# ORBIT DIRICHLET SERIES AND MULTISET PERMUTATIONS

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Abstract. We study Dirichlet series enumerating orbits of Cartesian products of maps whose orbit distributions are modelled on the distributions of finite index subgroups of free abelian groups of finite rank. We interpret Euler factors of such orbit Dirichlet series in terms of generating polynomials for statistics on multiset permutations, viz. descent and major index, generalizing Carlitz's $q$-Eulerian polynomials.

We give two main applications of this combinatorial interpretation. Firstly, we establish local functional equations for the Euler factors of the orbit Dirichlet series under consideration. Secondly, we determine these (global) Dirichlet series' abscissae of convergence and establish some meromorphic continuation beyond these abscissae. As a corollary, we describe the asymptotics of the relevant orbit growth sequences. For Cartesian products of more than two maps we establish a natural boundary for meromorphic continuation. For products of two maps, we prove the existence of such a natural boundary subject to a combinatorial conjecture.

## 1. Introduction and main results

Let $X$ be a space and $T : X \to X$ a map. A *closed orbit of length* $n \in \mathbb{N}$ is a set of the form

$$\{x, T(x), T^2(x), \dots, T^n(x) = x\}$$

of cardinality $n$. Assume that the number $\mathsf{O}_T(n)$ of closed orbits of length $n$ under $T$ is finite for all $n \in \mathbb{N}$. The *orbit Dirichlet series* of $T$ is the Dirichlet generating series

$$\mathsf{d}_T(s) = \sum_{n=1}^{\infty} \mathsf{O}_T(n) n^{-s},$$

where $s$ is a complex variable.

If $T$ has a single closed orbit of each length $n$, then $\mathsf{d}_T(s)$ is just Riemann's zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. If, more generally, $T = T_r$ is such that the number of closed orbits of length $n$ equals the number $a_n(\mathbb{Z}^r)$ of subgroups of $\mathbb{Z}^r$ of index $n$ for all $n \in \mathbb{N}$, then $\mathsf{d}_{T_r}(s)$ is the well known Dirichlet generating series (or "zeta function") $\zeta_{\mathbb{Z}^r}(s)$ enumerating subgroups of finite index of the free abelian group $\mathbb{Z}^r$ of rank $r$. More precisely,

$$\mathsf{d}_{T_r}(s) = \zeta_{\mathbb{Z}^r}(s) = \sum_{n=1}^{\infty} a_n(\mathbb{Z}^r) n^{-s} = \prod_{i=0}^{r-1} \zeta(s-i); \tag{1.1}$$

cf. [12, Proposition 1.1].

Let $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{N}_0^m$ with $\lambda_1 \geqslant \dots \geqslant \lambda_m \geqslant 1$ be a partition of $N = \sum_{i=1}^m \lambda_i$. For $i = 1, \dots, m$, let $T_{\lambda_i}$ be a map as above with $\mathsf{d}_{T_{\lambda_i}}(s) = \zeta_{\mathbb{Z}^{\lambda_i}}(s)$. We write

$$T_\lambda = T_{\lambda_1} \times \dots \times T_{\lambda_m}$$

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for the Cartesian product of the maps $T_{\lambda_i}$. Clearly, the arithmetic function $n \mapsto \mathsf{O}_{T_\lambda}(n)$ is multiplicative, whence

$$\mathsf{d}_{T_\lambda}(s) = \prod_{p \text{ prime}} \mathsf{d}_{T_\lambda,p}(s),$$

where, for a prime $p$,

$$\mathsf{d}_{T_\lambda,p}(s) = \sum_{k=0}^{\infty} \mathsf{O}_{T_\lambda}(p^k)p^{-ks}.$$

We remark that maps $T_{\lambda_i}$ as above exist: by a result of Windsor, any sequence $(a_n)_{n\geqslant 1}$ of nonnegative integers may be realized as the sequence $(\mathsf{O}_T(n))_{n\geqslant 1}$ for a suitable $C^\infty$-diffeomorphism $T$ of the 2-dimensional torus $X = \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$; cf. [25].

In this paper we prove and exploit combinatorial formulae for the Euler factors of orbit Dirichlet series of the form $\mathsf{d}_{T_\lambda}(s)$ above using generating polynomials for statistics on multiset permutations.

Our first main result is phrased in terms of the bivariate polynomial $C_\lambda \in \mathbb{Z}[x, q]$ giving the joint distribution of the statistics des and maj on $S_\lambda$, the set of multiset permutations of the multiset $\{\underbrace{\mathbf{1}, \dots \mathbf{1}}_{\lambda_1}, \underbrace{\mathbf{2}, \dots \mathbf{2}}_{\lambda_2} \dots, \underbrace{\mathbf{m}, \dots, \mathbf{m}}_{\lambda_m}\}$. See Section 2 for precise definitions.

**Theorem 1.1.** *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of $N$. Then*

$$\mathsf{d}_{T_\lambda}(s) = \prod_{p \text{ prime}} \frac{C_\lambda(p^{-1-s}, p)}{\prod_{i=1}^{N}(1 - p^{i-1-s})} = \prod_{p \text{ prime}} \frac{\sum_{w\in S_\lambda} p^{(-1-s)\operatorname{des}(w)+\operatorname{maj}(w)}}{\prod_{i=1}^{N}(1 - p^{i-1-s})}. \tag{1.2}$$

Key to Theorem 1.1 is an identity, essentially due to MacMahon, for Hadamard products of certain rational generating functions. Recall that if $A(x) = \sum_{i=0}^{\infty} a_i x^i$ and $B(x) = \sum_{i=0}^{\infty} b_i x^i \in \mathbb{Q}(x)$ are rational generating functions, then their Hadamard product $(A * B)(x) = \sum_{i=0}^{\infty} a_i b_i x^i$ is also a rational function; cf. [23, Proposition 4.2.5].

**Proposition 1.2** (MacMahon; cf. Remark 3.2)**.** *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of $N$. Then*

$$*_{i=1}^{m} \prod_{k=0}^{\lambda_i} \frac{1}{1 - q^k x} = \frac{C_\lambda(x, q)}{\prod_{i=0}^{N}(1 - xq^i)} \in \mathbb{Q}(x, q).$$

We call a partition of the form $\lambda = (r, \dots, r) = (r^m)$ a *rectangle.* We use Theorem 1.1 to prove that the Euler factors in (1.2) satisfy functional equations upon inversion of the prime if and only if the partition $\lambda$ is a rectangle.

**Theorem 1.3.** *Let $p$ be a prime. For all $r, m \in \mathbb{N}$,*

$$\mathsf{d}_{T_r^{\times m},p}(s)|_{p\to p^{-1}} = (-1)^{rm} p^{m\binom{r+1}{2}-r-rs} \mathsf{d}_{T_r^{\times m},p}(s). \tag{1.3}$$

*If $\lambda$ is not a rectangle, then $\mathsf{d}_{T_\lambda,p}(s)$ does not satisfy a functional equation of the form*

$$\mathsf{d}_{T_\lambda,p}(s)|_{p\to p^{-1}} = \pm p^{d_1-d_2 s} \mathsf{d}_{T_\lambda,p}(s) \tag{1.4}$$

*for $d_1, d_2 \in \mathbb{N}_0$.*

We prove Theorems 1.1 and 1.3 in Section 3. The functional equations (1.3) are deduced from the combinatorial properties of the polynomials $C_\lambda$ studied in Section 2.

In Section 4 we collect a number of corollaries about the analytic properties of the orbit Dirichlet series that we study. In particular, we determine the abscissa of convergence of $\mathsf{d}_{T_\lambda}(s)$ and establish meromorphic continuation beyond this abscissa. A standard application of a Tauberian theorem then yields an asymptotic result on the growth of the (partial sums of the) numbers $\mathsf{O}_{T_\lambda}(n)$; see Theorem 4.1.

If $\lambda = (r)$ or $\lambda = (1,1)$, then $\mathsf{d}_{T_\lambda}(s)$ has meromorphic continuation to the whole complex plane. In contrast, for partitions with more than two parts – pertaining to Cartesian products of more than two maps – or two parts of equal length greater than 1 we establish a natural boundary for meromorphic continuation at $(\sum_{i=1}^m \lambda_i) - 2$. For partitions with two parts of unequal lengths, we establish such a natural boundary subject to a combinatorial conjecture on some special values of the polynomials $C_\lambda$ discussed in Section 2.2; see Theorem 4.2.

In Section 5 we concentrate on partitions of the form $\lambda = (1^m)$, pertaining to the $m$-th Cartesian power of a map with orbit Dirichlet series $\mathsf{d}_{T_1}(s) = \zeta(s)$. Orbit Dirichlet series of products of such maps were previously studied, for very special cases, in [17]. The result [17, Theorem 4.1], for instance, is the special case $\lambda = (1,1,1)$ of our Theorem 4.2; see also Section 5. For partitions of the form $\lambda = (1^m)$ the polynomial $C_\lambda$ is the well-studied $q$-Carlitz polynomial, enumerating the elements of the symmetric group by the statistics des and maj. We also observe that in this case the Euler factors of (1.2) are Igusa functions in the terminology of [19].

In Section 6 we consider "reduced" orbit Dirichlet series and note some connections with the theory of $h$-vectors of simplicial complexes.

Dirichlet generating series are widely used in enumerative problems arising in algebra, geometry, and number theory. Orbit Dirichlet series as defined above are studied for instance in [7]. Local functional equations such as the ones established in Theorem 1.3 occur frequently in the theory of zeta functions of rings; see, for example, [24]. In the cases where they are explained combinatorially, they may often be traced back to functional equations satisfied by Igusa-type functions; see, for instance, [14, 19].

1.1. **Notation.** We write $\mathbb{N} = \{1, 2, \dots\}$ and, for a subset $I \subseteq \mathbb{N}$, set $I_0 = I \cup \{0\}$. Given $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$ and $n - I = \{n - i \mid i \in I\}$. For $I = \{i_1, \dots, i_r\} \subseteq [n-1]$ with $i_1 \leqslant i_1 \leqslant \dots \leqslant i_r$ we let

$$\binom{n}{I} = \frac{n!}{i_1!(i_2 - i_1)! \cdots (n - i_r)!}$$

denote the multinomial coefficient. Given $a \geqslant b \in \mathbb{N}_0$ and a variable $q$, we write

$$\binom{a}{b}_q = \prod_{i=1}^{b} \frac{q^{a-b+i} - 1}{q^i - 1} \in \mathbb{Z}[q]$$

for the $q$-binomial coefficient.

## 2. Permutations of multisets

In this section we set up notation and prove some basic facts regarding multiset permutations (see also [15, Section 5.1.2]).

2.1. **Multiset permutations.** Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of $N = \sum_{i=1}^m \lambda_i$. The *multiset*

$$A_\lambda = \{\underbrace{\mathbf{1}, \dots \mathbf{1}}_{\lambda_1}, \underbrace{\mathbf{2}, \dots \mathbf{2}}_{\lambda_2} \dots, \underbrace{\mathbf{m}, \dots, \mathbf{m}}_{\lambda_m}\}$$

comprises $\lambda_1$ (indistinguishable) copies of the "letter" $\mathbf{1}$, $\lambda_2$ copies of the "letter" $\mathbf{2}$ etc. A *multiset permutation* (or *multipermutation*) on $A_\lambda$ is a word $w = w_1 \dots w_N$ formed with all the $N$ elements of $A_\lambda$. We denote with $S_\lambda$ the set of all multiset permutations on $A_\lambda$. If $\lambda = (1, \dots, 1) = (1^m)$, then we recover the set $S_m$ of permutations of the set $A_{(1^m)} = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{m}\}$.

In general, $S_\lambda$ lacks a natural group structure, but a number of classical statistics on the Coxeter group $S_m$ have analogues for general partitions. For instance, one defines the *descent set* $\mathrm{Des}(w)$ of $w = \prod_{i=1}^N w_i \in S_\lambda$ as

$$\mathrm{Des}(w) = \{i \in [N-1] \mid w_i > w_{i+1}\},$$

where, of course, one uses the "natural" ordering $\mathbf{m} > \cdots > \mathbf{2} > \mathbf{1}$ on the letters of $A$. The *descent* and *major index* statistics on $S_\lambda$ are defined via

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)| \quad \text{and} \quad \mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

The "trivial word" $\mathbf{1}^{\lambda_1}\mathbf{2}^{\lambda_2}\ldots\mathbf{m}^{\lambda_m}$ is clearly the unique element in $S_\lambda$ with empty descent set.

*Example* 2.1. For $\lambda = (3,3,1)$, the element $w = \mathbf{1212312} \in S_\lambda$ has $\mathrm{Des}(w) = \{2,5\}$, whence $\mathrm{des}(w) = 2$ and $\mathrm{maj}(w) = 7$.

*Remark* 2.2. One may, more generally, consider multisets indexed by compositions, rather than partitions, of $N$. As we are interested in the joint distribution of des and maj, the order of the parts does not matter to us (cf. (2.3) below), so we only consider partitions.

Recall that we call $\lambda$ a rectangle if $\lambda_1 = \cdots = \lambda_m = r$, say, viz. $\lambda = (r^m)$. In this case, we write $S_{r,m}$ for $S_{(r^m)}$. If, moreover, $r = 1$, then we write $S_m$ for $S_{1,m} = S_{(1^m)}$, the (set underlying the) symmetric group of degree $m$.

**Lemma 2.3.** *The partition $\lambda$ is a rectangle if and only if there exists a unique element of $S_\lambda$ at which* des *attains its maximum. If $\lambda = (r^m)$, then both* des *and* maj *take their maximal values at $w_0 = (\mathbf{m}\ldots\mathbf{21})^r$ of $S_\lambda$, viz.* $\mathrm{des}(w_0) = r(m-1)$ *and* $\mathrm{maj}(w_0) = r^2\binom{m}{2}$.

*Proof.* Set $s = \lambda_1$ and write $\mu = (\mu_1, \ldots, \mu_s)$ for the dual partition of $\lambda$. Thus $m \geqslant \mu_1 \geqslant \cdots \geqslant \mu_s \geqslant 1$. The statistic $\mathrm{des}(w)$ attains its maximal value $\sum_{\sigma=1}^s(\mu_\sigma - 1)$ precisely at the word

$$w_0 = (\boldsymbol{\mu_1}\ldots\mathbf{21})(\boldsymbol{\mu_2}\ldots\mathbf{21})\ldots(\boldsymbol{\mu_s}\ldots\mathbf{21})$$

and all the elements of $S_\lambda$ obtained from $w_0$ by permuting the $s$ "blocks" $\boldsymbol{\mu_\sigma}\ldots\mathbf{21}$, $\sigma \in [s]$. All these elements coincide if and only if $\lambda$ is a rectangle, say $\lambda = (r^m)$. In this case, $\mu = (m^r)$ and $w_0$ satisfies $\mathrm{des}(w_0) = r(m-1)$ and $\mathrm{maj}(w_0) = \binom{rm}{2} - m\binom{r}{2} = r^2\binom{m}{2}$. $\square$

We define the involution

$$^\circ : S_{r,m} \to S_{r,m}, \quad w = w_1 \ldots w_N \mapsto w^\circ = (\mathbf{m}+\mathbf{1}-w_N)\ldots(\mathbf{m}+\mathbf{1}-w_1) \tag{2.1}$$

which "reverses and inverts" the elements of $S_{r,m}$.

*Remark* 2.4. If $r = 1$, then $w_0 \in S_m$ is the "longest element" (with respect to Coxeter length) and $^\circ$ is just conjugation by $w_0$.

We collect some properties of this involution in the following elementary and easy lemma, whose proof we omit.

**Lemma 2.5.** *For all $w \in S_{r,m}$ the following hold.*

1. $\mathrm{Des}(w^\circ) = rm - \mathrm{Des}(w)$,
2. $\mathrm{des}(w^\circ) = \mathrm{des}(w)$,
3. $\mathrm{maj}(w^\circ) = \mathrm{des}(w)rm - \mathrm{maj}(w)$.

2.2. **Generating polynomials.** Let $x$ and $q$ be variables and set

$$C_\lambda(x,q) = \sum_{w \in S_\lambda} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} \in \mathbb{Z}[x,q]. \tag{2.2}$$

A result of MacMahon ([16, §462, Vol. 2, Ch. IV, Sect. IX]) states that, in $\mathbb{Q}(x,q) \cap \mathbb{Q}(q)[\![x]\!]$,

$$\sum_{k=0}^{\infty} \left( \prod_{i=1}^{m} \binom{\lambda_j + k}{k}_q \right) x^k = \frac{C_\lambda(x,q)}{\prod_{i=0}^{N}(1-xq^i)}. \tag{2.3}$$

If $\lambda = (r^m)$ is a rectangle, then we write $C_{r,m}$ for $C_{(r^m)}$. If, moreover, $r = 1$, then we write $C_m$ for $C_{1,m} = C_{(1^m)}$. In this case, (2.2) defines Carlitz's $q$-Eulerian polynomial ([1, 2])

$$C_m(x,q) = \sum_{w \in S_m} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} \in \mathbb{Z}[x,q].$$

Note that

$$C_m(x,1) = \sum_{w \in S_m} x^{\mathrm{des}(w)} = A_m(x)/x \in \mathbb{Z}[x], \tag{2.4}$$

where $A_m(x)$ is the $m$-th Eulerian polynomial; cf. [23, Section 1.4].

*Example* 2.6. For $\lambda = (2,1)$, $S_\lambda = \{\mathbf{112}, \mathbf{121}, \mathbf{211}\}$, so

$$C_{(2,1)}(x,q) = 1 + xq + xq^2.$$

For $r = m = 2$, $S_{2,2} = \{\mathbf{1122}, \mathbf{1221}, \mathbf{1212}, \mathbf{2112}, \mathbf{2211}, \mathbf{2121}\}$, whence

$$C_{2,2}(x,q) = 1 + xq + 2xq^2 + xq^3 + x^2q^4.$$

Finally, for $m = 3$ resp. $m = 4$,

$$C_3(x,q) = 1 + 2xq + 2xq^2 + x^2q^3, \text{ resp.}$$
$$C_4(x,q) = (1 + xq^2)(1 + 3xq + 4xq^2 + 3xq^3 + x^2q^4).$$

To establish some of the analytic properties of $\mathsf{d}_{T_\lambda}(s)$ in Section 4, we need a description of the unitary factors of the bivariate polynomials $C_\lambda(x,q)$. Here, a polynomial $f \in \mathbb{Z}[x,q]$ is called *unitary* if it is non-constant and there exists $F \in \mathbb{Z}[Y]$ such that all complex roots of $F$ have absolute value 1 and $f(x,q) = F(x^a q^b)$ for some $a, b \in \mathbb{N}_0$.

As $\mathrm{maj}(w) > 0$ implies $\mathrm{des}(w) > 0$ for all $w \in S_\lambda$, unitary factors of $C_\lambda(x,q) \in \mathbb{Z}[x,q]$ give rise to unitary factors of

$$C_\lambda(x,1) = \sum_{w \in S_\lambda} x^{\mathrm{des}(w)} \in \mathbb{Z}[x]. \tag{2.5}$$

The following Lemma describes the occurrence of unitary factors of Carlitz $q$-Eulerian polynomials, pertaining to partitions of the form $\lambda = (1^m)$.

**Lemma 2.7.** *Carlitz's q-Eulerian polynomial $C_m(x,q) \in \mathbb{Z}[x,q]$ has a unitary factor if and only if $m$ is even. If $m = 2k$, then*

$$C_m(x,q) = (1 + xq^k) C'_m(x,q),$$

*where $C'_m(x,q) = \sum_{w \in S_m^{\{k\}}} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}$ and $S_m^{\{k\}}$ is the parabolic quotient $S_m^{\{k\}} = \{w \in S_m \mid \mathrm{Des}(w) \subseteq [m-1] \setminus \{k\}\}$. Moreover, $C'_m(x,q)$ has no unitary factor.*

*Proof.* By a result of Frobenius ([9, p. 829]), the roots of $C_m(x,1)$ are all real, simple, and negative; moreover, $-1$ is a root only for even $m$. Thus the $q$-Eulerian polynomials $C_m(x,q)$ have unitary factors only for even $m$. Let $m=2k$ and denote by $w_0$ the longest element of $S_m$. The map $S_m^{\{k\}} \to S_m \setminus S_m^{\{k\}}, w \mapsto ww_0$, is obviously a bijection. Hence

$$\begin{aligned} C_m(x,q) &= \sum_{w\in S_m} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w\in S_m} \prod_{j\in \mathrm{Des}(w)} xq^j \\ &= \sum_{w\in S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j + \sum_{w\in S_m\setminus S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j \\ &= \sum_{w\in S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j + xq^k \sum_{w\in S_m\setminus S_m^{\{k\}}} \prod_{\substack{j\in \mathrm{Des}(w)\\ j\neq k}} xq^j \\ &= \sum_{w\in S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j + xq^k \sum_{w\in S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j \\ &= (1+xq^k) \sum_{w\in S_m^{\{k\}}} \prod_{j\in \mathrm{Des}(w)} xq^j = (1+xq^k) C'_m(x,q). \end{aligned}$$

The non-existence of unitary factors of $C'_m(x,q)$ follows again from the simplicity of $x=-1$ as a root of $C_m(x,1)$. □

*Remark* 2.8. The polynomials $C_m(x,1)$, for $m$ odd, resp. $C_m(x,1)/(1+x)$, for $m$ even, have been conjectured to be irreducible for all $m$; for a discussion and proofs of irreducibility in various special cases, see [13].

*Remark* 2.9. Consider again a general partition $\lambda$. Generalizing the result of Frobenius referred to in the proof of Lemma 2.7, all zeros of the polynomials $C_\lambda(x,1)$ are real, simple, and negative; see [20, Corollary 2]. By the above discussion, a necessary condition for the occurrence of unitary factors of $C_\lambda(x,q)$ is hence that $C_\lambda(-1,1)=0$. We remark that in the case that $\lambda=(r^m)$ is a rectangle, $C_\lambda(-1,1)$ is, up to a sign, the so-called Charney-Davis quantity of the graded poset of the disjoint union of $m$ labelled chains of length $r$; see [18].

For our applications in Section 4 we require statements about the (non-)existence of unitary factors of $C_\lambda(x,q)$ principally in the case $m=2$, on which we focus for most of the reminder of this section. Recall that $C_{(\lambda_1,\lambda_2)}(x,1)$ is the descent polynomial of $S_{(\lambda_1,\lambda_2)}$; cf. (2.5). In [16, §144-146, Vol. 1, Ch. II, Sect. IV] MacMahon gives three proofs of the following lemma.

**Lemma 2.10** (MacMahon)**.** *Let $\lambda=(\lambda_1,\lambda_2)$. Then*

$$C_{(\lambda_1,\lambda_2)}(x,1) = \sum_{j=0}^{\lambda_2} \binom{\lambda_1}{j}\binom{\lambda_2}{j} x^j.$$

In terms of Jacobi polynomials,

$$C_{(\lambda_1,\lambda_2)}(x,1) = (1-x)^{\lambda_2} P_{\lambda_2}^{(0,\lambda_1-\lambda_2)}\left(\frac{1+x}{1-x}\right);$$

cf. [10, eq. (1.2.7)]. It follows from MacMahon's third proof of Lemma 2.10 (cf. [16, §146]) that the number of elements in $S_{(\lambda_1,\lambda_2)}$ with $k$ descents equals the number of elements with $k$ occurrences of $\mathbf{2}$ in the first $\lambda_1$ positions. We conjecture the following.

**Conjecture A.** *Let $\lambda_1 > \lambda_2$. Then the following equivalent statements hold:*

(1) $C_{(\lambda_1,\lambda_2)}(-1,1) \neq 0$,
(2) $P_{\lambda_2}^{(0,\lambda_1-\lambda_2)}(0) \neq 0$,

(3)

$$\#\{w \in S_{(\lambda_1,\lambda_2)} \text{ with an even number of } \mathbf{2}\text{s in the first } \lambda_1 \text{ positions}\} \neq$$
$$\#\{w \in S_{(\lambda_1,\lambda_2)} \text{ with an odd number of } \mathbf{2}\text{s in the first } \lambda_1 \text{ positions}\}.$$

*In particular, $C_{(\lambda_1,\lambda_2)}(x,q)$ has no unitary factor.*

Dennis Stanton pointed out to us that $C_{(\lambda_1,\lambda_2)}(-1,1) \neq 0$ for all $\lambda_2$ and $\lambda_1 > \lambda_2(\lambda_2+1)-1$, since the alternating summands have increasing absolute values. The quantity may also be expressed in terms of Krawtchouk polynomials; in the notation of [4], it is equal to $(-1)^{\lambda_2} k_{\lambda_2}(\lambda_2, 2, \lambda_1+\lambda_2)$.

If $\lambda_1 = \lambda_2$, then $C_{(\lambda_1,\lambda_2)}(-1,1) \neq 0$ if and only if the $\lambda_i$ are even (see [18, Proposition 2.4]), whence $C_{(\lambda_1,\lambda_2)}(x,q)$ has no unitary factors in this case. In the odd case, the following holds.

**Proposition 2.11** ([3, Proposition 2.3])**.** *Let $\lambda_1 = \lambda_2 = r \equiv 1 \pmod 2$. Then*

$$C_{r,2}(x,q) = (1+xq^r)C'_{r,2}(x,q),$$

*where $C'_{r,2}(x,q)$ has no unitary factor.*

Generalizing Lemma 2.7, Conjecture A and Proposition 2.11, we put forward the following

**Conjecture B.** *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition. Then $C_\lambda(x,q)$ has a unitary factor if and only if $\lambda = (r^m)$ is a rectangle, with $r$ odd and $m$ even. In this case,*

$$C_{r,m}(x,q) = (1+xq^{\frac{rm}{2}})C'_{r,m}(x,q)$$

*and $C'_{r,m}(x,q)$ has no unitary factor.*

## 2.3. **Functional equations.**

**Proposition 2.12.** *For all $r, m \in \mathbb{N}$,*

$$C_{r,m}(x^{-1},q^{-1}) = x^{-r(m-1)}q^{-r^2\binom{m}{2}}C_{r,m}(x,q).$$

*If $\lambda$ is not a rectangle, then $C_\lambda(x,q)$ does not satisfy a functional equation of the form*

$$C_\lambda(x^{-1},q^{-1}) = x^{-d_1}q^{-d_2}C_\lambda(x,q), \tag{2.6}$$

*for $d_1, d_2 \in \mathbb{N}_0$.*

*Proof.* As $C_\lambda(x,1) \in \mathbb{Z}[x]$ has constant term 1, a necessary condition for $C_\lambda$ to satisfy a functional equation of the form (2.6) is that $C_\lambda(x,1)$ is monic. By Lemma 2.3, this holds if and only if $\lambda$ is a rectangle. This proves the second statement.

To establish the first statement, let $r, m \in \mathbb{N}$. For $i \in [r(m-1)]_0$, we set

$$C^{(i)}_{r,m}(q) = \sum_{\{w \in S_{r,m} \mid \mathrm{des}(w)=i\}} q^{\mathrm{maj}(w)} \in \mathbb{Z}[q],$$

so that $C_{r,m}(x,q) = \sum_{i=0}^{r(m-1)} C^{(i)}_{r,m}(q)x^i$. With the map $^\circ$ defined in (2.1), Lemma 2.5 yields

$$C^{(i)}_{r,m}(q^{-1}) = q^{-irm} \sum_{\substack{\{w \in S_{r,m} \mid \\ \mathrm{des}(w)=i\}}} q^{irm-\mathrm{maj}(w)} = q^{-irm} \sum_{\substack{\{w \in S_{r,m} \mid \\ \mathrm{des}(w)=i\}}} q^{\mathrm{maj}(w^\circ)} = q^{-irm}C^{(i)}_{r,m}(q).$$

Using this and the relations

$$C^{(r(m-1)-i)}_{r,m}(q) = q^{r^2\binom{m}{2}-irm}C^{(i)}_{r,m}(q)$$

(cf. [16, §461, Vol. 2, Ch. IV, Sect. IX]) we obtain

$$\begin{aligned} C_{r,m}(x^{-1},q^{-1}) &= \sum_{i=0}^{r(m-1)} C_{r,m}^{(i)}(q^{-1})x^{-i} = \sum_{i=0}^{r(m-1)} q^{-irm} C_{r,m}^{(i)}(q)x^{-i} \\ &= \sum_{i=0}^{r(m-1)} q^{-irm} q^{-r^2\binom{m}{2}+irm} C_{r,m}^{(r(m-1)-i)}(q)x^{-i} \\ &= q^{-r^2\binom{m}{2}} \sum_{j=0}^{r(m-1)} C_{r,m}^{(j)}(q)x^{-r(m-1)+j} \\ &= x^{-r(m-1)} q^{-r^2\binom{m}{2}} C_{r,m}(x,q). \end{aligned}$$ □

## 3. Proofs of Theorems 1.1 and 1.3

3.1. **Proof of Theorem 1.1.** Let $r \in \mathbb{N}$. Recall that $\zeta_{\mathbb{Z}^r}(s) = \prod_{p \text{ prime}} \zeta_{\mathbb{Z}_p^r}(s)$, where, for a prime $p$, the Euler factor $\zeta_{\mathbb{Z}_p^r}(s) = \sum_{k=0}^{\infty} a_{p^k}(\mathbb{Z}_p^r)p^{-ks}$ enumerates the $\mathbb{Z}_p$-submodules of finite additive index in $\mathbb{Z}_p^r$. Here, $\mathbb{Z}_p$ denotes the ring of $p$-adic integers.

**Lemma 3.1** (cf., e.g., [11]). *For all $k \in \mathbb{N}_0$,*

$$a_{p^k}(\mathbb{Z}_p^r) = \binom{r-1+k}{k}_p.$$

*Proof.* For a variable $t$,

$$\sum_{k=0}^{\infty} a_{p^k}(\mathbb{Z}_p^r)t^k = \frac{1}{\prod_{i=1}^{r}(1-p^{i-1}t)} = \sum_{k=0}^{\infty} \binom{r-1+k}{k}_p t^k; \tag{3.1}$$

see (1.1) for the first equality and, for instance, [23, Ch. 1.8] for the second. □

*Remark* 3.2. Proposition 1.2 follows from combining the second equality in (3.1) with (2.3).

Recall that for a map $T : X \to X$ we denote with $\mathsf{O}_T(n)$ the number of closed orbits of $T$ of length $n$. Let $\mathsf{F}_T(n) = |\{x \in X \mid T^n(x) = x\}|$ denote the number of points of period $n$. Then

$$\mathsf{F}_T(n) = \sum_{d|n} d\mathsf{O}_T(d) \tag{3.2}$$

and, by Möbius inversion,

$$\mathsf{O}_T(n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) \mathsf{F}_T(d). \tag{3.3}$$

From (3.2),

$$\mathsf{p}_T(s) := \sum_{n=1}^{\infty} \mathsf{F}_T(n)n^{-s} = \zeta(s)\mathsf{d}_T(s-1). \tag{3.4}$$

Let now $r \in \mathbb{N}$ and $T_r$ be a map with orbit Dirichlet series $\mathsf{d}_{T_r}(s) = \zeta_{\mathbb{Z}^r}(s)$ as in (1.1).

**Corollary 3.3.** *For all $k \in \mathbb{N}_0$,*

$$\mathsf{F}_{T_r}(p^k) = \sum_{j=0}^{k} p^j a_{p^j}(\mathbb{Z}_p^r) = \binom{r+k}{k}_p.$$

*Proof.* By (1.1) and (3.4), $\mathsf{p}_{T_r}(s) = \zeta(s)\prod_{i=0}^{r-1}\zeta(s+1-i) = \zeta_{\mathbb{Z}^{r+1}}(s)$. Together with Lemma 3.1, this yields the result. □

Recall that $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition. For all $n \in \mathbb{N}$ we have $\mathsf{F}_{T_{\lambda_1} \times \cdots \times T_{\lambda_m}}(n) = \prod_{i=1}^m \mathsf{F}_{T_{\lambda_i}}(n)$, so that, by Corollary 3.3, for a prime $p$ and $k \in \mathbb{N}_0$,

$$\mathsf{F}_{T_\lambda}(p^k) = \prod_{i=1}^m \binom{\lambda_i + k}{k}_p.$$

Using (3.3) we deduce, as in the proof of [17, Proposition 3.1], that, for $k > 0$,

$$\begin{aligned}\mathsf{O}_{T_\lambda}(p^k) &= \frac{1}{p^k} \sum_{d \mid p^k} \mu\left(\frac{p^k}{d}\right) \mathsf{F}_{T_\lambda}(d) \\ &= \frac{1}{p^k}\left(\mathsf{F}_{T_\lambda}(p^k) - \mathsf{F}_{T_\lambda}(p^{k-1})\right) \\ &= \frac{1}{p^k}\left(\prod_{i=1}^m \binom{\lambda_i + k}{k}_p - \prod_{i=1}^m \binom{\lambda_i + k - 1}{k-1}_p\right).\end{aligned}$$

Hence

$$\begin{aligned}\mathsf{d}_{T_\lambda,p}(s) = \sum_{k=0}^\infty \mathsf{O}_{T_\lambda}(p^k) t^k &= 1 + \sum_{k=1}^\infty \frac{1}{p^k}\left(\prod_{i=1}^m \binom{\lambda_i + k}{k}_p - \prod_{i=1}^m \binom{\lambda_i + k - 1}{k-1}_p\right) t^k \\ &= \left(1 - \frac{t}{p}\right) \sum_{k=0}^\infty \left(\prod_{i=1}^m \binom{\lambda_i + k}{k}_p\right)\left(\frac{t}{p}\right)^k.\end{aligned}$$

By substituting $(t/p, p)$ for $(x, q)$ in (2.3) and setting $t = p^{-s}$, this may be rewritten as

$$\mathsf{d}_{T_\lambda,p}(s) = \frac{C_\lambda(p^{-1-s}, p)}{\prod_{i=1}^N (1 - p^{i-1-s})}.$$

The second statement in (1.2) follows from (2.2). This concludes the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.3.** Given the expression (1.2) in Theorem 1.1, it is clear that a functional equation of the form (1.4) holds if and only if $C_\lambda(x, q)$ satisfies a functional equation of the form (2.6). By Proposition 2.12, this holds if and only if $\lambda$ is a rectangle. If $\lambda = (r^m)$, then, substituting $(p^{-1-s}, p)$ for $(x, q)$, this result implies that

$$C_{r,m}(p^{1+s}, p^{-1}) = p^{-r^2\binom{m}{2} + r(m-1) + sr(m-1)} C_{r,m}(p^{-1-s}, p). \tag{3.5}$$

The functional equation (1.3) holds, as

$$\frac{1}{\prod_{i=1}^{rm}(1 - p^{-i+1+s})} = (-1)^{rm} p^{\binom{rm+1}{2} - rm - srm} \frac{1}{\prod_{i=1}^{rm}(1 - p^{i-1-s})}$$

combined with (3.5) gives

$$\begin{aligned}\mathsf{d}_{T_r^{\times m},p}(s)|_{p \to p^{-1}} &= (-1)^{rm} p^{\binom{rm+1}{2} - rm - r^2\binom{m}{2} + r(m-1) - srm + sr(m-1)} \mathsf{d}_{T_r^{\times m},p}(s) \\ &= (-1)^{rm} p^{m\binom{r+1}{2} - r - rs} \mathsf{d}_{T_r^{\times m},p}(s).\end{aligned}$$

## 4. Analytic properties and asymptotics

In this section we exploit the combinatorial description of the Dirichlet series $d_{T_\lambda}(s)$ given in (1.2) to deduce some of their key analytic properties. Recall that $\lambda$ is a partition of $N = \sum_{i=1}^m \lambda_i$. In the following, $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

**Theorem 4.1.** (1) *The orbit Dirichlet series $\mathsf{d}_{T_\lambda}(s)$ has abscissa of convergence $N$. If $m = 1$ or $\lambda = (1, 1)$, then it may be continued meromorphically to the whole complex plane; otherwise it has meromorphic continuation to*

$$\{s \in \mathbb{C} \mid \mathrm{Re}(s) > N - 2\}.$$

*In any case, the continued function is holomorphic on the line* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = N\}$ *except for a simple pole at* $s = N$.

(2) *There exists a constant* $K_\lambda \in \mathbb{R}_{>0}$ *such that*

$$\sum_{\nu \leqslant n} O_{T_\lambda}(\nu) \sim K_\lambda n^N \quad \text{as } n \to \infty.$$

*Proof.* Recall that, for all $i \in \mathbb{N}_0$, the translate $\zeta(s-i)$ of Riemann's zeta function converges for $\mathrm{Re}(s) > i+1$ and may be continued meromorphically to the whole complex plane. The continued function is holomorphic on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = i+1\}$ except for a simple pole at $s = i+1$. This establishes all claims in (1) if $m = 1$ or $\lambda = (1,1)$, as $\mathsf{d}_{T_r}(s) = \zeta_{\mathbb{Z}^r}(s)$ and $\mathsf{d}_{T_{(1,1)}}(s) = \frac{\zeta(s)^2\zeta(s-1)}{\zeta(2s)}$.

Assume thus that $m \geqslant 2$ and $\lambda \neq (1,1)$ and recall the expression (1.2) for $\mathsf{d}_{T_\lambda}(s)$. The product $\prod_{p \text{ prime}} C_\lambda(p^{-1-s}, p)$ has abscissa of convergence $N-1$ and may be meromorphically continued to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > N-2\}$. Indeed, an Euler product of the form

$$\prod_{p \text{ prime}} \left(1 + \sum_{(i,k) \in I} p^{i-ks}\right),$$

where $I \subset \mathbb{N}_0 \times \mathbb{N}$ is a finite (multi-)set, converges on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$, where

$$\alpha = \max\left\{\frac{i+1}{k} \mid (i,k) \in I\right\},$$

and has a meromorphic continuation to $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \beta\}$, where

$$\beta = \max\left\{\frac{i}{k} \mid (i,k) \in I\right\};$$

see [6, Lemmas 5.4 and 5.5]. It follows from inspection of the Euler product

$$\prod_{p \text{ prime}} C_\lambda(p^{-1-s}, p) = \prod_{p \text{ prime}} \left(\sum_{w \in S_\lambda} \prod_{j \in \mathrm{Des}(w)} p^{j-1-s}\right) \tag{4.1}$$

that the relevant maxima $\alpha = N-1$ resp. $\beta = N-2$ are both attained at the elements $w \in S_\lambda$ with $\mathrm{Des}(w) = \{N-1\}$. We note that

$$\#\{w \in S_\lambda \mid \mathrm{Des}(w) = \{N-1\}\} = m-1. \tag{4.2}$$

As

$$\prod_{p \text{ prime}} \frac{1}{\prod_{i=1}^N (1-p^{i-1-s})} = \prod_{i=1}^N \zeta(s-i+1)$$

has abscissa of convergence $N > \alpha$, this concludes the proof of (1).

(2) follows from (1), for instance via the Tauberian theorem [5, Theorem 4.20]. □

If $m > 1$ and $\lambda \neq (1,1)$, then the meromorphic continuation to $\beta = N-2$ is often – and conjecturally always – best possible, as we now prove.

**Theorem 4.2.** *Assume that* $\lambda \neq (1,1)$ *and that either*

(i) $m > 2$ *or*
(ii) $m = 2$ *and* $\lambda_1 = \lambda_2$ *or*
(iii) $m = 2$, $\lambda_1 > \lambda_2$, *and Conjecture A holds.*

*Then the orbit Dirichlet series* $\mathsf{d}_{T_\lambda}(s)$ *has a natural boundary at*

$$\{s \in \mathbb{C} \mid \mathrm{Re}(s) = N-2\}.$$

*Proof.* We keep the notation as in Theorem 4.1 and set

$$W^\lambda(X,Y) = C_\lambda(X^{-1}Y, X) = \sum_{(i,k)\in I_\lambda} c_{i,k} X^i Y^k \in \mathbb{Z}[X,Y]$$

for suitable $I_\lambda \subseteq \mathbb{N}_0^2$ and $c_{i,k} \in \mathbb{N}$. The Euler product (4.1) then reads

$$\prod_{p \text{ prime}} W^\lambda(p, p^{-s}). \tag{4.3}$$

To prove that under any of the assumptions (i)-(iii), the Euler product (4.3) has a natural boundary at $\beta = N-2$ we consider the *ghost polynomial* associated to $W^\lambda$ and prove that $W^\lambda$ is a polynomial of *Type I* (in case (i)) or *Type II* (in cases (ii) and (iii)) in the terminology of [6, Section 5.2].

We claim that the first factor of the ghost polynomial of $W^\lambda(X,Y)$ is, in notation close to the one used in [6, Section 5.2],

$$\widetilde{W}_1^\lambda(X,Y) = \sum_{(i,k)\in \mathfrak{l}_1\cap I_\lambda} c_{i,k} X^i Y^k = 1 + (m-1)X^\beta Y. \tag{4.4}$$

Here, $\mathfrak{l}_1$ is the line in $\mathbb{R}^2$ through $(0,0)$ and $(\beta,1)$. It is characterized by the fact that its gradient $1/\beta$ is minimal among the lines in $\mathbb{R}^2$ passing through $(0,0)$ and the points $(i,k) \in I_\lambda \backslash \{(0,0)\}$. Moreover, $\mathfrak{l}_1 \cap I_\lambda = \{(0,0),(\beta,1)\}$ and $c_{\beta,1} = m-1$ (cf. (4.2)), which proves (4.4). Setting $U = X^\beta Y$, we obtain

$$\widetilde{W}_1^\lambda(U) = 1 + (m-1)U \in \mathbb{Z}[U].$$

If $m > 2$, then $\widetilde{W}_1^\lambda(U)$ is not cyclotomic, whence $W^\lambda(X,Y)$ is a polynomial of Type I in the parlance of [6, p. 127]. Without loss of generality we may divide $W^\lambda$ by any unitary factors it may have; cf. Conjecture B. Indeed, if $W^\lambda = fV^\lambda$ for $f \in \mathbb{Z}[X,Y]$ unitary, then the Newton polygon of $W^\lambda$ is the Minkowski sum of the Newton polygons of $f$ and $V^\lambda$. The former, however, is a segment of a line in $\mathbb{R}^2$. As $\widetilde{W}_1^\lambda$ does not have a unitary factor, the slope of this line is strictly larger than $1/\beta$, whence $\widetilde{W}_1^\lambda = \tilde{V}_1^\lambda$, i.e. the first factors of the ghosts of $W^\lambda$ and $V^\lambda$ coincide.

Assuming thus, as we may, that $W^\lambda$ has no unitary factors, [6, Theorem 5.6] yields that $\beta$ is a natural boundary for (4.3) and thus for $\mathsf{d}_{T_\lambda}(s)$. This concludes the proof in case (i).

Turning to cases (ii) and (iii) we now assume that $m = 2$. Hence $\widetilde{W}_1^\lambda(U) = 1 + U$ *is* cyclotomic. In particular, $W^\lambda(X,Y)$ is not of Type I. We claim that it is a polynomial of Type II in the sense of [6, p. 127]. To prove this, we check that the hypotheses of [6, Corollary 5.15] are satisfied. To this end, we verify that $W^\lambda(X,Y)$ is such that Hypotheses 1 and 2 defined on [6, p. 134] are satisfiable. The polynomial $A(U) = 1 + \sum_{\frac{n_k}{k}=\beta} c_{n_k,k} U^k = 1 + U$ (cf. [6, p. 134]) obviously has a unique root $\omega = -1$. It is simple, so in particular satisfies Hypothesis 1.

Hypothesis 2 is equivalent to $\mathrm{Re}\left(-\frac{B_\gamma(\omega)}{\omega A'(\omega)}\right) < 0$ hence to $B_\gamma(\omega) < 0$, where

$$\gamma := \min\{n \in \mathbb{N}_0 \mid B_n(\omega) \neq 0\}$$

and, for $n \in \mathbb{N}_0$ and $(n_j, j) \in I_\lambda$ such that $n_j/j = \beta$ and $j$ is minimal with this property,

$$B_n(U) = \sum_{n_j k - ij = n} c_{i,k} U^k = \sum_{\beta k - i = n} c_{i,k} U^k;$$

cf. [6, (5.12)]. Note that $B_0(U) = A(U) = 1 + U$.

Recall that

$$W^\lambda(X,Y) = \sum_{w\in S_\lambda} X^{\mathrm{maj}(w)-\mathrm{des}(w)} Y^{\mathrm{des}(w)} = \sum_{(i,k)\in I_\lambda} c_{i,k} X^i Y^k.$$

For $(i,k) \in I_\lambda$, we thus have $c_{i,k} = \#\{w \in S_\lambda \mid k = \mathrm{des}(w),\ i = \mathrm{maj}(w) - k\}$.

We claim that $\gamma = 1$. If $(i,k)$ satisfies

$$\beta k - i = 1, \tag{4.5}$$

then clearly $k \neq 0$. If $k = 2$, then (4.5) necessitates $i = 2N - 5$, that is $\mathrm{maj}(w) = 2N - 3$. But there is no element $w \in S_\lambda$ such that $\mathrm{des}(w) = 2$ and $\mathrm{maj}(w) = 2N - 3$. Indeed, such an element would need to have descents at the consecutive positions $N-2$ and $N-1$ (as $\mathrm{maj}(w) = 2N - 3 = (N-1) + (N-2)$), which is clearly impossible for a word in $\{\underbrace{\mathbf{1},\dots,\mathbf{1}}_{\lambda_1},\underbrace{\mathbf{2},\dots,\mathbf{2}}_{\lambda_2}\}$. A similar argument excludes pairs $(i,k)$ that satisfy (4.5) and for which $k > 2$. To determine $B_1(U)$ we thus need to determine

$$c_{\beta-1,1} = \#\{w \in S_\lambda \mid \mathrm{des}(w) = 1,\ \mathrm{maj}(w) = N - 2\},$$

i.e. to enumerate the multiset permutations with no descent in the first $N-3$ positions and ending in $\dots\mathbf{212}$ or $\dots\mathbf{211}$. If $\lambda_2 > 1$, then there are exactly two such words; if $\lambda_2 = 1$, then only the second option occurs. So $B_1(U) = 2U$ resp. $B_1(U) = U$. In any case, $\gamma = 1$ and $B_\gamma(\omega) = -2 < 0$ resp. $B_\gamma(\omega) = -1 < 0$. Hence Hypothesis 2 is satisfied.

Since Hypotheses 1 and 2 are satisfied and $1 = \gamma \geqslant j = 1$, [6, Corollary 5.15] implies that $W^\lambda(X,Y)$ is of Type II. Thus in case (ii) for $\lambda_1 = \lambda_2$ odd, and in case (iii), as $W^\lambda(X,Y)$ has no unitary factors, [6, Theorem 5.6] yields that $\beta$ is a natural boundary for (4.3) and thus for $\mathsf{d}_{T_\lambda}(s)$. In case (ii) for $\lambda_1 = \lambda_2$ even, Proposition 2.11 asserts that a unique unitary factor exists: $W^\lambda(X,Y) = (1+X^{\lambda_1-1}Y)W'^\lambda(X,Y)$ for some $W'^\lambda \in \mathbb{Z}[X,Y]$. But $\beta = N-2 > \lambda_1 - 1$, so the minimal gradient for $W'^\lambda(X,Y)$ is still $\beta$. Thus also in this case [6, Theorem 5.6] implies that $\beta$ is a natural boundary. ☐

## 5. Connection with Igusa functions and the special case $\lambda = (1^m)$

Taking $\lambda = (1^m)$ corresponds to considering the $m$-th power of a map $T = T_1$ whose orbit Dirichlet series $\mathsf{d}_T(s)$ is the Riemann zeta function $\zeta(s)$. In this case, Theorem 1.1 reads

$$\mathsf{d}_{T^{\times m}}(s) = \prod_{p \text{ prime}} \frac{C_m(p^{-1-s},p)}{\prod_{i=1}^m (1-p^{i-1-s})} = \prod_{p \text{ prime}} \frac{\sum_{w\in S_m} p^{(-1-s)\,\mathrm{des}(w)+\mathrm{maj}(w)}}{\prod_{i=1}^m (1-p^{i-1-s})}.$$

where $C_m(x,q) = C_{1,m}(x,q)$ is Carlitz's $q$-Eulerian polynomial.

More generally one may, for $a \in \mathbb{R}_{\geqslant 0}$, consider a map ${}_aT$ such that $\mathsf{d}_{{}_aT}(s) = \zeta(s-a)$. Then

$$\mathsf{O}_{{}_aT}(p^k) = p^{ak} \quad \text{and} \quad \mathsf{F}_{{}_aT}(p^k) = \sum_{j=0}^k p^j p^{aj} = \binom{1+k}{1}_{p^{a+1}}.$$

The orbit Dirichlet series of the $m$-th power of ${}_aT$ is thus MacMahon's generating series (2.3) for $\lambda = (1^m)$ and $(x,q) = (p^{-1-s},p^{a+1})$:

$$\mathsf{d}_{{}_aT^{\times m}}(s) = \prod_{p \text{ prime}} \frac{C_m(p^{-1-s},p^{a+1})}{\prod_{i=1}^m (1-p^{(a+1)i-1-s})} = \prod_{p \text{ prime}} \frac{\sum_{w\in S_m} p^{(-1-s)\,\mathrm{des}(w)+(a+1)\,\mathrm{maj}(w)}}{\prod_{i=1}^m (1-p^{(a+1)i-1-s})}. \tag{5.1}$$

A formula for $\mathsf{d}_{{}_aT^{\times m}}(s)$ appears in [17, p. 41], where it is called $E_p(s)$ and suffers from a transcript error in the definition of the expression $A_b$. Moreover, no combinatorial interpretation is given. [17, Theorem 4.1] is Theorem 4.2 in the special case $\lambda = (1,1,1)$.

Each factor of the Euler product (5.1) is an instance of an Igusa function:

$$\begin{aligned}\mathsf{d}_{{}_aT^{\times m},p}(s) &= \frac{C_m(p^{-1-s},p^{a+1})}{\prod_{i=1}^m(1-p^{(a+1)i-1-s})} = \frac{\sum_{w\in S_m}\prod_{j\in\mathrm{Des}(w)}p^{(a+1)j-1-s}}{\prod_{i=1}^m(1-p^{(a+1)i-1-s})}\\ &= \frac{1}{1-p^{(a+1)m-1-s}}\sum_{I\subseteq[m-1]}\binom{m}{I}\prod_{i\in I}\frac{p^{(a+1)i-1-s}}{1-p^{(a+1)i-1-s}}\in\mathbb{Q}(p,p^{-s}).\end{aligned}$$

In the terminology of [19, Definition 2.5] it would be called $I_m(1;(p^{(a+1)i-1-s})_{i=1}^m)$ and so (1.3) in Theorem 1.3 follows in this case from [19, Proposition 4.2].

In the case of a general partition, we are not aware of a simple expression of the local factors of the orbit Dirichlet series of the product of such "shifted maps". Turning back to the case $a=0$ and general partition $\lambda$, the local factors of (1.2) may be rewritten as

$$\mathsf{d}_{T_\lambda,p}(s) = \frac{1}{1-p^{N-1-s}}\sum_{I\subseteq[N-1]}\nu_{\lambda,I}\prod_{i\in I}\frac{p^{i-1-s}}{1-p^{i-1-s}}\in\mathbb{Q}(p,p^{-s})$$

where $\nu_{\lambda,I}=\#\{w\in S_\lambda\mid \mathrm{Des}(w)\subseteq I\}$. We are not aware of a simple expression, say in terms of multinomial coefficients, for $\nu_{\lambda,I}$ if $\lambda$ is not of the form $(1^m)$.

## 6. Reduced orbit Dirichlet series: setting $p=1$

Viewing the Euler factors of (1.2) as bivariate rational functions in $p$ and $t=p^{-s}$, one may evaluate them at $p=1$ whilst leaving $t$ as an independent variable. Motivated by the notion of *reduced zeta functions of Lie algebras* introduced in [8] we thus define the *reduced orbit Dirichlet series*

$$\mathsf{d}_{T_\lambda,\mathrm{red}}(t) := \frac{C_\lambda(t,1)}{(1-t)^N}\in\mathbb{Q}(t).$$

It seems remarkable that for $\lambda=(1^m)$ the reduced orbit Dirichlet series $\mathsf{d}_{T_1^{\times m},\mathrm{red}}(t)$ is the Hilbert series of the Stanley-Reisner ring of a simplicial complex. Indeed, let $k$ be any field, write $\mathrm{sd}(\Delta_{m-1})$ for the barycentric subdivision of the $(m-1)$-simplex $\Delta_{m-1}$ — or, equivalently, the Coxeter complex of type $A_{m-1}$ — , with Stanley-Reisner (or face) ring $k[\mathrm{sd}(\Delta_{m-1})]$; see, for instance, [22, Ch. III, Sec. 4]. The fact that the $m$-th Eulerian polynomial (cf. (2.4)) is the generating function of the $h$-vector of $\mathrm{sd}(\Delta_{m-1})$ is reflected in the following fact.

**Proposition 6.1.**

$$\mathsf{d}_{T_1^{\times m},\mathrm{red}}(t) = \frac{A_m(t)/t}{(1-t)^m} = \mathrm{Hilb}(k[\mathrm{sd}(\Delta_{m-1})],t).$$

Similarly, we observe that for $\lambda=(r,r)$, the polynomial $C_{r,2}(t,1)$ may be viewed as the $h$-vector of the $r$-dimensional type-B simplicial associahedron $Q_n^{\mathrm{B}}$; cf. [21, Corollary 1].

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