} \cdot x_0$$

for some finite set of representatives $\zeta_r \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$. We then define from this decomposition (15) the corresponding analytic divisor

(16)
$$Z_A(\Phi, m, U_A) := \sum_r \Phi(\zeta_r^{-1} \cdot x_0) Z_A(x_0, \zeta_r, U_A).$$

If $U'_A \subset U_A$ is an inclusion of compact open subgroups of $\operatorname{GSpin}(V_A)(\mathbf{A}_f)$ and

$$\operatorname{pr}: \operatorname{Sh}_{U'_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \longrightarrow \operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$$

the corresponding covering of Shimura varieties, we have the projection formula

$$\operatorname{pr}^* Z_A(\Phi, m, U_A) = Z_A(\Phi, m, U'_A).$$

Hence, the analytic divisor is defined on the Shimura variety

$$\operatorname{Sh}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) = \lim_{U_A} \operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)),$$

and so we are justified in dropping the reference to the compact open subgroup U_A from the notation. We can also consider the right multiplication by $h \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$, which determines a morphism

$$[h] : \operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \longrightarrow \operatorname{Sh}_{hU_Ah^{-1}}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)).$$

This morphism [h] is defined over \mathbf{Q} , and its pushforward $[h]_*$ satisfies the relation

$$[h]_*: Z_A(\Phi, m, U_A) \longrightarrow Z(r_{\psi}(h)\Phi, m, hU_Ah^{-1}), \text{ where } r_{\psi}(h)\Phi(x) = \Phi(h^{-1}x).$$

In this way, we can deduce that these analytic divisors (16) are compatible with Hecke operators on $\operatorname{Sh}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$. Moreover, with respect to the decomposition (13), the result of [33, Proposition 5.3] (cf. [34, §1]) shows that the analytic divisor $Z_A(\Phi, m, U_A)$ decomposes as

$$Z_A(\Phi, m, U_A) = \sum_j Z_{A,j}(\Phi, m, U_A),$$

where for each factor j we write

$$Z_{A,j}(\Phi, m, U_A) = \sum_{x \in \Omega_{A,m}(\mathbf{Q}) \bmod \Gamma_{A,j}} \Phi(h_j^{-1}x) \operatorname{pr}_j(D^{\pm}(V_A)_x)$$

for $\operatorname{pr}_j : D^{\pm}(V_A) \longrightarrow \Gamma_{A,j} \setminus D^{\pm}(V_A)$ the natural projection. Writing $\Phi^{\vee}(x) = \Phi(-x)$, these analytic divisors also satisfy the functional equations $Z_A(\Phi, m, U_A) = Z_A(\Phi^{\vee}, m, U_A)$.

Definition 4.2. Given a positive rational number m > 0 for which $\Omega_{A,m}(\mathbf{Q}) \neq \emptyset$ and a coset $\mu \in \Lambda_A^{\#}/\Lambda_A$ with corresponding characteristic function $\mathbf{1}_{\mu} = \mathbf{1}_{\mu+\widehat{\Lambda}_A}$, we write $Z_A(\mu, m) = Z_A(\mathbf{1}_{\mu}, m) = Z_A(\mathbf{1}_{\mu}, m, U_A)$ for the corresponding analytic divisor on the spin Shimura surface $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$.

4.3.4. Relation to Hirzebruch-Zagier divisors. The special divisors $Z_A(\mu, m)$ of Definition 4.2 correspond to sums of Hirzebruch-Zagier divisors on the Hilbert modular surface $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \cong Y_0(N)^2$. This can be seen from the decomposition

$$Z_{A}(\mu,m)(\mathbf{C}) \cong \Gamma_{0}(N)^{2} \Big\langle \prod_{\substack{x \in \mu + \Lambda_{A} \\ Q_{A}(x) = m}} D(V_{A})_{x} = \Gamma_{0}(N)^{2} \Big\rangle \prod_{\substack{x \in \mu + \Lambda_{A} \\ Q_{A}(x) = m}} \left\{ z \in D^{\pm}(V_{A}) : (z,x)_{A} = 0 \right\}$$
$$\cong \Gamma_{0}(N)^{2} \Big\langle \prod_{\substack{x \in \mu + \Lambda_{A} \\ Q_{A}(x) = m}} \left\{ z = (z_{1},z_{2}) \in \mathfrak{H}^{2} : Q_{A}(z+x) - Q_{A}(z) = m \right\} \subset Y_{0}(N)(\mathbf{C}) \times Y_{0}(N)(\mathbf{C}).$$

Note that these special divisors $Z_A(\mu, x)$ can be viewed as embeddings of modular curves into the surface $Y_0(N) \times Y_0(N)$. Indeed, each positive definite point in the quadric $\Omega_{A,\mu,m}(\mathbf{Q}) = \{x \in \mu + \Lambda_A : Q_A(x) = x\}$ gives rise to a rational quadratic subspace $W_A = x^{\perp} \subset V_A$ of type (1, 2), with corresponding general spin group $\operatorname{GSpin}(W_A) \subset \operatorname{GSpin}(V_A)$, Grassmannian $D(W_A) \subset D(V_A)$, and quaternionic Shimura curve $\operatorname{Sh}_{U_A \cap \operatorname{GSpin}(W_A)}(\operatorname{GSpin}(W_A), D(W_A)) \longrightarrow \operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D(V_A)) \cong Y_0(N) \times Y_0(N).$

Recall that the classical Hirzebruch-Zagier divisor $T_m = T_m(\Lambda_A)$ of discriminant $m \in \mathbf{Q}_{>0}$ for the lattice $\Lambda_A \subset V_A$ is defined by

(17)
$$T_m = T_m(\Lambda_A) = \sum_{\substack{\lambda = (\lambda_1, \lambda_2) \in \Lambda_A^{\#}/\{\pm 1\} \\ Q_A(\lambda) = \frac{m}{\Delta}}} \left\{ z = (z_1, z_2) \in \mathfrak{H}^2 : Q_A(z + \lambda) - Q_A(z) - Q_A(\lambda) = 0 \right\},$$

where $\Delta = c^2 d_K$ denotes the discriminant of the order $\mathcal{O}_c = \mathbf{Z} + c \mathcal{O}_K$. Hence, we find the relation

$$T_m = T_m(\Lambda_A) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} Z_A(\mu, m/\Delta)$$

We refer to [6, Definition 2.27], [28, §3], and [8, §8] for more background on these Hirzebruch-Zagier divisors.

4.3.5. Geodesic spaces. Each of the subspaces $(V_{A,j}, Q_{A,j})$ of signature (1,1) gives rise to a geodesic set

$$Z(V_{A,j}) := \operatorname{GSpin}(V_{A,j})(\mathbf{Q}) \setminus \left(D^{\pm}(V_{A,j}) \times \operatorname{GSpin}(V_{A,j})(\mathbf{A}_f) / U_{A,j} \right), \quad U_{A,j} := U_A \cap \operatorname{GSpin}(V_{A,j})(\mathbf{A}_f).$$

We can embed each subset $Z(V_{A,2})$ as "geodesic" subset

(18)
$$Z(V_{A,2}) \longrightarrow \operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \cong \operatorname{GSpin}(V_A)(\mathbf{Q}) \setminus \left(\mathfrak{H}^2 \times \operatorname{GSpin}(V_A)(\mathbf{A}_f)/U_A\right).$$

That is, let us now consider the norm form $Q_{\mathfrak{a}}(z) = \mathbf{N}_{K/\mathbf{Q}}(z)/\mathbf{N}\mathfrak{a}$ as a binary quadratic form

$$Q_{A,2}(X,Y) := \mathbf{N}_{K/\mathbf{Q}}(X + z_{\mathfrak{a}}Y) / \mathbf{N}\mathfrak{a} = a_{\mathfrak{a}}X^2 + b_{\mathfrak{a}}XY + c_{\mathfrak{a}}Y^2.$$

The roots $\mathfrak{Z}^{\pm}_{\mathfrak{a}} = (-b_{\mathfrak{a}} \pm \sqrt{\Delta})/2a_{\mathfrak{a}}$ of the quadratic polynomial $Q_{A,2}(X,1) = 0$ or $Q_{A,2}(1,Y) = 0$ determine the endpoints of a geodesic arc $\gamma_{\mathfrak{a}}$ in \mathfrak{H} . In this way, through $D(V_{A,2}) \cong \mathfrak{H}^2$, we can view $Z(V_{A,2})$ as a "geodesic" subset of $Y_0(N) \hookrightarrow Y_0(N) \times Y_0(N) \cong \operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A))$. More generally, viewing each of the Hirzebruch-Zagier special divisors $Z_A(\mu, m) \subset \operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A)) \cong Y_0(N)^2$ as a modular curve, we view the geodesic sets $Z(V_{A,2})$ as subsets of $Z_A(\mu, m) \subset \operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A)) \cong Y_0(N)^2$. 4.3.6. Arithmetic automorphic forms. Let $\mathcal{L}_{D^{\pm}(V_A)}$ denote the restriction of $D^{\pm}(V_A) \cong Q(V_A)$ of the tautological bundle on $\mathbf{P}(V_A(\mathbf{C}))$. The natural action of $\mathrm{GO}(V_A)(\mathbf{R})$ on $V_A(\mathbf{C})$ induces one of $\mathrm{GSpin}(V_A)(\mathbf{R})^0$ on $\mathcal{L}_{D^{\pm}(V_A)}$. Hence, there is a holomorphic line bundle

$$\mathcal{L}_{A}^{\pm} := \operatorname{GSpin}(V_{A})(\mathbf{Q}) \setminus \left(\mathcal{L}_{D^{\pm}(V_{A})} \times \operatorname{GSpin}(V_{A})(\mathbf{A}_{f})/U_{A} \right) \longrightarrow \operatorname{Sh}_{U_{A}}(D^{\pm}(V_{A}), \operatorname{GSpin}(V_{A})).$$

This holomorphic line bundle \mathcal{L}_{A}^{\pm} is known by [25] to have a canonical model over **Q**. We can define a hermitian metric $h_{\mathcal{L}_{D^{\pm}(V_{A})}}$ on $\mathcal{L}_{D^{\pm}(V_{A})}$ by the rule

$$h_{\mathcal{L}_{D^{\pm}(V_A)}}(w_1, w_2)_A := \frac{1}{2} \cdot (w_1, \overline{w}_2)_A$$

This metric is invariant under the action by the orthogonal group $\operatorname{GO}(V_A)(\mathbf{R})$, and hence descends to \mathcal{L}_A^{\pm} . The map $z \mapsto w(z)$ used in to the tube domain model $D^{\pm}(V_A) \cong \mathcal{H}_{\pm}(V_A) \cong \mathfrak{H}_{\pm}^2$ can be viewed as a nowhere vanishing section of $\mathcal{L}_{D^{\pm}(V_A)}$, whose norm we define to be

$$||w(z)||_A = -\frac{1}{2} \cdot (w(z), \overline{w}(z))_A = -(y, y)_A =: |y|_A^2.$$

Moreover, given $h \in \mathrm{GO}(V_A)(\mathbf{R})$ or $h \in \mathrm{GSpin}(V_A)(\mathbf{R})$, we have an automorphy factor

$$j: \operatorname{GSpin}(V_A)(\mathbf{R}) \times D^{\pm}(V_A) \longrightarrow \mathbf{C}^{\diamond}$$

defined by $h \cdot w(z) = w(hz) \cdot j(h, z)$. In this way, the holomorphic sections of $\mathcal{L}_A^{\otimes k} = \mathcal{L}_A^{\pm \otimes k}$ for any integer k can be interpreted as holomorphic functions $\Psi : D^{\pm}(V_A) \times \operatorname{GSpin}(V_A)(\mathbf{A}_f) \longrightarrow C$ satisfying the following transformation properties: For any $z \in D^{\pm}(V_A)$ and $h \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)$,

•
$$\Psi(z,hu) = \Psi(z,h)$$
 for all $u \in U_A$,

• $\Psi(\gamma z, \gamma h) = j(\gamma, z)^k \cdot \Psi(z, h)$ for all $\gamma \in \operatorname{GSpin}(V_A)(\mathbf{Q})$.

We define the norm of a section $(z,h) \to \Psi(z,h) \cdot w(z)^{\otimes k}$ to be

$$||\Psi(z,h)||_A^2 = |\Psi(z,h)|_A^2 \cdot |y|_A^{2k},$$

we refer to this as the Petersson norm of the holomorphic section Ψ . Note that under the isomorphism (13), such a section Ψ corresponds to the collection $\{\Psi(\cdot, h_j)\}_j$ of holomorphic functions on the connected component $D^{\pm}(V_A)^0$ which are holomorphic of weight k for the corresponding arithmetic subgroup $\Gamma_{A,j}$. The latter forms have a classical interpretation as modular forms corresponding to congruence subgroups of lattices $\Lambda_A \subset V_A$, and correspond to holomorphic Hilbert modular forms of parallel weight k in this setting (see e.g. the discussion $[6, \S 2.7]$).

4.4. **Regularized theta lifts.** We now describe the construction of regularized theta lifts for the spaces we consider (V_A, A_A) . Here, we follow [34] and [8] for background.

4.4.1. Gaussian functions. Given $z \in D^{\pm}(V_A)$, let $\operatorname{pr}_z : V_A(\mathbf{R}) \longrightarrow z$ denote the projection, whose kernel defines the orthogonal complement $z^{\perp} := \ker(\mathrm{pr}_z)$. Given a vector $x \in V_A(\mathbf{R})$, we then define

$$R(x,z)_A := -\left(\mathrm{pr}_x(x), \mathrm{pr}_z(x)\right) = \left|(x,w(z))_A\right|_A^2 \cdot \left|y\right|_A^2$$

Using this definition, we can associate to a hyperplane $z \in D^{\pm}(V_A)$ and vector $x \in V_A(\mathbf{R})$ a majorant

$$(x,x)_{A,z} := (x,x)_A + 2 \cdot R(x,z)_A.$$

Writing $\mathcal{C}^{\infty}(D^{\pm}(V_A))$ to denote the space of smooth functions on $D^{\pm}(V_A)$, we use this majorant to define a Gaussian function $\Phi_{\infty} \in \mathcal{S}(V_A(\mathbf{R})) \otimes \mathcal{C}^{\infty}(D^{\pm}(V_A))$ by the rule

$$\Phi_{\infty}(x,z) := \exp\left(-\pi \cdot (x,x)_{A,z}\right).$$

It is known that $\Phi_{\infty}(hx, hz) = \Phi_{\infty}(x, z)$ for all $h \in SO(V_A)(\mathbf{R})$, and also that Φ_{∞} has weight 0 for the action of the maximal compact subgroup $SO_2(\mathbf{R})$ of $SL_2(\mathbf{R})$.

4.4.2. Construction of the theta kernel. Given $z \in D^{\pm}(V_A)$, $h_f \in SO(V_A)(\mathbf{A}_f)$ and $g \in SL_2(\mathbf{A})$, we write $\theta_{r_{sb}}^{\star}$ to denote the linear functional on $\Phi_f \in \mathcal{S}(V_A(\mathbf{A}_f))$ defined by

(19)

$$\Phi_{f} \longmapsto \theta_{r_{\psi}}^{\star}(z, h_{f}, g; \Phi_{f}) := \sum_{x \in V_{A}(\mathbf{Q})} r_{\psi}(g) \left(\Phi_{\infty}(\cdot, z) \otimes r_{\psi}(h_{f})\Phi_{f}\right)(x) \\
= \sum_{x \in V_{A}(\mathbf{Q})} r_{\psi}(1, g) \left(\Phi_{\infty}(\cdot, z) \otimes r_{\psi}(h_{f}, 1)\Phi_{f}\right)(x).$$

It is easy to see that this is automorphic for the orthogonal group: For all $\gamma \in SO(V_A)(\mathbf{Q})$, we have

$$\theta^{\star}_{r_{\psi}}(\gamma z,\gamma h_{f},g;\Phi_{f})=\theta^{\star}_{r_{\psi}}(z,h_{f},g;\Phi_{f}).$$

By Poisson summation (see [46], [34, (1.22)]), we can also see that the functional is automorphic for the symplectic group: For all $\gamma \in SL_2(\mathbf{Q})$, we have

$$\theta_{r_{\psi}}^{\star}(z, h_f, \gamma g; \Phi_f) = \theta_{r_{\psi}}^{\star}(z, h_f, g; \Phi_f).$$

Using properties of r_{ψ} , we can also see that for any $h'_f \in SO(V_A)(\mathbf{A}_f)$ and any $g' \in SL_2(\mathbf{A})$,

(20)
$$\theta_{r_{\psi}}^{\star}(z,h_fh_f',gg';\Phi_f) = \theta_{r_{\psi}}^{\star}(z,h_f,g;r_{\psi}(h_f',g')\Phi_f).$$

In this way, we see that for any compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ and decomposable U_A -invariant Schwartz function $\Phi \in \mathcal{S}(V_A(\mathbf{A}_f))^U$, the functional

$$(z, h_f) \longmapsto \theta^{\star}_{r_{\psi}}(z, h_f, g; \Phi_f)$$

on $(z, h_f) \in D^{\pm}(V_A) \times \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ descends to a function on the corresponding Shimura variety $\operatorname{Sh}_{U_A}(\operatorname{GSpin}(V_A), D^{\pm}(V_A))$. Although it is not holomorphic in the variable $z \in D^{\pm}(V_A)$, we obtain a function

$$\theta_{r_{\psi}}^{\star}: \operatorname{Sh}_{U_{A}}(D^{\pm}(V_{A}), \operatorname{GSpin}(V_{A})) \times \operatorname{SL}_{2}(\mathbf{Q}) \setminus \operatorname{SL}_{2}(\mathbf{A}) \longrightarrow \left(\mathcal{S}(V_{A}(\mathbf{A}_{f}))^{U_{A}} \right)^{\vee}$$

Extending to similitudes, we also obtain a function

$$\theta_{r_{\psi}}^{\star}: \operatorname{Sh}_{U_{A}}(D^{\pm}(V_{A}), \operatorname{GSpin}(V_{A})) \times \operatorname{GL}_{2}(\mathbf{Q}) \setminus \operatorname{GL}_{2}(\mathbf{A}) \longrightarrow \left(\mathcal{S}(V_{A}(\mathbf{A}_{f}))^{U_{A}} \right)^{\vee}$$

As explained in [34, §1], we can view the Gaussian Φ_{∞} as an eigenfunction for the action of the maximal compact subgroup $\operatorname{SO}_2(\mathbf{R}) \subset \operatorname{SL}_2(\mathbf{R})$, which for any $k_{\infty} \in \operatorname{SO}_2(\mathbf{R})$, $z \in D^{\pm}(V_A)$, and $h \in \operatorname{GSpin}(V_A)(\mathbf{A})$ satisfies the relation $r_{\psi}(k_{\infty})\Phi_{\infty}(x,z) = \Phi_{\infty}(x,z)$. Using the transformation property (20), we can then deduce that for all k_{∞} in the maximal compact subgroup $\operatorname{SO}_2(\mathbf{R})$ of $\operatorname{SL}_2(\mathbf{R})$ and all k in the maximal compact subgroup $\mathcal{K} = \operatorname{SL}_2(\widehat{\mathbf{Z}})$ of $\operatorname{SL}_2(\mathbf{A}_f)$, we have that

(21)
$$\theta_{r_{\psi}}^{\star}(z,h_f,gk_{\infty}k;\Phi_f) = (r_{\psi}(k)^{\vee})^{-1} \cdot \theta_{r_{\psi}}^{\star}(z,h_f,g;\Phi_f),$$

where $r_{\psi}(k)^{\vee}$ denotes the action of \mathcal{K} on the space $\mathcal{S}(V_A(\mathbf{A}_f))^{\mathcal{K}}$ dual to its action on $\mathcal{S}(V_A(\mathbf{A}_f))^{\mathcal{K}}$. In particular, this theta kernel $\theta_{r_{\psi}}^{\star}$ in the setting of quadratic spaces of signature (2, 2) as we consider has weight zero under the action of the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R})$.

4.4.3. Construction of the regularized theta lift. Suppose now that we fix any function

$$\phi: \operatorname{SL}_2(\mathbf{Q}) \setminus \operatorname{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(V_A(\mathbf{A}_f))^{U_A}$$

which for each $g \in SL_2(\mathbf{A}), k_{\infty} \in SO_2(\mathbf{R})$, and $k \in \mathcal{K}$ satisfies the transformation property

$$\phi(gkk_{\infty}) = r_{\psi}(k)^{-1} \cdot \phi(g).$$

It is then easy to check that the C-linear pairing $\{\cdot, \cdot\}$ defined as a function on $g \in SL_2(\mathbf{A})$ by the rule

$$\left\{\phi(g), \theta^{\star}_{r_{\psi}}(z, h_f, g)\right\} := \theta^{\star}_{r_{\psi}}(z, h_f, g; \phi(g))$$

is both left $SL_2(\mathbf{Q})$ -invariant and right $\mathcal{K} SO_2(\mathbf{R})$ -invariant. We can then consider the regularized theta lift

(22)
$$\vartheta_{\phi}^{\star}(z,h_{f}) := \int_{\mathrm{SL}_{2}(\mathbf{Q})\backslash \mathrm{SL}_{2}(\mathbf{A})}^{\star} \left\{ \phi(g), \theta_{r_{\psi}}^{\star}(z,h_{f},g) \right\} dg = \int_{\mathrm{SL}_{2}(\mathbf{Q})\backslash \mathrm{SL}_{2}(\mathbf{A})}^{\star} \theta_{r_{\psi}}^{\star}(z,h_{f},g;\phi(g)) dg,$$

as well as its extension to similitudes as described above, both as functions on $(z, h) \in Sh_U(D^{\pm}(V_A), GSpin(V_A))$.

To describe the integrals \int^{\star} in (22) defining these regularized theta lifts more explicitly, we first give semiclassical translation of the setup (cf. [34, §1]). Recall (see e.g. [21, Proposition 4.4.4] or [18]) that after fixing a standard fundamental domain $\mathcal{F} = \{\tau = u + iv \in \mathfrak{H} : |\Re(\tau)| \leq 1/2, \tau \overline{\tau} \geq 1\}$ for the action of $SL_2(\mathbb{Z})$ on \mathfrak{H} , each adelic matrix $g \in SL_2(\mathbb{A})$ can be expressed uniquely as a product

(23)
$$g = \gamma \cdot \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} v^{\frac{1}{2}} \\ & v^{-\frac{1}{2}} \end{pmatrix} \cdot k$$

for some $\gamma \in \mathrm{SL}_2(\mathbf{Q})$, $\tau = u + iv \in \mathcal{F}$, and $k \in \mathrm{SO}_2(\mathbf{R})$. Taking the decomposition (23) for granted, let us define for a given $g \in \mathrm{SL}_2(\mathbf{A})$ the corresponding matrix

$$g_{\tau} := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} \\ & v^{-\frac{1}{2}} \end{pmatrix}.$$

Similarly, fixing a standard fundamental domain $\mathcal{G} = \{\tau = u + iv : 0 \leq |\Re(\tau)| \leq 1/2, \tau \overline{\tau} \geq 1\}$ for the action of $\operatorname{GL}_2(\mathbf{Z})$ on $\operatorname{GL}_2(\mathbf{R})$, each element $g \in \operatorname{GL}_2(\mathbf{A})$ can be decomposed uniquely as as a product of matrices

(24)
$$g = \gamma \cdot \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} v \\ & 1 \end{pmatrix} \cdot k$$

for some $\gamma \in \mathrm{GL}_2(\mathbf{Q})$, $\tau = u + iv \in \mathcal{G}$, and $k \in O_2(\mathbf{R})$. Taking such a decomposition (24) for granted, we also define for a given $g \in \mathrm{GL}_2(\mathbf{A})$ the corresponding archimedean mirabolic matrix

$$g_{\tau} := \left(\begin{array}{cc} 1 & u \\ & 1 \end{array} \right) \left(\begin{array}{cc} v & \\ & 1 \end{array} \right).$$

Given a weight-zero L^2 -automorphic form ϕ on $\mathrm{SL}_2(\mathbf{Q}) \setminus \mathrm{SL}_2(\mathbf{A})$ or more generally $\mathrm{GL}_2(\mathbf{Q}) \setminus \mathrm{GL}_2(\mathbf{A})$, we shall write $f(\tau) := \phi(g_{\tau})$ to denote the corresponding weight-zero Maass form on $\tau = u + iv \in \mathfrak{H}$.

Suppose now that (ρ, \mathcal{V}) is a representation of the maximal compact subgroup $\mathcal{K} = \operatorname{SL}_2(\widehat{\mathbf{Z}}) \subset \operatorname{SL}_2(\mathbf{A}_f)$. Fix $\phi : \operatorname{SL}_2(\mathbf{Q}) \setminus \operatorname{SL}_2(\mathbf{A}) \longrightarrow \mathcal{V}$ a weight-zero automorphic form satisfying $\phi(gk_{\infty}k) = \rho(k)\phi(g)$ for all $g \in \operatorname{SL}_2(\mathbf{A}), k \in \mathcal{K}$, and $k_{\infty} \in \operatorname{SO}_2(\mathbf{R})$. Given $\gamma \in \operatorname{SL}_2(\mathbf{Z})$, we write $k\gamma$ to denote the unique lifting $k_{\gamma} \in \mathcal{K}$ determined by the diagonal embedding. We know that the weight-zero Maass form defined by $f(\tau) := \phi(g_{\tau})$ satisfies the following transformation law: For all $\gamma \in \operatorname{SL}_2(\mathbf{Z})$, we have $f(\gamma(\tau)) = \rho(k_{\gamma})f(\tau)$. We can proceed in the same way for the more general setting with $\phi : \operatorname{GL}_2(\mathbf{Q}) \setminus \operatorname{GL}_2(\mathbf{A}) \longrightarrow \mathcal{V}$ and automorphic form of weight zero, satisfying $\phi(gk_{\infty}k) = \rho(k)\phi(g)$ for all $g \in \operatorname{GL}_2(\mathbf{A}), k \in \mathcal{K} = \operatorname{GL}_2(\widehat{\mathbf{Z}}) \subset \operatorname{GL}_2(\mathbf{A}_f)$, and $k_{\infty} \in \operatorname{O}_2(\mathbf{R})$. Thus, the corresponding weight-zero form defined by $f(\tau) = \phi(g_{\tau})$ satisfies the transformation law $f(\gamma(\tau)) = \rho(k_{\gamma})f(\tau)$ for all $\gamma \in \operatorname{GL}_2(\mathbf{A})$, with k_{τ} the unique lift to \mathcal{K} via the diagonal embedding. Using these semiclassical notations, the regularized theta integral (22) can be expressed more concretely as

(25)
$$\vartheta_{\phi}^{\star}(z,h_{f}) = \vartheta_{f}^{\star}(z,h_{f}) = \int_{\mathrm{SL}_{2}(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \left\{ f(\tau), \theta_{r_{\psi_{0}}}^{\star}(z,h_{f},g_{\tau}) \right\} \frac{dudy}{v^{2}} \\
= \int_{\mathrm{SL}_{2}(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \theta_{r_{\psi_{0}}}^{\star}(z,h_{f},g_{\tau};f(\tau)) \frac{dudv}{v^{2}},$$

where the symbol \int^* denotes the regularized theta integral. To describe this more explicitly, let us write $\operatorname{CT}_{s=0} F(s)$ to denote the constant term in the Laurent series expansion around s = 0 of a function F(s) in $s \in \mathbb{C}$. Then, the regularized theta integral is given more explicitly by the constant terms

$$\int_{\mathrm{SL}_{2}(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \left\{ f_{0}(\tau), \theta_{r_{\psi}}^{\star}(z, h_{f}, g_{\tau}) \right\} \frac{dudy}{v^{2}} = \mathrm{CT}_{s=0} \left\{ \varinjlim_{T} \int_{\mathcal{F}_{T}} \left\{ f(\tau), \theta_{r_{\psi}}^{\star}(z, h_{f}, g_{\tau}) \right\} v^{-s} \frac{dudv}{v^{2}} \right\}$$
$$= \mathrm{CT}_{s=0} \left\{ \varinjlim_{T} \int_{\mathcal{F}_{T}} \theta_{r_{\psi}}^{\star}(z, h_{f}, g_{\tau}; f(\tau)) v^{-s} \frac{dudv}{v^{2}} \right\},$$

where the limits again are taken over the truncated fundamental domains

$$\mathcal{F}_T := \{ \tau = u + it \in \mathfrak{H} : |u| \le 1/2, \tau \overline{\tau} \ge 1, \text{ and } v \le T \}.$$

4.4.4. Harmonic weak Maass forms. We now consider S_{Λ_A} -valued harmonic weak Maass forms. Let $(\rho_{\Lambda_A}, \mathcal{V}_A)$ be the conjugate Weil representation on S_{Λ_A} , that is $\rho_{\Lambda_A}(\gamma) = \overline{r}_{\psi,\Lambda_A}(k_{\gamma}) = r_{\psi,-\Lambda_A}(g_{\gamma})$ for $\gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$ and its corresponding diagonal image $k_{\gamma} \in \mathcal{K} = \mathrm{SL}_2(\widehat{\mathbb{Z}})$ (cf. [8, (2.7)]). Suppose first that $k \in \mathbb{Z}$ is any integer weight; we shall later specialize to the case of k = 0. Let $|_{k,\rho_{\Lambda_A}}$ denote the Petersson weight k operator with respect to ρ_{Λ_A} , defined on a function f on $\Gamma \backslash \mathfrak{H}$ by the rule

$$f|_{k,\rho_{\Lambda_A}}(\gamma(\tau)) = (c\tau + d)^k \cdot \rho_{\Lambda_A}(\gamma) \cdot f(\tau) \quad \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let Δ_k denote the hyperbolic Laplacian of weight k, defined for $\tau = u + iv \in \mathfrak{H}$ by

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ik \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

Note that this Laplacian can be expressed in terms of the respective weight k Maass weight raising and lowering operators R_k and L_k as $-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k$, where

(26)
$$R_k = 2i \cdot \frac{\partial}{\partial \tau} + k \cdot v^-$$

denotes the Maass weight raising operator of weight k (which raises the weight by 2), and

(27)
$$L_k = -2iv^2 \cdot \frac{\partial}{\partial \overline{\tau}}$$

denotes the Maass lowering operator (which lowers the weight k by 2).

Definition 4.3. Fix an integer $k \leq 1$, and a lattice $\Lambda_A \subset V_A$ with corresponding subspace $S_{\Lambda_A} \subset S(V_A(\mathbf{A})))$. A twice differentiable function $f : \mathfrak{H} \longrightarrow S_{\Lambda_A}$ is a harmonic weak Maass form of weight k with respect to $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and representation ρ_{Λ_A} if:

- (i) The function is invariant under the Petersson weight-k operator: $f|_{k,\rho_{\Lambda_A}}\gamma = f$ for all $\gamma \in \Gamma$.
- (ii) There exists an S_{Λ_A} -valued Fourier polynomial

$$P_f(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \le 0} c_f^+(\mu, m) e(m\tau) \mathbf{1}_{\mu}$$

such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon v})$ as $v = \Im(\tau) \to \infty$ for some $\varepsilon > 0$.

(iii) The function is harmonic of weight k, i.e. $\Delta_k f = 0$.

We write $H_{k,\rho_{\Lambda_A}}$ for the vector space of such functions, and call the polynomial $P_f(\tau)$ the principal part of f. In the special case where we take the representation ρ_{Λ_A} to be the Weil representation r_{ψ,Λ_A} , we shall sometimes write $H_{k,\Lambda_A} = H_{k,\rho_{\Lambda_A}}$ for simplicity.

Recall that the Fourier series expansion of any weak harmonic Maass form $f \in H_{k,\rho_{\Lambda_A}}$ decomposes uniquely as the sum $f(\tau) = f^+(\tau) + f^-(\tau)$, where

$$f^+(\tau) := \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(\mu, m) e(m\tau) \mathbf{1}_{\mu}$$

is the holomorphic part, and

$$f^{-}(\tau) := \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \in \mathbf{Q} \atop m < 0} c_f^{-}(\mu, m) W_k(2\pi m v) e(m\tau) \mathbf{1}_{\mu},$$

for $W_k(a) := \int_{2a}^{\infty} e^{-t} t^{-k} dt = \Gamma(1-k, 2|a|)$ for a < 0 is the non-holomorphic part.

We consider the subspace $M_{k,\rho_{\Lambda_A}}^! \subset H_{k,\rho_{\Lambda_A}}$ of such weakly holomorphic forms, these being meromorphic modular functions whose poles are supported at the cusps. As explained in [8, §3], there is an antilinear differential operator ξ_k taking $H_{k,\rho_{\Lambda_A}}$ to the space $S_{2-k,\bar{\rho}_{\Lambda_A}}$ of holomorphic forms of weight 2-k with respect to Γ and $\overline{\rho}_{\Lambda_A}$, these forms being defined in the analogous way with $f = f^+$ for each $f \in S_{2-k,\overline{\rho}_{\Lambda_A}}$. This operator ξ_k can be defined explicitly as follows: We have an exact sequence of **C**-vector spaces

(28)
$$0 \longrightarrow M^!_{k,\rho_{\Lambda_A}} \longrightarrow H_{k,\rho_{\Lambda_A}} \xrightarrow{\xi_k} S_{2-k,\overline{\rho}_{\Lambda_A}} \longrightarrow 0,$$

where the map $\xi_k : H_{k,\rho_{\Lambda_A}} \longrightarrow S_{k-2,\overline{\rho}_{\Lambda_A}}$ is defined by

$$f(\tau) \mapsto \xi_k f(\tau) := v^{k-2} \overline{L_k f(\tau)}.$$

Here, the Petersson inner product $\langle \cdot, \cdot \rangle$ induces a bilinear pairing

$$\{\cdot,\cdot\}: M_{2-k,\overline{\rho}_{\Lambda_A}} \times H_{k,\rho_{\Lambda_A}} \longrightarrow \mathbf{C}, \quad \{g,f\}:=\langle g,\xi_k(f)\rangle.$$

By [7, Proposition 3.5] (cf. [8, § 3.1]), given $g \in M_{2-k,\overline{\rho}_{\Lambda_A}}$ with Fourier series expansion

$$g(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \ge 0} c_g(\mu, m) e(m\tau),$$

the pairing against a harmonic weak Maass form $f \in H_{k,\rho_{\Lambda_A}}$ with expansion as described above is given by

$$\{g,f\} = \langle \xi_k(f),g \rangle = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \le 0} c_f^+(\mu,m) c_g(\mu,-m)$$

In particular, his implies that $\{g, f\}$ depends only on the principal part $P_f(\tau)$ of f. We also deduce from the exactness of (28) that this pairing $\{\cdot, \cdot\}$ between $S_{2-k,\overline{\rho}_{\Lambda_A}}$ and $H_{k,\rho_{\Lambda_A}}/M_{k,\rho_{\Lambda_A}}^!$ is nondegenerate. Given $f \in H_{k,\rho_{\Lambda_A}}$ with constant principal part $P_f(\tau)$, it is known that f must be a holomorphic modular form $f \in M_{k,\rho_{\Lambda_A}}$ (see [8, Lemma 3.3]).

4.4.5. Theorems of Borcherds, Bruinier, and Howard-Madapusi Pera. Let us now return to the case of weight k = 0 we consider. We can define the regularized theta lift $\vartheta_{f_0}^{\star}(z, h_f)$ for any harmonic weak Maass form $f_0 \in H_{0,\Lambda_A}$ as

$$\vartheta_{f_0}^{\star}(z,h_f) = \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \langle \langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle \rangle \frac{dudv}{v^2} = \mathrm{CT}_{s=0} \left\{ \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle \rangle v^{-s} \frac{dudv}{v^2} \right\}$$

Here, for each $h_f \in \operatorname{GSpin}(V_A)(\mathbf{A}_f)/U_A$, we realize the theta kernel described above more concretely in terms of the Siegel theta series

$$\theta_{\Lambda_A}(\tau, z, h_f) : \mathfrak{H} \times D^{\pm}(V_A) \longrightarrow \mathcal{S}_{\Lambda_A}$$

determined by

$$heta_{\Lambda_A}(au,z,h_f) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} heta_{r_{\psi},\Lambda_A}^{\star}(z,h_f,g_{ au};\mathbf{1}_{\mu}).$$

When $f_{0,A} \in M_{0,\Lambda_A}^!$ is a weakly holomorphic form, the regularized theta lift $\vartheta_{f_{0,A}}^*(z, h_f)$ for the space (V_A, q_A) of signature (2, 2) can be computed by a theorem of Borcherds [4, Theorem 13.3] (cf. [34, Theorem 1.2]) as

$$\vartheta_{f_{0,A}}^{\star}(z,h_f) = -2\log|\Psi_{f_{0,A}}(z,h_f)|_A^2 - c_{f_{0,A}}^+(0,0) \cdot (2\log|y|_A + \Gamma'(1))$$

Here, $\Psi_{f_{0,A}}$ is a meromorphic form on $D^{\pm}(V_A) \times \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ of weight $k = c_{f_{0,A}}^+(0,0)/2$ known as the Borcherds product associated to $f_{0,A}$. Moreover, Borcherds [4] computed the divisor $\operatorname{Div}(\Psi_{f_{0,A}}^2)$ of this meromorphic function $\Psi_{f_{0,A}}$ explicitly in terms of the Fourier coefficients of $f_{0,A}$ and the special divisors $Z_A(m,\mu)$ of Definition (4.2) above. The subsequent theorem of Howard-Madapusi Pera [27, Theorem 9.1.1] shows that the Borcherds product $\Psi_{f_{0,A}}(z,h_f)$ takes algebraic values, so that the regularized theta lift $\vartheta_{f_{0,A}}^*(z,h_f)$ attached to any weakly holomorphic form $f_{0,A} \in M_{0,\Lambda_A}^!$ is seen to take values in logarithms of algebraic numbers (and hence in the ring of periods). To be more precise, given a weakly holomorphic form $f_{0,A} \in M_{0,\Lambda_A}^!$ with holomorphic part

$$f_{0,A}^{+}(\tau) = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} f_{0,A}^{+}(\tau) \mathbf{1}_{\mu} = \sum_{\mu \in \Lambda_{A}^{\#}/\Lambda_{A}} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_{f_{0,A}}^{+}(\mu, m) e(m\tau) \mathbf{1}_{\mu},$$

whose Fourier coefficients $c_{f_{A,0}}^+(\mu,m) \in \mathbf{Z}$ are integers, let us define the corresponding divisor

$$Z(f_{0,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \geq 0 \atop m > 0} c^+_{f_{0,A}}(\mu, -m) Z_A(\mu, m)$$

Note that in the special case where $f_{0,A} \in M_{0,\Lambda_A} \subset M_{0,\Lambda_A}^!$ is holomorphic, we have that $f_{0,A} = f_{0,A}^+$, and hence $c_{f_{0,A}}(\mu,m) = c_{f_{0,A}}^+(\mu,m)$ for each of the coefficients in the Fourier series expansion. As explained in [34] and [27], we consider the metrized line bundle $\hat{\omega} \in \widehat{\text{Pic}}(\text{Sh}_{U_A}(D^{\pm}(V_A), \text{GSpin}(V_A)))$ of modular forms of weight one, which under the complex uniformization of $\text{Sh}_{U_A}(D^{\pm}(V_A), \text{GSpin}(V_A))$ pulls back to the tautological line bundle on $D^{\pm}(V_A)$. Now, the Shimura varieties $\text{Sh}_{U_A}(D^{\pm}(V_A), \text{GSpin}(V_A))$ we consider have regular, flat integral models $Sh_{U_A}(D^{\pm}(V_A), \text{GSpin}(V_A)) \longrightarrow \text{Spec}(\mathbf{Z})$. The metrized line bundle $\hat{\omega}$ and the special divisors $Z(\mu, m)$ both extend in a natural way to $Sh_{U_A}(D^{\pm}(V_A), \text{GSpin}(V_A))$.

Theorem 4.4 (Borchards, Howard-Madapusi Pera). Let $f_{0,A} \in M_{0,\Lambda_A}^!$ be a weakly holomorphic form with integral Fourier coefficients $c_{f_{0,A}}^+(\mu, -m)$ for all $\mu \in \Lambda_A^{\#}/\Lambda_A$ and $m \in \mathbf{Q}_{>0}$. Replacing $f_{0,A}$ by a suitable integer multiple if needed, there exists a rational section $\Psi_{f_{0,A}}$ of the line bundle $\omega^{c_{f_{0,A}}^+(0,0)}$ on $Sh_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))$ whose norm under the metric defined by

$$|z||_A = \frac{(z,\overline{z})_A}{4\pi e^{\gamma}} = \frac{Q_A(z+\overline{z}) - Q_A(z) - Q_A(\overline{z})}{4\pi e^{\gamma}}$$

satisfies the relation

$$-2\log ||\Psi_{f_{0,A}}(z,h)||_A = \vartheta_{f_{0,A}}^{\star}(z,h)$$

for all $(z,h) \in D^{\pm}(V_A) \times \operatorname{GSpin}(V_A)(\mathbf{A}_f)$. Hence by Borcherds' theorem, we have that

$$\operatorname{Div}(\Psi_{f_{0,A}}) = Z(f_{0,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \in \mathbf{Q}_{>0}} c_{f_{0,A}}^+(\mu, -m) \cdot Z_A(m, \mu).$$

In particular, the Borcherds product is defined over \mathbf{Q} , from which we deduce that it takes algebraic values.

Proof. See [27, Theorem 9.1.1], which refines [4, Theorem 13.3], cf. [34, Theorem 1.2].

We have the following generalization when $f_{0,A} \in H_{0,-\Lambda_A}$ is not a weakly holomorphic form:

Theorem 4.5 (Borcherds, Bruinier). Let $f_{0,A} \in H_{0,-\Lambda_A}$ be a harmonic weak Maass form of weight 0 and representation $r_{\psi,-\Lambda_A}$. The regularized theta lift $\vartheta^*_{f_{0,A}}$ is a smooth function on $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)) \setminus Z(f_{0,A})$, with a logarithmic singularity along $-2 \log Z(f_{0,A})$. Moreover:

- The (1,1) form $dd^c \vartheta_{f_{0,A}}^*(z,h)$ has an analytic continuation to a smooth form on $\mathrm{Sh}_{U_A}(D(V_A)^{\pm}, \mathrm{GSpin}(V_A))$, and satisfies the Green current equation $dd^c[\vartheta_{f_{0,A}}^*(z,h)] + \delta_{Z(f_{0,A})} = [dd^c \vartheta_{f_{0,A}}^*(z,h)]$. Here, $\delta_{Z(f_{0,A})}$ denotes the Dirac current of the divisor $Z(f_{0,A})$.
- The regularized theta lift $\vartheta_{f_{0,A}}^{\star}$ is an eigenfunction for the generalized Laplacian operator Δ_z defined on $z \in D(V_A)$, with eigenvalue $c_{f_{0,A}}^+(0,0)/2$.

In particular, the regularized theta lift $\vartheta_{f_{0,A}}^{\star}$ can be identified with the automorphic Green's function $G_{Z(f_{0,A})}$ for the divisor $Z(f_{0,A})$, giving an arithmetic divisor $\widehat{Z}(f_{0,A}) = (Z(f_{0,A}), \vartheta_{f_{0,A}}^{\star})$ on $\mathrm{Sh}_{U_A}(D^{\pm}(V_A), \mathrm{GSpin}(V_A))$.

Proof. See [8, Theorems 4.2 and 4.3] and [6], as well as [7, Proposition 5.6, Theorem 6.1, Theorem 6.2]. As explained in [8, Theorem 4.3] and [5, Corollary 4.22], the difference $G_{Z(f_{0,A})}(z,h) - \vartheta_{f_{0,A}}^{\star}(z,h)$ can be viewed as a smooth subharmonic function on $\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A))(\mathbb{C})$ which is contained in the Hilbert space $L^{1+\varepsilon}(\operatorname{Sh}_{U_A}(D^{\pm}(V_A), \operatorname{GSpin}(V_A)), d\mu)$. The theorem of Yau [47] shows that such a function is constant. For the case of holomorphic forms due to Borcherds, see also [4, Theorem 13.3] and [34, Theorem 1.3].

4.5. Choice of harmonic weak Maass form. We choose the harmonic weak Maass form $f_{0,\eta,A} \in H_{0,\rho\Lambda_A}$ so that the cuspidal form $g_{A,\eta} = \xi_0(f_{0,\eta,A}) \in S_{2,\overline{\rho}\Lambda_A}$ is the canonical lift in the sense of Theorem 4.6 below of the twisted eigenform $f \otimes \eta \in S_2(\Gamma_0(d_K^2N), \eta)$. Here again, $f \in S_2(\Gamma_0(N))$ denotes the cuspidal newform parametrizing E/\mathbf{Q} , and $\eta = \eta_{K/\mathbf{Q}} = \left(\frac{d_K}{\cdot}\right)$ the even Dirichlet character associated K. For simplicity, we assume that $(N, d_K) = 1$. That is, we have the following relation to scalar-valued forms (cf. [8, §3]).

Theorem 4.6. Let us retain the setup described above with (V_A, Q_A) a quadratic space of type (2, 2). Let $\Lambda_A \subset V_A$ be the lattice associated to the compact open subgroup U_A of $\operatorname{GSpin}(V_A)(\mathbf{A}_f)$ described by (9) via (8). Let η denote the extension of the quadratic Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$ to a character of $\Gamma_0(d_K^2 N)$, with $f_\eta = f \otimes \eta \in S_2(\Gamma_0(d_K^2 N), \eta)$ the twisted cusp form having the Fourier series expansion

$$f_{\eta}(\tau) = (f \otimes \eta)(\tau) = \sum_{m \ge 1} c_f(m) \eta(m) e(m\tau).$$

There exists an S_{Λ_A} -valued modular form g_{η} of weight 2, determined canonically as the lifting of f_{η} defined in [49], whose Fourier series expansion is given by

$$g_{\eta}(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} g_{\eta,\mu}(\tau) \mathbf{1}_{\mu}, \quad \text{where} \quad g_{\eta,\mu}(\tau) = \sum_{\substack{m \in \mathbf{Q} \\ m \equiv d_K^2 N Q_A(\mu) \bmod (d_K^2 N)}} c_f(m) \eta(m) s(m) e\left(\frac{m\tau}{d_K^2 N}\right).$$

Here, s(m) denotes the function defined on each class $m \mod d_K N$ by $s(m) = 2^{\Omega(m, d_K^2 N)}$, where $\Omega(m, d_K^2 N)$ denotes the number of divisors of the greatest common divisor $(m, d_K^2 N)$.

Proof. This is a special case of [49, Theorem 4.15], adapted to match the setup of [8, p. 639, Lemma 3.1]. \Box

Observe from the Fourier series expansion described in Theorem 4.6 above that $f_{0,\eta,A}$ must be cuspidal, and hence that the corresponding regularized theta lift $\vartheta^*_{f_{0,\eta,A}}$ is annihilated by Δ_z . That is, the Green's function $\vartheta^*_{f_{0,\eta,A}}$ for the divisor $Z_A(f_{0,\eta,A})$ is a Laplacian eigenvector of eigenvalue 0 by Theorem 4.5:

Corollary 4.7. The regularized theta lift $\vartheta_{f_{0,\eta,A}}^{\star}$ is annihilated by the generalized Laplacian operator Δ_z . Hence, the automorphic Green's function $G_{Z(f_{0,\eta,A})} = \vartheta_{f_{0,\eta,A}}^{\star}$ is harmonic with respect to Δ_z .

4.6. Langlands Eisenstein series and the Siegel-Weil formula. Let us now record some special cases of the Siegel-Weil formula for our later calculations of averages over the subspaces $Z(V_{A,2})$ associated to the anisotropic subspaces $(V_{A,2}, Q_{A,2})$. We first introduce Langlands Eisenstein series and review the relevant Siegel-Weil formula abstractly following [34, Theorem 4.1] and [8, Theorem 2.1]. We then give a more arithmetic description of the vector-valued Siegel theta and Eisenstein series.

Recall we introduced the anisotropic subspaces $(V_{A,j}, Q_{A,j})$ of signature (1, 1). Let us temporarily write (V_0, Q_0) to denote the ambient quadratic space (V_A, A_A) of signature (2, 2), so that (V_j, Q_j) for j = 0, 1, 2 can denote any of these three spaces. In each case, we write $r_{\psi,j} : \mathrm{SO}(V_j)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(V_j(\mathbf{A}))$ to denote the corresponding (restriction of the) Weil representation $r_{\psi} : \mathrm{SO}(V_A)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(V_A(\mathbf{A}))$, with $\theta_{r_{\psi,j}}$ the corresponding theta kernel defined on $h \in \mathrm{SO}(V_j)(\mathbf{A}), g \in \mathrm{SL}_2(\mathbf{A})$ and $\Phi \in \mathcal{S}(V_j(\mathbf{A}))$ by

$$\theta_{r_{\psi,j}}(h,g;\Phi) = \sum_{x \in V_j(\mathbf{Q})} r_{\psi,j}(h,g)\Phi(x).$$

We now consider the associated Langlands Eisenstein series. Recall that we write $\mathcal{K} = \mathrm{SL}_2(\mathbf{Z})$ to denote the maximal compact subgroup of $\mathrm{SL}_2(\mathbf{A}_f)$. To describe this, we shall use the Iwasawa decomposition

(29)
$$\operatorname{SL}_2(\mathbf{A}) = N_2(\mathbf{A})M_2(\mathbf{A})\mathcal{K}\operatorname{SO}_2(\mathbf{R}),$$

with standard shorthand matrix notations

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N_2(\mathbf{A}), \quad m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in M_2(\mathbf{A}), \quad k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R}).$$

Let χ_{V_j} denote the idele class character of \mathbf{Q} defined on $x \in \mathbf{A}^{\times}/\mathbf{Q}^{\times}$ by the formula $\chi_{V_j}(x) = (x, \det(V_j))_{\mathbf{A}}$, where $(\cdot, \cdot)_{\mathbf{A}}$ denotes the Hilbert symbol on \mathbf{A} , and $\det(V_j)$ the Gram determinant. Writing $s \in \mathbf{C}$ to denote a complex parameter, let $I(s, \chi_{V_j})$ denote the corresponding principal series representation of $\mathrm{SL}_2(\mathbf{A})$ induced by the quasi-character $\chi_{V_j} |\cdot|^s$. This consists of all smooth (decomposable) functions $\varphi(g, s)$ on $g \in SL_2(\mathbf{A})$ and $s \in \mathbf{C}$ satisfying

$$\varphi(n(b)m(a)g,s) = \chi_{V_i}(a)|a|^{s+1}\varphi(g,s)$$

for all $b \in \mathbf{A}$ and $a \in \mathbf{A}^{\times}$. Note that $\mathrm{SL}_2(\mathbf{A})$ acts on the space $I(s, \chi_{V_j})$ by right translations. Writing $s_0(V_j) := \dim(V_j)/2 - 1$, there is an $\mathrm{SL}_2(\mathbf{A})$ -intertwining map

$$\lambda: \mathcal{S}(V_j(\mathbf{A})) \longrightarrow I(s_0(V_j), \chi_{V_j}), \quad \Phi \mapsto \lambda(\Phi)(g) := (r_{\psi,j}(g)\Phi)(0).$$

A section $\varphi = \varphi(g, s) \in I(s, \chi_{V_j})$ is said to be *standard* if its restriction to the maximal compact subgroup \mathcal{K} SO₂(**R**) does not depend on $s \in \mathbf{C}$. As explained in [8, § 2.1], using the Iwasawa decomposition, we deduce that $\lambda(\Phi) \in I(s_0(V_j), \chi_{V_j})$ has a unique extension to a standard section $\lambda(\Phi, s) \in I(s, \chi_{V_j})$ for which $\lambda(\Phi, s_0(V_j)) = \lambda(\Phi)$. Given any standard section $\varphi = \varphi(g, s) \in I(s, \chi_{V_j})$, and writing $P = N_2 M_2 \subset SL_2$ to denote the standard parabolic subgroup, we then consider the Eisenstein series defined by

$$E(g,s;\varphi) = E_{r_{\psi,j}}(g,s;\varphi) = \sum_{\gamma \in P(\mathbf{Q}) \setminus \operatorname{SL}_2(\mathbf{Q})} \varphi(\gamma g,s).$$

We can now state the following special case(s) of the Siegel-Weil formula in this setting.

Theorem 4.8 (Siegel-Weil). Let (V_j, Q_j) for j = 0, 1, 2 denote any of the quadratic spaces introduced above. We have for any $g \in SL_2(\mathbf{A})$ and decomposable Schwartz function $\Phi \in \mathcal{S}(V_j(\mathbf{A}))$ the average formula

$$\kappa \cdot \int_{\mathrm{SO}(V_j)(\mathbf{Q}) \setminus \mathrm{SO}(V_j)(\mathbf{A})} \theta_{r_{\psi,j}}(h,g;\Phi) dh = E_{r_{\psi,j}}(g,s_0,\lambda(\Phi)),$$

where

$$\kappa = \begin{cases} 1 & \text{if } \dim(V_j) > 2\\ 2 & \text{if } \dim(V_j) \le 2 \end{cases} \quad and \quad s_0 = s_0(V_j) = \frac{\dim(V_j)}{2} - 1.$$

Moreover, the Eisenstein series $E_{r_{\psi,j}}(g,s,\lambda(\Phi))$ in each case j=0,1,2 is holomorphic at $s=s_0$.

Proof. See [34, Theorem 4.1], and more generally [36, § I.4].

Let us now consider the following more explicit version of Theorem 4.8. We first describe the theta kernel $\theta_{r_{\psi,j}}$ and Eisenstein series $E_{r_{\psi,j}}$ in terms of vector-valued modular forms. Following [8, § 2.1], we can for any integer weight $l \in \mathbf{Z}$ consider the unique standard section $\Phi_{\infty}^{l}(s) \in I_{\infty}(s, \chi_{V_{0}})$ for which

(30)
$$\Phi^l_{\infty}(k(\theta), s) = \exp(il\theta)$$

In terms of the Iwasawa decomposition (29), this section also satisfies the transformation property

(31)
$$\Phi^l_{\infty}(n(b)m(a)k(\theta), s) = \chi_{V_0}(a)|a|^{s+1}\exp(il\theta)$$

for all $n(b) \in N_2(\mathbf{A})$, $m(a) \in M_2(\mathbf{A})$, and $k(\theta) \in SO_2(\mathbf{R})$. We shall use the same notation to denote the restriction to each of the subspaces $\Phi_{\infty}^l = \Phi_{\infty}^l(s) \in I(s, \chi_{V_i})$.

Following the discussion in [8, (2.15)], we deduce from our definition of the weight zero Gaussian function $\Phi_{\infty} \in \mathcal{S}(V_0(\mathbf{R})) \otimes C^{\infty}(D^{\pm}(V_0))$ that we have the relation

(32)
$$\lambda_{\infty}(\Phi_{\infty}) = \lambda_{\infty}(\Phi_{\infty}(\cdot, z)) = \Phi_{\infty}^{\frac{p(V_0) - q(V_0)}{2}}(s_0(V_0)) = \Phi_{\infty}^0(1) \in I_{\infty}(1, \chi_{V_0}).$$

Here, $(p(V_j), q(V_j))$ denotes the signature of any of the spaces V_j . We remark that each of the quadratic spaces V_j we consider leads to looking at an Eisenstein series of weight $k = k(V_j) = (p(V_j) - q(V_j))/2 = 0$. We know that (32) has a unique extension to a standard section $\Phi^0_{\infty}(s) \in I_{\infty}(s, \chi_{V_0})$ so that $\Phi^0_{\infty}(s_0(V_0)) = \lambda_{\infty}(\Phi_{\infty})$. We can restrict this section $\Phi^0_{\infty} = \Phi^0_{\infty}(s) \in I(s, \chi_{V_0})$ naturally to each of the subspaces V_j with j = 1, 2. Again, we shall use the same notations to denote each of these restrictions $\Phi^0_{\infty} = \Phi^0_{\infty}(s) \in I(s, \chi_{V_j})$.

Given any even lattice $\Lambda_j \subset V_j$, and writing λ_f to denote the finite component of the standard section $\lambda(\Phi) = \lambda(\Phi, s) \in I(s, \chi_{V_j})$ described above, we consider the corresponding S_{Λ_j} -valued Eisenstein series of weight k = 0 defined on $\tau = u + iv \in \mathfrak{H}$ and $s \in \mathbb{C}$ by

$$E_{\Lambda_j}(\tau,s;0) := \sum_{\mu \in \Lambda_j^{\#}/\Lambda_j} E_{r_{\psi,j}}(g_{\tau},s;\Phi_{\infty}^0 \otimes \lambda_f(\mathbf{1}_{\mu})) \cdot \mathbf{1}_{\mu}$$

We consider the \mathcal{S}_{Λ_i} -valued theta function defined on $\tau = u + iv \in \mathfrak{H}, z \in D^{\pm}(V_i)$, and $h_f \in \mathrm{GO}(V_i)(\mathbf{A})$ by

(33)
$$\theta_{\Lambda_j}(\tau, z, h) := \sum_{\mu \in \Lambda_j^{\#}/\Lambda_j} \theta_{r_{\psi,j}}^{\star}(z, h_f, g_{\tau}; \mathbf{1}_{\mu}) \cdot \mathbf{1}_{\mu}.$$

Theorem 4.9 (Siegel-Weil for S_{Λ_j} -valued forms). We have the identification of functions of $\tau \in \mathfrak{H}$:

$$\kappa \cdot \int_{\mathrm{SO}(V_j)(\mathbf{Q}) \setminus \mathrm{SO}(V_j)(\mathbf{A})} \theta_{\Lambda_j}(\tau, z, h_f) = E_{\Lambda_j}(\tau, s_0, k) = E_{\Lambda_j}(\tau, s_0(V_j); k(V_j)).$$

Here again, $s_0 = s_0(V_j) := \dim(V_j)/2 - 1$, and $k = k(V_j) := (p(V_j) - q(V_j))/2 = 0$.

Proof. Cf. [8, Proposition 2.2], and note that we deduce this from Theorem 4.8 with (30) and (32). \Box

4.7. Eisenstein series and Maass weight-raising operators. As preparation for our later calculations, let us also give the following more classical descriptions of the Eisenstein series appearing in Theorem 4.9, with relations to the Maass raising and lowering operators R_l, L_l introduced above for any integer l. We remark that these are *not* incoherent Eisenstein series in the sense of Kudla. We also use the same notational conventions with the three spaces $(V_i, q_i), j = 0, 1, 2$ as in our discussion of the Siegel-Weil theorem above.

Here, we take for granted the definition of the matrix g_{τ} for $\tau = u + iv \in \mathfrak{H}$ in the unique decomposition (23) above via the Iwasawa decomposition for $SL_2(\mathbf{A})$, also as described above in (29). Following the discussion in [8, § 2.2], we consider elements of $SL_2(\mathbf{A})$ of the form

$$\gamma \cdot g_{\tau} = n(\beta) \cdot m(\alpha) \cdot k(\theta) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbf{Z}), \ \beta \in \mathbf{R}, \ \alpha \in \mathbf{R}_{>0}, \ k(\theta) \in \mathrm{SO}_2(\mathbf{R}).$$

A direct calculation shows that

$$\alpha = v^{\frac{1}{2}} \cdot |c\tau + d|^{-1}, \quad \exp(i\theta) = \frac{c\overline{\tau} + d}{|c\tau + d|},$$

so that substituting into (31) gives us

$$\Phi^{l}_{\infty}(\gamma g_{\tau}, s) = v^{\frac{s}{2} + \frac{1}{2}} (c\tau + d)^{-l} |c\tau + d|^{l-s-1}.$$

Hence, writing $\Gamma_{\infty} = P(\mathbf{Q}) \cap \Gamma$ for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ as above, we find that

$$E_{r_{\psi,2}}(g_{\tau},s;\Phi_{\infty}^{l}\otimes\lambda_{f}(\mathbf{1}_{\mu})) = \sum_{\gamma\in\Gamma_{\infty}\backslash\Gamma}(c\tau+d)^{-l}\frac{v^{\frac{1}{2}+\frac{1}{2}}}{|c\tau+d|^{s+1-l}}\cdot\lambda_{f}(\mathbf{1}_{\mu})(\gamma)$$
$$= \sum_{\gamma\in\Gamma_{\infty}\backslash\Gamma}(c\tau+d)^{-l}\frac{v^{\frac{s}{2}+\frac{1}{2}}}{|c\tau+d|^{s+1-l}}\cdot\langle\mathbf{1}_{\mu},(r_{\psi_{0},j}^{-1}(\gamma)\mathbf{1}_{0})\rangle$$

where $\langle \cdot, \cdot \rangle$ here denotes the L^2 inner product on S_{Λ_j} . In this way, we find that the vector-valued Eisenstein series we considered above can be written classically as

(34)
$$E_{\Lambda_j}(\tau,s;l) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \left[\Im(\tau)^{\frac{(s+1-l)}{2}} \mathbf{1}_0 \right] \Big|_{l,\rho_{\Lambda_j}} \gamma,$$

where $|_{l,\rho_{\Lambda_{i}}}$ again denotes the Petersson weight-*l* slash operator for $\rho_{\Lambda_{j}}$.

4.7.1. Eisenstein series associated to the anisotropic subspaces. Let us now say more about the Eisenstein series associated to the lattices $\Lambda_{A,2} = \Lambda_A \cap V_{A,2}$ in the signature (1, 1) subspace $V_{A,2} = (V_{A,2}, Q_{A,2})$. Writing $\mathfrak{d}_{\mathfrak{a}}$ to denote the different of the integer ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ of the class $A = [\mathfrak{a}]$, with inverse different $\mathfrak{d}_{\mathfrak{a}}^{-1} = \{\lambda \in \mathfrak{a} : \operatorname{Tr}(\lambda \mathfrak{a}) \in \mathbb{Z}\}$, we have $\Lambda_{A,2}^{\#} \cong \mathfrak{d}_{\mathfrak{a}}^{-1} \cap \Lambda_{A,2}$ and $\Lambda_{A,2}^{\#}/\Lambda_{A,2} \cong (\mathfrak{d}_{\mathfrak{a}}^{-1} \cap \Lambda_{A,2})/\Lambda_{A,2}$. We can also identify $\chi_{V_{A,2}} = \eta = \eta_K$ with the quadratic Dirichlet character $\eta_K(\cdot) = (\frac{d_K}{\cdot})$. Writing

$$\Lambda(s,\eta) = d_K^{\frac{s}{2}} \Gamma_{\mathbf{R}}(s+1)L(s,\eta), \quad \Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$$

to denote its corresponding completed L-function, we consider the completed Eisenstein series defined by

$$E^{\star}_{\Lambda_{A,2}}(\tau,s) := \Lambda(s+1,\eta) E_{\Lambda_{A,2}}(\tau,s)$$

Proposition 4.10. The Eisenstein series $E^{\star}_{\Lambda_{A,2}}(\tau, s)$ has a meromorphic continuation to all $s \in \mathbf{C}$, and satisfies the symmetric functional equation $E^{\star}_{\Lambda_{A,2}}(\tau, s) = E^{\star}_{\Lambda_{A,2}}(\tau, -s)$.

Proof. See the proof of [8, Proposition 2.5] or more generally [11, Theorem 3.7.2]. We deduce this in a more straightforward way from the Langlands functional equation for the (coherent) Eisenstein series

$$E_{\Lambda_{A,2}}(\tau,s) = E_{\Lambda_{A,2}}(\tau,s;0) = \sum_{\mu \in \Lambda_{A,2}^{\#}/\Lambda_{A,2}} E(g_{\tau},s,\Phi_{\infty}^{0} \otimes \lambda_{f}(\mathbf{1}_{\mu})) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \left[\Im(\tau)^{\frac{(s+1)}{2}} \mathbf{1}_{0}\right] \Big|_{0,\rho_{\Lambda_{A,2}}} \gamma.$$

To be more precise, it will suffice to prove the functional equation for each of the Langlands Eisenstein series $E(g_{\tau}, s, \Phi_{0}^{0} \otimes \lambda_{f}(\mathbf{1}_{\mu})) = E_{r_{\psi,2}}(g_{\tau}, s, \Phi_{\infty}^{0} \otimes \lambda_{f}(\mathbf{1}_{\mu}))$. Let us write the Euler product decomposition of $\Lambda(s, \eta) = \Lambda(s, \eta_{D})$ as $\Lambda(s, \eta) = \prod_{v \leq \infty} L(s, \eta_{v})$. Let us also for simplicity write $\Phi_{\mu} = \lambda_{f}(\mathbf{1}_{\mu})$ for the nonarchimedean part of our chosen global section $\varphi = \Phi_{\infty}^{0} \otimes \lambda_{f}(\mathbf{1}_{\mu}) \in I(s, \chi_{V_{A,2}}) = I(s, \eta)$. Given any standard section $\varphi = \varphi(s) \in I(s, \eta)$ and $g \in SL_{2}(\mathbf{A})$, the Langlands functional equation implies that

$$E(g, s; \varphi) = E(g, -s; M(s)\varphi)$$

for $M(s) = \prod_{v \leq \infty} M_v(s) : I(s,\eta) \to I(s,\eta)$ the global intertwining operator. Recall that for $\Re(s) \gg 0$ sufficiently large, each of the local intertwining operators $M_v(s) : I_v(s,\eta) \to I_v(s,\eta)$ is given by the formula

$$M_{v}(s)\varphi_{v}(g,s) = \int_{\mathbf{Q}_{v}} \varphi_{v}(wn(b)g,s)db, \quad w := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

for φ_v in the local principal series representation $I_v(s,\eta)$. At the real place $v = \infty$, it is well-known that

$$M_{\infty}(s)\Phi_{\infty}^{0}(g,s) = C_{\infty}(s)\Phi_{\infty}^{0}(g,-s), \quad C_{\infty}(s) = \gamma_{\infty}(V_{A,2}) \cdot \frac{\Gamma_{\mathbf{R}}(s+1)}{\Gamma_{\mathbf{R}}(s+1)}.$$

Here, $\gamma_{\infty}(V_{A,2}) = 1$ denotes the local Weil index for the representation $r_{\psi,\Lambda_{A,2}}$ of SO($V_{A,2}$) × SL₂(**R**) acting on $\mathcal{S}(V(\mathbf{R}))$ associated to the signature (1, 1) lattice $\Lambda_{A,2}$. At finite places $v \nmid d_K \infty$, is also well-known that

$$M_{v}(s)\Phi_{\mu}(g,s) = C_{v}(s)\Phi_{\mu}^{0}(g,-s), \quad C_{v}(s) = \frac{L(s,\eta_{v})}{L(s+1,\eta_{v})}$$

For the remaining finite places $v \mid d_K$, we can use the same computation of the local intertwining operators Φ_{μ} given in [8, Proposition 2.5] to show that

$$M_v(s)\Phi_\mu(g,s) = \gamma_v(V_{A,2})\operatorname{vol}(\Lambda_{A,2,v})\Phi_\mu(g,-s)$$

where $\gamma_v(V_{A,2})$ is the local Weil index, and $\operatorname{vol}(\Lambda_{A,2,v}) = [\Lambda_{A,2,v}^{\#} : \Lambda_{A,2,v}]^{-\frac{1}{2}}$ is the measure of $\Lambda_{A,2,v}$ with respect to the self-dual Haar measure on $\Lambda_{A,2,v}$ for the local additive character ψ_v . Combining the previous local functional equations with the product formulae

$$\prod_{v|d_K} \operatorname{vol}(\Lambda_{A,2,v}) = d_K^{-\frac{1}{2}}, \quad \prod_{v \le \infty} \gamma_v(V_{A,2}) = 1,$$

we then obtain the global functional equation

$$E(g, s, \Phi^0_{\infty} \otimes \Phi_{\mu}) = \frac{\Lambda(s, \eta)}{\Lambda(s+1, \eta)} \cdot E(g, -s, \Phi^0_{\infty} \otimes \Phi_{\mu}).$$

Using the classical (Dirichlet) functional equation $\Lambda(s,\eta) = \Lambda(1-s,\eta)$, we then deduce the claim.

4.7.2. Maass weight raising and lowering operators. Recall that we defined the Maass weight raising and lowering operators R_l and L_l in (26) and (27) above. These operators raise and lower the weights of these Eisenstein series by two respectively. To be more precise, it is easy to check from the definitions that

$$L_{l}E_{\Lambda_{j}}(\tau, s; l) = \frac{1}{2} \cdot (s + 1 - l) \cdot E_{\Lambda_{j}}(\tau, s; l - 2),$$

$$R_{l}E_{\Lambda_{j}}(\tau, s; l) = \frac{1}{2} \cdot (s + 1 + l) \cdot E_{\Lambda_{j}}(\tau, s; l + 2).$$

We refer to [34, Proposition 2.7] and [8, Lemma 2.3] for details. Here, we have for the Eisenstein series corresponding to our signature (1, 1) subspace V_2 that

(35)
$$L_2 E_{\Lambda_2}(\tau, s; 2) = \frac{1}{2} \cdot (s - 1) \cdot E_{\Lambda_2}(\tau, s; 0)$$

Observe that the Eisenstein series $E_{\Lambda_2}(\tau, s; 0)$ is holomorphic at $s = s_0 = s_0(V_2) := \dim(V_2)/2 - 1 = 0$ thanks to Siegel-Weil, Theorem 4.8 (cf. Corollary 4.9). It follows that at s = 0, we have the identity

(36)
$$L_2 E_{\Lambda_2}(\tau, 0; 2) = -\frac{1}{2} \cdot E_{\Lambda_2}(\tau, 0; 0).$$

Now, taking the first derivative with respect to s on each side of (35) we get

$$L_2 E'_{\Lambda_2}(\tau, s; 2) = \frac{1}{2} \cdot (s - 1) \cdot E'_{\Lambda_2}(\tau, s; 0) + \frac{1}{2} \cdot E_{\Lambda_2}(\tau, s; 0).$$

Evaluating this identity at s = 0 gives us

$$L_2 E'_{\Lambda_2}(\tau, 0; 2) = \frac{1}{2} \cdot E_{\Lambda_2}(\tau, 0; 0) - \frac{1}{2} \cdot E'_{\Lambda_2}(\tau, 0; 0)$$

and hence

(37)
$$2L_2 E'_{\Lambda_2}(\tau, 0; 2) = E_{\Lambda_2}(\tau, 0; 0) - E'_{\Lambda_2}(\tau, 0; 0).$$

Let ∂ and $\overline{\partial}$ denote the Dolbeault operators, so that the exterior derivative on differential forms on \mathfrak{H} is given by $d = \partial + \overline{\partial}$. We also write $d\mu(\tau) = \frac{dudv}{v^2}$ for $\tau = u + iv \in \mathfrak{H}$. We have the following useful relation.

Lemma 4.11. The weight-lowering operator L_l can be described in terms of differential forms as

$$\overline{\partial}(fd\tau) = -v^{2-l}\xi_l(f)d\mu(\tau) = -L_lfd\mu(\tau).$$

Proof. See [16, Lemma 2.5] (cf. [8, Lemma 2.3]).

We now derive the following result for later use.

Proposition 4.12. We have that $E'_{\Lambda_2}(\tau, 0; 0) = 0$, and hence via (37) that $-2L_2E'_{\Lambda_2}(\tau, 0; 2) = -E_{\Lambda_2}(\tau, 0; 0)$. Expressed equivalently in terms of differential forms via Lemma 4.11, we obtain the relation

$$-2L_2 E'_{\Lambda_2}(\tau,0;2)d\mu(\tau) = 2\overline{\partial} \left(E'_{\Lambda_2}(\tau,0;2)d\tau \right) = -E_{\Lambda_2}(\tau,0;0)d\mu(\tau),$$

equivalently

(38)
$$E_{\Lambda_2}(\tau, 0; 0) d\mu(\tau) = -2\overline{\partial} \left(E'_{\Lambda_2}(\tau, 0; 2) d\tau \right)$$

Proof. We know by the Siegel-Weil formula (Theorem 4.9) that the Eisenstein series $E_{\Lambda_2}(\tau, s; 0)$ is analytic at s = 0. Hence, $E_{\Lambda_2}(\tau, s; 0)$ and its derivatives with respect to s are analytic at s = 0. This implies, for instance, that the values $E_{\Lambda_2}(\tau, 0; 0)$ and $E'_{\Lambda_2}(\tau, 0; 0)$ are defined and finite, and that we can expand $E_{\Lambda_2}(\tau, s; 0)$ into its Taylor series expansion around s = 0. Now, we know from the discussion of Proposition 4.10 that the Eisenstein series $E_{\Lambda_2}(\tau, 0; 0)$ associated to the signature (1, 1) lattice Λ_2 has an analytic continuation $E^*_{\Lambda_2}(\tau, s; 0)$ to all $s \in \mathbf{C}$ which satisfies an even functional equation $E^*_{\Lambda_2}(\tau, s) = E^*_{\Lambda_2}(\tau, -s)$. Comparing the corresponding Taylor series expansions around s = 0 as we may, we then see that for any $s \in \mathbf{C}$ with $0 \leq \Re(s) < 1$ we have the relation

$$E^{\star}_{\Lambda_2}(\tau,0) + E^{\star\prime}_{\Lambda_2}(\tau,0)s + O(s^2) = E^{\star}_{\Lambda_2}(\tau,0) - E^{\star\prime}_{\Lambda_2}(\tau,0)s + O(s^2)$$

equivalently

$$E_{\Lambda_2}^{\star\prime}(\tau, 0)s + O(s^2) = -E_{\Lambda_2}^{\star\prime}(\tau, 0)s + O(s^2).$$

Taking the limit as $\Re(s) \to 0$, we then see that $E_{\Lambda_2}^{\star\prime}(\tau, 0)$ must vanish, and hence that $E_{\Lambda_2}^{\prime}(\tau, 0; 0) = 0$. \Box

Let us now consider the Fourier series expansion of the Eisenstein series

$$E_{\Lambda_2}(\tau, s; 2) = \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} A_{\Lambda_2}(s, \mu, m, v) e(m\tau) \mathbf{1}_{\mu}.$$

We can use⁸ the discussion in Kudla [34, §2] (cf. [8, § 2.2]) to show that the Laurent series expansions of each of the Fourier coefficients $A_{\Lambda_2}(s, \mu, m, v)$ around s = 0 takes the form

(39)
$$A_{\Lambda_2}(s,\mu,m,v) = a_{\Lambda_2}(\mu,m) + b_{\Lambda_2}(\mu,m,v)s + O(s^2),$$

and deduce that the corresponding derivative Eisenstein series at s = 0 has the Fourier series expansion

(40)
$$E'_{\Lambda_2}(\tau,0;2) = \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} b_{\Lambda_2}(\mu,m,v) e(m\tau) \mathbf{1}_{\mu}$$

Following the argument of [34, Theorem 2.12], we then consider the limiting values

(41)
$$\kappa_{\Lambda_{2}}(\mu, m) = \begin{cases} \lim_{v \to \infty} b_{\Lambda_{2}}(\mu, m, v) & \text{if } \mu \neq 0 \text{ or } m \neq 0\\ \lim_{v \to \infty} b_{\Lambda_{2}}(\mu, m, v) - \log(v) & \text{if } \mu = 0 \text{ and } m = 0 \end{cases}$$

We define from these coefficients the S_{Λ_2} -valued periodic function $\mathcal{E}_{\Lambda_2}(\tau)$ on $\tau = u + iv \in \mathfrak{H}$ via

(42)
$$\mathcal{E}_{\Lambda_2}(\tau) := \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} \kappa_{\Lambda_2}(\mu, m) e(m\tau) \mathbf{1}_{\mu}.$$

Observe (cf. [8, Remark 2.4, (3.5)]) that we can view this form $\mathcal{E}_{\Lambda_2}(\tau)$ defined by (42) as the holomorphic part of derivative Eisenstein series $E'_{\Lambda_2}(\tau, 0; 2)$, i.e. $\mathcal{E}_{\Lambda_2}(\tau) = E'^+_{\Lambda_2}(\tau, 0; 2)$. We shall return to this point later.

4.8. Summation along anisotropic subspaces of type (1, 1). We now calculate the regularized theta lifts $\vartheta_{f_0}^*(z,h)$ along the anisotropic subspace of type (1, 1) corresponding to the ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ of the class $A = [\mathfrak{a}] \in \operatorname{Pic}(\mathcal{O}_c)$. Let us simplify notations in writing $(V,q) = (V_A, Q_A)$ to denote the ambient quadratic space of signature (2, 2). We then write (V_j, Q_j) for j = 1, 2 to denote the respective subspaces $(V_{A,1}, Q_{A,1})$, and $(V_{A,2}, Q_{A,2})$ of signature (1, 1). We also write $\Lambda = \Lambda_A, \Lambda_1 = \Lambda_A \cap V_{A,1}$, and $\Lambda_2 = \Lambda_A \cap V_{A,2}$ for the corresponding lattices. Let $f_0 \in H_{0,\Lambda}$ be any harmonic weak Maass form. We develop the ideas of [8, Theorem 4.7] and [16] to calculate the values of the regularized theta lift $\vartheta_{f_0}^*(z,h)$ along the geodesic subset corresponding to the subspace $(V_2, Q_2) = (V_{A,2}, Q_{A,2})$ in terms of the central derivative values of some related Rankin-Selberg *L*-function. Let us note again that we do not encounter incoherent Eisenstein series in this setup, and so our arguments differ from those of [8], [34], and [16] (for instance).

We again write $D^{\pm}(V) = D^{\pm}(V_A)$ for the Grassmannian of oriented hyperplanes $z \subset V(\mathbf{R})$, with $D^{\pm}(V_2) = D^{\pm}(V_{A,2})$ the subdomain of hyperbolic lines. Hence, each $z \in D^{\pm}(V)$ gives rise to a pair of hyperbolic lines $z_{V_2}^{\pm} \in D^{\pm}(V_2)$. Again, we consider $\operatorname{GSpin}(V_2)$ as a subgroup of $\operatorname{GSpin}(V)$ acting trivially on V_1 . Fixing a compact open subgroup $U \subset \operatorname{GSpin}(V)(\mathbf{A}_f)$ as above, let $U_2 := U \cap \operatorname{GSpin}(V_2)(\mathbf{A}_f)$. We then consider the corresponding "geodesic" set

$$Z(V_2) = \operatorname{GSpin}(V_2)(\mathbf{Q}) \setminus \{D^{\pm}(V_2)\} \times \operatorname{GSpin}(V_2)(\mathbf{A}_f)/U_2$$

associated to

$$\operatorname{Sh}_U(\operatorname{GSpin}(V), D^{\pm}(V)) := \operatorname{GSpin}(V)(\mathbf{Q}) \setminus D^{\pm}(V) \times \operatorname{GSpin}(V)(\mathbf{A}_f) / U$$

Given a point $(z_{V_2}^{\pm}, h) \in Z(V_2)$ and a harmonic weak Maass form $f_0 \in H_{0,\Lambda}$, we now compute the summation of the regularized theta lift $\vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h)$ defined above over $Z(V_2)$,

$$\vartheta_{f_0}^{\star}(Z(V_2)) := \sum_{\substack{(z_{V_2}^{\pm}, h) \in Z(V_2)}} \vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h).$$

Fix a Tamagawa measure on $SO(V_2)(\mathbf{A})$ for which $vol(SO(V_2)(\mathbf{R})) = 1$ and $vol(SO(V_2)(\mathbf{Q}) \setminus SO(V_2)(\mathbf{A})) = 2$. Fix a Haar measure on \mathbf{A}_f^{\times} with the property that $vol(\mathbf{Z}_p^{\times}) = 1$ for each finite place p, and $vol(\mathbf{A}_f^{\times}/\mathbf{Q}^{\times}) = 1/2$. We obtain from these choices a Haar measure on $GSpin(V_2)(\mathbf{A}_f)$ via the short exact sequence

$$1 \longrightarrow \mathbf{A}_{f}^{\times} \longrightarrow \operatorname{GSpin}(V_{2})(\mathbf{A}_{f}) \longrightarrow \operatorname{SO}(V_{2})(\mathbf{A}_{f}) \to 1.$$

⁸Note that no assumption is made on the signature of the quadratic space (V, Q) underlying the Eisenstein series in [34, §4].

Lemma 4.13. Let $U \subset \operatorname{GSpin}(V)(\mathbf{A}_f)$ be any compact open subgroup, and $U_2 = U \cap \operatorname{GSpin}(V_2)(\mathbf{A}_f)$. Then,

$$\vartheta_{f_0}^{\star}(Z(V_2)) = \frac{1}{\operatorname{vol}(U_2)} \cdot \int_{\operatorname{SO}(V_2)(\mathbf{Q}) \setminus \operatorname{SO}(V_2)(\mathbf{A})} \vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h) dh.$$

Proof. Cf. [8, Lemma 4.5], we apply [41, Lemma 2.13] to the function $B(h) = \vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h)$. This result shows that for any function B(h) on $\operatorname{GSpin}(V_2)(\mathbf{A})$ which (i) depends only on the image of h in $\operatorname{SO}(V_2)(\mathbf{A}_f)$, (ii) is left $\operatorname{GSpin}(V_2)(\mathbf{Q})$ -invariant, and (iii) is right invariant under the compact open subgroup U_2 , we have that

$$\int_{\mathrm{SO}(V_2)(\mathbf{Q})\setminus\mathrm{SO}(V_2)(\mathbf{A})} B(h)dh = \mathrm{vol}(U_2) \sum_{h\in\mathrm{GSpin}(V_2)(\mathbf{Q})\setminus\mathrm{GSpin}(V_2)(\mathbf{A})/U_2} B(h)$$

Here, the sum on the right-hand side is finite. In this way, we compute the sum over the subset $Z(V_2)$ as

$$\vartheta_{f_0}^{\star}(Z(V_2)) = \frac{1}{\operatorname{vol}(U_2)} \cdot \int_{\operatorname{SO}(V_2)(\mathbf{Q}) \setminus \operatorname{SO}(V_2)(\mathbf{A})} \vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h) dh.$$

Fix an S_{Λ} -valued harmonic weak Maass form $f_0 \in H_{0,\Lambda}$, with decomposition $f_0 = f_0^+ + f_0^-$ into holomorphic part f_0^+ and non-holomorphic part f_0^- . We consider the even lattice $\Lambda \subset V$ with its corresponding S_{Λ} -valued Siegel theta series $\theta_{\Lambda}(\tau, z, h)$ defined on $z \in D_V^+$, $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, and $\tau = u + iv \in \mathfrak{H}$ by

$$\theta_{\Lambda}(\tau,z,h) = \theta_{\Lambda,r_{\psi_0}}(\tau,z,h) = \sum_{\mu \in \Lambda^{\#}/\Lambda} \theta^{\star}_{r_{\psi_0}}(z,h,g_{\tau};\mathbf{1}_{\mu}) \cdot \mathbf{1}_{\mu}.$$

Following [8, (3.3), Lemma 3.1], we argue that after replacing f_0 by its restriction $f_{0,\Lambda_1\oplus\Lambda_2}$, we may also replace the theta series $\theta_{\Lambda}(\tau, z, h)$ of the lattice Λ with the theta series $\theta_{\Lambda_1\oplus\Lambda_2}(\tau, z, h)$ of the finite-index sublattice $\Lambda_1 \oplus \Lambda_2 \subset \Lambda$. That is, we use the relation $(\theta_{\Lambda})^{\Lambda_1\oplus\Lambda_2} = \theta_{\Lambda_1+\Lambda_2}$ to derive the identity

$$\langle\langle f_0(\tau), \theta_{\Lambda}(\tau)\rangle\rangle = \langle\langle f_{0,\Lambda_1 \oplus \Lambda_2}(\tau), \theta_{\Lambda_1 \oplus \Lambda_2}(\tau)\rangle\rangle.$$

Let us henceforth write $f_0(\tau)$ to denote the restriction $f_{0,\Lambda_1\oplus\Lambda_2}$ of $f_0(\tau)$ to the finite-index sublattice $\Lambda_1\oplus\Lambda_2$ of Λ (see [8, Lemma 3.1]). We shall then work with the corresponding theta series $\theta_{\Lambda_1\oplus\Lambda_2}(\tau, z, h)$, which has the following convenient decomposition: For $(z_{V_2}^{\pm}, h) \in Z(V_2)$ and $\tau = u + iv \in \mathfrak{H}$,

(43)
$$\theta_{\Lambda}(z_{V_2}^{\pm},\tau) = \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}^{\pm}, h) = \theta_{\Lambda_1}(\tau, 1, 1) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}^{\pm}, h).$$

To proceed, we first give the following standard expression for the regularized theta lift

$$\vartheta_{f_0}^{\star}(z_{V_2}^{\pm},h) = \mathrm{CT}_{s=0} \left\{ \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda}(\tau, z_{V_2}^{\pm}, h) \rangle \rangle v^{-s} d\mu(\tau) \right\}$$

as a limit of truncated integrals.

Lemma 4.14. Let $\theta_{\Lambda_1}^+(\tau)$ denote the holomorphic part of the Siegel theta series $\theta_{\Lambda_1}(\tau)$. We have that

$$\vartheta_{f_0}^{\star}(z_{V_2}^{\pm},h) = \left[\lim_{T \to \infty} \int\limits_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}^{\pm},h) \rangle \rangle d\mu(\tau) - A_0 \log(T) \right],$$

where

$$A_0 = \operatorname{CT}\langle\langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathbf{1}_{0+\Lambda_2}\rangle\rangle$$

denotes the constant term in the Fourier series expansion of the modular form $\langle\langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathbf{1}_{0+\Lambda_2} \rangle\rangle$.

Proof. See [34, Proposition 2.5], [8, Lemma 4.5], and [16, Lemma 3.4]. We first split the regularized theta integral into two parts according to the decomposition $f_0 = f_0^+ + f_0^-$ to get

$$\vartheta_{f_0}^{\star}(z_{V_2}^{\pm},h) = \int_{\mathcal{F}}^{\star} \langle \langle f_0^{+}(\tau), \theta_{\Lambda}(\tau, z_{V_2}^{\pm},h) \rangle \rangle d\mu(\tau) + \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0^{-}(\tau), \theta_{\Lambda}(\tau, z_{V_2}^{\pm},h) \rangle \rangle d\mu(\tau).$$

Here, the second integral is absolutely convergent. We then decompose the first integral similarly according to the corresponding decomposition $\theta_{\Lambda} = \theta_{\Lambda}^{+} + \theta_{\Lambda}^{-}$ for the Siegel theta series θ_{Λ} to get

$$\int_{\mathcal{F}}^{\star} \langle \langle f_0^+(\tau), \theta_{\Lambda}(\tau, z_{V_2}^{\pm}, h) \rangle \rangle d\mu(\tau) = \int_{\mathcal{F}}^{\star} \langle \langle f_0^+(\tau), \theta_{\Lambda}^+(\tau, z_{V_2}^{\pm}, h) \rangle \rangle d\mu(\tau) + \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle \langle f_0^+(\tau), \theta_{\Lambda}^-(\tau, z_{V_2}^{\pm}, h) \rangle \rangle d\mu(\tau).$$

Here again, the second integral is absolutely convergent. To evaluate the remaining first integral, we use the decomposition of theta series described in (43) with the calculation⁹ of Kudla [34, Proposition 2.5] to find

$$\int_{\mathcal{F}}^{\star} \langle \langle f_0^+(\tau), \theta_{\Lambda}^+(\tau, z_{V_2}^{\pm}, h) \rangle \rangle d\mu(\tau) = \lim_{T \to \infty} \left| \int_{\mathcal{F}_T} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \theta_{\Lambda_2}^+(\tau, z_{V_2}^{\pm}, h) \rangle \rangle d\mu(\tau) - A_0 \log(T) \right|.$$

Putting together the three pieces (expressed as limits of truncated integrals), we derive the stated formula. \Box

Corollary 4.15. Using the Siegel-Weil formula of Theorem 4.8 and Corollary 4.9, we have that

$$\vartheta_{f_0}^{\star}(Z(V_2)) = \frac{2}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 0) \rangle \rangle d\mu(\tau) - \frac{1}{2} \cdot A_0 \log(T) \right].$$

Proof. We expand the definition using Lemma 4.13, Lemma 4.14 and the decomposition (43); we then switch the order of summation, and apply Corollary 4.9 (with $\kappa = 2$) to evaluate the inner integral over $\theta_{\Lambda_2}(z_{V_2}^{\pm}, h)$. In this way, we compute

$$\begin{split} \vartheta_{f_0}^{\star}(Z(V_2)) &= \frac{1}{\operatorname{vol}(U_2)} \cdot \int_{\operatorname{SO}(V_2)(\mathbf{Q}) \setminus \operatorname{SO}(V_2)(\mathbf{A})} \vartheta_{f_0}^{\star}(z_{V_2}^{\pm}, h) dh \\ &= \frac{1}{\operatorname{vol}(U_2)} \cdot \int_{\operatorname{SO}(V_2)(\mathbf{Q}) \setminus \operatorname{SO}(V_2)(\mathbf{A})} \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(z_{V_2}^{\pm}, h, \tau) \rangle \rangle d\mu(\tau) - A_0 \log(T) \right] dh \\ &= \frac{1}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \left(\int_{\operatorname{SO}(V_2)(\mathbf{Q}) \setminus \operatorname{SO}(V_2)(\mathbf{A}_f)} \theta_{\Lambda_2}(z_{V_2}^{\pm}, h, \tau) dh \right) \rangle \rangle d\mu(\tau) - A_0 \log(T) \right] \\ &= \frac{2}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[\int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 0) \rangle \rangle d\mu(\tau) - \frac{1}{2} \cdot A_0 \log(T) \right]. \end{split}$$

Given $g \in S_{2,\Lambda}$ a cuspidal holomorphic modular form of weight 2 and representation $r_{\psi,\Lambda}$, let us now consider the Rankin-Selberg *L*-function given by the integral presentation

$$L(s,g,V_2) := \langle g(\tau), \theta_{\Lambda_1}(\cdot) \otimes E_{\Lambda_2}(\tau,s;2) \rangle = \int_{\mathcal{F}} \langle \langle g(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau,s,2) \rangle \rangle v^2 d\mu(\tau).$$

We shall take $g = \xi_0(f_0)$, and write $L'(s, g, V) = \frac{d}{ds}L(s, g, V)$ to denote the derivative with respect to s. Recall that we write $\mathcal{E}_{\Lambda_2}(\tau)$ by the Fourier expansion (42), with coefficients defined in (41).

Theorem 4.16. Writing $\theta_{\Lambda_1}^+(\tau)$ to denote the holomorphic part of the Siegel theta series $\theta_{\Lambda_1}(\tau)$, and $\mathcal{E}_{\Lambda_2}(\tau) = E_{\Lambda_2}^+(\tau, 0; 2)$ the holomorphic part of the derivative Eisenstein series $E'_{\Lambda_2}(\tau, 0; 2)$, we obtain

$$\vartheta_{f_0}^{\star}(Z(V_2)) = -\frac{4}{\operatorname{vol}(U_2)} \cdot \left(\operatorname{CT}\langle\langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau)\rangle\rangle + L'(0, \xi_0(f_0), V_2)\right)$$

Proof. We derive a variation of [8, Theorem 4.7] and [16, Theorem 3.5] via Proposition 4.12 above. Here, Lemma 4.13, Lemma 4.14, and Corollary 4.15 imply that

$$\vartheta_{f_0}^{\star}(Z(V_2)) = \frac{2}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[I_T(f_0) - \frac{1}{2} \cdot A_0 \log(T) \right], \quad I_T(f_0) := \int_{\mathcal{F}_T} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 0) \rangle \rangle d\mu(\tau).$$

⁹Formally, we replace the weakly holomorphic form $f(\tau)$ in [34, Proposition 2.5] with $f_0^+(\tau) \otimes \theta_{\Lambda_1}^+(\tau)$.

Using the identity (38) for the Eisenstein series $E_{\Lambda_2}(\tau, s, 0)$ at s = 0, we find that

(45)
$$I_{T}(f_{0}) = \int_{\mathcal{F}_{T}} \langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E_{\Lambda_{2}}(\tau, 0; 0) \rangle \rangle d\mu(\tau) = -2 \int_{\mathcal{F}_{T}} \langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes \overline{\partial} E'_{\Lambda_{2}}(\tau, 0; 2) d\tau \rangle \rangle$$
$$= -2 \int_{\mathcal{F}_{T}} d\langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E'_{\Lambda_{2}}(\tau, 0; 2) d\tau \rangle \rangle + 2 \int_{\mathcal{F}_{T}} \langle \langle \overline{\partial} f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E'_{\Lambda_{2}}(\tau, 0; 2) d\tau \rangle \rangle.$$

To compute the first integral on the right-hand side of (45), we apply Stokes' theorem¹⁰ to find that (46)

$$-2\int_{\mathcal{F}_{T}} d\langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E'_{\Lambda_{2}}(\tau, 0; 2)d\tau \rangle \rangle = -2\int_{\partial \mathcal{F}_{T}} \langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E'_{\Lambda_{2}}(\tau, 0; 2)d\tau \rangle \rangle$$
$$= -2\int_{\tau=iT}^{iT+1} \langle \langle f_{0}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E'_{\Lambda_{2}}(\tau, 0; 2) \rangle \rangle d\tau = -2\int_{0}^{1} \langle \langle f_{0}(u+iT), \theta_{\Lambda_{1}}(u+iT) \otimes E'_{\Lambda_{2}}(u+iT, 0; 2) \rangle \rangle du.$$

To compute the second integral on the right-hand side of (45), we use the relation of differential forms

$$\overline{\partial}(f_0(\tau)d\tau) = -v^2 \overline{\xi_0(f_0)(\tau)} d\mu(\tau) = -L_0 f_0(\tau) d\mu(\tau)$$

implied by Lemma 4.11 to deduce that

(47)
$$2\int_{\mathcal{F}_T} \langle \langle \overline{\partial} f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) d\tau \rangle \rangle = -2 \int_{\mathcal{F}_T} \langle \langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle v^2 d\mu(\tau).$$

Hence, we obtain the identity

$$(48) I_T(f_0) = -2 \int_{t=iT}^{iT+1} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau - 2 \int_{\mathcal{F}_T} \langle \langle \overline{\xi_0(f_0)}, \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle v^2 d\mu(\tau)$$

Inserting this identity (48) back into the initial formula (44) then gives us the preliminary formula

(49)
$$\vartheta_{f_0}^{\star}(Z(V_2)) = -\frac{2}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[2 \int_{\tau=iT}^{iT+1} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau - \frac{1}{2} \cdot A_0 \log(T) \right] \\ - \frac{2}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} 2 \int_{\mathcal{F}_T} \langle \langle \overline{\xi_0(f_0)}, \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle v^2 d\mu(\tau).$$

We now argue as in [16, Theorem 3.5, (3.12), (3.11)] that we may replace the $f_0(\tau)$ in the first integral on the right of (48) with its holomorphic part $f_0^+(\tau)$, as the remaining non-holomorphic part $f_0^-(\tau)$ is rapidly decreasing as $v \to \infty$. That is, we first split the constant coefficient term in (49) into three parts as

(50)
$$\lim_{T \to \infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau$$
$$= \lim_{T \to \infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau$$
$$+ \lim_{T \to \infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0^-(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau$$
$$+ \lim_{T \to \infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0^-(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau.$$

¹⁰Note that this does not require a change of sign after identifying the boundary $\partial \mathcal{F}_T$ with the interval [iT, iT+1], and that there is a sign error in the first integral on the right-hand side of the second identity stated in [8, p. 655, proof of Theorem 4.7]. There is also a sign error in the second integral, c.f. [2, Theorem 5.7.1]. This latter error appears to come from the differential forms identity $\overline{\partial}(fd\tau) = -v^{l-2}\xi_k(f)d\mu(\tau) = -L_lfd\mu(\tau)$, cf. [16, Lemma 2.5], which is used implicitly without the sign change in the first identification of [8, p. 655].

Let us first consider the third integral on the right-hand side of (50), writing the Fourier series expansion as

$$\langle\langle f_0^-(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle\rangle = \sum_{n \in \mathbf{Z}} a(n, iv) e(n\tau)$$

Opening up this expansion in the corresponding integral, then using the orthogonality of additive characters on the torus $\mathbf{R}/\mathbf{Z} \cong [0,1]$ to evaluate, we find that

$$\begin{split} &\int_{\tau=iT}^{iT+1} \langle \langle f_0^-(\tau), \theta_{\Lambda_1}(\tau, 1, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau = \int_0^1 \langle \langle f_0^-(u+iT), \theta_{\Lambda_1}(u+iT, 1, 1) \otimes E'_{\Lambda_2}(u+iT, 0; 2) \rangle \rangle du \\ &= \sum_{n \in \mathbf{Z}} a(n, iT) e(inT) \int_0^1 e(nu) du = a(0, iT) = \sum_{\mu \in \Lambda^{\#}/\Lambda} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^-(-\mu, m) W_0(-2\pi m v) c_g(\mu, m, v). \end{split}$$

Here, we write $c_g(m, \mu, v)$ to denote the Fourier series coefficients of $g(\tau) = \theta_{\Lambda_1}(\tau, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2)$, i.e.

$$g(\tau) = \theta_{\Lambda_1}(\tau, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2) = \sum_{\mu \in (\Lambda_1 \oplus \Lambda_2)^{\#} / (\Lambda_1 \oplus \Lambda_2)} \sum_{m \in \mathbf{Q}} c_g(\mu, m, v) \mathbf{1}_{\mu} e(m\tau)$$

We can now use the rapid decay for the Whittaker coefficients $W_0(y) = \int_{-2y}^{\infty} e^{-t} dt = \Gamma(1, 2|y|)$ for $y \to -\infty$ in the Fourier series expansions of $f_0^-(\tau)$ with standard bounds for the Fourier coefficients of $f_0^-(\tau)$ and $g(\tau)$ to deduce that for some integer M > 0 and some constant C > 0, we have for each $m \ge M$ that

$$c_{f_0}^{-}(\mu, -m)W_0(-2\pi mv)c_g(\mu, m, v) = O\left(e^{-mCv}\right)$$

We deduce from this that for some constants c, C > 0, we have the upper bound

$$|a(0, iT)| \le c \cdot \frac{e^{-CT}}{(1 - e^{-CT})},$$

from which it follows that $\lim_{T\to\infty} |a(0,iT)| = 0$. Hence, the third integral on the right-hand side of (50) vanishes in the limit with $T \to 0$. A similar argument (cf. [16, 3.11]) shows that the second integral on the right-hand side of (50) vanishes,

$$\lim_{T \to \infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^-(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau = 0.$$

Hence, the first term on the right-hand side of (49) can be simplified to the expression

(51)
$$\frac{4}{\operatorname{vol}(U_2)} \cdot \lim_{T \to \infty} \left[\int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau - \frac{1}{4} \cdot A_0 \log(T) \right].$$

To evaluate this, we follow the approach of [8, Theorem 4.7] with the calculations (41) and (42) to find that (52)

$$\lim_{T \to \infty} \left[\int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau - A_0 \log(T) \right]$$

$$= \lim_{T \to \infty} \int_0^1 \langle \langle f_0^+(u+iT), \theta_{\Lambda_1}^+(u+iT) \otimes \sum_{\mu \in \Lambda_2^\#/\Lambda_2} \sum_{m \in \mathbf{Q}} \left(b_{\Lambda_2}(\mu, m, T) - \delta_{\mu,0} \delta_{m,0} \log(T) \right) e(m(u+iT)) \mathbf{1}_{\mu} \rangle \rangle du$$

$$= \lim_{T \to \infty} \int_0^1 \langle \langle f_0^+(u+iT), \theta_{\Lambda_1}^+(u+iT) \otimes \sum_{\mu \in \Lambda_2^\#/\Lambda_2} \sum_{m \in \mathbf{Q}} \kappa_{\Lambda_2}(\mu, m) e(m(u+iT)) \mathbf{1}_{\mu} \rangle \rangle du = \operatorname{CT} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle \rangle.$$

To use (52) to evaluate (51), we first pair off one of the integrals with $\lim_{T\to\infty} -A_0 \log(T)$, then argue that the contributions from the nonholomorphic part $E'_{\Lambda_2}(\tau, 0; 2)$ of the derivative Eisenstein series $E'_{\Lambda_2}(\tau, 0; 2)$

in each of the three remaining integrals vanishes (cf. [34, Proposition 2.11]). That is, we first evaluate

$$\begin{split} &\lim_{T\to\infty} \left[4 \int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau - A_0 \log(T) \right] \\ &= \operatorname{CT} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle \rangle + \lim_{T\to\infty} 3 \int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau \\ &= 4 \operatorname{CT} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle \rangle + 3 \lim_{T\to\infty} \int_{\tau=iT}^{iT+1} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2) \rangle \rangle d\tau. \end{split}$$

We then argue that the limit

$$3\lim_{T\to\infty}\int_{\tau=iT}^{iT+1}\langle\langle f_0^+(\tau),\theta_{\Lambda_1}^+(\tau)\otimes E_{\Lambda_2}'^-(\tau,0;2)\rangle\rangle d\tau = 3\lim_{T\to\infty}\int_0^1\langle\langle f_0^+(u+iT),\theta_{\Lambda_1}^+(u+iT)\otimes E_{\Lambda_2}'^-(u+iT,0;2)\rangle\rangle du$$

on the right-hand side vanishes. Indeed, opening up the Fourier series expansions and evaluating the unipotent integral via orthogonality of additive characters, we see that this limit has the Fourier series decomposition

$$\begin{split} &\lim_{T \to \infty} 3 \sum_{\mu \in (\Lambda_1 + \Lambda_2)^{\#} / (\Lambda_1 + \Lambda_2)} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^+(\mu, m) c_{\theta_{\Lambda_1}^+ \otimes E_{\Lambda_2}^{'-}}(-\mu, -m) W_2(-2\pi m T) \\ &= \lim_{T \to \infty} 3 \sum_{\mu \in (\Lambda_1 + \Lambda_2)^{\#} / (\Lambda_1 + \Lambda_2)} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^+(\mu, m) \sum_{\substack{\mu_1 \in \Lambda_1^{\#} / \Lambda_1 \\ \mu_2 \in \Lambda_2^{\#} / \Lambda_2 \\ \mu_1 + \mu_2 \equiv -\mu \mod (\Lambda_1 + \Lambda_2)}} \sum_{\substack{m_1 \in \mathbf{Q}_{\geq 0} \\ m_2 \in \mathbf{Q}_{< 0} \\ m_1 + m_2 = -m}} c_{\theta_{\Lambda_1}}^+(\mu_1, m_1) c_{E_{\Lambda_2}^{'}}^-(\mu_2, m_2) W_2(-2\pi m_2 T). \end{split}$$

We then use the rapid decay of the Whittaker function $W_2(y) = \int_{-2y}^{\infty} e^{-t} t^{-2} dt = \Gamma(-1, 2|y|)$ with $y \to -\infty$ to deduce that each inner sum tends to zero with $T \to \infty$. Hence, we find that (51) can be identified with $4 \operatorname{CT} \langle \langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle \rangle$. Substituting this identification back into (49), we then derive the formula

$$\vartheta_{f_0}^{\star}(Z(V_2)) = -\frac{4}{\operatorname{vol}(U_2)} \cdot \left(\operatorname{CT}\langle\langle f_0^+(\tau), \theta_{\Lambda_1}^+(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle\rangle + \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)}, \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 2) \rangle\rangle v^2 d\mu(\tau) \right)$$

Taking the limit with $T \to \infty$ gives the stated formula.

Taking the limit with $T \to \infty$ gives the stated formula.

4.9. Application to the central derivative value $\Lambda'(1/2, \Pi \otimes \chi)$. Recall that we write $\eta = \otimes_v \eta_v$ to denote the idele class character of Q associated to the quadratic extension K/Q, which we can and do identify with its corresponding Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$. Recall as well that $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ denotes the quadratic basechange of the cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $GL_2(\mathbf{A})$ corresponding to our elliptic curve E/\mathbf{Q} to $\operatorname{GL}_2(\mathbf{A}_K)$. As a consequence of the theory of cyclic basechange, we then have an equivalence of the $\operatorname{GL}_2(\mathbf{A}_K) \times \operatorname{GL}_1(\mathbf{A}_K)$ -automorphic L-function $\Lambda(s, \Pi \otimes \chi)$ with the $\operatorname{GL}_2(\mathbf{A}) \times \operatorname{GL}_2(\mathbf{A})$ Rankin-Selberg L-function $\Lambda(s, \pi \times \pi(\chi))$. Let us now consider the following classical integral representations of the Rankin-Selberg *L*-functions relevant to the discussion above.

To describe this setup in classical terms, recall that we consider the cuspidal newform of weight 2 associated to the elliptic curve E/\mathbf{Q} , with Fourier series expansion

$$f(\tau) = f_E(\tau) = \sum_{m \ge 1} c_f(m) e(m\tau) = \sum_{m \ge 1} a_f(n) n^{\frac{1}{2}} e(n\tau) \in S_2^{\text{new}}(\Gamma_0(N)), \quad \tau = u + iv \in \mathfrak{H}$$

Hence, the finite part L(s, f) of the standard L-function $\Lambda(s, f) = \Lambda(s, \pi) = L(s, \pi_{\infty})L(s, \pi)$ has the Dirichlet series expansion $L(s, f) = \sum_{m \ge 1} a_f(n) n^{-s} = \sum_{m \ge 1} c_f(n) n^{-(s+1/2)}$ (first for $\Re(s) > 1$). Recall that we fix a ring class character χ of some conductor $c \in \mathbf{Z}_{\ge 1}$ of K. Hence, $\chi = \bigotimes_x \chi_w$ is a character of the class group

$$\operatorname{Pic}(\mathcal{O}_c) = \mathbf{A}_K^{\times} / \mathbf{A}^{\times} K_{\infty}^{\times} K^{\times} \widehat{\mathcal{O}}_c^{\times}, \quad \widehat{\mathcal{O}}_c^{\times} = \prod_{w < \infty} \mathcal{O}_{c, u}^{\times}$$

of the **Z**-order $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K. We consider the corresponding Hecke theta series defined by the twisted linear combination (see e.g. [24, (5.4)])

(53)
$$\theta(\chi)(\tau) = \sum_{A \in \operatorname{Pic}(\mathcal{O}_c)} \chi(A) \theta_A(\tau),$$

where each of the partial theta series $\theta_A(\tau)$ can be defined classically as follows. Let $w_K = \mu(K)/2$ denote half the number of roots of unity in k. Since the unit group $\mathcal{O}_K^{\times} \cong \mathbf{Z} \times \mu(K) = \langle \varepsilon_K \rangle \times \mu(K)$ is not torsion by Dirichlet's unit theorem, we fix a fundamental domain $\mathfrak{a}^* = [1, z_\mathfrak{a}]^*$ for the action of $\mathcal{O}_K^{\times}/\mu(K) = \langle \varepsilon_K \rangle$ on \mathfrak{a} . The corresponding theta series can then be described more explicitly via the expansion

$$\theta_A(\tau) = \frac{1}{w_K} \sum_{\lambda \in \mathfrak{a}^\star} e\left(\frac{\mathbf{N}_{K/\mathbf{Q}}(\lambda)}{\mathbf{N}\mathfrak{a}} \cdot \tau\right) = \sum_{m \ge 0} r_A(m) e(m\tau),$$

where $r_A(m)$ denotes the corresponding counting function

$$r_A(m) = \frac{1}{w_K} \cdot \# \left\{ \lambda \in \mathfrak{a}^* = [1, z_\mathfrak{a}]^* : \frac{\mathbf{N}_{K/\mathbf{Q}}(\lambda)}{\mathbf{N}\mathfrak{a}} = m \right\}.$$

A classical theorem of Hecke shows that each $\theta(\chi)(\tau)$ is a modular form of weight zero, level $\Gamma_0(d_K)$ and character $\eta = \eta_K$. We consider the corresponding Rankin-Selberg presentation

$$\Lambda(s,\pi\times\pi(\chi))=\Lambda(s,f\times\theta(\chi))=\sum_{A\in\operatorname{Pic}(\mathcal{O}_c)}\chi(A)\Lambda(s,f\times\theta_A),$$

given as a twisted linear combination of the partial Rankin-Selberg L-functions (cf. e.g. [24, § IV (0.1)])¹¹

(54)

$$\begin{aligned} \Lambda(s, f \times \theta_A) &:= \langle f, \theta_A E^{\star}(\cdot, s; 2) \rangle = \frac{\Gamma(s)}{(4\pi)^s} \cdot \Lambda(2s, \eta) \cdot \sum_{m \ge 1} \frac{c_f(m) r_A(m)}{m^s} \\ &= \frac{\Gamma(s)}{(4\pi)^s} \cdot \Lambda(2s, \eta) \cdot \frac{1}{w_K} \sum_{\substack{\lambda \in \mathfrak{a}^{\star} \\ [\mathfrak{a}] = A \in \operatorname{Pic}(\mathcal{O}_c)}} \frac{c_f(\mathbf{N}(\lambda))}{\mathbf{N}(\lambda)^s} \qquad (\Re(s) > 1)
\end{aligned}$$

associated to each class $A \in \text{Pic}(\mathcal{O}_c)$. We also consider the quadratic twist $f \otimes \eta = f_E \otimes \eta_{K/\mathbb{Q}}$ given by

$$(f \otimes \eta)(\tau) = \sum_{m \ge 1} c_f(m)\eta(m)e(m\tau) = \sum_{m \ge 1} a_f(m)m^{\frac{1}{2}}\eta(m)e(m\tau) \in S_2^{\text{new}}(\Gamma_0(d_K^2N),\eta),$$

along with its corresponding Rankin-Selberg L-function

$$\Lambda(s,(\pi\otimes\eta)\times\pi(\chi))=\Lambda(s,(f\otimes\eta)\times\theta(\chi))=\sum_{A\in\operatorname{Pic}(\mathcal{O}_c)}\chi(A)\Lambda(s,f\otimes\eta\times\theta_A),$$

where each partial L-series $\Lambda(s, (f \otimes \eta) \times \theta_A)$ is given by the expansion

$$\begin{split} \Lambda(s,(f\otimes\eta)\times\theta_A) &:= \langle f\otimes\eta, \theta_A E^{\star}(\cdot,s;2) \rangle = \frac{\Gamma(s)}{(4\pi)^s} \cdot \Lambda(2s,\eta^2) \cdot \sum_{m\geq 1} \frac{c_f(m)\eta(m)r_A(m)}{m^s} \\ &= \frac{\Gamma(s)}{(4\pi)^s} \cdot \Lambda(2s) \cdot \frac{1}{w_K} \sum_{\substack{\lambda \in \mathfrak{a} \\ [\mathfrak{a}] = A \in \operatorname{Pic}(\mathcal{O}_c)}} \frac{c_f(\mathbf{N}(\lambda))\eta(\mathbf{N}(\lambda))}{\mathbf{N}(\lambda)s} \quad (\Re(s) > 1). \end{split}$$

Lemma 4.17. We have the equivalent Rankin-Selberg integral presentations

$$\Lambda(s,\pi\times\pi(\chi))=\Lambda(s,f\times\theta(\chi))=\Lambda(s,(f\otimes\eta)\times\theta(\chi))=\Lambda(s,(\pi\otimes\eta)\times\pi(\chi))$$

for the basechange L-function $\Lambda(s, \Pi \otimes \chi) = \Lambda(s, BC_{K/\mathbf{Q}} \otimes \chi)$, for χ any ring class character of K.

Proof. Consider the basechange $\Pi' = \operatorname{BC}_{K/\mathbf{Q}}(\pi \otimes \chi)$ of the cuspidal automorphic representation $\pi \otimes \eta$ of $\operatorname{GL}_2(\mathbf{A})$ to $\operatorname{GL}_2(\mathbf{A}_K)$ generated by $f \otimes \eta$, whose corresponding standard *L*-function $\Lambda(s, \Pi')$ decomposes as $\Lambda(s, \Pi') = \Lambda(s, \pi \otimes \eta)\Lambda(s, \pi \otimes \eta^2) = \Lambda(s, \pi \otimes \eta)\Lambda(s, \pi)$. Here again, we use that the quadratic Dirichlet character $\eta(\cdot) = \left(\frac{d_K}{\cdot}\right)$ has order 2 to deduce that $\Lambda(s, \Pi') = \Lambda(s, \pi \otimes \eta)\Lambda(s, \pi) = \Lambda(s, \Pi)$. Hence, we deduce that the equivalent $\operatorname{GL}_2(\mathbf{A}_K) \times \operatorname{GL}_1(\mathbf{A}_K)$ -automorphic *L*-functions $\Lambda(s, \Pi \otimes \chi) = \Lambda(s, \Pi' \otimes \chi)$ have the same $\operatorname{GL}_2(\mathbf{A}) \times \operatorname{GL}_2(\mathbf{A})$ Rankin-Selberg presentation $\Lambda(s, \pi \times \pi(\chi)) = \Lambda(s, (\pi \otimes \eta) \times \pi(\chi))$, equivalently that $\Lambda(s, f \times \theta(\chi)) = \Lambda(s, f \otimes \eta \times \theta(\chi))$.

¹¹Observe that since $\theta_A(\tau)$ has weight zero, the arithmetic normalization of the Rankin-Selberg *L*-function $L(2s,\eta)\sum_{m\geq 1}c_f(m)c_{\theta_A}(m)m^{-\left(s+\frac{2+0}{2}-1\right)} = L(2s,\eta)\sum_{m\geq 1}c_f(m)c_{\theta_A}(m)m^{-s} = L(2s,\eta)\sum_{m\geq 1}c_f(m)r_A(m)m^{-s}$ coincides with the unitary normalization $L(2s,\eta)\sum_{m\geq 1}a_f(m)a_{\theta_A}(m)m^{-s} = L(2s,\eta)\sum_{m\geq 1}c_f(m)m^{-\frac{1}{2}}c_{\theta_A}(m)m^{\frac{1}{2}}m^{-s}$.

Recall that Theorem 4.6 gives us a relation between the scalar-valued form $f_{\eta} := f \otimes \eta$ and its vector-valued avatar g_{η} . Let us for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ fix an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ with **Z**-basis $[1, z_{\mathfrak{a}}]$. We again consider for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ the corresponding quadratic space (V_A, Q_A) described in Definition 3.1, with vector space $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathfrak{a}_{\mathbf{Q}}$, and quaratic form $Q_A(z) = Q_A((z_1, z_2)) = Q_{\mathfrak{a}}(z_1) - Q_{\mathfrak{a}}(z_2)$. As well, we consider the anisotropic subspaces $(V_{A,j}, Q_{A,j})$ of type (1,1) defined by $V_{A,1} = \mathfrak{a}_{\mathbf{Q}}$ with $Q_{A,1} = -Q_{\mathfrak{a}}$ and $V_{A,2} = \mathfrak{a}_{\mathbf{Q}}$ with $Q_{A,2} = Q_{\mathfrak{a}}$. Recall we write $\Lambda_A \subset V_A$ for the lattice determined by the compact open subgroup $U_A \subset \operatorname{GSpin}(V_A)(\mathbf{A}_f)$ described in (9) via (8). We write $\Lambda_{A,j} := \Lambda_A \cap V_{A,j}$ for each of j = 1, 2 to denote the signature (1,1) sublattice determined by restriction to $V_{A,j}$. By Theorem 4.6, we can associate to the quadratic twist $f \otimes \eta \in S_2^{\operatorname{new}}(\Gamma_0(d_K^2N), \eta)$ an \mathcal{S}_{Λ_A} -valued modular form g_{η} of weight 2. Recall as well that we consider the (incomplete, partial) Rankin-Selberg *L*-functions given by the Petersson inner products

$$L(s, g_{\eta}, V_{A,2}) := \langle g_{\eta}(\cdot), \theta_{\Lambda_{A,1}}(\cdot) \otimes E_{\Lambda_{A,2}}(\cdot, s; 2) \rangle = \langle g_{\eta}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E_{\Lambda_{A,2}}(\tau, s; 2) \rangle.$$

We also consider the completed version, given with respect to the completed Eisenstein series $E^{\star}_{\Lambda_2}(\tau,s;2)$:

$$L^{\star}(s, g_{\eta}, V_{A,2}) := \langle g_{\eta}(\cdot), \theta_{\Lambda_{A,1}}(\cdot) \otimes E^{\star}_{\Lambda_{A,2}}(\cdot, s; 2) \rangle = \langle g_{\eta}(\tau), \theta_{\Lambda_{1}}(\tau) \otimes E^{\star}_{\Lambda_{A,2}}(\tau, s; 2) \rangle.$$

Corollary 4.18. We have in the setup described the equivalent presentations

$$\Lambda(s-1/2,\Pi\otimes\chi) = \sum_{A\in\operatorname{Pic}(\mathcal{O}_c)}\chi(A)\Lambda(s-1/2,f\otimes\eta\times\theta_A) = \frac{1}{2}\cdot\sum_{A\in\operatorname{Pic}(\mathcal{O}_c)}\chi(A)L^{\star}(2s-2,g_{\eta},V_{A,2}).$$

In particular, we have that

$$\Lambda'(1/2,\Pi\otimes\chi) = \sum_{A\in\operatorname{Pic}(\mathcal{O}_c)} \chi(A)\Lambda'(1/2,f\otimes\eta\times\theta_A) = \frac{1}{2} \cdot \sum_{A\in\operatorname{Pic}(\mathcal{O}_c)} \chi(A)L^{\star\prime}(0,g_{\eta},V_{A,2}).$$

Proof. In the same way as for [8, §4, (4.24)] (with Fourier coefficient notations as described above), each partial Rankin-Selberg product $L(s, g_{\eta}, V_{A,2})$ has the Dirichlet series expansion

$$L(s, g_{\eta}, V_{A,2}) = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in \Lambda_{A,1}^{\#}/\Lambda_{A,1}} \sum_{m \in \mathbf{Q}_{>0}} \frac{c_{g_{\eta}}(\mu, m)c_{\theta_{\Lambda_{A,1}}}^{+}(\mu, m)}{m^{\frac{s+2}{2}}}$$
$$= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in \Lambda_{A,1}^{\#}/\Lambda_{A,1}} \sum_{m \in \mathbf{Q}_{>0}} \frac{c_{g_{\eta}}(\mu, m)c_{\Lambda_{A,1}}^{+}(\mu, m)}{m^{\frac{s+2}{2}}},$$

where $r_{\Lambda_{A,1}}(\mu, m)$ denotes the counting function

$$r_{\Lambda_{A,1}}(\mu,m) = \frac{1}{w_K} \cdot \# \left\{ \lambda \in \mu + \Lambda_{A,1} : Q_{A,1}(\lambda) = m \right\} / \langle \varepsilon_K \rangle$$

Here again, we fix a fundamental domain for the action of the fundamental unit $\langle \varepsilon_K \rangle \cong \mathcal{O}_K^{\times}/\mu(K)$. Now, since the lattice $\Lambda_{A,1}$ will form a **Z**-basis for the ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ of $A = [\mathfrak{a}]$, we see that $Q_{A,1}(x, y)$ is a binary quadratic form representative. Hence, $r_{\Lambda_{A,1}}(\mu, m)$ counts the number of ideals in $\mu + \mathfrak{a}^*$ of norm m. It then follows as a relatively formal consequence that we can identify the partial Rankin-Selberg *L*-function $L(s, g_\eta, V_{A,2})$ with the classical partial Rankin-Selberg *L*-function $L(s, f_\eta \times \theta_A)$, as we can expand

$$L(s, g_{\eta}, V_{A,2}) = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \sum_{\mu \in \Lambda_{A,1}^{\#}/\Lambda_{A,1}} \sum_{m \in \mathbf{Q}_{>0}} \frac{c_{g_{\eta}}(\mu, m)c_{\theta_{\Lambda_{A,1}}}^{+}(\mu, m)}{m^{\frac{s+2}{2}}}$$
$$= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \cdot \frac{1}{w_{K}} \sum_{\mu \in \Lambda_{A,1}^{\#}/\Lambda_{A,1}} \sum_{\lambda \in \mu + \mathfrak{a}^{\star}} \frac{c_{g_{\eta}}(\mu, Q_{A,1}(\lambda))}{Q_{A,1}(\lambda)^{\frac{s+2}{2}}}$$
$$= \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \cdot \frac{2}{w_{K}} \sum_{\lambda \in \mathfrak{a}^{\star}} \frac{c_{f_{\eta}}(\mathbf{N}(\lambda))}{\mathbf{N}(\gamma)^{\frac{s+2}{2}}} = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}} \cdot \frac{2}{w_{K}} \sum_{\lambda \in \mathfrak{a}^{\star}} \frac{c_{f_{\eta}}(\mathbf{N}(\lambda))}{\mathbf{N}(\gamma)^{\frac{s+2}{2}}}$$

Here, we use the relation of coefficients described in Theorem 4.6 and that the Dirichlet series expansion is taken over rational integers $m \ge 1$ coprime to $d_K N$. We then deduce that we have for each class $A \in \text{Pic}(\mathcal{O}_c)$

the relation $L^*(2s - 1, g_\eta, V_{A,2}) = 2\Lambda(s, f_\eta \times \theta_A)$ (cf. [24, § IV (0.1), p. 271]). The stated relations follow, with the analytic continuation and functional equations determined by the underlying Eisenstein series. \Box

Theorem 4.19 (Twisted linear combinations of regularized theta integrals). Let us retain the setup above, with $f = f_E \in S_2^{new}(\Gamma_0(N))$ the cuspidal eigenform parametrizing our elliptic curve E/\mathbf{Q} , π the corresponding cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A})$, and $\Pi = \operatorname{BC}_{K/\mathbf{Q}}(\pi)$ its quadratic basechange lifting to a cuspidal automorphic representation of $\operatorname{GL}_2(\mathbf{A}_K)$. Let χ be any ring class character of the real quadratic field K of conductor c coprime to $d_K N$. Let $f_{0,\eta,A} \in H_{0,\rho_{\Lambda_A}}$ for each class $A \in \operatorname{Pic}(\mathcal{O}_c)$ denote the harmonic weak Maass form of weight zero with image $\xi_0(f_{0,\eta,A}) = g_{\eta,A} \in S_{2,\overline{\rho}_{\Lambda_A}}$ where $g_{\eta,A}$ denotes the lifting our our quadratic twist $f \otimes \eta \in S_2^{new}(\Gamma_0(d_K^2N), \eta)$ the space vector-valued forms $S_{2,\overline{\rho}_{\Lambda_A}}$ as described in Theorem 4.6 above. Then, we have the formula

$$\frac{\Lambda'(1/2,\Pi\otimes\chi)}{L(1,\eta)} = -\frac{1}{2} \cdot \sum_{A\in\operatorname{Pic}(\mathcal{O}_c)} \chi(A) \left(\operatorname{CT}\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle + \frac{\operatorname{vol}(U_{A,2})}{4} \cdot \vartheta_{f_{0,\eta,A}}^\star(Z(V_{A,2}))\right).$$

Here, for each class $A \in \text{Pic}(\mathcal{O}_c)$, we write $U_{A,2} := U \cap \text{GSpin}_{V_{A,2}}(\mathbf{A}_f)$ as in Lemma 4.13 above.

Proof. Formally, this is a consequence of Lemma 4.17 and Corollary 4.18 after applying Theorem 4.16 to each of the partial Rankin-Selberg *L*-series $L(s, g_{\eta}, V_{A,2}) = L(s, \xi_0(f_{0,\eta,A}), V_{A,2})$, which together imply that

$$\sum_{A \in \operatorname{Pic}(\mathcal{O}_c)} \chi(A) \cdot \frac{\operatorname{vol}(U_{A,2})}{4} \cdot \vartheta_{f_{0,\eta,A}}^{\star}(Z(V_{2,A}))$$

= $-\sum_{A \in \operatorname{Pic}(\mathcal{O}_c)} \chi(A) \cdot \left(\operatorname{CT}\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}}^+(\tau) \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle + L'(0, \xi_0(f_{0,\eta,A}), V_{A,2}) \right).$

It is then easy to identify the second term in this latter expression in terms of the central derivative value $L'(1/2, \Pi \otimes \chi)$ via Corollary 4.18. Let us thus consider the first term, which according to the expansions implied by Theorem 4.6 and the discussions in [8, §§ 4-5] can be evaluated as

$$\sum_{A \in \operatorname{Pic}(\mathcal{O}_{c})} \chi(A) \operatorname{CT} \langle \langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}^{+}(\tau) \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \rangle$$

$$= \sum_{A \in \operatorname{Pic}(\mathcal{O}_{c})} \chi(A) \operatorname{CT} \left(\sum_{\substack{\mu_{1} \in \Lambda_{A,1}^{\#}/\Lambda_{A,1} \\ \mu_{2} \in \Lambda_{A,2}^{\#}/\Lambda_{A,2} \\ \mu_{1}+\mu_{2} \equiv \mu \mod \Lambda_{A}}} f_{0,A,\mu}^{+}(\tau) \theta_{\Lambda_{A,1},\mu_{1}}^{+}(\tau) \otimes \mathcal{E}_{\Lambda_{A,2},\mu_{2}}(\tau) \right)$$

$$= \sum_{A \in \operatorname{Pic}(\mathcal{O}_{c})} \chi(A) \left(\sum_{\substack{\mu_{1} \in \Lambda_{A,1}^{\#}/\Lambda_{A,1} \\ \mu_{2} \in \Lambda_{A,2}^{\#}/\Lambda_{A,2} \\ \mu_{1}+\mu_{2} \equiv \mu \mod \Lambda_{A}}} \sum_{\substack{m,m_{2} \in \mathbf{Q}_{\geq 0}, m_{1} \in \mathbf{Q} \\ m_{1}+m_{2}=m}} c_{f_{0,\eta,A}}^{+}(-m,\mu) c_{\theta_{\Lambda_{A,1}}}^{+}(m_{1},\mu_{1}) \kappa_{\Lambda_{A,2}}(m_{2},\mu_{2})} \right).$$

Note that the analogous constant term for the CM setting is the subject of [8, Conjectures 5.1 and 5.2], and that this has now been improved in important special cases by [2, Theorem A]. \Box

Now, recall that the Dirichlet analytic class number formula gives us the following classical arithmetic description of the value $L(1,\eta)$. Writing d_K again to denote the fundamental discriminant associated to $K = \mathbf{Q}(\sqrt{d})$, let $h_K = \# \operatorname{Pic}(\mathcal{O}_K)$ denote the class number, and $\epsilon_K = \frac{1}{2}(t + u\sqrt{d_K})$ for the smallest solution t, u > 0 (with u minimal) to Pell's equation $t^2 - d_K u^2 = 4$. We can then express the formula derived above for the central derivative value $L'(1/2, \Pi \otimes \chi)$ in terms of Dirichlet's analytic class number formula

(56)
$$L(1,\eta) = \frac{\log \epsilon_K \cdot h_K}{\sqrt{d_K}}.$$

Corollary 4.20. We have that

$$\begin{split} \Lambda'(1/2,\Pi\otimes\chi) &= \Lambda'(1/2,\pi\times\pi(\chi)) = \Lambda'(1/2,f\times\theta(\chi)) = \Lambda'(E/K,\chi,1) \\ &= -\frac{\sqrt{d_K}}{\log\epsilon_K\cdot h_K} \cdot \frac{1}{2} \sum_{A\in\operatorname{Pic}(\mathcal{O}_c)} \chi(A) \left(\operatorname{CT}\langle\langle f^+_{0,\eta,A}(\tau), \theta^+_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle + \frac{\operatorname{vol}(U_{A,2})}{4} \cdot \vartheta^*_{f_{0,\eta,A}}(Z(V_{A,2}))\right). \end{split}$$

Moreover, if we assume Hypothesis 2.1 that the inert level N^+ is the squarefree product of an odd number of primes, then this central derivative value is not forced by the functional equation (6) to vanish identically.

Proof. This simply restates Theorem 4.19 in terms of the Dirichlet analytic class number formula (56). \Box

5. Relation to the conjecture of Birch and Swinnerton-Dyer

Let us now consider Theorem 4.19 from the point of view of the refined conjecture of Birch and Swinnerton-Dyer, comparing with the Gross-Zagier formula [24]. To date, there is no known or conjectural construction of points on the corresponding elliptic curve E(K[c]) or modular curve $X_0(N)(K[c])$ analogous to Heegner points¹², where K[c] denotes the ring class extension of conductor c of the real quadratic field K. We can consider the implications for arithmetic terms in the refined Birch and Swinnerton-Dyer formula for $L^{\star'}(E/K, \chi, 1)$ here, in the style of the comparison given in Popa [39, §6.4]. Taking for granted the refined conjecture of Birch and Swinnerton-Dyer for E(K[c])) in this setting – particularly for the case of rank one corresponding to Hypothesis 2.1 – we shall then derive "automorphic" interpretations of the corresponding Tate-Shafarevich group III(E/K[c]) and regulator Reg(E/K[c]). We also derive an unconditional result in special cases to illustrate surprising connections here.

Again, we fix χ a primitive ring class character of some conductor $c \geq 1$ prime to $d_K N$, and view this as a character of the class group $\operatorname{Pic}(\mathcal{O}_c)$. Recall that the reciprocity map of class field theory gives us an isomorphism $\operatorname{Pic}(\mathcal{O}_c) := \mathbf{A}_K^{\times}/\mathbf{A}^{\times}K_{\infty}^{\times}K^{\times}\widehat{\mathcal{O}}_c^{\times} \longrightarrow \operatorname{Gal}(K[c]/K)$, where K[c] is (by definition) the ring class extension of conductor c of K. Recall as well that by the theory of cyclic basechange of [38] and more generally [3] with Artin formalism, we can write the completed Hasse-Weil *L*-function $\Lambda(E/K[c], s)$ of Ebasechanged to K[c]/K as the product

(57)

$$\begin{aligned}
\Lambda(E/K[c],s) &= \prod_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \Lambda(E/K,\chi,s) \\
&= \prod_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \Lambda(s-1/2,\Pi \otimes \chi) \\
&= \prod_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \Lambda(s-1/2,\operatorname{BC}_{K/\mathbf{Q}}(\pi) \otimes \chi) \\
&= \prod_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \Lambda(s-1/2,\pi \times \pi(\chi)) \\
&= \prod_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \Lambda(s-1/2,f \times \theta(\chi)).
\end{aligned}$$

Here, we use all of the same conventions and definitions as established above with $\Pi = \text{BC}_{K/\mathbf{Q}}(\pi(f))$. Writing $\text{ord}_{s=s_0}$ as usual to denote the order of vanishing at a given $s_0 \in \mathbf{C}$, it then follows as a formal consequence of (57) that we have the relation(s)

(58)
$$\operatorname{ord}_{s=1} \Lambda(E/K[c], s) = \sum_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \operatorname{ord}_{s=1/2} \Lambda(s, \Pi \otimes \chi),$$

so that the conjecture of Birch and Swinnerton-Dyer predicts the rank equivalence

(59)
$$\operatorname{rk}_{\mathbf{Z}} E(K[c]) = \operatorname{ord}_{s=1} \Lambda(E/K[c], s) = \sum_{\chi \in \operatorname{Pic}(\mathcal{O}_c)^{\vee} \cong \operatorname{Gal}(K[c]/K)^{\vee}} \operatorname{ord}_{s=1/2} \Lambda(s, \Pi \otimes \chi).$$

 $^{^{12}}$ There is however a *p*-adic construction due to Darmon [14].

Let us now assume Hypothesis 2.1, so that for each ring class character χ on the right hand side of (59), we know by the symmetric functional equation (6) that $\operatorname{ord}_{s=1/2} \Lambda(s, \Pi \otimes \chi) \geq 1$. Let us also assume for the moment that the rank equality predicted by the conjecture of Birch and Swinnerton-Dyer holds, so that

(60)
$$\operatorname{rk}_{\mathbf{Z}} E(K[c]) = h(\mathcal{O}_c) := \#\operatorname{Pic}(\mathcal{O}_c) = \#\operatorname{Gal}(K[c]/K).$$

Let $r_E(K[c])$ denote the Mordell-Weil rank of E over the ring class extension K[c] of conductor c over K. The refined conjecture of Birch and Swinnerton-Dyer predicts that the leading term in the Taylor series expansion around $\Lambda^{(r_E(K[c]))}(E/K[c],s)/(r_E(K[c]))!$ around s = 1 is given by the following formula. Let $\operatorname{III}_E(K[c])$ denote the Tate-Shafarevich group of E over K[c],

$$\operatorname{III}_{E}(K[c]) = \ker \left(H^{1}(K, E) \longrightarrow \prod_{w} H^{1}(K_{w}, E) \right),$$

which we shall assume is known to be finite. Let $R_E(K[c])$ denote the regulator of E over K[c]. Hence, fixing a basis $(e_j)_{j=1}^{r_E(K[c])}$ of $E(K[c])/E(K[c])_{\text{tors}}$, and writing $[\cdot, \cdot]$ to denote the Néron-Tate height pairing,

$$R_E(K[c]) = \det\left(\left[e_i, e_j\right]\right)_{i,j}$$

Let us also write $T_E(K[c])$ to denote the product over local Tamagawa factors, so

$$T_E(K[c]) = \prod_{\substack{\nu < \infty \\ \text{primes of } \mathcal{O}_{K[c]}}} \left[E(K[c]_{\nu}) : E_0(K[c]_{\nu}) \right] \cdot \left| \frac{\omega}{\omega_{\nu}^*} \right|_{\nu},$$

where $\omega = \omega_E$ is a fixed invariant differential for E/K[c], and each ω_{ν}^* the Néron differential at ν . The refined conjecture of Birch and Swinnerton-Dyer then predicts that the leading term in the Taylor series expansion around s = 1 of $\Lambda^{(r_E(K[c]))}(E/K[c], s)/(r_E(K[c]))!$ around s = 1 is given by the formula

(61)
$$\frac{\#\operatorname{III}_{E}(K[c]) \cdot R_{E}(K[c]) \cdot T_{E}(K[c])}{\sqrt{d_{K}} \cdot \#E(K[c])_{\operatorname{tors}}^{2}} \cdot \prod_{\substack{\mu \mid \infty \\ \mu:K[c] \to \mathbf{R} \\ \operatorname{real places}}} \int_{E(K[c]_{\mu})} |\omega| \cdot \prod_{\substack{\sigma \mid \infty \\ \sigma, \sigma: K[c] \to \mathbf{C} \\ \operatorname{pairs of complex places}}} 2 \int_{E(K[c]_{\sigma})} \omega \wedge \overline{\omega}.$$

Let us first assume for simplicity that the class number is one: $h(\mathcal{O}_c) = h_K = 1$. Then, assuming the conjecture of Birch and Swinnerton-Dyer (60) and (61), we derive via Theorem 4.19 and Corollary 4.5 the (conditional) identifications

$$\begin{split} \Lambda'(E/K,1) &= \Lambda'(1/2,\Pi) = \Lambda'(1/2,\Pi) = \frac{\# \amalg_E(K) \cdot R_E(K) \cdot T_E(K)}{\sqrt{d_K} \cdot \# E(K)_{\text{tors}}^2} \cdot \prod_{\mu \mid \infty \atop \mu: K \to \mathbf{R}} \int_{E(K_{\mu})} |\omega| \\ &= -\frac{\sqrt{d_K}}{\log \epsilon_K} \cdot \frac{1}{2} \left(\text{CT} \left(\langle \langle f_{0,\eta,\mathcal{O}_K}^+(\tau), \theta_{\Lambda_{\mathcal{O}_K,1}}^+ \otimes \mathcal{E}_{\Lambda_{\mathcal{O}_K,2}}(\tau) \rangle \rangle \right) + \frac{\text{vol}(U_{\mathcal{O}_K,2})}{4} \sum_{(z_{V_{\mathcal{O}_K,2}}^\pm, h) \in Z(V_{\mathcal{O}_K,2})} \vartheta_{f_{0,\eta,\mathcal{O}_K}}^*(z_{V_{\mathcal{O}_K,2}}^\pm, h) \right) \end{split}$$

This suggests that the regulator $R_E(K) = [e_{??}, e_{??}]$ should be given by the formula (62)

$$R_{E}(K) = [e_{??}, e_{??}] = -\frac{\#E(K)^{2}_{\text{tors}} \cdot d_{K} \left(\operatorname{CT} \left(\langle \langle f^{+}_{0,\eta,\mathcal{O}_{K}}(\tau), \theta^{+}_{\Lambda_{\mathcal{O}_{K},1}} \otimes \mathcal{E}_{\Lambda_{\mathcal{O}_{K},2}}(\tau) \rangle \rangle \right) + \frac{\operatorname{vol}(U_{\mathcal{O}_{K},2})}{4} \sum_{\substack{(z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \in Z(V_{\mathcal{O}_{K},2}) \\ (z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \in Z(V_{\mathcal{O}_{K},2})}} \vartheta^{\star}_{f_{0,\eta,\mathcal{O}_{K}}}(z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \right)}{2 \log \epsilon_{K} \cdot \# \operatorname{III}_{E}(K) \cdot T_{E}(K) \cdot \prod_{\mu \mid \infty \atop \mu: K \to \mathbf{R}} \int_{E(K_{\mu})} |\omega|} .$$

Similarly, the cardinality $\# \amalg_E(K)$ of Tate-Shafarevich group $\amalg_E(K)$ should be given by the formula (63) $\# \amalg_E(K)$

$$= -\frac{\#E(K)^{2}_{\text{tors}} \cdot d_{K} \left(\operatorname{CT} \left(\langle \langle f^{+}_{0,\eta,\mathcal{O}_{K}}(\tau), \theta^{+}_{\Lambda_{\mathcal{O}_{K},1}} \otimes \mathcal{E}_{\Lambda_{\mathcal{O}_{K},2}}(\tau) \rangle \rangle \right) + \frac{\operatorname{vol}(U_{\mathcal{O}_{K},2})}{4} \sum_{\substack{(z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \in Z(V_{\mathcal{O}_{K},2}) \\ 0 \\ (z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \in Z(V_{\mathcal{O}_{K},2})}} \vartheta^{\star}_{f_{0,\eta,\mathcal{O}_{K}}}(z^{\pm}_{V_{\mathcal{O}_{K},2}}, h) \right)}{2 \log \epsilon_{K} \cdot R_{E}(K) \cdot T_{E}(K) \cdot \prod_{\substack{\mu \mid \infty \\ \mu \mid K \to \mathbf{R}}} \int_{E(K_{\mu})} |\omega|} .$$

Note that we can also derive similar albeit more intricate conditional arithmetic expressions for $\# III_E(K[c])$ and $R_E(K[c])$ in the more general setting where $h_K \geq 1$, e.g. after specializing our main result to the principal character $\chi = \chi_0$ of the class group of K, and summing over classes. We leave the details as an exercise to the reader. Finally, we can also establish the following unconditional result.

Theorem 5.1. Assume that $\operatorname{ord}_{s=1} \Lambda(E/K, 1) = 1$, so that either $\Lambda(E, 1) = \Lambda(1/2, \pi)$ or the quadratic twist $\Lambda(E^{(d_K)}, 1) = \Lambda(1/2, \pi \otimes \eta)$ vanishes. Let us also assume that E has semistable reduction so that its conductor N is squarefree, with N coprime to the discriminant d_K of K, and for each prime $p \geq 5$:

- The residual Galois representations E[p] and $E^{(d_K)}[p]$ attached to E and $E^{(d_K)}$ are irreducible.
- There exists a prime divisor $l \parallel N$ distinct from p where the residual representation E[p] is ramified, and a prime divisor $q \parallel Nd_K$ distinct from p where the residual representation $E^{(d_K)}[p]$ is ramified.

Writing [e, e] to denote the regulator of either E or $E^{(d_k)}$ according to which factor vanishes, we have the following unconditional identity, up to powers of 2 and 3:

$$\frac{\#\operatorname{III}_{E}(\mathbf{Q}) \cdot \#\operatorname{III}_{E^{(d_{K})}}(\mathbf{Q}) \cdot [e,e] \cdot T_{E}(\mathbf{Q}) \cdot T_{E^{(d_{K})}}(\mathbf{Q})}{\#E(\mathbf{Q})_{\operatorname{tors}}^{2} \cdot \#E^{(d_{k})}(\mathbf{Q})_{\operatorname{tors}}^{2}} \cdot \int_{E(\mathbf{R})} |\omega_{E}| \cdot \int_{E^{(d_{K})}(\mathbf{R})} |\omega_{E^{(d_{k})}}| \\
= -\frac{\sqrt{d_{K}}}{\log \epsilon_{K}} \cdot \frac{1}{2} \sum_{A \in \operatorname{Pic}(\mathcal{O}_{K})} \left(\operatorname{CT}\left(\langle \langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}^{+} \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle \rangle \right) + \frac{\operatorname{vol}(U_{A,2})}{4} \sum_{(z_{V_{A,2}}^{\pm},h) \in Z(V_{A,2})} \vartheta_{f_{0,\eta,A}}^{\star}(z_{V_{A,2}}^{\pm},h) \right)$$

Proof. Assuming as we do that $\operatorname{ord}_{s=1} \Lambda(E/K, 1) = 1$, we deduce from the Artin formalism that

$$\Lambda'(E/K,1) = \Lambda'(E,1)\Lambda(E^{(d_K)},1) + \Lambda'(E^{(d_K)},1)\Lambda(E,1),$$

or equivalently that

$$\Lambda'(1/2,\Pi) = \Lambda'(1/2,\pi)\Lambda(1/2,\pi\otimes\eta) + \Lambda'(1/2,\pi\otimes\eta)\Lambda(1/2,\pi)$$

where precisely one of the summands on the right-hand side in each version does not vanish. Note that we can take for granted the refined conjecture of Birch and Swinnerton-Dyer (61) for the nonvanishing summand up to powers of 2 and 3 by our hypotheses, using the combined works of Kato [30], Kolyvagin [31], Rohrlich [40], and Skinner-Urban [43] with the corresponding Euler characteristic calculations of Burungale-Skinner-Tian [9] (cf. [9], [12]) for the analytic rank zero part, together with Jetchev-Skinner-Wan [29], Skinner-Zhang [44], and Zhang [50] for the analytic rank one part. We refer to the summary given in [9, Theorem 3.10] for the current status of these deductions confirming the *p*-part of the conjectural Birch-Swinnerton-Dyer formula via Iwasawa-Greenberg main conjectures. Applying (61) to each factor, we can then deduce (up to powers of 2 and 3) that we have the refined product formula

$$\begin{split} \Lambda'(E/K,1) &= \Lambda'(1/2,\Pi) \\ &= \frac{\# \mathrm{III}_E(\mathbf{Q}) \cdot \# \mathrm{III}_{E^{(d_K)}}(\mathbf{Q}) \cdot [e,e] \cdot T_E(\mathbf{Q}) \cdot T_{E^{(d_K)}}(\mathbf{Q})}{\# E(\mathbf{Q})^2_{\mathrm{tors}} \cdot \# E^{(d_k)}(\mathbf{Q})^2_{\mathrm{tors}}} \cdot \int_{E(\mathbf{R})} |\omega_E| \cdot \int_{E^{(d_K)}(\mathbf{R})} |\omega_{E^{(d_k)}}|. \end{split}$$

The stated identity then follows from Theorem 4.19 and Corollary 4.5.

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