ENUMERATING TRACELESS MATRICES OVER COMPACT DISCRETE VALUATION RINGS

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Abstract. We enumerate traceless square matrices over finite quotients of compact discrete valuation rings by their image sizes. We express the associated rational generating functions in terms of statistics on symmetric and hyperoctahedral groups, viz. Coxeter groups of types A and B, respectively. These rational functions may also be interpreted as local representation zeta functions associated to the members of an infinite family of finitely generated class-2-nilpotent groups.

As a byproduct of our work, we obtain descriptions of the numbers of traceless square matrices over a finite field of fixed rank in terms of statistics on the hyperoctahedral groups.

1. Introduction and statement of main results

Let $n \in \mathbb{N}$. Given a ring R, we write $\mathfrak{sl}_n(R)$ for the set of $n \times n$-matrices over R of trace zero. In the case that R is a finite field $\mathbb{F}_q$, we write, for $i \in \{0, 1, \ldots, n\}$,

$$B_{n,n-i}(\mathbb{F}_q) = \{\boldsymbol{x} \in \mathfrak{sl}_n(\mathbb{F}_q) \mid \mathrm{rk}(\boldsymbol{x}) = n - i\}$$

for the set of traceless $n \times n$-matrices over $\mathbb{F}_q$ of rank $n - i$. Formulae expressing the cardinalities of these sets as polynomials in the field's cardinality q are well-known, e.g. by the work [3] of Bender; cf. Lemma 2.2.

In the current paper, we generalize these enumerative formulae to traceless matrices over more general finite quotients of compact discrete valuation rings. As a byproduct we obtain an interpretation of the polynomials $|B_{n,n-i}(\mathbb{F}_q)|$ in terms of statistics on hyperoctahedral groups, viz. finite Coxeter groups of type B; cf. Proposition 1.4.

1.1. **Counting traceless matrices over compact discrete valuation rings.** We now state our generalized counting problem. Let $\mathfrak{o}$ be a compact discrete valuation ring of arbitrary characteristic, with unique maximal ideal $\mathfrak{p}$, residue field cardinality q, and residue field characteristic p. Given a *level* $N \in \mathbb{N}_0$, we set $\mathfrak{o}_N = \mathfrak{o}/\mathfrak{p}^N$. Clearly, traceless matrices of level $N = 1$ decompose in a disjoint union as follows:

$$\mathfrak{sl}_n(\mathfrak{o}_1) = \bigcup_{i=0}^{n} B_{n,n-i}(\mathbb{F}_q).$$

Noting that $\boldsymbol{x} \in \mathfrak{sl}_n(\mathfrak{o}_1)$ has rank $\mathrm{rk}(\boldsymbol{x}) = n - i$ if and only if $|\mathrm{im}(\boldsymbol{x})| = q^{n-i}$ when $\boldsymbol{x}$ is viewed as an endomorphism of $\mathfrak{o}_1^n$ suggests a natural generalization of this decomposition to arbitrary levels, viz. by image sizes. For any $N \in \mathbb{N}$ and $\boldsymbol{x} \in \mathfrak{sl}_n(\mathfrak{o}_N)$, write $\mathrm{im}(\boldsymbol{x})$ for the image of $\boldsymbol{x}$, viewed as an endomorphism of $\mathfrak{o}_N^n$. For $N \in \mathbb{N}$, we write $\mathfrak{sl}_n(\mathfrak{o}_N)^* =$

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$\mathfrak{sl}_n(\mathfrak{o}_N) \setminus (\mathfrak{p} \cdot \mathfrak{sl}_n(\mathfrak{o}_N))$ for the *primitive* traceless $n \times n$-matrices of level N, i.e. those which are not zero modulo $\mathfrak{p}$. We set $\mathfrak{sl}_n(\mathfrak{o}_0)^* = \{0\}$.

Definition 1.1. The *image zeta function of* $\mathfrak{sl}_n(\mathfrak{o})$ is the ordinary generating function

$$\mathcal{P}_{n,\mathfrak{o}}(s) := \sum_{N=0}^{\infty} \sum_{\boldsymbol{x} \in \mathfrak{sl}_n(\mathfrak{o}_N)^*} |\operatorname{im}(\boldsymbol{x})|^{-s} \in \mathbb{Q}[\![q^{-s}]\!],$$

where s is a complex variable.

The series $\mathcal{P}_{n,\mathfrak{o}}(s)$ is well-defined as summation is restricted to primitive matrices.

One of our main results shows that the power series $\mathcal{P}_{n,\mathfrak{o}}(s)$ is a rational function over $\mathbb{Q}$ in q and q^{-s}, and describes this rational function explicitly. In the following, S_n denotes the symmetric group of degree n, whereas ℓ and Des denote the standard Coxeter length and the descent set statistics on S_n, respectively; see Section 3.1 for details.

Theorem 1.2. *For all compact discrete valuation rings* $\mathfrak{o}$*, with residue field cardinality* q*, say, the image zeta function of* $\mathfrak{sl}_n(\mathfrak{o})$ *satisfies*

$$\mathcal{P}_{n,\mathfrak{o}}(s) = \left(\sum_{w \in S_n} q^{-\ell(w)} \prod_{j \in \operatorname{Des}(w)} q^{n^2-j^2-1-s(n-j)} \right) \prod_{j=0}^{n-1} \frac{1-q^{j-s}}{1-q^{n^2-j^2-1-s(n-j)}}.$$

Example 1.3. Throughout, we write $t = q^{-s}$.

(1)

$$\mathcal{P}_{1,\mathfrak{o}}(s) = 1.$$

(2)

$$\mathcal{P}_{2,\mathfrak{o}}(s) = (1+qt)\frac{(1-t)(1-qt)}{(1-q^2t)(1-q^3t^2)} = \frac{(1-t)(1-q^2t^2)}{(1-q^2t)(1-q^3t^2)}.$$

(3)

$$\begin{aligned}\mathcal{P}_{3,\mathfrak{o}}(s) &= (1+q^2t)(1+q^3t+q^6t^2)\frac{(1-t)(1-qt)(1-q^2t)}{(1-q^4t)(1-q^7t^2)(1-q^8t^3)} \\ &= \frac{(1-q^4t^2)(1-q^9t^3)(1-t)(1-qt)(1-q^2t)}{(1-q^2t)(1-q^3t)(1-q^4t)(1-q^7t^2)(1-q^8t^3)}.\end{aligned}$$

(4)

$$\mathcal{P}_{4,\mathfrak{o}}(s) = V_4(q,t)\frac{(1-t)(1-qt)(1-q^2t)(1-q^3t)}{(1-q^6t)(1-q^{11}t^2)(1-q^{14}t^3)(1-q^{15}t^4)},$$

where $V_4(q,t) = (1+q^4t)V_4'(q,t)$ and

$$\begin{aligned}V_4'(X,Y) = X^{21}Y^5 + X^{18}Y^4 + X^{16}Y^4 + X^{13}Y^3 + X^{12}Y^3 + X^{11}Y^3 + \\ X^{10}Y^2 + X^9Y^2 + X^8Y^2 + X^5Y + X^3Y + 1.\end{aligned}$$

The polynomial $V_4'(X,Y) \in \mathbb{Q}[X,Y]$ appears to be irreducible.

(5)

$$\mathcal{P}_{5,\mathfrak{o}}(s) = V_5(q,t)\frac{(1-t)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)}{(1-q^8t)(1-q^{15}t^2)(1-q^{20}t^3)(1-q^{23}t^4)(1-q^{24}t^5)},$$

where $V_5(X,Y) \in \mathbb{Q}[X,Y]$ has degree 56 in X and 10 in Y, which appears to be irreducible.

To prove Theorem 1.2—itself a reformulation of Theorem 3.12—we first organize the enumeration of primitive traceless matrices of given level by the matrices' elementary divisor types; cf. Section 2.2. Proposition 2.3 reduces this problem to the problems of enumerating the sets $B_{n,n-i}(\mathbb{F}_q)$ and enumerating sets of arbitrary (viz. not necessarily traceless) matrices of smaller dimension and level. A first formula for $\mathcal{P}_{n,\mathfrak{o}}(s)$ is obtained in Theorem 2.6 by combining the solutions of the former problem by Bender's formula (cf. Lemma 2.2) and the latter problem by [14, Proposition 3.4].

1.2. **Counting traceless matrices over finite fields via (signed) permutation statistics.** A key step in the proof of Theorem 1.2 is the formulation of the numbers $|B_{n,n-i}(\mathbb{F}_q)|$ in terms of statistics on signed permutation groups. In the following result,—which follows directly by combining Lemmas 2.2 and 3.1—we denote by B_n the hyperoctahedral (or signed permutation) group of degree n. The precise definitions of this group, its *quotients* $B_n^{\{i\}^c}$ as well as the statistics neg, ε_n, and ℓ on B_n are given in Section 3.1.

Proposition 1.4. *For $i \in [n-1]_0$,*

$$|B_{n,n-i}(\mathbb{F}_q)| = q^{n^2-i^2-1} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} q^{(\varepsilon_n - \ell)(w)}.$$

Proposition 1.4 complements similar results on the numbers of square matrices over finite fields of fixed ranks which are symmetric, antisymmetric, or satisfy no further restrictions on their entries; cf. Remark 3.2. It allows us to rewrite our "first formula" for the image zeta function $\mathcal{P}_{n,\mathfrak{o}}(s)$ given in Theorem 2.6 in terms of a generating polynomial on the Weyl group B_n, leading to a "second formula" for $\mathcal{P}_{n,\mathfrak{o}}(s)$ in Proposition 3.11. That this generating polynomial on B_n factorizes as a product of $\prod_{j=0}^{n-1}(1-q^{j-s})$ and a generating polynomial on S_n is a consequence of the general Proposition 3.4, establishing a factorization of a generating polynomial controlling the joint distribution of four statistics on B_n. Given this factorization, Theorem 3.12 and hence Theorem 1.2 follow swiftly.

1.3. **Applications to representation zeta functions of nilpotent groups.** The Poincaré series $\mathcal{P}_{n,\mathfrak{o}}(s)$ have an interpretation as local representation zeta functions of a unipotent group scheme. Let, more precisely, K_n denote the unipotent group scheme over $\mathbb{Q}$ associated to the $\mathbb{Z}$-Lie lattice

$$\mathcal{K}_n = \langle x_1, \dots, x_{2n}, y_{ij}, 1 \leq i,j \leq n \mid [x_i, x_{n+j}] - y_{ij}, \mathrm{Tr}(\mathbf{y}) \rangle_{\mathbb{Z}}; \tag{1.1}$$

see [14, Section 2.1.2] and compare with the unipotent group schemes $F_{n,\delta}$, G_n, and H_n defined analogously in [14, Section 1.3]. We refer to the finitely generated, class-2-nilpotent groups of the form $K_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field, as *groups of type* K. The *representation zeta function* $\zeta_{K_n(\mathcal{O})}(s)$ enumerates the irreducible finite-dimensional complex representations of $K_n(\mathcal{O})$ up to twists by one-dimensional representations; cf. [14, Section 1.1]. By [14, Proposition 2.2], it satisfies an Euler product of the form

$$\zeta_{K_n(\mathcal{O})}(s) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})} \zeta_{K_n(\mathcal{O}_\mathfrak{p})}(s), \tag{1.2}$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of $\mathcal{O}$. By design of the relations in (1.1) and [14, Proposition 2.18],

$$\zeta_{K_n(\mathcal{O}_\mathfrak{p})}(s) = \mathcal{P}_{n,\mathcal{O}_\mathfrak{p}}(s)$$

for all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$. The formula in Theorem 1.2 thus yields an explicit expression for the local representation zeta functions of groups of type K, in analogy to [14, Theorems B and C]. We note, however, that, in contrast to their siblings of type F, G, and H, groups of type K do not, in general, have representation zeta functions that factorize completely as finite products of translates of Dedekind zeta functions and their inverses; cf. Example 1.3. In Section 4 we deduce some fundamental properties of the Euler product (1.2), such as its abscissa of convergence and some meromorphic continuation. In Section 4.2 we study the topological representation functions associated to the group schemes K_n.

Remark 1.5. The $\mathbb{Z}$-Lie lattice $\mathcal{K}_2$ is a $\mathbb{Z}$-form of the 7-dimensional complex Lie algebra called $3, 7_D$ on [12, p. 483]. Both the generic local representation zeta functions of the groups of type K_2 (cf. Example 1.3 (2)) and their topological counterparts (cf. Proposition 4.3) have first been computed by means of the computer-algebra package Zeta (cf. [9]). We thank T. Rossmann for pointing this out to us.

1.4. **Related results.** Counting matrices over finite fields of given rank satisfying some (linear) restrictions on their entries is a classical theme that continues to pique researchers' interest to this day. The paper [7], for instance, studies matrices over finite fields of fixed rank with specific entries equal to zero. One particularly rich context for the study of such questions is work in the wake of a speculation of Kontsevich, disproved by Belkale and Brosnan in [2], about the polynomiality in q of the numbers of $\mathbb{F}_q$-rational points of certain such rank-varieties arising from finite graphs.

As explained above in the special case of traceless matrices, every rank-distribution problem for matrices satisfying ($\mathbb{Q}$-)linear restrictions over finite fields has analogues over finite quotients $\mathfrak{o}_N = \mathfrak{o}/\mathfrak{p}^N$ of compact discrete valuation rings $\mathfrak{o}$: one may enumerate (primitive) such matrices over these finite rings by the sizes of their images, or—more finely—by their elementary divisor types. By general, deep results, the ordinary generating functions—or Poincaré series—encoding these numbers, for $N \in \mathbb{N}_0$, as coefficients of powers of a variable $t = q^{-s}$, say, are rational functions in t, whose denominators are finite products of terms of the form $1 - q^b t^a$ for some $a \in \mathbb{N}$, $b \in \mathbb{N}_0$. It is, however, in general a hard problem to compute these rational functions or only just to describe how they depend on the local rings $\mathfrak{o}$. The work [2] suggests that rationality in q^{-s} <u>and</u> q as exhibited in Theorem 1.2 for traceless matrices (and in [14, Theorem B] for antisymmetric, symmetric and general square matrices) should be the exception rather than the rule. The functional equations

$$\mathcal{P}_{n,\mathfrak{o}}(s)\big|_{q\to q^{-1}} = q^{n^2-1}\mathcal{P}_{n,\mathfrak{o}}(s),$$

however, are general features of Poincaré series enumerating matrices of linear forms by elementary divisor types; cf. [17, Proposition 2.2].

A common feature observed for the four families of matrices discussed above (and possibly others, cf. [15]) is that the relevant rational Poincaré series have interpretations in terms of statistics on Weyl groups of type A and B; cf. Remark 3.2.

1.5. **Notation.** We write $\mathbb{N} = \{1, 2, \dots\}$ for the set of natural numbers. For a subset $X \subseteq \mathbb{N}$ we write $X_0 = X \cup \{0\}$. Given $m, n \in \mathbb{Z}$, we set $[n] = \{1, \dots, n\}$ and $[m, n] = \{m, \dots, n\}$. The notation $I = \{i_1, \dots, i_\ell\}_<$ for a finite subset $I \subseteq \mathbb{N}_0$ indicates that $i_1 < \dots < i_\ell$. For a property P, we write δ_P for the "Kronecker delta" of P, which is 1 if P holds and 0 otherwise. We write $\mathrm{Mat}_n(R)$ for the ring of $n \times n$-matrices over a ring R. The notation $\mathrm{Mat}_{n,m}(\mathbb{F}_q)$, however, is reserved for the set of $n \times n$-matrices over the finite field $\mathbb{F}_q$ of rank m. By $R^\times$ we denote the group of units of a (commutative) ring R. Given $i \in \mathbb{N}$ and I as above, we write $\binom{n}{i}_X \in \mathbb{Z}[X]$ for the X-binomial coefficient and $\binom{n}{I}_X = \binom{n}{i_\ell}_X \binom{i_\ell}{i_{\ell-1}}_X \cdots \binom{i_2}{i_1}_X$ for the X-multinomial coefficient. We use the notation $\mathrm{gp}(X)$ for the geometric progression $\sum_{r=1}^{\infty} X^r = \frac{X}{1-X} \in \mathbb{Q}(X)$ and write $\mathrm{Tr}(\boldsymbol{x})$ for the trace of a square matrix $\boldsymbol{x}$.

2. Counting traceless matrices

2.1. **Counting traceless matrices over finite fields by rank.** Formulae for the cardinalities $|B_{n,n-i}(\mathbb{F}_q)| = |\{\boldsymbol{x} \in \mathfrak{sl}_n(\mathbb{F}_q) \mid \mathrm{rk}(\boldsymbol{x}) = n-i\}|$, for $i \in [n-1]_0$, are given, for example, in [3]. We recall them here, suitably rephrased. Recall the definition

$$f_{n,i}(X) := f_{G_n,\{i\}}(X) = \binom{n}{i}_X \prod_{j=i+1}^{n} (1 - X^j) \in \mathbb{Z}[X] \tag{2.1}$$

from [14, Theorem C]. [14, Lemma 3.1 (3.3)] asserts that

$$q^{n^2-i^2} f_{n,i}(q^{-1})$$

is the number $|\mathrm{Mat}_{n,n-i}(\mathbb{F}_q)|$ of $n \times n$-matrices over $\mathbb{F}_q$ of rank $n-i$. Set

(2.2)

$$\begin{aligned} b_{n,i}(X) &= f_{n,i}(X) + (-1)^{n-i} \binom{n}{i}_X (1-X) X^{\binom{n+1}{2} - \binom{i+1}{2} - 1} \\ &= \binom{n}{i}_X \left(\left(\prod_{j=i+1}^{n} (1 - X^j) \right) + (-1)^{n-i} (1-X) X^{\binom{n+1}{2} - \binom{i+1}{2} - 1} \right) \in \mathbb{Z}[X]. \end{aligned}$$

Remark 2.1. Informally, we obtain $b_{n,i}(X)$ from $f_{n,i}(X)$ by lowering by one the exponent in the leading term in the factor $\prod_{j=i+1}^{n}(1 - X^j)$ of $f_{n,i}(X)$.

Lemma 2.2 ([3]). *For $i \in [n-1]_0$,*

$$|B_{n,n-i}(\mathbb{F}_q)| = q^{n^2-i^2-1} b_{n,i}(q^{-1}).$$

Proof. In eq. (1) of his paper [3], Bender states that

$$|B_{n,n-i}(\mathbb{F}_q)| = q^{-1} q^{\binom{n-i}{2}} \left(\prod_{j=1}^{n-i} \frac{(q^{n-j+1} - 1)^2}{q^j - 1} \right) + (1 - q^{-1}) \prod_{j=1}^{n-i} \frac{q^{j-1} - q^n}{q^j - 1}. \tag{2.3}$$

Using the identities

$$\binom{n}{n-i}_X = \binom{n}{i}_X = \prod_{j=1}^{n-i} \frac{X^{i+j} - 1}{X^j - 1} = \prod_{j=1}^{n-i} \frac{X^{n-j+1} - 1}{X^j - 1} \in \mathbb{Z}[X]$$

it is a triviality to see that the first summand on the right-hand side of (2.3) is equal to $q^{n^2-i^2-1} f_{n,i}(q^{-1})$, whereas the second summand is equal to

$$q^{n^2-i^2-1} (-1)^{n-i} \binom{n}{i}_{q^{-1}} (1 - q^{-1}) q^{-\binom{n+1}{2} + \binom{i+1}{2} + 1}. \tag{2.4}$$

□

We note that (2.4) gives the "error term" $|B_{n,n-i}(\mathbb{F}_q)| - |\mathrm{Mat}_{n,n-i}(\mathbb{F}_q)|/q$.

2.2. **Counting traceless matrices over quotients of compact discrete valuation rings by elementary divisor type.** Our aim is to generalize the enumeration of traceless matrices over finite fields by their ranks to traceless matrices over larger quotients of compact discrete valutation rings by their image sizes. The latter, in turn, are controlled by the matrices' elementary divisor types. More precisely, given $I = \{i_1, \dots, i_\ell\}_< \subseteq [n-1]_0$ and $\mathbf{r}_I \in \mathbb{N}^I$, we set $\mu_j = i_{j+1} - i_j$ with $i_{\ell+1} = n$ and $i_0 = 0$, and $N = \sum_{\iota \in I} r_\iota$. Recall, e.g. from [14, §3], that a (primitive) $n \times n$-matrix $\boldsymbol{x}$ over $\mathfrak{o}_N = \mathfrak{o}/\mathfrak{p}^N$ is said to be of elementary divisor type $(I, \mathbf{r}_I)$ if

$$\nu(\boldsymbol{x}) = (\underbrace{0, \dots, 0}_{\mu_\ell}, \underbrace{r_{i_\ell}, \dots, r_{i_\ell}}_{\mu_{\ell-1}}, \underbrace{r_{i_\ell} + r_{i_{\ell-1}}, \dots, r_{i_\ell} + r_{i_{\ell-1}}}_{\mu_{\ell-2}}, \dots, \underbrace{N, \dots, N}_{\mu_0}),$$

where $\nu(\boldsymbol{x})$ is the n-tuple of valuations of the elementary divisors of $\boldsymbol{x}$, in nondescending order. In analogy with the notation (introduced in [14, Section 3])

$$\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(G_n)$$

for the set of $n \times n$-matrices over $\mathfrak{o}_N$ of elementary divisor type $(I, \mathbf{r}_I)$, we write

$$\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)$$

for the set of traceless $n \times n$-matrices over $\mathfrak{o}_N$ of elementary divisor type $(I, \mathbf{r}_I)$.

Let $t = q^{-s}$. Noting that a matrix $\boldsymbol{x} \in \mathfrak{sl}_n(\mathfrak{o}_N)$ of elementary divisor type $(I, \mathbf{r}_I)$ has image size $|\operatorname{im}(\boldsymbol{x})| = q^{\sum_{i \in I} r_i(n-i)}$, equation [14, (3.1)] gives

$$\mathcal{P}_{n,\mathfrak{o}}(s) = \sum_{I \subseteq [n-1]_0} \sum_{\mathbf{r}_I \in \mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| t^{\sum_{i \in I} r_i(n-i)}, \tag{2.5}$$

reducing the problem of computing $\mathcal{P}_{n,\mathfrak{o}}(s)$ to the one of effectively describing the numbers $|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)|$ for varying I and $\mathbf{r}_I$. The following result reduces the latter problem further to the problems of counting traceless matrices over the residue field $\mathbb{F}_q$ and counting smaller, but not necessarily traceless matrices over a proper quotient of $\mathfrak{o}_N$.

Proposition 2.3. *Given $\varnothing \neq I \subseteq [n-1]_0$ and $\mathbf{r}_I \in \mathbb{N}^I$, with $i_\ell = \max I$,*

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| = |B_{n,n-i_\ell}(\mathbb{F}_q)| q^{(N-1)(n^2 - i_\ell^2 - 1)} \cdot |\mathrm{N}^{\mathfrak{o}}_{I \setminus \{i_\ell\}, \mathbf{r}_{I \setminus \{i_\ell\}}}(G_{i_\ell})|.$$

Let us fix some notation for the proof of Proposition 2.3. Given $1 \leq k, m \leq n$ and $N \in \mathbb{N}$ we define the injection

$$\imath_k : \mathbb{A}_{n-1}(\mathfrak{o}_N) \to \mathbb{A}_n(\mathfrak{o}_N), \quad (t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{k-1}, 1, t_k, \dots, t_{n-1})$$

and let

$$\jmath_{k,m} : \mathrm{Mat}_{n-1}(\mathfrak{o}_N) \to \mathrm{Mat}_n(\mathfrak{o}_N), \quad \boldsymbol{x} \mapsto \boldsymbol{x}'$$

be the map sending an $(n-1) \times (n-1)$-matrix $\boldsymbol{x}$ to the $n \times n$-matrix $\boldsymbol{x}'$ whose k-th row and m-th column are zero, and such that the submatrix obtained by deleting the k-th row and m-th column is $\boldsymbol{x}$.

The proof of Proposition 2.3 also requires the following lemma.

Lemma 2.4. *Let $\mathbf{a} = (a_{i,j}) \in \mathfrak{sl}_n(\mathbb{F}_q)$. Assume that $\mathbf{a}$ is noncentral. There exist $1 \leq k < n$ and $u \in \mathrm{SL}_n(\mathbb{F}_q)$ such that the (k,n)-entry of $u\mathbf{a}u^{-1}$ is nonzero.*

Proof. The assumption that $\mathbf{a}$ is not central implies that one of the following holds:

(1) $\mathbf{a}$ has a nonzero entry $a_{k,r}$ with $r \neq k$ or
(2) $\mathbf{a}$ is a diagonal nonscalar matrix.

In case (1), we may take $u = -\sigma_{r,n}$ where $\sigma_{r,n}$ is the permutation matrix interchanging the r-th and n-th elements of the standard basis and fixing all others.

In case (2), up to conjugation by an element of $\mathrm{SL}_n(\mathbb{F}_q)$, we may assume $a_{n,n} \neq 0$, and let $1 \leq k < n$ be such that $a_{k,k} \neq a_{n,n}$ is also nonzero. In this case, we may take u to be the elementary matrix with 1 in position (k, n). □

Proof of Proposition 2.3. We denote by

$$\varphi : \mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n) \to B_{n,n-i_\ell}(\mathbb{F}_q)$$

the reduction modulo $\mathfrak{p}$. We will prove that, for any $\mathbf{a} = (a_{i,j}) \in B_{n,n-i_\ell}(\mathbb{F}_q)$, the fibre of φ over $\mathbf{a}$ has size

$$\left|\varphi^{-1}(\mathbf{a})\right| = q^{(N-1)(n^2-i_\ell^2-1)} \left|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})\right|.$$

This clearly suffices to prove the proposition. Given $1 \leq i, j \leq n$, we define a map

$$\begin{aligned}\Phi_{i,j} : \mathfrak{o}_N^\times \times \mathbb{A}_{n-1}(\mathfrak{o}_N) \times \mathbb{A}_{n-1}(\mathfrak{o}_N) \times \mathrm{Mat}_{n-1}(\mathfrak{o}_N) &\to \mathrm{Mat}_n(\mathfrak{o}_N)\\ (x, \mathbf{c}, \mathbf{r}, \mathbf{y}) &\mapsto x\left(\imath_i(\mathbf{c})^{\mathrm{t}}\imath_j(\mathbf{r}) + \jmath_{i,j}(\mathbf{y})\right).\end{aligned}$$

Note that $\Phi_{i,j}$ is a bijection onto the set of matrices $\boldsymbol{x} \in \mathrm{Mat}_n(\mathfrak{o}_N)$ with invertible (i, j)-entry. In particular, $\mathrm{im}(\Phi_{i_0,j_0}) \supseteq \varphi^{-1}(\mathbf{a})$ for any $1 \leq i_0, j_0 \leq n$ such that $a_{i_0,j_0} \neq 0$.

Given $a_{i_0,j_0} \neq 0$, the requirement $\Phi_{i_0,j_0}(x, \mathbf{c}, \mathbf{r}, \mathbf{y}) \equiv \mathbf{a} \pmod{\mathfrak{p}}$ is equivalent to the conjunction of the following conditions:

(i) $x \equiv a_{i_0,j_0} \pmod{\mathfrak{p}}$,
(ii) $\imath_{i_0}(\mathbf{c})^{\mathrm{t}}$ is congruent modulo $\mathfrak{p}$ to the j_0-th column of the matrix $a_{i_0,j_0}^{-1} \cdot \mathbf{a}$,
(iii) $\imath_{j_0}(\mathbf{r})$ is congruent modulo $\mathfrak{p}$ to the i_0-th row of the matrix $a_{i_0,j_0}^{-1} \cdot \mathbf{a}$, and
(iv) the matrix of $\jmath_{i_0,j_0}(\mathbf{y})$ is congruent modulo $\mathfrak{p}$ to the matrix $\mathbf{b} = (b_{i,j})$, where $b_{i,j} = \left(\frac{a_{i,j}}{a_{i_0,j_0}}\right) - \left(\frac{a_{i,j_0}}{a_{i_0,j_0}}\right)\left(\frac{a_{i_0,j}}{a_{i_0,j_0}}\right)$ for all $1 \leq i, j \leq n$.

In particular, since the elementary divisor type of a matrix is invariant under elementary row and column operations, for any choice of $x \in \mathfrak{o}_N^\times$ and $\mathbf{c} = (c_1, \ldots, c_{n-1}), \mathbf{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{A}_{n-1}(\mathfrak{o}_N)$ satisfying conditions (i),(ii), and (iii) above, there exists a matrix $\mathbf{y} \in \mathrm{Mat}_{n-1}(\mathfrak{o}_N)$, satisfying condition (iv) such that $\Phi_{i_0,j_0}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is of elementary divisor type $(I, \mathbf{r}_I)$. We now compute the number of such tuples which also satisfy the condition of being traceless. We proceed by a case distinction depending on the diagonal entries of $\mathbf{a}$.

Case 1: $\mathbf{a}$ is noncentral and has a nonzero diagonal entry. In this case, since $\left|\varphi^{-1}(\mathbf{a})\right|$ is invariant under conjugating $\mathbf{a}$ by an element of $\mathrm{SL}_n(\mathbb{F}_q)$ we may, for simplicity, assume that $i_0 = j_0 = n$. By the same token and Lemma 2.4, we may assume that there exists $1 \leq k_0 \leq n-1$ such that $a_{k_0,n} \neq 0$ as well. In particular, it follows that the k_0-th entry c_{k_0} of any element $\mathbf{c} \in \mathbb{A}_{n-1}(\mathfrak{o}_N)$ satisfying condition (ii) is invertible in $\mathfrak{o}_N$.

A direct computation yields

$$\mathrm{Tr}(\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})) = x\left(\imath_n(\mathbf{r})\imath_n(\mathbf{c})^{\mathrm{tr}} + \mathrm{Tr}(\jmath_{n,n}(\mathbf{y}))\right) = x\left(\sum_{k=1}^{n-1} c_k r_k + 1 + \mathrm{Tr}(\mathbf{y})\right).$$

As $x \in \mathfrak{o}_N^\times$, the requirement $\mathrm{Tr}(\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})) = 0$ is equivalent to

$$\sum_{k=1}^{n-1} c_k r_k = -1 - \mathrm{Tr}(\mathbf{y}).$$

Recalling that $c_{k_0} \in \mathfrak{o}_N^\times$, one easily verifies that any choice of $\mathbf{y} \in \mathrm{Mat}_{n-1}(\mathfrak{o}_N)$ which satisfies condition (iv) admits exactly $q^{(N-1)(2n-2)}$ triples $(x, \mathbf{c}, \mathbf{r}) \in \mathfrak{o}_N^\times \times \mathbb{A}_{n-1}(\mathfrak{o}_N) \times \mathbb{A}_{n-1}(\mathfrak{o}_N)$ such that $\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is traceless and reduces modulo $\mathfrak{p}$ to $\mathbf{a}$. Furthermore, the elementary divisor type of $\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is determined by that of $\mathbf{y}$ as follows.

(1) Suppose that $\mathbf{a} \in \mathrm{Mat}_{n,1}(\mathbb{F}_q)$, i.e. $\mathrm{rk}(\mathbf{a}) = 1$ and $i_\ell = \max I = n-1$. Let $\varpi \in \mathfrak{p}$ be a uniformizer. The matrix $\mathbf{b}$ in condition (iv) is zero, in which case $\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is of elementary divisor type $(I, \mathbf{r}_I)$ if and only if $\mathbf{y} \in \mathfrak{p}^{r_{i_\ell}} \mathrm{Mat}_{n-1}(\mathfrak{o}_N)$ is such that $\varpi^{-r_{i_\ell}}\mathbf{y}$ is of elementary divisor type $(I \setminus \{i_\ell\}, \mathbf{r}_{I\setminus\{i_\ell\}})$. Thus

$$\left|\varphi^{-1}(\mathbf{a})\right| = q^{(N-1)(2n-2)} \left|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{n-1})\right| = q^{(N-1)(n^2-i_\ell^2-1)} \left|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})\right|.$$

(2) Otherwise, if $i_\ell = \max I < n-1$, the matrix $\mathbf{b}$ is nonzero, and $\Phi_{n,n}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is of elementary divisor type $(I, \mathbf{r}_I)$ if and only if $\mathbf{y}$ is an $(n-1)\times(n-1)$-matrix of elementary divisor type $(I, \mathbf{r}_I)$. Arguing as in [11, Proposition 3.4], the number of such lifts $\mathbf{y}$ of $\mathbf{b}$ is $q^{(N-1)((n-1)^2-i_\ell^2)}|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})|$. Combined with the $q^{(N-1)(2n-2)}$ possibilities to choose $(x, \mathbf{c}, \mathbf{r})$ for any such $\mathbf{y}$, we obtain

$$\left|\varphi^{-1}(\mathbf{a})\right| = q^{(N-1)(n^2-i_\ell^2-1)} \left|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})\right|,$$

as wanted.

Case 2: all diagonal entries of $\mathbf{a}$ *are zero.* In this case $i_0 \neq j_0$ and

$$\mathrm{Tr}(\Phi_{i_0,j_0}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})) = x(\imath_{j_0}(\mathbf{r})\imath_{i_0}(\mathbf{c})^{\mathrm{tr}} + \mathrm{Tr}(\jmath_{i_0,j_0}(\mathbf{y}))) = x(c_{j_0} + r_{i_0} + \theta_{i_0,j_0}(\mathbf{r}, \mathbf{c}) + \tau_{i_0,j_0}(\mathbf{y}))$$

where $\theta_{i_0,j_0}(\mathbf{r}, \mathbf{c})$ is some quadratic function in $\mathbf{r} = (r_1, \dots, r_{n-1}), \mathbf{c} = (c_1, \dots, c_{n-1})$ which does not involve r_{i_0} or c_{j_0}, and $\tau_{i_0,j_0}(\mathbf{y})$ is some linear function in the entries of $\mathbf{y}$. In particular, the condition $\mathrm{Tr}(\Phi_{i_0,j_0}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})) = 0$ is equivalent to

$$c_{j_0} = -r_{i_0} - \theta_{i_0,j_0}(\mathbf{r}, \mathbf{c}) - \tau_{i_0,j_0}(\mathbf{y}).$$

Consequently, any choice of $\mathbf{y}$ which satisfies condition (iv) admits exactly $q^{(N-1)(2n-2)}$ triples $(x, \mathbf{c}, \mathbf{r}) \in \mathfrak{o}_N^\times \times \mathbb{A}_{n-1}(\mathfrak{o}_N) \times \mathbb{A}_{n-1}(\mathfrak{o}_N)$ such that $\Phi_{i_0,j_0}(x, \mathbf{c}, \mathbf{r}, \mathbf{y})$ is traceless and reduces to $\mathbf{a}$ modulo $\mathfrak{p}$. Considering the same two possibilities for $\mathbf{a}$ as in the previous case and applying the same arguments, we obtain

$$\left|\varphi^{-1}(\mathbf{a})\right| = q^{(N-1)(n^2-i_\ell^2-1)} \left|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})\right|$$

in this case as well.

Case 3: $\mathbf{a}$ *is central.* Note that this case is only possible if p divides n and $I = \{0\}$. Thus, our goal in this case is to prove

$$\left|\varphi^{-1}(\mathbf{a})\right| = q^{(N-1)(n^2-1)}.$$

Since the lift of an element of $\mathrm{Mat}_{n,n}(\mathbb{F}_q)$ (viz. an invertible matrix over $\mathbb{F}_q$) to $\mathrm{Mat}_n(\mathfrak{o}_N)$ is always an element of elementary divisor type $(I, \mathbf{r}_I) = (\{0\}, (N))$ (viz. an invertible matrix over $\mathfrak{o}_N$), we simply need to note that such an element $\mathbf{a}$ admits $q^{(N-1)(n^2-1)}$ traceless lifts to $\mathrm{Mat}_n(\mathfrak{o}_N)$.

This concludes the proof of Proposition 2.3. □

Recall from [14, Theorem C] the definition, for $I \subseteq [n-1]_0$, of the polynomials

$$f_{n,I}(X) := f_{G_{n,I}}(X) = \binom{n}{I}_X \prod_{j=\min I+1}^{n} (1-X^j) \in \mathbb{Z}[X], \tag{2.6}$$

generalizing those defined in (2.1) for singletons $I = \{i\}$.

Proposition 2.5. *Given $\varnothing \neq I \subseteq [n-1]_0$ and $\mathbf{r}_I \in \mathbb{N}^I$, with $i_\ell = \max I$,*

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| = b_{n,i_\ell}(q^{-1}) f_{i_\ell, I\setminus\{i_\ell\}}(q^{-1}) q^{\sum_{\iota\in I} r_\iota(n^2-\iota^2-1)}.$$

Proof. Recall that $N = \sum_{\iota\in I} r_\iota$. Using [14, Proposition 3.4 (3.8)], which asserts that

$$|\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})| = f_{i_\ell,I\setminus\{i_\ell\}}(q^{-1}) q^{\sum_{\iota\in I\setminus\{i_\ell\}} r_\iota(i_\ell^2-\iota^2)},$$

and Lemma 2.2, Proposition 2.3 yields

$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| = |B_{n,n-i_\ell}(\mathbb{F}_q)| q^{(N-1)(n^2-i_\ell^2-1)} \cdot |\mathrm{N}^{\mathfrak{o}}_{I\setminus\{i_\ell\},\mathbf{r}_{I\setminus\{i_\ell\}}}(G_{i_\ell})| =$$

$$|B_{n,n-i_\ell}(\mathbb{F}_q)| q^{(r_{i_\ell}-1)(n^2-i_\ell^2-1)} q^{\left(\sum_{\iota\in I\setminus\{i_\ell\}} r_\iota\right)(n^2-i_\ell^2-1)} \cdot f_{i_\ell,I\setminus\{i_\ell\}}(q^{-1}) q^{\sum_{\iota\in I\setminus\{i_\ell\}} r_\iota(i_\ell^2-\iota^2)} =$$

$$b_{n,i_\ell}(q^{-1}) f_{i_\ell,I\setminus\{i_\ell\}}(q^{-1}) q^{\sum_{\iota\in I} r_\iota(n^2-\iota^2-1)}. \qquad \square$$

We apply Proposition 2.5 to obtain a first formula for the Poincaré series $\mathcal{P}_{n,\mathfrak{o}}(s)$. Recall that $\mathrm{gp}(X) = \frac{X}{1-X}$ and $t = q^{-s}$.

Theorem 2.6. *Setting $x_{n,i} = q^{n^2-i^2-1} t^{n-i}$ for $i \in [n-1]_0$, the image zeta function of $\mathfrak{sl}_n(\mathfrak{o})$ satisfies*

$$\mathcal{P}_{n,\mathfrak{o}}(s) = 1 + \sum_{i=0}^{n-1} b_{n,i}(q^{-1})\, \mathrm{gp}(x_{n,i}) \left(\sum_{J\subseteq[i-1]_0} f_{i,J}(q^{-1}) \prod_{j\in J} \mathrm{gp}(x_{n,j}) \right).$$

Proof. Rewriting (2.5) yields

$$\begin{aligned}\mathcal{P}_{n,\mathfrak{o}}(s) &= \sum_{I\subseteq[n-1]_0} \sum_{\mathbf{r}_I\in\mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| t^{\sum_{i\in I} r_i(n-i)} \\ &= 1 + \sum_{i_\ell=0}^{n-1} \sum_{\substack{I\subseteq[n-1]_0 \\ \max I = i_\ell}} \sum_{\mathbf{r}_I\in\mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| t^{\sum_{i\in I} r_i(n-i)}.\end{aligned}$$

For fixed $i_\ell \in [n-1]_0$ we find, using Proposition 2.5, that

$$\sum_{\substack{I\subseteq[n-1]_0 \\ \max I = i_\ell}} \sum_{\mathbf{r}_I\in\mathbb{N}^I} |\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| t^{\sum_{i\in I} r_i(n-i)} =$$

$$b_{n,i_\ell}(q^{-1})\, \mathrm{gp}(x_{n,i_\ell}) \sum_{J\subseteq[i_\ell-1]_0} f_{i_\ell,J}(q^{-1}) \prod_{j\in J} \mathrm{gp}(x_{n,j}). \qquad \square$$

3. Traceless matrices and signed permutation statistics

In this section we express the Poincaré series $\mathcal{P}_{n,\mathfrak{o}}(s)$ in terms of certain signed permutation statistics (Proposition 3.11) and Igusa functions (Theorem 3.12).

3.1. **Preliminaries on signed permutation groups.** We collect a few definitions and notation regarding (signed) permutation groups, mostly standard and covered in general references such as [4].

We write S_n for the symmetric group of degree n, viz. the group of permutations of the set $[n]$. It is a Coxeter group with Coxeter generating set $\{s_1, \dots, s_{n-1}\}$, consisting of the standard transpositions s_i interchanging letters i and $i+1$.

We denote further by B_n the group of signed permutations of degree n, viz. permutations w of $\{-n, -n+1, \dots, n-1, n\}$ satisfying $w(-i) = -w(i)$ for all $i \in [n]_0$. Signed permutations are uniquely determined by their restrictions to $[n]$. This is exploited in the one-line notation, representing $w \in B_n$ by its values $w(1)\, w(2) \, \dots \, w(n)$. When using the one-line notation we write $\bar{a}$ instead of $-a$ for $a \in \mathbb{Z}$.

The group B_n is a Coxeter group with Coxeter generating set $S^{B_n} = \{s_0, s_1, \dots s_{n-1}\}$, where $s_0 = \bar{1}23\dots n$ and, for $i \in [n-1]$, $s_i = 12\dots(i+1)i\dots n$ are the standard transpositions. Let ℓ denote the Coxeter length with respect to S^{B_n}.

We identify S_n with the (parabolic) subgroup of B_n consisting of elements w satisfying $w(i) > 0$ for all $i \in [n]$, generated by the transpositions s_i for $i \in [n-1]$. The restriction of the Coxeter length ℓ on B_n to S_n coincides with the Coxeter length on S_n, so the use of the terminology 'ℓ' is unambigious.

Let $w \in B_n$. The *negative set* of w is

$$\operatorname{Neg}(w) = \{i \in [n] \mid w(i) < 0\}$$

and the *descent set* of w is

$$\operatorname{Des}(w) = \{i \in [n-1]_0 \mid \ell(ws_i) < \ell(w)\}.$$

Recall also the following statistics on B_n:

$$\begin{aligned}
\operatorname{rmaj}(w) &= \sum_{i \in \operatorname{Des}(w)} (n-i) && \text{(reverse major index)}\\
\operatorname{des}(w) &= |\operatorname{Des}(w)| && \text{(descent number)}\\
\operatorname{inv}(w) &= |\{(i,j) \in [n]^2 \mid i<j,\, w(i) > w(j)\}| && \text{(inversion number)}\\
\operatorname{neg}(w) &= |\operatorname{Neg}(w)| && \text{(negative number)}\\
\operatorname{nsp}(w) &= |\{(i,j) \in [n]^2 \mid i<j,\, w(i)+w(j) < 0\}| && \text{(negative sum pair number)}
\end{aligned}$$

It is well-known (cf. [4, Proposition 8.1.1]) that the Coxeter length ℓ on B_n satisfies

$$\ell = \operatorname{inv} + \operatorname{neg} + \operatorname{nsp}. \tag{3.1}$$

We further set

$$\begin{aligned}
\sigma_B(w) &= \sum_{i \in \operatorname{Des}(w)} (n^2 - i^2) && \text{for } w \in B_n \text{ and}\\
\sigma_A(w) &= \sum_{i \in \operatorname{Des}(w)} i(n-i) && \text{for } w \in S_n.
\end{aligned}$$

The statistics σ_A on S_n and σ_B on B_n have a uniform definition in terms of simple root coefficients of the sum over all (positive) roots in the pertinent root systems; cf. [14, p. 510]. The statistics $\sigma_A - \ell$ and $\sigma_B - \ell$ are examples of Weyl group statistics that "ought to be better known", as the authors of [16] argue. In any case, they both feature in one of this section's main results, viz. Proposition 3.4.

In [1, Section 1.2.1], the statistic ε_n on B_n is defined by setting, for $w \in B_n$,

$$\varepsilon_n(w) = \begin{cases} 1 & \text{if } w(n) < 0, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } w(i) < 0 \text{ for all } i \in [\max \mathrm{Des}(w)+1, n], \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

For $I \subseteq [n-1]_0$, the corresponding *quotient* of B_n is defined as

$$B_n^{I^c} = \{w \in B_n \mid \mathrm{Des}(w) \subseteq I\}.$$

Quotients of S_n are defined analogously.

3.2. **Signed permutation statistics and enumeration of matrices.** It is known by [14, Proposition 4.6]—essentially a result of Reiner's; cf. [8]—that the polynomials $f_{n,I}(X)$ defined in (2.6) satisfy

$$f_{n,I}(X) = \sum_{w \in B_n^{I^c}} (-1)^{\mathrm{neg}(w)} X^{\ell(w)}. \tag{3.3}$$

In particular, by [14, Lemma 3.1 (3.3)], for $i \in [n-1]_0$, the number $|\operatorname{Mat}_{n,n-i}(\mathbb{F}_q)|$ of $n \times n$-matrices over $\mathbb{F}_q$ of rank $n-i$ satisfies

$$|\operatorname{Mat}_{n,n-i}(\mathbb{F}_q)| = q^{n^2-i^2} \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} q^{-\ell(w)}.$$

Key to a similar interpretation of the numbers $|B_{n,n-i}(\mathbb{F}_q)|$ in terms of signed permutation statistics in Proposition 1.4 is the following lemma.

Lemma 3.1. *For $i \in [n-1]_0$, the polynomial $b_{n,i}(X)$ defined in* (2.2) *satisfies*

$$b_{n,i}(X) = \sum_{w \in B_n^{\{i\}^c}} (-1)^{\mathrm{neg}(w)} X^{(\ell - \varepsilon_n)(w)}.$$

Proof. Thanks to Remark 2.1 it suffices to prove that

$$\sum_{\{w \in B_n^{\{i\}^c} \mid w(n) < 0\}} (-1)^{\mathrm{neg}(w)} X^{\ell(w)} = \binom{n}{i}_X (-1)^{n-i} X^{\binom{n+1}{2} - \binom{i+1}{2}}. \tag{3.4}$$

To prove (3.4) we first note that, for $w \in B_n^{\{i\}^c}$, the condition $w(n) < 0$ is equivalent to $w(j) < 0$ for $j = [i+1, n]$; cf. (3.2). Clearly $\mathrm{neg}(w) = n - i$ in this case. The set $\{w \in B_n^{\{i\}^c} \mid w(n) < 0\}$ is in bijection with $S_n^{\{n-i\}^c}$ through the map $w \mapsto \widetilde{w} := \overline{w(n)} \dots \overline{w(i+1)} w(1) \dots w(i)$. In light of the well-known identity

$$\sum_{\widetilde{w} \in S_n^{\{n-i\}^c}} X^{\ell(\widetilde{w})} = \binom{n}{n-i}_X = \binom{n}{i}_X$$

it suffices to prove that

$$\ell(w) - \ell(\widetilde{w}) = \binom{n+1}{2} - \binom{i+1}{2} \text{ for all } w \in B_n^{\{i\}^c} \text{ with } w(n) < 0. \tag{3.5}$$

By (3.1), the quantity $\ell(w)$ is the sum of the quantities

$$\mathrm{inv}(w) = i(n-i), \qquad \mathrm{neg}(w) = n - i, \qquad \mathrm{nsp}(w) = \binom{n-i}{2} + r(w),$$

where $r(w) := |\{(\sigma, \tau) \in [i] \times [i+1, n] \mid w(\sigma) + w(\tau) < 0\}|$. Clearly $\ell(\widetilde{w}) = \mathrm{inv}(\widetilde{w}) = r(w)$. Eq. (3.5) follows, as $i(n-i) + (n-i) + \binom{n-i}{2} + r(w) - r(w) = \binom{n+1}{2} - \binom{i+1}{2}$. □

Remark 3.2. Formulae similar to the one given in Proposition 1.4 hold for the numbers of antisymmetric resp. symmetric matrices over finite fields of fixed ranks. Indeed, let $\delta \in \{0,1\}$ and $i \in [n-1]_0$.

(1) In the notation of [14, Section 3], the number $|\operatorname{Alt}_{2n+\delta,2(n-i)}(\mathbb{F}_q)|$ of antisymmetric $(2n+\delta)\times(2n+\delta)$-matrices over $\mathbb{F}_q$ of rank $2(n-i)$ satisfies
$$|\operatorname{Alt}_{2n+\delta,2(n-i)}(\mathbb{F}_q)| = q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}} \sum_{w\in B_n^{\{i\}^c}} (-1)^{\operatorname{neg}(w)} q^{-(2\ell+(2\delta-1)\operatorname{neg})(w)};$$
cf. [14, Lemma 3.1 (3.2) and Proposition 4.6].

(2) Likewise, the numbers $|\operatorname{Sym}_{n,n-i}(\mathbb{F}_q)|$ of symmetric $n\times n$-matrices over $\mathbb{F}_q$ of rank $n-i$ are given in terms of the *odd length* statistic L on B_n defined, for $w\in B_n$, by
$$L(w) = \frac{1}{2}|\{(i,j)\in[-n,n]^2 \mid i<j,\ w(i)>w(j),\ i\not\equiv j \pmod 2\}|, \tag{3.6}$$
(cf. [14, Eq. (1.14)]): combining [14, Lemma 3.1 (3.4)] with [5, Theorem 5.4] yields
$$|\operatorname{Sym}_{n,n-i}(\mathbb{F}_q)| = q^{\binom{n+1}{2}-\binom{i+1}{2}} \sum_{w\in B_n^{\{i\}^c}} (-1)^{\ell(w)} q^{-L(w)}.$$

For $I\subseteq[n-1]_0$, let
$$f_{K_n,I}(X) = \sum_{w\in B_n^{I^c}} (-1)^{\operatorname{neg}(w)} X^{(\ell-\varepsilon_n)(w)}. \tag{3.7}$$

It is easy to see that
$$f_{K_n,I}(X) = b_{n,\max I}(X) f_{\max I, I\setminus\{\max I\}}(X). \tag{3.8}$$

The following is analogous to [14, Proposition 3.4].

Proposition 3.3. *Let $I\subseteq[n-1]_0$ and $\mathbf{r}_I\in\mathbb{N}^I$. Then*
$$|\mathrm{N}^{\mathfrak{o}}_{I,\mathbf{r}_I}(K_n)| = f_{K_n,I}(q^{-1}) q^{\sum_{i\in I} r_i(n^2-i^2-1)}.$$

Proof. This follows by combining Proposition 2.5 and (3.8). □

3.3. **A joint distribution result for signed permutation statistics.** In our further analysis of the Poincaré series $\mathcal{P}_{n,\mathfrak{o}}(s)$ we will make use of the following formal identity.

Proposition 3.4. *In $\mathbb{Q}[W,X,Y,Z]$, the following identity holds:*
$$\sum_{w\in B_n} W^{(\operatorname{des}-\varepsilon_n)(w)} X^{(\sigma_B-\ell)(w)} Y^{\operatorname{neg}(w)} Z^{\operatorname{rmaj}(w)} = \left(\sum_{w\in S_n} W^{\operatorname{des}(w)} X^{(\sigma_A-\ell)(w)} (X^n Z)^{\operatorname{rmaj}(w)}\right) \prod_{j=0}^{n-1}(1+X^jYZ). \tag{3.9}$$

To prove Proposition 3.4 we decompose B_n as the disjoint union of subsets of signed permutations of fixed *inverse negative set*, i.e. the negative set of the inverse element. Note that, for $w\in B_n$, the set $\operatorname{Neg}(w^{-1})$ simply encodes, with inverted sign, the negative entries in the one-line notation for w. For $w = 1\bar{3}\bar{4}2 \in B_4$, for example, $\operatorname{Neg}(w^{-1}) = \operatorname{Neg}(14\bar{2}\bar{3}) = \{3,4\}$. Trivially,
$$B_n = \bigcup_{J\subseteq[n]} \{w\in B_n \mid \operatorname{Neg}(w^{-1}) = J\}.$$

For the proof of Proposition 3.4 we shall need the following lemma.

Lemma 3.5. *For* $J \subseteq [n]$ *and* $\max J < j \leq n$,

$$\sum_{\{w \in B_n \mid \mathrm{Neg}(w^{-1}) = J \cup \{j\}\}} W^{(\mathrm{des} - \varepsilon_n)(w)} X^{(\sigma_B - \ell)(w)} Y^{\mathrm{neg}(w)} Z^{\mathrm{rmaj}(w)} = X^{n-j} Y Z \sum_{\{w \in B_n \mid \mathrm{Neg}(w^{-1}) = J\}} W^{(\mathrm{des} - \varepsilon_n)(w)} X^{(\sigma_B - \ell)(w)} Y^{\mathrm{neg}(w)} Z^{\mathrm{rmaj}(w)}.$$

Proof. Let J and j be as in the Lemma. We define a bijective map

$$\bar{\ }^j : \{w \in B_n \mid \mathrm{Neg}(w^{-1}) = J\} \to \{w \in B_n \mid \mathrm{Neg}(w^{-1}) = J \cup \{j\}\}, \quad w \mapsto \bar{w}^j$$

and control its effect on the relevant statistics. Write $[n] \setminus J = \{a_1, \dots, a_s\}_<$, note that $a_s = n$, and set $\{b_1, \dots, b_{s-1}\}_< = \{a_1, \dots, a_s\} \setminus \{j\}$. Given $w \in B_n$ with $\mathrm{Neg}(w^{-1}) = J$, the signed permutation $\bar{w}^j$ is obtained by replacing, in the one-line notation for w, the letters $a_1, \dots, a_{s-1}$ with $b_1, \dots, b_{s-1}$ and $a_s = n$ with $\bar{\jmath}$. Informally speaking, the signed permutation matrix associated with $\bar{w}^j$ is obtained by cyclically permuting the last $n - (j-1)$ rows of the signed permutation matrix associated with w and then switching the sign in the j-th row. That the map $\bar{\ }^j$ is well-defined and bijective is clear.

Let $k = w^{-1}(n)$, which is to say that the letter $\bar{\jmath}$ appears in $\bar{w}^j$ in position k. We claim that

$$\ell(\bar{w}^j) = \ell(w) + 2k - n - 1 + j, \tag{3.10}$$

$$\mathrm{Des}(\bar{w}^j) = (\mathrm{Des}(w) \setminus \{k\}) \cup \{k-1\}. \tag{3.11}$$

These claims, which are proved below, suffice to prove the lemma. Indeed, observe that (3.11) implies that $\sigma_B(\bar{w}^j) = \sum_{i \in \mathrm{Des}(\bar{w}^j)} (n^2 - i^2) = \left(\sum_{i \in \mathrm{Des}(w)} (n^2 - i^2)\right) - (n^2 - k^2) + (n^2 - (k-1)^2) = \sigma_B(w) + 2k - 1$, whence it follows, with (3.10), that

$$(\sigma_B - \ell)(\bar{w}^j) = (\sigma_B - \ell)(w) + n - j.$$

Eq. (3.11) also implies that $\mathrm{des}(\bar{w}^j) = \mathrm{des}(w) + \delta_{\{k=n\}}$, because $\mathrm{des}(w) = \mathrm{des}(\bar{w}^j)$ unless $\bar{\jmath}$ appears in $\bar{w}^j$ in position n. Also, $\varepsilon_n(\bar{w}^j) = \varepsilon_n(w) + \delta_{\{k=n\}}$, whence

$$(\mathrm{des} - \varepsilon_n)(\bar{w}^j) = (\mathrm{des} - \varepsilon_n)(w).$$

Clearly

$$\mathrm{neg}(\bar{w}^j) = \mathrm{neg}(w) + 1$$

and, again by (3.11),

$$\mathrm{rmaj}(\bar{w}^j) = \mathrm{rmaj}(w) + 1.$$

To prove claim (3.10) recall the identity $\ell = \mathrm{inv} + \mathrm{neg} + \mathrm{nsp}$ (cf. (3.1)) and observe that inversions and negative sum pairs—enumerated respectively by the statistics inv and nsp—involving positions (r, t) with $k < t \leq n$ are the same for w and $\bar{w}^j$. As $w(k) = n$, there are $n - k$ inversions in w indexed by the pairs (k, t) with $t \in [k+1, n]$; they all disappear in $\bar{w}^j$. Likewise, as $\bar{w}^j(k) = \bar{\jmath}$ (which, by hypothesis, is the minimum of the entries of $\bar{w}^j$ in one-line notation), the pairs (i, k) for $i \in [k-1]$ are inversions for $\bar{w}^j$ but not for w. Moreover, the (negative) entry $\bar{\jmath}$ in $\bar{w}^j$ gives rise to $j - 1$ additional negative sum pairs compared with w. Hence

$$\begin{aligned}\ell(\bar{w}^j) &= (\mathrm{inv} + \mathrm{neg} + \mathrm{nsp})(\bar{w}^j) \\ &= \mathrm{inv}(w) - (n-k) + (k-1) + \mathrm{neg}(w) + 1 + \mathrm{nsp}(w) + (j-1) \\ &= \ell(w) + 2k - n - 1 + j.\end{aligned}$$

The claim (3.11) follows easily from the fact that the operation $w \mapsto \bar{w}^j$ preserves all relative positions of the letters of w other than n and j. In position k, it replaces the letter n by $\bar{\jmath}$. If $k < n$, then $k \in \mathrm{Des}(w)$ but $k-1 \notin \mathrm{Des}(w)$ and the operation shifts this descent of w from position k to a descent of $\bar{w}^j$ in position $k-1$; if $k = n$, then it produces a descent at $n-1$. □

Proof of Proposition 3.4. The claim follows by observing that, for $w \in S_n = \{w \in B_n \mid \mathrm{Neg}(w^{-1}) = \varnothing\}$,

$$(\sigma_A + n\,\mathrm{rmaj})(w) = \sum_{i \in \mathrm{Des}(w)} (i(n-i) + n(n-i)) = \sigma_B(w)$$

and repeated applications of Lemma 3.5. □

Remark 3.6. Setting $W = 1$ in (3.9) yields a polynomial which factors further:

$$\begin{aligned}\sum_{w \in B_n} X^{(\sigma_B - \ell)(w)} Y^{\mathrm{neg}(w)} Z^{\mathrm{rmaj}(w)} &= \left(\sum_{w \in S_n} X^{(\sigma_A - \ell)(w)} (X^n Z)^{\mathrm{rmaj}(w)} \right) \prod_{j=0}^{n-1} (1 + X^j Y Z) \\ &= \prod_{j=0}^{n-1} \left(\frac{1 - (X^{n+j} Z)^{n-j}}{1 - X^{n+j} Z} \right) \left(1 + X^j Y Z\right);\end{aligned}$$

cf. [14, Propositions 1.7 and 4.8]. In the setup of Proposition 3.4, however, further factorization of the sum over S_n is not to be expected in general. For $n = 3$, for instance, the polynomial

$$\sum_{w \in S_3} W^{\mathrm{des}(w)} X^{(\sigma_A - \ell)(w)} (X^3 Z)^{\mathrm{rmaj}(w)} = 1 + W X^3 Z (1 + X)(1 + X^3 Z) + W^2 X^{10} Z^3$$

is irreducible. It factors after the specific substitutions $((q^{-1}, q, -1, q^{-s})$ for $(W, X, Y, Z))$ we perform in our applications to counting traceless matrices; cf. the proof of Theorem 3.14. For $n > 3$, however, there appears to be no "systematic" factorization; cf. Example 1.3.

We record here a conjectural factorization for a twisted joint distribution on B_n of several statistics, including one involving the odd length function L defined in (3.6).

Conjecture 3.7.

$$\sum_{w \in B_n} (-1)^{\ell(w)} X^{\left(\frac{\sigma_B + \mathrm{rmaj}}{2} - L\right)(w)} Z^{\mathrm{rmaj}(w)} = \left(\sum_{w \in S_n} X^{\left(\frac{\sigma_B + \mathrm{rmaj}}{2} - \ell\right)(w)} Z^{\mathrm{rmaj}(w)} \right) \prod_{i=0}^{n-1} (1 - X^i Z).$$

Remark 3.8. Conjecture 3.7 is slightly weaker than its analogue Proposition 3.4; replacing the character $(-1)^{\ell(w)}$ by $Y^{\ell(w)}$ on the left hand side does not lead to a similar factorization. Note that [14, Proposition 5.5] and [5, Theorem 5.4] yield another factorization formula for the left-hand side which is not, however, expressed in terms of statistics on the symmetric group. Moreover, [13, Lemma 8] implies that the sum on the left-hand side remains unchanged when restricted to chessboard elements; for definitions and properties of these see [13] and [5].

3.4. **Igusa functions.** Recall, e.g. from [11, Definition 2.5], the definition of the *Igusa function* (of degree n)

$$I_n(Y;X_1,\dots,X_n) = \frac{1}{1-X_n}\sum_{I\subseteq[n-1]}\binom{n}{I}_Y\prod_{i\in I}\mathrm{gp}\left(X_i\right)$$
$$= \frac{\sum_{w\in S_n}Y^{\ell(w)}\prod_{j\in\mathrm{Des}(w)}X_j}{\prod_{j=1}^{n}(1-X_j)} \in \mathbb{Q}(Y,X_1,\dots,X_n).$$

Specific choices of "numerical data" to be substituted for the variables $Y,X_1,\dots,X_n$ may lead to factorizations or cancellations. An extremal example is the following.

Example 3.9. [14, Proposition 4.2]

$$I_n\left(X^{-1};\left(X^iZ\right)^{n-i}\right)_{i=n-1}^{0}\Big) = \frac{1}{\prod_{j=0}^{n-1}(1-X^jZ)}. \tag{3.12}$$

We record an application to the rational function

$$\mathcal{B}_n(X,Y,Z) = \sum_{j=0}^{n}\binom{n}{j}_X\frac{Z^{n-j}\prod_{i=0}^{n-j-1}(1-X^{-i-j-1}Y)}{\prod_{i=0}^{n-j-1}(1-X^{i+j}Z)}$$

defined in [14, eq. (4.7)].

Corollary 3.10.

$$\mathcal{B}_n(X,-Y,X^nZ) = \frac{\prod_{j=0}^{n-1}(1+X^jYZ)}{\prod_{j=0}^{n-1}(1-X^{n+j}Z)}$$
$$= I_n\left(X^{-1},\left(X^{n^2-i^2}Z^{n-i}\right)_{i=n-1}^{0}\right)\prod_{j=0}^{n-1}\left(1+X^jYZ\right).$$

Proof. The first equality is proven in [14, Section 4.2.2], the second follows from (3.12). □

3.5. **Traceless matrices, signed permutation statistics, and Igusa functions.** We recast the formula for the image zeta function $\mathcal{P}_{n,\mathfrak{o}}(s)$ given in Theorem 2.6 in terms of signed permutation statistics; cf. Proposition 3.11. In Theorem 3.12 we establish an expression for $\mathcal{P}_{n,\mathfrak{o}}(s)$ in terms of Igusa functions.

Proposition 3.11. *The following identities hold in the field* $\mathbb{Q}(Y,X_0,\dots,X_{n-1})$*:*

$$1+\sum_{i=0}^{n-1}b_{n,i}(Y)\,\mathrm{gp}(X_i)\left(\sum_{J\subseteq[i-1]_0}f_{i,J}(Y)\prod_{j\in J}\mathrm{gp}(X_j)\right) = \tag{3.13}$$
$$\sum_{I\subseteq[n-1]_0}f_{K_n,I}(Y)\prod_{j\in I}\mathrm{gp}(X_j) = \frac{\sum_{w\in B_n}(-1)^{\mathrm{neg}(w)}Y^{(\ell-\varepsilon_n)(w)}\prod_{j\in\mathrm{Des}(w)}X_j}{\prod_{j=0}^{n-1}(1-X_j)}.$$

Proof. Eq. (3.8) yields

$$\sum_{I\subseteq[n-1]_0}f_{K_n,I}(Y)\prod_{j\in I}\mathrm{gp}(X_j) = 1+\sum_{i=0}^{n-1}\sum_{i\in J\subseteq[i]_0}f_{K_n,J}(Y)\prod_{j\in J}\mathrm{gp}(X_j)$$
$$= 1+\sum_{i=0}^{n-1}b_{n,i}(Y)\,\mathrm{gp}(X_i)\left(\sum_{J\subseteq[i-1]_0}f_{i,J}(Y)\prod_{j\in J}\mathrm{gp}(X_j)\right),$$

establishing the first equality. The second one follows from (3.7) and [14, Lemma 4.4]. □

Theorem 2.6 shows that the Poincaré series $\mathcal{P}_{n,\mathfrak{o}}(s)$ may be obtained from the rational function on the left-hand side of (3.13) by substituting q^{-1} for the variable Y and $x_{n,i} = q^{n^2-i^2-1}t^{n-i}$ for the variables X_i. Our next result shows that, under this substitution, the numerator of the rational function on the right-hand side of (3.13) factorizes partly.

Theorem 3.12.

$$\mathcal{P}_{n,\mathfrak{o}}(s) = I_n\left(q^{-1}; \left(q^{n^2-i^2-1}t^{n-i}\right)_{i=n-1}^{0}\right)\prod_{j=0}^{n-1}(1-q^j t). \tag{3.14}$$

Proof. Using Theorem 2.6 and Propositions 3.11 and 3.4 (where we substitute $(q^{-1}, q, -1, t)$ for (W, X, Y, Z)), we obtain

$$\begin{aligned}
\mathcal{P}_{n,\mathfrak{o}}(s) &= 1 + \sum_{i=0}^{n-1} b_{n,i}(q^{-1})\,\mathrm{gp}(x_{n,i})\left(\sum_{J\subseteq[i-1]_0} f_{i,J}(q^{-1})\prod_{j\in J}\mathrm{gp}(x_{n,j})\right)\\
&= \frac{\sum_{w\in B_n}(-1)^{\mathrm{neg}(w)}q^{(-\ell+\varepsilon_n)(w)}\prod_{j\in\mathrm{Des}(w)}q^{n^2-j^2-1}t^{n-j}}{\prod_{j=0}^{n-1}(1-q^{n^2-j^2-1}t^{n-j})}\\
&= \frac{\sum_{w\in B_n}(-1)^{\mathrm{neg}(w)}q^{(\sigma_B-\ell+\varepsilon_n-\mathrm{des})(w)}t^{\mathrm{rmaj}(w)}}{\prod_{j=0}^{n-1}(1-q^{n^2-j^2-1}t^{n-j})}\\
&= \frac{\left(\sum_{w\in S_n}q^{(\sigma_A-\ell-\mathrm{des})(w)}(q^n t)^{\mathrm{rmaj}(w)}\right)\prod_{j=0}^{n-1}(1-q^j t)}{\prod_{j=0}^{n-1}(1-q^{n^2-j^2-1}t^{n-j})}\\
&= \frac{\left(\sum_{w\in S_n}q^{-\ell(w)}\prod_{j\in\mathrm{Des}(w)}q^{j(n-j)-1+n(n-j)}t^{n-j}\right)\prod_{j=0}^{n-1}(1-q^j t)}{\prod_{j=0}^{n-1}(1-q^{n^2-j^2-1}t^{n-j})}\\
&= I_n\left(q^{-1}; \left(q^{n^2-i^2-1}t^{n-i}\right)_{i=n-1}^{0}\right)\prod_{j=0}^{n-1}(1-q^j t).
\end{aligned}$$

□

Remark 3.13. The inverse of the second factor $\prod_{j=0}^{n-1}(1-q^j t)$ on the right-hand side of (3.14) is itself an Igusa function of degree n, viz. $I_n(q^{-1}; ((q^i t)^{n-i})_{i=n-1}^{0})$; cf. Example 3.9. Numerical evidence for small n suggests that there is no further "systematic" factorization of the first factor; cf. Example 1.3.

4. Representation zeta functions of groups of type K

In this section we consider applications of the formulae obtained in the previous sections to representation zeta functions of finitely generated nilpotent groups of type K, in particular their global analytic properties. In the sequel we assume that $n > 1$, ensuring that the group schemes K_n are nonabelian.

4.1. Global analytic properties of Euler products.

We consider Euler products of Poincaré series of the form $\mathcal{P}_{n,\mathfrak{o}}(s)$, where $\mathfrak{o}$ runs through the completions $\mathcal{O}_\mathfrak{p}$ of the ring of integers $\mathcal{O}$ of a number field F at its nonzero prime ideals $\mathfrak{p}$. As mentioned in Section 1.3, the (global) representation zeta function of the group $K_n(\mathcal{O})$ satisfies

$$\zeta_{K_n(\mathcal{O})}(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})}\zeta_{K_n(\mathcal{O}_\mathfrak{p})}(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O})}\mathcal{P}_{n,\mathcal{O}_\mathfrak{p}}(s).$$

Proposition 4.1.

(1) *The abscissa of convergence of $\zeta_{K_n(\mathcal{O})}(s)$ is $\alpha(K_n(\mathcal{O})) = 2n-1$.*
(2) *The zeta function $\zeta_{K_n(\mathcal{O})}(s)$ may be continued meromorphically to $\{s \in \mathbb{C} \mid \Re(s) > 2n-3\}$. For $n \in \{2,3\}$ it may be continued meromorphically to the whole complex plane.*

Proof. According to Theorem 3.12, the representation zeta function $\zeta_{K_n(\mathcal{O})}(s)$ is the product of a finite number of (inverses of) translates of the Dedekind zeta function $\zeta_F(s)$ of the number field F and an Euler product of Igusa functions. Equivalently, writing

$$I_n\left(q^{-1}; \left(q^{n^2-i^2-1}t^{n-i}\right)_{i=n-1}^{0}\right) = \frac{V_n(q,t)}{\prod_{j=0}^{n-1}(1-q^{n^2-j^2-1}t^{n-j})}$$

with

$$V_n(q,t) = \sum_{w\in S_n} q^{-\ell(w)} \prod_{j\in \mathrm{Des}(w)} q^{n^2-j^2-1}t^{n-j}$$

(cf. Theorem 1.2), we have

$$\zeta_{K_n(\mathcal{O})}(s) = \left(\prod_{j=0}^{n-1} \frac{\zeta_F((n-j)s-(n^2-j^2-1))}{\zeta_F(s-j)}\right) \prod_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O})} V_n(q,t).$$

Here as throughout, $q = |\mathcal{O} : \mathfrak{p}|$ denotes the residue field cardinality at $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. Recall that, for $a, b \in \mathbb{R}$, with $a \neq 0$, the translate $\zeta_F(as-b)$ converges uniformly on $\{s \in \mathbb{C} \mid \Re(s) > \frac{b+1}{a}\}$ and has meromorphic continuation to the whole complex plane. Obviously, $\max\left\{\frac{n^2-j^2-1+1}{n-j} \mid j \in [n-1]_0\right\} = 2n-1$. Given the explicit formulae for $\zeta_{K_n(\mathcal{O}_\mathfrak{p})}(s)$ for $n \in \{2,3\}$ in Example 1.3, the proposition thus holds for $n \in \{2,3\}$ and we may assume that $n \geq 4$. In this general case, the proposition will follow if we can show that the Euler product

$$\prod_{\mathfrak{p}\in \mathrm{Spec}(\mathcal{O})} V_n(q,t)$$

may be meromorphically continued to $\{s \in \mathbb{C} \mid \Re(s) > 2n-3\}$.

By [6, Lemma 5.5], it suffices to prove that

$$\beta := \max\left\{\frac{\left(\sum_{j\in \mathrm{Des}(w)}(n^2-j^2-1)\right)-\ell(w)}{\sum_{j\in \mathrm{Des}(w)}(n-j)} \mid w \in S_n \setminus \{1\}\right\} = 2n-3.$$

We will prove, more precisely, that this maximum is attained exactly twice, viz. at the Coxeter generators $w = s_{n-1}$ and $w = s_{n-2}$. We first show that, for $w \in S_n \setminus \{1\}$, the quantity

$$\frac{\left(\sum_{j\in \mathrm{Des}(w)}(n^2-j^2-1)\right)-\ell(w)}{\sum_{j\in \mathrm{Des}(w)}(n-j)}$$

does not exceed $2n-3$. Indeed, suppose that it is $\geq 2n-3$, i.e.

$$\left(\sum_{j\in \mathrm{Des}(w)}((n-j)(n+j)-1)\right) - \ell(w) - (2n-3)\sum_{j\in \mathrm{Des}(w)}(n-j) =$$

$$\left(\sum_{j\in \mathrm{Des}(w)}(-(n-j)^2+3(n-j))\right) - \mathrm{des}(w) - \ell(w) \geq 0.$$

Writing $i = n - j$ and setting $\rho(w) = \ell(w) - \mathrm{des}(w) \in \mathbb{N}_0$, this is equivalent to

$$\sum_{n-i \in \mathrm{Des}(w)} (3i-2) \geq \left(\sum_{n-i \in \mathrm{Des}(w)} i^2 \right) + \rho(w).$$

The latter inequality has solutions only if $\mathrm{Des}(w) \subseteq \{n-2, n-1\}$. It is an equality if, and only if, additionally $\rho(w) = 0$. The elements $w = s_{n-1}$ and $w = s_{n-2}$ clearly satisfy this condition and are the only solutions whose descent sets are singletons; no permutation with $\mathrm{Des}(w) = \{n-2, n-1\}$ satisfies $\rho(w) = 0$.

We conclude that the maximum $\beta = 2n-3$ is attained at s_{n-1} and s_{n-2} as claimed. □

Remark 4.2. Proposition 4.1 leaves open the interesting question whether or not the line $\{s \in \mathbb{C} \mid \Re(s) = 2n-3\}$ is actually a natural boundary for meromorphic continuation of $\zeta_{K_n(\mathcal{O})}(s)$ for $n \geq 4$. It may be of interest to note that the proof of its part (2) implies that, in the terminology of [6, Section 5.2], the first factor of the ghost polynomial of $V_n(q,t)$ is the unitary polynomial $\widetilde{V_{n,1}}(q,t) = 1 + q^{2n-3}t + q^{4n-6}t^2$.

4.2. **Topological representation zeta functions.** In [10], Rossmann initiated the study of topological representation zeta functions associated to unipotent group schemes defined over number fields. Very roughly speaking, these are rational functions encapturing the "limit $q \to 1$" of the local representation zeta functions occurring as Euler factors in the global representation zeta functions associated to the groups of rational points over number rings of the unipotent group schemes in question. The latter are rational functions in the parameters q^{-s} and the topological representation zeta function may be defined as the leading coefficients of the series expansions of these rational functions in $q-1$. For precise definitions and instructive examples see [10, Section 3]. A number of intriguing open questions regarding topological representation zeta functions of unipotent group schemes are raised in [10, Section 7]. We use the explicit formulae derived in the current paper to show that some of these questions have positive answers for the unipotent group schemes K_n. The following is an immediate consequence of Theorem 3.12.

Proposition 4.3. *The topological zeta function of K_n is equal to*

$$\zeta_{K_n,\mathrm{top}}(s) = \prod_{i=0}^{n-1} \frac{s-i}{s - (n + i - \frac{1}{n-i})}.$$

We note that $\zeta_{K_n,\mathrm{top}}(s)$ has a simple zero at $s = 0$ and that $\zeta_{K_n,\mathrm{top}}(s) - 1$ has degree -1 in s; cf. [10, Questions 7.4 and 7.1]. Following [10] we consider the invariant

$$\omega(K_n) = s\,(\zeta_{K_n,\mathrm{top}}(s) - 1)\Big|_{s=\infty} \in \mathbb{Q}.$$

In the pertinent special cases, [10, Question 7.2] is answered positively by the following result.

Proposition 4.4.

$$\omega(K_n) > n(n-1).$$

Proof. The quantity $\omega(K_n)$ is the quotient of the leading coefficients of the polynomials in s occurring in numerator and denominator of the right-hand side of the expression

$$\zeta_{K_n,\mathrm{top}}(s) - 1 = \frac{\left(\prod_{i=0}^{n-1}(s-i)\right) - \left(\prod_{i=0}^{n-1}\left(s - \left(n + i - \frac{1}{n-i}\right)\right)\right)}{\prod_{i=0}^{n-1}\left(s - \left(n + i - \frac{1}{n-i}\right)\right)}.$$

As the denominator is monic, $\omega(K_n)$ is in fact just the difference between the coefficients of s^{n-1} in $\prod_{i=0}^{n-1}(s-i)$ and $\prod_{i=0}^{n-1}\left(s-\left(n+i-\frac{1}{n-i}\right)\right)$, respectively. The former is $-\binom{n}{2}$, the latter satisfies

$$-\sum_{i=0}^{n-1}\left(n+i-\frac{1}{n-i}\right) < -\sum_{i=0}^{n-1}(n+i-1) = -\binom{2n-1}{2}+\binom{n-1}{2}.$$

It follows that $\omega(K_n) > -\binom{n}{2}+\binom{2n-1}{2}-\binom{n-1}{2} = n(n-1)$ as claimed. □

Remark 4.5. The topological representation zeta functions the unipotent group schemes $F_{n,\delta}$, G_n, and H_n are easily read off from [14, Theorem B]:

$$\zeta_{F_{n,\delta},\mathrm{top}}(s) = \prod_{i=0}^{n-1}\frac{s-2i}{s-2(n+i+\delta)+1},$$
$$\zeta_{G_n,\mathrm{top}}(s) = \prod_{i=0}^{n-1}\frac{s-i}{s-n-i},$$
$$\zeta_{H_n,\mathrm{top}}(s) = \prod_{i=0}^{n-1}\frac{s-i}{s-\frac{n+i+1}{2}}.$$

[10, Questions 7.1 and 7.4] are easily seen to have positive answers in these cases, too. One computes easily that

$$\omega(F_{n,\delta}) = 2n^2+(2\delta-1)n, \qquad \omega(G_n) = n^2, \qquad \omega(H_n) = \frac{n^2+3n}{4}.$$

In contrast, the invariant $\omega(K_n)$ does not appear to be a (half-)integer.

4.3. **Representation zeta functions and Igusa functions.** Theorem 3.12 describes local representation zeta functions associated to groups of type K as quotients of two Igusa functions. We note similar factorizations for groups of type F and G and observe that Conjecture 3.7 yields an analogous conjectural expression for groups of type H.

Proposition 4.6. *Let $\delta \in \{0,1\}$ and $\mathfrak{o}$ be of characteristic zero. The representation zeta functions $\zeta_{F_{n,\delta}(\mathfrak{o})}(s)$ and $\zeta_{G_n(\mathfrak{o})}(s)$ described in* [14, Theorem C] *satisfy*

$$\zeta_{F_{n,\delta}(\mathfrak{o})}(s) = I_n\left(q^{-2};\left(q^{\binom{2n+\delta}{2}-\binom{2i+\delta}{2}}t^{n-i}\right)_{i=n-1}^{0}\right)\prod_{i=0}^{n-1}(1-q^{2i}t), \tag{4.1}$$

$$\zeta_{G_n(\mathfrak{o})}(s) = I_n\left(q^{-1};\left(q^{n^2-i^2}t^{n-i}\right)_{i=n-1}^{0}\right)\prod_{i=0}^{n-1}(1-q^{i}t) \tag{4.2}$$

Proof. Using that $\zeta_{F_{n,\delta}(\mathfrak{o})}(s) = \mathcal{B}_n(q^2, q^{-2\delta+1}, q^{2(n+\delta)-1-s})$ and $\zeta_{G_n(\mathfrak{o})}(s) = \mathcal{B}_n(q,1,q^{n-s})$ (cf. proof of [14, Proposition 5.1]), this follows from Corollary 3.10. □

Conjecture 4.7. *Let $\mathfrak{o}$ be of characteristic zero. The representation zeta function $\zeta_{H_n(\mathfrak{o})}(s)$ described in* [14, Theorem C] *satisfies*

$$\zeta_{H_n(\mathfrak{o})}(s) = I_n\left(q^{-1};\left(q^{\binom{n+1}{2}-\binom{i+1}{2}}t^{n-i}\right)_{i=n-1}^{0}\right)\prod_{i=0}^{n-1}(1-q^{i}t). \tag{4.3}$$

Regarding the "numerical data" of the Igusa functions in (4.1), (4.2), and (4.3), we remark that, in the notation of [14, Theorem C].

$$\binom{2n+\delta}{2}-\binom{2i+\delta}{2} = a(F_{n,\delta},i), \quad n^2-i^2 = a(G_n,i), \quad \binom{n+1}{2}-\binom{i+1}{2} = a(H_n,i).$$

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References

[1] R. M. Adin, F. Brenti, and Y. Roichman, *Descent representations and multivariate statistics*, Trans. Amer. Math. Soc. **357** (2005), no. 8, 3051–3082.
[2] P. Belkale and P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. **116** (2003), no. 1, 147–188.
[3] E. A. Bender, *On Buckhiester's enumeration of* $n \times n$ *matrices*, J. Combinatorial Theory Ser. A **17** (1974), 273–274.
[4] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
[5] F. Brenti and A. Carnevale, *Proof of a conjecture of Klopsch-Voll on Weyl groups of type A*, Trans. Amer. Math. Soc. **369** (2017), no. 10, 7531–7547.
[6] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer-Verlag, Berlin, 2008.
[7] J. B. Lewis, R. I. Liu, A. H. Morales, G. Panove, S. V. Sam, and Y. X. Zhang, *Matrices with restricted entries and* q*-analogues of permutations*, J. Comb. **2** (2011), no. 3, 355–395.
[8] V. Reiner, *Signed permutation statistics*, European J. Combin. **14** (1993), no. 6, 553–567.
[9] T. Rossmann, Zeta, Version 0.3.2, see `https://www.math.uni-bielefeld.de/~rossmann/Zeta/`.
[10] T. Rossmann, *Topological representation zeta functions of unipotent groups*, J. Algebra **448** (2016), 210–237.
[11] M. M. Schein and C. Voll, *Normal zeta functions of the Heisenberg groups over number rings I – the unramified case*, J. Lond. Math. Soc. (2) **91** (2015), no. 1, 19–46.
[12] C. Seeley, 7*-dimensional nilpotent Lie algebras*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 479–496.
[13] A. Stasinski and C. Voll, *A new statistic on the hyperoctahedral groups*, Electron. J. Combin. **20** (2013), no. 3, Paper 50, 23.
[14] ———, *Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B*, Amer. J. Math. **136** (2014), no. 2, 501–550.
[15] ———, *Representation zeta functions of some nilpotent groups associated to prehomogeneous vector spaces*, Forum Math. **29** (2017), no. 3, 717 – 734.
[16] J. R. Stembridge and D. J. Waugh, *A Weyl group generating function that ought to be better known*, Indag. Math. (N.S.) **9** (1998), no. 3, 451–457.
[17] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.

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