L-FUNCTIONS OF ELLIPTIC CURVES IN RING CLASS EXTENSIONS OF REAL QUADRATIC FIELDS VIA REGULARIZED THETA LIFTINGS

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Abstract. We derive integral presentations for central derivative values of L-functions of elliptic curves defined over the rationals, basechanged to a real quadratic field K, and twisted by ring class characters of K. In particular, we derive an explicit formula for the central derivative value in terms of special values of certain automorphic Green's functions for Hirzebruch-Zagier divisors on the Hilbert modular surface associated with the quadratic basechange form. In special cases, we can also describe these central derivative values as periods, and more generally reinterpret the refined conjecture of Birch and Swinnerton-Dyer in terms of special values of automorphic Green's functions.

Contents

1. Introduction

Let E be an elliptic curve of conductor N defined over the rational number field $\mathbf{Q}$, with corresponding Hasse-Weil L-function denoted by $L(E,s)$. The modularity theorem of Wiles, Taylor-Wiles, and Breuil-Conrad-Diamond-Taylor implies that $L(E,s)$ has an analytic continuation $L^{\star}(E,s)$ as the Mellin transform

$$L^{\star}(E,s) = L(s,f) = \int_0^{\infty} f\left(\frac{iy}{\sqrt{N}}\right) y^s \frac{dy}{y} = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(E,s) \tag{1}$$

of some weight-two newform $f = f_E \in S_2^{\mathrm{new}}(\Gamma_0(N))$. That is, writing $\pi = \otimes_v \pi_v$ to denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated to f, with $L(s,\pi) = \prod_{v \leq \infty} L(s,\pi_v)$ its standard L-function (using the unitary normalization so that $s = 1/2$ is the central value) we have equivalences of L-functions

$$L^{\star}(E,s) = L(s,f) = L(s-1/2,\pi).$$

Suppose now k is any number field. The Mordell-Weil theorem implies that the group of k-rational points $E(k)$ has the structure of a finitely generated abelian group $E(k) \cong \mathbf{Z}^{r_E(k)} \oplus E(k)_{\mathrm{tors}}$. It is a fundamental

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open problem to characterize the rank $r_E(k) = \mathrm{rk}_{\mathbf{Z}} E(k)$. Writing $L(E/k, s)$ to denote the Hasse-Weil L-function of E/k, Birch and Swinnerton-Dyer conjectured that this generating series $L(E/k, s)$, defined a priori only for $\Re(s) > 3/2$, has an analytic continuation to all $s \in \mathbf{C}$, and satisfies a functional equation relating values at s to $2 - s$ (so that $s = 1$ is the central point). Taking for granted this preliminary hypothesis[1], the conjecture of Birch and Swinnerton-Dyer predicts that the rank $r_E(k)$ is given by the order of vanishing $\mathrm{ord}_{s=1} L(E/k, s)$ at this central point. Although this conjecture has been verified over the past several decades for $r_E(k) \leq 1$ with $k = \mathbf{Q}$ or k an imaginary quadratic field, it remains open at large, without a single known example for $r_E(k) \geq 2$. The most stunning progress to date has come through the Iwasawa theory of elliptic curves, using as a starting point special value formulae for the values $L^{(r_E(k))}(E/k, 1)$. In particular, the celebrated theorem of Gross-Zagier [23] (with generalizations such as [52] and [8]) for the central derivative value $L'(E/k, \chi, 1)$, with χ a class group character of an imaginary quadratic field k, has played a major role underlying most of this progress for rank one. This tour de force makes use of all that is known about the theory of complex multiplication and explicit class field theory for imaginary quadratic fields, and especially a construction of points $e_H \in E(k[1])$ dating back to Heegner to relate the central derivative values $L'(E/k, \chi, 1)$ for χ a character of the class group $\mathrm{Pic}(\mathcal{O}_k) \cong \mathrm{Gal}(k[1]/k)$ (with $k[1]/k$ the Hilbert class field) to the regulator term $R_E(k) = [e_H, e_H]$ (with $[\cdot, \cdot]$ the Néron-Tate height pairing).

Here, we return to the wilderness by taking $k = K$ to be a real quadratic field $K = \mathbf{Q}(\sqrt{d})$ of discriminant

$$d_K = \begin{cases} d & \text{if } d \equiv 1 \bmod 4 \\ 4d & \text{if } d \equiv 2, 3 \bmod 4 \end{cases}$$

prime to N, and corresponding even Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$. Let χ be any ring class character of K of conductor $c \in \mathbf{Z}_{\geq 1}$ prime to $d_K N$. More precisely, we consider χ a character of the class group $\mathrm{Pic}(\mathcal{O}_c) \cong \mathrm{Gal}(K[c]/K)$ of the $\mathbf{Z}$-order $\mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K,

$$\chi : \mathrm{Pic}(\mathcal{O}_c) := \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_c^\times \longrightarrow \mathbf{S}^1, \quad \widehat{\mathcal{O}}_c^\times := \prod_{v < \infty} \mathcal{O}_{c,v}^\times.$$

As we explain below, the theories of Rankin-Selberg convolution and cyclic basechange allow us to deduce from the modularity theorem (3.2) that the corresponding Hasse-Weil L-function $L(E/K, \chi, s)$ has an analytic continuation $L^\star(E/K, \chi, s)$ to all $s \in \mathbf{C}$, and has a functional equation relating values at s to $2 - s$. To be more precise, writing $\pi(\chi)$ to denote the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ of level $d_K c^2$ and character η induced from the ring class character χ, this $L^\star(E/K, \chi, s)$ is equivalent to the corresponding shifted $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-function $L(s - 1/2, \pi \times \pi(\chi))$. Writing $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ to denote the quadratic basechange lifting of π to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$, this $L^\star(E/K, \chi, s)$ is also equivalent to the corresponding shifted $\mathrm{GL}_2(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$ automorphic L-function $L(s - 1/2, \Pi \otimes \chi)$. In this way, we deduce that that completed Hasse-Weil L-function $L^\star(E/K, \chi, s)$ has an analytic continuation through its equivalent presentations

$$L^\star(E/K, \chi, s) = L(s - 1/2, \pi \times \pi(\chi)) = L(s - 1/2, \Pi \otimes \chi)$$

and in particular that it satisfies a symmetric functional equation (see (8)). The immediate consequence of this is the following, whose proof we explain in the discussion leading to Hypothesis 2.1 below:

Lemma 1.1. *Let E be an elliptic curve of conductor N defined over $\mathbf{Q}$, and $\pi = \pi(f)$ the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by the eigenform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizing E. Let K be a real quadratic field of discriminant d_K prime to N, so that we can write $N = N^+ N^-$ for N^+ the product of prime divisors $q \mid N$ which split in K, and N^- the product of prime divisors $q \mid N$ which remain inert in K, and $\eta(-N) = \eta(N) = \eta(N^-)$. If N^- is the squarefree product of an odd number of primes, then*

$$L^\star(E/K, \chi, 1) = L(1/2, \pi \times \pi(\chi)) = L(1/2, \Pi \otimes \chi) = 0$$

for any ring class character χ of K of conductor c prime to $d_K N$.

Taking the conditions of Lemma 1.1 for granted, it makes sense to consider the corresponding central derivative values $L^{\star\prime}(E/K, \chi, 1) = L'(1/2, \pi \times \pi(\chi)) = L'(1/2, \Pi \otimes \chi)$. The aim of this work is to use the same setup with theta correspondence underlying the calculations of Gross-Zagier [23], Yuan-Zhang-Zhang

[1] which remains open in general for non-abelian number fields k which are neither totally real nor solvable

[52], and especially Bruinier-Yang [8] to calculate these central derivative values in terms of theta liftings from certain orthogonal groups connected to the basechange representation Π of $\mathrm{GL}_2(\mathbf{A}_K)$. Here, we derive several presentations, including connections to the Shimura-Shintani-Waldspurger correspondence (see (28)).

The main result is the following connection to automorphic Green's functions evaluated along certain geodesic orbits on spin Shimura varieties associated to Hilbert modular surfaces. While there is no known or conjectural global analogue of the Heegner point construction in this setting, at least beyond well-known local conjectures made by Darmon [15], we present some depiction of the provenance of such points $e_{??} \in E(K[c])$ in these geodesic orbits. As we explain, these Shimura varieties correspond to Hilbert modular surfaces related to the basechange representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$. In the discussion below, we first recall some general theory leading up to (11), starting with how we may find a certain cuspidal automorphic form $\varphi \in \Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ on $\mathrm{GL}_2(\mathbf{A}_K)$ constructed from a pure tensor in the cuspidal automorphic representation Π so that

$$L'(1/2, \Pi \otimes \chi) = \frac{d}{ds}\bigg|_{s=1/2} \int_{\mathbf{A}_K^\times/K^\times} \varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \chi(y)|y|^{s-\frac{1}{2}} dy. \tag{2}$$

We consider realizations of this integral presentation via the theta correspondence, first abstractly via symplectic-orthogonal pairs for Theorem 4.2 via Proposition 4.1, then in terms of the Shimura-Shintani-Waldspurger correspondence for (28) via Proposition 4.3. We focus on the orthogonal symplectic pair for the rest of this work. The setup we consider here is special in that we have the following accidental isomorphism to the spin group Spin_V of the quadratic space (V, q) with $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$ with $q(x, y, \lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$. More generally, for any class $A \in \mathrm{Pic}(\mathcal{O}_c)$, we fix an integral representative $\mathfrak{a} \subset \mathcal{O}_K$ so that $A = [\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$, and write $\mathfrak{a}_\mathbf{Q} := \mathfrak{a} \otimes_\mathbf{Z} \mathbf{Q}$ to denote the corresponding vector space. We also consider the quadratic space $(V, q) = (V_A, q_A)$ of signature $(2, 2)$ defined by $V_A = \mathbf{Q} \oplus \mathbf{Q} \oplus \mathfrak{a}_\mathbf{Q}$ with quadratic form $q_A = \mathbf{N}_{K/\mathbf{Q}}(\lambda)\mathbf{N}\mathfrak{a}^{-1} - xy$. Here, $\mathbf{N}_{K/\mathbf{Q}}(\lambda) = \lambda\lambda^\tau$ denotes the norm homomorphism, with τ the conjugation in K. In each case, we have for the underlying rational quadratic space V an isomorphism of algebraic groups

$$\mathrm{Spin}_V \cong \mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_2(K)$$

over $\mathbf{Q}$ (see (13)). These spin groups are related to the orthogonal groups $\mathrm{SO}(V)$ via the short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V \longrightarrow \mathrm{SO}(V) \to 1.$$

Now, we can view our cuspidal automorphic form φ of trivial central character on $\mathrm{GL}_2(\mathbf{A}_K)$ as an automorphic form on $\mathrm{SL}_2(\mathbf{A}_K)$. Via the identification $\mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_2(\mathbf{A}_K) \cong \mathrm{Spin}_V(\mathbf{A})$, we then obtain from φ an automorphic form φ' on $\mathrm{Spin}_V(\mathbf{A})$, and via the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V(\mathbf{A}) \longrightarrow \mathrm{SO}(V)(\mathbf{A}) \to 1$$

an automorphic form φ'' on $\mathrm{SO}(V)(\mathbf{A})$. These extend respectively to an automorphic form φ' on the general spin group $\mathrm{GSpin}_V(\mathbf{A})$ and to an automorphic form φ'' on the general orthogonal group $\mathrm{GO}(V)(\mathbf{A})$.

For any rational quadratic space (V, q) of signature $(2, 2)$, there is a corresponding hermitian symmetric domain D_V which is in bijective correspondence with oriented positive-definite[2] hyperplanes in $V_\mathbf{R}$. This leads us to the spin Shimura variety $\mathrm{Sh}(D_V, \mathrm{GSpin}_V)$ associated to this space, where the regularized theta lifts can be seen as an arithmetic geometric development of the standard theta kernel from the $\mathrm{Sp}_2 \cong \mathrm{SL}_2$. The heart of our approach, which develops the distinct proof of the Gross-Zagier formula [23] given by Bruinier-Yang [8], is to realize the pure tensor $\varphi \in \Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ on $\mathrm{GL}_2(\mathbf{A}_K)$ or its incarnations φ' on $\mathrm{GSpin}_V(\mathbf{A})$ and φ'' on $\mathrm{GO}(V)(\mathbf{A})$ via the accidental $\mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_2(\mathbf{A}_K) \cong \mathrm{Spin}_V(\mathbf{A})$ as a theta lifting for the reductive dual pair $(\mathrm{GO}(V)(\mathbf{A}), \mathrm{GSp}(W)(\mathbf{A})) = (\mathrm{GO}(V)(\mathbf{A}), \mathrm{GL}_2(\mathbf{A}))$. In the preliminary abstract formula derived in Theorem 4.2 below for this setup, we first find an explicit realization of the quadratic basechange lifting $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$, which allows us to recover the standard Rankin-Selberg integral presentation $L'(1/2, \pi \times \pi(\chi))$ in a novel way. Broadly, we develop this preliminary formula by realizing the vectors that appear as certain arithmetic automorphic forms on the ambient Shimura variety, and in particular as certain regularized theta lifts.

To describe the Shimura varieties in more detail, let $H \subset \mathrm{GL}_2(\mathbf{A}_{K,f})$ be the compact open subgroup determined by the level of the basechange representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$, with $U = U(\Pi)$ the corresponding compact open subgroup of $\mathrm{GSpin}_V(\mathbf{A}_f)$. We use the same notation to denote the corresponding compact

[2] or negative-definite

open subgroup of $\mathrm{GO}(V)(\mathbf{A}_f)$. Let D_V denote the Grassmannian of oriented positive definite hyperplanes in $V(\mathbf{R})$. We can then consider φ' and φ'' as arithmetic automorphic forms on the respective Shimura varieties

$$\mathrm{Sh}_U(\mathrm{GSpin}_V, D_V) = \mathrm{GSpin}_V(\mathbf{Q})\backslash \left(D \times \mathrm{GSpin}_V(\mathbf{A}_f)/U\right)$$

and

$$\mathrm{Sh}(\mathrm{GO}(V), D_V) = \mathrm{GO}(V)(\mathbf{Q})\backslash \left(D_V \times \mathrm{GO}(V)(\mathbf{A}_f)/H\right).$$

As we explain below, the latter Shimura variety can be identified with the non-compact Hilbert modular surface $Y(\Gamma) = \Gamma\backslash\mathfrak{H}^2$, where $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$ denotes the congruence subgroup determined by $U \cap \mathrm{GO}(V)(\mathbf{Q})$.

Fix a primitive ring class character χ of some conductor c prime to $d_K N$, which we view as a character of the class group $\mathrm{Pic}(\mathcal{O}_c)$ of the order $\mathcal{O}_c \subset \mathcal{O}_K$ of conductor c in K. Fix an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_K$ for each class $A = [\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$, and consider the rational quadratic space (V_A, q_A) of signature $(2,2)$ given by the vector space $V_A = \mathbf{Q} \oplus \mathbf{Q} \oplus \mathfrak{a}_{\mathbf{Q}}$ and the quadratic form $q_A(x,y,\lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda)\mathbf{N}\mathfrak{a}^{-1} - xy$. Observe that the embedding of quadratic spaces $V_A \longrightarrow V$ induces and embedding of algebraic groups $\mathrm{GSpin}_{V_A} \longrightarrow \mathrm{GSpin}_V$. Writing $U_A = U \cap \mathrm{GSpin}_{V_A}(\mathbf{A}_f)$ to denote the corresponding compact open subgroup in $\mathrm{GSpin}_{V_A}(\mathbf{A}_f)$, and D_{V_A} the corresponding Grassmannian of positive hyperplanes, we have an embedding of Shimura varieties $\mathrm{Sh}_{U_A}(\mathrm{GSpin}_{V_A}, D_{V_A}) \longrightarrow \mathrm{Sh}_U(\mathrm{GSpin}_V, D_V)$. Similarly, we have an embedding of Shimura varieties $\mathrm{Sh}_{U_A}(\mathrm{GO}(V_A), D_{V_A}) \longrightarrow \mathrm{Sh}_U(\mathrm{GO}(V), D_V)$. The subspace $(V_{A,2}, q_{A,2})$ of signature $(1,1)$ given by the fractional ideal $V_{A,2} = \mathfrak{a}_{\mathbf{Q}}$ and quadratic form $q_{A,2}(\lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda)\mathbf{N}\mathfrak{a}^{-1}$ gives rise to a "point"

$$z_{V_{A,2}} \in D_{V_{A,2}} = \{z \in V_{A,2}(\mathbf{R}) : \dim(z) = 1,\ q_{A,2}|_z > 0\}$$

in the corresponding subdomain $D_{V_{A,2}}$ for $V_{A,2} \subset V_A$ (see [7, §2, cf. § 5-6]). That is, each symmetric space $D_{V_{A,2}}$ determines an open subset of real projective space of dimension one, and each "point" $z_{V_{A,2}} \in D_{V_{A,2}}$ is a real curve of dimension one – equivalent to a real geodesic on a quaternionic Shimura curve which is embedded into the ambient Hilbert modular variety as a Hirzebruch-Zagier divisor. We consider for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$ the corresponding "geodesics" defined by

$$Z(V_{A,2}) = \mathrm{GSpin}_{V_{A,2}}(\mathbf{Q})\backslash \left(D_{V_{A,2}} \times \mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)/\left(U \cap \mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)\right)\right)$$

on the spin Shimura variety $\mathrm{Sh}_{U_A}(\mathrm{GSpin}_{V_A}, D_{V_A}) \subset \mathrm{Sh}_U(\mathrm{GSpin}_V, D_V)$, and similarly

$$Z(V_{A,2}) = O(V_{A,2})(\mathbf{Q})\backslash \left(D_{V_{A,2}} \times O(V_{A,2})(\mathbf{A}_f)/\left(U \cap O(V_{A,2})(\mathbf{A}_f)\right)\right)$$

on the orthogonal Shimura variety $\mathrm{Sh}_{U_A}(\mathrm{GO}(V_A), D_{V_A}) \subset \mathrm{Sh}_U(\mathrm{GO}(V), D_V)$. Again, these "geodesics" are equivalent to real geodesics on quaternionic Shimura curves, embedded into the ambient Hilbert modular surfaces as Hirzebruch-Zagier divisors. We refer to them as "geodesics" in this way for simplicity.

We realize the preliminary formula (2) more explicitly in terms of certain regularized theta lifts evaluated along these "geodesic" subsets $Z(V_{A,2})$, in the style of Bruinier-Yang [8, Theorem 4.7]. Let us first emphasize that several distinct features appear in deriving such an integral presentation for $L'(1/2, \Pi \otimes \chi) = L'(1/2, \pi \times \pi(\chi))$. Beyond the fact that the quadratic subspaces $(V_{A,2}, q_{A,2})$ have signature $(1,1)$ – as opposed to $(0,2)$ when K is replaced by an imaginary quadratic field[3] – we also work class-by-class $A \in \mathrm{Pic}(\mathcal{O}_c)$ to obtain a novel variation of the Rankin-Selberg integral presentation for the twist by $\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee$, and also take into account features of the non-holomorphic Siegel theta series that appear. Moreover, as an indirect consequence of the fact that we do not work with "incoherent" Eisenstein series after taking the sum along the geodesic – and applying the Siegel-Weil formula to identify special value of Eisenstein series – we obtain a surprising formula for the sum $L(1/2, \Pi \otimes \chi) + L'(1/2, \Pi \otimes \chi)$ of the central value $L(1/2, \Pi \otimes \chi)$ plus the central derivative value $L'(1/2, \Pi \otimes \chi)$.

Let us now describe the regularized theta lifts that appear, starting with those constructed abstractly and "classically" for the ambient quadratic spaces (V, q) of signature $(2,2)$, Let $\psi_0 = \otimes_v \psi_{0,v}$ denote the standard additive character of $\mathbf{A}/\mathbf{Q}$. Let r_{ψ_0} denote the Weil representation (cf. [32], [8], [33]) associated to the reductive dual pair $(\mathrm{SO}(V)(\mathbf{A}), \mathrm{Sp}(W)(\mathbf{A}))$,

$$r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{Sp}(W)(\mathbf{Q}) \longrightarrow \mathcal{S}(\mathbb{W}(\mathbf{A})), \quad \mathbb{W} = V \otimes W,$$

[3] with a minor alternation to the definition of the quadratic form, cf. [8]

as well as its extension to similitudes

$$R(\mathbf{A}) = \{(h,g) \in \mathrm{O}(V)(\mathbf{A}) \times \mathrm{GSp}(W)(\mathbf{A}) : \nu(h) = \det(g)\} \subset \mathrm{GO}(V)(\mathbf{A}) \times \mathrm{GSp}(W)(\mathbf{A}).$$

We have natural identifications $\mathrm{Sp}(W) \cong \mathrm{SL}_2$ and $\mathrm{GSp}(W) \cong \mathrm{GL}_2$. Given an even lattice $\Lambda \subset V$ with dual lattice $\Lambda^{\#}$, we consider the subspaces $\mathcal{S}_\Lambda \subset \mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ of decomposable Schwartz functions with support on the profinite completion $\widehat{\Lambda}^{\#} = \Lambda^{\#} \otimes \widehat{\mathbf{Z}}$ which are constant on cosets of $\widehat{\Lambda} = \Lambda \otimes \widehat{\mathbf{Z}}$. Note that $\mathcal{S}(\mathbb{W}(\mathbf{A}_f)) = \varinjlim_\Lambda \mathcal{S}_\Lambda$ can be decomposed into a basis of such functions (see (39)). We write $\mathbf{1}_\mu$ to denote such a characteristic function associated to a given coset $\mu \in \Lambda^{\#}/\Lambda$. We shall consider certain automorphic forms taking values in these spaces $\mathcal{S}_\Lambda$, with $\Lambda \subset V$ the lattice corresponding to the fixed choice of compact open subgroup $U = U(\varphi) \subset \mathrm{GSpin}_V(\mathbf{A}_f)$. That is, we take $\Lambda \subset V$ to be the lattice whose profinite completion $\widehat{\Lambda} = \Lambda \otimes \mathbf{Z}$ and discriminant group $\Lambda^{\#}/\Lambda$ are both fixed by the action of U. As explained below (Theorem 4.9) these can be viewed as canonical liftings of the scalar-valued eigenforms introduced above. In particular, we shall consider the twist $f \otimes \eta \in S_2^{\mathrm{new}}(\Gamma_0(d_K N))$ of our eigenform f by the quadratic character $\eta = \eta_{K/\mathbf{Q}}$ associated to the real quadratic field K. Hence, writing the Fourier series expansion of f as $f(\tau) = \sum_{m \geq 1} a_f(m) e(m\tau)$ for $\tau = u + iv \in \mathfrak{H}$ and $e(z) = \exp(2\pi i z)$, this quadratic twist $f \otimes \eta$ has the series expansion $f \otimes \eta(\tau) = \sum_{m \geq 1} a_f(m)\eta(m) e(m\tau)$. According to [53, Theorem 4.15], as summarized in Theorem 4.9 below for our setup, we can associate to this quadratic twist $f \otimes \eta$ an $\mathcal{S}_\Lambda$-valued cusp form $g_\eta \in S_{2,\Lambda}$. We can also view the usual theta kernel $\theta_{r_{\psi_0}}$ associated to the Weil representation r_{ψ_0} by the rule

$$\theta_{r_{\psi_0}}(h,g;\Phi) = \sum_{x \in \mathbb{W}(\mathbf{A})} r_{\psi_0}(h,g)\Phi(x)$$

as such a vector-valued form $\theta_{r_{\psi_0},\Lambda}$ defined on $z \in D_V$, $\tau = u + iv \in \mathfrak{H}$, and $h \in \mathrm{GO}(V)(\mathbf{A}_f)$ according to the discussion in [32] (see (50) below) as

$$\theta_\Lambda(z,h,\tau) = \theta_{\Lambda, r_{\psi_0}}(z,h,\tau) = \sum_{\mu \in \Lambda^{\#}/\Lambda} \theta^\star_{r_{\psi_0}}(z,h,g_\tau;\mathbf{1}_\mu) \cdot \mathbf{1}_\mu.$$

Here, $\mathbf{1}_\mu = \mathrm{char}(\mu + \widehat{\Lambda})$ is the characteristic function of $\mu + \widehat{\Lambda}$, and g_τ the matrix

$$g_\tau = \begin{pmatrix} v & u \\ & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}) \cong \mathrm{GSp}(W)(\mathbf{R}) \quad \text{or } g_\tau = \begin{pmatrix} v^{\frac{1}{2}} & uv^{-\frac{1}{2}} \\ & v^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}) \cong \mathrm{Sp}(W)(\mathbf{R}).$$

More specifically, we shall consider the regularized theta lifts associated to these quadratic spaces as follows. Fix a class $A \in \mathrm{Pic}(\mathcal{O}_c)$, together with an integral ideal representative $\mathfrak{a} \subset \mathcal{O}_K$. We work with the corresponding quadratic space (V_A, q_A) of signature $(2,2)$ and its hermitian symmetric domain D_{V_A}, as described above. Taking $U_A = U_A(\Pi) \subset \mathrm{GSpin}_{V_A}(\mathbf{A}_f)$ to be the compact open subgroup determined by the level $U = U(\Pi) \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ of the basechange representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ on $\mathrm{GL}_2(\mathbf{A}_K)$ (hence $U_A = U \cap \mathrm{GSpin}_{V_A}(\mathbf{A}_f)$), we consider the corresponding spin Shimura variety

$$\mathrm{Sh}_{U_A}(\mathrm{GSpin}_{V_A}, D_{V_A}) = \mathrm{GSpin}_{V_A}(\mathbf{Q}) \backslash D_{V_A} \times \mathrm{GSpin}_{V_A}(\mathbf{A}_f)/U_A.$$

Fix a lattice $\Lambda_A \subset V_A$ whose profinite completion $\widehat{\Lambda}_A = \Lambda_A \otimes \widehat{\mathbf{Z}}$ and discriminant group $\Lambda_A^{\#}/\Lambda_A$ are both stabilized by the action of U_A. We shall fix a certain left $\mathrm{SO}(V_A)(\mathbf{R})$-invariant Gaussian function $\Phi_\infty \in \mathcal{S}(V_A(\mathbf{R})) \otimes \mathcal{C}^\infty(D_{V_A})$ as in [32, § 1] for the ambient space (V_A, q_A). More generally, we can choose such a function Φ_∞ for any case on the signature according to Bruinier-Funke [7, Proposition 5.6], and in particular for our distinguished subspace $(V_{A,2}, q_{A,2})$ of signature $(1,1)$.[4] Given a hyperplane $z \in D_{V_A}$, a finite element $h_f \in \mathrm{SO}(V_A)(\mathbf{A}_f)$, and a matrix $g \in \mathrm{SL}_2(\mathbf{A})$, we then consider the linear function defined on decomposable Schwartz functions $\Phi_f \in \mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ by

$$\Phi_f \longmapsto \theta^\star_{r_{\psi_0}}(z, h_f, g; \Phi_f) := \sum_{x \in \mathbb{W}(\mathbf{Q})} r_{\psi_0}(1,g) \left(\Phi_\infty(\cdot, z) \otimes r_{\psi_0}(h_f, 1)\Phi_f\right)(x).$$

[4]The exact form of this archimedean component Φ_∞ makes no difference for our subsequent discussion.

Given any $\mathcal{S}_{\Lambda_A}$-valued modular form f_0 of weight $0(=1-2/2)$, we then define the corresponding regularized theta lift $\vartheta_{f_0}^{\star}$ on $\tau = u + iv \in \mathfrak{H}$ by the limit

$$\vartheta_{f_0}^{\star}(z, h_f) := \int_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}}^{\star} \theta_{r_{\psi_0}}^{\star}(z, h_f, g_\tau, f_0)\frac{dudv}{v^2} = \lim_{\overrightarrow{T\rightarrow\infty}} \int_{\mathcal{F}_T} \theta_{r_{\psi_0}}^{\star}(z, h_f, g_\tau, f_0)\frac{dudv}{v^2},$$

where each $\mathcal{F}_T$ denotes the truncated fundamental domain

$$\mathcal{F}_T = \left\{\tau = u + iv \in \mathfrak{H} : |u| \leq 1/2, \tau\overline{\tau} \geq 1, \text{ and } v \leq T\right\}.$$

A well-known theorem of Borcherds [4] (cf. [32, Theorem 1.2]) computes these regularized theta lifts in terms of twists of meromorphic line bundles on the Shimura variety $\mathrm{Sh}_{U_A}(\mathrm{GSpin}_{V_A}, D_{V_A}) \subset \mathrm{Sh}_U(\mathrm{GSpin}_V, D_V)$, and moreover relates these to automorphic Green's functions. This theorem was refined by Bruinier [5] and Bruinier-Funke [7] to a level of generality that applies here, and in particular to allow for the input form f_0 to be a harmonic weak Maass form. To describe this, let us for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$ take $f_{0,\eta,A} \in H_{0,-\Lambda_A}$ to be the harmonic weak Maass form of weight 0 and (Weil) representation $\overline{r}_{\psi_0,\Lambda_A} = r_{\psi_0,-\Lambda_A}$ whose image $g_{\eta,A} = \xi_0(f_{0,\eta,A}) \in S_{2,-\Lambda_A}$ under the antilinear differential operator $\xi_0 : H_{0,\Lambda_A} \longrightarrow S_{2,-\Lambda_A}$ described below (cf. [8, (3.5)]) has a canonical lift as described in Theorem 4.9 to the twisted scalar-valued eigenform $f \otimes \eta$. Here, we write $-L_A$ to denote the quadratic space determined by $(L_A, -q_A)$, and note that each of the vector-valued cusp forms $g_{A,\eta}$ has Fourier series expansion given explicitly in terms of the Fourier coefficients of the scalar-valued cuspidal eigenform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizing the elliptic curve E. To be more precise, we have for each class $A = [\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$ that

$$g_{\eta,A}(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} g_{\eta,A,\mu}(\tau)\mathbf{1}_\mu = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \left(\sum_{\substack{m \in \mathbf{Q}_{>0} \\ m \equiv d_K N q_A(m) \bmod d_K N}} a_f(m)\eta(m)s(m)e\left(\frac{m\tau}{d_K N}\right)\right)\mathbf{1}_\mu.$$

Here, writing $\tau = u + iv \in \mathfrak{H}$ and $e(\tau) = \exp(2\pi i\tau)$ as usual, we let s denote the function defined on classes $m \bmod d_K N$ by $s(m) = 2^{\Omega(m, d_K N)}$, where $\Omega(m, d_K N)$ is the number of divisors of the greatest comment divisor $(m, d_K N)$ of m and $d_K N$. Recall that this harmonic weak Maass form $f_{0,\eta,A} \in H_{0,\Lambda_A}$ determined by the condition $\xi_0 f_{0,\eta,A}(\tau) = g_{\eta,A}(\tau)$ has a decomposition $f_{0,\eta,A}(\tau) = f_{0,\eta,A}^+(\tau) + f_{0,\eta,A}^-(\tau)$ into a holomorphic part $f_{0,\eta,A}^+(\tau)$ and an antiholomorphic part $f_{0,\eta,A}^-(\tau)$. We write the respective Fourier series expansions as

$$f_{0,\eta,A}^+(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} f_{0,\eta,A,\mu}^+(\tau)\mathbf{1}_\mu = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \left(\sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_{f_{0,\eta,A}}^+(m,\mu)e(m\tau)\right)\mathbf{1}_\mu,$$

and

$$f_{0,\eta,A}^-(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} f_{0,\eta,A,\mu}^-(\tau)\mathbf{1}_\mu = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \left(\sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_{f_{0,\eta,A}}^-(m,\mu)W_0(2\pi m v)\right)\mathbf{1}_\mu,$$

with Whittaker function $W_0(n) = \int_{-2n}^{\infty} e^{-t}dt = \Gamma(1, 2|n|)$ defined for $n < 0$. More generally, for any half-integer $k \in \frac{1}{2}\mathbf{Z}$, we can consider the space H_{k,Λ_A} of harmonic weak Maass forms of weight k and representation r_{ψ_0,Λ_A} (defined below), with $M_{k,\Lambda_A}^! \subset H_{k,\Lambda_A}$ the subspace of weakly holomorphic forms, $M_{k,\Lambda_A} \subset M_{k,\Lambda_A}^!$ the subspace of holomorphic forms, and $S_{k,\Lambda_A} \subset M_{k,\Lambda_A}$ the subspace of cuspidal forms:

$$S_{k,\Lambda_A} \subset M_{k,\Lambda_A} \subset M_{k,\Lambda_A}^! \subset H_{k,\Lambda_A}.$$

Bruinier-Funke [7] define an antilinear differential operator

$$\xi_k : H_{k,\Lambda_A} \longrightarrow S_{2-k,\Lambda_A}, \quad \xi_k(\phi) := 2iv^k\overline{\left(\frac{\partial\phi}{\partial\overline{\tau}}\right)},$$

which is related to the classical weight-lowering operator

$$L_k = -2iv^2\frac{\partial}{\partial\overline{\tau}}$$

by $\xi_k(\phi)(\tau) = v^{k-2}\overline{L_k\phi(\tau)}$. In particular, this determines a short exact sequence of $\mathbf{C}$-vector spaces

$$0 \longrightarrow M^!_{k,\Lambda_A} \longrightarrow H_{k,\Lambda_A} \xrightarrow{\xi_k} S_{2-k,-\Lambda_A} \longrightarrow 0,$$

where the subspace of weakly holomorphic forms

$$\phi(\tau) = \sum_{\mu \in \Lambda_A^\#/\Lambda_A} \phi_\mu(\tau) = \sum_{\mu \in \Lambda_A^\#/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_\phi(\mu, m) e(m\tau)\mathbf{1}_\mu \in M^!_{k,\Lambda_A} \subset H_{k,\Lambda_A}$$

is identified with $\ker(\xi_k)$. We can also define from the Petersson inner product $\langle \cdot, \cdot \rangle$ a pairing

$$\{\cdot, \cdot\} : H_{k,\Lambda_A} \times M_{2-k,-\Lambda_A} \longrightarrow \mathbf{C}.$$

That is, we can consider the space A_{k,Λ_A} of all smooth modular forms of weight k and representation r_{ψ_0,Λ_A}, so that we have the inclusions $M^!_{k,\Lambda_A} \subset H_{k,\Lambda_A} \subset A_{k,\Lambda_A}$. Given forms $f \in A_{k,\Lambda_A}$ and $g \in A_{-k,-\Lambda_A}$ with

$$f(\tau) = \sum_{\mu \in \Lambda_A^\#/\Lambda_A} f_\mu(\tau)\mathbf{1}_\mu, \quad g(\tau) \sum_{\mu \in \Lambda_A^\#/\Lambda_A} g_\mu(\tau)\mathbf{1}_\mu,$$

we define the scalar product

$$\langle\langle f, g \rangle\rangle = \sum_{\mu \in \Lambda_A^\#/\Lambda_A} f_\mu(\tau) g_\mu(\tau).$$

Writing $\mathcal{F} = \{\tau = u + iv \in \mathfrak{H} : |u| \leq 1/2, u^2 + v^2 \geq 1\}$ to denote a standard fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$, we then define the Petersson inner product (if it converges) by

$$\langle f, g \rangle = \int_{\mathcal{F}} \langle\langle f(\tau), \overline{g(\tau)} \rangle\rangle v^k \frac{dudv}{v^2}.$$

Given $f_k(\tau) = f_k^+(\tau) + f_k^-(\tau) \in H_{k,\Lambda_A}$ any weakly holomorphic form of weight k and representation r_{ψ_0,Λ_A}, and g any modular form $M_{2-k,-\Lambda_A}$ of weight $2-k$ and representation $r_{\psi_0,-\Lambda_A}$ we then define

$$\{f_k, g\} := \langle \xi_k(f_k), g \rangle.$$

Returning to the case of weight $k = 0$ we consider, we can also define the regularized theta lift $\vartheta^\star_{f_0}(z, h_f)$ equivalently in terms of this pairing via the limited of truncated integrals

$$\vartheta^\star_{f_0}(z, h_f) = \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \langle\langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle\rangle \frac{dudv}{v^2} = \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_A}(\tau) \rangle\rangle \frac{dudv}{v^2}.$$

Here, for each $h_f \in \mathrm{GSpin}_{V_A}(\mathbf{A}_f)/U_A$, we realize the theta kernel described above more concretely in terms of its corresponding Siegel theta series

$$\theta_{\Lambda_A}(\tau, z, h_f) : \mathfrak{H} \times D_{V_A} \longrightarrow \mathcal{S}_{\Lambda_A}$$

(see [8, (2.6)]). In the special case where $f_{0,A} \in M^!_{0,\Lambda_A}$ is a weakly holomorphic form, the regularized theta lift $\vartheta^\star_{f_{0,A}}(z, h_f)$ for the quadratic space (V_A, q_A) of signature $(2,2)$ can be computed thanks to a fundamental theorem of Borcherds [4, Theorem 13.3] (cf. [32, Theorem 1.2]) as

$$\vartheta^\star_{f_{0,A}}(z, h_f) = -2\log|\Psi_{f_{0,A}}(z, h_f)|^2 - c^+_{f_{0,A}}(0,0) \cdot (2\log|y| + \Gamma'(1)),$$

where $\Psi_{f_{0,A}}$ is a meromorphic form on $D_V \times \mathrm{GSpin}_V(\mathbf{A}_f)$ of weight $k = c^+_{f_{0,A}}(0,0)/2$ (see [5, § 3]). Moreover, Borcherds computes the divisor $\mathrm{Div}(\Psi^2_{f_{0,A}})$ of explicitly in terms of the Fourier coefficients of $f_{0,A}$ and certain "special divisors" $Z_A(m,\mu)$, which in the setting we consider correspond to Hirzebruch-Zagier divisors on Hilbert modular surfaces corresponding to the spin Shimura varieties $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$. Adding to this, the theorem of Howard-Madapusi Pera [26, Theorem 9.1.1] allows us to deduce that this so-called Borcherds product $\Psi_{f_0}(z, h_f)$ takes algebraic values, so that the regularized theta lift $\vartheta^\star_{f_{0,A}}(z, h_f)$ attached to any weakly holomorphic form $f_{0,A} \in M^!_{0,\Lambda_A}$ is seen to take values in logarithms of algebraic numbers. To give more detail, we first define the special divisors $Z_A(m,\mu)$. For each $m \in \mathbf{Q}$, consider the quadric defined by

$$\Omega_{m,A}(\mathbf{Q}) = \{x \in V_A : q_A(x) = m\}.$$

Consider the natural projection $\mathrm{pr}: D_{V_A} \times \mathrm{GSpin}_{V_A}(\mathbf{A}_f) \longrightarrow \mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$. Given a point $x \in D_{V_A}$, consider the orthogonal projection $D_{V_A,x} = \{z \in D_{V_A} : z \perp x\}$ of the Grassmannian D_{V_A}. We then define for each coset $\mu \in \Lambda_A^{\#}/\Lambda_A$ the divisor

$$Z_A(\mu, m) = \sum_{x \in (\mathrm{GSpin}\, V_A(\mathbf{Q}) \cap U_A) \backslash \Omega_m(\mathbf{Q})} \mathbf{1}_\mu(x)\, \mathrm{pr}(D_{V_A,x}).$$

Given a weakly holomorphic form $f_{0,A} \in M^!_{0,\Lambda_A}$ with holomorphic part

$$f^+_{0,A}(\tau) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} f^+_0(\tau)\mathbf{1}_\mu = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c^+_{f_0}(\mu, m) e(m\tau)\mathbf{1}_\mu,$$

we then define the corresponding divisor

$$Z(f_{0,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c^+_{f_{0,A}}(\mu, -m) Z(\mu, m).$$

Note that in the special case where $f_{0,A} \in M_{0,\Lambda_A} \subset M^!_{0,\Lambda_A}$ is holomorphic, we have $f_{0,A} = f^+_{0,A}$, and hence $c_{f_{0,A}}(\mu, m) = c^+_{f_{0,A}}(\mu, m)$ for each of the coefficients in the Fourier expansion. In this case, as explained in [32] and [26], we consider the metrized line bundle

$$\widehat{\omega} \in \widehat{\mathrm{Pic}}\left(\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})\right)$$

of modular forms of weight one, which under the complex uniformization of $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$ pulls back to the tautological line bundle on D_{V_A}. Now, the Shimura varieties $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$ we consider have regular, flat integral models $\mathit{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A}) \longrightarrow \mathrm{Spec}(\mathbf{Z})$. The metrized line bundle $\widehat{\omega}$ and the special divisors $Z(\mu, m)$ both extend in a natural way to the integral model $\mathit{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$.

Theorem 1.2 (Borchards, Howard-Madapusi Pera)**.** *Let $f_{0,A} \in M^!_{0,\Lambda_A}$ be a weakly holomorphic form with integral holomorphic Fourier coefficients $c^+_{f_{0,A}}(\mu, -m) \in \mathbf{Z}$ for all $\mu \in \Lambda_A^{\#}/\Lambda_A$ and $m \in \mathbf{Q}_{>0}$. After replacing $f_{0,A}$ by a suitable integer multiple if necessary, there exists a rational section $\Psi_{f_{0,A}}$ of the line bundle $\omega^{c^+_{f_{0,A}}(0,0)}$ on $\mathit{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$ whose norm under the metric defined by*

$$||z||_A = \frac{(z, \overline{z})_A}{4\pi e^\gamma} = \frac{q_A(z + \overline{z}) - q_A(z) - q_A(\overline{z})}{4\pi e^\gamma}$$

satisfies the relation

$$-2 \log ||\Psi_{f_{0,A}}(z, h)||_A = \vartheta^\star_{f_{0,A}}(z, h)$$

for all $(z, h) \in D_{V_A} \times \mathrm{GSpin}_{V_A}(\mathbf{A}_f)$. Hence by Borcherds' theorem, we have that

$$\mathrm{Div}(\Psi_{f_{0,A}}) = Z(f_{0,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{m \in \mathbf{Q}_{>0}} c^+_{f_{0,A}}(\mu, -m) \cdot Z_A(m, \mu).$$

Proof. See Howard-Madapusi Pera [26, Theorem 9.1.1], which gives a refinement of the original theorem of Borcherds [4, Theorem 13.3], as described in [32, Theorem 1.2]. In particular, [26, Theorem 9.1.1] shows that the Borcherds product is defined over $\mathbf{Q}$, from which we deduce that it takes algebraic values. $\square$

We have the following relevant generalization when $f_{0,A} \in H_{0,-\Lambda_A}$ is not a weakly holomorphic form:

Theorem 1.3 (Bruinier, Bruinier-Funke)**.** *Let $f_{0,A} \in H_{0,-\Lambda_A}$ be a harmonic weak Maass form of weight 0 and representation $r_{\psi_0,-\Lambda_A}$. The regularized theta lift $\vartheta^\star_{f_{0,A}}$ is a smooth function on $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A}) \backslash Z(f_{0,A})$, with a logarithmic singularity along $-2\log Z(f_{0,A})$. Moreover,*

- *The $(1,1)$ form $dd^c\vartheta^\star_{f_{0,A}}(z, h)$ has an analytic continuation to a smooth form on $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$, and satisfies the Green current equation $dd^c[\vartheta^\star_{f_{0,A}}(z, h)] + \delta_{Z(f_{0,A})} = [dd^c\vartheta^\star_{f_{0,A}}(z, h)]$, where $\delta_{Z(f_{0,A})}$ denotes the Dirac current of the divisor $Z(f_{0,A})$.*
- *The regularized theta lift $\vartheta^\star_{f_{0,A}}$ is an eigenfunction for the generalized Laplacian operator Δ_z defined on $z \in D_{V_A}$, with eigenvalue $c^+_{f_{0,A}}(0,0)/2$.*

In particular, the regularized theta lift $\vartheta^{\star}_{f_{0,A}}$ can be identified with the automorphic Green's function $G_{Z(f_{0,A})}$ for the divisor $Z(f_{0,A})$, giving us an arithmetic divisor $\widehat{Z}(f_{0,A}) = (Z(f_{0,A}), \vartheta^{\star}_{f_{0,A}})$ on $\mathrm{Sh}_{U_A}(D_{V_A}, \mathrm{GSpin}_{V_A})$.

Proof. See [8, Theorems 4.2 and 4.3] and [6], as well as [7, Proposition 5.6, Theorem 6.1, Theorem 6.2]. $\square$

We develop this setup by evaluating these automorphic Green's functions $G_{Z(f_{0,A})}$ along the "geodesic" anisotropic subspaces of signature $(1,1)$ corresponding to the fractional ideals $\mathfrak{a}_{\mathbf{Q}}$ of the real quadratic field K to derive a novel integral presentation for the sum $L(1/2, \Pi \otimes \chi) + L'(1/2, \Pi \otimes \chi)$. In this way, we recover a formula for the central derivative value $L'(1/2, \Pi \otimes \chi) = L'(E/K, \chi, 1)$ when the ersatz Heegner hypothesis (Hypothesis 2.1) is assumed – so that the central value $L(1/2, \Pi \otimes \chi) = L(E/K, \chi, 1)$ is forced to vanish by the symmetric functional equation. We also derive a novel formula for the central value in the complementary setting for the central value $L(1/2, \Pi \otimes \chi) = L(E/K, \chi, 1)$, giving a new realization the toric period formula derived by Popa [37] (and more generally/abstractly by Waldspurger [49]).

Let us now summarize the main results, given in more detail in Theorem 4.19 and Corollary 4.20. We first decompose the Siegel theta series $\theta_{\Lambda_A}(\tau, z, h_f)$. To be more precise, recall that we split each quadratic space $V_A = \mathbf{Q} \oplus \mathbf{Q} \oplus \mathfrak{a}_{\mathbf{Q}}$ with $q_A(x, y, \lambda) = \mathbf{N}\mathfrak{a}^{-1}\mathbf{N}(\lambda) - xy$ into subspaces $V_{A,1} := \mathbf{Q} \oplus \mathbf{Q}$ with $q_{A,1}(x, y) = -xy$ and $V_{A,2} = \mathfrak{a}_{\mathbf{Q}}$ with $q_{A,2}(\lambda) = \mathbf{N}\mathfrak{a}^{-1}\mathbf{N}(\lambda)$. Note that each subspace $(V_{A,j}, q_{A,j})$ has signature $(1,1)$. Let us for each index $j = 1, 2$ consider the corresponding sublattice

$$\Lambda_{A,j} := \Lambda_A \cap V_{A,j}$$

of signature $(1,1)$. We then have for each index $j = 1, 2$ the corresponding Siegel theta series

$$\theta_{\Lambda_{A,j}}(\tau, z, h_f) : \mathfrak{H} \times D_{V_{A,j}} \longrightarrow \mathcal{S}_{\Lambda_{A,j}},$$

of weight zero and representation $r_{\psi_0, \Lambda_{A,j}}$, where $D_{V_{A,j}}$ denotes the corresponding subdomain of D_{V_A}. Since we evaluate at points $z_{V_{A,2}} \in D_{V_{A,2}}$ and $h_f \in \mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)$ in our main calculation, and since we have a natural splitting $\Lambda_A = \Lambda_{A,1} + \Lambda_{A,2}$, we decompose each Siegel theta series $\theta_{\Lambda_A}(\tau, z_{V_{A,2}}, h_f)$ as

$$\theta_{\Lambda_A}(\tau, z_{V_{A,2}}, h_f) = \theta_{\Lambda_{A,1}}(\tau, 1, 1) \otimes \theta_{\Lambda_{A,2}}(\tau, z_{V_{A,2}}, h_f).$$

The Siegel-Weil theorem (Theorem 4.10 and Corollary 4.11) allows us to interpret the average

$$2 \int\limits_{\mathrm{SO}(V_{A,2})(\mathbf{Q}) \backslash \mathrm{SO}(V_{A,2})(\mathbf{A})} \theta_{\Lambda_{A,2}}(\tau, z_{V_{A,2}}, h_f) dh$$

as the value at $s = 0$ of a certain $\mathcal{S}_{\Lambda_{A,2}}$-valued Eisenstein series $E_{\Lambda_{A,2}}(\tau, s; 0)$ of weight 0, which in turn can be interpreted in terms of the image under the antilinear differential weight-lowering operator ξ_2 of a related derivative Eisenstein series $E'_{\Lambda_{A,2}}(\tau, 0; 2)$ of weight two. Writing L_2 to denote the standard weight lowering operator, we have the relation

$$L_2 E'_{\Lambda_{A,2}}(\tau, 0; 2) = \frac{1}{2} \cdot E_{\Lambda_{A,2}}(\tau, 0; 0) - \frac{1}{2} \cdot E'_{\Lambda_{A,2}}(\tau, 0; 0).$$

We consider the Fourier series expansion

$$E'_{\Lambda_{A,2}}(\tau, 0; 2) = \sum_{\mu \in \Lambda_{A,2}^{\#}/\Lambda_{A,2}} \sum_{m \in \mathbf{Q}} b_{\Lambda_{A,2}}(\mu, m, v) e(m\tau) \cdot \mathbf{1}_{\mu},$$

we can then consider the coefficients for each pair (μ, m) by

$$\kappa_{\Lambda_{A,2}}(\mu, m) = \lim_{v \to \infty} \begin{cases} b_{\Lambda_{A,2}}(\mu, m, v) & \text{if } \mu \neq 0 \text{ or } m \neq 0 \\ b_{\Lambda_{A,2}}(\mu, m, v) - \log(v) & \text{if } \mu = m = 0 \end{cases},$$

and form from these the series

$$\mathcal{E}_{\Lambda_{A,2}}(\tau) = \sum_{\mu \in \Lambda_{A,2}^{\#}/\Lambda_{A,2}} \sum_{m \in \mathbf{Q}} \kappa_{\Lambda_{A,2}}(\mu, m) e(m\tau) \cdot \mathbf{1}_{\mu}.$$

Note that $\mathcal{E}_{\Lambda_{A,2}}(\tau)$ is the holomorphic part of $E'_{\Lambda_{A,2}}(\tau,0;2)$. As we shall see, this holomorphic part $\mathcal{E}_{\Lambda_{A,2}}(\tau)$ plays a role analogous to that of the holomorphic projection in the theorem of Gross-Zagier [23] for the corresponding integral presentation of $L'(1/2,\Pi\otimes\chi)$ with K an imaginary quadratic field. Let

$$\begin{aligned}
&\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right) \\
(3)\qquad &= \mathrm{CT}\left(\sum_{\substack{\mu_1\in\Lambda_{A,1}^{\#}/\Lambda_{A,1},\mu_2\in\Lambda_{A,1}^{\#}/\Lambda_{A,2}\\ \mu_1+\mu_2\equiv\mu \bmod \Lambda_A}}\ \sum_{m,m_1,m_2\in\mathbf{Q}_{\geq 0}} c_{f_{0,\eta,A}}^{+}(-m,\mu)c_{\theta_{\Lambda_{A,1}}}(m_1,\mu_1)\kappa_{\Lambda_{A,2}}(m_2,\mu_2)\right) \\
&= \sum_{\substack{\mu_1\in\Lambda_{A,1}^{\#}/\Lambda_{A,1},\mu_2\in\Lambda_{A,1}^{\#}/\Lambda_{A,2}\\ \mu_1+\mu_2\equiv\mu \bmod \Lambda_A}}\ \sum_{\substack{m,m_1,m_2\in\mathbf{Q}_{\geq 0}\\ m_1+m_2=m}} c_{f_{0,\eta,A}}^{+}(-m,\mu)c_{\theta_{\Lambda_{A,1}}}(m_1,\mu_1)\kappa_{\Lambda_{A,2}}(m_2,\mu_2)
\end{aligned}$$

denote the constant coefficient of $\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle$. Observe that since there are only a finite number of $m\geq 0$ for which $c_{f_{0,\eta,A}}^{+}(-m,\mu)\neq 0$ for any $\mu\in\Lambda_A^{\#}/\Lambda_A$ and $\kappa_{\Lambda_{A,2}}(m_2,\mu_2)\neq 0$ only for $m_2\geq 0$ for any $\mu_2\in\Lambda_{A,2}^{\#}/\Lambda_{A,2}$, this sum (3) is finite. Let h_K denote the class number of K, and ϵ_K the fundamental unit, so that $\epsilon_K=\frac{1}{2}(t+u\sqrt{d_K})$ is the least integral solution (with u minimal) to Pell's equation $t^2-d_Ku^2=4$.

Theorem 1.4 (Theorem 4.19, Corollary 4.7)**.** *Let us retain the setup above. Hence, let E be an elliptic curve of conductor N defined over $\mathbf{Q}$, with $\pi=\pi(f)$ the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by the cuspidal eigenform $f(\tau)=\sum_{m\geq 1}a_f(m)e(m\tau)\in S_2^{\mathrm{new}}(\Gamma_0(N))$ parametrizing E. Let K be a real quadratic field of discriminant d_K prime to N and even quadratic Dirichlet character $\eta=\eta_{K/\mathbf{Q}}$. Writing $V=(V,q)$ for the rational quadratic space of signature $(2,2)$ determined by the vector space $\mathbf{Q}\oplus\mathbf{Q}\oplus K$ and quadratic form $q(x,y,\lambda)=\mathbf{N}(\lambda)-xy$, let $U=U(\Pi)$ denote the compact open subgroup of $\mathrm{GSpin}_V(\mathbf{A}_f)$ determined by the level $H=H(\Pi)\subset\mathrm{GL}_2(\mathbf{A}_{K,f})$ of the quadratic basechange lifting $\Pi=\mathrm{BC}_{K/\mathbf{Q}}(\pi)$ of $\pi=\pi(f)$ to $\mathrm{GL}_2(\mathbf{A}_K)$. Let $\Lambda\subset V$ denote the maximal lattice corresponding to this compact open subgroup $U\subset\mathrm{GSpin}_V(\mathbf{A})$.*

Let χ be any ring class character of K of conductor c prime to d_KN (assuming such a character exists), which we view as a character $\chi:\mathrm{Pic}(\mathcal{O}_c)\to\mathbf{S}^1$ on the class group $\mathrm{Pic}(\mathcal{O}_c)$ of the order $\mathcal{O}_c\subset\mathcal{O}_K$ of conductor c in K. Fixing an integral ideal representative $\mathfrak{a}\subset\mathcal{O}_K$ for each class $A=[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)$, consider the rational quadratic space $V_A=(V_A,q_A)$ of signature $(2,2)$ defined by the vector space $V_A=\mathbf{Q}\oplus\mathbf{Q}\oplus\mathfrak{a}_{\mathbf{Q}}$ and the quadratic form $q_A(x,y,\lambda)=\mathbf{N}\mathfrak{a}^{-1}\mathbf{N}(\lambda)-xy$. Let GSpin_{V_A} denote corresponding spin group, with $U_A=U\cap\mathrm{GSpin}_{V_A}(\mathbf{A}_f)$ the corresponding level structure, and $\Lambda_A=\Lambda\cap V_A$ the corresponding lattice. Let $\mathrm{Sh}_{U_A}(D_{V_A},\mathrm{GSpin}_{V_A})$ denote the corresponding spin Shimura variety, which can be viewed classically as the Hilbert modular surface with congruence subgroup determined by Λ_A. For each index $j=1,2$, we consider the subspace $V_{A,j}=(V_{A,j},q_{A,j})$ of signature $(1,1)$ with corresponding lattice $\Lambda_{A,j}=\Lambda_A\cap V_{A,j}$ and level $U_{A,j}=U_A\cap\mathrm{GSpin}(V_{A,j})(\mathbf{A}_f)$ defined as follows: For $j=1$, we restrict to $V_{A,1}=\mathbf{Q}\oplus\mathbf{Q}$ with $q_{A,1}=q_A|_{V_{A,1}}$, and for $j=2$ we restrict to the fractional ideal representative $V_{A,2}=\mathfrak{a}_{\mathbf{Q}}$ with $q_{A,2}=q_A|_{V_{A,2}}$. Let

$$Z(V_{A,2})=\mathrm{GSpin}_{V_{A,2}}(\mathbf{Q})\backslash D_{V_{A,2}}\times\mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)/U_{A,2}\subset\mathrm{Sh}_{U_A}(D_{V_A},\mathrm{GSpin}_{V_A})$$

denote the corresponding geodesic of points. Let $\theta_{\Lambda_{A,1}}(\tau,z,h)$ denote the Siegel theta series associated to the lattice $\Lambda_{A,1}$ as above, and $\mathcal{E}_{\Lambda_{A,2}}$ the holomorphic part of the Eisenstein series $E'_{\Lambda_{A,2}}(\tau,0;2)$ constructed via Siegel-Weil after averaging the Siegel theta series $\theta_{\Lambda_{A,2}}(\tau,z,h)$ over the geodesic $(z,h)\in Z(V_{A,2})$ as a lifting under the weight-lowering operator L_2. Let $f_{0,\eta,A}=f_{0,\eta,A}^{+}+f_{0,\eta,A}^{-}\in H_{0,-\Lambda_A}$ be the harmonic weak Maass form of weight 0 and representation $r_{\psi_0,-\Lambda_A}$ whose image $g_{A,\eta}(\tau)=\xi_0(f_{0,\eta,A})(\tau)\in S_{2,\Lambda_A}$ under the antilinear differential operator $\xi_0:H_{0,-\Lambda_A}\to S_{2,\Lambda_A}$ is the vector-valued lift of the twisted eigenform $f\otimes\eta(\tau)=\sum_{m\geq 1}a_f(m)\eta(m)e(m\tau)\in S_2^{\mathrm{new}}(\Gamma_0(d_K^2N))$. Writing $\mathrm{Vol}(U_{A,2})$ to denote the volume of $U_{A,2}$ with *respect to a certain chioce of Haar measure on* $\mathrm{GSpin}_{V_A}(\mathbf{A})$*, we derive the integral presentation*

$$L'(1/2,\Pi\otimes\chi)+L(1/2,\Pi\otimes\chi)=L^{\star\prime}(E/K,\chi,1)+L^{\star}(E/K,\chi,1)$$
$$=\frac{\sqrt{d_K}}{\log\epsilon_K\cdot h_K}\sum_{\substack{A\in\mathrm{Pic}(\mathcal{O}_c)\\ A=[\mathfrak{a}]}}\chi(A)\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau),\theta_{\Lambda_{A,1}}\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right)-\mathrm{vol}(U_{A,2})\sum_{(z_{V_{A,2}},h)\in Z(V_{A,2})}\vartheta^{\star}_{f_{0,\eta,A}}(z_{V_{A,2}},h)\right).$$

Equivalently, writing $G_{Z(f_{0,\eta,A})}$ *for each class* A *to denote the automorphic Green's function for the divisor*

$$Z(f_{0,\eta,A})=\sum_{\mu\in\Lambda_A^{\#}/\Lambda_A}\sum_{m\in\mathbf{Q}_{>0}}c^{+}_{f_{0,\eta,A}}(\mu,-m)\cdot Z_A(\mu,m)$$

given by linear combination of special (Hirzebruch-Zagier) divisors $Z_A(\mu,m)$ *on* $\mathrm{Sh}_{U_A}(D_{V_A},\mathrm{GSpin}_{V_A})$*, let*

$$G_{Z(f_{0,\eta,A})}(V_{A,2})=\sum_{(z,h)\in D_{V_{A,2}}\times\mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)}\vartheta^{\star}_{f_{0,\eta,A}}(z,h)$$

denote the sum along the geodesic $Z(V_{A,2})$ *in* $\mathrm{Sh}_{U_A}(D_{V_A},\mathrm{GSpin}_{V_A})$*. We obtain the integral presentation*

$$L'(1/2,\Pi\otimes\chi)+L(1/2,\Pi\otimes\chi)=L^{\star\prime}(E/K,\chi,1)+L^{\star}(E/K,\chi,1)$$
$$=\frac{\sqrt{d_K}}{\log\epsilon_K\cdot h_K}\sum_{\substack{A\in\mathrm{Pic}(\mathcal{O}_c)\\ A=[\mathfrak{a}]}}\chi(A)\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau),\theta_{\Lambda_{A,1}}\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right)-\mathrm{vol}(U_{A,2})G_{Z(f_{0,\eta,A})}(V_{A,2})\right).$$

In particular, if we assume the ersatz Heegner hypothesis (Lemma 1.1, Hypothesis 2.1) that the inert level N^- is given by the squarefree product of an odd number of primes, then $L(1/2,\Pi\otimes\chi)=0$ by symmetric functional equation (8), and we obtain a formula for the central derivative value $L'(1/2,\Pi\otimes\chi)$.

Corollary 1.5. *Assume the ersatz Heegner hypothesis (Lemma 1.1, Hypothesis 2.1) that the inert level* N^- *is given by the squarefree product of an odd number of primes. We have the central derivative value formula*

$$L'(1/2,\Pi\otimes\chi)=L^{\star\prime}(E/K,\chi,1)$$
$$=\frac{\sqrt{d_K}}{\log\epsilon_K\cdot h_K}\sum_{\substack{A\in\mathrm{Pic}(\mathcal{O}_c)\\ A=[\mathfrak{a}]}}\chi(A)\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau),\theta_{\Lambda_{A,1}}\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right)-\mathrm{vol}(U_{A,2})G_{Z(f_{0,\eta,A})}(V_{A,2})\right).$$

The analogous formula for central values $L(1/2,\Pi\otimes\chi)=L(1/2,\pi\times\pi(\chi))=L^{\star}(E/K,\chi,1)$ in the setting where the generic root number $\eta(-N)=\eta(N)=+1$ is given by Popa [37, § 1, Theorem 6.3.1], which develops Waldspurger's theorem [49] into an exact toric period formula for these central values, and moreover generalizes the corresponding formula of Gross [22] for for the analogous setup with K an imaginary quadratic field. Roughly speaking, Waldspurger's theorem [49] equates the nonvanishing of the central value $L(1/2,\pi\times\pi(\chi))$ with that of the toric period integral

$$\int_{\mathbf{A}_K^{\times}/K^{\times}}\varphi(t)\chi(t)dt,$$

for $\varphi\in\pi^{\mathrm{JL}}$ a vector in the Jacquet-Langlands lift π^{JL} of π to an *indefinite* quaternion algebra B over $\mathbf{Q}$ with ramification given by the inert level: $\mathrm{Ram}(B)=\{q\mid N^-\}$. Popa [37] gives an exact and even classical formula for $L(1/2,\pi\times\pi(\chi))$ as such as toric integral, which according to the discussion in [37, § 6] can be viewed as a twisted sum over geodesic on the modular curve $X_0(N)$ parametrizing E. Our Theorem 4.19 can be viewed as an analogue of Popa's theorem for the central derivative values $L'(1/2,\Pi\otimes\chi)=L'(1/2,\pi\times\pi(\chi))$ when the generic root number is $\eta(-N)=\eta(N)=-1$ (i.e. when Hypothesis 2.1 holds), and moreover as a variant for the central values $L(1/2,\Pi\otimes\chi)=L(1/2,\pi\times\pi(\chi))$ when the generic root number is $\eta(-N)=\eta(N)=+1$ (i.e. when Hypothesis 2.1 does not hold). Let us also remark that in the setting we consider here (when the ersatz Heegner hypothesis 2.1 holds), the corresponding Jacquet-Langlands lift π^{JL} is an automorphic form on the *definite* quaternion algebra D over $\mathbf{Q}$ with ramification given by the inert level N^-, so $\mathrm{Ram}(D)=\{q\mid\infty N^-\}$. It is tempting to ask if one could develop an arithmetic formula resembling that of Gross [22] to describe the central derivative values $L'(1/2,\pi\times\pi(\chi))$ in this way. However, we remark that there is no connection a priori[5] to the modular curve $X_0(N)$ via the Jacquet-Langlands correspondence in this way, nor the Hilbert modular variety associated to the basechange Π. That is, the double coset space $D^\times(\mathbf{Q})\backslash D^\times(\mathbf{A})/U(\pi)$ through which any $\varphi \in \pi^{\mathrm{JL}}$ factors is a finite set. On the other hand, working with the basechange L-function $L(1/2, \Pi \otimes \chi)$ and its connection to spin Shimura varieties as outlined above, we also obtain from Theorem 4.19 the following result when Hypothesis 2.1 does not hold.

Corollary 1.6. *Assume that the inert level N^- is given by the squarefree product of an even number of primes, so that the sign $\eta(-N) = \eta(N) = \eta(N^-) = 1$ of the functional equation is "even", and hence that $L'(1/2, \Pi \otimes \chi) = L^{\star,\prime}(E/K, \chi, 1)$ vanishes. Then, we have the central value formula*

$$\begin{aligned} &L(1/2, \Pi \otimes \chi) = L^\star(E/K, \chi, 1) \\ &= \frac{\sqrt{d_K}}{\log \epsilon_K \cdot h_K} \sum_{\substack{A \in \mathrm{Pic}(\mathcal{O}_c) \\ A = [\mathfrak{a}]}} \chi(A) \left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}} \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle \right) - \mathrm{vol}(U_{A,2}) G_{Z(f_{0,\eta,A})}(V_{A,2}) \right). \end{aligned}$$

Moreover, comparing Corollary 1.6 with the formula of Popa [37, Theorem 6.3.1], we derive the following relation in the special case where conductor N is squarefree with trivial inert level $N^- = 1$, so that each prime dividing $N = N^+$ splits in K, and χ is a ring class character of conductor $c = 1$ factoring through the narrow class group of K. Recall that in this setting with $N = N^+$ split, there exist optimal embeddings

$$\alpha : K \longrightarrow M_2(\mathbf{Q}), \quad \alpha(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for integers $a, b, c \in \mathbf{Z}$ satisfying the constraints

$$a^2 + bc = d_K, \quad 2N \mid c, \quad 2 \mid b, \quad \gcd(a, b/2, c/2) = 1,$$

with $\alpha(K) \cap B_0(N) = \alpha(\mathcal{O}_K)$ for $B_0(N)$ the subgroup of upper-triangular matrices whose lower left entry is divisible by N. Fixing a square root a_0 of $d_K \bmod 4N$, such an optimal embedding $\alpha : K \to M_2(\mathbf{Q})$ is said to be *oriented* if $a_0^2 \equiv d_K \bmod 4N$. We write Ξ_N to denote the set of such oriented optimal embeddings, noting that each $\alpha \in \Xi_N$ corresponds to a binary quadratic form

$$Q_\alpha(x, y) = -\frac{c}{2}x^2 + axy + \frac{b}{2}y^2,$$

and that there is a natural action of the congruence subgroup $\Gamma_0(N)$ on Ξ_N for which quotient $\Xi_N/\Gamma_0(N)$ can be identified with the narrow class group of K. In this setting we have the central value formula

$$\begin{aligned} &L(1/2, \Pi \otimes \chi) = L^\star(E/K, \chi, 1) \\ &= \frac{\sqrt{d_K}}{\log \epsilon_K \cdot h_K} \sum_{\substack{A \in \mathrm{Pic}(\mathcal{O}_c) \\ A = [\mathfrak{a}]}} \chi(A) \left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^+(\tau), \theta_{\Lambda_{A,1}} \otimes \mathcal{E}_{\Lambda_{A,2}}(\tau) \rangle\rangle \right) - \mathrm{vol}(U_{A,2}) G_{Z(f_{0,\eta,A})}(V_{A,2}) \right) \\ &= 4\sqrt{d_K} \cdot \left| \sum_{\alpha \in \Xi_N/\Gamma_0(N)} \chi^{-1}(\alpha) \int_{z_\alpha}^{M_\alpha z_\alpha} f(z) dz \right|^2. \end{aligned} \tag{4}$$

Here, fixing a square root $i = \sqrt{-1} \in \mathfrak{H}$, we write $z_\alpha \in X_0(N)$ to denote the point determined by

$$z_\alpha = \begin{pmatrix} \frac{a + d_K}{c} & \frac{a - d_K}{c} \end{pmatrix} i.$$

Expanding the fundamental unit ϵ_K explicitly as $\epsilon_K = m + n\sqrt{d_K}$ for some $m, n \in \mathbf{Z}/2\mathbf{Z}$, we also write $M_\alpha \in \Gamma_0(N)$ to denote the matrix determined by

$$M_\alpha = \begin{pmatrix} m + na & nb \\ nc & m - na \end{pmatrix} \in \Gamma_0(N).$$

That is, the integrals on the right hand side of (4) are taken over the geodesic cycles on $X_0(N)$ determined by semicircles of $\mathfrak{H}$ connecting z_α and $M_\alpha z_\alpha = \alpha(\epsilon_K) z_\alpha$, and passing through the real points $(a \pm \sqrt{d_K})/2$. Moreover, the modular curve $X_0(N)$ can be realized as the compactification of the spin Shimura variety

[5] Note that K does not embed into the quaternion algebra D in this setting.

$Y_0(N) = \mathrm{Sh}_{U'}(D_{V'}, \mathrm{GSpin}_{V'})$ associated to the quadratic space $V' = (V', q')$ of signature $(1, 2)$ given by the vector space $V' = M_2^{\mathrm{tr}=0}(\mathbf{Q})$ and the quadratic form $q'(x) = N \det(x)$, with level $U' = U'(\pi) \subset \mathrm{GSpin}_{V'}(\mathbf{A}_f)$ corresponding to $\Gamma_0(N)$ by the explicit description given in [8, §7.1]. It is tempting to ask if in general, the various ring classes $A \in \mathrm{Pic}(\mathcal{O}_c)$ give rise to embeddings of rational quadratic spaces $j_A : (V', q') \longrightarrow (V_A, q_A)$, and hence the corresponding Shimura varieties $j_A : Y_0(N) = \mathrm{Sh}_{U'}(D_{V'}, \mathrm{GSpin}_{V'}) \longrightarrow \mathrm{Sh}_U(D_{V_A}, \mathrm{GSpin}_{V_A})$, in such a way that the pullbacks of the divisors $Z(f_{0,\eta,A})$ and more generally the Hirzebruch-Zagier divisors $Z_A(\mu, m)$ might give new constructions of points in the Mordell-Weil group $E(K[c])$, including the provenance of the elusive "Start-Heegner" points in the setting of Corollary 1.5. We expect that such embeddings exist in the split case $N = N^+$ corresponding to central values, and that a suitable application of level-raising congruences for the Jacquet-Langlands lifts $\varphi \in \pi(f)^{\mathrm{JL}}$ might always allow us to reduce to this setting, in the style of the argument of Bertolini-Darmon [3]. However, we refrain from posing any conjecture.

1.0.1. *Some remarks.* Let us now make a few more comments about Theorem 1.4.

(i) Proposition 4.3 below suggests there should be an integral presentation for $L'(1/2, \Pi \otimes \chi)$ in terms of Fourier coefficients of half-integral weight forms. For instance, we expect that some application of the derivations shown in Bump-Friedberg-Hoffstein [12] links these central derivative values to the constant coefficients of certain Eisenstein series on the metaplectic cover of $\mathrm{GSp}_4(\mathbf{A})$. We intend to develop these connections in subsequent work.

(ii) The presentation of Theorem 4.19 applies to both totally odd and totally even ring class characters. In fact, our main derivations would apply to any Hecke character of K, but restrict our focus to the interesting arithmetic setup where the corresponding functional equation is symmetric.

(iii) The regularized theta lifts $\vartheta^{\star}_{f_{0,\eta,A}} = G_{Z(f_{0,\eta,A})}$ can be related to the theta lifts constructed by Kudla-Millson in [35] by the arguments of Bruinier-Funke [7, Theorems 1.4 and 1.5]. Such relations, which hold for any signature (p, q), suggest another potential geometric development of this formula.

(iv) The role played by the holomorphic projection in Gross-Zagier [23] is replaced here by the holomorphic part $\mathcal{E}_{L_{A,2}}(s, \tau)$ of the derivative Eisenstein series $E'_{L_{A,2}}(s, \tau; 2)$ appearing in our formula. More precisely, this derivative Eisenstein series appears after applying the Siegel-Weil formula to the theta series $\theta_{L_{A,2}}(\tau)$ corresponding to each signature $(1, 1)$ quadratic subspace $(V_{A,2}, q_{A,2})$, and then after interpreting the corresponding value at $s_0 = 0$ of the weight zero Eisenstein series $E_{L_{A,2}}(s_0, \tau; 0)$ appearing in this way under some Maass weight lowering operator of $E'_{L_{A,2}}(s, \tau; 2)$. Although the underlying weight-zero Eisenstein series $E_{L_{A,2}}(\tau, s; 0)$ in our setup is coherent, i.e. not incoherent in the sense of Kudla (e.g. [32]), we also find it in our main calculation for Theorem 4.19.

(v) Recall that a complex number is said to be a *period* if its real and imaginary parts can be expressed as integrals of rational functions with rational coefficients, over domains in $\mathbf{R}^n$ given by polynomials inequalities with rational coefficients. We expect that the central derivative values $L'(1/2, \Pi \otimes \chi) = L^{\star\prime}(E/K, \chi, 1)$ are always periods (cf. [30, Question 4]), as this would be implied refined conjecture of Birch and Swinnerton-Dyer, and note that this can be deduced in the special cases described in Corollary 1.6 and Corollary 6.1 from Theorem 4.19. We expect more generally that the central derivative values appearing in [8, Theorem 4.7] are periods, as well as the values taken by the regularized theta lifts $\vartheta^{\star}_{f_0}$. Although we can only deduce this in special cases such as (4) above, the following heuristic calculation suggests that the values of the regularized theta lift $\vartheta^{\star}_{f_0}$ at special divisors should always be periods. We can decompose any cuspidal harmonic weak Maass form f_0 into a linear combination of Hejhal-Maass Poincaré series $F_{\mu,m}$ as in [5, Theorem 2.14]. Ignoring issues of absolute convergence, we obtain a corresponding decomposition for the regularized theta lift $\vartheta^{\star}_{f_0}$ into a linear combination of its Poincar series constituents $\vartheta^{\star}_{F_{\mu,m}}$. Evaluated at the "points" we consider, these constituents $\vartheta^{\star}_{F_{\mu,m}}$ can be computed equivalently as rational linear combination of the Gaussian hypergeometric function ${}_2F_1$ at rational (which are known to be periods).

(vi) Although we do not give a formula in terms of cycles on the spin Shimura varieties we consider, we do give a formula for $L'(1/2, \Pi \otimes \chi) = L'(1/2, \pi \times \pi(\chi)) = L^{\star,\prime}(E/K, \chi, 1)$ in terms of the "geodesics" $Z(V_{A,2})$. These are the locally symmetric spaces associated to the signature $(1,1)$ quadratic subspaces $(V_{A,2}, q_{A,2})$, which via the theta correspondence can be related to the locally symmetric space associated to $\mathrm{GL}_2(\mathbf{A})$. This latter characterization should have a natural interpretation in terms of the Borel-Serre compactification of a Shimura variety for $\mathrm{GSp}_4(\mathbf{A})$. This idea of realizing locally symmetric spaces in the boundaries of Borel-Serre compactifications of ambient Shimura varieties, which seems to go back to Clozel, is used crucially in the constructions by [25] and [41] of Galois representations for GL_n over totally real fields. It would be very interesting if we could describe the sums of regularized theta lifts $\sum_{(z,h)\in Z(V_{A,2})} \vartheta^{\star}_{f_{0,\eta,A}}(z,h) = G_{Z(f_{0,\eta,A})}(Z(V_{A,2}))$ explicitly in terms cycles on these boundaries of the Borel-Serre compactifications of such ambient Shimura varieties. We hope to return to this idea.

1.0.2. *Applications towards Birch-Swinnerton-Dyer.* Theorem 4.19 also suggests a possible origin for certain "real multiplication" points in the $K[c]$-rational Mordel-Weil groups $E(K[c])$ coming from geodesic cycles $Z(V_{A,2})$ related to the spin Shimura varieties $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$ and orthogonal Shimura varieties $\mathrm{Sh}_U(O(V), D)$ corresponding to the Hilbert modular variety $Y(\Gamma) = \Gamma\backslash\mathfrak{H}^2$. In this spirit, we also describe interpretations of these formulae in terms of the corresponding homology groups (cf. [37, § 6.4], [**?**, §8]), and also how the refined Birch and Swinnerton-Dyer conjecture suggests new characterizations of the Tate-Shafarevich group $Ш_E(K[c])$ and regulator term $R_E(K[c])$. We refer to (99), (100) and below for more details of what can be deduced here. One consequence is the following result.

Corollary 1.7 (Theorem 6.1)**.** *Assume the ersatz Heegner hypothesis (Lemma 1.1, Hypothesis 2.1) that the inert level N^- is given by the squarefree product of an odd number of primes, then $L(1/2, \Pi \otimes \chi) = 0$ by symmetric functional equation* (8)*. Writing E again to denote the underlying elliptic curve over $\mathbf{Q}$, we write $E^{(d_K)}$ to denote its quadratic twist. Let us also assume that E has semistable reduction so that its conductor N is squarefree, with N coprime to the discriminant d_K of K, and for each prime $p \geq 5$:*

- *The residual Galois representations $E[p]$ and $E^{(d_K)}[p]$ attached to E and $E^{(d_K)}$ are irreducible,*
- *There exists a prime divisor $l \mid\mid N$ distinct from p where the residual representation $E[p]$ is ramified, and a prime divisor $q \mid\mid Nd_K$ distinct from p where the residual representation $E^{(d_K)}[p]$ is ramified.*

For either elliptic curve $A = E, E^{(d_K)}$, let us write $Ш_A(\mathbf{Q})$ to denote the Tate-Shafarevich group, with $T_A(\mathbf{Q})$ the product over local Tamagawa factors, and ω_A a fixed invariant differential for $A/\mathbf{Q}$. Suppose that $\mathrm{ord}_{s=1} L^\star(E/K, 1) = 1$*, so that either $L^\star(E, 1) = L(1/2, \pi)$ or the quadratic twist $L^\star(E^{(d_K)}, 1) = L(1/2, \pi \otimes \eta)$ vanishes. Writing $[e, e]$ to denote the regulator of either E or $E^{(d_k)}$ according to which factor vanishes, we have the unconditional identity up to powers of* 2 *and* 3*:*

$$\frac{\#Ш_E(\mathbf{Q}) \cdot \#Ш_{E^{(d_K)}}(\mathbf{Q}) \cdot [e,e] \cdot T_E(\mathbf{Q}) \cdot T_{E^{(d_K)}}(\mathbf{Q})}{\#E(\mathbf{Q})^2_{\mathrm{tors}} \cdot \#E^{(d_k)}(\mathbf{Q})^2_{\mathrm{tors}}} \cdot \int_{E(\mathbf{R})} |\omega_E| \cdot \int_{E^{(d_K)}(\mathbf{R})} |\omega_{E^{(d_k)}}|$$

$$= \frac{\sqrt{d_K}}{\log \epsilon_K} \sum_{A \in \mathrm{Pic}(\mathcal{O}_K)} \left(\mathrm{CT}\left(\langle\langle f^+_{0,\eta,A}(\tau), \theta_{L_{A,1}} \otimes \mathcal{E}_{L_{A,2}}(\tau) \rangle\rangle \right) - \mathrm{vol}(U_{A,2}) \sum_{(z_{V_{A,2}}, h) \in Z(V_{A,2})} \vartheta^\star_{f_{0,\eta,A}}(z_{V_{A,2}}, h) \right)$$

$$= \frac{\sqrt{d_K}}{\log \epsilon_K} \sum_{A \in \mathrm{Pic}(\mathcal{O}_K)} \left(\mathrm{CT}\left(\langle\langle f^+_{0,\eta,A}(\tau), \theta_{L_{A,1}} \otimes \mathcal{E}_{L_{A,2}}(\tau) \rangle\rangle \right) - \mathrm{vol}(U_{A,2}) G_{Z(f_{0,\eta,A})}(Z_{A,2}) \right).$$

In particular, the value on the left-hand side is known to be a period (see [30, §4]*).*

It would be interesting to develop these connections further, especially in connection to the real quadratic Borcherds products studied by [16], perhaps leading to a global analogue of Darmon's [15, Conjecture 5.6].

Outline. We first describe the setup with L-functions and their functional equations with more detail in §2, followed by Eulerian integral presentations in §3, then connections to spin and spin Shimura varieties in §4. We give some relevant abstract discussion of the theta correspondence leading to Proposition 4.1 and Theorem 4.2 realizing the quadratic basechange lifting in an explicit way. We also present Proposition 4.3 and (28), explaining the connection to the Shimura-Shintani-Waldspurger correspondence. We describe regularized theta liftings following [32] and [8, §§2-4] in §4.4, leading to the main Theorem 4.19 and Corollary 4.7. Our main results are then derived in Theorem 4.16, Theorem 4.19, and Corollary 4.20. We describe the connection to the setting of Hilbert modular forms in §5. Finally, we describe the connection to homology classes in §6.1 and arithmetic invariants associated with Mordell-Weil groups via the refined Birch and Swinnerton-Dyer conjecture in §6.2.

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2. Review of equivalent L-functions and their functional equations

Recall that we let E be an elliptic curve defined over the rationals, corresponding via modularity to a cuspidal newform $f \in S_2(\Gamma_0(N))$. Let us write $\pi = \otimes_v \pi_v$ to denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ generated by f, so that we have identifications of completed L-functions

$$L^\star(E, s) = L(s - 1/2, \pi) = \prod_{v \leq \infty} L(s - 1/2, \pi_v) \tag{5}$$

Again, we fix K a real quadratic field of discriminant d_K prime to the conductor N, and write $\eta = \eta_{K/\mathbf{Q}}$ to denote the corresponding Dirichlet character. As well, we fix a ring class character χ of K of some conductor $c \in \mathbf{Z}_{\geq 1}$ coprime to the product $d_K N$, and let us write $K[c]$ for the ring class extension of K of conductor c. Inspired by the conjecture of Darmon [15, Conjecture 5.6] and the theorem of Gross-Zagier [23], as well as the various related theorems (e.g. those of [49] and [37]) we seek to detect Heegner-like points in the Mordell-Weil group $E(K[c])$ of $K[c]$-rational points through the study of integral presentations of the central derivative value $L^{\star\prime}(E/K, \chi, 1)$ of the completed Hasse-Weil L-function $L^\star(E/K, \chi, s)$ of E basechanged to K and twisted by χ. By the theory of Rankin-Selberg convolution (cf. e.g. [23], [11, Ch. 3]), we can deduce from modularity, or rather from the corresponding identification of L-functions (5), that $L(E/K, \chi, s)$ has an analytic continuation to all $s \in \mathbf{C}$ given through its identification with the Rankin-Selberg L-function $L(s, \pi \times \pi(\chi))$ of π times the representation $\pi = \otimes_v \pi(\chi)_v$ of $\mathrm{GL}_2(\mathbf{A})$ induced by π:

$$L^\star(E/K, \chi, s) = L(s - 1/2, \pi \times \pi(\chi)) = \prod_{v \leq \infty} L(s - 1/2, \pi_v \times \pi(\chi)_v). \tag{6}$$

On the other hand, recall that by the theory of cyclic basechange (in the sense of [36], [2]), we can attach to π a cuspidal automorphic representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}$ of $\mathrm{GL}_2(\mathbf{A}_K)$. It is then well-known that the Rankin-Selberg L-function $L(s, \pi \times \pi(\chi))$ on $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ is equivalent to the (twisted) standard or Godement-Jacquet L-function $L(s, \Pi \otimes \chi)$ on $\mathrm{GL}_2(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$. This gives us another equivalence of L-functions

$$L^\star(E/K, \chi, s) = L(s - 1/2, \Pi \otimes \chi) = \prod_{w \leq \infty} L(s - 1/2, \Pi_w \otimes \chi_w), \tag{7}$$

where we view χ as an idele class character $\chi = \otimes_w \chi_w$ of K having trivial archimedean component $\chi_\infty \equiv 1$.

In each of these presentations (6) and (7), the automorphic L-function $L(s, \pi \times \pi(\chi)) = L(s, \Pi \otimes \chi)$ has a well-known analytic continuation to all $s \in \mathbf{C}$, and satisfies a functional equation relating values at s to $1 - s$. Moreover, since $\pi \cong \widetilde{\pi}$ is self-dual, and ring class characters equivariant under complex conjugation, it is well-known that the Rankin-Selberg L-function $L(s, \pi \times \pi(\chi))$ satisfies a symmetric functional equation

$$L(s, \pi \times \pi(\chi)) = \epsilon(s, \pi \times \pi(\chi)) L(1 - s, \pi \times \pi(\chi)) \tag{8}$$

with epsilon factor

$$\epsilon(s, \pi \times \pi(\chi)) = c(\pi \times \pi(\chi))^{\frac{1}{2} - s} \cdot \epsilon(1/2, \pi \times \pi(\chi)) = (d_K^2 N^2 c^4)^{\frac{1}{2} - s} \cdot \epsilon(1/2, \pi \times \pi(\chi))$$

and root number $\epsilon(1/2, \pi \times \pi(\chi)) \in \{\pm 1\} \subset \mathbf{S}^1$ given by the simple formula

$$\epsilon(1/2, \pi \times \pi(\chi)) = \eta(-N) = \eta(N). \tag{9}$$

Here, we write $c(\pi \times \pi(\chi)) = d_K^2 N^2 c^4$ to denote the conductor of the L-function $L(s, \pi \times \pi(\chi))$, and use that the quadratic Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$ is even (as K is a real quadratic extension). Note that this formula (9) holds for any choice of ring class character χ of K of conductor c coprime to the product $d_K N$, and moreover that this does not depend on this choice. Since the functional equation (8) is symmetric, we deduce that must be forced vanishing of the central value $L(1/2, \pi \times \pi(\chi)) = L(1/2, \Pi \otimes \chi) = 0$ in the event that $\eta(N) = -1$. We can therefore impose the following condition on the level N of π, equivalently the conductor N of f and E, to ensure this forced vanishing. Here, since we assume that N is coprime to the disciminant d_K, we can assume that the conductor N factorizes as $N = N^+ N^-$, where for each prime $q \mid N$,

$$q \mid N^+ \quad \iff \quad \eta(q) = 1 \quad \iff \quad q \text{ splits in } K$$
$$q \mid N^- \quad \iff \quad \eta(q) = -1 \quad \iff \quad q \text{ is inert } K.$$

Hypothesis 2.1 (Ersatz Heegner hypothesis). *Let us assume that the inert level N^- is the squarefree product of an odd number of primes, and hence that the root number of $L(s, \pi \times \pi(\chi))$ for χ any ring class character of K of conductor c prime to $d_K N$ is given by $\epsilon(1/2, \pi \times \pi(\chi)) = \eta(-N) = \eta(N) = \eta(N^-) = -1$.*

If the condition of Hypothesis 2.1 is met, as we shall assume henceforth, then the corresponding central value $L(1/2, \pi \times \pi(\chi))$ is forced by the functional equation (8) to vanish: $L(1/2, \pi \times \pi(\chi)) = L(1/2, \Pi \otimes \chi) = 0$. It then makes sense to derive integral presentations for the central derivative values in this case,

$$L'(1/2, \pi \times \pi(\chi)) = L'(1/2, \pi_K \otimes \chi) = \text{?}$$

That is, if one believes in the conjectures of Birch-Swinnerton-Dyer, Darmon (e.g. [15, Conjecture 5.6]), Kudla, and also Bruinier-Yang [8, Conjecture 1.1] (for instance), then this central derivative value should be related to the height of some CM-type point or arithmetic divisor on some Shimura variety.

3. Eulerian integral presentations

Let us now work with the basechange $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ of π to $\mathrm{GL}_2(\mathbf{A}_K)$, which exists by the theory of Langlands [36] and more generally Arthur-Clozel [2]. In addition to Hypothesis 2.1, we shall also require that the basechange representation $\Pi = \otimes_w \Pi_w$ of $\mathrm{GL}_2(\mathbf{A}_K)$ is cuspidal. Happily, this follows from the modularity theorem of Freitas-Le Hung-Siksek [18]:

Proposition 3.1. *The quadratic basechange $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ of the cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$ corresponding to our elliptic curve $E/\mathbf{Q}$ to an automorphic representation $\mathrm{GL}_2(\mathbf{A}_K)$ is cuspidal.*

Proof. We consider the basechange of the our fixed elliptic curve E to the quadratic field K, and its corresponding Mordell-Weil group $E(K)$. The main theorem [18, Theorem 1] implies that $E(K)$ is modular, and hence that its completed L-function $L^\star(E/K, s)$ is equivalent to the shift by 1/2 of the corresponding L-function $L(s, \sigma)$, wth $\sigma = \otimes_w \sigma_w$ a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$ determined uniquely by $E(K)$. On the other hand, by modularity of $E(\mathbf{Q})$ with the Artin basechange decomposition described above (which implies that $L(s, \Pi) = L(s, \pi) L(s, \pi \otimes \eta))$, we already have the formal relations

$$L^\star(E/K, s) = L(s - 1/2, \pi) L(s - 1/2, \pi \otimes \eta) = L(s - 1/2, \Pi).$$

Hence, we deduce that $\sigma = \Pi$, whence Π must be cuspidal by [18, Theorem 1]. □

In classical terms, this gives the following immediate consequence, which we record for future reference:

Corollary 3.2. *Our modular elliptic curve $E(\mathbf{Q})$ can be associated to some cuspidal newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2, trivial character, and level N equal to the conductor of $E/\mathbf{Q}$, with $\pi = \otimes_v \pi_v$ the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ of level $c(\pi) = N$ and trivial central character whose archimedean component is a holomorphic discrete series of weight 2. Using the unitary normalization for the automorphic L-functions (so that $s = 1/2$ is the central value), we have the equivalences of L-functions*

$$L^\star(E, s) = L(s - 1/2, f) = L(s - 1/2, \pi).$$

The basechanged elliptic curve $E(K)$ can be associated to some cuspidal Hilbert newform $\mathbf{f}$ of parallel weight two, trivial central character, and level $\mathfrak{N} \subset \mathcal{O}_K$ equal to the conductor of E/K, with $\Pi = \otimes_w \Pi_w$ the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$ of level $c(\Pi) = \mathfrak{N} \subset \mathcal{O}_K$ and trivial central *character whose archimedean component is a holomorphic discrete series of parallel weight two. We then have the corresponding equivalences of L-functions*

$$\begin{aligned}L^\star(E/K, s) &= L(s-1/2, \mathbf{f}) = L(s-1/2, \Pi)\\ &= L(s-1/2,\pi)L(s-1/2,\pi\otimes\eta) = L(s-1/2, f)L(s-1/2, f\otimes\eta).\end{aligned}$$

Let us now consider the following vector in our cuspidal automorphic representation $\Pi = \otimes_w \Pi_w$ of $\mathrm{GL}_2(\mathbf{A}_K)$, which will play a role analogous to a "newform" or new vector in this setup. Hence let us fix a pure tensor $\varphi = \otimes_w \varphi_w \in V_\Pi$ whose nonarchimedean local components are each "essential Whittaker vectors", we have the corresponding Whittaker function defined on $g \in \mathrm{GL}_2(\mathbf{A}_K)$ by

$$W_\varphi(g) = \int_{\mathbf{A}_K/K} \varphi\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right)\psi(-x)dx = \int_{N_2(K)\backslash N_2(\mathbf{A}_K)} \varphi(ng)\psi(-n)dn,$$

i.e. so that φ has the Fourier-Whittaker expansion

$$\varphi(g) = \sum_{\gamma\in K^\times} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right).$$

Here, we write $N_2 \subset \mathrm{GL}_2$ as usual to denote the standard unipotent subgroup of upper triangular matrices. We also have the classical "Eulerian" integral presentation (as detailed e.g. in the lectures of [14])

$$\begin{aligned}L(s, \Pi\otimes\chi) &= \int_{\mathbf{A}_K^\times/K^\times} \varphi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy\\ &= \int_{\mathbf{A}_K^\times/K^\times} \sum_{\gamma\in K^\times} W_\varphi\left(\begin{pmatrix} \gamma y & \\ & 1\end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy \qquad (10)\\ &= \int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy.\end{aligned}$$

In particular, this gives us the preliminary integral presentation

$$\begin{aligned}L'(1/2, \Pi\otimes\chi) &= \frac{d}{ds}\Big|_{s=1/2}\left(\int_{\mathbf{A}_K^\times/K^\times} \varphi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy\right) \qquad (11)\\ &= \frac{d}{ds}\Big|_{s=1/2}\left(\int_{\mathbf{A}_K^\times} W_\varphi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\chi(y)|y|^{s-\frac{1}{2}}dy\right).\end{aligned}$$

4. Connection to spin groups and orthogonal groups

Let us now explain how to view this preliminary Eulerian integral presentation (11) for the central derivative value in terms of Shimura varieties associated to spin groups and orthogonal groups, as well as how to use two corresponding versions of the theta correspondence to derive distinct integral presentations for the central derivative value $L'(1/2, \Pi\otimes\chi)$. Here, we follow [6, §§ 2.3, 2.7] and [8, §§ 2, 4].

4.1. An accidental isomorphism.

Let us first explain how for a certain hermitian quadratic space (V, q) over $\mathbf{Q}$, we can view our automorphic form $\varphi \in V_\Pi$ on $\mathrm{GL}_2(\mathbf{A}_K)$ from (11) as an automorphic form φ' on the spin group $\mathrm{GSpin}_V(\mathbf{A})$ as well as an automorphic form φ'' on the orthogonal group $O(V)(\mathbf{A})$.

4.1.1. *The quadratic space* (V, q). Writing our quadratic extension as $K = \mathbf{Q}(\sqrt{d})$ for $d > 0$, let $V = (V, q)$ be the quadratic space with underlying vector space

$$V = \mathbf{Q}\oplus\mathbf{Q}\oplus K, \qquad (12)$$

and associated quadratic form q defined by

$$q(x, y, \lambda) = \lambda\lambda^\tau - xy \quad \text{ for } x, y \in \mathbf{Q} \text{ and } \lambda \in K.$$

Here, λ^τ denotes the image of $\lambda \in K$ under the conjugation in K. Hence, we can see by inspection that (V, q) is a rational quadratic space of type $(2, 2)$ as $d > 0$ is positive[6]. We shall write $(\cdot, \cdot) : V \times V \to \mathbf{Q}$ to denote the corresponding hermitian bilinear form defined on $v_1, v_2 \in V$ by $(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$.

Remark We shall later take (V, q) to be any quadratic space of signature $(2, 2)$ with underlying vector space $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$, as well as slightly more general choices of quadratic form associated to the ring class group $\mathrm{Pic}(\mathcal{O}_c)$ of the $\mathbf{Z}$-order $\mathcal{O}_c := \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K through which our fixed ring class character χ factors. In this more general setting, the quadratic form q will be given by $q(x, y, \lambda) = c \cdot \mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$ for some positive constant $c > 0$ determined by the conductor, namely $c = \mathbf{N}\mathfrak{a}^{-1}$ for $\mathfrak{a} \subset \mathcal{O}_K$ an integral ideal representative of a given class $A \in \mathrm{Pic}(\mathcal{O}_c)$. Hence for this reason, we shall consider a slightly more general choice of quadratic space (V, q) in the preliminary discussion.

4.1.2. *Relation to spin and orthogonal groups of V.* As explained in [6, §2.7], fixing the orthogonal basis

$$v_1 = (1, 1, 0), \quad v_2 = (1, -1, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (0, 0, \sqrt{d})$$

so that $\delta := v_1 v_2 v_3 v_4$ satisfies $\delta^2 = d$, the centre $Z(C_V^0)$ of the $\mathbf{Q}$-submodule C_V^0 of the Clifford algebra C_V of V spanned by products of even numbers of basis vectors (see [6, § 2.2]) can be identified with $Z(C_V^0) \cong \mathbf{Q} + \mathbf{Q}\delta \cong K$. Moreover, as explained in [6, Example 2.10, § 2.7], we have the identification

$$C_V^0 \cong Z(C_V^0) + Z(C_V^0)v_1v_2 + Z(C_V^0)v_2v_3 + Z(C_V^0)v_1v_3 \cong M_2(K)$$

of C_V^0 with the split quaternion algebra $M_2(K)$ over K via the assignment

$$1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad v_1v_2 \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad v_2v_3 \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad v_1v_3 \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

The canonical involution on C_V^0 corresponds to the conjugation $\star$ in $M_2(K)$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and the Clifford norm is seen in this way to correspond to the determinant in $M_2(K)$. In particular, using the known characterization of elements of Clifford algebras in lower dimensions in terms of the Clifford norm $\mathbf{N}_{C_V}$ (see e.g. [6, Lemma 2.14]), we then deduce that we have the accidental isomorphism

$$\mathrm{Spin}_V := \{x \in C_V^0 : \mathbf{N}_{C_V}(x) = 1\} \cong \mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_2(K) \tag{13}$$

as algebraic groups over $\mathbf{Q}$. On the other hand, we also have the short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V \longrightarrow \mathrm{SO}(V) \longrightarrow 1, \tag{14}$$

with $\mathrm{SO}(V)$ the special orthogonal group of V,

$$\mathrm{SO}(V) = \{\sigma \in \mathrm{Aut}(V) : q(\sigma(v)) = q(v) \quad \forall v \in V, \quad \det(\sigma) = 1\}.$$

Note that we can derive from (13) and (14) the corresponding relations for the general orthogonal group

$$\mathrm{GO}(V) = \left\{\sigma \in \mathrm{Aut}(V) : q(\sigma(v)) = q(v) \quad \forall v \in V, \quad \det(\sigma) \in \mathbf{Q}^\times \right\}.$$

To be more precise, we also have the identification

$$\mathrm{GSpin}_V := \{x \in C_V^0 : \mathbf{N}_{C_V}(x) \in \mathbf{Q}^\times\},$$

as well as the injection

$$\mathrm{GSpin}_V \longrightarrow \mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_2(K), \tag{15}$$

of algebraic groups over $\mathbf{Q}$. We also have the short exact sequence

$$1 \longrightarrow \mathbf{Q}^\times \longrightarrow \mathrm{GC}_V \longrightarrow \mathrm{GO}(V) \longrightarrow 1, \tag{16}$$

where $\mathrm{GC}_V \supset \mathrm{GSpin}_V := \mathrm{GC}_V \cap C_V^0$ denotes the Clifford group of V. We refer to [6, Lemma 2.14] for details.

[6]Note that when $d < 0$ (so that $K = \mathbf{Q}(\sqrt{d})$ is imaginary quadratic), the corresponding space (V, q) is of type $(3, 1)$. That the space has signature $(2, 2)$ when $d > 0$ can be seen more directly after putting the quadratic form into diagonal form, say with respect to the basis $(1, 1), (1, -1)$, with x replaced by $(u + v)$ and y replaced by $(u - v)$, so that the quadratic form q is given equivalently by $q(u, v, \lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda) - (u + v)(u - v) = \lambda\lambda^\tau - u^2 + v^2$. Expanding this in terms of an integral basis for the real quadratic field $K = \mathbf{Q}(\sqrt{d})$, it is then easy to see by inspection that this space has signature $(1, 1)$.

4.1.3. *Realizations of the automorphic form $\varphi \in V_\Pi$.* Since the basechange representation Π has trivial central character in our setup, we can view our cuspidal automorphic form $\varphi \in V_\Pi$ of trivial central character on $\mathrm{GL}_2(\mathbf{A}_K)$ as a cuspidal automorphic form on $\mathrm{SL}_2(\mathbf{A}_K)$. Via the isomorphism (13), we can then view this form $\varphi \in V_\Pi$ as an automorphic form φ' on the spin group $\mathrm{Spin}_V(\mathbf{A})$, and via the short exact sequence (14) as an automorphic form φ'' on the orthogonal group $\mathrm{O}(V)(\mathbf{A})$. It is easy to see that we can view these latter forms on the corresponding general spin and general orthogonal groups of trivial central character. In this way, we can view this form $\varphi \in V_\Pi$ as an automorphic form φ' on $\mathrm{GSpin}_V(\mathbf{A})$, and via the short exact sequence (16) as an automorphic form φ'' on $\mathrm{GO}(V)(\mathbf{A})$.

4.1.4. *The quadratic subspace (V_2, q_2).* We shall sometimes consider the subspace $V_2 \subset V$ of signature $(1,1)$ corresponding to the quadratic field $V_2 = K$ with restricted form $q_2 = q|_{V_2}$. We write GSpin_{V_2} to denote the corresponding general spin group, and $\mathrm{GO}(V_2)$ the corresponding orthogonal group. Note that we have natural identifications $\mathrm{Spin}_{V_2} \cong \mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_1(K)$ and $\mathrm{SO}(V_2) \cong \mathrm{Res}_{K/\mathbf{Q}} \mathrm{SL}_1(K)$ of algebraic groups over $\mathbf{Q}$ by a variation of the discussion above (see [6, §2.3]). In particular, we obtain natural identifications of the idele class group $\mathbf{A}_K^\times/K^\times$:

$$\mathrm{GSpin}_{V_2}(\mathbf{Q})\backslash \mathrm{GSpin}_{V_2}(\mathbf{A}) \cong \mathrm{GO}(V_2)(\mathbf{Q})\backslash \mathrm{GO}(V_2)(\mathbf{A}) \cong \mathbf{A}_K^\times/K^\times. \tag{17}$$

Here, strictly speaking, we fix one of the two connected components $\mathrm{GO}^\pm(V)$ of $\mathrm{GO}(V)$, so that the identification (17) should read

$$\mathrm{GSpin}_{V_2}(\mathbf{Q})\backslash \mathrm{GSpin}_{V_2}(\mathbf{A}) \cong \mathrm{GO}^\pm(V_2)(\mathbf{Q})\backslash \mathrm{GO}^\pm(V_2)(\mathbf{A}) \cong \mathbf{A}_K^\times/K^\times.$$

We shall drop the superscript $\mathrm{GO}^\pm(V)$ to simplify notations in the subsequent discussion. We refer to the discussion in [37, Theorem 2.3.3] for more background leading to this identification.

Given an idele class character $\chi = \otimes_w \chi_w$ of $\mathbf{A}_K^\times/K^\times$, we shall write χ' to denote the corresponding automorphic form on $\mathrm{GSpin}_{V_2}(\mathbf{A})$ and χ'' the corresponding automorphic form on $\mathrm{GO}(V_2)(\mathbf{A})$ under (17). We shall also assume that the natural embeddings $\mathrm{GSpin}_{V_2}(\mathbf{A}) \to \mathrm{GSpin}_V(\mathbf{A})$ and $\mathrm{GO}(V_2)(\mathbf{A}) \to \mathrm{GO}(V)(\mathbf{A})$ induced by the subspace inclusion $V_2 \subset V$ coincide under the identification (15) with the natural embedding

$$\mathbf{A}_K^\times \cong \mathrm{GL}_1(\mathbf{A}_K) \longrightarrow \mathrm{GL}_2(\mathbf{A}_K), \quad y \longmapsto \begin{pmatrix} y & \\ & 1 \end{pmatrix}.$$

4.2. **Relations to spin and orthogonal Shimura varieties.** We obtain the following immediate relation to spin and orthogonal Shimura varieties from the discussion of the accidental isomorphism (13) above.

Let use write $D = D_V = D_V^+$ to denote the Grassmannian of oriented 2-dimensional subspaces of $V(\mathbf{R})$ on which the quadratic form q is positive definite,

$$D = D_V^+ = \{W \subset V(\mathbf{R}) : \dim(W) = 2, q|_W > 0\},$$

and D^- the Grassmannian of oriented 2-dimensional subspaces of $V(\mathbf{R})$ on which the q is negative definite,

$$D^- = \{W \subset V(\mathbf{R}) : \dim(W) = 2, q|_W < 0\}.$$

Again, we write $D_2^\pm = D_{V_2}^\pm$ to denote the corresponding Grassmannians for the signature $(1,1)$ subspaces $V_2 \subset V$, as introduced above (see e.g. [7, §]).

4.2.1. *Relation to GSpin_V Shimura varieties.* Let $H = H(\varphi') \subset \mathrm{GSpin}_V(\mathbf{A})$ denote the compact open subgroup corresponding to the form φ' on $\mathrm{GSpin}_V(\mathbf{A})$ and hence to the form $\varphi \in V_\Pi$ on $\mathrm{GL}_2(\mathbf{A}_K)$. We can consider the automorphic form φ' as a function on the corresponding spin Shimura variety

$$\mathrm{Sh}_U(\mathrm{GSpin}_V, D^\pm) = \mathrm{GSpin}_V(\mathbf{Q})\backslash \left(D^\pm \times \mathrm{GSpin}_V(\mathbf{A}_f)/U\right).$$

Note that this determines a variety of dimension 2 defined over $\mathbf{Q}$[7], and that the quadratic subspace (V_2, q_2) described above gives rise to a "point" defined with respect to the sub-Grassmannian $D_2^\pm = D_{V_2}^\pm$ by

$$\mathrm{Sh}_{\overline{U}}(\mathrm{GSpin}_{V_2}, D_2^\pm) = \mathrm{GSpin}_{V_2}(\mathbf{Q})\backslash \left(D_2^\pm \times \mathrm{GSpin}_{V_2}(\mathbf{A}_f)/\overline{U}\right), \quad \overline{U} := U \cap \mathrm{GSpin}_{V_2}(\mathbf{A}_f).$$

[7] In general, for a hermitian space (V, q) of signature $(n, 2)$, the corresponding Shimura variety $\mathrm{Sh}_U(\mathrm{Spin}_V, \mathbb{D}^\pm)$ is a quasi-projective variety of dimension n over $\mathbf{Q}$, projective if and only if V is anisotropic.

Writing χ' again to denote the automorphic form on $\mathrm{GSpin}_{V_2}(\mathbf{A})$ corresponding to a ring class character $\chi = \otimes_w \chi_w$ of $\mathbf{A}_K^\times / K^\times$, we can rewrite the preliminary integral presentation (11) equivalently as

$$L'(1/2, \Pi \otimes \chi) = \frac{d}{ds}\Big|_{s=1/2} \left(\int\limits_{\mathrm{GSpin}_{V_2}(\mathbf{Q}) \backslash \mathrm{GSpin}_{V_2}(\mathbf{A})} \varphi'(y) \chi'(y) |y|^{s-\frac{1}{2}} dy \right). \tag{18}$$

4.2.2. *Relation to* $\mathrm{GO}(V)$ *Shimura varieties.* Let $U = U(\varphi'')$ denote the compact open subgroup of $\mathrm{GO}(V)(\mathbf{A}_f)$ corresponding to the form φ'' on $\mathrm{GO}(V)(\mathbf{A}_f)$, and hence to the form $\varphi \in V_\Pi$ on $\mathrm{GL}_2(\mathbf{A}_K)$. We can also view φ'' as an automorphic form on the orthogonal Shimura variety

$$\mathrm{Sh}_U(\mathrm{GO}(V), D^\pm) = \mathrm{GO}(V)(\mathbf{Q}) \backslash \left(D^\pm \times \mathrm{GO}(V)(\mathbf{A}_f)/U \right),$$

which for the special case of quadratic space (V, q) of signature $(2,2)$ which we consider is a Hilbert modular surface (see e.g. [6, §2]). Again, the quadratic subspace (V_2, q_2) of signature $(1,1)$ gives rise to a "point" defined with respect to the corresponding Grassmannian $D_2^\pm = D_{V_2}^\pm$,

$$\mathrm{Sh}_{\overline{U}}(\mathrm{GO}(V_2), D^\pm) = \mathrm{GO}(V_2)(\mathbf{Q}) \backslash \left(D^\pm \times \mathrm{GO}(V_2)(\mathbf{A}_f)/\overline{U} \right), \quad \overline{U} := U \cap \mathrm{GO}(V_2)(\mathbf{A}_f).$$

Writing χ'' to denote the automorphic form on $\mathrm{GO}(V_2)(\mathbf{A})$ corresponding to a ring class character $\chi = \otimes_w \chi_w$ of $\mathbf{A}_K^\times / K^\times$, we can also write the preliminary integral presentation (11) equivalently as

$$L'(1/2, \Pi \otimes \chi) = \frac{d}{ds}\Big|_{s=1/2} \left(\int\limits_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A})} \varphi''(y) \chi''(y) |y|^{s-\frac{1}{2}} dy \right). \tag{19}$$

4.3. **Realization of the basechange form as a theta lifting.** We now explain how we can use the identification (15) and short exact sequence (16) to give an realization of our automorphic form $\varphi \in V_\Pi$ on $\mathrm{GL}_2(\mathbf{A}_K)$ as a theta lifting from $\mathrm{GSp}_2(\mathbf{A})$ corresponding to the basechange lifting. This little-known classical construction is detailed in Bruinier [6, §2.7], which we follow. Here, we give representation theoretic descriptions of the setup, and defer giving a more arithmetic description in terms of regularized theta liftings and Borcherds products (or automorphic Green's functions) until the next section.

Remark Note that we could also realize our cuspidal form φ on $\mathrm{GL}_2(\mathbf{A}_K)$ as a Shimura lifting from from a genuine automorphic form on the two-fold metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{A}_K)$ of $\mathrm{SL}_2(\mathbf{A}_K)$ in the sense of the theorems of Shimura [42], [43] and Waldspurger [47], and that the preliminary setup would be formally almost identical. We omit giving details of this variation for simplicity.

4.3.1. *Abstract lifting via the orthogonal symplectic pair* $(\mathrm{GO}(V), \mathrm{GSp}_2)$. We refer to [33, § II.1] and [34] for more background. Let $W \cong \mathbf{Q}^4$ denote the standard symplectic space over $\mathbf{Q}$ with pairing $[\cdot,\cdot] : W \times W \to \mathbf{Q}$ given by the determinant, and write $\mathrm{Sp}(W) \cong \mathrm{Sp}_2 \cong \mathrm{SL}_2$ to denote corresponding symplectic group. Writing $\mathbb{W} = V \otimes_{\mathbf{Q}} W \cong V^2$ to denote the corresponding product space with $\mathbf{Q}$-bilinear form $[[\cdot,\cdot]] : \mathbb{W} \times \mathbb{W} \to \mathbf{Q}$ defined by the rule $[[v_1 \otimes w_2, v_1 \otimes w_2]] = q(v_1, v_2) \cdot [w_1, w_2]$, we can consider the corresponding reductive dual pair $(\mathrm{SO}(V), \mathrm{Sp}(W))$ in $\mathrm{Sp}(\mathbb{W})$. Extending to similitudes then gives us the reductive dual pair $(\mathrm{GO}(V), \mathrm{GSp}(W))$, where $\mathrm{GSp}(W)(\mathbf{A}) \cong \mathrm{GL}_2(\mathbf{A})$ and $\mathrm{GO}(V)(\mathbf{A})$ can be identified with $\mathrm{GL}_2(\mathbf{A}_K)$ for our purposes via the accidental isomorphism (13) with the exact sequence (16).

We consider the global theta correspondence for $(\mathrm{SO}(V), \mathrm{SL}_2)$ and its extension to similitudes $(\mathrm{GO}(V), \mathrm{GL}_2)$. Hence, let us now write $\psi_0 = \otimes_v \psi_{0,v}$ to denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, and

$$r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathrm{Aut}(V(\mathbf{A})^2)$$

the Weil representation extended to the similitude group

$$R(\mathbf{A}) = \{(h, g) \in \mathrm{GO}(V)(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A}) : \nu(h) = \det(g)\} \subset \mathrm{GO}(V)(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A}).$$

We can then consider the corresponding theta kernel defined on $(h, g) \in R(\mathbf{A})$ and a decomposable Schwartz function $\Phi = \otimes_v \Phi_v \in \mathcal{S}(V(\mathbf{A})^2)$ by

$$\theta_{r_{\psi_0}}(h, g; \Phi) = \sum_{x \in V(\mathbf{Q})^2} r_{\psi_0}(h, g)\Phi(x).$$

Taking ϕ to be an automorphic form on $\mathrm{GL}_2(\mathbf{A}) \cong \mathrm{GSp}(W)(\mathbf{A})$, let us now consider the automorphic form defined on $h \in \mathrm{GO}(V)(\mathbf{A})$ by the theta integral

$$\varphi(h) = \vartheta_\phi(h; \Phi) := \int_{\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A})} \theta_{r_{\psi_0}}(h, \sigma g; \Phi)\phi(\sigma g) d\sigma. \tag{20}$$

Here, we choose some $g \in \mathrm{GL}_2(\mathbf{A})$ with matching similitude factor $\det(g) = \nu(h)$. We also claim that we can use our chosen cuspidal pure tensor $\varphi \in V_\Pi$ above to be as such a theta lifting (20) as follows. Recall that any automorphic form φ on $\mathrm{GL}_2(\mathbf{A}_K)$ can be viewed as an automorphic form φ'' on $\mathrm{GO}(V)(\mathbf{A})$ via the short exact sequence (14) with the accidental isomorphism (13). Recall as well that by a "new vector" $\phi \in V_\pi$ or $\varphi \in V_\Pi$, we mean a pure tensor $\phi = \otimes_v \phi_v \in V_\pi$ or $\varphi = \otimes_v \varphi_v \in V_\Pi$ whose nonarchimedean local components are each essential Whittaker vectors. We can derive the following explicit relation here.

Proposition 4.1 (Realization of the basechange form $\varphi \in V_\Pi$ as a theta lift from $\mathrm{GSp}_2(\mathbf{A})$)**.** *Write $\pi = \otimes_v \pi_v$ again to denote our cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ (of trivial central character), with $\Pi = \otimes_w \Pi_w$ its quadratic basechange lifting $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ to a cuspidal automorphic representation $\mathrm{GL}_2(\mathbf{A}_K)$. Let $\phi \in V_\pi$ be any new vector. Then, the corresponding theta lift ϑ_ϕ defined in (20) determines an automorphic form $\varphi'' = \vartheta_\phi$ on $h \in \mathrm{GO}(V)(\mathbf{A})$, which via the discussion above with the accidental isomorphism (15) and short exact sequence (16) can be identified as a cuspidal automorphic form φ on $g \in \mathrm{GL}_2(\mathbf{A}_K)$. This latter form φ can be identified with our chosen pure tensor $\varphi \in V_\Pi$ as in the discussion above, i.e. which is also a new vector in the sense that it corresponds to a pure tensor $\varphi = \otimes_w \varphi_w \in V_\Pi$ whose nonarchimedean local components φ_w are each essential Whittaker vectors.*

Proof. Cf. [6, Theorem 2.23] for a classical description of this setup. That such a vector exists is a consequence of the existence of the quadratic basechange lifting of Langlands [36] and more generally Arthur-Clozel [2], together with the global theta correspondence for the reductive dual pair $(\mathrm{GO}(V), \mathrm{GSp}_2)$ described above. $\square$

This realization of the basechange lifting Π as a theta lifting (20) according to Proposition 4.1 allows us to derive the following "seesaw" identity via the substitution

$$\begin{aligned} L(s, \Pi \otimes \chi) &= \int_{\mathbf{A}_K^\times/K^\times} \vartheta_\phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}; \Phi\right) \chi(y)|y|^{s-\frac{1}{2}} dy \\ &= \int_{\mathbf{A}_K^\times/K^\times} \left(\int_{\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A})} \theta_{r_{\psi_0}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \sigma g; \Phi\right) \phi(\sigma g) d\sigma\right) \chi(y)|y|^{s-\frac{1}{2}} dy \\ &= \int_{\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A})} \phi(\sigma g) \left(\int_{\mathbf{A}_K^\times/K^\times} \theta_{r_{\psi_0}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \sigma g; \Phi\right) \chi(y)|y|^{s-\frac{1}{2}} dy\right) d\sigma. \end{aligned} \tag{21}$$

Observe now (cf. [33, § IV.I]) that we can consider the orthogonal decomposition $V = V_1 \oplus V_2$, where $V_1 = \mathbf{Q} \oplus \mathbf{Q}$ with associated quadratic form $q|_{V_1}$ and $V_2 = K$ with associated quadratic form $q|_{V_2} = \mathbf{N}_{K/\mathbf{Q}}$. We can then express (21) equivalently via the seesaw dual pair

$$\begin{tikzcd} \mathrm{GO}(V) \arrow[dr, leftrightarrow] & {[\mathrm{GL}_2 \times \mathrm{GL}_2]} \arrow[dl, leftrightarrow] \\ {[\mathrm{GO}(V_1) \times \mathrm{GO}(V_2)]} \arrow[u] & \mathrm{GL}_2 \arrow[u] \end{tikzcd}$$

as

$$L(s, \Pi \otimes \chi) = \int_{\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A})} \phi(\sigma g) \left(\int_{\mathrm{GO}(V_2)(\mathbf{Q})\backslash \mathrm{GO}(V_2)(\mathbf{A})} \theta_{r_{\psi_0}}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \sigma g; \Phi\right) \chi(y)|y|^{s-\frac{1}{2}} dy\right) d\sigma. \tag{22}$$

Here again, we use the natural identification $\mathrm{GO}(V_2)(\mathbf{Q})\backslash \mathrm{GO}(V_2)(\mathbf{A}) \cong \mathbf{A}_K^\times/K^\times$.

As explained in [37, Theorem 2.3.3], setting $s = 1/2$, the inner integral in (22) can be identified as an explicit realization of a vector in the space of the induced representation $\pi(\chi)$ of $\mathrm{GL}_2(\mathbf{A})$. That is, we find a distinct variation of the equivalent Rankin-Selberg integral presentation of $L(s, \pi \times \pi(\chi)) = L(s, \Pi \otimes \chi)$ in this way, i.e. where $L(s, \pi \times \pi(\chi))$ denotes the $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ corresponding to the cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbf{A})$ corresponding to our elliptic curve $E/\mathbf{Q}$ times the induced automorphic representation $\pi(\chi) = \otimes_v \pi(\chi)_v$ of $\mathrm{GL}_2(\mathbf{A})$ corresponding to our ring class character $\chi = \otimes_w \chi_w$ of $\mathbf{A}_K^\times / K^\times$. Here, using the notations and conventions introduced above, we deduce that the Eisenstein series appears indirectly through in the inner Siegel theta-Eisenstein series defined by

$$Z_{\chi''}(s, g; \Phi) := \int_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A}) \cong \mathbf{A}_K^\times / K^\times} \theta_{r_{\psi_0}}(h, g; \Phi)\, \chi''(y) |y|^{s - \frac{1}{2}} dy. \tag{23}$$

Observe that this integral (23) is seen via the global theta correspondence (as described above) to be an automorphic form on $g \in \mathrm{GL}_2(\mathbf{A}) \cong \mathrm{GSp}(W)(\mathbf{A})$. Writing $\langle \cdot, \cdot \rangle$ to denote the inner product on the space $L^2(\mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A}), \mathbf{1})$ of L^2-automorphic forms on $\mathrm{GL}_2(\mathbf{A})$ with trivial central character $\mathbf{1}$, we can thus derive the corresponding integral presentations of our L-function,

$$L(s, \Pi \otimes \chi) = \langle \phi, Z_{\chi''}(s, *; \Phi) \rangle, \quad L'(1/2, \Pi \otimes \chi) = \langle \phi, Z'_{\chi''}(1/2, *; \Phi) \rangle. \tag{24}$$

In the terminology of [52], this automorphic form $Z_\chi(s, g; \Phi)$ on $g \in \mathrm{GL}_2(\mathbf{A})$ could be viewed as our "analytic kernel function". Putting this together with the discussion leading to (19), we derive the preliminary result.

Theorem 4.2. *Let E be an elliptic curve defined over $\mathbf{Q}$, parametrized by a cuspidal newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$ of weight 2, trivial character, and level N equal to the conductor of E, with $\pi = \otimes_v \pi_v$ the correponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$. Let $\Pi = \otimes_w \Pi_w$ denote the quadratic basechange $\Pi = \mathrm{BC}_{K/\mathbf{Q}}$ of π to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$, which exists by [36] and more generally [2]. Fix $\varphi = \otimes_w \varphi_w \in V_\Pi$ a pure tensor whose nonarchimedean local components are essential Whittaker vectors, which we view as an automorphic form φ on $g \in \mathrm{GL}_2(\mathbf{A}_K)$. Recall we consider the quadratic space (V, q) with $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$ with $q(x, y, \lambda) = \lambda\lambda^\tau - xy$, with $(V_2, q_2) = (K, \mathbf{N}_{K/\mathbf{Q}})$ the subspace of signature $(1,1)$ corresponding to the real quadratic field K. Let φ'' denote the corresponding automorphic form on $h \in \mathrm{GO}(V)(\mathbf{A})$ determined by (15) and (16). By Proposition 4.1, we can and do take this φ'' to be a theta lift ϑ_ϕ of an automorphic form ϕ on $\mathrm{GSp}_2(\mathbf{A})$ corresponding to a pure tensor $\phi = \otimes_v \phi_v \in V_\pi$ whose nonarchimedean local components are essential Whittaker vectors. By the discussion above, we can also view this φ'' as an automorphic form on the orthogonal Shimura variety $\mathrm{Sh}_U(\mathrm{GO}(V), D_V^\pm)$, where $H = H(\varphi) = H(\Pi)$ denotes the compact open subgroup $\mathrm{GO}(V)(\mathbf{A}_f)$ determined by the level of Π. Given $\chi = \otimes_w \chi_w$ any ring class character of $\mathbf{A}_K^\times / K^\times$, let us write χ'' to denote the corresponding automorphic representation of $\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A}) \cong \mathbf{A}_K^\times / K^\times$. We then have for this setup the preliminary integral presentations*

$$L'(1/2, \Pi \otimes \chi) = \frac{d}{ds}\bigg|_{s=1/2} \left(\int_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A})} \varphi''(y)\, \chi''(y) |y|^{s - \frac{1}{2}} dy \right) = \frac{d}{ds}\bigg|_{s=1/2} \langle \phi, Z_{\chi''}(s, *; \Phi) \rangle.$$

Here again, $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A}), \mathbf{1})$, and $Z_{\chi''}(s, g; \Phi)$ the automorphic (Siegel theta-Eisenstein) generating series on $g \in \mathrm{GL}_2(\mathbf{A})$ as defined in (23) above. Moreover, assuming our ersatz Heegner Hypothesis 2.1, this integral presentation is not forced to vanish by the symmetric functional equation (8) for $L(s, \Pi \otimes \chi)$ – and conjecturally should account for some point on the Mordell-Weil group $E(K[c])$, where $K[c]$ denotes the ring class extension of K of conductor c equal to that of the ring class character χ.

4.3.2. *Comparison with the Rankin-Selberg integral presentation.* Let us for the sake of completeness describe the equivalent formula derived via the Rankin-Selberg integral presentation $L(s, \Pi \otimes \chi) = L(s, \pi \otimes \pi(\chi))$. Recall that by the theory of Rankin-Selberg convolution (see e.g. [11, §3.8.2] or [37, (4.0.1)]), we have the following well-known integral presentation for the L-function $L(s, \pi \times \pi(\chi))$. Given decomposable vectors $\phi_1 \in V_\pi$ and $\phi_2 \in V_{\pi(\chi)}$, let us for each of $j = 1, 2$ write W_{ϕ_j} to denote the corresponding Whittaker function defined on $g \in \mathrm{GL}_2(\mathbf{A})$ by

$$W_{\phi_j}(g) = \int_{\mathbf{A}/\mathbf{Q}} \phi_j \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) \psi(-x) dx = \int_{N_2(\mathbf{Q})\backslash N_2(\mathbf{A})} \phi_j(ng)\psi(-n)dn.$$

Here, we fix $\psi = \otimes_v \psi_v$ to be the standard additive character on $\mathbf{A}/\mathbf{Q} \cong N_2(\mathbf{A})/N_2(\mathbf{Q})$, writing $N_2 \subset \mathrm{GL}_2$ to denote the unipotent subgroup of upper triangular matrices. Fixing a suitable decomposable section $f(s,g)$ in the induced representation space $\mathrm{Ind}(|\cdot|^{s-\frac{1}{2}}, \eta^{-1}|\cdot|^{\frac{1}{2}-s})$, we consider the corresponding Eisenstein series

$$E(s,g) = \sum_{\gamma \in B(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{Q})} f(s, \gamma g).$$

Here, we write $B \subset \mathrm{GL}_2$ to denote the Borel subgroup of upper triangular matrices. Let us also write $Z_2 \subset \mathrm{GL}_2$ to denote the centre. The theory of Rankin-Selberg convolution or unfolding gives us for any pure tensors $\phi_1 \in V_\pi$ and $\phi_2 \in V_{\pi(\chi)}$ the integral presentation

$$\begin{aligned} L(s, \pi \times \pi(\chi)) &= \int_{\mathrm{GL}_2(\mathbf{Q})Z_2(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} \phi_1(g)\phi_2(g)E(s,g)dg \\ &= \int_{Z_2(\mathbf{A})N_2(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} W_{\phi_1}(g)W_{\phi_2}(g)f(s,g)\delta_B(g)dg. \end{aligned}$$

Here (as in [11, Proposition 3.8.3]), we write δ_B to denote the modular quasicharacter defined on $B(\mathbf{A})$ or rather on the subgroup $T_1(\mathbf{A}) = \left\{ \begin{pmatrix} t & \\ & 1 \end{pmatrix} : t \in \mathbf{A}^\times \right\} \subset B(\mathbf{A})$ by

$$\delta_B \left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} \right) = |t|.$$

Note that in this setting, we take $\phi = \phi_1 \in V_\pi$ to be the pure tensor described above for our underlying cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$ corresponding to the elliptic curve $E/\mathbf{Q}$, and the theta series $\phi_2 = \theta_\chi \in V_{\pi(\chi)}$ corresponding to the ring class character χ. Note that this theta series $\theta_\chi(g)$ on $g \in \mathrm{GL}_2(\mathbf{A})$ can also be realized as the theta lifting from an orthogonal group $\mathrm{GO}(V_2)(\mathbf{A})$,

$$\theta_\chi(g) = \int_{\mathrm{SO}(V_2)(\mathbf{Q})\backslash \mathrm{SO}(V_2)(\mathbf{A})} \theta_{r_{\psi_0}}(\sigma h, g; \Phi)\chi(\sigma h)d\sigma$$

for $h \in \mathrm{GO}(V_2)(\mathbf{A})$ chosen to have matching similitude factor $\nu(h) = \det(g)$. See [37, §2.3, Theorem 2.3.3]. We omit details for brevity, however keep in mind that the preliminary formula (4.2) is equivalent to the more familiar Rankin-Selberg integral presentation

$$L'(1/2, \Pi \otimes \chi) = L'(1/2, \pi \times \pi(\chi)) \quad = \frac{d}{ds}\bigg|_{s=1/2} \langle \phi, \theta_\chi E(s,*)\rangle, \tag{25}$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product on $L^2(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}), \eta)$. We shall revisit this equivalent abstract presentation (25) in classical terms in the discussion below, specifically in relating to arithmetic theta liftings and automorphic Green functions in the style of Bruinier-Yang [8, §§2-4].

4.3.3. *Subsequent abstract lifting via the Shimura-Shintani-Waldspurger pair* $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}}_2)$. Let us also for the record explain how we could consider the following separate application of the theta correspondence, namely the reinterpretation due to Waldspurger [47] of the Shimura/Shintani correspondence (see e.g. [42]) to make a similar formal substitution of the cuspidal automorphic form $\phi \in V_\pi$ on $\mathrm{GL}_2(\mathbf{A})$ in the discussion above. We include this discussion only for completeness only, and do not use or pursue further in this work.

Let us now write $\mathcal{V}$ to denote the vector space of trace-zero matrices in $M_2(\mathbf{Q})$, equipped with the quadratic form $Q(v) = -\det(v)$. Note that $\mathrm{GL}_2(\mathbf{Q})$ acts on $\mathcal{V}$ by the rule $g \cdot v = gvg^{-1}$ for $g \in \mathrm{GL}_2(\mathbf{Q})$ and $v \in \mathcal{V}$. As a well-known consequence, we can identify corresponding special orthogonal group $\mathrm{SO}(\mathcal{V})$ with the linear group PGL_2. In this way, we shall view our cuspidal automorphic form $\phi \in L^2(\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}), \mathbf{1})$ as an automorphic form on $\mathrm{SO}(\mathcal{V})(\mathbf{A}) \cong \mathrm{PGL}_2(\mathbf{A})$. Let us now consider the standard symplectic vector space $\mathcal{W} = \mathbf{Q}^2$ with skew symmetric pairing $[\cdot,\cdot]$, so that the corresponding isometry group $\mathrm{Sp}(\mathcal{W})$ is isomorphic to SL_2. We again consider the corresponding tensor product $\mathbb{W}_0 = \mathcal{V} \otimes_{\mathbf{Q}} \mathcal{W}$, with form $[[\cdot,\cdot]] : \mathbb{W}_0 \times \mathbb{W}_0 \to \mathbf{Q}$ given by the product $[[v_1 \otimes w_1, v_2 \otimes w_2]] = Q(v_1, v_2) \cdot [w_1, w_2]$. Let us now write $\widetilde{\mathrm{Sp}}(\mathcal{W}) \cong \widetilde{\mathrm{SL}}_2$ to denote the two-fold metaplectic cover of $\mathrm{Sp}(\mathcal{W}) \cong \mathrm{SL}_2$. We refer to the discussion in [20] for more background about automorphic forms on this group. In this setting, the corresponding Weil representation ω_{ψ_0} factors through this metaplectic cover, and can be seen as a genuine representation $\mathrm{SO}(\mathcal{V})(\mathbf{A}) \times \widetilde{\mathrm{Sp}}(\mathcal{W})(\mathbf{A})$ on the corresponding space of Schwartz functions $\mathcal{S}(\mathbb{W}_0(\mathbf{A}))$,

$$\omega_{\psi_0} : \mathrm{SO}(\mathcal{V})(\mathbf{A}) \times \widetilde{\mathrm{Sp}}(\mathcal{W})(\mathbf{A}) \cong \mathrm{PGL}_2(\mathbf{A}) \times \widetilde{\mathrm{SL}}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W}_0(\mathbf{A})),$$

and we can consider for any decomposable Schwartz function $\Phi_0 \in \mathcal{S}(\mathbb{W}_0(\mathbf{A}))$ the corresponding theta kernel

$$\theta_{\omega_{\psi_0}}(h, g; \Phi_0) = \sum_{x \in \mathbb{W}_0(\mathbf{Q})} \omega_{\phi_0}(h, g)\Phi_0(x).$$

Now, given $\widetilde{\phi}$ an automorphic form on $g \in \widetilde{\mathrm{Sp}}(\mathcal{W})(\mathbf{A}) \cong \widetilde{\mathrm{SL}}_2(\mathbf{A})$, we consider the corresponding automorphic form defined on $h \in \mathrm{SO}(\mathcal{V})(\mathbf{A}) \cong \mathrm{PGL}_2(\mathbf{A})$ by the theta integral

$$\vartheta_{\widetilde{\phi}}(h) = \vartheta_{\widetilde{\phi}}(h; \Phi_0) = \int_{\mathrm{SL}_2(\mathbf{A}) \backslash \widetilde{\mathrm{SL}}_2(\mathbf{A})} \theta_{\omega_{\psi_0}}(h, g; \Phi_0)\widetilde{\phi}(g)dg. \tag{26}$$

Here, we also write $\mathrm{SL}_2(\mathbf{Q})$ to denote the full inverse image of $\mathrm{SL}_2(\mathbf{Q})$ in $\widetilde{\mathrm{SL}}_2(\mathbf{A})$ (cf. [20]).

Proposition 4.3 (Realization of the cusp form $\phi \in V_\pi$ on $\mathrm{PGL}_2(\mathbf{A})$ as a Shimura theta lift from $\widetilde{\mathrm{SL}}_2(\mathbf{A})$)**.** *Write $\pi = \otimes_v \pi_v$ again to denote our cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbf{A})$*. Let $\phi \in V_\pi$ be any new new vector, i.e. any pure tensor $\varphi = \otimes_v \varphi_v \in V_\pi$ whose nonarchimedean local components are each essential Whittaker vectors, which we can and do view as a cuspidal automorphic form on* $\mathrm{GL}_2(\mathbf{A})$ *with trivial central character, and hence as a cuspidal automorphic form on the quotient* $\mathrm{PGL}_2(\mathbf{A})$*. Assume that the central value $L(1/2, \pi \otimes \eta)$ of the standard L-function of π twisted by the quadratic character $\eta = \eta_{K/\mathbf{Q}}$ associated to the real quadratic field K does not vanish*[8]*. Then, there exists a genuine automorphic form $\widetilde{\phi}$ on* $\widetilde{\mathrm{SL}}_2(\mathbf{A})$ *corresponding to modular form of weight* $3/2$ *such that, as functions of $h \in \mathrm{SO}(\mathcal{V}) \cong \mathrm{PGL}_2(\mathbf{A})$, we have*

$$\phi(h) = \vartheta_{\widetilde{\phi}}(h) = \vartheta_{\widetilde{\phi}}(h; \Phi_0).$$

In this way, we can realize ϕ as a Shintani-Shimura-Waldspurger theta lift (26) *from this form $\widetilde{\phi}$ on* $\widetilde{\mathrm{SL}}_2(\mathbf{A})$*.*

Proof. That such a vector exists is a consequence of the existence of the Shimura correspondence shown in [43], together with the representation theoretic interpretation given by Waldspurger [47]. Note that the precise conditions required to ensure the nontriviality the corresponding central character of the genuine metaplectic form $\widetilde{\phi}$ on $\widetilde{\mathrm{SL}}_2(\mathbf{A})$ corresponding to a modular form on weight $3/2$ are determined precisely in Waldspurger [48, Proposition], cf. also [38, Proposition 3.3]. □

[8]Hence, if we take for granted the conjecture of Birch and Swinnerton-Dyer (which is known for this special case), then we assume that the Mordell-Weil group $E^{(d_K)}(\mathbf{Q})$ of the quadratic twist $E^{(d_K)}$ of E determined by the discriminant d_K of the character $\eta_{K/\mathbf{Q}}$ is finite. At the same time, in order to ensure that our standing condition on the root number $\eta_{K/\mathbf{Q}}(-N)$ of the basechange L-function $L^{\star}(E/K, 1) = L(s - 1/2, \pi)L(s - 1/2, \pi \otimes \eta_{K_{\mathbf{Q}}})$ is met, we would also have to assume the vanishing of the base L-function $L^{\star}(E, s) = L(1/2, \pi)$ corresponding (via Birch-Swinnerton-Dyer) to the Mordell-Weil group $E(\mathbf{Q})$ having odd (positive) rank.

Note that we can use this realization $\phi = \vartheta_{\widetilde{\phi}}$ in Proposition 4.2 above: Writing $\zeta = (g, \pm 1) \in \widetilde{\mathrm{SL}}_2(\mathbf{A})$ to denote the metaplectic variable, and taking the extension to similitudes in the same way, we then derive
(27)

$$\begin{aligned}
&L(s, \Pi \otimes \chi) \\
&= \int_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \phi(\sigma g) \left(\int_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A})} \theta_{r_{\psi_0}} \left(\left(\begin{matrix} y & \\ & 1 \end{matrix} \right), \sigma g; \Phi \right) \chi(y) |y|^{s - \frac{1}{2}} dy \right) d\sigma \\
&= \int\limits_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int\limits_{\mathrm{SL}_2(\mathbf{Q}) \backslash \widetilde{\mathrm{SL}}_2(\mathbf{A})} \theta_{\omega_{\psi_0}}(\sigma g, \zeta; \Phi_0) \widetilde{\phi}(\zeta) d\zeta \int\limits_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A})} \theta_{r_{\psi_0}} \left(\left(\begin{matrix} y & \\ & 1 \end{matrix} \right), \sigma g; \Phi \right) \chi''(y) |y|^{s - \frac{1}{2}} dy d\sigma \\
&= \int\limits_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \int\limits_{\mathrm{Sp}(\mathcal{W})(\mathbf{Q}) \backslash \widetilde{\mathrm{Sp}}(\mathcal{W})(\mathbf{A})} \theta_{\omega_{\psi_0}}(\sigma g, \zeta; \Phi_0) \widetilde{\phi}(\zeta) d\zeta \int\limits_{\mathrm{GO}(V_2)(\mathbf{Q}) \backslash \mathrm{GO}(V_2)(\mathbf{A})} \theta_{r_{\psi_0}} \left(\left(\begin{matrix} y & \\ & 1 \end{matrix} \right), \sigma g; \Phi \right) \chi''(y) |y|^{s - \frac{1}{2}} dy d\sigma.
\end{aligned}$$

Thus, in the notations of Propositions 4.2 and 4.3 above, we also derive the abstract integral presentation

$$L'(1/2, \Pi \otimes \chi) = \frac{d}{ds}\Big|_{s=1/2} \langle \vartheta_{\widetilde{\phi}}(*, \Phi_0), Z_{\chi''}(s, *, \Phi) \rangle. \tag{28}$$

To be clear, we use two subsequent theta liftings to derive this relation (28). First, we realize the cuspidal automorphic form φ on $\mathrm{GL}_2(\mathbf{A}_K)$ via its corresponding form φ'' on $\mathrm{GO}(V)(\mathbf{A})$ as a theta lifting $\varphi = \varphi_\phi$ from some cuspidal form $\phi \in V_\pi$ on $\mathrm{GL}_2(\mathbf{A}) \cong \mathrm{GSp}_2(W)(\mathbf{A})$ via the quadratic basechange lifting of Langlands-Shintani. Viewing this ϕ in terms of its corresponding form on $\mathrm{PGL}_2(\mathbf{A}) \cong \mathrm{SO}(\mathcal{V})(\mathbf{A})$, and taking for granted that the requisite nontriviality or existence conditions detailed in [48, Proposition] (cf. [38, Proposition 3.3]) are met, we can then realize the cuspidal form ϕ as a theta lifting $\vartheta_{\widetilde{\phi}}$ from some genuine metaplectic form $\widetilde{\phi}$ on $\widetilde{\mathrm{SL}}_2(\mathbf{A})$ via the Shimura-Shintani-Waldspurger correspondence.

4.4. **Regularized theta lifts and automorphic Green's functions.** We now give a more arithmetic treatment of the discussion above via regularized theta lifts following Borcherds [4], Kudla [32], and Bruinier-Funke [7]. Here, we shall first describe the setup for arbitrary quadratic spaces of signature $(n, 2)$ following [33], although our main interest is the quadratic space (V, q) of signature $(2, 2)$ introduced above[9]. That is, we shall usually fix $V = (V, q)$ to be the quadratic space of signature $(2, 2)$ with vector space $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$, and associated quadratic form $q(x, y, \lambda) = \lambda\lambda^\tau - xy$ for $x, y \in \mathbf{Q}$ and $\lambda \in K$. Here again, λ^τ denotes the image of $\lambda \in K$ under the conjugation in K. We then describe the distinct setup of regularized theta lifts for the anisotropic quadratic subspaces (V_2, q_2) of signature $(1, 1)$ determined by the quadratic field $V_2 = K$ with quadratic form $q_2(\lambda) = q|_{V_2}(\lambda) = \mathbf{N}_{K/\mathbf{Q}}(\lambda)$ given by the norm $\mathbf{N}_{K/\mathbf{Q}}$ following [7] for our subsequent calculations. More generally, given a ring class $A \in \mathrm{Pic}(\mathcal{O}_c)$ with integral ideal representative $\mathfrak{a}$ (so that $A = [\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$), and writing $\mathfrak{a}_{\mathbf{Q}} = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{Q}$ for the the corresponding vector space, we shall later consider the corresponding quadratic spaces (V_A, q_A) of signature $(2, 2)$ given by $V_A = \mathfrak{a}_{\mathbf{Q}} \oplus \mathbf{Q} \oplus \mathbf{Q}$ and $q_A(x, y, \lambda) := \mathbf{N}\mathfrak{a}^{-1}\mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$, together with the quadratic subspaces $(V_{A,2}, q_{A,2})$ of signature $(1, 1)$ given by $V_{A,2} = \mathfrak{a}_{\mathbf{Q}}$ and $q_{A,2}(\lambda) = \mathbf{N}\mathfrak{a}^{-1}\mathbf{N}_{K/\mathbf{Q}}(\lambda)$.

4.4.1. *Arithmetic automorphic forms.* Let first explain how the vector φ' defined above can be viewed as an arithmetic automorphic form on the Shimura variety $\mathrm{Sh}_U(\mathrm{GSpin}_V, D_V^{\pm})$. We now take $D = D_V$ to be the Grassmannian of oriented negative definite planes $z \subset V(\mathbf{R})$ as in [32], and consider

$$\mathrm{Sh}_U(\mathrm{GSpin}_V, D) = \mathrm{GSpin}_V(\mathbf{Q}) \backslash (D \times \mathrm{GSpin}_V(\mathbf{A}_f)/U).$$

Note that we could also take the Grassmannian $D = D_V^+$ instead to get the equivalent result (cf. [6, §2]). In fact, we shall make this choice in a consistent way later. Let us also note that the Grassmannian D can be

[9]In fact, we could work more generally with (V, q) or (V_A, q_A) any rational quadratic space of signature $(n, 2)$ with $n \geq 2$ admitting a distinguished quadratic subspace $(V_2, q_2) = (K, \mathbf{N}_{K/\mathbf{Q}}(\cdot))$ or $(V_{A,2}, q_{A,2}) = (\mathfrak{a}_{\mathbf{Q}}, \mathbf{N}\mathfrak{a}^{-1}\mathbf{N}_{K/\mathbf{Q}}(\cdot))$ respectively. For instance, we could just as well take (V, q) with $V = K' \oplus K$ for K' another real quadratic field, and quadratic form $q(\lambda', \lambda) = \mathbf{N}_{K'/\mathbf{Q}}(\lambda') + \mathbf{N}_{K/\mathbf{Q}}(\lambda)$, then consider the corresponding spin Shimura varietes. We restrict to the simplest possible setting to keep the exposition as clear and explicit as possible.

identificed with the open subset Q_- of the quadric defined by

$$Q_- = \{w \in V(\mathbf{C}) : (w, w) = 0, (w, \overline{w}) > 0\} / \mathbf{C}^\times \subset \mathbb{P}(V(\mathbf{C}))$$

via the map sending $z \in D$ to $v_1 - iv_2 = w$ for v_1, v_2 a properly-oriented standard basis for D with $(v_1, v_1) = (v_2, v_2) = -1$ and $(v_1, v_2) = 0$. We henceforth take this identification $D = D_V \cong Q^-$ for granted.

Fix a compact open subgroup $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$, which later we shall take to be that corresponding to the level of the basechange form $\varphi \in V_\Pi$ on $\mathrm{GL}_2(\mathbf{A}_K)$. Fixing a connected component D^+ of D, and writing $\mathrm{GSpin}_V(\mathbf{R})^+$ to denote the corresponding component of $\mathrm{GSpin}_V(\mathbf{R})$, we have for some fixed set of representatives $h_j \in \mathrm{GSpin}_V(\mathbf{Q}) \backslash \mathrm{GSpin}_V(\mathbf{A}_f)/U$ the decomposition

$$\mathrm{GSpin}_V(\mathbf{A}) = \coprod_j \mathrm{GSpin}_V(\mathbf{Q}) \, \mathrm{GSpin}_V(\mathbf{R})^+ h_j U. \tag{29}$$

This gives us the corresponding decomposition of the Shimura variety as

$$\mathrm{Sh}_U(\mathrm{GSpin}_V, D) = \coprod_j X_j, \quad \text{where } X_j = \Gamma_j \backslash D^+ \text{ for } \Gamma_j := \mathrm{GSpin}_V(\mathbf{Q}) \cap \left(\mathrm{GSpin}_V(\mathbf{R})^+ h_j U h_j^{-1} \right). \tag{30}$$

Let $\mathcal{L}_D$ denote the restriction of $D \cong Q_-$ of the tautological or universal bundle on $\mathbb{P}(V(\mathbf{C}))$. The natural action of $V(\mathbf{C})$ on $\mathrm{GO}(V)(\mathbf{R})$ induces one of $\mathcal{L}_D$ on $\mathrm{GSpin}_V(\mathbf{R})^+$. Hence, there is a holomorphic line bundle

$$\mathcal{L} := \mathrm{GSpin}_V(\mathbf{Q}) \backslash (\mathcal{L}_D \times \mathrm{GSpin}_V(\mathbf{A}_f)/U) \longrightarrow \mathrm{Sh}_U(\mathrm{GSpin}_V, D).$$

In fact, this bundle $\mathcal{L}$ is known by work of Harris [24] to have a canonical model over $\mathbf{Q}$. After restriction to one of components $\Gamma_j \backslash D^+$ in (30), it has the form $\Gamma_j \backslash \mathcal{L}_D$. We can define a hermitian metric $h_\mathcal{L}$ on $\mathcal{L}$ by

$$h_\mathcal{L}(w_1, w_2) := \frac{1}{2} \cdot (w_1, \overline{w}_2).$$

This metric is invariant under the action by $\mathrm{GO}(V)(\mathbf{R})$, and hence descends to $\mathcal{L}$.

Fix a Witt decomposition

$$V(\mathbf{R}) = V_0 + \mathbf{R} \cdot e + \mathbf{R} \cdot f$$

with e and f chosen so that $(e, e) = (f, f) = 0$ and $(e, f) = 1$, and

$$C = \{y \in V_0 : (y, y) < 0\}$$

its negative cone. We can then identify the Grassmannian $D \cong Q_-$ with the corresponding tube domain

$$\mathcal{H} := \{z \in V_0(\mathbf{C}) : \Im(z_0) \in C\} \cong \mathfrak{H}^2$$

via the map $\mathcal{H} \longrightarrow V(\mathbf{C})$ sending $z \longmapsto w(z) := z + e - q(z) f$ composed with the projection to Q_-. The map $z \mapsto w(z)$ can be viewed as a nowhere vanishing section of $\mathcal{L}_D$, whose norm we define to be

$$||w(z)|| = -\frac{1}{2} \cdot (w(z), \overline{w}(z)) = -(y, y) =: |y|^2.$$

Moreover, given $h \in \mathrm{GO}(V)(\mathbf{R})$ or $h \in \mathrm{GSpin}_V(\mathbf{R})$, we have an automorphy factor

$$j : \mathrm{GSpin}_V(\mathbf{R}) \times D \longrightarrow \mathbf{C}^\times.$$

defined by

$$h \cdot w(z) = w(hz) \cdot j(h, z).$$

In this way, the holomorphic sections of $\mathcal{L}^{\otimes k}$ for any integer $k \in \mathbf{Z}$ can be interpreted as those holomorphic functions $\Psi : D \times \mathrm{GSpin}_V(\mathbf{A}_f) \longrightarrow C$ on $z \in D$ and $h \in \mathrm{GSpin}(\mathbf{A}_f)$ satisfying the transformation properties

- $\Psi(z, hu) = \Psi(z, h)$ for all $u \in U$,
- $\Psi(\gamma z, \gamma h) = j(\gamma, z)^k \cdot \Psi(z, h)$ for all $\gamma \in \mathrm{GSpin}_V(\mathbf{Q})$.

We define the norm of a section $(z, h) \to \Psi(z, h) \cdot w(z)^{\otimes k}$ to be

$$||\Psi(z, h)||^2 = |\Psi(z, h)|^2 \cdot |y|^{2k},$$

we refer to this as the *Petersson norm* of the holomorphic section Ψ. Note that under the isomorphism (30), such a section Ψ corresponds to the collection $\{\Psi(\cdot, h_j)\}_j$ of holomorphic functions on D^+ which are holomorphic of weight k on the corresponding Γ_j. The latter forms have a classical interpretation as modular forms corresponding to congruence subgroups of lattices $\Lambda \subset V$, and correspond to holomorphic Hilbert modular forms of parallel weight k in this setting (see e.g. the discussion [6, § 2.7], as recalled below).

4.4.2. *Regularized theta lifts for quadratic spaces of signature* $(2,2)$. Let us now take (V,q) to be any rational quadratic space of signature $(2,2)$, although the discussion that follows carries over more generally for any quadratic space of signature $(n,2)$. We outline the construction due to Borcherds [4] of sections of twists of $\mathcal{L}^{\otimes k}$ using the theta correspondence for the reductive dual pair $(\mathrm{GO}(V), \mathrm{Sp}(W))$, following the adelic description given by Kudla [32].

Given a hyperplane $z \in D = D_V$, let $\mathrm{pr}_z : V(\mathbf{R}) \longrightarrow z$ denote the corresponding projection, whose kernel defines the orthogonal complement $z^{\perp} := \ker(\mathrm{pr}_z)$. Given a vector $x \in V(\mathbf{R})$, let us then define

$$R(x,z) := -\left(\mathrm{pr}_x(x), \mathrm{pr}_z(x)\right) = \left|(x, w(z))\right|^2 \cdot |y|^2.$$

Using this definition, we can associate to a plane $z \in D = D_V$ and vector $x \in V(\mathbf{R})$ a majorant

$$(x,x)_z := (x,x) + 2 \cdot R(x,z).$$

Writing $\mathcal{C}^{\infty}(D)$ to denote the space of smooth functions on the Grassmannian $D = D_V^{-}$, we now use this majorant as follows to define a Gaussian function $\Phi_{\infty} \in \mathcal{S}(V(\mathbf{R})) \otimes \mathcal{C}^{\infty}(D)$ by the rule

$$\Phi_{\infty}(x,z) := \exp\left(-\pi \cdot (x,x)_z\right).$$

It is easy to deduce that

$$\Phi_{\infty}(hx, hz) = \Phi_{\infty}(x,z) \quad \text{ for all } h \in \mathrm{SO}(V)(\mathbf{R}).$$

Recall that we write $\psi_0 = \otimes_v \psi_{0,v}$ to denote the standard additive character of $\mathbf{A}/\mathbf{Q}$, with $r_{\psi_0} = \otimes_v r_{\psi_{0,v}}$ the corresponding Weil representation

$$r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{Sp}(W)(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(V(\mathbf{A})),$$

as well as its extension to the similitude group

$$R(\mathbf{A}) := \{(h,g) \in \mathrm{GO}(V)(\mathbf{A}) \times \mathrm{GSp}(W)(\mathbf{A}) : \nu(h) = \det(g)\} \subset \mathrm{GO}(V)(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A}).$$

Note that in this setting where $\dim(V) = 4$ is even, the Weil representation r_{ψ_0} factors through $\mathrm{Sp}(\mathbb{W})$ rather than its metaplectic cover $\widetilde{\mathrm{Sp}}(\mathbb{W})$. The action of $\mathrm{Sp}(W)(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A})$ on $\mathcal{S}(\mathbb{W}(\mathbf{A}))$ commutes with that of $\mathrm{SO}(V)(\mathbf{A})$. Let us write $r_{\psi_0}(h)\Phi(x) = \Phi(h^{-1}x)$ for $h \in \mathrm{SO}(V)(\mathbf{A})$ and $\Phi \in \mathcal{S}(\mathbb{W}(\mathbf{A}))$ to denote the former action. We now make the following modification to the definition of the corresponding theta kernel $\theta_{r_{\psi_0}}$ defined above. Let us for a given hyperplane $z \in D$, $h_f \in \mathrm{SO}(V)(\mathbf{A}_f)$ and $g \in \mathrm{Sp}(W)(\mathbf{A}) \cong \mathrm{SL}_2(\mathbf{A})$ write $\theta^{\star}_{r_{\psi_0}}$ to denote the linear functional on $\Phi_f \in \mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ defined by

$$\begin{aligned}\Phi_f \longmapsto \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f) &:= \sum_{x \in \mathbb{W}(\mathbf{Q})} r_{\psi_0}(g)\left(\Phi_{\infty}(\cdot, z) \otimes r_{\psi_0}(h_f)\Phi_f\right)(x) \\ &= \sum_{x \in \mathbb{W}(\mathbf{Q})} r_{\psi_0}(1,g)\left(\Phi_{\infty}(\cdot, z) \otimes r_{\psi_0}(h_f,1)\Phi_f\right)(x).\end{aligned} \tag{31}$$

Note that this functional (31) can be thought of as an arithmetic modification of the theta kernel $\theta_{r_{\psi_0}}(h,g;\Phi)$ for a decomposable Schwartz function in $\mathcal{S}(\mathbb{W}(\mathbf{A}))$ as introduced above. To be more precise (cf. [8, (2.3)]), fixing a base hyperplane $z_0 \in D$ and an element $h_z \in \mathrm{GSpin}_V(\mathbf{R})$ so that $h_z z_0 = z$, we have the relation

$$\theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f) := \theta_{r_{\psi_0}}\left(h_f h_z, g; \Phi_{\infty}(\cdot, z) \otimes \Phi_f(\cdot)\right). \tag{32}$$

It is easy to see that this is automorphic for the special orthogonal group: For all $\gamma \in \mathrm{SO}(V)(\mathbf{Q})$, we have

$$\theta^{\star}_{r_{\psi_0}}(\gamma z, \gamma h_f, g; \Phi_f) = \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f).$$

By Poisson summation (see [50], [32, (1.22)]), we can also deduce that the functional is automorphic for the symplectic group $\mathrm{Sp}(W) \cong \mathrm{SL}_2$: For all $\gamma \in \mathrm{SL}_2(\mathbf{Q})$, we have

$$\theta^{\star}_{r_{\psi_0}}(z, h_f, \gamma g; \Phi_f) = \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f).$$

Using properties of the Weil representation r_{ψ_0}, we see that for any $h'_f \in \mathrm{SO}(V)(\mathbf{A}_f)$ and any $g' \in \mathrm{SL}_2(\mathbf{A})$,

$$\theta^{\star}_{r_{\psi_0}}(z, h_f h'_f, g g'; \Phi_f) = \theta^{\star}_{r_{\psi_0}}(z, h_f, g; r_{\psi_0}(h'_f, g')\Phi_f). \tag{33}$$

In this way, we see that for any compact open subgroup $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ and decomposable U-invariant Schwartz function $\Phi \in \mathcal{S}(\mathbb{W}(\mathbf{A}_f))^U$, the functional

$$(z, h_f) \longmapsto \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f)$$

on $(z, h_f) \in D \times \mathrm{GSpin}_V(\mathbf{A}_f)$ descends to a function on the corresponding Shimura variety $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$. Although it is not holomorphic in the variable $z \in D$, we obtain in this way a function

$$\theta^{\star}_{r_{\psi_0}} : \mathrm{Sh}_U(\mathrm{GSpin}_V, D) \times \mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A}) \longrightarrow \left(\mathcal{S}(\mathbb{W}(\mathbf{A}_f))^U\right)^{\vee}.$$

Extending to similitudes as in the discussion above, we also obtain a function

$$\theta^{\star}_{r_{\psi_0}} : \mathrm{Sh}_U(\mathrm{GSpin}_V, D) \times \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A}) \longrightarrow \left(\mathcal{S}(\mathbb{W}(\mathbf{A}_f))^U\right)^{\vee}.$$

As explained in [32, §1], we can view the Gaussian Φ_{∞} as an eigenfunction for the action of the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R})$, which for any $k_{\infty} \in \mathrm{SO}_2(\mathbf{R})$, $z \in D$, and $h \in \mathrm{GSpin}_V(\mathbf{A})$ satisfies

$$r_{\psi_0}(k_{\infty})\Phi_{\infty}(x, z) = \Phi_{\infty}(x, z).$$

Using the transformation property (33), we can then deduce that for all k_{∞} in the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R})$ of $\mathrm{SL}_2(\mathbf{R})$ and all k in the maximal compact subgroup $\mathcal{K} = \mathrm{SL}_2(\widehat{\mathbf{Z}})$ of $\mathrm{SL}_2(\mathbf{A}_f)$, we have that

$$\theta^{\star}_{r_{\psi_0}}(z, h_f, gk_{\infty}k; \Phi_f) = \left(r_{\psi_0}(k)^{\vee}\right)^{-1} \cdot \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \Phi_f), \tag{34}$$

where $r_{\psi_0}(k)^{\vee}$ denotes the action of $\mathcal{K}$ on the space $\mathcal{S}(\mathbb{W}(\mathbf{A}_f))^{\mathcal{K}}$ dual to its action on $\mathcal{S}(\mathbb{W}(\mathbf{A}_f))^{\mathcal{K}}$. In particular, this theta kernel $\theta^{\star}_{r_{\psi_0}}$ in the setting of quadratic spaces of signature $(2, 2)$ as we consider has weight zero under the action of the maximal compact subgroup $\mathrm{SO}_2(\mathbf{R}) \subset \mathrm{SL}_2(\mathbf{R})$.

Suppose now that we fix any function

$$\phi : \mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W}(\mathbf{A}_f))^U$$

which for each $g \in \mathrm{SL}_2(\mathbf{A})$, $k_{\infty} \in \mathrm{SO}_2(\mathbf{R})$, and $k \in \mathcal{K}$ satisfies the transformation property

$$\phi(gkk_{\infty}) = r_{\psi_0}(k)^{-1} \cdot \phi(g).$$

It is then easy to check that the $\mathbf{C}$-linear pairing $\{\cdot, \cdot\}$ defined as a function on $g \in \mathrm{SL}_2(\mathbf{A})$ by the rule

$$\left\{\phi(g), \theta^{\star}_{r_{\psi_0}}(z, h_f, g)\right\} := \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \phi(g))$$

is both left $\mathrm{SL}_2(\mathbf{Q})$-invariant and right $\mathcal{K}\,\mathrm{SO}_2(\mathbf{R})$-invariant. We can then consider the *regularized theta lift*

$$\vartheta^{\star}_{\phi}(z, h_f) := \int^{\star}_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \left\{\phi(g), \theta^{\star}_{r_{\psi_0}}(z, h_f, g)\right\} dg = \int^{\star}_{\mathrm{SL}_2(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{A})} \theta^{\star}_{r_{\psi_0}}(z, h_f, g; \phi(g)) dg, \tag{35}$$

as well as its extension to similitudes as described above, both as functions on $(z, h) \in \mathrm{Sh}_U(\mathrm{GSpin}_V, D)$. Here, the notation $\int^{\star}$ denotes the regularization, which is obtained after using the Iwasawa decomposition by taking a limit over truncated fundamental domains (as described e.g.in [32, §§1-2], [8, § 4], and [4]). To describe these regularized theta integrals $\int^{\star}$ in (35) more explicitly, we first give semiclassical translation of the setup (cf. [32, §1]). Recall (see e.g. [21, Proposition 4.4.4] or [19]) that after fixing a standard fundamental domain $\mathcal{F} = \{\tau = u + iv \in \mathfrak{H} : |\Re(\tau)| \leq 1/2, \tau\overline{\tau} \geq 1\}$ for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$, each adelic matrix $g \in \mathrm{SL}_2(\mathbf{A})$ can be expressed uniquely as a product

$$g = \gamma \cdot \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} v^{\frac{1}{2}} & \\ & v^{-\frac{1}{2}} \end{pmatrix} \cdot k \tag{36}$$

for some $\gamma \in \mathrm{SL}_2(\mathbf{Q})$, $\tau = u + iv \in \mathcal{F}$, and $k \in \mathrm{SO}_2(\mathbf{R})$. Taking the decomposition (36) for granted, let us define for a given $g \in \mathrm{SL}_2(\mathbf{A})$ the corresponding matrix

$$g_{\tau} := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac{1}{2}} & \\ & v^{-\frac{1}{2}} \end{pmatrix}.$$

Similarly, fixing a standard fundamental domain $\mathcal{G} = \{\tau = u + iv : 0 \leq |\Re(\tau)| \leq 1/2, \tau\overline{\tau} \geq 1\}$ for the action of $\mathrm{GL}_2(\mathbf{Z})$ on $\mathrm{GL}_2(\mathbf{R})$, each element $g \in \mathrm{GL}_2(\mathbf{A})$ can be decomposed uniquely as as a product of matrices

$$g = \gamma \cdot \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} v & \\ & 1 \end{pmatrix} \cdot k \tag{37}$$

for some $\gamma \in \mathrm{GL}_2(\mathbf{Q})$, $\tau = u + iv \in \mathcal{G}$, and $k \in O_2(\mathbf{R})$. Taking such a decomposition (37) for granted, we also define for a given $g \in \mathrm{GL}_2(\mathbf{A})$ define the corresponding archimedean mirabolic matrix

$$g_\tau := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & 1 \end{pmatrix}.$$

Given a weight-zero L^2-automorphic form ϕ_0 on $\mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A})$ or more generally $\mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A})$, we shall write $f_0(\tau) := \phi_0(g_\tau)$ to denote the corresponding weight-zero automorphic form on $\tau = u + iv \in \mathfrak{H}$.

Suppose now that $(\rho, \mathcal{V})$ is a representation of the maximal compact subgroup $\mathcal{K} = \mathrm{SL}_2(\widehat{\mathbf{Z}}) \subset \mathrm{SL}_2(\mathbf{A}_f)$, and that $\phi_0 : \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{V}$ is a weight-zero automorphic form for which the transformation law $\phi_0(gk_\infty k) = \rho(k)\phi_0(g)$ holds for all $g \in \mathrm{SL}_2(\mathbf{A})$, $k \in \mathcal{K}$, and $k_\infty \in \mathrm{SO}_2(\mathbf{R})$. Then, writing $k_\gamma \in \mathcal{K}$ for a given matrix $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ to denote the unique lifting $k_\gamma \in \mathcal{K}$ (determined by the diagonal embedding[10]), it is easy to check (see e.g. [32, Lemma 1.1]) that the weight-zero automorphic function defined by $f_0(\tau) := \phi_0(g_\tau)$ satisfies the following transformation law: For all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, we have that $f(\gamma(\tau)) = \rho(k_\gamma)f(\tau)$. Note that we can proceed in the same way for the more general setting with ϕ_0 a weight-zero automorphic form $\phi_0 : \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}) \longrightarrow \mathcal{V}$ satisfying $\phi_0(gk_\infty k) = \rho(k)\phi_0(g)$ for all $g \in \mathrm{GL}_2(\mathbf{A})$ and $k \in \mathcal{K}$, where $\mathcal{K} = \mathrm{GL}_2(\widehat{\mathbf{Z}}) \subset \mathrm{GL}_2(\mathbf{A}_f)$ denotes the maximal compact subgroup, and $k_\infty \in O_2(\mathbf{R})$. Thus, the corresponding weight-zero automorphic form defined by $f_0(\tau) = \phi_0(g_\tau)$ satisfies the transformation law $f_0(\gamma(\tau)) = \rho(k_\gamma)f_0(\tau)$ for all $\gamma \in \mathrm{GL}_2(\mathbf{A})$, with k_τ the unique lift to $\mathcal{K}$ via the diagonal embedding. Let us also note that we have a bijective correspondence between these vector-valued automorphic forms and scalar-valued automorphic forms as shown in [53, Theorem 4.15]; we shall return to this later. Taking for granted these correspondences, the regularized theta integral (35) can be written in semiclassical terms as

$$\begin{aligned} \vartheta^\star_{\phi_0}(z, h_f) = \vartheta^\star_{f_0}(z, h_f) &= \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \left\{ f_0(\tau), \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau) \right\} \frac{dudy}{v^2} \\ &= \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau; f_0(\tau)) \frac{dudv}{v^2}. \end{aligned} \tag{38}$$

Here, the regularized integral $\int^\star$ is defined more precisely as the limit over partial integrals

$$\begin{aligned} \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \left\{ f_0(\tau), \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau) \right\} \frac{dudy}{v^2} &= \varinjlim_T \int_{\mathcal{F}_T} \left\{ f_0(\tau), \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau) \right\} \frac{dudy}{v^2} \\ &= \varinjlim_T \int_{\mathcal{F}_T} \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau; f_0(\tau)) \frac{dudv}{v^2}, \end{aligned}$$

where each $\mathcal{F}_T$ denotes the truncated fundamental domain defined by

$$\mathcal{F}_T := \{\tau = u + it \in \mathfrak{H} : |u| \leq 1/2, \tau\overline{\tau} \geq 1, \text{ and } v \leq T\}.$$

Let us now take $\Lambda \subset V$ to be an even lattice (cf. [8, §2], [32, §1]), with $\Lambda^\#$ the corresponding dual lattice, and $\mathcal{S}_\Lambda \subset \mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ the subspace of Schwartz functions with support on $\widehat{\Lambda}^\# := \Lambda^\# \otimes \widehat{\mathbf{Z}}$ which are constant on cosets of $\widehat{\Lambda} := \Lambda \otimes \widehat{\mathbf{Z}}$. Note that this space $\mathcal{S}_\Lambda$ admits a basis of characteristic functions $\mathbf{1}_{\mu+\widehat{\Lambda}}$ of the form

$$\mathcal{S}_\Lambda = \bigoplus_{\mu \in \Lambda^\#/\Lambda} \mathbf{C} \cdot \mathbf{1}_{\mu+\widehat{\Lambda}} \subset \mathcal{S}(\mathbb{W}(\mathbf{A}_f)), \tag{39}$$

and also that it is stable under the action of our fixed compact open subgroup $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ through the Weil representation $r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W}(\mathbf{A}))$. Note as well that the space of Schwartz functions $\mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ can be written as a direct limit of such subspaces $\mathcal{S}(\mathbb{W}(\mathbf{A}_f)) = \varinjlim_\Lambda \mathcal{S}_\Lambda$. Given f_0 an $\mathcal{S}_\Lambda$-valued weight-zero automorphic form on $\mathfrak{H}$ as above, hence determined its corresponding $\mathcal{S}_\Lambda$-valued

[10] So that γk_γ is identified uniquely as an element of $\mathrm{SL}_2(\mathbf{Q})$

weight-zero automorphic form ϕ_0 on $\mathrm{SL}_2(\mathbf{A})$, which we shall assume is meromorphic at the cusp, let us first decompose f_0 into components with respect to the basis (39) as

$$f_0(\tau) = \sum_{\mu \in \Lambda^\#/\Lambda} f_{0,\mu}(\tau) \cdot \mathbf{1}_{\mu + \widehat{\Lambda}}.$$

We then write the Fourier series expansion of each component $f_{0,\mu}(\tau)$ as

$$f_{0,\mu}(\tau) = \sum_{m \in \mathbf{Q}} c_{f_0}(m,\mu) e(m\tau) = \sum_{m \in \mathbf{Q}} c_{f_0}(m,\mu) \exp(2\pi i \cdot m\tau).$$

As explained in [32], we can then write the pairing $\{\cdot,\cdot\}$ more explicitly as

$$\left\{ f_0(\tau), \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau) \right\} = \sum_{\mu \in \Lambda^\#/\Lambda} f_{0,\mu}(\tau) \theta^\star_{r_{\psi_0}}(z, h_f, g_\tau; f_0(\tau)).$$

We can now give the following explicit description of the theorems of Borcherds [4, Theorem 13.3] following [32, §1] for the setting we consider here.

Theorem 4.4 (Borcherds)**.** *Let $\phi_0 : \mathrm{SL}_2(\mathbf{Q})\backslash \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}_\Lambda$ be an $\mathcal{S}_\Lambda$-valued L^2-automorphic form of weight 0 on $\mathrm{SL}_2(\mathbf{A})$, with corresponding modular form $f_0(\tau) = \phi_0(g_\tau)$ determined by the unique decomposition* (36) *above. Assume f_0 is weakly holomorphic. Then for $(z,h) \in D \times \mathrm{GSpin}_V(\mathbf{A}_f)$, the regularized theta integral*

$$\begin{aligned} \vartheta^\star_{f_0}(z,h) &= \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \left\{ f_0(\tau), \theta^\star_{r_{\psi_0}}(z,h,g_\tau) \right\} \frac{dudv}{v^2} \\ &= \int^\star_{\mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{H}} \theta^\star_{r_{\psi_0}}(z,h,g_\tau; f_0(\tau)) \frac{dudv}{v^2} \\ &= \varinjlim_{T} \int_{\mathcal{F}_T} \theta^\star_{r_{\psi_0}}(z,h,g_\tau; f_0(\tau)) \frac{dudv}{v^2} \end{aligned}$$

is equivalent to the expression

$$\vartheta^\star_{f_0}(z,h) = -2 \log |\Psi_{f_0}(z,h)|^2 - c_{f_0}(0,0) \cdot (2 \log|y| + \log(2\pi) + \Gamma'(1))$$

for some meromorphic modular form Ψ_{f_0} on $(z,h) \in D \times \mathrm{GSpin}_V(\mathbf{A}_f)$ of weight $k = \frac{1}{2} c_{f_0}(0,0) \in \mathbf{Z}$. More precisely, for some choice of unitary character ξ of $\mathrm{GSpin}_V(\mathbf{Q})$, this latter function satisfies for each $\gamma \in \mathrm{GSpin}_V(\mathbf{Q})$ the transformation property $\Psi_{f_0}(\gamma z, \gamma h) = \xi(\gamma) \cdot j(\gamma,z)^k \cdot \Psi_{f_0}(z,h)$ for $j(\gamma,z)$ the standard automorphy factor of weight k. In other words, the function $\Psi_{f_0}(z,h)$ determines a meromorphic section of the twisted line bundle $\mathcal{L}^{\otimes k} \otimes \mathcal{V}_\xi$, where $\mathcal{V}_\xi$ denotes the flat bundle determined by the character ξ.

The divisor $Z(f_0)$ of $\Psi^2_{f_0}$ describing the logarithmic singularity of $\vartheta^\star_{f_0}$ is also determined by Borcherds [4]. This leads to the more general characterization of the regularized theta lift $\vartheta^\star_{f_0}(z,h)$ attached to any harmonic weak Maass form f_0 of weight 0 as an automorphic Green's function for this divisor $Z(f_0)$ shown by Bruinier (see [8, Theorem 4.2]). To describe this in more detail, we first introduce certain special cycles and analytic divisors on $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$. Here, we follow the discussion Kudla [32] (cf. [31]), which we note applies to any codimension. Let us also write h rather than h_f to denote a generic element of $\mathrm{GSpin}_V(\mathbf{A}_f)$ to simplify notation. Given a vector $x \in V(\mathbf{Q})$ with $q(x) > 0$, let $V_x := x^\perp \subset V$ denote the orthogonal complement, and $D_x = \{z \in D : x \perp z\}$ the corresponding sub-Grassmannian. Let us also write $\mathrm{GSpin}_{V,x}(\mathbf{A}_f)$ to denote the stabilizer in $\mathrm{GSpin}_V(\mathbf{A}_f)$ of x, which we can and do identify as $\mathrm{GSpin}_{V,x}(\mathbf{A}_f) \cong \mathrm{GSpin}_{V_x}(\mathbf{A}_f)$. We then have a natural map defined on $h \in \mathrm{GSpin}_V(\mathbf{A}_f)$ by

$$\begin{aligned} \mathrm{GSpin}_{V,x}(\mathbf{Q})\backslash D_x \times \mathrm{GSpin}_{V,x}(\mathbf{A}_f)/\left(\mathrm{GSpin}_{V,x}(\mathbf{A}_f) \cap hUh^{-1}\right) &\longrightarrow \mathrm{Sh}_U(\mathrm{GSpin}_V, D) \\ [z, h_1] &\longmapsto [z, h_1 h]. \end{aligned} \tag{40}$$

Definition 4.5. *Given $x \in V(\mathbf{Q})$ with $q(x) > 0$ and $h \in \mathrm{GSpin}_V(\mathbf{A}_f)$, let*

$$Z(x,h) = Z(x,h,U)$$

denote the image of the corresponding map (40)*. Here, we drop the fixed compact open subgroup $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ from the notation when the context is clear.*

This image $Z(x,h) = Z(x,h,U)$ determines a special cycle $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$, and is known to be rational over $\mathbf{Q}$. As explained in [32, §1] and [31], these special cycles satisfy many nice properties, including compatibility with Hecke operators. To illustrate a couple of these relevant properties before going on, let is for a given positive rational number $m \in \mathbf{Q}_{>0}$ write Ω_m to denote the corresponding quadric

$$\Omega_m(\mathbf{Q}) = \{x \in V : q(x) = m\}.$$

If $\Omega_m(\mathbf{Q})$ is not the empty set, in which case we fix a point $x_0 \in \Omega_m(\mathbf{Q})$, the corresponding finite adelic points $\Omega_m(\mathbf{A}_f)$ determine a closed subgroup of $V(\mathbf{A}_f)$. Given $\Phi \in \mathcal{S}(\mathbb{W}(\mathbf{A}_f))^U$, we then write

$$\mathrm{supp}(\Phi) \cap \Omega_m(\mathbf{A}_f) = \coprod_r U \cdot \zeta_r^{-1} \cdot x_0 \tag{41}$$

for some finite set of representatives $\zeta_r \in \mathrm{GSpin}_V(\mathbf{A}_f)$. We then define from this decomposition (41) the corresponding analytic divisor

$$Z(m, \Phi, U) := \sum_r \Phi(\zeta_r^{-1} \cdot x_0) Z(x_0, \zeta_r, U). \tag{42}$$

If $U' \subset U$ is an inclusion of compact open subgroups of $\mathrm{GSpin}_V(\mathbf{A}_f)$ and

$$\mathrm{pr} : \mathrm{Sh}_{U'}(\mathrm{GSpin}_V, D) \longrightarrow \mathrm{Sh}_U(\mathrm{GSpin}_V, D)$$

the corresponding covering of Shimura varieties, we have the projection formula

$$\mathrm{pr}^* Z(m, \Phi, U) = Z(m, \Phi, U').$$

Hence, the analytic divisor is defined on full Shimura variety $\mathrm{Sh}(\mathrm{GSpin}_V, D) = \varprojlim_U \mathrm{Sh}(\mathrm{GSpin}_V, D)$, and so we are justified in dropping the reference to the compact open subgroup from the notation altogether. We can also consider the right multiplication by $h \in \mathrm{GSpin}_V(\mathbf{A}_f)$, which determines a morphism

$$[h] : \mathrm{Sh}_U(\mathrm{GSpin}_V, D) \longrightarrow \mathrm{Sh}_{hUh^{-1}}(\mathrm{GSpin}_V, D).$$

This morphism $[h]$ is defined over $\mathbf{Q}$, and its pushforward $[h]_*$ satisfies the relation

$$[h]_* : Z(m, \Phi, U) \longrightarrow Z(m, r_{\psi_0}(h)\Phi, hUh^{-1}), \quad \text{where } r_{\psi_0}(h)\Phi(x) = \Phi(h^{-1}x).$$

In this way, we can deduce that these analytic divisors are compatible with Hecke operators on $\mathrm{Sh}(\mathrm{GSpin}, D)$. Moreover, with respect to the decomposition (30) above, the result of [31, Proposition 5.3] (cf. also [32, §1]) shows that the analytic divisor $Z(m, \Phi, U)$ decomposes as

$$Z(m, \Phi, U) = \sum_j Z_j(m, \Phi, U),$$

where for each factor j we write

$$Z_j(m, \Phi, U) = \sum_{x \in \Omega_m(\mathbf{Q}) \bmod \Gamma_j} \Phi(h_j^{-1}x)\, \mathrm{pr}_j(D_x) \qquad \text{for } \mathrm{pr}_j : D^+ \longrightarrow \Gamma_j \backslash D^+ \text{ the natural projection.}$$

Writing $\Phi^\vee(x) = \Phi(-x)$, these analytic divisors also satisfy the functional equations $Z(m, \Phi, U) = Z(m, \Phi^\vee, U)$.

Theorem 4.6 (Borcherds/Bruinier)**.** *Given $f_0 \in M^!_{0,\Lambda}$ any weakly holomorphic form of weight zero and representation $r_{\phi_0,\Lambda}$ with Fourier expansion*

$$f_0(\tau) = \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_{f_0}(m, \mu) e(m\tau),$$

the regularized theta lift $\vartheta^\star_{f_0}$ is smooth on $\mathrm{Sh}_U(\mathrm{GSpin}_V, D) \backslash Z(f_0)$, where $Z(f_0)$ is the divisor defined by

$$Z(f_0) = \mathrm{div}\left(\Psi^2_{f_0}\right) = \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}(-m, \mu) Z(m, \mathbf{1}_{\mu + \widehat{L}}, U).$$

More generally, given $f_0 = f_0^+ + f_0^- \in H_{0,\Lambda}$ any harmonic weak Maass form of weight 0 and representation $r_{\phi_0,\Lambda}$ the regularized theta lift $\vartheta_{f_0}^\star$ determines a smooth function on $\mathrm{Sh}_U(\mathrm{GSpin}, D)$ with logarithmic singularity along the divisor $-2Z(f_0)$, where

$$Z(f_0) := \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^+(-m, \mu) Z(m, \mathbf{1}_{\mu + \widehat{L}}, U)$$

depends only on the holomorphic part f_0^+. The $(1,1)$-form $dd^c \vartheta_{f_0}^\star = -(2\pi i)^{-1} \partial\overline{\partial} \vartheta_{f_0}^\star$ can be continued to a smooth function on all of $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$, and writing $\delta_{Z(f_0)}$ to denote the Dirac current, we have the Green current equation $dd^c[\vartheta_{f_0}^\star] + \delta_{Z(f_0)} = [dd^c \vartheta_{f_0}^\star]$. Writing Δ_z to denote the invariant Laplacian operator on D (normalized as in [5] *and* [8, Theorem 4.2]*), we have that $\Delta_z \vartheta_{f_0}^\star = 2 \cdot c_{f_0}(0,0)$.*

Proof. See [4, Theorem 13.3], as well as the descriptions in [32, Theorem 1.3] and [8, Theorem 4.2]. □

Theorem 4.6 can be used to show that $\vartheta_{f_0}^\star$ is the automorphic Green's function for the divisor $Z(f_0)$:

Theorem 4.7 (Bruinier)**.** *Let $G : \mathrm{Sh}_U(\mathrm{GSpin}_V, D) \backslash Z(f_0) \longrightarrow \mathbf{R}$ be any smooth function such that:*

(i) *G has logarithmic singularities along $-2Z(f_0)$,*

(ii) *$\Delta_z G$ is a constant,*

(iii) *$G \in L^{1+\varepsilon}(\mathrm{Sh}_U(\mathrm{GSpin}_V, D), d\mu)$ for some $\varepsilon > 0$, where $d\mu$ denotes the measure on $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$ induced from some choice of Haar measure on $\mathrm{GSpin}(\mathbf{A})$.*

Then, this function G – which defines the Green's function for the divisor $Z(f_0)$ – differs from $\vartheta_{f_0}^\star$ by a constant. In other words, the regularized theta lift $\vartheta_{f_0}^\star$ constructed from any harmonic weak Maass form f_0 of weight zero is the automorphic Green's function $G_{Z(f_0)}$ for the divisor $Z(f_0)$.

Proof. See [8, Theorems 4.2 and 4.3], as well as [7, Theorem 1.5]. As explained in [8, Theorem 4.3] and more generally [5, Corollary 4.22, cf. Theorem 4.23], the difference $G(z,h) - \vartheta_{f_0}^\star(z,h)$ can be viewed as a smooth subharmonic function on the complex manifold $\mathrm{Sh}_U(\mathrm{GSpin}_V, D)$ which is contained in the Hilbert space $L^{1+\varepsilon}(\mathrm{Sh}_U(\mathrm{GSpin}_V, D), d\mu)$. It then follows from a theorem of Yau [51] that such a function is constant. □

Remark on generality. Note that the work of Borcherds [4] applies to any rational quadratic spaces (V, q) of signature (p, r), with $D = D_V$ the corresponding symmetric space of oriented r-planes. The special case of $(n, 2)$ corresponding to hermitian symmetric spaces is better-known because of its applications to cycles on orthogonal Shimura varieties. The general setup is explained also in [7], with relation to the construction of Kudla-Millson [35]. In general, we have the identification $D \cong \mathrm{SO}(V)(\mathbf{R})/\mathcal{K}$ for $\mathcal{K} \subset \mathrm{SO}(V)(\mathbf{R})$ the maximal compact subgroup. Given an r-plane $z \in D_V$, we have a majorant $(\ ,\ )_z$ defined on $x \in V(\mathbf{R})$ by

$$(x, x)_z = (x_{z^\perp}, x_{z^\perp}) - (x_z, x_z).$$

Here, we decompose x as $x = x_z + x_{z^\perp}$ according to the orthogonal decomposition $z \oplus z^\perp = V(\mathbf{R})$, and write $(\ ,\ )$ to denote the bilinear form associated to q. Given a vector $x \in V(\mathbf{R})$, we write $D_x = \{z \in D : z \perp x\}$ to denote the corresponding subsymmetric space, with $\mathrm{SO}(V)(\mathbf{R})_x \subset \mathrm{SO}(V)(\mathbf{R})$ the corresponding stabilizer subgroup. Fixing a compact open subgroup $\Gamma \subset \mathrm{SO}(V)(\mathbf{R})$, we consider the corresponding intersection $\Gamma_x = \Gamma \cap \mathrm{SO}(V)(\mathbf{R})_x$. Given an even lattice $\Lambda \subset V$ and any element $x \in \Lambda^\#$ the quotient

$$Z(x) = \Gamma_x \backslash D_x \longrightarrow \Gamma \backslash D$$

determines a general relative cycle. For $\mu \in \Lambda^\#/\Lambda$ and $m \in \mathbf{Q}$, the group Γ acts on the set $\Lambda_{\mu,m} := \{x \in \Lambda + \mu, q(x) = m\}$ with finitely many orbits, giving rise to the so-called composite cycle

$$Z(\mu, m) = \sum_{x \in \Gamma \backslash \Lambda_{\mu,m}} Z(x).$$

As explained in [7, §4-6], we can replace the Gaussian Φ_∞ in our discussion above with a certain Schwartz function $\psi = \psi_{\mathrm{KM}}$ following the construction of Kudla-Millson [35]. By [35, Proposition 5.6, cf. Theorems 6.1

and 6.2], the corresponding regularized theta lift $\vartheta_{f_0}^{\star}(z,h)$ – defined as above for $z \in D$ and $h \in \mathrm{GSpin}_V(\mathbf{A}_f)$ – converges to a smooth function on D with singularities along

$$Z(f_0) = \sum_{\mu \in \Lambda^{\#}/\Lambda} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^{+}(-m,\mu) Z(\mu,m).$$

These singularities are described in greater detail in [7, Proposition 5.6], with relations between the corresponding Borcherds and Kudla-Millson lifts in [7, Theorems 6.1 and 6.2].

4.4.3. *Choice of vector-valued input form.* Henceforth, we shall choose $f_0 = f_{0,\eta,A}$ to be a harmonic weak Maass form of weight 0 and representation $\overline{r}_{\psi_0,\Lambda_A} = r_{\psi_0,-\Lambda_A}$, similar to the analogous setting of Bruinier-Yang [8] (where the corresponding harmonic weak Maass form has weight $k = 1 - n/2$). We shall also assume henceforth that this f_0 has integral principal part, so $c_{f_0}^{+}(-m,\mu) \in \mathbf{Z}$ for all $m \geq 0$ and $\mu \in \Lambda_A^{\#}/\Lambda$. As explained in [8, Theorem 4.2], we know that for the corresponding analytic divisor

$$Z(f_{0,\eta,A}) = \sum_{\mu \in \Lambda_A^{\#}/\Lambda_A} \sum_{\substack{m \in \mathbf{Q} \\ m > 0}} c_{f_{0,\eta,A}}^{+}(-m,\mu) Z(m,\mu),$$

the corresponding regularized theta lift $\vartheta_{f_{0,\eta,A}}^{\star}$ determines a smooth function on

$$\mathrm{Sh}_{U_A}(\mathrm{GSpin}_{V_A}, D_{V_A}) \backslash Z(f_{0,\eta,A}) \subset \mathrm{Sh}_U(\mathrm{GSpin}_V, D_V)$$

with logarithmic singularities along $-2\log Z(f_{0,\eta,A})$. Again, $(1,1)$-form $dd^c \vartheta_{f_{0,\eta,A}}^{\star} = -(2\pi i)^{-1}\partial\overline{\partial}\vartheta_{f_{0,\eta,A}}^{\star}$ can be continued to a smooth function on the ambient Shimura variety, we have the Green current equation

$$dd^c[\vartheta_{f_{0,\eta,A}}^{\star}] + \delta_{Z(f_{0,\eta,A})} = [dd^c \vartheta_{f_{0,\eta,A}}^{\star}].$$

The regularized theta lift $\vartheta_{f_{0,\eta,A}}^{\star}$ is moreover an eigenfunction for the Laplacian operator Δ_z, with eigenvalue

$$\Delta_z \vartheta_{f_{0,\eta,A}}^{\star} = 2 \cdot c_{f_{0,\eta,A}}^{+}(0,0). \tag{43}$$

For our subsequent applications to integral presentations of $L'(1/2, \Pi \otimes \chi) = L'(1/2, \pi \times \pi(\chi))$, we shall also assume that for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$, the harmonic weak Maass form $f_{0,\eta,A} \in H_{0,\rho_{\Lambda_A}}$ is chosen so that the corresponding cuspidal form $g_{A,\eta} = \xi_0(f_{0,\eta,A}) \in S_{2,\overline{\rho}_{\Lambda_A}}$ is the canonical lift in the sense of Theorem 4.9 below of the twisted eigenform $f \otimes \eta$. Here again, $f \in S_2(\Gamma_0(N))$ denotes the scalar-valued eigenform parametrizing an elliptic curve $E/\mathbf{Q}$, and $\eta = \eta_{K/\mathbf{Q}} = \left(\frac{d_K}{\cdot}\right)$ the even Dirichlet character associated to our fixed real quadratic field $K/\mathbf{Q}$ of discriminant d_K. Putting together the description of the Fourier series expansion from Theorem 4.9, we deduce that this this $f_{0,\eta,A}$ must be cuspidal, and hence by (43) that the corresponding regularized theta lift $\vartheta_{f_{0,\eta,A}}^{\star}$ is annihilated by Δ_z, and hence determines a Laplacian eigenvector of eigenvalue 0.

To describe this choice in more detail, we first give some more details about harmonic weak Maass forms in general. Hence, we now say more about the choice of L^2-automorphic form ϕ_0 and corresponding form f_0 on $\mathfrak{H}$ for the construction above implicit in the abstract theta lifting of Proposition 4.2. Let us thus revert to the more general setting of a quadratic space (V,q) of signature $(2,2)$. Recall we write r_{ψ_0} to denote the Weil representation $r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W}(\mathbf{A}))$, as well as its extension to similitudes. Given a matrix $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, we write $k_\gamma \in \mathrm{SL}_2(\widehat{\mathbf{Z}})$ to denote the image (cf. [8, (2.1)] and [32, § 1][11]).

Suppose first that $k \in \mathbf{Z}$ is any integer weight; we shall later specialize to the case of $k = 0$ as above. Again, we fix a lattice $\Lambda \subset V$, and consider the corresponding subspace of Schwartz functions $\mathcal{S}_\Lambda \subset \mathcal{S}(\mathbb{W}(\mathbf{A}_f))$ as in (39) above. We consider $\mathcal{S}_\Lambda$-valued harmonic weak Maass forms, defined more generally as follows. Let $(\rho_\Lambda, \mathcal{V})$ be the conjugate Weil representation on $\mathcal{S}_\Lambda$, that is $\rho_\Lambda(\gamma) = \overline{r}_{\psi_0,\Lambda}(k_\gamma) = r_{\psi_0,-\Lambda}(g_\gamma)$ for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and $k_\gamma \in \widetilde{\mathrm{SL}}_2(\mathbf{Z})$ (cf. [8, (2.7)]). Let us also write $|_{k,\rho_\Lambda}$ to denote the Petersson weight k operator with respect to ρ_Λ, defined on a function f on $\Gamma\backslash\mathfrak{H}$ by the rule

$$f|_{k,\rho_\Lambda}(\gamma(\tau)) = (c\tau + d)^k \cdot \rho_\Lambda(\gamma) \cdot f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

[11] i.e. noting that we do not have to work with metaplectic covers here as our quadratic space (V,q) has signature $(2,n)$ with $n = 2$ even.

Let us also write Δ_k to denote the hyperbolic Laplacian of weight k, defined for $\tau = u + iv \in \mathfrak{H}$ by

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ik \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

Note that this Laplacian can be expressed in terms of the respective weight k Maass weight raising and lowering operators R_k and L_k as $-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k$, where

$$R_k = 2i \cdot \frac{\partial}{\partial \tau} + k \cdot v^{-1} \tag{44}$$

denotes the Maass weight raising operator of weight k (which raises the weight by 2), and

$$L_k = -2iv^2 \cdot \frac{\partial}{\partial \overline{\tau}} \tag{45}$$

denotes the Maass lowering operator (which lowers the weight by 2).

Definition 4.8. *Fix an integer $k \leq 1$, and a lattice $\Lambda \subset V$ with corresponding subspace $\mathcal{S}_\Lambda \subset \mathcal{S}(\mathbb{W}(\mathbf{A})))$. A twice differentiable function $f : \mathfrak{H} \longrightarrow \mathcal{S}_\Lambda$ is a harmonic weak Maass form of weight k with respect to $\mathrm{SL}_2(\mathbf{Z})$ and representation ρ_Λ if:*

(i) *The function is invariant under the Petersson weight-k operator: $f|_{k,\rho_\Lambda}\gamma = f$ for all $\gamma \in \Gamma$.*

(ii) *There exists an $\mathcal{S}_\Lambda$-valued Fourier polynomial*

$$P_f(\tau) = \sum_{\mu \in \Lambda^\# / \Lambda} \sum_{m \leq 0} c_f^+(\mu, m) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}}$$

such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon v})$ as $v = \Im(\tau) \to \infty$ for some $\varepsilon > 0$.

(iii) *The function is harmonic of weight k, i.e. $\Delta_k f = 0$.*

We write H_{k,ρ_Λ} for the vector space of such functions, and call the polynomial $P_f(\tau)$ the principal part of f. In the special case where we take the representation ρ_Λ to be the Weil representation $r_{\psi_0,\Lambda}$, we shall sometimes write $H_{k,\Lambda} = H_{k,\rho_\Lambda}$ for simplicity.

Recall that the Fourier series expansion of any weak harmonic Maass form $f \in H_{k,\rho_\Lambda}$ decomposes uniquely as the sum $f(\tau) = f^+(\tau) + f^-(\tau)$, where

$$f^+(\tau) := \sum_{\mu \in \Lambda^\# / \Lambda} \sum_{\substack{m \in \mathbf{Q} \\ m \gg -\infty}} c_f^+(m, \mu) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}}$$

is the holomorphic part, and

$$f^-(\tau) := \sum_{\mu \in \Lambda^\# / \Lambda} \sum_{\substack{m \in \mathbf{Q} \\ m < 0}} c_f^-(m, \mu) W_k(2\pi m v) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}},$$

for $W_k(a) := \int_{2a}^\infty e^{-t} t^{-k} dt = \Gamma(1-k, 2|a|)$ for $a < 0$ is the non-holomorphic part. We consider the subspace $M^!_{k,\rho_\Lambda} \subset H_{k,\rho_\Lambda}$ of such weakly holomorphic forms, these being meromorphic modular functions whose poles are supported at the cusps. As explained in [8, §3], there is an antilinear differential operator ξ_k taking H_{k,ρ_Λ} to the space $S_{2-k,\overline{\rho}_\Lambda}$ of holomorphic forms of weight $2-k$ with respect to Γ and $\overline{\rho}_\Lambda$, these forms being defined in the analogous way with $f = f^+$ for each $f \in S_{2-k,\overline{\rho}_\Lambda}$. This operator ξ_k can be defined explicitly via the Maass lowering operator as follows. We have an exact sequence of vector spaces

$$0 \longrightarrow M^!_{k,\rho_\Lambda} \longrightarrow H_{k,\rho_\Lambda} \xrightarrow{\xi_k} S_{2-k,\overline{\rho}_\Lambda} \longrightarrow 0, \tag{46}$$

where $\xi_k : H_{k,\rho_\Lambda} \longrightarrow S_{k-2,\overline{\rho}_\Lambda}$ is defined by

$$f(\tau) \mapsto \xi_k f(\tau) := v^{k-2} \overline{L_k f(\tau)}.$$

Note that the natural inner product $\langle\langle\cdot,\cdot\rangle\rangle$ here induces a bilinear pairing

$$\{\cdot,\cdot\}: M_{2-k,\overline{\rho}_\Lambda} \times H_{k,\rho_\Lambda} \longrightarrow \mathbf{C}, \quad \{g,f\} := \langle\langle g, \xi_k(f)\rangle\rangle.$$

To be clear, using conventions as in [8, § 3.1], given $g \in M_{2-k,\overline{\rho}_\Lambda}$ with Fourier series expansion

$$g(\tau) = \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{m \geq 0} c_g(\mu, m) e(m\tau),$$

the pairing against a harmonic weak Maass form $f \in H_{k,\rho_\Lambda}$ with expansion as described above is given by

$$\{g,f\} = \langle \xi_k(f), g\rangle = \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{m \leq 0} c_f^+(\mu, m) c_g(\mu, -m).$$

This in particular implies that $\{g,f\}$ depends only on the principal part $P_f(\tau)$ of f. It is also simple to deduce from the exactness of (46) that this pairing $\{\cdot,\cdot\}$ between $S_{2-k,\overline{\rho}_\Lambda}$ and $H_{k,\rho_\Lambda}/M^!_{k,\rho_\Lambda}$ is nondegenerate. Moreover, given a harmonic weak Maass form $f \in H_{k,\rho_\Lambda}$ with constant principal part $P_f(\tau)$, it is not hard to deduce that f must be a holomorphic modular form $f \in M_{k,\rho_\Lambda}$ (see [8, Lemma 3.3]).

Finally, we have the following relation to scalar-valued forms for the setting we consider here (cf. [8, §3]). Here, we are interested in working later with a vector-valued modular form g_η of weight 2 and level $\Gamma_0(d_K N)$ corresponding to the twist $f_\eta = f \otimes \eta \in S_2(\Gamma_0(d_K N))$ by the quadratic Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$ of our initial weight 2 cusp form $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$, and hence describe only this case. Such a lifting is known to exist, and more precisely we have the following.

Theorem 4.9. *Let us keep the setup described above with (V,q) a quadratic space of signature $(2,2)$, and $\Lambda \subset V$ any lattice associated to the compact open subgroup of $\mathrm{GSpin}_V(\mathbf{A}_f)$ giving the level of the cuspidal automorphic basechange representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi) = \mathrm{BC}_{K/\mathbf{Q}}(\pi(f))$, where $\pi = \pi(f)$ denotes the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_f)$ corresponding to our initial cusp form $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$. Let us write the Fourier series expansion of this latter cusp form as*

$$f(\tau) = \sum_{m \geq 1} a_f(m) e(m\tau).$$

Let us also also write η to denote the extension of the quadratic Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$ to a character of $\Gamma_0(d_K N)$, with $f_\eta = f \otimes \eta \in S_2(\Gamma_0(d_K N))$ the cusp form with Fourier expansion

$$f_\eta(\tau) = (f \otimes \eta)(\tau) = \sum_{m \geq 1} a_f(m)\eta(m) e(m\tau).$$

There exists an $\mathcal{S}_\Lambda$-valued modular form g_η of weight 2, determined canonically as the lifting of f_η defined in [53], whose Fourier series expansion is given by

$$g_\eta(\tau) = \sum_{\mu \in \Lambda^\#/\Lambda} g_{\eta,\mu}(\tau) \mathbf{1}_{\mu + \widehat{\Lambda}}, \quad \text{where} \quad g_{\eta,\mu}(\tau) = \sum_{\substack{m \in \mathbf{Q} \\ m \equiv d_K N q(\mu) \bmod (d_K N)}} a_f(m)\eta(m)s(m) e\left(\frac{m\tau}{d_K N}\right).$$

Here, $s(m)$ denotes the function defined on each class $m \bmod d_K N$ by $s(m) = 2^{\Omega(m, d_K N)}$, where $\Omega(m, d_K N)$ denotes the number of divisors of the greatest common divisor $(m, d_K N)$.

Proof. This is a special case of [53, Theorem 4.15], adapted to match the setup of [8, p. 639, Lemma 3.1]. Note that we identify the nebentype character η^2 with the principal character of $\Gamma_0(d_K N)$. □

4.4.4. *The Siegel-Weil formula, and a vector-valued variant.* Let us now record some special cases of the Siegel-Weil formula for our later calculations of averages of regularized theta lifts over the quadratic subspaces $V_2 \subset V$ and $V_{A,2} \subset V_A$. We now take (V,q) to be any quadratic space over $\mathbf{Q}$ with signature $(2,2)$ and underlying vector space $V = \mathbf{Q} \otimes \mathbf{Q} \oplus K$, as described above. We shall also consider the signature $(1,1)$ subspace (V_2, q_2) given by the real quadratic field $V_2 = K = \mathbf{Q}(\sqrt{d})$ with restriction $q_2 = q|_{V_2}$ of the quadratic form, i.e. which for us will always be of the form $c \cdot \mathbf{N}(\cdot)$ for some integer $c \geq 1$, where $\mathbf{N} = \mathbf{N}_{K/\mathbf{Q}} : K \longrightarrow \mathbf{Q}, \lambda \mapsto \lambda\lambda^\tau$ denotes the norm homomorphism. We first review the Siegel-Weil formula abstractly following [32, Theorem 4.1] and [8, Theorem 2.1]. We then give a more arithmetic description of the vector-valued Siegel theta and Eisenstein series that appear after taking averages over the subspaces V_2, in preparation for our subsequent calculations.

More generally, we consider the subspaces (V_j, q_j) for $j = 1, 2$ defined by $V_1 = \mathbf{Q} \oplus \mathbf{Q}$ with $q_1 = q|_{V_1}$ and $V_2 = K$ with $q_2 = q|_{V_2}$. Moreover, to treat all cases in this discussion uniformly, let us also write $(V_0, q_0) = (V, q)$, so that (V_j, q_j) for $j = 0, 1, 2$ refers to any of the three quadratic spaces $(V_0, q_0) = (V, q)$, $(V_1, q_1) = (\mathbf{Q} \oplus \mathbf{Q}, q_1)$, or $(V_2, q_2) = (K, q_2)$. Write $r_{\psi_0, j} : \mathrm{SO}(V_j)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W}|_{V_j})$ to denote the corresponding restriction of the Weil representation $r_{\psi_0} : \mathrm{SO}(V)(\mathbf{A}) \times \mathrm{SL}_2(\mathbf{A}) \longrightarrow \mathcal{S}(\mathbb{W})$, and $\theta_{r_{\psi_0, j}}$ the corresponding theta kernel defined on $h \in \mathrm{SO}(V_j)(\mathbf{A})$, $g \in \mathrm{SL}_2(\mathbf{A})$ and $\Phi \in \mathcal{S}(\mathbb{W}|_{V_j})$ by

$$\theta_{\psi_0, j}(h, g; \Phi) = \sum_{x \in \mathbb{W}|_{V_j}(\mathbf{Q})} r_{\psi_0, j}(h, g)\Phi(x).$$

We also consider the associated Langlands Eisenstein series as follows. Let χ_{V_j} denote the idele class character of $\mathbf{Q}$ defined on $x \in \mathbf{A}^\times / \mathbf{Q}^\times$ by the formula $\chi_{V_j}(x) = (x, \det(V_j))_{\mathbf{A}}$, where $(\cdot, \cdot)_{\mathbf{A}}$ denotes the Hilbert symbol on $\mathbf{A}$, and $\det(V_j)$ the Gram determinant. Writing $s \in \mathbf{C}$ to denote a complex parameter, let $I(s, \chi_{V_j})$ denote the corresponding principal series representation of $\mathrm{SL}_2(\mathbf{A})$ given by right translation. Writing $s_0 = \dim(V_j)/2 - 1 = 0$, there is an $\mathrm{SL}_2(\mathbf{A})$-intertwining representation

$$\lambda : \mathcal{S}(\mathbb{W}|_{V_j}) \longrightarrow I(0, \chi_{V_j}), \quad \lambda(\Phi)(g) := (r_{\psi_0, j}(g)\Phi)(0).$$

Recall that a section $\Phi(s) \in I(s, \chi_{V_j})$ is said to be *standard* if its restriction to the maximal compact subgroup $\mathcal{K}\,\mathrm{SO}_2(\mathbf{R})$ of $\mathrm{SL}_2(\mathbf{A})$ does not depend on the complex parameter $s \in \mathbf{C}$. As explained in [8, § 2.1], using the Iwasawa decomposition

$$\mathrm{SL}_2(\mathbf{A}) = N_2(\mathbf{A}) M_2(\mathbf{A}) \mathcal{K}\,\mathrm{SO}_2(\mathbf{R}), \tag{47}$$

we deduce that $\lambda(\Phi)(g) \in I(0, \chi_{V_j})$ has a unique extension to a standard section $\lambda(\Phi, s) \in I(s, \chi_{V_j})$ for which $\lambda(\Phi, 0) = \lambda(\Phi)$. Given any standard section $\varphi \in I(s, \chi_{V_j})$, and writing $P \subset \mathrm{SL}_2$ to denote the standard parabolic subgroup, we then consider the Eisenstein series defined on $g \in \mathrm{SL}_2(\mathbf{A})$ by

$$E_{r_{\psi_0, j}}(g, s; \varphi) = \sum_{\gamma \in P(\mathbf{Q}) \backslash \mathrm{SL}_2(\mathbf{Q})} \varphi(\gamma g, s).$$

We can now state the following special case of the Siegel-Weil formula in this setting.

Theorem 4.10 (Siegel Weil)**.** *Let (V_j, q_j) for $j = 0, 1, 2$ denote any of the quadratic spaces introduced above. We have for any $g \in \mathrm{SL}_2(\mathbf{A})$ the average formula*

$$\kappa \cdot \int_{\mathrm{SO}(V_j)(\mathbf{Q}) \backslash \mathrm{SO}(V_j)(\mathbf{A})} \theta_{\psi_0, j}(h, g; \Phi) dh = E_{r_{\psi_0, j}}(g, s_0, \lambda(\Phi)),$$

where

$$\kappa = \begin{cases} 1 & \text{if } \dim(V_j) > 2 \\ 2 & \text{if } \dim(V_j) \leq 2 \end{cases} \quad \text{and} \quad s_0 = s_0(V_j) = \frac{\dim(V_j)}{2} - 1.$$

Moreover, the Eisenstein series $E_{r_{\psi_0, j}}(g, s, \lambda(\Phi))$ in each case $j = 0, 1, 2$ is holomorphic at $s = s_0$.

Proof. See [32, Theorem 4.1], and more generally [34, § I.4]. □

Let us now consider the following more explicit version of Theorem 4.10. Here, we shall give a more precise formula by describing the theta kernel $\theta_{r_{\psi_0, j}}$ and Langlands Eisenstein series $E_{r_{\psi_0, j}}$ in terms of vector-valued modular forms as follows. As explained in [8, § 2.1], given any integer weight $k \in \mathbf{Z}$ there is a unique section $\sigma_\infty^k = \sigma_\infty^k(s) \in I_\infty(s, \chi_{V_j})$ for which

$$\sigma_\infty^k(n(n) m(a) k(\theta), s) = \chi_{V_j}(a) |a|^{s+1} \exp(ik\theta) \tag{48}$$

with respect to the Iwasawa decomposition (47), i.e. with coordinates

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in N_2(\mathbf{A}), \quad m(a) = \begin{pmatrix} a^{\frac{1}{2}} & \\ & a^{-\frac{1}{2}} \end{pmatrix} \in M_2(\mathbf{A}), \quad k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbf{R}).$$

We deduce from our definition of the Gaussian function $\Phi_\infty \in \mathcal{S}(\mathbb{W}(\mathbf{R})|_{V_j}) \otimes C^\infty(D^\pm)$ above that

$$\lambda_\infty(\Phi_\infty(\cdot, z)) = \sigma_\infty^{\frac{p(V_j) - q(V_j)}{2}}(0) = \sigma_\infty^0(0), \tag{49}$$

where $(p(V_j), q(V_j))$ denotes the signature of the quadratic space V_j. Let us remark that each of the quadratic spaces V_j we consider here leads to looking at an Eisenstein series of weight $k = k(V_j) = (p(V_j) - q(V_j))/2 = 0$.

Given any even lattice $\Lambda_j \subset V_j$, and writing λ_f to denote the finite component of the standard section $\lambda(\Phi) = \lambda(\Phi, s) \in I(s, \chi_{V_j})$ described above, we consider the corresponding $\mathcal{S}_{\Lambda_j}$-valued Eisenstein series of weight k defined on $\tau = u + iv \in \mathfrak{H}$ and $s \in \mathbf{C}$ by

$$E_{\Lambda_j}(\tau, s; k) = E_{\Lambda_j, r_{\psi_0, j}}(\tau, s, k) := v^{\frac{k}{2}} \sum_{\mu \in \Lambda_j^{\#}/\Lambda_j} E_{r_{\psi_0,j}}(g_\tau, s; \sigma_\infty^k \otimes \lambda_f(\mathbf{1}_{\mu + \widehat{\Lambda}_j})) \cdot \mathbf{1}_{\mu + \widehat{\Lambda}_j}.$$

We also consider the $\mathcal{S}_{\Lambda_j}$-valued theta kernel defined on $\tau = u + iv \in \mathfrak{H}$, $z \in \mathbb{D}^{\pm}$, and $h_f \in \mathrm{GO}(V_j)(\mathbf{A})$ by

$$\theta_{\Lambda_j}(\tau, z, h) = \theta_{\Lambda_j, r_{\psi_0, j}}(\tau, z, h_f) = \sum_{\mu \in \Lambda_j^{\#}/\Lambda_j} \theta^{\star}_{r_{\psi_0,j}}(z, h_f, g_\tau; \mathbf{1}_{\mu + \widehat{\Lambda}_j}) \cdot \mathbf{1}_{\mu + \widehat{\Lambda}_j}. \tag{50}$$

Corollary 4.11 (Siegel-Weil for $\mathcal{S}_{\Lambda_j}$-valued forms)**.** *We have the identification of functions of $\tau \in \mathfrak{H}$:*

$$\kappa \cdot \int_{\mathrm{SO}(V_j)(\mathbf{Q}) \backslash \mathrm{SO}(V_j)(\mathbf{A})} \theta_{\Lambda_j}(\tau, z, h_f) = E_{\Lambda_j}(\tau, s_0, k) = E_{\Lambda_j}(\tau, s_0(V_j); k(V_j)).$$

Here again, $s_0 = s_0(V_j) := \dim(V_j)/2 - 1$, and $k = k(V_j) := (p(V_j) - q(V_j))/2$.

Proof. Cf. [8, Proposition 2.2], and note that we deduce this from Theorem 4.10 with (48) and (49). $\square$

4.4.5. *Eisenstein series and Maass weight-raising operators.* As preparation for our later calculations, let us also give the following more classical descriptions of the Eisenstein series appearing in the Siegel-Weil theorem above, with relations to the Maass raising and lowering operators R_k, L_k introduced above. Here, we take for granted the definition of the matrix g_τ for $\tau = u + iv \in \mathfrak{H}$ in the unique decomposition (36) above via the Iwasawa decomposition for $\mathrm{SL}_2(\mathbf{A})$, also as described above in (47). We then consider

$$\gamma \cdot g_\tau = n(\beta) \cdot m(\alpha) \cdot k(\theta) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \beta \in \mathbf{R}, \ \alpha \in \mathbf{R}^\times, \ k(\theta) \in \mathrm{SO}_2(\mathbf{R}).$$

Direct calculations show that

$$\alpha = v^{\frac{1}{2}} \cdot |c\tau + d|^{-1}, \quad \exp(i\theta) = \frac{c\overline{\tau} + d}{|c\tau + d|},$$

so that substituting into (47) gives

$$\sigma_\infty^k(\gamma g_\tau, s) = v^{\frac{s}{2} + \frac{1}{2}} (c\tau + d)^{-k} |c\tau + d|^{k - s - 1}.$$

Hence, writing $\Gamma_\infty = P(\mathbf{Q}) \cap \Gamma$ for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ as above, we find that

$$\begin{aligned} E(g_\tau, s; \sigma_\infty^k \otimes \lambda_f(\mathbf{1}_{\mu + \widehat{\Lambda}_j})) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} \frac{v^{\frac{s}{2} + \frac{1}{2}}}{|c\tau + d|^{s + 1 - k}} \cdot \lambda_f(\mathbf{1}_{\mu + \widehat{\Lambda}_j})(\gamma) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} \frac{v^{\frac{s}{2} + \frac{1}{2}}}{|c\tau + d|^{s + 1 - k}} \cdot \langle \mathbf{1}_{\mu + \widehat{\Lambda}_j}, (r_{\psi_0, j}^{-1}(\gamma) \mathbf{1}_{0 + \widehat{\Lambda}_j}) \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ here denotes the L^2 inner product on $\mathcal{S}_{\Lambda_j}$. In this way, we find that the vector-valued Eisenstein series we considered above can be written classically as

$$E_{\Lambda_j}(\tau, s; k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\Im(\tau)^{\frac{(s + 1 - k)}{2}} \right] \Big|_{k, \rho_{\Lambda_j}} \gamma, \tag{51}$$

where $|_{k, \rho_{\Lambda_j}}$ again denotes the Petersson weight-k slash operator for ρ_{Λ_j}.

Recall that we defined the Maass weight raising and lowering operators R_k and L_k in (44) and (45) above. These operators raise and lower respectively the weights of the these Eisenstein by two. To be more precise, it is easy to check from the definitions that

$$L_k E_{\Lambda_j}(\tau, s; k) = \frac{1}{2} \cdot (s + 1 - k) \cdot E_{\Lambda_j}(\tau, s; k - 2),$$
$$R_k E_{\Lambda_j}(\tau, s; k) = \frac{1}{2} \cdot (s + 1 + k) \cdot E_{\Lambda_j}(\tau, s : k + 2).$$

We refer to [32, Proposition 2.7] and [8, Lemma 2.3] for more details, but note that these works consider distinct setups where the Eisenstein series are always incoherent (in the sense of Kudla [32]). Here, we have for the Eisenstein series corresponding to our signature $(1, 1)$ subspace V_2 that

$$L_2 E_{\Lambda_2}(\tau, s; 2) = \frac{1}{2} \cdot (s - 1) \cdot E_{\Lambda_2}(\tau, s; 0). \tag{52}$$

Observe that the Eisenstein series $E_{\Lambda_2}(\tau, s; 0)$ is holomorphic at $s = s_0 = s_0(V_2) := \dim(V_2)/2 - 1 = 0$ thanks to Siegel-Weil, Theorem 4.10 (cf. Corollary 4.11). It follows that at $s = 0$, we have the identity

$$L_2 E_{\Lambda_2}(\tau, 0; 2) = -\frac{1}{2} \cdot E_{\Lambda_2}(\tau, 0; 0).$$

Hence, taking the first derivative with respect to s on each side of (52) to get

$$L_2 E'_{\Lambda_2}(\tau, s; 2) = \frac{1}{2}(s - 1) E'_{\Lambda_2}(\tau, s; 0) - \frac{1}{2} E'_{\Lambda_2}(\tau, s; 0),$$

which after evaluating at $s = 0$ gives

$$L_2 E'_{\Lambda_2}(\tau, 0; 2) = -\frac{1}{2} \cdot E_{\Lambda_2}(\tau, 0; 0) - \frac{1}{2} \cdot E'_{\Lambda_2}(\tau, 0; 0).$$

and hence

$$-2 L_2 E'_{\Lambda_2}(\tau, 0; 2) = E_{\Lambda_2}(\tau, 0; 0) + E'_{\Lambda_2}(\tau, 0; 0). \tag{53}$$

Writing (53) in terms of differential forms as in [8, Lemma 2.3], we then find that

$$-2 L_2 E'_{\Lambda_2}(\tau, 0; 2) d\mu(\tau) = 2\overline{\partial}\left(E'_{\Lambda_2}(\tau, 0; 2)\right) = \left(E_{\Lambda_2}(\tau, 0; 0) + E'_{\Lambda_2}(\tau, 0; 0)\right) d\mu(\tau)$$

and hence

$$E_{\Lambda_2}(\tau, 0; 0) d\mu(\tau) = -2\overline{\partial}\left(E'_{\Lambda_2}(\tau, 0; 2) d\tau\right) - E'_{\Lambda_2}(\tau, 0; 0) d\mu(\tau). \tag{54}$$

Here, we write $d\mu(\tau) = \frac{dudv}{v^2}$ for $\tau = u + iv \in \mathfrak{H}$.

Let us now consider the Fourier series expansion of the Eisenstein series

$$E'_{\Lambda_2}(\tau, s; 2) = \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} A_{\Lambda_2}(s, \mu, m, v) e(m\tau) \cdot \mathbf{1}_{\mu + \widehat{\Lambda}_2}.$$

We alter the discussion given in [32, Proposition 2.7] (cf. [8, § 2.2]) for this subspace V_2 of signature $(1, 1)$ as follows. That is, we apply the discussion of [32, Proposition 2.7] to each of the derivative Eisenstein series to deduce that the Laurent series expansions of each the Fourier coefficients around $s = s_0 = 0$ has the form

$$A_{\Lambda_2}(s, \mu, m, v) = b_{\Lambda_2}(\mu, m, v)(s - s_0) + O\left((s - s_0)^2\right) = b_{\Lambda_2}(\mu, m, v) s + O(s^2). \tag{55}$$

It follows that we have the expansion

$$E'_{\Lambda_2}(\tau, 0; 2) = \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} b_{\Lambda_2}(\mu, m, v) e(m\tau) \cdot \mathbf{1}_{\mu + \widehat{\Lambda}_2}. \tag{56}$$

Let us for future reference define the coefficients

$$\kappa_{\Lambda_2}(\mu, m) = \begin{cases} \lim_{v \to \infty} b_{\Lambda_2}(\mu, m, v) & \text{if } \mu \neq 0 \text{ or } m \neq 0 \\ \lim_{v \to \infty} b_{\Lambda_2}(\mu, m, v) - \log(v) & \text{if } \mu = 0 \text{ and } m = 0. \end{cases} \tag{57}$$

Note that these limits are shown to exist by the argument of Kudla [32, Theorem 2.12]. Let us define from these coefficients the $\mathcal{S}_{\Lambda_2}$-valued periodic function $\mathcal{E}_{\Lambda_2}(\tau)$ on $\tau = u + iv \in \mathfrak{H}$ by the expansion

$$\mathcal{E}_{\Lambda_2}(\tau) := \sum_{\mu \in \Lambda_2^{\#}/\Lambda_2} \sum_{m \in \mathbf{Q}} \kappa_{\Lambda_2}(\mu, m) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}_2}. \tag{58}$$

Remark Observe (cf. [8, Remark 2.4, (3.5)]) that we can view this form $\mathcal{E}_{\Lambda_2}(\tau)$ defined by the Fourier series expansion (58) as the holomorphic part of the Maass form defined by $E'_{\Lambda_2}(\tau, 0; 2)$,

$$\mathcal{E}_{\Lambda_2}(\tau) = E'^{+}_{\Lambda_2}(\tau, 0; 2).$$

4.4.6. *Eisenstein series associated to the signature (1,1) subspaces* (V_2, q_2). Let us now say more about the Eisenstein series associated to the lattices $\Lambda_{A,2} \subset V_2$. Again, we fix a class $A \in \mathrm{Pic}(\mathcal{O}_c)$, where $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ is the $\mathbf{Z}$-order of K of conductor c (equal to that of our chosen ring class character χ). Let $\mathfrak{a} \subset \mathcal{O}_K$ be any integral representative, so that $[\mathfrak{a}] = A \in \mathrm{Pic}(\mathcal{O}_c)$. Starting with the ambient lattices $\Lambda_A \subset V_A$ and $\Lambda \subset V$, we consider the signature (1,1) sublattices defined by $\Lambda_{A,2} = \mathfrak{a} \subset V_{A,2}$ and $\Lambda_2 = \mathcal{O}_K \subset V_2$.

Writing $\mathfrak{d}_K$ to denote the different of K, with

$$\mathfrak{d}_K^{-1} = \{\lambda \in K : \mathrm{Tr}(\lambda \mathcal{O}_K) \in \mathbf{Z}\} = \begin{cases} \left(\frac{1}{\sqrt{d}}\right) & \text{if } d \equiv 1 \bmod 4 \\ \frac{1}{2}\mathbf{Z} + \frac{\sqrt{d}}{2d}\mathbf{Z} & \text{if } d \equiv 2, 3 \bmod 4 \end{cases}$$

the inverse different, we then have the identifications $\mathfrak{a}^{\#} \cong \mathfrak{d}_K^{-1} \cap \mathfrak{a}$ and

$$\Lambda_{A,2}^{\#}/\Lambda_{A,2} \cong (\mathfrak{d}_K^{-1} \cap \Lambda_{A,2})/\Lambda_{A,2}.$$

Writing d_K again to denote the discriminant of K,

$$d_K = \begin{cases} d & \text{if } d \equiv 1 \bmod 4 \\ 4d & \text{if } d \equiv 2, 3 \bmod 4, \end{cases}$$

recall that we write $\eta = \eta_{K/\mathbf{Q}}$ to denote corresponding Dirichlet character. Here, we have $\chi_{V_2} = \eta$. Writing

$$\Lambda(s, \eta) = |d|^{\frac{s}{2}} \Gamma_{\mathbf{R}}(s+1) L(s, \eta), \quad \Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$$

to denote its corresponding completed L-function, we defined the completed Eisenstein series

$$E^{\star}_{\Lambda_2}(\tau, s) := \Lambda(s+1, \eta) E_{\Lambda_2}(\tau, s).$$

Proposition 4.12. *The Eisenstein series $E^{\star}_{\Lambda_2}(\tau, s)$ has a meromorphic continuation to all $s \in \mathbf{C}$, and satisfies the symmetric functional equation $E^{\star}_{\Lambda_2}(\tau, s) = E^{\star}_{\Lambda_2}(\tau, -s)$. Consequently, the Eisenstein series $E^{\star}_{\Lambda_{A,2}}(\tau, s) := \Lambda(s+1, \eta) E_{\Lambda_{A,2}}(\tau, s)$ associated to each sublattice $\Lambda_{A,2}$ has a meromorphic continuation to all $s \in \mathbf{C}$, and satisfies the symmetric functional equation $E^{\star}_{\Lambda_{A,2}}(\tau, s) = E_{\Lambda_{A,2}}(\tau, -s)$.*

Proof. See [8, Proposition 2.5]. We deduce this in the same way from the Langlands functional equation $E(g, s; \sigma) = E(g, -s, M(s)\sigma)$ for each of the summands $E(g, s, \sigma) = E(\tau, s, \sigma_{\infty}^{k(V_2)} \otimes \lambda_f(\mathbf{1}_{\mu} + \widehat{\Lambda}_2))$. Note however the sign change in switching from the imaginary to the real quadratic case. That is, the sign is determined by the root number $\tau(\eta)|D|^{-\frac{1}{2}}$ of the completed Dirichlet L-series $\Lambda(s, \eta)$. Moreover, we can deduce this in a more direct way (than [8, Proposition 2.5]) from the Langlands functional equation. This is because the Eisenstein series we consider here is not incoherent, but rather the naturally-appearing Eisenstein series defined from the local data at each place $v \leq \infty$. $\square$

4.4.7. *Summation along anisotropic quadratic subspaces of signature* $(1,1)$. Let (V, q) be the quadratic space of signature $(2,2)$ with underlying vector space $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$. Again, we consider the subspaces V_j of signature $(1,1)$ introduced above for $j = 1, 2$, with induced quadratic forms $q_j = q|_{V_j}$. We fix even lattices $\Lambda_j \subset V_j$ for each of these spaces. Here, we alter the general discussion given in [33, § 2], [5], and [8, § 4] to calculate the regularized theta lifts $\vartheta^{\star}_{f_0}(z, h)$ defined above. In particular, we shall link the values of these regularized theta lifts along the geodesic corresponding to subspace (V_2, q_2) to central (derivative) values of some related Rankin-Selberg L-function. Let us note that we do not meet incoherent Eisenstein series in this setup, and moreover that we derive an integral presentation for the sum of a central value plus a central derivative value. Hence, the integral presentation is something of a deviation from the style of Kudla's programme, as well as the analogous formula for the CM case derived by Bruinier-Yang [8, Theorem 4.7].

Let $D_V = D_V^+$ be the Grassmannian of oriented hyperplanes $z \subset V(\mathbf{R})$ with $q|_Z > 0$. Hence, the subspace $V_2 \subset V$ gives rise to a pair of points in the corresponding subgrassmannian $z_{V_2} \in D_{V_2}^+$. Again, we consider

GSpin_{V_2} as a subgroup of GSpin_V acting trivially on V_1. Fixing a compact open subgroup $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ as above, let $U_2 := U \cap \mathrm{GSpin}_{V_2}(\mathbf{A}_f)$. We then consider the geodesic

$$Z(V_2) := \mathrm{GSpin}_{V_2}(\mathbf{Q}) \backslash \{z_{V_2}\} \times \mathrm{GSpin}_{V_2}(\mathbf{A}_f)/U_2 \tag{59}$$

on the corresponding Shimura variety

$$\mathrm{Sh}_U(\mathrm{GSpin}_V, D_V^+) := \mathrm{GSpin}_V(\mathbf{Q}) \backslash D_V^+ \times \mathrm{GSpin}_V(\mathbf{A}_f)/U.$$

Note that for our calculations, we count each point $(z_{V_2}, h) \in Z(V_2)$ with multiplicity $\frac{2}{\#(\mathrm{GSpin}_{V_2}(\mathbf{Q}) \cap U_2)}$.

Given a point $(z_{V_2}, h) \in Z(V_2)$ and a harmonic weak Maass form $f_0 \in H_{0,\rho_{\Lambda_2}}$, we now compute the regularized theta lift $\vartheta^\star_{f_0}(z_{V_2}, h)$ defined above, and then the sum over the "geodesic" subset $Z(V_2)$,

$$\vartheta^\star_{f_0}(Z(V_2)) := \sum_{(z_{V_2}, h) \in Z(V_2)} \vartheta^\star_{f_0}(z_{V_2}, h).$$

Fix a Tamagawa measure on $\mathrm{SO}(V_2)(\mathbf{A})$ for which $\mathrm{vol}(\mathrm{SO}(V_2)(\mathbf{R})) = 1$ and $\mathrm{vol}\left(\mathrm{SO}(V_2)(\mathbf{Q}) \backslash \mathrm{SO}(V_2)(\mathbf{A})\right) = 2$. Fix a Haar measure on $\mathbf{A}_f^\times$ with the property that $\mathrm{vol}(\mathbf{Z}_p^\times) = 1$ for each finite place p, and $\mathrm{vol}(\mathbf{A}_f^\times / \mathbf{Q}^\times) = 1/2$. We obtain from these choices a Haar measure on $\mathrm{GSpin}_{V_2}(\mathbf{A}_f)$ via the short exact sequence

$$1 \longrightarrow \mathbf{A}_f^\times \longrightarrow \mathrm{GSpin}_{V_2}(\mathbf{A}_f) \longrightarrow \mathrm{SO}(V_2)(\mathbf{A}_f) \to 1.$$

Lemma 4.13. *Let $U \subset \mathrm{GSpin}_V(\mathbf{A}_f)$ be any compact open subgroup, and $U_2 = U \cap \mathrm{GSpin}_{V_2}(\mathbf{A}_f)$. Then,*

$$\vartheta^\star_{f_0}(Z(V_2)) = \frac{2}{\mathrm{vol}(U_2)} \cdot \int_{\mathrm{SO}(V_2)(\mathbf{Q}) \backslash \mathrm{SO}(V_2)(\mathbf{A})} \vartheta^\star_{f_0}(z_{V_2}, h) dh.$$

Proof. As in [8, Lemma 4.5], we use the general result of [40, Lemma 2.13] applied to the specific functions $B(h) = \vartheta^\star_{f_0}(z_{V_2}, h)$ and $B(h) = 1$. This general result [40, Lemma 2.13] implies that for $B(h)$ any function on $\mathrm{GSpin}_{V_2}(\mathbf{A})$ which (i) depends only on the image of h in $\mathrm{SO}(V_2)(\mathbf{A}_f)$, (ii) is left $\mathrm{GSpin}_{V_2}(\mathbf{Q})$-invariant, and (iii) is right invariant under the compact open subgroup U_2, we have the relation

$$\int_{\mathrm{SO}(V_2)(\mathbf{Q}) \backslash \mathrm{SO}(V_2)(\mathbf{A})} B(h) dh = \mathrm{vol}(U_2) \cdot \sum_{h \in \mathrm{GSpin}_{V_2}(\mathbf{Q}) \backslash \mathrm{GSpin}_{V_2}(\mathbf{A})/U_2} B(h).$$

Here, the sum on the right hand side is finite. In this way, we compute the sum over the subset $Z(V_2)$ as

$$\begin{aligned} \vartheta^\star_{f_0}(Z(V_2)) &= \frac{2}{\#(\mathrm{GSpin}_{V_2}(\mathbf{Q}) \cap U_2)} \sum_{z_{V_2} \in \mathrm{supp}(Z(V_2))} \vartheta^\star_{f_0}(z_{V_2}, 1) \\ &= \frac{2}{\mathrm{vol}(U_2)} \cdot \int_{\mathrm{SO}(V_2)(\mathbf{Q}) \backslash \mathrm{SO}(V_2)(\mathbf{A})} \vartheta^\star_{f_0}(z_{V_2}, h) dh. \end{aligned}$$

□

Let us now consider the even lattice $\Lambda = \Lambda_1 \oplus \Lambda_2 \subset V$ with its corresponding $\mathcal{S}_\Lambda$-valued Siegel theta series $\theta_\Lambda(\tau, z, h)$ defined on $z \in D_V^+$, $h \in \mathrm{GSpin}_V(\mathbf{A}_f)$, and $\tau = u + iv \in \mathfrak{H}$ by

$$\theta_\Lambda(\tau, z, h) = \theta_{\Lambda, r_{\psi_0}}(\tau, z, h) = \sum_{\mu \in \Lambda^\# / \Lambda} \theta^\star_{r_{\psi_0}}(z, h, g_\tau; \mathbf{1}_\mu) \cdot \mathbf{1}_\mu.$$

Again, it is not hard to see ([8, (3.3), Lemma 3.1]) that we have the decomposition of theta series

$$\theta_\Lambda = \theta_{\Lambda_1 \oplus \Lambda_2} = \theta_{\Lambda_1} \otimes \theta_{\Lambda_2},$$

and more specifically for this setup with $(z_{V_2}, h) \in Z(V_2)$ and $\tau = u + iv \in \mathfrak{H}$ the decomposition

$$\theta_\Lambda(z_{V_2}, \tau) = \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}, h) = \theta_{\Lambda_1}(\tau, 1, 1) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}, h). \tag{60}$$

That is, for the Grassmannian variable $z_V = (z_{V_1}, z_{V_2}) \in D_V^+$ (with each z_{V_j} the projection to $D_{V_j}^+$), any $\tau = u + iv \in \mathfrak{H}$, and $h \in \mathrm{GSpin}(V)(\mathbf{A}_f)$, we have the decomposition

$$\theta_\Lambda(\tau, z_V, 1) = \theta_{\Lambda_1}(\tau, z_{V_1}, 1) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}, h).$$

Since we fix first variable $z_{V_1} = 1$ to be trivial, we suppress this from the notation henceforth. Let us also fix an $\mathcal{S}_\Lambda$-valued harmonic weak Maass form $f_0 \in H_{0,\Lambda}$, as described above, with decomposition $f_0 = f_0^+ + f_0^-$ into holomorphic part f_0^+ and non-holomorphic part f_0^-. Let us write A_0 to denote the constant coefficient

$$A_0 = \mathrm{CT}\left(\langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes \mathbf{1}_{\Lambda_2}\rangle\rangle\right).$$

Lemma 4.14. *We have for each $(z_{V_2}, h) \in Z(V_2)$ that*

$$\vartheta^\star_{f_0}(z_{V_2}, h) = \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) - A_0 \log(T)\right).$$

Proof. Cf. [8, Lemma 4.5]. Opening the definition of $\vartheta^\star_{f_0}(z_{V_2}, h)$, we first decompose $f_0 = f_0^+ + f_0^-$ into its holomorphic and nonholomorphic parts, and split the integral accordingly

$$\begin{aligned}\vartheta^\star_{f_0}(z_{V_2}; h) &= \lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau)\\ &= \lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) + \lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_0^-(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau).\end{aligned}$$

Using the fact that the initial Siegel theta series $\theta_\Lambda(\tau, z_{V_2}, h) \in M_{0,\Lambda}$ is holomorphic, we can also argue as in [8, Lemma 4.5] that the second integral is absolutely convergent due to the rapid decay of $f_0^-(\tau)$. We then use the argument of [32, Proposition 2.5] to deduce that the first integral in this latter expression equals

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) = \lim_{T\to\infty}\left[\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) - A_0\log(T)\right],$$

which after decomposing the Siegel theta series $\theta_\Lambda(\tau, z_{V_2}, h) = \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}, 1)$ gives

$$\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_\Lambda(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) = \lim_{T\to\infty}\left[\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(\tau, z_{V_2}, 1)\rangle\rangle d\mu(\tau) - A_0\log(T)\right].$$

We then deduce the result in the same way as [8, Lemma 4.5]. □

Corollary 4.15. *Using the Siegel-Weil formula of Theorem 4.10 and Corollary 4.11, we have that*

$$\vartheta^\star_{f_0}(Z(V_2)) = \frac{1}{\mathrm{vol}(U_2)} \cdot \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 0)\rangle\rangle d\mu(\tau) - A_0 \log(T)\right).$$

Proof. We expand the definition using Lemmas 4.13 and 4.14, switch the order of summation, then use Corollary 4.11 (with $\kappa = 2$) to evaluate the inner integral over $\theta_{\Lambda_2}(z_{V_2}, h)$. In this way, we compute

$$\begin{aligned}&\vartheta^\star_{f_0}(Z(V_2))\\ &= \frac{2}{\mathrm{vol}(U_2)} \cdot \int_{\mathrm{SO}(V_2)(\mathbf{Q})\backslash \mathrm{SO}(V_2)(\mathbf{A})} \vartheta^\star_{f_0}(z_{V_2}, h) dh\\ &= \frac{2}{\mathrm{vol}(U_2)} \cdot \int_{\mathrm{SO}(V_2)(\mathbf{Q})\backslash \mathrm{SO}(V_2)(\mathbf{A})} \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \theta_{\Lambda_2}(z_{V_2}, h, \tau)\rangle\rangle d\mu(\tau) - A_0\log(T)\right) dh\\ &= \frac{2}{\mathrm{vol}(U_2)} \cdot \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes \left(\int_{\mathrm{SO}(V_2)(\mathbf{Q})\backslash \mathrm{SO}(V_2)(\mathbf{A}_f)} \theta_{\Lambda_2}(z_{V_2}, h, \tau) dh\right)\rangle\rangle d\mu(\tau) - A_0\log(T)\right)\\ &= \frac{1}{\mathrm{vol}(U_2)} \cdot \lim_{T\to\infty}\left(\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 0)\rangle\rangle d\mu(\tau) - A_0\log(T)\right).\end{aligned}$$

□

Given $g \in S_{2,\Lambda}$ a cuspidal holomorphic modular form of weight 2 and representation $r_{\phi_0,\Lambda}$, let us now consider the Rankin-Selberg L-function given by the integral presentation

$$L(s, g, V_2) := \langle g(\tau), \theta_{\Lambda_1}(\cdot) \otimes E_{\Lambda_2}(\tau, s; 2)\rangle = \int\limits_{\mathcal{F}} \langle\langle g(\tau), \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, s, 2)\rangle\rangle v^2 d\mu(\tau).$$

We shall take $g = \xi_0(f_0)$, and write $L'(s, g, V) = \frac{d}{ds}L(s, g, V)$ to denote the derivative with respect to s. Recall that we write $\mathcal{E}_{\Lambda_2}(\tau)$ by the Fourier expansion (58), with coefficients defined in (57).

Theorem 4.16. *Writing $\mathcal{E}_{\Lambda_2}(\tau) = E^+_{\Lambda_2}(\tau,0;2)$ for the holomorphic part of $E'_{\Lambda_2}(\tau,0;2)$, we obtain*

$$\vartheta^\star_{f_0}(Z(V_2)) = \frac{1}{\mathrm{vol}(U_2)} \cdot \left(\mathrm{CT}\left(\langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau)\otimes \mathcal{E}_{\Lambda_2}(\tau)\rangle\rangle\right) + L(0,\xi_0(f_0),V_2) - L'(0,\xi_0(f_0),V_2)\right).$$

Proof. We derive a variation of [8, Theorem 4.7]. Here, Lemma 4.13, Lemma 4.14, and Corollary 4.15 imply

$$\vartheta^\star_{f_0}(Z(V_2)) = \frac{1}{\mathrm{vol}(U_2)} \cdot \lim_{T\to\infty}\left(I_T(f_0) - A_0\log(T)\right), \tag{61}$$

where

$$I_T(f_0) := \int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;0)\rangle\rangle d\mu(\tau).$$

Using the identity (54) for the non-incoherent Eisenstein series $E_{\Lambda_2}(\tau,s,0)$ at $s=0$, we deduce that

$$\begin{aligned}
I_T(f_0) &= \int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;0)\rangle\rangle d\mu(\tau)\\
&= -2\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes \overline{\partial}\left(E'_{\Lambda_2}(\tau,0;2)\right)d\tau\rangle\rangle - \int_{\mathcal{F}_T}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle\\
&= -2\int_{\mathcal{F}_T} d\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)d\tau\rangle\rangle + 2\int_{\mathcal{F}_T}\langle\langle \overline{\partial} f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes \left(E'_{\Lambda_2}(\tau,0;2)\right)d\tau\rangle\rangle\\
&\quad - \int_{\mathcal{F}_T}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle
\end{aligned} \tag{62}$$

To compute the first integral on the right hand side of (62), we apply Stokes' theorem to find

$$\begin{aligned}
&-2\int_{\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes \overline{\partial}\left(E'_{\Lambda_2}(\tau,0;2)\right)d\tau\rangle\rangle = -2\int_{\mathcal{F}_T} d\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)d\tau\rangle\rangle\\
&= -2\int_{\partial\mathcal{F}_T} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)d\tau\rangle\rangle = -2\int_{\tau=iT}^{iT+1}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle d\tau.
\end{aligned} \tag{63}$$

To compute the second integral on the right hand side of (62), we use the relation of differential forms

$$\overline{\partial}(f_0(\tau)d\tau) = -v^2\overline{\xi_0(f_0)(\tau)}d\mu(\tau) = -L_0 f_0(\tau)d\mu(\tau)$$

to deduce that

$$2\int_{\mathcal{F}_T}\langle\langle \overline{\partial} f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)d\tau\rangle\rangle = -2\int_{\mathcal{F}_T}\langle\langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2 d\mu(\tau). \tag{64}$$

To compute the third integral on the right hand side of (62), recall that we have the relation

$$2L_2 E_{\Lambda_2}(\tau,s;2) = (s-1)\cdot E_{\Lambda_2}(\tau,s;0) \tag{65}$$

and hence

$$E'_{\Lambda_2}(\tau,0;0) = -2L_2E'_{\Lambda_2}(\tau,0;2) + E_{\Lambda_2}(\tau,0;0).$$

We then find after substitution that

$$\begin{aligned}
&\int_{\mathcal{F}_T}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle\\
&= -2\int_{\mathcal{F}_T}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes L_2E'_{\Lambda_2}(\tau,0;2)d\mu(\tau)\rangle\rangle + \int_{\mathcal{F}_T}\langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle.
\end{aligned}$$

Applying the relation

$$L_2 = -2iv^2\frac{\partial}{\partial\overline{\tau}} = L_0;\quad \xi_0(f_0)(\tau) = v^{-2}\overline{L_0f_0(\tau)} \implies \overline{L_0f_0(\tau)} = \overline{L_2f_0(\tau)} = v^2\xi_0(f_0)(\tau) \tag{66}$$

to the first integral in this latter expression, we find that

$$-2\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes L_2E'_{\Lambda_2}(\tau,0;2)d\mu(\tau)\rangle\rangle = -2\int\limits_{\mathcal{F}_T}\langle\langle L_0f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle d\mu(\tau)$$
$$= -2\int\limits_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)(\tau)},\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau),$$

and hence

$$\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle$$
$$= -2\int\limits_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)(\tau)},\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau) + \int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle.$$

Now, to evaluate the second integral in this latter expression, recall that we have

$$E_{\Lambda_2}(\tau,0;0) = -2L_2E_{\Lambda_2}(\tau,0;2) \tag{67}$$

so that

$$\begin{aligned}\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle &= -2\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes L_2E_{\Lambda_2}(\tau,0;2)d\mu(\tau)\rangle\rangle\\ &= -2\int\limits_{\mathcal{F}_T}\langle\langle L_0f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;2)\rangle\rangle d\mu(\tau)\\ &= -2\int\limits_{\mathcal{F}_T}\langle\langle\xi_0(f_0)(\tau),\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau)\end{aligned} \tag{68}$$

and hence

$$\begin{aligned}&-\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;0)d\mu(\tau)\rangle\rangle\\ &= 2\int\limits_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes L_2E'_{\Lambda_2}(\tau,0;2)\rangle\rangle d\mu(\tau) + 2\int_{\mathcal{F}_T}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes L_2E_{\Lambda_2}(\tau,0;2)\rangle\rangle d\mu(\tau)\\ &= 2\int\limits_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)(\tau)},\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau) + 2\int\limits_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)(\tau)},\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau).\end{aligned} \tag{69}$$

Substituting the expressions for the first integral (63), the second integral (64), and the third integral (69) into (62), we obtain

$$\begin{aligned}I_T(f_0) = 2\int\limits_{t=iT}^{iT+1}&\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)d\tau\rangle\rangle - 2\int_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)},\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau)\\ &+ 2\int_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)},\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau) + 2\int_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)},\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau).\end{aligned} \tag{70}$$

Inserting (70) back into the initial formula (61) then gives us the preliminary formula

$$\begin{aligned}\vartheta^{\star}_{f_0}(Z(V_2)) = \frac{1}{\mathrm{vol}(U_2)}\cdot\lim_{T\to\infty}2\int_{\tau=iT}^{iT+1}\langle\langle f_0(\tau),\theta_{\Lambda_1}(\tau)\otimes E'_{\Lambda_2}(\tau,0;2)\rangle\rangle d\tau - A_0\log(T)\\ - \frac{1}{\mathrm{vol}(U_2)}\cdot\lim_{T\to\infty}2\int_{\mathcal{F}_T}\langle\langle\overline{\xi_0(f_0)},\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_2}(\tau,0;2)\rangle\rangle v^2d\mu(\tau).\end{aligned} \tag{71}$$

Now, we argue as in Ehlen [17, Theorem 3.5, (3.12)] that we may replace the $f_0(\tau)$ in the first integral on the right of (70) with its holomorphic part $f_0^+(\tau)$, as the remaining non-holomorphic part $f_0^-(\tau)$ is rapidly decreasing as $v \to \infty$. To be clear, let us write the Fourier series expansion as

$$\langle\langle f_0^-(\tau), \theta_{\Lambda_1}(\tau, 1, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2)\rangle\rangle = \sum_{n \in \mathbf{Z}} a(n, v) e(n\tau).$$

Expanding the integral, opening the Fourier series expansion, then using the orthogonality of additive characters on the torus $\mathbf{R}/\mathbf{Z} \cong [0, 1]$ to evaluate, we find that

$$\begin{aligned}\int_{iT}^{iT+1} \langle\langle f_0^-(\tau), \theta_{\Lambda_1}(\tau, 1, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2)\rangle\rangle &= \int_0^1 \langle\langle f_0^-(u + iT), \theta_{\Lambda_1}(u + iT, 1, 1) \otimes E'_{\Lambda_2}(u + iT, 0; 2)\rangle\rangle du \\ &= \sum_{n \in \mathbf{Z}} a(n, iT) e(inT) \int_0^1 e(nu) du = a(0, iT) \\ &= \sum_{\mu \in \Lambda^\#/\Lambda} \sum_{m \in \mathbf{Q}_{>0}} c_{f_0}^-(\mu, -m) W_0(-2\pi m v) c_g(\mu, m, v),\end{aligned}$$

where we write $c_g(m, \mu, v)$ to denote the Fourier series coefficients of $g(\tau) = \theta\Lambda_1(\tau, 1) \otimes E'_{\Lambda_2}(\tau, 0; 2)$. Noting that the Fourier series expansion of the specialized Siegel theta function $\theta_{\Lambda_1}(\tau, 1, 1)$ is determined by

$$\theta_{\Lambda_1}(\tau, 1, 1) = \sum_{\mu \in \Lambda_1^\#/\Lambda_1} \theta_{\Lambda_1, \mu}(\tau) \mathbf{1}_\mu = \sum_{\mu \in \Lambda_1^\#/\Lambda_1} \left(\sum_{\lambda \in \Lambda_1 + \mu} e(q_1(\lambda)\tau) \right) \mathbf{1}_\mu,$$

we can use standard bounds for the Whittaker coefficients in the Fourier series expansion of $f_0^-(\tau)$ to deduce that for some integer $M > 0$ and some constant $C > 0$, we have for each $m \geq M$ the bounds

$$c_{f_0}^-(\mu, -m) W_0(-2\pi m v) c_g(\mu, m, v) = O\left(e^{-mCv}\right).$$

We deduce from this that for some constants $c, C > 0$, we have the upper bound

$$|a(0, iT)| \leq c \cdot \frac{e^{-CT}}{(1 - e^{-CT})},$$

from which it follows that

$$\lim_{T \to \infty} |a(0, iT)| = 0.$$

This justifies our claim that we may replace the $f_0(\tau)$ in the first integral on the right of (70) by its holomorphic part $f_0^+(\tau)$. Writing $\delta_{\star,\star}$ to denote the Kronecker delta function, and using the Fourier expansion

$$E'_{\Lambda_2}(\tau, 0; 2) = \sum_{\mu \in \Lambda_2^\#/\Lambda_2} \sum_{m \in \mathbf{Q}} b_{\Lambda_2}(\mu, m, v) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}_2}$$

as in (56), we deduce from the expansion (58) of $\mathcal{E}_{\Lambda_2}(\tau) = E^{+\prime}_{\Lambda_2}(\tau, 0; 2)$ and Lemma 4.14 that

$$\begin{aligned}&\lim_{T \to \infty} \left(\int_{\tau = iT}^{iT+1} \langle\langle f_0(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2)\rangle\rangle d\tau - A_0 \log(T) \right) \\ &= \lim_{T \to \infty} \left(\int_{\tau = iT}^{iT+1} \langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes E'_{\Lambda_2}(\tau, 0; 2)\rangle\rangle d\tau - A_0 \log(T) \right) \\ &= \lim_{T \to \infty} \left(\int_{\mathcal{F}_T} \langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes \sum_{\mu \in \Lambda_2^\#/\Lambda_2} \sum_{m \in \mathbf{Q}} \left(b_{\Lambda_2}(\mu, m, v) - \delta_{\mu,0} \cdot \delta_{m,0} \log(v) \right) e(m\tau) \mathbf{1}_{\mu + \widehat{\Lambda}_2} \rangle\rangle d\tau \right) \\ &= \mathrm{CT}\left(\langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle\rangle \right).\end{aligned}$$

After substitution into (71), we then derive the formula

$$\vartheta^{\star}_{f_0}(Z(V_2)) \tag{72}$$
$$= \frac{1}{\mathrm{vol}(U_2)} \cdot \left(\mathrm{CT}\left(\langle\langle f_0^+(\tau), \theta_{\Lambda_1}(\tau) \otimes \mathcal{E}_{\Lambda_2}(\tau) \rangle\rangle \right) - \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)}, \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0; 2) \rangle\rangle v^2 d\mu(\tau) \right).$$

To evaluate (72) further, we use the weight raising operator R_0 as follows. Recall we have the relation

$$R_0 E_{\Lambda_2}(\tau, s; 0) = \frac{1}{2}(s+1) E_{\Lambda_2}(\tau, s; 2) \tag{73}$$

and hence

$$R_0 E'_{\Lambda}(\tau, s; 0) = \frac{1}{2}(s+1) E'_{\Lambda_2}(\tau, 0; 2) + \frac{1}{2} E_{\Lambda_2}(\tau, 0; 2),$$

which after taking the value at $s=0$ gives us

$$R_0 E_{\Lambda}(\tau, 0; 0) = \frac{1}{2} E'_{\Lambda_2}(\tau, 0; 2) + \frac{1}{2} E_{\Lambda_2}(\tau, 0; 2). \tag{74}$$

Using this identity in (72), we evaluate

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0, 2) \rangle\rangle v^2 d\mu(\tau)$$
$$= 2 \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau) \otimes R_0 E_{\Lambda_2}(\tau, 0, 0) \rangle\rangle v^2 d\mu(\tau) - L'(\xi_0(f_0), V_2, 0),$$

which after taking (73) at $s=0$ to evaluate

$$R_0 E_{\Lambda_2}(\tau, s; 0) = \frac{1}{2} E_{\Lambda_2}(\tau, 0; 2)$$

in the first integral gives us

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0, 2) \rangle\rangle v^2 d\mu(\tau) \tag{75}$$
$$= \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle\langle \overline{\xi_0(f_0)(\tau)}, \theta_{\Lambda_1}(\tau) \otimes E_{\Lambda_2}(\tau, 0, 2) \rangle\rangle v^2 d\mu(\tau) - L'(\xi_0(f_0), V_2, 0) = L(\xi_0(f_0), V_2, 0) - L'(\xi_0(f_0), V_2, 0).$$

Substituting back into (72), we then derive the claimed formula. □

4.4.8. *Application to the central derivative value $L'(1/2, \Pi \otimes \chi)$.* Let us now finally explain how we can use these calculations to derive a formula analogous to the preliminary integral presentation of Proposition 4.2 above for the central derivative value $L'(1/2, \Pi \otimes \chi)$ via the classical Rankin-Selberg integral presentation.

Recall that we write $\eta = \otimes_v \eta_v$ to denote the idele class character of $\mathbf{Q}$ associated to the quadratic extension $K/\mathbf{Q}$, which we can and do identify with its corresponding Dirichlet character $\eta = \eta_{K/\mathbf{Q}}$. Recall as well that $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ denotes the quadratic basechange of our cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbf{A})$ corresponding to our elliptic curve $E/\mathbf{Q}$ to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$. As a consequence of the theory of cyclic basechange, we then have an equivalence of the $\mathrm{GL}_2(\mathbf{A}_K) \times \mathrm{GL}_1(\mathbf{A}_K)$-automorphic L-function $L(s, \Pi \otimes \chi)$ with the $\mathrm{GL}_2(\mathbf{A}) \times \mathrm{GL}_2(\mathbf{A})$ Rankin-Selberg L-function $L(s, \pi \times \pi(\chi))$, i.e. $L(s, \Pi \otimes \chi) = L(s, \pi \times \pi(\chi))$. Let us now consider the following classical integral representations of the Rankin-Selberg L-functions relevant to the discussion above.

To describe this setup in classical terms, recall that we write

$$f(\tau) = f_E(\tau) = \sum_{m \geq 1} a_f(m) e(m\tau) \in S_2^{\mathrm{new}}(\Gamma_0(N)), \quad \tau = u + iv \in \mathfrak{H}$$

to denote the cuspidal newform of weight 2 associated to the elliptic curve $E/\mathbf{Q}$. Recall that we fix a ring class character χ of some conductor $c \in \mathbf{Z}_{\geq 1}$ of K. Hence, $\chi = \otimes_x \chi_w$ is a character of the class group

$$\mathrm{Pic}(\mathcal{O}_c) = \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_c^\times, \quad \widehat{\mathcal{O}}_c^\times = \prod_{w < \infty} \mathcal{O}_{c,w}^\times$$

of the $\mathbf{Z}$-order $\mathcal{O}_c = \mathbf{Z} + c\mathcal{O}_K$ of conductor c in K. We consider the corresponding Hecke theta series defined by the twisted linear combination (see e.g. [23, (5.4)])

$$\theta(\chi)(\tau) = \sum_{A \in \mathrm{Pic}(\mathcal{O}_c)} \chi(A) \theta_A(\tau), \tag{76}$$

where each of the nonholomorphic partial theta series $\theta_A(\tau)$ is defined explicitly following the classical definitions. We have the classical Rankin-Selberg presentation

$$L(s - 1/2, \pi \times \pi(\chi)) = L(s, f \times \theta(\chi)) = \sum_{A \in \mathrm{Pic}(\mathcal{O}_c)} \chi(A) L(s, f \times \theta_A),$$

given as a twisted linear combination of the (completed) partial L-functions (cf. e.g. [23, § IV (0.1), p. 271])

$$\begin{aligned} L(s, f \times \theta_A) := \langle f, \theta_A E(\cdot, s) \rangle &= \frac{\Gamma(s-1)}{(4\pi)^{s-1}} \cdot L(2s-1, \eta) \cdot \sum_{m \geq 1} a_f(m) r_A(m) m^{-s} \\ &= \frac{\Gamma(s-1)}{(4\pi)^{s-1}} \cdot L(2s-1, \eta) \cdot \frac{1}{w} \sum_{\substack{\lambda \in \mathfrak{a} \\ [\mathfrak{a}] = A \in \mathrm{Pic}(\mathcal{O}_c)}} a_f(\mathbf{N}(\lambda)) \mathbf{N}(\lambda)^{-s} \end{aligned} \tag{77}$$

associated to each class $A \in \mathrm{Pic}(\mathcal{O}_c)$, where w denotes the number of automorphs of the corresponding binary quadratic form $q_A(x, y)$ after restriction to a certain fundamental domain, and $r_A(m)$ denotes the function counting the number of ideals of norm m in the class A.

Now, note that we can also consider the quadratic twist $f \otimes \eta = f_E \otimes \eta_{K/\mathbf{Q}}$ just as well for these integral presentations. That is, keeping in mind that $\eta^2 = 1$ as η is a quadratic character, we consider the cusp form

$$(f \otimes \eta)(\tau) = f_E \otimes \eta_{K/\mathbf{Q}}(\tau) = \sum_{m \geq 1} a_f(m) \eta(m) e(m\tau) \in S_2^{\mathrm{new}}(\Gamma_0(d_K N)), \quad \tau = u + iv \in \mathfrak{H}.$$

We can then also consider the Rankin-Selberg L-function

$$L(s - 1/2, (\pi \otimes \eta) \times \pi(\chi)) = L(s, (f \otimes \eta) \times \theta(\chi)) = \sum_{A \in \mathrm{Pic}(\mathcal{O}_c)} \chi(A) L(s, f \otimes \eta \times \theta_A),$$

where each partial Rankin-Selberg L-series $L(s, (f \otimes \eta) \times \theta_A)$ is given by the corresponding expansion

$$\begin{aligned} L(s, (f \otimes \eta) \times \theta_A) := \langle f \otimes \eta, \theta_A E(*, s) \rangle &= \frac{\Gamma(s-1)}{(4\pi)^{s-1}} \cdot L(2s-1, \eta) \cdot \sum_{m \geq 1} a_f(m) \eta(m) r_A(m) m^{-s} \\ &= \frac{\Gamma(s-1)}{(4\pi)^{s-1}} \cdot L(2s-1, \eta) \cdot \frac{1}{w} \sum_{\substack{\lambda \in \mathfrak{a} \\ [\mathfrak{a}] = A \in \mathrm{Pic}(\mathcal{O}_c)}} a_f(\mathbf{N}(\lambda)) \eta(\mathbf{N}(\lambda)) \mathbf{N}(\lambda)^{-s} \\ &= \frac{\Gamma(s-1)}{(4\pi)^{s-1}} \cdot L(2s-1, \eta) \cdot \frac{1}{w} \sum_{\substack{\lambda \in \mathfrak{a} \\ [\mathfrak{a}] = A \in \mathrm{Pic}(\mathcal{O}_c)}} a_f(\mathbf{N}(\lambda)) \mathbf{N}(\lambda)^{-s}. \end{aligned}$$

Lemma 4.17. *We have the equivalent Rankin-Selberg integral presentations*

$$L(s, \pi \times \pi(\chi)) = L(s + 1/2, f \times \theta(\chi)) = L(s + 1/2, (f \otimes \eta) \times \theta(\chi)) = L(s, (\pi \otimes \eta) \times \pi(\chi))$$

for the basechange L-function $L(s, \Pi \otimes \chi) = L(s, \mathrm{BC}_{K/\mathbf{Q}} \otimes \chi)$.

Proof. In classical terms, this can be seen by inspection the expansions of each of the partial Rankin-Selberg L-functions. In representation theoretic terms, let us consider the basechange $\Pi' = \mathrm{BC}_{K/\mathbf{Q}}(\pi \otimes \chi)$ of the cuspidal automorphic representation $\pi \times \eta$ of $\mathrm{GL}_2(\mathbf{A})$ to $\mathrm{GL}_2(\mathbf{A}_K)$, whose corresponding L-function $L(s, \Pi')$ then decomposes as $L(s, \Pi') = L(s, \pi \otimes \eta) L(s, \pi \otimes \eta^2) = L(s, \pi \otimes \eta) L(s, \pi)$. Here again, we use that the quadratic character η has order 2 to deduce that $L(s, \Pi') = L(s, \pi \otimes \eta) L(s, \pi) = L(s, \Pi)$. Hence, we deduce that the equivalent L-functions $L(s,\Pi\otimes\chi) = L(s,\Pi'\otimes\chi)$ have the same Rankin-Selberg integral presentation $L(s,\pi\times\pi(\chi)) = L(s,(\pi\otimes\eta)\times\pi(\chi))$. $\square$

Now, recall that we have the relation described in Theorem 4.9 between scalar-valued modular forms such as $f\otimes\eta$ and θ_A and their vector-valued analogues. To fix ideas, let us for each class $A\in\mathrm{Pic}(\mathcal{O}_c)$ fix an integral ideal representative $\mathfrak{a}\subset\mathcal{O}_K$ as in the expansions above. We then consider for each class $A\in\mathrm{Pic}(\mathcal{O}_c)$ the corresponding quadratic space (V_A,q_A) defined by

$$V_A := \mathbf{Q}\oplus\mathbf{Q}\oplus\mathfrak{a}_{\mathbf{Q}},\quad q_A(x,y,\lambda) := \frac{\mathbf{N}(\lambda)}{\mathbf{N}\mathfrak{a}} - xy = \frac{\lambda\lambda^\tau}{\mathbf{N}\mathfrak{a}} - xy.$$

We consider the subspaces $(V_{A,j},q_{A,j})$ $V_{A,1} = \mathbf{Q}\otimes\mathbf{Q}$ with $q_{A,1} = q_A|_{V_{A,1}}$ and $V_{A,2} = K$ with $q_{A,2} = q_A|_{V_2}$. We also consider the lattice Λ_A as above, determined by the compact open subgroup $U_A = U_A(\Pi)$ of $\mathrm{GSpin}_{V_A}(\mathbf{Q}_f)$ corresponding to the level of the basechange representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$, with sublattices $\Lambda_{A,1} = \Lambda_A\cap V_{A,1}$ and $\Lambda_{A,2} = \Lambda_A\cap V_{A,2}$. Note that these can be described in terms of the rational splitting $V = V_{A,1}\oplus V_{A,2}$ as in [8, (4.12)]. Using the bijection shown in [53, Theorem 4.15] (Theorem 4.9), we can associate to the quadratic twist $f\otimes\eta\in S_2^{\mathrm{new}}(\Gamma_0(d_KN))$ an $\mathcal{S}_{\Lambda_A}$-valued modular form g_η of weight 2. We can also associate to the partial theta series $\theta_A(\tau)$ an $\mathcal{S}_{\Lambda_{A,2}}$-valued Eisenstein series of weight two by the discussion above. To be clear, applying the Siegel-Weil formula as described in Theorem 4.10 and Corollary 4.11 to the theta series associated to $\Lambda_{A,2}$ gives the value at $s_0 = 0$ of a corresponding Eisenstein series $E_{\Lambda_{A,2}}(\tau,s,0)$ of weight zero. This in turn gives rise to the uniquely-determined Eisenstein series $E_{\Lambda_{A,2}}(\tau,s;2)$ related to $E_{\Lambda_{A,2}}(\tau,s;0)$ via the identities (53) and (54) above,

$$L_2E'_{\Lambda_{A,2}}(\tau,0;2) = \frac{1}{2}\cdot E_{\Lambda_{A,2}}(\tau,0;0) - \frac{1}{2}\cdot E_{\Lambda_{A,2}}(\tau,0;0).$$

Recall as well that we consider the (partial) Rankin-Selberg L-functions given by the Petersson inner products

$$L(s,g_\eta,V_{A,2}) := \langle g_\eta(\cdot),\theta_{\Lambda_{A,1}}(\cdot)\otimes E_{\Lambda_{A,2}}(\cdot,s;2)\rangle = \langle g_\eta(\tau),\theta_{\Lambda_1}(\tau)\otimes E_{\Lambda_{A,2}}(\tau,s;2)\rangle.$$

Let us for the sake of comparison also define the completed version of this latter L-series, i.e. with respect to the completed Eisenstein series $E^\star_{\Lambda_2}(\tau,s;2)$ introduced above:

$$L^\star(s,g_\eta,V_{A,2}) := \langle g_\eta(\cdot),\theta_{\Lambda_{A,1}}(\cdot)\otimes E^\star_{\Lambda_{A,2}}(\cdot,s;2)\rangle = \langle g_\eta(\tau),\theta_{\Lambda_1}(\tau)\otimes E^\star_{\Lambda_{A,2}}(\tau,s;2)\rangle.$$

Corollary 4.18. *We have in the setup described the equivalent presentations*

$$L(s,\Pi\otimes\chi) = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)}\chi(A)L(s+1/2,f\otimes\eta\times\theta_A) = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)}\chi(A)L^\star(2s-1,g_\eta,V_{A,2}).$$

In particular, we have that

$$L'(1/2,\Pi\otimes\chi) = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)}\chi(A)L'(1,f\otimes\eta\times\theta_A) = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)}\chi(A)L^{\star\prime}(0,g_\eta,V_{A,2}).$$

Proof. As explained in [8, §4, (4.24)] (with notations for Fourier coefficients as described above), each partial Rankin-Selberg product $L(s,g_\eta,V_{A,2})$ has the Dirichlet series expansion

$$L(s,g_\eta,V_{A,2}) = \frac{\Gamma\left(\frac{s+2}{2}\right)}{(4\pi)^{\frac{s+2}{2}}}\sum_{\mu\in\Lambda_{A,1}^\#/\Lambda_{A,1}}\sum_{m\in\mathbf{Q}_{>0}} c_{g_\eta}(\mu,m)c_{\theta_{\Lambda_{A,1}}}(\mu,m)m^{-\left(\frac{s+2}{2}\right)}.$$

We then deduce that we have for each class $A\in\mathrm{Pic}(\mathcal{O}_c)$ the relation $L^\star(2s-2,g_\eta,V_{A,2}) = L(s,f_\eta\times\theta_A)$ (cf. [23, § IV (0.1), p. 271]). The stated relations then follow as a formal consequence, with the analytic continuation and functional equations determined by the underlying Eisenstein series. $\square$

Theorem 4.19 (Twisted linear combinations of regularized theta integrals)**.** *Let us retain the setup above, with $f = f_E\in S_2^{\mathrm{new}}(\Gamma_0(N))$ the cuspidal eigenform parametrizing our elliptic curve $E/\mathbf{Q}$, π the corresponding cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$, and $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ its quadratic basechange lifting to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$. Let us also assume that the ersatz Heegner Hypothesis 2.1 holds. Let χ be any ring class character of the real quadratic field K of conductor c coprime to d_KN. Let $f_{0,\eta,A}\in H_{0,\rho_{\Lambda_A}}$ for each class $A\in\mathrm{Pic}(\mathcal{O}_c)$ denote the harmonic weak Maass form of weight zero with image*

$\xi_0(f_{0,\eta,A}) = g_{\eta,A} \in S_{2,\overline{\rho}_{\Lambda_A}}$ where $g_{\eta,A}$ denotes the lifting our our quadratic twist $f \otimes \eta \in S_2^{\mathrm{new}}(\Gamma_0(d_K N))$ the space vector-valued forms $S_{2,\overline{\rho}_{\Lambda_A}}$ as described in Theorem 4.9 above. Then, we have the formula

$$\begin{aligned}&\frac{L'(1/2,\Pi\otimes\chi)}{L(1,\eta)}\\&= \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right) - \mathrm{vol}(U_{A,2})\cdot\vartheta^{\star}_{f_{0,\eta,A}}(Z(V_{A,2}))\right).\end{aligned}$$

Here, for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$, we write $U_{A,2} := U \cap \mathrm{GSpin}_{V_{A,2}}(\mathbf{A}_f)$ as in Lemma 4.13 above.

Proof. Formally, this is a consequence of Lemma 4.17 and Corollary 4.18 after applying Theorem 4.16 to each of the partial Rankin-Selberg L-series $L(s, g_\eta, V_{A,2}) = L(s, \xi_0(f_{0,\eta,A}), V_{A,2})$, which together imply that

$$\begin{aligned}&\sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\cdot\mathrm{vol}(U_{A,2})\cdot\vartheta^{\star}_{f_{0,\eta,A}}(Z(V_{2,A}))\\&= \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\cdot\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right) + L(0,\xi_0(f_{0,\eta,A},V_{A,2})) - L'(0,\xi_0(f_{0,\eta,A}),V_{A,2})\right).\end{aligned}$$

It is then easy to identify the second term in this latter expression in terms of the central derivative value $L'(1/2, \Pi\otimes\chi)$ via Corollary 4.18. Let us thus consider the first term, which according to the expansions implied by Theorem 4.9 and the discussions in [8, §§ 4-5] can be evaluated as

$$\begin{aligned}&\sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\,\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right)\\&= \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\,\mathrm{CT}\left(\sum_{\substack{\mu_1\in\Lambda_{A,1}^{\#}/\Lambda_{A,1}\\ \mu_2\in\Lambda_{A,2}^{\#}/\Lambda_{A,2}\\ \mu_1+\mu_2\equiv\mu \bmod \Lambda_A}} f_{0,A,\mu}^{+}(\tau)\theta_{\Lambda_{A,1},\mu_1}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2},\mu_2}(\tau)\right)\\&= \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\left(\sum_{\substack{\mu_1\in\Lambda_{A,1}^{\#}/\Lambda_{A,1}\\ \mu_2\in\Lambda_{A,2}^{\#}/\Lambda_{A,2}\\ \mu_1+\mu_2\equiv\mu \bmod \Lambda_A}} \sum_{\substack{m,m_2\in\mathbf{Q}_{\geq 0}, m_1\in\mathbf{Q}\\ m_1+m_2=m}} c^{+}_{f_{0,\eta,A}}(-m,\mu)c_{\theta_{\Lambda_{A,1}}}(m_1,\mu_1)\kappa_{\Lambda_{A,2}}(m_2,\mu_2)\right).\end{aligned}\tag{78}$$

Note that the analogous constant term for the CM setting is the subject of [8, Conjectures 5.1 and 5.2], and that this has now been improved in important special cases by [1, Theorem A]. ☐

Now, recall that the Dirichlet analytic class number formula gives us the following classical arithmetic description of the value $L(1,\eta)$. Writing d_K again to denote the fundamental discriminant associated to $K = \mathbf{Q}(\sqrt{d})$, let $h_K = \#\,\mathrm{Pic}(\mathcal{O}_K)$ denote the class number, and $\epsilon_K = \frac{1}{2}(t + u\sqrt{d_K})$ for the smallest solution $t, u > 0$ (with u minimal) to Pell's equation $t^2 - d_K u^2 = 4$. We can then express the formula derived above for the central derivative value $L'(1/2,\Pi\otimes\chi)$ in terms of Dirichlet's analytic class number formula

$$L(1,\eta) = \frac{\log\epsilon_K\cdot h_K}{\sqrt{d_K}}. \tag{79}$$

Corollary 4.20. *We have that*

$$\begin{aligned}&L'(1/2,\Pi\otimes\chi) = L'(1/2,\pi\times\pi(\chi)) = L'(1, f\times\theta(\chi)) = L^{\star,\prime}(E/K,\chi,1)\\&= \frac{\sqrt{d_K}}{\log\epsilon_K\cdot h_K}\sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\left(\mathrm{CT}\left(\langle\langle f_{0,\eta,A}^{+}(\tau), \theta_{\Lambda_{A,1}}(\tau)\otimes\mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right) - \mathrm{vol}(U_{A,2})\cdot\vartheta^{\star}_{f_{0,\eta,A}}(Z(V_{A,2}))\right).\end{aligned}$$

Moreover, if we assume the ersatz Heegner hypothesis (Hypothesis 2.1) that the inert level N^+ is the squarefree product of an odd number of primes, then this central derivative value is not forced by the functional equation (8) *to vanish identically.*

Proof. This simply restates Theorem 4.19 in terms of the Dirichlet analytic class number formula (79). ☐

5. Classical description

We now describe the central derivative value formulae of Theorem 4.2, Theorem 4.19, and Corollary 4.7 in more classical terms, i.e. in terms of geodesic cycles of Hilbert modular surfaces. Our discussion here parallels that of Popa [37, §6] for the distinct setting of central values $L(1/2, \Pi \otimes \chi) = L(1/2, \pi \times \pi(\chi)) = L(E/K, \chi, 1)$ (when Hypothesis 2.1 fails) given in terms of geodesic cycles on the modular curve $X_0(N)$. Here, we shall follow [6] and also [11] for the relevant background on Hilbert modular forms.

5.1. Hilbert modular forms and varieties.

Recall that any pure tensor $\varphi = \otimes_w \varphi_w \in V_\Pi$ of the type we consider above, which we can and do view as an automorphic form on $\mathrm{GL}_2(\mathbf{A}_K)$, corresponds to a Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$ of parallel weight 2 and trivial central character over K for some congruence subgroup $\Gamma = \Gamma(\varphi)$ of the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_K)$, as can be determined explicitly using strong approximation for $\mathrm{GL}_2(\mathbf{A}_K)$ with the Iwasawa decomposition for $\mathrm{GL}_2(K_\infty) \cong \mathrm{GL}_2(\mathbf{R})^2$. To be more precise, let us first recall the following constructions of a Hilbert Hecke-Maass eigenform $\mathbf{g} = \mathbf{g}(\varphi)$ and a holomorphic Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$ associated to our chosen $\varphi \in V_\Pi$. We then explain to associate to φ' a Siegel modular form $F = F(\varphi')$ of weight 2 and trivial central character associated to the corresponding congruence subgroup Γ for the orthogonal group $O(\Lambda)$ of the lattice $\Lambda = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathcal{O}_K$ in $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$.

5.1.1. *Strong approximation for* $\mathrm{GL}_2(\mathbf{A}_K)$.

Let us write $C(\mathcal{O}_K)$ to denote the ideal class group of $\mathcal{O}_K$, which recall can be described adelically via the identification

$$\mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_K^\times = \mathbf{A}_K^\times / K_\infty^\times K^\times \prod_{w < \infty} \mathcal{O}_{K_w}^\times \cong C(\mathcal{O}_K). \tag{80}$$

We can therefore fix a set Δ of idele representatives $\zeta \in \mathbf{A}_K^\times$ for this class group $C(\mathcal{O}_K)$. Let us for each $\zeta \in \mathbf{A}_K^\times$ write $z_\zeta \in Z_2(\mathbf{A}_K) \cong \mathbf{A}_K^\times$ to denote the corresponding diagonal matrix in the centre $Z_2 \subset \mathrm{GL}_2$, i.e. with respect to the natural identification

$$\mathbf{A}_K^\times \cong Z_2(\mathbf{A}_K), \quad \zeta \longmapsto h_\zeta := \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix}.$$

Let us also write $U \subset \mathrm{GL}_2(\mathbf{A}_K)$ to denote the maximal compact subgroup,

$$U = \prod_{w \leq \infty} = O_2(K_\infty) \cdot \prod_{w < \infty} \mathrm{GL}_2(\mathcal{O}_{K_w}).$$

Now, observe that we may view $\mathrm{GL}_2(K)\,\mathrm{GL}_2(K_\infty) U$ as a subgroup of $\mathrm{GL}_2(\mathbf{A}_K)$, and moreover that the determinant map $\det : \mathrm{GL}_2(\mathbf{A}_K) \longrightarrow \mathbf{A}_K^\times$ allows us to derive from (80) an isomorphism

$$\mathrm{GL}_2(\mathbf{A}_K) / \mathrm{GL}_2(K) \cdot \mathrm{GL}_2(K_\infty) \cdot U \cong C(\mathcal{O}_K)$$

and from this the strong approximation decomposition for $\mathrm{GL}_2(\mathbf{A}_K)$

$$\mathrm{GL}_2(\mathbf{A}_K) / \mathrm{GL}_2(K) \cdot \mathrm{GL}_2(K_\infty) \cdot U \cong \coprod_{\zeta \in \Delta} \mathrm{GL}_2(K) \cdot \mathrm{GL}_2(K_\infty) \cdot h_\zeta \cdot U. \tag{81}$$

5.1.2. *Unique decomposition of* $\mathrm{GL}_2(\mathbf{A}_K)$ *via Iwasawa decomposition for* $\mathrm{GL}_2(K_\infty)$.

Let us now write P_2 to denote the mirabolic subgroup of GL_2, which is given explicitly on adelic points by

$$P_2(\mathbf{A}_K) = \left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix} : \ y \in \mathbf{A}_K^\times, x \in \mathbf{A}_K \right\} \subset \mathrm{GL}_2(\mathbf{A}_K).$$

In particular, fixing $i = \sqrt{-1}$, we have the natural identification

$$P_2(K_\infty) \cong \mathfrak{H}^2, \quad \begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \longmapsto x_\infty + i y_\infty \tag{82}$$

of $P_2(K_\infty) \cong P_2(\mathbf{R})$ with the two-fold upper-half plane $\mathfrak{H}^2$.

We shall consider the Iwasawa decomposition for $\mathrm{GL}_2(K_\infty)$ (cf. [21, § 4.1], [19]),

$$\mathrm{GL}_2(K_\infty) = P_2(K_\infty) \cdot Z_2(K_\infty) \cdot O_2(K_\infty),$$

which via the strong approximation decomposition (81) allows us the decompose $\mathrm{GL}_2(\mathbf{A}_K)$ accordingly as

$$\mathrm{GL}_2(\mathbf{A}_K) = \coprod_{\zeta \in \Delta} \mathrm{GL}_2(K) \cdot P_2(K_\infty) \cdot Z_2(K_\infty) \cdot h_\zeta \cdot U. \tag{83}$$

Now if we fix a fundamental domain for the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on $\mathrm{GL}_2(K_\infty)$, then (83) allows us to express each matrix $g \in \mathrm{GL}_2(\mathbf{A}_K)$ uniquely accordingly as follows (cf. [21, Proposition 4.12.1], [19]). Let us fix a standard fundamental domain $\mathcal{G}$ for the action of the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_K)$ on $\mathfrak{H}^2$, see e.g. [46, § I.3]. Roughly, this can be given by a product $\mathcal{G} = \coprod_{\zeta \in \Delta} D_\zeta$ of Siegel domains D_ζ indexed by equivalence classes of cusps ζ for $\mathbb{P}^1(K)$, where each domain D_ζ can be thought of loosely as a translate or subregion of the standard fundamental domain $\mathcal{D} = \{z = x + iy \in \mathfrak{H} : |\Re(z)| \leq 1/2, z\overline{z} \geq 1\}$ for the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{H}$. We can then construct from $\mathcal{G}$ a fundamental domain $\mathcal{F}$ for the action of $\mathrm{GL}_2(\mathcal{O}_K)$ of $\mathrm{GL}(K_\infty)$ after using the elementary matrix identity

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} y_\infty & -x_\infty \\ & 1 \end{pmatrix}$$

to deduce that this fundamental domain $\mathcal{F}$ will be "one half" of the fundamental domain $\mathcal{G}$ in the archimedean adele variable. In this way, we can deduce that any $g \in \mathrm{GL}_2(\mathbf{A}_K)$ can be decomposed uniquely as

$$g = \coprod_{\zeta \in \Delta} \gamma \cdot \begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix} \cdot h_\zeta \cdot u \in \mathrm{GL}_2(\mathbf{A}_K) \tag{84}$$

for $\gamma \in \mathrm{GL}_2(K)$, $r_\infty \in K_{\infty,+}^\times$ totally positive, $u \in U$, and $x_\infty + iy_\infty \in \mathfrak{H}^2$ contained strictly within the chosen fundamental domain $\mathcal{F}$. Again, we refer to the relevant discussions in [21, §4] and [19] (for instance) for more details on this unique decomposition.

5.1.3. *Hecke-Maass and holomorphic Hilbert modular eigenforms associated to $\varphi \in V_\Pi$.* We can use the unique decomposition (84) to construct from our automorphic form φ on $g \in \mathrm{GL}_2(\mathbf{A}_K)$ a Hecke-Maass eigenform $\mathbf{g} = \mathbf{g}(\varphi)$ and corresponding holomorphic Hilbert form $\mathbf{f} = \mathbf{f}(\varphi)$ as follows. Here again, we refer to the relevant discussions in [21, Proposition 4.12.1-13] and [19] for more details. In brief, we can consider the archimedean component $O_2(K_\infty)$ of the maximal compact subgroup $U \subset \mathrm{GL}_2(\mathbf{A}_K)$ in the Iwasawa decomposition and (84) more precisely as follows. Let us for each $\vartheta = (\vartheta_j)_{j=1}^2 \in (\mathbf{R}/2\pi\mathbf{Z})^2$ write

$$u(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in O_2(K_\infty) \cong O_2(\mathbf{R})^2.$$

We can then write the $u \in U$ in the unique decomposition (84) more explicitly with $u = u(\vartheta)u_f$, where

$$u(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in O_2(K_\infty), \quad u_f \in U_f := \prod_{w < \infty} \mathrm{GL}_2(\mathcal{O}_{K_w}).$$

Essentially, we can use this more precise version of the unique decomposition (84) of $g \in \mathrm{GL}_2(\mathbf{A}_K)$ to construct from any Hecke-Maass eigenform form $\mathbf{g} : \mathfrak{H}^2 \longrightarrow \mathbf{C}$ of a given weight $k = (k_1, k_2)$, level Γ, and trivial central character (say) its corresponding lifting $\widetilde{\mathbf{g}}$ defined by the rule $\widetilde{\mathbf{g}}(g) := (\mathbf{g}|_k)(x_\infty + iy_\infty)$, where $|_k$ denotes the corresponding weight operator. In this way, our chosen pure tensor $\varphi \in V_\Pi$, viewed as a cuspidal automorphic form on $\mathrm{GL}_2(\mathbf{A}_K)$ of parallel weight 2 (so $(k_1, k_2) = (2, 2)$) and trivial central character gives rise to a vector-valued Hecke-Maass cusp form $\mathbf{g} = \mathbf{g}(\varphi)$ determined by the assignment

$$\varphi(g) = \mathbf{g}|_{(2,2)}(x_\infty + iy_\infty). \tag{85}$$

Writing $|\cdot|$ to denote the idele norm, we can also define the corresponding holomorphic Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$ of parallel weight 2 on $x_\infty + iy_\infty \in \mathfrak{H}^2 \cong P_2(K_\infty) \cong \mathrm{GL}_2(K_\infty)/O_2(K_\infty) \cdot K_\infty^\times$ via the relation

$$\mathbf{f}(x_\infty + iy_\infty) = |y_\infty| \cdot \mathbf{g}(x_\infty + iy_\infty). \tag{86}$$

5.1.4. *Hilbert modular varieties associated to the orthogonal group.* Consider the lattice $\Lambda = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathcal{O}_K$ in $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$; we have a natural identification of the corresponding spin group Spin_Λ with $\mathrm{SL}_2(\mathcal{O}_K)$,

$$\mathrm{Spin}_\Lambda \cong \mathrm{SL}_2(\mathcal{O}_K), \tag{87}$$

and can identify finite-index subgroups via the fixed isomorphism (87) accordingly. More precisely, let

$$\mathrm{SO}(\Lambda) = \{\sigma \in \mathrm{Aut}(\Lambda) : q(\sigma(\gamma)) = q(\gamma) \quad \forall \gamma \in \Lambda, \quad \det(\sigma) = 1\}$$

denote the corresponding orthogonal group, and $\mathrm{SO}^+(V)(\mathbf{R}) \subset \mathrm{SO}(V)(\mathbf{R})$ the subgroup of elements whose spinor norm is given by the determinant (see [6, § 2.4]). Any finite index subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ can be identified via the isomorphism (87) as a finite index subgroup of the modular group $\mathrm{SL}_2(\mathcal{O}_K)$. Conversely, any finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$ can be identified with one of $\mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ in this way, and we shall use the notation Γ interchangeably to denote each subgroup.

Let us now consider modular forms for the orthogonal group $\mathrm{SO}(V)$ according to [6, §2.5]. Hence, we fix a finite index subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$, and consider the Hilbert modular variety defined by the quotient $Y(\Gamma) = \Gamma \backslash \mathfrak{H}^2$. We first give some background on realizations of the rational quadratic space (V, q) via the Grassmannian, projective, and tube domain models following [6, § 2.4]. Put $V(\mathbf{R}) = V \otimes_{\mathbf{Q}} \mathbf{R}$, and let $U_\infty \subset \mathrm{SO}(V)(\mathbf{R})$ denote the maximal compact subgroup. The quotient $\mathrm{SO}(V)(\mathbf{R})/U_\infty$ is known to be a hermitian symmetric space (as (V, q) has signature $(2, 2)$), and again we write $(\cdot, \cdot) : V \times V \to \mathbf{Q}$ to denote the hermitian form defined on $v_1, v_2 \in V$ by the bilinear form $(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$.

The Grassmannian model. Let us first describe the Grassmannian model of $\mathrm{SO}(V)(\mathbf{R})$. Recall[12] that we consider the Grassmannian $D_V = D_V^+$ of positive definite 2-dimensional subspaces $W \subset V(\mathbf{R})$:

$$D_V^+ = \{W \subset V(\mathbf{R}) : \dim(W) = 2, q|_W > 0\}.$$

By Witt's theorem, D_V^+ acts transitively on $\mathrm{SO}(V)(\mathbf{R})$. Moreover, given a subspace $W_0 \in D_V^+$, the corresponding stabilizer is the maximal compact subgroup:

$$\mathrm{Stab}_{D_V^+}(W_0) = U_\infty \subset \mathrm{SO}(V)(\mathbf{R}).$$

We can then view $D_V^+ \cong \mathrm{SO}(V)(\mathbf{R})/U_\infty = \mathrm{SO}(V)(\mathbf{R})/\mathrm{Stab}_{D_V^+}(W_0)$ as a realization of the hermitian symmetric space $\mathrm{SO}(V)(\mathbf{R})/U_\infty$. This description has the advantage of abstract simplicity, but at the same time hides the complex structure.

The projective model. To illustrate the complex structure, we now describe the projective model of $\mathrm{SO}(V)$. We consider the complexification $V(\mathbf{C}) = V \otimes_{\mathbf{Q}} \mathbf{C}$, as well as the corresponding projective space

$$\mathbb{P}(V(\mathbf{C})) = (V(\mathbf{C}) \backslash \{0\}) / \mathbf{C}^\times.$$

We can then consider the zero quadric defined by

$$\mathcal{N} := \{[Z] \in \mathbb{P}(V(\mathbf{C})) : (Z, Z) = 0\}.$$

This zero quadric $\mathcal{N} \subset V(\mathbf{C})$ determines a closed algebraic subvariety. Moreover, the subset $\mathcal{K}$ defined by

$$\mathcal{K} := \left\{[Z] \in \mathbb{P}(V(\mathbf{C})) : (Z, Z) = 0, (Z, \overline{Z}) > 0\right\} \subset \mathcal{N}$$

determines a complex manifold of dimension 2 having two connected components which we denote by $\mathcal{K}^\pm$. As explained in [6, § 2.4], the group $\mathrm{SO}(V)(\mathbf{R})$ acts transitively on $\mathcal{K}$, with $\mathrm{SO}(V)^+(\mathbf{R}) \subset \mathrm{SO}(V)(\mathbf{R})$ preserving these connected components $\mathcal{K}^\pm$, and with the the complement $\mathrm{SO}(V)(\mathbf{R}) \backslash \mathrm{SO}(V)^+(\mathbf{R})$ interchanging them. Let us now fix one of these connected components, $\mathcal{K}^+ \subset \mathcal{K}$ say. Given $Z \in V(\mathbf{C})$, let us write the corresponding decomposition into real and imaginary parts in the usual way as $Z = X + iY$ for $X, Y \in V(\mathbf{R})$. It is then simply to see ([6, Lemma 2.17]) that we have a real analytic isomorphism

$$\mathcal{K}^+ \cong D_V^+, \quad [Z] = [X + iY] \longmapsto \mathbf{R}X + \mathbf{R}Y.$$

This description has the advantage of revealing the complex structure, although it is not in general a direct analogue of the standard hermitian symmetric space $\mathfrak{H}$ for $\mathrm{SL}_2(\mathbf{R})$.

[12] Here, to be consistent with [6, § 2.7], we do not assume that the subspaces are oriented.

The tube domain model. Finally, let us describe the tube domain model. Here, we fix a nonzero isotropic vector $e_1 \in V$, together with any vector $e_2 \in V$ for which $(e_1, e_2) = 1$. We then consider the rational subspace $W \subset V$ defined by $W := V \cap \left(e_1^\perp \cap e_2^\perp\right) \subset V$. Hence, W is a Lorentian space of type $(1, 1)$, and we have the decomposition $V = W \oplus \mathbf{Q}e_1 \oplus \mathbf{Q}e_2$. We can then consider the corresponding tube domain $\mathcal{H}$ defined by

$$\mathcal{H} = \{Z \in W(\mathbf{C}) : q(\Im(Z)) > 0\} \subset W(\mathbf{C}).$$

As explained for [6, Lemma 2.18], we have a biholomorphic map

$$\mathcal{H} \longrightarrow \mathcal{K}, Z \longmapsto [(Z, 1, -q(Z) - q(e_2))]. \tag{88}$$

As well, the domain $\mathcal{H} \subset W(\mathbf{C}) \cong \mathbf{C}^2$ has two connected components $\mathcal{H}^\pm$ corresponding to the two cases of positive norm vectors in $W(\mathbf{R})$, and we write $\mathcal{H}^+$ to denote the component which is mapped to our fixed connected component $\mathcal{K}^+$ of $\mathcal{K}$ under (88). This can be viewed in a natural way as a generalization of the generalized upper-half plane, with $\mathrm{SO}(V)^+(\mathbf{R})$ acting transitively on $\mathcal{H}^+$. Moreover, in the case we consider here, there is a natural isomorphism $\mathcal{H}^+ \cong \mathfrak{H}^2$.

Now recall that we fix a finite volume subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$. Let us now consider the corresponding quotient $Y(\Gamma) = \Gamma\backslash\mathcal{H}^+ \cong \Gamma\backslash\mathfrak{H}^2$. Hence, $Y(\Gamma)$ is a normal complex space, compact if and only if the underlying space V is anisotropic – which is not the case here. We shall thus consider the Baily-Borel compactification $X(\Gamma)$ of $Y(\Gamma)$, which we describe briefly as follows. The quotient $Y(\Gamma)$ can be compactified by adding certain rational boundary components. These boundary components are easiest to describe in the projective model $\mathcal{K}^+$, where they arise as boundary points of $\mathcal{K}^+ \subset \mathcal{N}$ corresponding to nontrivial isotropic subspaces $\mathcal{F}$ of $V(\mathbf{R})$. In this description, an isotropic line $\mathcal{L} \subset V(\mathbf{R})$ represents a "special" boundary point of $\mathcal{K}$, and all other isotropic subspaces $\mathcal{F} \subset V(\mathbf{R})$ determine "generic" boundary points. The set consisting of special boundary points is called the "zero dimensional" boundary components, while those corresponding to two-dimensional isotropic subspaces of $V(\mathbf{R})$ as the "one dimensional" boundary components. As explained in [6, Lemma 2.20], there is a bijective correspondence between the boundary components of $\mathcal{K}^+ \subset \mathcal{N}$ and the nonzero isotropic subspaces $\mathcal{F} \subset V(\mathbf{R})$ (of matching dimensions), and moreover the boundary of $\mathcal{K}^+$ can be identified with the disjoint union of its boundary components in this sense. Now, a boundary component of $\mathcal{K}^+$ is said to be *rational* if its corresponding nonzero isotropic subspace $\mathcal{F} \subset V(\mathbf{R})$ is defined over $\mathbf{Q}$.

Definition 5.1. *Let $(\mathcal{K}^+)^\star$ denote the union of $\mathcal{K}^+$ with the disjoint union of rational boundary components of $\mathcal{K}^+$ corresponding to nonzero isotropic subspaces $\mathcal{F} \subset V(\mathbf{R})$ defined over $\mathbf{Q}$.*

Now, the rational orthogonal subgroup $\mathrm{SO}(V)(\mathbf{Q}) \cap \mathrm{SO}(V)^+(\mathbf{R})$ acts on $(\mathcal{K}^+)^\star$. It follows that we can consider the action of any finite index subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ on $(\mathcal{K}^+)^\star$. In this direction, we can now describe the following well-known compactification result.

Theorem 5.2 (Baily-Borel)**.** *Let $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ be any finite index subgroup. The quotient $X(\Gamma)$ defined by $X(\Gamma) = (\mathcal{K}^+)^\star/\Gamma$ is a compact Hausdorff space (in the Baily-Borel topology), and admits a natural complex structure as a normal complex space. In particular, $X(\Gamma)$ determines a projective algebraic variety.*

This so-called *Hilbert modular variety* $X(\Gamma)$ associated to the subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ also has a canonical model defined over some number field. Note that this is a surface in our setting.

5.1.5. *Modular forms associated to the Hilbert modular variety.* We also have a natural notion of modular forms here. To define this precisely, we consider the cone over $\mathcal{K}^+$ defined by

$$\widetilde{\mathcal{K}}^+ := \left\{Z \in V(\mathbf{C})\backslash\mathbf{C} : [Z] \in \mathcal{K}^+\right\}$$

Definition 5.3. *Fix a finite index subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$, an integer $k \in \mathbf{Z}$, and a character ξ pf Γ. A modular form of weight k and character ξ of Γ is a meromorphic function $F : \widetilde{\mathcal{K}}^+ \to \mathbf{C}$ such that:*

(i) *F is homogeneous of degree k: $F(cZ) = c^{-k}F(z)$ for all $c \in \mathbf{C}^\times$,*

(ii) *F is "invariant under Γ" in the sense that $F(\gamma Z) = \xi(\gamma)F(Z)$ for all $\gamma \in \Gamma$,*

(iii) *F is meromorphic at the boundary*[13].

Now, is it apparent from the identifications (87) and $\mathcal{H}^+ \cong \mathfrak{H}^2$ that we have a natural identification and any modular form F of weight k and character ξ on a congruence subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$ with a (uniquely-determined) Hilbert modular form $\mathbf{f} = \mathbf{f}(F)$ of parallel weight (k, k) and character ξ on the corresponding finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$. Conversely, to any such Hilbert modular form $\mathbf{f}$ of parallel integer weight (k, k) and character ξ on a finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$, there is a uniquely-determined modular form $F = F(\mathbf{f})$ of weight k and character ξ on the corresponding finite index subgroup $\Gamma \subset \mathrm{SO}(\Lambda) \cap \mathrm{SO}(V)^+(\mathbf{R})$. We refer to the discussion in [6, § 2.7] for a more precise account of the identifications. In what follows, we let $F = F(\varphi) = F(\mathbf{f}(\varphi))$ denote the modular form for Γ corresponding to the holomorphic Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$ of parallel weight 2 and trivial central character on the congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$ corresponding to the level of the cuspidal automorphic basechange representation Π of $\mathrm{GL}_2(\mathbf{A}_K)$, as described by the relation (86) above.

5.1.6. *Summary.* Our elliptic curve $E/\mathbf{Q}$ with corresponding cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbf{A})$ gives rise to the following cuspidal automorphic forms and cuspidal Hilbert modular forms:

$$\begin{array}{ccc} \varphi \in V_\Pi \text{ automorphic form on } \mathrm{GL}_2(\mathbf{A}_K) & \xrightarrow{(15)} & \varphi'' \text{ automorphic form on } \mathrm{SO}(V)(\mathbf{A}) \\ \downarrow & & \downarrow \\ \mathbf{f} = \mathbf{f}(\varphi) \text{ Hilbert modular form on } \Gamma & \xrightarrow{(87)} & F = F(\varphi) \text{ modular form on } X(\Gamma). \end{array} \tag{89}$$

Here, viewing our pure tensor $\varphi \in V_\Pi$ as an automorphic form on $\mathrm{GL}_2(\mathbf{A}_K)$ as we may, and according to the discussion above as an automorphic form φ'' on $\mathrm{SO}(V)(\mathbf{A})$ via the isomorphism (15) with the exact sequence (16) (cf. (13) and (14)), $\mathbf{f} = \mathbf{f}(\varphi)$ denotes the corresponding Hilbert modular form of parallel weight 2 and trivial central character on the associated congruence subgroup $\Gamma = \Gamma(\varphi)$ of $\mathrm{SL}_2(\mathcal{O}_K)$ (determined by Π). Writing Γ again to denote the congruence subgroup of $O(\Lambda) \cap O(V)^+(\mathbf{R})$ under the fixed isomorphism (87), $F = F(\varphi)$ denotes the modular form of weight 2 and trivial central character on Γ corresponding to this Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$.

5.2. **Geodesic formulae.** We now give more explicit, classical interpretations of Theorem 4.2, Theorem 4.19, and Corollary 4.7 above. Here, we shall take for granted the discussion above leading to the description (89) of the various realizations of the basechange pure tensor $\varphi \in V_\Pi$.

5.2.1. *Interpretations of* (18) *and* (19). Let us first describe the following more explicit variants in terms of the underlying Hilbert modular surface $X(\Gamma) = \Gamma \backslash (\mathcal{K}^+)^\star \cong \Gamma \backslash (\mathfrak{H}^2)^\star$. Hence, recall from the setup above that we fix the quadratic space (V, q) with $V = \mathbf{Q} \oplus \mathbf{Q} \oplus K$ and quadratic form $q(x, y, \lambda) := \mathbf{N}_{K/\mathbf{Q}}(\lambda) - xy$. Writing D_V^+ to denote the Grassmannian of oriented positive definite hyperplanes in $V(\mathbf{R})$, and taking $U \subset \mathrm{GO}(V)(\mathbf{A}_f)$ to be the compact oven subgroup determined by that of $\mathrm{GL}_2(\mathbf{A}_{K,f})$ corresponding to the representation $\Pi = \mathrm{BC}_{K/\mathbf{Q}}(\pi)$ (via (15)), we consider the corresponding orthogonal Shimura variety

$$\mathrm{Sh}_U(\mathrm{GO}(V), D_V^+) = \mathrm{GO}(V)(\mathbf{Q}) \backslash \left(D_V^+ \times \mathrm{GO}(V)(\mathbf{A}_f)/U \right),$$

which is equivalent to non-compact Hilbert modular surface $Y(\Gamma)$ introduced above, where Γ denotes the congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$ corresponding to the intersection $\Gamma = \mathrm{GO}(\Lambda) \cap U$ introduced above. In fact (cf. [32, (1.2)]), fixing a component $D_V \cong \mathcal{K}^+ \cong \mathfrak{H}^2$ of D_V^+, we can use the strong approximation theorem (81) for $\mathrm{GL}_2(\mathbf{A}_K)$ with the accidental isomorphism (15) to deduce that for elements $\zeta_j \in \mathrm{GO}(V)(\mathbf{A}_f)$ corresponding to representatives of the class group $\mathbf{A}_K / K_\infty^\times K^\times \widehat{\mathcal{O}}_K^\times$ of K, we have the decomposition

$$\mathrm{GO}(V)(\mathbf{A}) = \coprod_{j=1}^{h_K} \mathrm{GO}(V)(\mathbf{Q})\, \mathrm{GO}(V)^+(\mathbf{R}) \zeta_j U,$$

[13] In general, by the Koecher principle, this condition is always satisfied when the Witt index of V (i.e. the dimension of the maximal isotropic subspace of V) is less than 2.

so that

$$\mathrm{Sh}_U(\mathrm{GO}(V), D_V) = Y(\Gamma) = \coprod_j \Gamma_j \backslash D_V, \quad \text{where } \Gamma_j = \mathrm{GO}(V)(\mathbf{Q}) \cap \left(\mathrm{GO}(V)^+(\mathbf{R})\zeta_j U \zeta_j^{-1}\right).$$

The compactification $X(\Gamma)$ is then given according to the description above for each of the Hilbert modular surfaces $X_j := \Gamma_j\backslash D_V \cong \Gamma_j\backslash\mathfrak{H}^2$. In this way, we see explicitly how our automorphic form φ on $g \in \mathrm{GL}_2(\mathbf{A}_K)$ gives rise via the accidental isomorphism (15) to a modular form $F = F(\varphi)$ of weight two on the orthogonal Shimura variety $\mathrm{Sh}_U(\mathrm{GO}(V), D_V^+)$, and also to a Hilbert modular form $\mathbf{f} = \mathbf{f}(\varphi)$ of parallel weight two on the compactified modular surface $X(\Gamma)$. Note that for any complex variable $s \in \mathbf{C}$, we can also consider the automorphic form defined on $g \in \mathrm{GL}_2(\mathbf{A}_K)$ by $\varphi(g)|\det(g)|^{s-1/2}$, writing $F_s = F_s(\varphi)$ to denote the corresponding automorphic form on $\mathrm{Sh}_U(\mathrm{GO}(V), D_V^+)$, and also $\mathbf{f}_s = \mathbf{f}_s(\varphi)$ the corresponding form on $X(\Gamma)$.

Let us again for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$ fix an integral representative $\mathfrak{a} \subset \mathcal{O}_K$, which we can and do identify as an idele representative. We then consider the quadratic subspace (V_A, q_A) with $V_A = \mathbf{Q} \oplus \mathbf{Q} \oplus \mathfrak{a}$, and quadratic form $q_A(x, y, \lambda) = \mathbf{N}(\lambda)\mathbf{N}\mathfrak{a}^{-1} - xy = \lambda\lambda^\tau \mathbf{N}\mathfrak{a}^{-1} - xy$. We argue that there is a natural identification $\mathrm{GO}(V) \cong \mathrm{GO}(V_A)$ of algebraic groups over $\mathbf{Q}$. Moreover, taking $V_{A,2}$ to be the subspace $V_{A,2} = \mathfrak{a}$ with quadratic form $q_{A,2}(\lambda) = \mathbf{N}(\lambda)\mathbf{N}\mathfrak{a}^{-1}$, we argue that we may view $\mathrm{GO}(V_{A,2})$ as a subgroup of $\mathrm{GO}(V)$ in the natural way. Again, we consider the corresponding "geodesic" subset

$$Z(V_{A,2}) := \mathrm{GO}(V_{A,2})(\mathbf{Q})\backslash\{z_{V_{A,2}}^{\pm}\} \times \mathrm{GO}(V_{A,2})(\mathbf{A}_f)/U_{A,2}, \quad U_{A,2} := U_A \cap \mathrm{GO}(V_{A,2})(\mathbf{A}_f)$$

of the Hilbert modular surface $\mathrm{Sh}_U(\mathrm{GO}(V), D_V^+)$. Since we take the conductor c of the order $\mathcal{O}_c \subset \mathcal{O}_K$ to be coprime to the level N of the eigenform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$, we argue that any ring class character χ of conductor c of $\mathbf{A}_K^\times/K^\times \cong \mathrm{GO}(V_2)(\mathbf{A})/\mathrm{GO}(V_2)(\mathbf{Q})$ can be viewed as a right U_2-invariant automorphic from on $\mathrm{GO}(V_2)(\mathbf{Q})\backslash\mathrm{GO}(V_2)(\mathbf{A}_f)$ (cf. [37, §2.3]). We also argue that $\mathrm{vol}(U_2) = \mathrm{vol}(U_{A,2})$ for each class $A \in \mathrm{Pic}(\mathcal{O}_c)$. Using the result of [40, Lemma 2.13] again, we see after a decomposition of the integral over $\mathbf{A}_K^\times/K^\times$ that the integral expression (19) can be described more explicitly as

$$\begin{aligned} L'(1/2, \Pi\otimes\chi) &= \frac{d}{ds}\Big|_{s=1/2}\left(\int_{\mathrm{GO}(V_2)(\mathbf{Q})\backslash\mathrm{GO}(V_2)(\mathbf{A})} \varphi''(y)\,\chi''(y)|y|^{s-\frac{1}{2}}dy\right) \\ &= \frac{d}{ds}\Big|_{s=1/2}\left(\mathrm{vol}(U_2)\sum_{h\in\mathrm{GO}(V_2)(\mathbf{Q})\backslash\mathrm{GO}(V_2)(\mathbf{A})/U_2} \varphi''(h)\,\chi''(h)|h|^{s-\frac{1}{2}}\right) \\ &= \mathrm{vol}(U_2)\sum_{\substack{A\in\mathrm{Pic}(\mathcal{O}_c)\\ [\mathfrak{a}]=A}}\chi(A)\sum_{(z_{V_{A,2}},h_A)\in Z(V_{A,2})}\frac{d}{ds}\Big|_{s=1/2}F_s\left((z_{V_{A,2}},h_A)\right) \end{aligned} \tag{90}$$

for F_s on the orthogonal Shimura variety $\mathrm{Sh}_U(\mathrm{GO}(V), D_V^+) = Y(\Gamma)$, and similarly

$$L'(1/2, \Pi\otimes\chi) = \mathrm{vol}(U_2)\sum_{\substack{A\in\mathrm{Pic}(\mathcal{O}_c)\\ [\mathfrak{a}]=A}}\chi(A)\sum_{(z_{V_{A,2}},h_A)\in Z(V_{A,2})}\frac{d}{ds}\Big|_{s=1/2}\mathbf{f}_s\left((z_{V_{A,2}},h_A)\right) \tag{91}$$

for the Hilbert modular form $\mathbf{f}_s$ on $\Gamma = \Gamma(\varphi) \subset \mathrm{SL}_2(\mathcal{O}_K)$.

5.2.2. *Interpretations of Theorem 4.2, Theorem 4.19 and Corollary 4.7.* In view of the expressions (90) and (91), we see that the abstract characterization of the form φ'' on $\mathrm{GO}(V)(\mathbf{A})$ corresponding to φ as a theta lift in Theorem 4.2 can be realized explicitly in terms of regularized theta liftings via Theorem 4.19 and 4.7 above, which also allows us to recover the Rankin-Selberg integral presentation. To restate these results in more explicit terms as in (90) and (91), we have the following result.

Corollary 5.4. *Assume Hypothesis 2.1. Then, we have by Theorem 4.19 and Corollary 4.7 above that*

$$L'(1/2,\Pi\otimes\chi) = \mathrm{vol}(U_2) \sum_{\substack{A\in \mathrm{Pic}(\mathcal{O}_c)\\ [\mathfrak{a}]=A}} \chi(A) \sum_{(z_{V_{A,2}},h_A)\in Z(V_{A,2})} \frac{d}{ds}\Big|_{s=1/2} \mathbf{f}_s\left((z_{V_{A,2}},h_A)\right)$$

$$= \frac{\sqrt{d_K}}{\log \epsilon_K\cdot h_K} \sum_{\substack{A\in \mathrm{Pic}(\mathcal{O}_c)\\ A=[\mathfrak{a}]}} \chi(A) \left(\mathrm{CT}\left(\langle\langle f^+_{0,\eta,A}(\tau), \theta_{\Lambda_{A,1}}\otimes \mathcal{E}_{\Lambda_{A,2}}(\tau)\rangle\rangle\right) - \mathrm{vol}(U_{A,2}) \sum_{(z_{V_{A,2}},h_A)\in Z(V_{A,2})} \vartheta^\star_{f_{0,\eta,A}}(z_{V_{A,2}},h_A)\right).$$

6. Relation to the conjecture of Birch and Swinnerton-Dyer

Let us now consider Theorem 4.19 from the point of view of the refined conjecture of Birch and Swinnerton-Dyer, comparing with the Gross-Zagier formula [23]. To date, there is no known or conjectural construction of points on the corresponding elliptic curve $E(K[c])$ or modular curve $X_0(N)(K[c])$ analogous to Heegner points[14], where $K[c]$ denotes the ring class extension of conductor c of the real quadratic field K. We can consider the implications for arithmetic terms in the refined Birch and Swinnerton-Dyer formula for $L^{\star\prime}(E/K,\chi,1)$ here, in the style of the comparison given in Popa [37, §6.4]. Here, we first interpret our formula crudely in terms of homology groups of the Shimura varieties $\mathrm{Sh}_U(\mathrm{GO}(V),D_V^+) = Y(\Gamma)$ and $\mathrm{Sh}_U(\mathrm{GSpin}_V,D_V^+)$. Taking for granted the refined conjecture of Birch and Swinnerton-Dyer for $E(K[c]))$ in this setting – particularly for the case of rank one corresponding to Hypothesis 2.1 – we shall then derive "automorphic" interpretations of the corresponding Tate-Shafarevich group $Ш(E/K[c])$ and regulator $\mathrm{Reg}(E/K[c])$. We also derive an unconditional result in special cases to illustrate surprising connections here.

6.1. **Expressions in terms of geodesic homology classes.** Let us first note that the formulae (90) and (91) can be viewed crudely in terms of homology classes as follows. Given a class $A\in\mathrm{Pic}(\mathcal{O}_c)$, let us write Ψ_A to denote the class in either of the homology groups $H_1(Y(\Gamma),\mathbf{Z})\subset H_1(X(\Gamma),\mathbf{Z})$ or $H_1(\mathrm{Sh}_U(\mathrm{GO}(V),D_V^+),\mathbf{Z})$ determined by the locus of "geodesic points" determined by the symmetric subspace $D^+_{V_2}$ (which can be defined via restriction hyperplanes in D_V^+), and again consider the "geodesic" subset

$$Z(V_{A,2}) = \mathrm{GO}(V_{A,2})(\mathbf{Q})\backslash\{z^{\pm}_{V_{A,2}}\}\times \mathrm{GO}(V_{A,2})(\mathbf{A})/\left(U\cap \mathrm{GO}(V)(\mathbf{A}_f)\right).$$

We can then consider the $\mathbf{C}$-valued 1-cycles defined by the linear combinations

$$\alpha_\chi = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\cdot\deg(V_{A,2})\cdot\Psi_A \in H_1(X(\Gamma),\mathbf{Z})\otimes\mathbf{C}$$

and

$$\beta_\chi = \sum_{A\in\mathrm{Pic}(\mathcal{O}_c)} \chi(A)\cdot\deg(V_{A,2})\cdot\Psi_A \in H_1(\mathrm{Sh}_U(\mathrm{GO}(V),D_V^+),\mathbf{Z})\otimes\mathbf{C},$$

which allow us to represent (90) and (91) respectively (formally) as

$$L'(1/2,\Pi\otimes\chi) = \int_{\alpha_\chi} \frac{d}{ds}\Big|_{s=1/2} F_s \tag{92}$$

and

$$L'(1/2,\Pi\otimes\chi) = \int_{\beta_\chi} \frac{d}{ds}\Big|_{s=1/2} \mathbf{f}_s \tag{93}$$

Remark Note that the forms F_s and $\mathbf{f}_s$ here – corresponding to the automorphic form on $g\in\mathrm{GL}_2(\mathbf{A}_K)$ defined by $\varphi(g)|\det(g)|^{s-1/2}$ – should have interpretations as images under weight raising and lowering operators analogous[15] to those defined above for vector-valued forms above. Note as well that these forms F_s and $\mathbf{f}_s$ associated to the Hilbert modular surface $X(\Gamma)$ should have a more direct relation to the Mordell-Weil group $E(K)$ by the modularity theorem of Freitas-Le Hung-Siksek [18], although one does *not* expect to have any analogue of the modular parametrization $\varphi: X_0(N)\longrightarrow E$ here.

[14] There is however a p-adic construction due to Darmon [15].

[15] The theory of Maass weight raising and lowering operators is not yet well-understood for Hilbert modular forms.

6.2. **Comparison with the refined conjecture of Birch and Swinnerton-Dyer.** Finally, we consider these descriptions of the central derivative values $L'(1/2, \Pi \otimes \chi) = L^{\star\prime}(E/K, \chi, 1)$. Here again, we fix χ a primitive ring class character of some conductor $c \geq 1$ prime to $d_K N$, and view this as a character of the class group $\mathrm{Pic}(\mathcal{O}_c)$. Recall that the reciprocity map of class field theory gives us an isomorphism

$$\mathrm{Pic}(\mathcal{O}_c) := \mathbf{A}_K^\times / K_\infty^\times K^\times \widehat{\mathcal{O}}_c^\times \longrightarrow \mathrm{Gal}(K[c]/K),$$

where $K[c]$ is (by definition) the ring class extension of conductor c of K. Recall as well that by the theory of cyclic basechange of [36] and more generally [2] with Artin formalism, we can write the completed Hasse-Weil L-function $L^\star(E/K[c], s)$ of E basechanged to $K[c]/K$ as the product

$$\begin{aligned} L^\star(E/K[c], s) &= \prod_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} L^\star(E/K, \chi, s) \\ &= \prod_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} L(s - 1/2, \Pi \otimes \chi) \\ &= \prod_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} L(s - 1/2, \mathrm{BC}_{K/\mathbf{Q}}(\pi) \otimes \chi) \\ &= \prod_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} L(s - 1/2, \pi \times \pi(\chi)) \\ &= \prod_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} L(s, f \times \theta(\chi)). \end{aligned} \tag{94}$$

Here, we use all of the same conventions and definitions as established above, and will continue to use these without extra comment. Writing $\mathrm{ord}_{s=s_0}$ as usual to denote the order of vanishing at a given complex argument $s_0 \in \mathbf{C}$, it then follows as a formal consequence of (94) that we have the relation(s)

$$\mathrm{ord}_{s=1} L^\star(E/K[c], s) = \sum_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} \mathrm{ord}_{s=1/2} L(s, \Pi \otimes \chi), \tag{95}$$

so that the conjecture of Birch and Swinnerton-Dyer predicts the rank equivalence

$$\mathrm{rk}_{\mathbf{Z}} E(K[c]) = \mathrm{ord}_{s=1} L^\star(E/K[c], s) = \sum_{\chi \in \mathrm{Pic}(\mathcal{O}_c)^\vee \cong \mathrm{Gal}(K[c]/K)^\vee} \mathrm{ord}_{s=1/2} L(s, \Pi \otimes \chi). \tag{96}$$

Let us now assume Hypothesis 2.1, so that for each ring class character χ on the right hand side of (96), we know by the symmetric functional equation (8) that $\mathrm{ord}_{s=1/2} L(s, \Pi \otimes \chi) \geq 1$. Let us also assume for the moment that the rank equality predicted by the conjecture of Birch and Swinnerton-Dyer holds, so that

$$\mathrm{rk}_{\mathbf{Z}} E(K[c]) \geq h(\mathcal{O}_c) := \# \mathrm{Pic}(\mathcal{O}_c) = \# \mathrm{Gal}(K[c]/K). \tag{97}$$

Let $r_E(K[c]) \geq h(\mathcal{O}_c)$ denote the Mordell-Weil rank of E over the ring class extension $K[c]$ of conductor c over K. The refined conjecture of Birch and Swinnerton-Dyer predicts that the leading term in the Taylor series expansion around $L^{(r_E(K[c]))}(E/K[c], s)$ around $s = 1$ is given by the following formula. Let $\text{Ш}_E(K[c])$ denote the Tate-Shafarevich group of E over $K[c]$,

$$\text{Ш}_E(K[c]) = \ker \left(H^1(K, E) \longrightarrow \prod_w H^1(K_w, E) \right),$$

which we shall assume is known to be finite. Let $R_E(K[c])$ denote the regulator of E over $K[c]$. Hence, fixing a basis $(e_j)_{j=1}^{r_E(K[c])}$ of $E(K[c])/E(K[c])_{\mathrm{tors}}$, and writing $[\cdot, \cdot]$ to denote the Néron-Tate height pairing,

$$R_E(K[c]) = \det \left([e_i, e_j] \right)_{i,j}.$$

Let us also write $T_E(K[c])$ to denote the product over local Tamagawa factors, so

$$T_E(K[c]) = \prod_{\substack{\nu < \infty \\ \text{primes of } \mathcal{O}_{K[c]}}} [E(K[c]_\nu) : E_0(K[c]_\nu)] \cdot \left| \frac{\omega}{\omega_\nu^*} \right|_\nu,$$

where $\omega = \omega_E$ is a fixed invariant differential for $E/K[c]$, and each ω_ν^* the Néron differential at ν. The refined conjecture of Birch and Swinnerton-Dyer then predicts that the Taylor series expansion of $L^{(r_E(K[c])))}(E/K[c], s)$ around $s = 1$ has leading term given by the analytic-class-number-like formula

$$\frac{\#\text{Ш}_E(K[c]) \cdot R_E(K[c]) \cdot T_E(K[c])}{\sqrt{d_K} \cdot \#E(K[c])_{\text{tors}}^2} \cdot \prod_{\substack{\mu|\infty \\ \mu: K[c] \to \mathbf{R} \\ \text{realplaces}}} \int_{E(K[c]_\mu)} |\omega| \cdot \prod_{\substack{\sigma|\infty \\ \sigma, \overline{\sigma}: K[c] \to \mathbf{C} \\ \text{pairs of complex places}}} \omega \wedge \omega. \tag{98}$$

Let us first assume for simplicity that the class number is one: $h(\mathcal{O}_c) = h_K = 1$. Then, assuming the conjecture of Birch and Swinnerton-Dyer (97) and (98), we derive via Theorem 4.19, Corollary 4.7, and the relations (92) and (93) the (conditional) identifications

$$\begin{aligned} L^{\star\prime}(E/K,1) &= L'(1/2,\Pi) = L'(1/2,\Pi\otimes\mathbf{1}) = \int_{\alpha_1} \frac{d}{ds}\Big|_{s=1/2} F_s = \int_{\beta_1} \frac{d}{ds}\Big|_{s=1/2} \mathbf{f}_s \\ &= \frac{\#\text{Ш}_E(K) \cdot R_E(K) \cdot T_E(K)}{\sqrt{d_K} \cdot \#E(K)_{\text{tors}}^2} \cdot \prod_{\substack{\mu|\infty \\ \mu: K \to \mathbf{R}}} \int_{E(K_\mu)} |\omega| \\ &= \frac{\sqrt{d_K}}{\log \epsilon_K} \left(\text{CT}\left(\langle\langle f_{0,\eta,\mathcal{O}_K}^+(\tau), \theta_{L_{\mathcal{O}_K,1}}^+ \otimes \mathcal{E}_{L_{\mathcal{O}_K,2}}(\tau) \rangle\rangle \right) - \text{vol}(U_{\mathcal{O}_K,2}) \sum_{(z_{V_{\mathcal{O}_K,2}}^\pm, h) \in Z(V_{\mathcal{O}_K,2})} \vartheta_{f_{0,\eta,\mathcal{O}_K}}^\star(z_{V_{\mathcal{O}_K,2}}^\pm, h) \right) \end{aligned}$$

This suggests that the regulator $R_E(K) = [e_{??}, e_{??}]$ should be given by the formula

$$\begin{aligned} R_E(K) &= [e_{??}, e_{??}] \\ &= \frac{\#E(K)_{\text{tors}}^2 \left(\text{CT}\left(\langle\langle f_{0,\eta,\mathcal{O}_K}^+(\tau), \theta_{L_{\mathcal{O}_K,1}}^+ \otimes \mathcal{E}_{L_{\mathcal{O}_K,2}}(\tau) \rangle\rangle \right) - \text{vol}(U_{\mathcal{O}_K,2}) \sum_{(z_{V_{\mathcal{O}_K,2}}^\pm, h) \in Z(V_{\mathcal{O}_K,2})} \vartheta_{f_{0,\eta,\mathcal{O}_K}}^\star(z_{V_{\mathcal{O}_K,2}}^\pm, h) \right)}{\log \epsilon_K \cdot \#\text{Ш}_E(K) \cdot T_E(K) \cdot \prod_{\substack{\mu|\infty \\ \mu: K \to \mathbf{R}}} \int_{E(K_\mu)} |\omega|}. \end{aligned} \tag{99}$$

Similarly, the cardinality $\#\text{Ш}_E(K)$ of Tate-Shafarevich group $\text{Ш}_E(K)$ should be given by the formula

$$\begin{aligned} \#\text{Ш}_E(K) &= \frac{\#E(K)_{\text{tors}}^2 \left(\text{CT}\left(\langle\langle f_{0,\eta,\mathcal{O}_K}^+(\tau), \theta_{L_{\mathcal{O}_K,1}}^+ \otimes \mathcal{E}_{L_{\mathcal{O}_K,2}}(\tau) \rangle\rangle \right) - \text{vol}(U_{\mathcal{O}_K,2}) \sum_{(z_{V_{\mathcal{O}_K,2}}^\pm, h) \in Z(V_{\mathcal{O}_K,2})} \vartheta_{f_{0,\eta,\mathcal{O}_K}}^\star(z_{V_{\mathcal{O}_K,2}}^\pm, h) \right)}{\log \epsilon_K \cdot R_E(K) \cdot T_E(K) \cdot \prod_{\substack{\mu|\infty \\ \mu: K \to \mathbf{R}}} \int_{E(K_\mu)} |\omega|}. \end{aligned} \tag{100}$$

Note that we can also derive similar albeit more intricate conditional arithmetic expressions for $\#\text{Ш}_E(K[c])$ and $R_E(K[c])$ in the more general setting where $h_K \geq 1$, e.g. after specializing our main result to the principal character $\chi = \chi_0$ of the class group of K, and summing over classes. We leave the details as an exercise to the reader. Finally, we can also establish the following unconditional result.

Theorem 6.1. *Assume that* $\text{ord}_{s=1} L^\star(E/K,1) = 1$*, so that either* $L^\star(E,1) = L(1/2,\pi)$ *or the quadratic twist* $L^\star(E^{(d_K)},1) = L(1/2,\pi\otimes\eta)$ *vanishes. Let us also assume that* E *has semistable reduction so that its conductor* N *is squarefree, with* N *coprime to the discriminant* d_K *of* K*, and for each prime* $p \geq 5$*:*

- *The residual Galois representations* $E[p]$ *and* $E^{(d_K)}[p]$ *attached to* E *and* $E^{(d_K)}$ *are irreducible,*
- *There exists a prime divisor* $l \mid\mid N$ *distinct from* p *where the residual representation* $E[p]$ *is ramified, and a prime divisor* $q \mid\mid Nd_K$ *distinct from* p *where the residual representation* $E^{(d_K)}[p]$ *is ramified.*

Writing $[e,e]$ to denote the regulator of either E or $E^{(d_k)}$ according to which factor vanishes, we have the following unconditional identity, up to powers of 2 and 3:

$$\frac{\#Ш_E(\mathbf{Q})\cdot\#Ш_{E^{(d_K)}}(\mathbf{Q})\cdot[e,e]\cdot T_E(\mathbf{Q})\cdot T_{E^{(d_K)}}(\mathbf{Q})}{\#E(\mathbf{Q})^2_{\mathrm{tors}}\cdot\#E^{(d_k)}(\mathbf{Q})^2_{\mathrm{tors}}}\cdot\int_{E(\mathbf{R})}|\omega_E|\cdot\int_{E^{(d_K)}(\mathbf{R})}|\omega_{E^{(d_k)}}|$$
$$=\frac{\sqrt{d_K}}{\log\epsilon_K}\sum_{A\in\mathrm{Pic}(\mathcal{O}_K)}\left(\mathrm{CT}\left(\langle\langle f^+_{0,\eta,A}(\tau),\theta_{L_{A,1}}\otimes\mathcal{E}_{L_{A,2}}(\tau)\rangle\rangle\right)-\mathrm{vol}(U_{A,2})\sum_{(z^\pm_{V_{A,2}},h)\in Z(V_{A,2})}\vartheta^\star_{f_{0,\eta,A}}(z^\pm_{V_{A,2}},h)\right).$$

Proof. Assuming as we do that $\mathrm{ord}_{s=1}L^\star(E/K,1)=1$, we deduce from the Artin formalism that

$$L^{\star\prime}(E/K,1)=L^{\star\prime}(E/K,\mathbf{1},1)=L^{\star\prime}(E,1)L^\star(E^{(d_K)},1)+L^{\star\prime}(E^{(d_K)},1)L(E,1),$$

or equivalently that

$$L'(1/2,\Pi)=L'(1/2,\Pi\otimes\mathbf{1})=L'(1/2,\pi)L(1/2,\pi\otimes\eta)+L'(1/2,\pi\otimes\eta)L(1/2,\pi),$$

where precisely one of the summands on the right hand side in each version does not vanish. Note that we can take for granted the refined conjecture of Birch and Swinnerton-Dyer (98) for the nonvanishing summand up to powers of 2 and 3 by our hypotheses, using the combined works of Kato [28], Kolyvagin [29], Rohrlich [39], and Skinner-Urban [44] with Burungale-Skinner-Tian [9] (cf. [9], [13], [45]) for the analytic rank zero oart, together with the work of Jetchev-Skinner-Wan [27] or Zhang [54] for the analytic rank one part. We refer to the summary given in [9, Theorem 3.10] for the current status of these deductions confirming the p-part of the conjectural Birch-Swinnerton-Dyer formula via Iwasawa-Greenberg main conjectures. Applying (98) to each factor, we can then deduce unconditionally that we have the refined product formula

$$L^{\star\prime}(E/K,1)=L^{\star\prime}(E/K,\mathbf{1},1)=L'(1/2,\Pi\otimes\mathbf{1})$$
$$=\frac{\#Ш_E(\mathbf{Q})\cdot\#Ш_{E^{(d_K)}}(\mathbf{Q})\cdot[e,e]\cdot T_E(\mathbf{Q})\cdot T_{E^{(d_K)}}(\mathbf{Q})}{\#E(\mathbf{Q})^2_{\mathrm{tors}}\cdot\#E^{(d_k)}(\mathbf{Q})^2_{\mathrm{tors}}}\cdot\int_{E(\mathbf{R})}|\omega_E|\cdot\int_{E^{(d_K)}(\mathbf{R})}|\omega_{E^{(d_k)}}|.$$

The stated identity then follows from Theorem 4.19 and Corollary 4.7. ☐

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