The s-tame dimension vectors for stars

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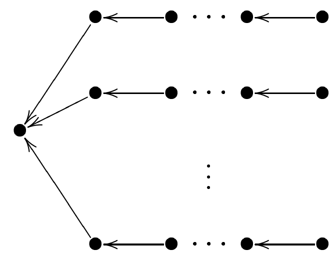
1 Introduction

A *quiver* $Q = (Q_0, Q_1, s, t)$ is given by a set Q_0 of vertices, a set Q_1 of arrows and two maps $s, t : Q_1 \to Q_0$ assigning to every arrow $\alpha \in Q_1$ its *starting point* $s(\alpha)$ and its *terminating point* $t(\alpha)$.

A *star* is a quiver of the following shape:



This means taking k linearly ordered quivers of type $\mathbb{A}$ with all arrows going in one direction and identifying them at the end points.

A star of type $T_{p_1,\dots,p_k}$ has k arms of type $\mathbb{A}$, where the i-th arm $(i = 1, \dots, k)$ contains exactly p_i points (including the central point).

A *representation* $(V_i, V_\alpha)_{i\in Q_0,\alpha\in Q_1}$ of a quiver Q is given by vector spaces V_i for each $i \in Q_0$ over a particular field K and K-linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ for each $\alpha \in Q_1$. A representation can be viewed as an element in

$$\prod_{\alpha\in Q_1} \mathrm{Hom}_K(V_{s(\alpha)}, V_{t(\alpha)}).$$

A representation $(V_i, V_\alpha) \neq 0$ is called *indecomposable*, if it is not a sum of two non zero representations, i. e. for all $(V_i^{(1)}, V_\alpha^{(1)})$, $(V_i^{(2)}, V_\alpha^{(2)})$ with $(V_i, V_\alpha) = (V_i^{(1)}, V_\alpha^{(1)}) \oplus (V_i^{(2)}, V_\alpha^{(2)})$ either $(V_i^{(1)}, V_\alpha^{(1)}) = 0$ or $(V_i^{(2)}, V_\alpha^{(2)}) = 0$.

By a *subspace representation* of a star Q we mean a representation that contains only injective maps. Every representation of a star which contains only injective maps is isomorphic to a representation, where the vector spaces along the quiver's arms are contained in one another by using the natural isomorphisms of n-dimensional vector spaces over K with K^n.

A *morphism* $f = (f_i)_{i\in Q_0}$ between two representations $V = (V_i, V_\alpha)_{i\in Q_0,\alpha\in Q_1}$ and $W = (W_i, W_\alpha)_{i\in Q_0,\alpha\in Q_1}$ of Q is given by K-linear maps $f_i : V_i \to W_i$, $i \in Q_0$, satisfying the conditions

$$f_{t(\alpha)} \circ V_\alpha = W_\alpha \circ f_{s(\alpha)} \ \forall \alpha \in Q_1$$

and can be viewed as an element in

$$\prod_{i\in Q_0} \mathrm{Hom}_K(V_i, W_i).$$

The set of representations of a quiver Q is denoted by $\operatorname{rep} Q$ and the set of subspace representations by $\operatorname{rep}_{\text{inj}} Q$.

From now on we consider only quivers with finitely many vertices, finitely many arrows and all vector spaces being finite dimensional over the field K.

The map

$$\underline{\dim} : \operatorname{rep} Q \to \mathbb{Z}^{Q_0}$$

assigns to every representation $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ its *dimension vector* $(\dim_K V_i)_{i \in Q_0}$.

We know by Gabriel (see [6]) that there are only finitely many isomorphism classes of indecomposable representations for a star Q if and only if the underlying graph $\overline{Q}$ of Q is contained in Table 1. And for Q contained in Table 2 there are infinitely many isomorphism classes of indecomposable representations, but one can characterise the behaviour of the representations in a nice way:

There are no two parameter families of indecomposable representations, and the dimension vectors of the one parameter families of indecomposable representations are exactly the integral multiples of the critical dimension vector corresponding to the quiver (see Table 4). The classification (up to isomorphism) of all indecomposable representations of the tame quivers was done by Dlab and Ringel in 1976 (see [5]).

A dimension vector $\mathbf{d}$ of representations of a star Q is called an *s-vector*, if there is a subspace representation with this dimension vector. Every s-vector must be increasing along its arms, and for every dimension vector of a representation of a star that is increasing along its arms, there exists a (not necessarily indecomposable) subspace representation. So it is equivalent to say that $\mathbf{d}$ is increasing along its arms.

The set of all dimension vectors for a quiver Q is denoted by $D(\operatorname{rep} Q)$ and the set of all s-vectors for Q by $D(\operatorname{rep}_{\text{inj}} Q)$.

The following can be shown (see Chapter 2): If the central dimension of a dimension vector for a star is not zero and we have an indecomposable representation with this dimension vector, then this is already a subspace representation. Therefore, the indecomposable representations of s-vectors are always subspace representations.

An *s-decomposition* of an s-vector $\mathbf{d}$ is a decomposition $\mathbf{d} = \sum_{i=1}^{r} \mathbf{d_i}$ (the addition is carried out componentwise) such that each of the summands $\mathbf{d_i}$ is an s-vector.

The s-vectors are compared as follows: If there is an s-vector $\mathbf{d_2} \neq 0$ with $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$, then $\mathbf{d_1}$ is called *smaller* than $\mathbf{d}$, and $\mathbf{d}$ is called *bigger* than $\mathbf{d_1}$ and $\mathbf{d_2}$.

The maximal number of parameters for families of representations of a certain dimension vector and the existence of indecomposable representations for a dimension vectors are closely related to its Tits form.

The Tits form $q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ of a quiver $Q = (Q_0, Q_1, s, t)$ is given by

$$q(\mathbf{d}) := \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}, \ \ \mathbf{d} \in \mathbb{Z}^{Q_0}.$$

For an algebraically closed field, Kac has shown the following:

If $\mathbf{d}$ is a positive root (see Section 8.2), the number $1 - q(\mathbf{d})$ is exactly the maximal number of parameters needed for families of indecomposable representations with dimension vector $\mathbf{d}$ and there is a unique indecomposable representation with dimension vector $\mathbf{d}$ if and only if $q(\mathbf{d}) = 1$. And if $\mathbf{d}$ is not a root, there is no indecomposable representation with this dimension vector.

By an *s-finite* dimension vector we mean an s-vector with the property that there are only finitely many isomorphism classes of subspace representations with this s-vector. The classification of the s-finite dimension vectors has been done by Magyar, Weyman, and Zelevinsky, and can be found in [13]. In particular, an s-vector $\mathbf{d}$ is s-finite if and only if the following two conditions hold:

1. $q(\mathbf{d}) = 1$, and
2. if $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ is an s-decomposition, then $q(\mathbf{d_1}) \geq 1$ and $q(\mathbf{d_2}) \geq 1$, where q is the Tits form corresponding to the star.

The next question arising from this is: If there are infinitely many isomorphism classes of subspace representations, what is the maximal number of parameters needed for families of subspace representations with this dimension vector? Our aim is to find all s-vectors $\mathbf{d}$ with the following two properties:

(i) There is a one parameter family of indecomposable subspace representations for $\mathbf{d}$, and

(ii) for every s-decomposition there is never an n-parameter family of indecomposable subspace representations with $n \geq 2$ for either of the summands.

An s-vector $\mathbf{d}$ is called *s-hypercritical*, if

1. $q(\mathbf{d}) < 0$, and
2. for every non trivial s-decomposition $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ the conditions $q(\mathbf{d_1}) \geq 0$ and $q(\mathbf{d_2}) \geq 0$ are fulfilled.

These will turn out to be the minimal dimension vectors (w.r.t. s-decompositions) for which there exists an n-parameter family of subspace representations with $n \geq 2$.

So if we want to find the s-vectors with the properties (i) and (ii), we have to search them among the s-tame ones, where we say that an s-vector $\mathbf{d}$ is *s-tame*, if the following two conditions hold:

1. $q(\mathbf{d}) = 0$, and
2. if $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ is an s-decomposition, then $q(\mathbf{d_1}) \geq 0$ and $q(\mathbf{d_2}) \geq 0$.

It will turn out that the s-tame dimension vectors are exactly the s-vectors with the properties (i) and (ii).

We see that (by definition) every s-vector which is smaller than an s-tame one has either Tits form ≥ 1 or is also s-tame.

s-finite, s-tame and s-hypercritical dimension vectors will always be dimension vectors of *subspace representations.*

This text is organised as follows: In Chapter 2 we are going to show that all indecomposable representations of the s-vectors are already subspace representations. Then — in Chapter 3 — we rewrite the s-vectors in terms of tuples of compositions and calculate the Tits form for the tuples of compositions. Chapter 4 shows the main properties of the Tits form for the tuples of compositions and gives the basics for the classifications of all s-hypercritical and s-tame vectors which are done in Chapter 5.

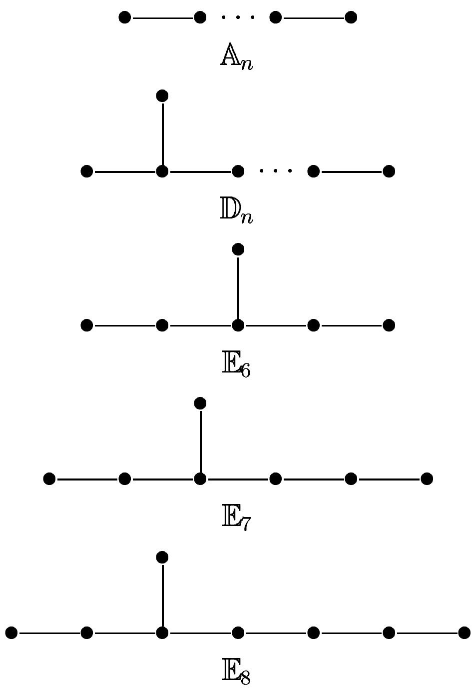
In Chapter 6 we will see how in the cases of non tame quivers families of decomposable representations can be constructed for the s-tame dimension vectors. Here it is shown that — in contrast to the situation of tame quivers — it is not possible to decompose an s-tame dimension vector into the sum of two s-tame dimension vectors.

Chapter 7 gives an overview over the reflection functors introduced by Bernstein, Gel'fand and Ponomarev (in [2]) and shows how one can reduce the problem of finding families of representations. In Chapter 8 one can find a summary of results of Kac (from [9], [10], [12] and [11]) and the proof for the existence of one parameter families of indecomposable representations for the s-tame vectors and n-parameter families for the s-hypercritical ones with $n \geq 2$. (Basically, one has to show that the s-tame and s-hypercritical dimension vectors are indeed roots.) In Chapter 9 the main result of this text is proven:

Theorem. *Let K be an algebraically closed field. Then the following assertions are equivalent:*

- *A dimension vector $\mathbf{d}$ is s-tame.*
- *There is a one parameter family of indecomposable subspace representations (over K) for $\mathbf{d}$, but for every s-decomposition there is never an n-parameter family of indecomposable subspace representations (over K) with $n \geq 2$ for either of the summands.*

Table 1: Stars of finite type – Stars of Dynkin type



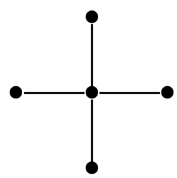
Furthermore, a characterisation of the s-hypercritical dimension vectors is given.

Chapter 10 shows that not all s-tame dimension vectors are tame. In Chapter 11 there are listed some construction methods for families of indecomposable representations. Basically, the methods work as follows: One restricts the given dimension vector to a dimension vector of a smaller quiver, decomposes the new one into a sum of dimension vectors of indecomposable (subspace) representations, chooses a suitable realisation of the representation restricted to the smaller quiver and then embeds a vector space into this realisation in an appropriate way.

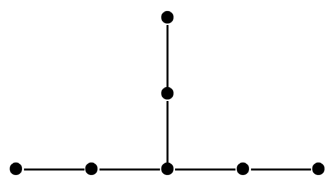
After that (in Chapter 12) the s-hypercritical and the s-tame vectors are sorted into orbits under the action of the reflection functors introduced in Chapter 7. And finally, in Chapter 13, the families of indecomposable representations for the s-hypercritical and the s-tame dimension vectors are constructed explicitly.

The appendices show that the results for the values of the Tits form for stars of finite and tame type can also be obtained by using the methods introduced in Chapter 4.

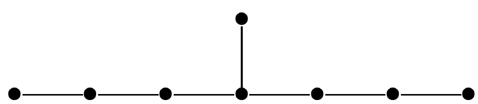
Table 2: Tame stars – Stars of Euclidean type



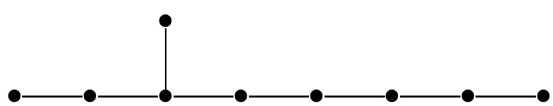
$\tilde{\mathbb{D}}_4$



$\tilde{\mathbb{E}}_6$

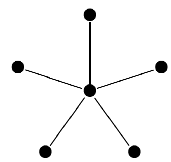


$\tilde{\mathbb{E}}_7$

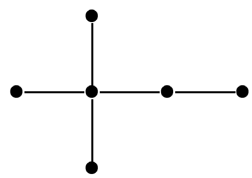


$\tilde{\mathbb{E}}_8$

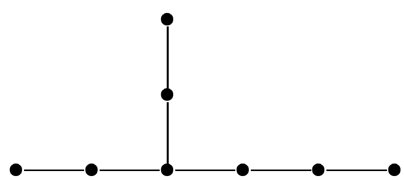
Table 3: Hyperbolic stars



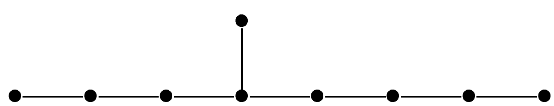
$T_{2,2,2,2,2}$



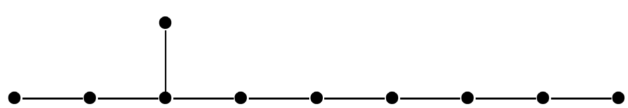
$T_{2,2,2,3}$



$T_{3,3,4}$

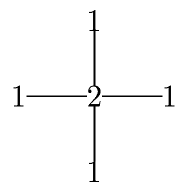


$T_{2,4,5}$

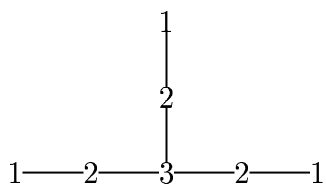


$T_{2,3,7}$

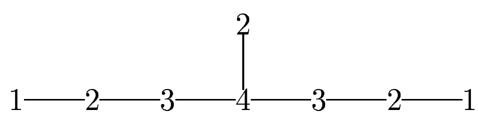
Table 4: Critical dimension vectors for the tame stars



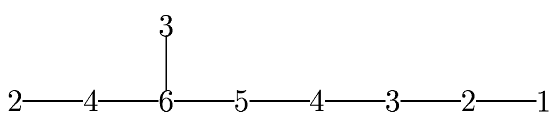
$\tilde{\mathbb{D}}_4$



$\tilde{\mathbb{E}}_6$



$\tilde{\mathbb{E}}_7$

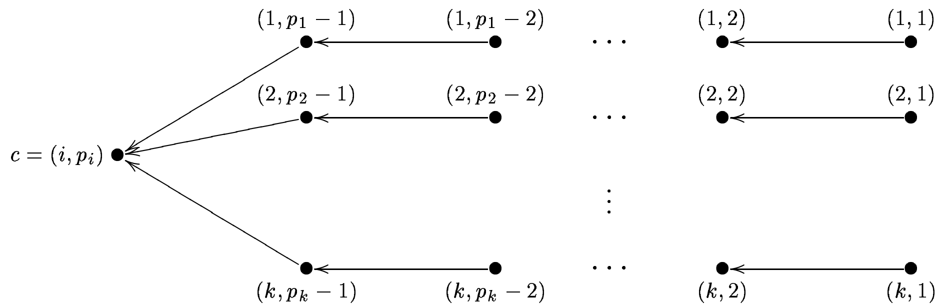


$\tilde{\mathbb{E}}_8$

2 Subspace representations

Lemma 2.1. *Let* $\mathbf{d} \in D(\operatorname{rep} Q)$*, where* Q *is a star, and let* d_c *denote the central dimension of* $\mathbf{d}$*. If* $d_c \neq 0$ *and* (V_i, V_α) *is an indecomposable representation of* Q *with dimension vector* $\mathbf{d}$*, then* (V_i, V_α) *is already a subspace representation.*

Proof. We enumerate the points in the quiver as follows:



It suffices to show that a representation $(V_{(i,j)}, V_\alpha)_{(i,j)\in Q_0, \alpha\in Q_1}$ with exactly one non injective map V_β for some $\beta \in Q_1$ has to be decomposable.

Let $V_\beta : V_{(l,m)} \to V_{(l,m+1)}$. If V_β is not injective, then $\operatorname{Ker} V_\beta \neq 0$ and

$$V_{(l,m)} \cong V_{(l,m)}/\operatorname{Ker} V_\beta \oplus \operatorname{Ker} V_\beta.$$

Then every vector space $V_{(l,o)}$ which is mapped into $V_{(l,m)}$ by a sequence of injective maps $V_{\beta_o}, \dots, V_{\beta_{m-1}}$ has also a decomposition

$$V_{(l,o)} \cong V_{(l,o)}/\operatorname{Ker}(V_\beta \circ V_{\beta_{m-1}} \circ \dots \circ V_{\beta_o}) \oplus \operatorname{Ker}(V_\beta \circ V_{\beta_{m-1}} \circ \dots \circ V_{\beta_o}).$$

So

$$(V_{(i,j)}, V_\alpha) \cong (W^{(1)}_{(i,j)}, W^{(1)}_\alpha) \oplus (W^{(2)}_{(i,j)}, W^{(2)}_\alpha)$$

with

$$W^{(1)}_{(i,j)} := V_{(i,j)} \text{ for } (i,j) \notin \{(l,1), \dots, (l,m)\},$$

$$W^{(1)}_{(i,j)} := V_{(i,j)}/\operatorname{Ker}(V_\beta \circ V_{\beta_{m-1}} \circ \dots \circ V_{\beta_j}) \text{ for } (i,j) \in \{(l,1), \dots, (l,m)\},$$

$$W^{(2)}_{(i,j)} := 0 \text{ for } (i,j) \notin \{(l,1), \dots, (l,m)\},$$

$$W^{(2)}_{(i,j)} := \operatorname{Ker}(V_\beta \circ V_{\beta_{m-1}} \circ \dots \circ V_{\beta_j}) \text{ for } (i,j) \in \{(l,1), \dots, (l,m)\}$$

as vector spaces and maps acting as

$$W^{(1)}_\alpha := V_\alpha \text{ for } s(\alpha) \notin \{(l,1), \dots, (l,m)\},$$

$$W^{(1)}_\alpha := \overline{V_\alpha} \text{ for } s(\alpha) \in \{(l,1), \dots, (l,m)\},$$

$$W_\alpha^{(2)} := 0 \text{ for } s(\alpha) \notin \{(l,1),\dots,(l,m)\},$$
$$W_\alpha^{(2)} := V_\alpha \text{ for } s(\alpha) \in \{(l,1),\dots,(l,m)\},$$

where $\overline{V_\alpha}$ denotes the induced map on the corresponding quotient space. □

Remark 2.2. *For this lemma it is essential that the arrows in the arms of the star are ordered linearly. We can for example choose the standard bases in K and K^2 and take the representation*

$$\begin{array}{ccccc} K & \xrightarrow{\binom{1}{0}} & K^2 & \xleftarrow{\binom{0}{1}} & K \\ & & \big\downarrow{\scriptstyle (1,1)} & & \\ & & K & & , \end{array}$$

which is indecomposable, but the map $K^2 \xrightarrow{(1,1)} K$ *is obviously* not *injective.*

3 Correspondence of s-vectors and tuples of compositions

3.1 s-vectors and tuples of compositions

The set of the natural numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$, whereas the set of natural numbers including zero is denoted by $\mathbb{N}_0$.

Definition 3.1. Let $p \in \mathbb{N}$. A tuple $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{N}_0^p$ is called a *composition* of a number $n \in \mathbb{N}_0$, if $\sum_{j=1}^{p} a_j = n$. If $\mathbf{a} \in \mathbb{N}^p$, we say that it is a *strict composition* of n.

If $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{N}_0^p$, we write for short

$$|\mathbf{a}| = \sum_{j=1}^{p} a_j,$$

$$\|\mathbf{a}\| = \sum_{j=1}^{p} a_j^2$$

and

$$\min \mathbf{a} = \min\{a_j \mid j = 1, \ldots, p\}.$$

The set of k-tuples of compositions of a number, where the compositions have lengths $p_1, \ldots, p_k \in \mathbb{N}$, $k \in \mathbb{N}$, is denoted by $D_{p_1,\ldots,p_k}$:

$$D_{p_1,\ldots,p_k} = \{(\mathbf{a_1}, \ldots, \mathbf{a_k}) \in \mathbb{N}_0^{p_1} \times \cdots \times \mathbb{N}_0^{p_k} \mid |\mathbf{a_1}| = \ldots = |\mathbf{a_k}|\}$$

Proposition 3.2. *There is a bijection $\sigma : D_{p_1,\ldots,p_k} \to D(\mathrm{rep_{inj}}\, Q)$ from the k-tuples of compositions to the s-vectors of a star Q of type $T_{p_1,\ldots,p_k}$ which is given by the following formula*

$$\sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}) = \begin{array}{c} \\ \\ |\mathbf{a_i}| \swarrow \leftarrow \nwarrow \\ \\ \\ \end{array} \begin{array}{cccccc} \sum_{j=1}^{p_1-1} a_{1j} & \cdots & a_{11} + a_{12} & \longleftarrow & a_{11} \\ \sum_{j=1}^{p_2-1} a_{2j} & \cdots & a_{21} + a_{22} & \longleftarrow & a_{21} \\ & \vdots & & & \\ \sum_{j=1}^{p_k-1} a_{kj} & \cdots & a_{k1} + a_{k2} & \longleftarrow & a_{k1} \end{array}$$

with the property

$$\sigma((\mathbf{a_1}, \ldots, \mathbf{a_k}) + (\mathbf{b_1}, \ldots, \mathbf{b_k})) = \sigma((\mathbf{a_1}, \ldots, \mathbf{a_k})) + \sigma((\mathbf{b_1}, \ldots, \mathbf{b_k})).$$

Proof. The inverse map is given by

$$\sigma^{-1}\left(\begin{array}{ccccc} & & d_{1,p_1-1} & \cdots & d_{12} \longleftarrow d_{11} \\ & \swarrow & & & \\ d_{k,p_k} & \longleftarrow & d_{2,p_2-1} & \cdots & d_{22} \longleftarrow d_{21} \\ & \nwarrow & & \vdots & \\ & & d_{k,p_k-1} & \cdots & d_{k2} \longleftarrow d_{k1} \end{array}\right)$$

$$= ((d_{11}, \dots, d_{1,p_1-1} - d_{1,p_1-2}, d_{k,p_k} - d_{1,p_1-1}), \dots, (d_{k1}, \dots, d_{k,p_k-1} - d_{k,p_k-2}, d_{k,p_k} - d_{k,p_k-1})).$$

Since the s-vectors are increasing along their arms, the right hand side is again a tuple of compositions.

It remains to show the additivity of σ:

$$\sigma((\mathbf{a_1}, \dots, \mathbf{a_k}) + (\mathbf{b_1}, \dots, \mathbf{b_k})) = \sigma((\mathbf{a_1} + \mathbf{b_1}, \dots, \mathbf{a_k} + \mathbf{b_k}))$$

$$= \begin{array}{ccccc} & & \sum_{j=1}^{p_1-1}(a_{1j}+b_{1j}) & \cdots & a_{11}+b_{11} \\ & \swarrow & & & \\ \sum_{j=1}^{p_k}(a_{kj}+b_{kj}) & \longleftarrow & \sum_{j=1}^{p_2-1}(a_{2j}+b_{2j}) & \cdots & a_{21}+b_{21} \\ & \nwarrow & & \vdots & \\ & & \sum_{j=1}^{p_k-1}(a_{kj}+b_{kj}) & \cdots & a_{k1}+b_{k1} \end{array}$$

$$= \begin{array}{ccccc} & & \sum_{j=1}^{p_1-1} a_{1j} & \cdots & a_{11} \\ & \swarrow & & & \\ \sum_{j=1}^{p_k} a_{kj} & \longleftarrow & \sum_{j=1}^{p_2-1} a_{2j} & \cdots & a_{21} \\ & \nwarrow & & \vdots & \\ & & \sum_{j=1}^{p_k-1} a_{kj} & \cdots & a_{k1} \end{array}$$

$$+ \begin{array}{ccccc} & & \sum_{j=1}^{p_1-1} b_{1j} & \cdots & b_{11} \\ & \swarrow & & & \\ \sum_{j=1}^{p_k} b_{kj} & \longleftarrow & \sum_{j=1}^{p_2-1} b_{2j} & \cdots & b_{21} \\ & \nwarrow & & \vdots & \\ & & \sum_{j=1}^{p_k-1} b_{kj} & \cdots & b_{k1} \end{array}$$

$$= \sigma((\mathbf{a_1}, \dots, \mathbf{a_k})) + \sigma((\mathbf{b_1}, \dots, \mathbf{b_k})).$$

□

Comparison of tuples of compositions of a number means comparison of the corresponding s-vectors. It follows from Proposition 3.2 that a tuple $(\mathbf{a'_1}, \ldots, \mathbf{a'_k})$ of compositions of the same number is smaller than a second one $(\mathbf{a_1}, \ldots, \mathbf{a_k})$, if and only if $(\mathbf{a_1}, \ldots, \mathbf{a_k}) -(\mathbf{a'_1}, \ldots, \mathbf{a'_k}) \geq 0$. (Addition is again carried out componentwise.)

3.2 The Tits form for tuples of compositions

The Tits form for tuples of compositions can be easily calculated.
Let $(\mathbf{a_1}, \ldots, \mathbf{a_k}) \in D_{p_1,\ldots,p_k}$. The corresponding Tits form $\overline{q}$ — defined by

$$\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) = \frac{1}{2}(\sum_{i=1}^{k} \|\mathbf{a_i}\| + (2-k)n^2),$$

where $n = |\mathbf{a_1}| = \ldots = |\mathbf{a_k}|$, — shows, that the original Tits form q depends only on the (unordered) dimension jumps of the dimension vectors along their arms, but not on the dimensions themselves.

In order to show that the Tits form $\overline{q}$ of the tuples of compositions actually coincides with the usual Tits form for the corresponding s-vectors we have to prove the following lemma.

Lemma 3.3. *Let* $\mathbf{a} = (a_1, \ldots, a_p) \in \mathbb{N}_0^p$. *Then*

$$\sum_{j=1}^{p} \sum_{l=1}^{j-1} a_l a_j = \frac{1}{2}\left(|\mathbf{a}|^2 - \|\mathbf{a}\|^2\right). \tag{3.1}$$

Proof. (by induction on the length p of $\mathbf{a}$)
For $p = 1$ the formula holds because

$$0 = \frac{1}{2}(a_1^2 - a_1^2).$$

$p \rightsquigarrow p+1$:
We have

$$\begin{aligned}
\sum_{j=1}^{p+1} \sum_{l=1}^{j-1} a_l a_j &= \sum_{j=1}^{p} \sum_{l=1}^{j-1} a_l a_j + a_{p+1} \sum_{l=1}^{p} a_l \\
&\overset{\text{by assumption}}{=} \frac{1}{2}\left(\left(\sum_{j=1}^{p} a_j\right)^2 - \sum_{j=1}^{p} a_j^2\right) + a_{p+1} \sum_{l=1}^{p} a_l \\
&= \frac{1}{2}\left(\left(\sum_{j=1}^{p} a_j\right)^2 - \sum_{j=1}^{p} a_j^2 + 2\sum_{l=1}^{p} a_l a_{p+1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\left(\sum_{j=1}^{p} a_j\right)^2 + 2\sum_{l=1}^{p} a_l a_{p+1} + a_{p+1}^2 - \left(\sum_{j=1}^{p} a_j^2 + a_{p+1}^2\right)\right) \\
&= \frac{1}{2}\left(\left(\sum_{j=1}^{p+1} a_j\right)^2 - \sum_{j=1}^{p+1} a_j^2\right) \\
&= \frac{1}{2}\left(|\mathbf{a}|^2 - \|\mathbf{a}\|^2\right).
\end{aligned}
$$

□

Now it is easy to show the Proposition, which shows that the form $\overline{q}$ coincides for a tuple of compositions with the Tits form for the corresponding dimension vector.

Proposition 3.4. *Let* $(\mathbf{a_1}, \ldots, \mathbf{a_k}) \in D_{p_1,\ldots,p_k}$ *and* σ *as in Proposition 3.2. Then*

$$\overline{q}((\mathbf{a_1}, \ldots, \mathbf{a_k})) = q(\sigma((\mathbf{a_1}, \ldots, \mathbf{a_k}))).$$

Proof.

$$
\begin{aligned}
\overline{q}(\mathbf{a_1}, \ldots \mathbf{a_k}) &= \frac{1}{2}\left(\sum_{i=1}^{k} \|\mathbf{a_i}\|^2 + (2-k)n^2\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} \|\mathbf{a_i}\|^2 - \sum_{i=1}^{k} |\mathbf{a_i}|^2\right) + n^2 \\
&\overset{(3.1)}{=} -\sum_{i=1}^{k}\sum_{j=1}^{p_i}\sum_{l=1}^{j-1} a_{il}a_{ij} + n^2 \\
&= -\sum_{i=1}^{k}\sum_{j=2}^{p_i}\sum_{l=1}^{j-1} a_{il}a_{ij} + n^2 \\
&= -\sum_{i=1}^{k}\sum_{j=1}^{p_i-1}\sum_{l=1}^{j} a_{il}a_{i,j+1} + n^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{p_i-1}\left(\sum_{l=1}^{j} a_{il}\right)\left(\sum_{l=1}^{j} a_{il} - \sum_{l=1}^{j+1} a_{il}\right) + n^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{p_i-1}\left(\sum_{l=1}^{j} a_{il}\right)^2 + \left(\sum_{j=1}^{p_k} a_{kj}\right)^2 - \sum_{i=1}^{k}\sum_{j=1}^{p_i-1}\left(\sum_{l=1}^{j} a_{il}\right)\left(\sum_{l=1}^{j+1} a_{il}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= q\left(\begin{array}{cccc}
 & \sum_{j=1}^{p_1-1} a_{1j} & \cdots & a_{11} \\
\sum_{j=1}^{p_k} a_{kj} & \sum_{j=1}^{p_2-1} a_{2j} & \cdots & a_{21} \\
 & & \vdots & \\
 & \sum_{j=1}^{p_k-1} a_{kj} & \cdots & a_{k1}
\end{array}\right) \\
&= q(\sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}))
\end{aligned}
$$

□

Definition 3.5. Given a composition $\mathbf{a}$ of a number $n \in \mathbb{N}_0$, the *reduced composition* $\mathbf{a^{red}}$ is the composition which we obtain by deleting all zero entries in $\mathbf{a}$ and ordering the entries increasingly.

We say that a tuple of compositions of a number is *equivalent* to another tuple of compositions, if the reduced tuples of compositions coincide. The notions of s-finite, s-tame and s-hypercritical are used for the tuples of compositions as well. A tuple $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ of compositions is called $\left\{\begin{array}{l} \textit{s-finite} \\ \textit{s-tame} \\ \textit{s-hypercritical} \end{array}\right\}$ if the corresponding s-vector $\sigma(\mathbf{a_1}, \ldots, \mathbf{a_k})$ is $\left\{\begin{array}{l} \text{s-finite} \\ \text{s-tame} \\ \text{s-hypercritical} \end{array}\right\}$.

4 Overview of properties of strict tuples of compositions

The following is a complete list of the tuples of strict compositions of a number with properties of their Tits forms.

Proposition 4.1. *Let* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *be a tuple of strict compositions of a number. Then:*

1. *If* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *belongs to a case labelled with* $(*)$, *then its Tits form is positive.*
2. *If* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *belongs to a case labelled with* $(\star)$, *then its Tits form is non negative.*
3. *In each of the cases labelled with* $(\circ)$ *the Tits form is indefinite, and the tuples of compositions with smallest central dimension and these properties have Tits form* < 0. *If for these tuples of compositions there is no restriction on the dimension jumps along one arm, then the entries in the corresponding composition are* 1 *except for at most one position.*

- $k = 1$: quiver of type $\mathbb{A}_n$ $(*)$
- $k = 2$: quiver of type $\mathbb{A}_n$ $(*)$
- $k = 3$:
  - $p_1 = 2$
    - $p_2 = 2$: quiver of type $\mathbb{D}_n$ $(*)$
    - $p_2 = 3$
      - $3 \leq p_3 \leq 5$: quiver of type $\mathbb{E}_{678}$ $(*)$
      - $p_3 = 6$: quiver of type $\tilde{\mathbb{E}}_8$ $(\star)$
      - $p_3 = 7$
        - $\min \mathbf{a_1} \leq 5$ $(\star)$
        - $\min \mathbf{a_1} \geq 6$
          - $\min \mathbf{a_2} \leq 3$ $(\star)$
          - $\min \mathbf{a_2} \geq 4$
            - at least three times 1 in $\mathbf{a_3}$ $(\star)$
            - at most two times 1 in $\mathbf{a_3}$ $(\circ)$
      - $p_3 \geq 8$
        - $\min \mathbf{a_1} \leq 3$ $(\star)$
        - $\min \mathbf{a_1} \geq 4$

        - at least one 1 in $\mathbf{a_2}$ $(*)$
        - at least two times 2 in $\mathbf{a_2}$ $(\star)$
        - at most one 2 in $\mathbf{a_2}$ $(\circ)$

- $p_2 = 4$
    - $p_3 = 4$: quiver of type $\tilde{\mathbb{E}}_7$ $(\star)$
    - $p_3 = 5$
        - $\min \mathbf{a_1} \leq 3$ $(\star)$
        - $\min \mathbf{a_1} = 4$
            - $\min \mathbf{a_2} = 1$ $(\star)$
            - $\min \mathbf{a_2} \geq 2$
                - at least three times 1 in $\mathbf{a_3}$ $(\star)$
                - at most two times 1 in $\mathbf{a_3}$ $(\circ)$, $\oplus$
        - $\min \mathbf{a_1} \geq 5$
            - at least two times 1 in $\mathbf{a_2}$ $(*)$
            - at least one 1 in $\mathbf{a_2}$
                - $\min \mathbf{a_3} = 1$ $(\star)$
                - $\min \mathbf{a_3} \geq 2$ $(\circ)$
            - at least one 1 and one 2 in $\mathbf{a_2}$ $(\star)$
            - $\min \mathbf{a_2} \geq 2$
                - at least three times 1 in $\mathbf{a_3}$ $(\star)$
                - at most two times 1 in $\mathbf{a_3}$ $\ominus$
    - $p_3 \geq 6$
        - $\min \mathbf{a_1} \leq 2$ $(\star)$
        - $\min \mathbf{a_1} \geq 3$
            - at least three times 1 in $\mathbf{a_2}$ $(\star)$
            - at most two times 1 in $\mathbf{a_2}$ $(\circ)$
- $p_2 \geq 5$
    - $\min \mathbf{a_1} = 1$ $(*)$
    - $\min \mathbf{a_1} \geq 2$ $(\circ)$

- $p_1 = 3$
  - $p_2 = 3$
    - $p_3 = 3$: quiver of type $\tilde{\mathbb{E}}_6$ $(\star)$
    - $p_3 = 4$
      - $\min \mathbf{a_1} = 1$ $(\star)$
      - $\min \mathbf{a_1} \geq 2$
        - $\min \mathbf{a_2} = 1$ $(\star)$
        - $\min \mathbf{a_2} \geq 2$
          - at least three times 1 in $\mathbf{a_3}$ $(\star)$
          - at most two times 1 in $\mathbf{a_3}$ $(\circ)$
    - $p_3 \geq 5$
      - at least two times 1 in $\mathbf{a_1}$ $(\star)$
      - at most one 1 in $\mathbf{a_1}$
        - at least two times 1 in $\mathbf{a_2}$ $(\star)$
        - at most one 1 in $\mathbf{a_2}$ $(\circ)$
  - $p_2 \geq 4$ $(\circ)$
- $p_1 \geq 4$ $(\circ)$

- $k = 4$:
  - $p_1 = 2$
    - $p_2 = 2$
      - $p_3 = 2$
        - $p_4 = 2$: quiver of type $\tilde{\mathbb{D}}_4$ $(\star)$
        - $p_4 = 3$
          - $\min \mathbf{a_1} = 1$ $(\star)$
          - $\min \mathbf{a_1} \geq 2$
            - $\min \mathbf{a_2} = 1$ $(\star)$
            - $\min \mathbf{a_2} \geq 2$
              - $\min \mathbf{a_3} = 1$ $(\star)$
              - $\min \mathbf{a_3} \geq 2$ $(\circ)$

- $p_4 \geq 4$
  - $\min \mathbf{a_1} = 1$
    - $\min \mathbf{a_2} = 1$ $(\star)$
    - $\min \mathbf{a_2} \geq 2$
      - $\min \mathbf{a_3} = 1$ $(\star)$
      - $\min \mathbf{a_3} \geq 2$ $(\circ)$
  - $\min \mathbf{a_1} \geq 2$
    - $\min \mathbf{a_2} = 1$
      - $\min \mathbf{a_3} = 1$ $(\star)$
      - $\min \mathbf{a_3} \geq 2$ $(\circ)$
    - $\min \mathbf{a_2} \geq 2$ $(\circ)$
- $p_3 \geq 3$ $(\circ)$
- $p_2 \geq 3$ $(\circ)$
- $p_1 \geq 3$ $(\circ)$
- $k \geq 5$ $(\circ)$

In order to prove this proposition we have to prove some lemmas which give a lower bound for the Tits form (depending on the central dimension n) and show part the behaviour of the Tits form for tuples of compositions. The main consequences of these lemmas are that for a fixed star the Tits form for a dimension vector is always bigger than a polynomial of degree 2 in the central dimension n of this dimension vector and that it becomes minimal for fixed central dimension if and only if the dimension jumps along the arms are distributed as evenly as possible.

Lemma 4.2. *Let* $(a_1, \ldots, a_p) \in \mathbb{N}_0^p$ *with* $\sum_{j=1}^p a_j = n = pm + r$, $m, r \in \mathbb{N}_0$, $p \in \mathbb{N}$, *and* $r < p$. *Then*

$$\sum_{j=1}^{p} a_j^2 \geq pm^2 + r(2m+1) = mn + (m+1)r \geq \frac{n^2}{p}.$$

Proof. Every a_j, $j = 1, \ldots, p$, can be written in the shape $a_j = m + b_j$, where $b_j \in \mathbb{Z}$ and $\sum_{j=1}^p b_j = r$.

$$
\begin{aligned}
\sum_{j=1}^{p} a_j^2 &= \sum_{j=1}^{p} (m+b_j)^2 \\
&= \sum_{j=1}^{p} (m^2 + 2mb_j + b_j^2) \\
&= pm^2 + 2m\sum_{j=1}^{p} b_j + \sum_{j=1}^{p} b_j^2 \\
&= pm^2 + 2mr + \sum_{j=1}^{p} b_j^2 \\
&\geq pm^2 + 2mr + \sum_{j=1}^{p} |b_j| \\
&\geq pm^2 + 2mr + \sum_{j=1}^{p} b_j \\
&= pm^2 + 2mr + r \\
&= pm^2 + r(2m+1) \\
&= mn + (m+1)r
\end{aligned}
$$

The first equality holds exactly if and only if $|b_j| \leq 1$ for all $j = 1, \ldots, p$, and the second holds exactly if and only if $|b_j| = b_j$ for all $j = 1, \ldots, p$. Using $\sum_{j=1}^{p} b_j = r$, we get equality signs, if and only if $b_j = 1$ for r indices and $b_j = 0$ for the remaining.

If $n = pm + r$, and $p \neq 0$, then $m = \frac{n-r}{p}$. This leads to the following equation:

$$mn + (m+1)r = \frac{n^2 - rn}{p} + \frac{rn - r^2}{p} + r \geq \frac{n^2}{p}.$$

The last inequality holds, because $r < p$ and therefore $\frac{r^2}{p} \leq r$. □

Remark 4.3. *The proof of this lemma shows that the Tits form for s-vectors with a fixed central dimension becomes smallest, if and only if the dimension jumps along the arms are distributed as evenly as possible.*

Lemma 4.4. *Let* $a, b, c \in \mathbb{N}$ *with* $c \leq a \leq b$. *Then*

$$a^2 + b^2 < (a-c)^2 + (b+c)^2.$$

Proof.

$$(a-c)^2 + (b+c)^2 = a^2 + b^2 + \underbrace{2\underbrace{(b-a)}_{\geq 0}c + 2c^2}_{>0} > a^2 + b^2$$

□

Remark 4.5. *By induction (on the number of changes in the tuple of compositions) one gets that the Tits form becomes strictly bigger, if one changes the dimension jumps along the arms of a dimension vector in such a way that they are distributed more unevenly, even if they were already distributed unevenly.*

Proof of Proposition 4.1. It suffices to show the properties in the Proposition for the cases in which the tuples of compositions are ordered increasingly along their arms, since the value of the Tits form does not change for any reordering by Proposition 3.4.

First we prove the claim for the cases labelled with $(\circ)$.

We have the following tuples of compositions with smallest central dimension and their Tits forms are always negative:

$k = 3$:

- $\overline{q}((6,6),(4,4,4),(1,1,2,2,2,2,2)) = \frac{1}{2}(2\cdot 1^2 + 5\cdot 2^2 + 3\cdot 4^2 + 2\cdot 6^2 + (2-3)\cdot 12^2) = -1$
- $\overline{q}((4,4),(2,3,3),(1,1,1,1,1,1,1,1)) = \frac{1}{2}(8\cdot 1^2 + 2^2 + 2\cdot 3^2 + 2\cdot 4^2 + (2-3)\cdot 8^2) = -1$
- $\overline{q}((4,4),(2,2,2,2),(1,1,2,2,2)) = \frac{1}{2}(2\cdot 1^2 + 7\cdot 2^2 + 2\cdot 4^2 + (2-3)\cdot 8^2) = -1$
- $\overline{q}((5,5),(1,3,3,3),(2,2,2,2,2)) = \frac{1}{2}(1^2 + 5\cdot 2^2 + 3\cdot 3^2 + 2\cdot 5^2 + (2-3)\cdot 10^2) = -1$
- $\overline{q}((3,3),(1,1,2,2),(1,1,1,1,1,1)) = \frac{1}{2}(8\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2 + (2-3)\cdot 6^2) = -1$
- $\overline{q}((2,3),(1,1,1,1,1),(1,1,1,1,1)) = \frac{1}{2}(10\cdot 1^2 + 2^2 + 3^2 + (2-3)\cdot 5^2) = -1$
- $\overline{q}((2,2,2),(2,2,2),(1,1,2,2)) = \frac{1}{2}(2\cdot 1^2 + 8\cdot 2^2 + (2-3)\cdot 6^2) = -1$
- $\overline{q}((1,2,2),(1,2,2),(1,1,1,1,1)) = \frac{1}{2}(7\cdot 1^2 + 4\cdot 2^2 + (2-3)\cdot 5^2) = -1$
- $\overline{q}((1,1,2),(1,1,1,1),(1,1,1,1)) = \frac{1}{2}(10\cdot 1^2 + 2^2 + (2-3)\cdot 4^2) = -1$
- $\overline{q}((1,1,1,1),(1,1,1,1),(1,1,1,1)) = \frac{1}{2}(12\cdot 1^2 - (2-3)\cdot 4^2) = -2$

$k = 4$:

- $\overline{q}((2,2),(2,2),(2,2),(1,1,2)) = \frac{1}{2}(2\cdot 1^2 + 7\cdot 2^2 + (2-4)\cdot 4^2) = -1$
- $\overline{q}((2,2),(2,2),(2,2),(1,1,1,1)) = \frac{1}{2}(4\cdot 1^2 + 6\cdot 2^2 + (2-4)\cdot 4^2) = -2$
- $\overline{q}((1,3),(2,2),(2,2),(1,1,1,1)) = \overline{q}((2,2),(1,3),(2,2),(1,1,1,1))$
  $= \overline{q}((2,2),(2,2),(1,3),(1,1,1,1)) = \frac{1}{2}(5\cdot 1^2 + 4\cdot 2^2 + 3^2 + (2-4)\cdot 4^2) = -1$
- $\overline{q}((1,2)(1,2)(1,1,1)(1,1,1)) = \frac{1}{2}(8\cdot 1^2 + 2\cdot 2^2 + (2-4)\cdot 3^2) = -1$

- $\overline{q}((1,2),(1,1,1),(1,1,1),(1,1,1)) = \frac{1}{2}(10 \cdot 1^2 + 2^2 + (2-4) \cdot 3^2) = -2$
- $\overline{q}((1,1,1),(1,1,1),(1,1,1),(1,1,1)) = \frac{1}{2}(12 \cdot 1^2 + (2-4) \cdot 3^2) = -3$

$k = 5$:

$$\overline{q}((1,1),(1,1),(1,1),(1,1),(1,1)) = \frac{1}{2}(10 \cdot 1^2 + (2-5) \cdot 2^2) = -1$$

Now we will prove the claims for the cases labelled with $(*)$ and $(\star)$ (except for the ones with underlying quiver of finite or tame type). [1]

We are going to show that in these cases the Tits form is bounded from below by a polynomial in the central dimension of degree 2 with a *positive* leading coefficient. (For this we will use Lemma 4.2.) Once we have shown this, the Tits form will certainly become positive, if the central dimension becomes big or small enough. So there are only finitely many cases left to check whether or not the Tits form is non negative. The central dimensions to be considered can be computed by computing the zeros of the polynomials.

By Remark 4.3 we know that we can restrict ourselves to the cases in which the dimension jumps along the arms are distributed as evenly as possible. Then we have:

- $k = 3$:
  - $p_1 = 2$, $p_2 = 3$, $p_3 = 7$ with $7 \leq n \leq 11$ or $\mathbf{a_1} = (5, n-5)$, $n \geq 12$
    (a) $*$ $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(5^2 + (n-5)^2 + \frac{n^2}{3} + \frac{n^2}{7} - n^2) = \frac{1}{2}(\frac{10}{21}n^2 - 10n + 50) > 0$, if $n \geq 13$
    (b) $*$ $\overline{q}((3,4),(2,2,3),(1,1,1,1,1,1,1)) = \frac{1}{2}(7 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 4^2 + (2-3) \cdot 7^2) = 0$ $(n = 7)$
    $*$ $\overline{q}((4,4),(2,3,3),(1,1,1,1,1,1,2)) = \frac{1}{2}(6 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + (2-3) \cdot 8^2) = 0$ $(n = 8)$
    $*$ $\overline{q}((4,5),(3,3,3),(1,1,1,1,1,2,2)) = \frac{1}{2}(5 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 3^2 + 4^2 + 5^2 + (2-3) \cdot 9^2) = 0$ $(n = 9)$
    $*$ $\overline{q}((5,5),(3,3,4),(1,1,1,1,2,2,2)) = \frac{1}{2}(4 \cdot 1^2 + 3 \cdot 2^2 + 2 \cdot 3^2 + 4^2 + 2 \cdot 5^2 + (2-3) \cdot 10^2) = 0$ $(n = 10)$
    $*$ $\overline{q}((5,6),(3,4,4),(1,1,1,2,2,2,2)) = \frac{1}{2}(3 \cdot 1^2 + 4 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 6^2 + (2-3) \cdot 11^2) = 0$ $(n = 11)$
    $*$ $\overline{q}((5,7),(4,4,4),(1,1,2,2,2,2,2)) = \frac{1}{2}(2 \cdot 1^2 + 5 \cdot 2^2 + 3 \cdot 4^2 + 5^2 + 7^2 - 12^2) = 0$ $(n = 12)$
  - $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $\mathbf{a_2} = (3, \underbrace{-,-}_{n-3})$, $n \geq 12$

[1] In the finite cases it was shown by Gabriel (see [6]) that the Tits form is always positive, and in the tame cases Dlab and Ringel showed that the Tits form is non negative (see [5]). This can also be shown be using the lower bound for the Tits form we have by Lemma 4.2 (see Appendices A and B).

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + 3^2 + \frac{(n-3)^2}{2} + \frac{n^2}{7} - n^2) = \frac{1}{2}(\frac{n^2}{7} - 3n + \frac{27}{2}) > 0$, if $n \geq 15$

(b) * $\overline{q}((6,6),(3,4,5),(1,1,2,2,2,2,2)) = \frac{1}{2}(2 \cdot 1^2 + 5 \cdot 2^2 + 3^2 + 4^2 + 5^2 + 2 \cdot 6^2 - 12^2) = 0$ $(n = 12)$

* $\overline{q}((6,7),(3,5,5),(1,2,2,2,2,2,2)) = \frac{1}{2}(1^2 + 6 \cdot 2^2 + 3^2 + 2 \cdot 5^2 + 6^2 + 7^2 - 13^2) = 0$ $(n = 13)$

* $\overline{q}((7,7),(3,5,6),(2,2,2,2,2,2,2)) = \frac{1}{2}(7 \cdot 2^2 + 3^2 + 5^2 + 6^2 + 2 \cdot 7^2 - 14^2) = 0$ $(n = 14)$

– $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $\mathbf{a_3} = (1,1,1,\underbrace{-,-,-,-}_{n-3})$, $n \geq 12$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + \frac{n^2}{3} + 3 \cdot 1^2 + \frac{(n-3)^2}{4} - n^2) = \frac{1}{2}(\frac{n^2}{12} - \frac{3}{2}n + \frac{21}{4}) > 0$, if $n \geq 14$

* $\overline{q}((6,7),(4,4,5),(1,1,1,2,2,3,3)) = \frac{1}{2}(3 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + 5^2 + 6^2 + 7^2 - 13^2) = 1$ $(n = 13)$

(b) * $\overline{q}((6,6),(4,4,4),(1,1,1,2,2,2,3)) = \frac{1}{2}(3 \cdot 1^2 + 3 \cdot 2^2 + 3^2 + 3 \cdot 4^2 + 2 \cdot 6^2 - 12^2) = 0$ $(n = 12)$

– $p_1 = 2$, $p_2 = 3$, $p_3 \geq 8$, $\mathbf{a_1} = (3, n-3)$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(3^2 + (n-3)^2 + \frac{n^2}{3} + n - n^2) = \frac{1}{2}(\frac{n^2}{3} - 5n + 18) > 0$, if $n \geq 10$

(b) * $\overline{q}((3,5),(2,3,3),(1,1,1,1,1,1,1,1)) = \frac{1}{2}(8 \cdot 1^2 + 2^2 + 3 \cdot 3^2 + 5^2 - 8^2) = 0$ $(n = 8)$

* $\overline{q}((3,6),(3,3,3),(1,1,1,1,1,1,1,1,1)) = \frac{1}{2}(9 \cdot 1^2 + 4 \cdot 3^2 + 6^2 - 9^2) = 0$ $(n = 9)$

– $p_1 = 2$, $p_2 = 3$, $p_3 \geq 8$, $\mathbf{a_2} = (1,\underbrace{-,-}_{n-1})$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + 1^2 + \frac{(n-1)^2}{2} + n - n^2) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} > 0 \ \forall n \in \mathbb{N}$

– $p_1 = 2$, $p_2 = 3$, $p_3 \geq 8$, $\mathbf{a_2} = (2, 2, n-4)$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + 2 \cdot 2^2 + (n-4)^2 + n - n^2) = \frac{1}{2}(\frac{n^2}{2} - 7n + 24) > 0$, if $n \geq 9$

(b) * $\overline{q}((4,4),(2,2,4),(1,1,1,1,1,1,1,1)) = \frac{1}{2}(8 \cdot 1^2 + 2 \cdot 2^2 + 3 \cdot 4^2 - 8^2) = 0$ $(n = 8)$

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$ with $5 \leq n \leq 7$ or $\mathbf{a_1} = (3, n-3)$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(3^2 + (n-3)^2 + \frac{n^2}{4} + \frac{n^2}{5} - n^2) = \frac{1}{2}(\frac{9}{20}n^2 - 6n + 18) > 0$, if $n \geq 9$

(b) * $\overline{q}((2,3),(1,1,1,2),(1,1,1,1,1)) = \frac{1}{2}(8 \cdot 1^2 + 2 \cdot 2^2 + 3^2 + (2-3) \cdot 5^2) = 0$ $(n = 5)$

* $\overline{q}((3,3),(1,1,2,2),(1,1,1,1,2)) = \frac{1}{2}(6 \cdot 1^2 + 3 \cdot 2^2 + 2 \cdot 3^2 + (2-3) \cdot 6^2) = 0$ $(n = 6)$

* $\overline{q}((3,4),(1,2,2,2),(1,1,1,2,2)) = \frac{1}{2}(4\cdot 1^2+5\cdot 2^2+3^2+4^2+(2-3)\cdot 7^2) = 0$ $(n=7)$
* $\overline{q}((3,5),(2,2,2,2),(1,1,2,2,2)) = \frac{1}{2}(2\cdot 1^2+7\cdot 2^2+3^2+5^2-8^2) = 0$ $(n=8)$

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_1} = (4, n-4)$, $\mathbf{a_2} = (1, \underbrace{-,-,-}_{n-1})$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) \geq \frac{1}{2}(4^2+(n-4)^2+1^2+\frac{(n-1)^2}{3}+\frac{n^2}{5}-n^2) = \frac{1}{2}(\frac{8}{15}n^2-\frac{26}{3}n+\frac{100}{3}) > 0$, if $n \geq 11$

(b) * $\overline{q}((4,4),(1,2,2,3),(1,1,2,2,2)) = \frac{1}{2}(3\cdot 1^2+5\cdot 2^2+3^2+2\cdot 4^2-8^2) = 0$ $(n=8)$
* $\overline{q}((4,5),(1,2,3,3),(1,2,2,2,2)) = \frac{1}{2}(2\cdot 1^2+5\cdot 2^2+2\cdot 3^2+4^2+5^2-9^2) = 0$ $(n=9)$
* $\overline{q}((4,6),(1,3,3,3),(2,2,2,2,2)) = \frac{1}{2}(1^2+5\cdot 2^2+3\cdot 3^2+4^2+6^2-10^2) = 0$ $(n=10)$

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_1} = (4, n-4)$, $\mathbf{a_3} = (1,1,1, \underbrace{-,-}_{n-3})$, $n \geq 8$

(a) * $\overline{q}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) \geq \frac{1}{2}(4^2+(n-4)^2+\frac{n^2}{4}+3\cdot 1^2+\frac{(n-3)^2}{2}-n^2) = \frac{1}{2}(\frac{3}{4}n^2-11n+\frac{79}{2}) > 0$, if $n \geq 9$

(b) * $\overline{q}((4,4),(2,2,2,2),(1,1,1,2,3)) = \frac{1}{2}(3\cdot 1^2+5\cdot 2^2+3^2+2\cdot 4^2-8^2) = 0$ $(n=8)$

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_2} = (1,1, \underbrace{-,-}_{n-2})$, $n \geq 10$

(a) * $\overline{q}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2}+2\cdot 1^2+\frac{(n-2)^2}{2}+\frac{n^2}{5}-n^2) = \frac{1}{2}(\frac{n^2}{5}-2n+4) > 0$, if $n \geq 10$ (even if $n \geq 8$)

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_2} = (1, \underbrace{-,-,-}_{n-1})$, $\mathbf{a_3} = (1, \underbrace{-,-,-,-}_{n-1})$, $n \geq 10$

(a) * $\overline{q}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2}+1^2+\frac{(n-1)^2}{3}+1^2+\frac{(n-1)^2}{4}-n^2) = \frac{1}{2}(\frac{n^2}{12}-\frac{7}{6}n+\frac{31}{12}) > 0$, if $n \geq 12$
* $\overline{q}((5,6),(1.3,3,4),(1,2,2,3,3)) = \frac{1}{2}(2\cdot 1^2+2\cdot 2^2+4\cdot 3^2+4^2+5^2+6^2-11^2) = 0 (n=11)$

(b) * $\overline{q}((5,5),(1.3,3,3),(1,2,2,2,3)) = \frac{1}{2}(2\cdot 1^2+3\cdot 2^2+4\cdot 3^2+2\cdot 5^2-10^2) = 0$ $(n=10)$

– $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_2} = (1,2, \underbrace{-,-}_{n-3})$, $n \geq 10$

(a) * $\overline{q}(\mathbf{a_1},\mathbf{a_2},\mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2}+1^2+2^2+\frac{(n-3)^2}{2}+\frac{n^2}{5}-n^2) = \frac{1}{2}(\frac{n^2}{5}-3n+\frac{19}{2}) > 0$, if $n \geq 11$

(b) * $\overline{q}((5,5),(1,2,3,4),(2,2,2,2,2)) = \frac{1}{2}(1^2+6\cdot 2^2+3^2+4^2+2\cdot 5^2-10^2) = 0$ $(n=10)$

- $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\mathbf{a_3} = (1, 1, 1, \underbrace{-, -}_{n-3})$, $n \geq 10$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + \frac{n^2}{4} + 3 \cdot 1^2 + \frac{(n-3)^2}{2} - n^2) = \frac{1}{2}(\frac{n^2}{4} - 3n + \frac{15}{2}) > 0$, if $n \geq 10$ (even if $n \geq 9$)
- $p_1 = 2$, $p_2 = 4$, $p_3 \geq 6$, $\mathbf{a_1} = (2, n-2)$, $n \geq 6$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(2^2 + (n-2)^2 + \frac{n^2}{4} + n - n^2) = \frac{1}{2}(\frac{n^2}{4} - 3n + 8) > 0$, if $n \geq 9$
  (b) * $\overline{q}((2, 4), (1, 1, 2, 2), (1, 1, 1, 1, 1, 1)) = \frac{1}{2}(8 \cdot 1^2 + 3 \cdot 2^2 + 4^2 - 6^2) = 0$ $(n = 6)$
  * $\overline{q}((2, 5), (1, 2, 2, 2), (1, 1, 1, 1, 1, 1, 1)) = \frac{1}{2}(8 \cdot 1^2 + 4 \cdot 2^2 + 5^2 - 7^2) = 0$ $(n = 7)$
  * $\overline{q}((2, 6), (2, 2, 2, 2), (1, 1, 1, 1, 1, 1, 1, 1)) = \frac{1}{2}(8 \cdot 1^2 + 5 \cdot 2^2 + 6^2 - 8^2) = 0$ $(n = 8)$
- $p_1 = 2$, $p_2 = 4$, $p_3 \geq 6$, $\mathbf{a_2} = (1, 1, 1, n-3)$, $n \geq 6$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{2} + 3 \cdot 1^2 + (n-3)^2 + n - n^2) = \frac{1}{2}(\frac{n^2}{2} - 5n + 12) > 0$, if $n \geq 7$
  (b) * $\overline{q}((3, 3), (1, 1, 1, 3), (1, 1, 1, 1, 1, 1)) = \frac{1}{2}(9 \cdot 1^2 + 3 \cdot 3^2 - 6^2) = 0$ $(n = 6)$
- $p_1 = 2$, $p_2 \geq 5$, $p_3 \geq 5$, $\mathbf{a_1} = (1, n-1)$, $n \geq 5$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(1^2 + (n-1)^2 + n + n - n^2) = 1 > 0$ $\forall n \in \mathbb{N}$
- $p_1 = 3$, $p_2 = 3$, $p_3 = 4$, $\mathbf{a_1} = (1, \underbrace{-, -}_{n-1})$ or $\mathbf{a_2} = (1, \underbrace{-, -}_{n-1})$, $n \geq 6$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(1^2 + \frac{(n-1)^2}{2} + \frac{n^2}{3} + \frac{n^2}{4} - n^2) = \frac{1}{2}(\frac{n^2}{12} - n + \frac{3}{2}) > 0$, if $n \geq 11$
  * $\overline{q}((1, 4, 5), (3, 3, 4), (2, 2, 3, 3)) = \overline{q}((3, 3, 4), (1, 4, 5), (2, 2, 3, 3)) = \frac{1}{2}(1^2 + 2 \cdot 2^2 + 4 \cdot 3^2 + 2 \cdot 4^2 + 5^2 - 10^2) = 1$ $(n = 10)$
  (b) * $\overline{q}((1, 2, 3), (2, 2, 2), (1, 1, 2, 2)) = \overline{q}((2, 2, 2), (1, 2, 3), (1, 1, 2, 2)) = \frac{1}{2}(3 \cdot 1^2 + 6 \cdot 2^2 + 3^2 - 6^2) = 0$ $(n = 6)$
  * $\overline{q}((1, 3, 3), (2, 2, 3), (1, 2, 2, 2)) = \overline{q}((2, 2, 3), (1, 3, 3), (1, 2, 2, 2)) = \frac{1}{2}(2 \cdot 1^2 + 5 \cdot 2^2 + 3 \cdot 3^2 - 7^2) = 0$ $(n = 7)$
  * $\overline{q}((1, 3, 4), (2, 3, 3), (2, 2, 2, 2)) = \overline{q}((2, 3, 3), (1, 3, 4), (2, 2, 2, 2)) = \frac{1}{2}(1^2 + 5 \cdot 2^2 + 3 \cdot 3^2 + 4^2 - 8^2) = 0$ $(n = 8)$
  * $\overline{q}((1, 4, 4), (3, 3, 3), (2, 2, 2, 3)) = \overline{q}((3, 3, 3), (1, 4, 4), (2, 2, 2, 3)) = \frac{1}{2}(1^2 + 3 \cdot 2^2 + 4 \cdot 3^2 + 2 \cdot 4^2 - 9^2) = 0$ $(n = 9)$
- $p_1 = 3$, $p_2 = 3$, $p_3 = 4$, $\mathbf{a_3} = (1, 1, 1, n-3)$, $n \geq 6$
  (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(\frac{n^2}{3} + \frac{n^2}{3} + 3 \cdot 1^2 + (n-3)^2 - n^2) = \frac{1}{2}(\frac{2}{3}n^2 - 6n + 12) > 0$, if $n \geq 7$
  (b) * $\overline{q}((2, 2, 2), (2, 2, 2), (1, 1, 1, 3)) = \frac{1}{2}(3 \cdot 1^2 + 6 \cdot 2^2 + 3^2 - 6^2) = 0$ $(n = 6)$
- $p_1 = 3$, $p_2 = 3$, $p_3 \geq 5$, $\mathbf{a_1} = (1, 1, n-2)$ or $\mathbf{a_2} = (1, 1, n-2)$, $n \geq 5$

(a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}(2 \cdot 1^2 + (n-2)^2 + \frac{n^2}{3} + n - n^2) = \frac{1}{2}(\frac{n^2}{3} - 3n + 6) > 0$, if $n \geq 7$

(b) * $\overline{q}((1,1,3),(1,2,2),(1,1,1,1,1)) = \overline{q}((1,2,2),(1,1,3),(1,1,1,1,1)) = \frac{1}{2}(8 \cdot 1^2 + 2 \cdot 2^2 + 3^2 - 5^2) = 0$ $(n=5)$

* $\overline{q}((1,1,4),(2,2,2),(1,1,1,1,1,1)) = \overline{q}((2,2,2),(1,1,4),(1,1,1,1,1,1)) = \frac{1}{2}(8 \cdot 1^2 + 3 \cdot 2^2 + 4^2 - 6^2) = 0$ $(n=6)$

- $k = 4$:
  - $p_1 = 2, p_2 = 2, p_3 = 2, p_4 = 3$, $\mathbf{a_1} = (1, n-1)$ or $\mathbf{a_2} = (1, n-1)$ or $\mathbf{a_3} = (1, n-1)$, $n \geq 3$

    (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) \geq \frac{1}{2}(1^2 + (n-1)^2 + \frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{3} + (2-4) \cdot n^2) = \frac{1}{2}(\frac{n^2}{3} - 2n + 2) > 0$, if $n \geq 5$

    (b) * $\overline{q}((1,3),(2,2),(2,2),(1,1,2)) = \overline{q}((2,2),(1,3),(2,2),(1,1,2)) = \overline{q}((2,2),(2,2),(1,3),(1,1,2)) = \frac{1}{2}(3 \cdot 1^2 + 5 \cdot 2^2 + 3^2 + (2-4) \cdot 4^2) = 0$ $(n=4)$

  - $p_1 = 2, p_2 = 2, p_3 = 2, p_4 \geq 4$, $\mathbf{a_1} = (1, n-1)$, $\mathbf{a_2} = (1, n-1)$ or $\mathbf{a_1} = (1, n-1)$, $\mathbf{a_3} = (1, n-1)$ or $\mathbf{a_2} = (1, n-1)$, $\mathbf{a_3} = (1, n-1)$, $n \geq 4$

    (a) * $\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) \geq \frac{1}{2}(2 \cdot 1^2 + 2 \cdot (n-1)^2 + \frac{n^2}{2} + n + (2-4) \cdot n^2) = \frac{1}{2}(\frac{n^2}{2} - 3n + 4) > 0$, if $n \geq 5$

    (b) * $\overline{q}((1,3),(1,3),(2,2),(1,1,1,1)) = \overline{q}((1,3),(2,2),(1,3),(1,1,1,1)) = \overline{q}((2,2),(1,3),(1,3),(1,1,1,1)) = \frac{1}{2}(6 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 - 2 \cdot 4^2) = 0$ $(n=4)$

□

5 Classification of the s-hypercritical and s-tame vectors

As we have seen in Chapter 3, the dimension vectors of subspace representations are in one-to-one correspondence with the tuples of compositions of a number.

The aim of this chapter is to provide two lists: The first one (in Proposition 5.3) contains (up to reordering) all s-hypercritical tuples of strict compositions, and in the second one (in Proposition 5.4) one can find (up to reordering) all s-tame tuples of strict compositions.

It suffices to give the classification up to reordering since the following holds:

Remark 5.1. *Let* $\sigma_i \in S_{p_i}$, $i = 1, \ldots, k$, *where* S_p *denotes the group of permutations of the set* $\{1, \ldots, p\}$. *Let* $\mathbf{a}^\sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(p)})$ *be the reordering of* $\mathbf{a} = (a_1, \ldots, a_p)$ *w.r.t.* $\sigma \in S_p$. *Then:*

- $q(\mathbf{a_1}^{\sigma_1}, \ldots, \mathbf{a_k}^{\sigma_\mathbf{k}}) = q(\mathbf{a_1}, \ldots, \mathbf{a_k})$, *and*
- $(\mathbf{a_1}, \ldots, \mathbf{a_k}) < (\mathbf{b_1}, \ldots, \mathbf{b_k}) \Leftrightarrow (\mathbf{a_1}^{\sigma_1}, \ldots, \mathbf{a_k}^{\sigma_\mathbf{k}}) < (\mathbf{b_1}^{\sigma_1}, \ldots, \mathbf{b_k}^{\sigma_\mathbf{k}})$.

This means that the tuples of compositions which are smaller than an unordered tuple of composition are exactly the reorderings of the tuples of compositions which are smaller than the corresponding tuple of compositions which is ordered increasingly along its arms.

Furthermore, we can restrict ourselves for the classification to tuples of strict compositions, because monomorphisms between two vector spaces of the same dimension already have to be isomorphisms, and therefore, the subspace representations of tuples of non strict compositions have the same decomposition properties as the subspace representations for the corresponding tuples of strict compositions.

Remark 5.2. *As already mentioned in Chapter 3, the partial order given on the s-vectors carries over to the corresponding partial order on the tuples of compositions, where it is given in the following way:*

$$(\mathbf{a_1}, \ldots, \mathbf{a_k}) < (\mathbf{b_1}, \ldots, \mathbf{b_k}) \quad \Leftrightarrow \quad \forall i = 1, \ldots, k \ \forall j = 1, \ldots p_i : a_{ij} \leq b_{ij} \text{ and } (\mathbf{a_1}, \ldots, \mathbf{a_k}) \neq (\mathbf{b_1}, \ldots, \mathbf{b_k})$$

5.1 Classification of the s-hypercritical vectors

Proposition 5.3. *A reduced tuple* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *of compositions is contained in* **List 1** *if and only if*

(1) $\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) < 0$ *and*

(2) $\overline{q}(\mathbf{a'_1}, \ldots, \mathbf{a'_k}) \geq 0$ *for all* $(\mathbf{a'_1}, \ldots, \mathbf{a'_k}) < (\mathbf{a_1}, \ldots, \mathbf{a_k})$.

List 1

$k = 5$:

$p_1 = p_2 = p_3 = p_4 = p_5 = 2$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}) = ((1,1), (1,1), (1,1), (1,1), (1,1))$

$k = 4$:

1. $p_1 = p_2 = p_3 = 2$, $p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2,2), (2,2), (2,2), (1,1,2))$
2. $p_1 = p_2 = p_3 = 2$, $p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2,2), (2,2), (2,2), (1,1,1,1))$
3. $p_1 = p_2 = p_3 = 2$, $p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1,3), (2,2), (2,2), (1,1,1,1))$
4. $p_1 = p_2 = p_3 = 2$, $p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2,2), (1,3), (2,2), (1,1,1,1))$
5. $p_1 = p_2 = p_3 = 2$, $p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2,2), (2,2), (1,3), (1,1,1,1))$
6. $p_1 = p_2 = 2$, $p_3 = p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1,2), (1,2), (1,1,1), (1,1,1))$
7. $p_1 = 2$, $p_2 = p_3 = p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1,2), (1,1,1), (1,1,1), (1,1,1))$
8. $p_1 = p_2 = p_3 = p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1,1,1), (1,1,1), (1,1,1), (1,1,1))$

$k = 3$:

1. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((6,6), (4,4,4), (1,1,2,2,2,2,2))$
2. $p_1 = 2$, $p_2 = 3$, $p_3 = 8$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4), (2,3,3), (1,1,1,1,1,1,1,1))$
3. $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4), (2,2,2,2), (1,1,2,2,2))$
4. $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5,5), (1,3,3,3), (2,2,2,2,2))$
5. $p_1 = 2$, $p_2 = 4$, $p_3 = 6$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,3), (1,1,2,2), (1,1,1,1,1,1))$
6. $p_1 = 2$, $p_2 = p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,3), (1,1,1,1,1), (1,1,1,1,1))$
7. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,2,2), (2,2,2), (1,1,2,2))$
8. $p_1 = p_2 = 3$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,2,2), (1,2,2), (1,1,1,1,1))$
9. $p_1 = 3$, $p_2 = p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,2), (1,1,1,1), (1,1,1,1))$
10. $p_1 = p_2 = p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,1,1), (1,1,1,1), (1,1,1,1))$

5.2 Classification of the s-tame vectors

Proposition 5.4. *A reduced tuple* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *of compositions is contained in* **List 2** *if and only if*

(1) $\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) = 0$ *and*

(2) $\overline{q}(\mathbf{a'_1}, \ldots, \mathbf{a'_k}) \geq 0$ *for all* $(\mathbf{a'_1}, \ldots, \mathbf{a'_k}) < (\mathbf{a_1}, \ldots, \mathbf{a_k})$.

List 2

$k = 4$:

1. $p_1 = p_2 = p_3 = p_4 = 2$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((m, m), (m, m), (m, m), (m, m))$, $m \in \mathbb{N}$
2. $p_1 = p_2 = p_3 = 2, p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1, 2), (1, 2), (1, 2), (1, 1, 1))$
3. $p_1 = p_2 = p_3 = 2, p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1, 3), (2, 2), (2, 2), (1, 1, 2))$
4. $p_1 = p_2 = p_3 = 2, p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2, 2), (1, 3), (2, 2), (1, 1, 2))$
5. $p_1 = p_2 = p_3 = 2, p_4 = 3$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2, 2), (2, 2), (1, 3), (1, 1, 2))$
6. $p_1 = p_2 = p_3 = 2, p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1, 3), (1, 3), (2, 2), (1, 1, 1, 1))$
7. $p_1 = p_2 = p_3 = 2, p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1, 3), (2, 2), (1, 3), (1, 1, 1, 1))$
8. $p_1 = p_2 = p_3 = 2, p_4 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((2, 2), (1, 3), (1, 3), (1, 1, 1, 1))$

$k = 3$:

1. $p_1 = 2$, $p_2 = 3$, $p_3 = 6$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3m, 3m), (2m, 2m, 2m), (m, m, m, m, m, m))$, $m \in \mathbb{N}$
2. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3, 4), (2, 2, 3), (1, 1, 1, 1, 1, 1, 1))$
3. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4, 4), (2, 3, 3), (1, 1, 1, 1, 1, 1, 2))$
4. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4, 5), (3, 3, 3), (1, 1, 1, 1, 1, 2, 2))$
5. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5, 5), (3, 3, 4), (1, 1, 1, 1, 2, 2, 2))$
6. $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5, 6), (3, 4, 4), (1, 1, 1, 2, 2, 2, 2))$
7. $p_1 = 2$, $p_2 = p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2m, 2m), (m, m, m, m), (m, m, m, m))$, $m \in \mathbb{N}$
8. $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2, 3), (1, 1, 1, 2), (1, 1, 1, 1, 1))$

9. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,3),(1,1,2,2),(1,1,1,1,2))$
10. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,4),(1,2,2,2),(1,1,1,2,2))$
11. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4),(1,2,2,3),(1,1,2,2,2))$
12. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,5),(1,2,3,3),(1,2,2,2,2))$
13. $p_1 = p_2 = p_3 = 3,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((m,m,m),(m,m,m),(m,m,m)),\ m \in \mathbb{N}$
14. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,2),(1,1,2),(1,1,1,1))$
15. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,2,2),(1,2,2),(1,1,1,2))$
16. $p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5,7),(4,4,4),(1,1,2,2,2,2,2))$
17. $p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((6,6),(3,4,5),(1,1,2,2,2,2,2))$
18. $p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((6,7),(3,5,5),(1,2,2,2,2,2,2))$
19. $p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((7,7),(3,5,6),(2,2,2,2,2,2,2))$
20. $p_1 = 2,\ p_2 = 3,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((6,6),(4,4,4),(1,1,1,2,2,2,3))$
21. $p_1 = 2,\ p_2 = 3,\ p_3 = 8,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,5),(2,3,3),(1,1,1,1,1,1,1,1))$
22. $p_1 = 2,\ p_2 = 3,\ p_3 = 9,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,6),(3,3,3),(1,1,1,1,1,1,1,1,1))$
23. $p_1 = 2,\ p_2 = 3,\ p_3 = 8,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4),(2,2,4),(1,1,1,1,1,1,1,1))$
24. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,5),(2,2,2,2),(1,1,2,2,2))$
25. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4),(2,2,2,2),(1,1,1,2,3))$
26. $p_1 = 2,\ p_2 = 4,\ p_3 = 5,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5,5),(1,2,3,4),(2,2,2,2,2))$
27. $p_1 = 2,\ p_2 = 4,\ p_3 = 6,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,4),(1,1,2,2),(1,1,1,1,1,1))$
28. $p_1 = 2,\ p_2 = 4,\ p_3 = 7,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,5),(1,2,2,2),(1,1,1,1,1,1,1))$
29. $p_1 = 2,\ p_2 = 4,\ p_3 = 8,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,6),(2,2,2,2),(1,1,1,1,1,1,1,1))$
30. $p_1 = 2,\ p_2 = 4,\ p_3 = 6,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,3),(1,1,1,3),(1,1,1,1,1,1))$
31. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,2,3),(2,2,2),(1,1,2,2))$
32. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,2,2),(1,2,3),(1,1,2,2))$
33. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,3,3),(2,2,3),(1,2,2,2))$
34. $p_1 = p_2 = 3,\ p_3 = 4,\ (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,2,3),(1,3,3),(1,2,2,2))$

35. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,3,4),(2,3,3),(2,2,2,2))$

36. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,3,3),(1,3,4),(2,2,2,2))$

37. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,4,4),(3,3,3),(2,2,2,3))$

38. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,3,3),(1,4,4),(2,2,2,3))$

39. $p_1 = p_2 = 3$, $p_3 = 4$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,2,2),(2,2,2),(1,1,1,3))$

40. $p_1 = p_2 = 3$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,3),(1,2,2),(1,1,1,1,1))$

41. $p_1 = p_2 = 3$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,2,2),(1,1,3),(1,1,1,1,1))$

42. $p_1 = p_2 = 3$, $p_3 = 6$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,4),(2,2,2),(1,1,1,1,1,1))$

43. $p_1 = p_2 = 3$, $p_3 = 6$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,2,2),(1,1,4),(1,1,1,1,1,1))$

44. $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,6),(1,3,3,3),(2,2,2,2,2))$

45. $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((5,5),(1,3,3,3),(1,2,2,2,3))$

5.3 Proofs of Propositions 5.3 and 5.4

We can use the results of Chapter 4 to prove Proposition 5.3:

Proof of Proposition 5.3. We know that in the cases labelled with $(*)$ and $(\star)$ (in Proposition 4.1) there cannot be any tuples of compositions which fulfil condition (1), since the Tits form is non negative in these cases. If a tuple of compositions belongs to a case $(\circ)$ and has a central dimension bigger than the smallest central dimension possible in this case, then condition (2) cannot be fulfilled. And if a tuple of compositions belongs to case $\ominus$, then condition (2) is not fulfilled either: A tuple of compositions in case $\ominus$ is always bigger than $((4,4),(2,2,2,2),(1,1,2,2,2))$ (from case $\oplus$) which has negative Tits form.

Now we want to show that the tuples of compositions with smallest central dimension in the cases labelled with $(\circ)$ fulfil both conditions (1) and (2). All of them have negative Tits form and therefore fulfil condition (1). If a tuple of compositions is smaller than one of these, the central dimension also becomes smaller. Then we have exactly two possibilities: either the smaller tuple of compositions belongs to a case labelled with $(*)$ or $(\star)$, since the conditions on the arms cannot be fulfilled anymore, or the underlying quiver becomes smaller. If the quiver becomes smaller, then a case by case inspection gives that one of the following conditions has to be fulfilled:

- the underlying quiver is of finite or tame type

- $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $n = 7$ ($\Rightarrow \min \mathbf{a_1} \leq 5$)
- $k = 3$, $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $n = 5$ ($\Rightarrow \min \mathbf{a_1} \leq 3$)
- $k = 3$, $p_1 = 3$, $p_2 = 3$, $p_3 = 4$, $n = 4$ ($\Rightarrow \min \mathbf{a_1} = 1$)
- $k = 4$, $p_1 = 2$, $p_2 = 2$, $p_3 = 2$, $p_4 = 3$, $n = 3$ ($\Rightarrow \min \mathbf{a_1} = 1$)

Each of these cases is labelled with ($*$) or ($\star$). If a tuple of compositions belongs to a case labelled with ($*$) or ($\star$), then one of the following holds for each smaller tuple of compositions: either the same restrictions on the arms have to be fulfilled — thus, the Tits form is non negative — or the underlying quiver becomes smaller. By a case by case inspection we get that in the cases the quiver becomes smaller one of the following conditions has to be fulfilled:

- the underlying quiver is of finite or tame type
- $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $\min \mathbf{a_1} \leq 5$
- $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, $\min \mathbf{a_2} \leq 3$
- $k = 3$, $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\min \mathbf{a_1} \leq 3$
- $k = 3$, $p_1 = 2$, $p_2 = 4$, $p_3 = 5$, $\min \mathbf{a_2} = 1$
- $k = 3$, $p_1 = 3$, $p_2 = 3$, $p_3 = 4$, $\min \mathbf{a_1} = 1$
- $k = 3$, $p_1 = 3$, $p_2 = 3$, $p_3 = 4$, $\min \mathbf{a_2} = 1$
- $k = 4$, $p_1 = 2$, $p_2 = 2$, $p_3 = 2$, $p_4 = 3$, $\min \mathbf{a_1} = 1$
- $k = 4$, $p_1 = 2$, $p_2 = 2$, $p_3 = 2$, $p_4 = 3$, $\min \mathbf{a_1} \geq 2$, $\min \mathbf{a_2} = 1$

But in all cases the Tits form stays non negative — these cases are all labelled with ($*$) or ($\star$) from Proposition 4.1 —, and therefore also condition (2) is fulfilled. So we have to list all tuples of compositions with minimal central dimension from the cases labelled with ($\circ$). □

And here is the proof for Proposition 5.4:

Proof of Proposition 5.4. If a tuple of compositions belongs to a case labelled with ($\circ$), then it is always bigger than a tuple of compositions with negative Tits form or has negative Tits form itself, because the tuples of compositions with smallest central dimension in the cases labelled with ($\circ$) have negative Tits form. So condition (2) is not fulfilled.

We also know that a tuple of compositions from case $\ominus$ cannot fulfil condition (2), since it is bigger than $((4,4),(2,2,2,2),(1,1,2,2,2))$ (from case $\oplus$) which has negative Tits form.

Furthermore, if a tuple of compositions has positive Tits form, then condition (1) is not fulfilled.

If a tuple of compositions belongs to a case labelled with $(\star)$ and has Tits form zero, then we already know (by the last part of the proof of Proposition 5.3) that all smaller tuples of compositions have non negative Tits form. That means we have to list all tuples of compositions in the cases labelled with $(\star)$ with Tits form zero.

The tuples of compositions in the cases labelled with $(\star)$ and Tits form zero are exactly the ones which occur in the proof of Proposition 4.1 and are labelled with (b). $\square$

6 Decomposition properties of the s-tame vectors

In this chapter the underling quivers for the tuples of compositions are always assumed to be stars and *not* of tame type.

Lemma 6.1. *For every s-tame vector* $\mathbf{d}$ *except for the cases 27., 28., 29., and 30. there exists exactly one tame quiver which is smaller than the underlying quiver of* $\mathbf{d}$.

Proof. We obtain the result by comparing the lengths of the arms of the quivers with those of the tame ones. □

Lemma 6.2. *In the cases 27., 28., 29., and 30. the tuples of compositions are not bigger than a tuple of compositions equivalent to one belonging to the critical dimension vector of* $\tilde{\mathbb{E}}_8$.

Proof. In the cases 27., 28., and 29. we have

$$\min(\mathbf{a_1}) = 2,$$

but

$$\min(\mathbf{a_1'}) = 3,$$

for a tuple $(\mathbf{a_1'}, \mathbf{a_2'}, \mathbf{a_3'})$ of compositions equivalent to one belonging to the critical dimension vector of $\tilde{\mathbb{E}}_8$.

In case 30. we have

$$|\{a_{2j} \mid j = 1, \dots, 4;\ a_{2j} \geq 2\}| = 1,$$

but

$$|\{a_{2j}' \mid j = 1, \dots, 3;\ a_{2j}' \geq 2\}| = 3$$

in a tuple $(\mathbf{a_1'}, \mathbf{a_2'}, \mathbf{a_3'})$ of compositions equivalent to one belonging to the critical dimension vector of $\tilde{\mathbb{E}}_8$. □

Labels in Proposition 5.4	tame quiver
$k = 4$, 2. – 8.	$\tilde{\mathbb{D}}_4$
$k = 3$, 2. – 6., 16. – 23.	$\tilde{\mathbb{E}}_8$
$k = 3$, 8. – 12., 24. – 30., 44., 45.	$\tilde{\mathbb{E}}_7$
$k = 3$, 14., 15., 31. – 43.	$\tilde{\mathbb{E}}_6$

Figures 1 – 4 show how the s-tame tuples of compositions can be compared with each other. The tuples at the tops of the figures are the biggest ones, those at the bottoms the smallest ones, and they can be compared if there is a line between them. (Sometimes one has to extend the underlying quiver of the smaller one by one point and one arrow at the end of one arm and then add a zero for the corresponding entry in the tuple of compositions. This is indicated by dotted lines between the particular tuples.) The numbers are those of Proposition 5.4.

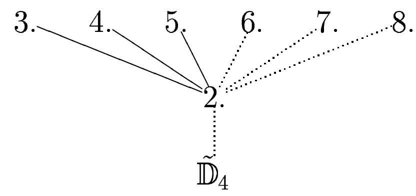


Figure 1: $\geq \tilde{\mathbb{D}}_4$

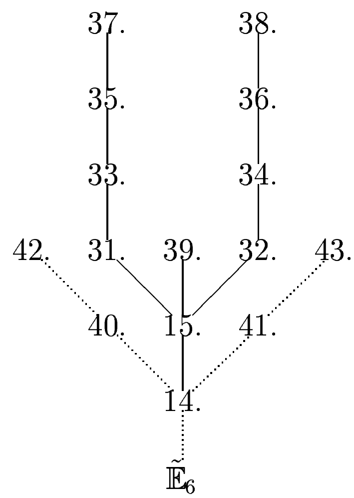


Figure 2: $\geq \tilde{\mathbb{E}}_6$

Proposition 6.3. *Every s-tame vector* $\mathbf{d}$ *can be decomposed into a sum of s-vectors* $\mathbf{d_1}$ *and* $\mathbf{d_2}$*, where* $\mathbf{d_1}$ *is equivalent to a critical dimension vector of a tame quiver and* $\mathbf{d_2}$ *is s-finite.*

Proof. Firstly we see by Figures 1 – 4 that every s-tame vector $\mathbf{d}$ can be decomposed into a sum of an s-vector $\mathbf{d_1}$ equivalent to a critical dimension vector of a tame quiver and another s-vector $\mathbf{d_2}$. Secondly it is not possible to split off more than one s-vector equivalent to a critical dimension vector of a tame quiver.

If it were possible to split off more than one s-vector equivalent to a critical dimension vector of a tame quiver, the central dimension would be at least twice as much as the central dimension of the critical dimension vector of the tame quiver.

This happens only for $k = 4$ in the cases 3. – 8. and for $k = 3$ in the cases 11., 12., 16. – 20., 24. – 26., 29., 31. – 39., and 42. – 45.

If one takes the sum of two s-vectors both equivalent to a critical dimension vector of the tame quiver which is uniquely determined by the above lemmas, the dimension jumps along the first two arms (for $k = 3$) or the first three arms (for $k = 4$) are uniquely determined.

The following possibilities occur:

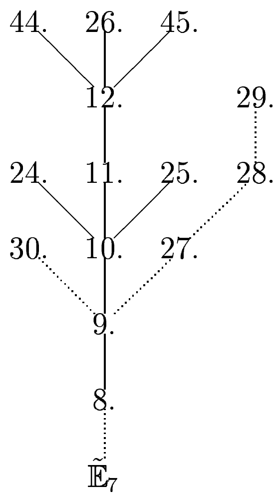


Figure 3: $\geq \tilde{\mathbb{E}}_7$

- $k = 4$, $\tilde{\mathbb{D}}_4$: $\mathbf{a_1} = \mathbf{a_2} = \mathbf{a_3} = (2,2)$

  This shows that splitting off more than one s-vector equivalent to a critical dimension vector of $\tilde{\mathbb{D}}_4$ is not possible in the cases 3. – 8.

- $k = 3$, $\tilde{\mathbb{E}}_6$: $\mathbf{a_1} = \mathbf{a_2} = (2,2,2)$

  This shows that splitting off more than one s-vector equivalent to a critical dimension vector of $\tilde{\mathbb{E}}_6$ is not possible in the cases 31. – 38., 42., and 43.

- $k = 3$, $\tilde{\mathbb{E}}_7$: $\mathbf{a_1} = (4,4), \mathbf{a_2} = (2,2,2,2)$

  This shows that splitting off more than one s-vector equivalent to a critical dimension vector of $\tilde{\mathbb{E}}_7$ is not possible in the cases 11., 12., 24., 26., 29., 44., and 45.

- $k = 3$, $\tilde{\mathbb{E}}_8$: $\mathbf{a_1} = (6,6), \mathbf{a_2} = (2,2,2)$

  This shows that splitting off more than one s-vector equivalent to a critical dimension vector of $\tilde{\mathbb{E}}_8$ is not possible in the cases 16. – 19.

The remaining cases are 20., 25., and 39.

But here we have the following properties:

$$|\{a_{3j} \mid j = 1, \ldots, 7;\ a_{3j} \geq 2\}| = 1,$$

but

$$|\{a'_{3j} \mid j = 1, \ldots, 7;\ a'_{3j} \geq 2\}| \geq 2$$

in a sum $(\mathbf{a'_1}, \mathbf{a'_2}, \mathbf{a'_3})$ of two tuples of compositions equivalent to one belonging to the critical dimension vector of $\tilde{\mathbb{E}}_6$ (case 39.),

$$|\{a_{3j} \mid j = 1, \ldots, 7;\ a_{3j} \geq 2\}| = 2,$$

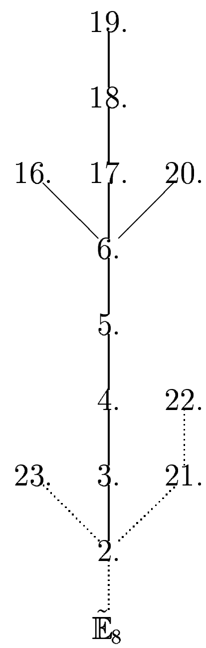


Figure 4: $\geq \tilde{\mathbb{E}}_8$

but

$$|\{a'_{3j} \mid j = 1, \ldots, 7;\ a'_{3j} \geq 2\}| \geq 3$$

in a sum $(\mathbf{a_1'}, \mathbf{a_2'}, \mathbf{a_3'})$ of two tuples of compositions equivalent to one belonging to the critical dimension vector of $\tilde{\mathbb{E}}_7$ (case 25.),

$$|\{a_{3j} \mid j = 1, \ldots, 7;\ a_{3j} \geq 2\}| = 4,$$

but

$$|\{a'_{3j} \mid j = 1, \ldots, 7;\ a'_{3j} \geq 2\}| \geq 5$$

in a sum $(\mathbf{a_1'}, \mathbf{a_2'}, \mathbf{a_3'})$ of two tuples of compositions equivalent to one belonging to the critical dimension vector of $\tilde{\mathbb{E}}_8$ (case 20.).

This shows (along with the classification of the s-finite vectors, see [13]) that $\mathbf{d_2}$ must be s-finite. □

Corollary 6.4. *If* $\mathbf{d}$ *is an s-tame vector and* $\mathbf{d} = \mathbf{d_1} + \mathbf{d_2}$ *is a decomposition of* $\mathbf{d}$ *with* $\mathbf{d_1}$ *s-tame, then* $\mathbf{d_2}$ *is s-finite.*

Proof. Proposition 6.3 shows that if $\mathbf{d} = \mathbf{d_1}' + \mathbf{d_2}'$ with $\mathbf{d_1}'$ equivalent to a critical dimension vector of a tame quiver, then $\mathbf{d_2}'$ is s-finite. Since $\mathbf{d_1} > \mathbf{d_1}'$, we have $\mathbf{d_2} < \mathbf{d_2}'$, and hence $\mathbf{d_2}$ is s-finite. □

This means that the one parameter families of *decomposable* subspace representations for the s-tame vectors can be constructed by using the one parameter families of indecomposable subspace representations for the smaller s-tame vectors and adding some subspace representations for s-finite vectors. Furthermore, in contrast to the case of the underlying quiver being of tame type, there are also no n-parameter families of decomposable subspace representations with $n \geq 2$.

In the tame case we can for example decompose

$$\mathbf{d} := \begin{matrix} & 2 & \\ 2 & 4 & 2 \\ & 2 & \end{matrix}$$

as

$$\begin{matrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{matrix}$$

and find a two parameter family of decomposable subspace representations for $\mathbf{d}$.

7 Reflection functors, Coxeter functors and the Auslander-Reiten translate

7.1 Reflections for representations of quivers

In 1973, Bernstein, Gel'fand, and Ponomarev introduced reflection functors for representations of quivers (see [2]).

Definition 7.1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A point $j \in Q_0$ is called a *source* (resp. a *sink*) if $t(\alpha) \neq j$ (resp. $s(\alpha) \neq j$) for all $\alpha \in Q_1$. It is called *accessible* if it is either a source or a sink.

Given a quiver $Q = (Q_0, Q_1, s, t)$ with a $\left\{ \begin{array}{c} \text{source} \\ \text{sink} \end{array} \right\}$ $j \in Q_0$, one constructs a new quiver $\left\{ \begin{array}{c} S_j^-(Q) = (Q_0, Q_1, s', t') \\ S_j^+(Q) = (Q_0, Q_1, s', t') \end{array} \right\}$ by changing the orientation of the arrows $\left\{ \begin{array}{c} \text{starting} \\ \text{terminating} \end{array} \right\}$ in j.

This means the new maps $s' : Q_1 \to Q_0$ and $t' : Q_1 \to Q_0$ are given by

$$s'(\alpha) = s(\alpha), t'(\alpha) = t(\alpha), \text{ if } s(\alpha) \neq j$$

and

$$s'(\alpha) = t(\alpha), t'(\alpha) = s(\alpha), \text{ if } s(\alpha) = j$$

in the first case (for S_j^-) and

$$s'(\alpha) = s(\alpha), t'(\alpha) = t(\alpha), \text{ if } t(\alpha) \neq j$$

and

$$s'(\alpha) = t(\alpha), t'(\alpha) = s(\alpha), \text{ if } t(\alpha) = j$$

in the second one (for S_j^+).

In [2], it is shown how one can construct a new representation $\left\{ \begin{array}{c} S_j^-(V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1} \\ S_j^+(V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1} \end{array} \right\}$ of the quiver $\left\{ \begin{array}{c} S_j^-(Q) \\ S_j^+(Q) \end{array} \right\}$ by calculating the cokernel (resp. kernel) of the sum of the maps involved in point $j \in Q_0$ in the given representation $(V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$.

With these reflection functors it is possible to construct a family of indecomposable representations from another family of indecomposable representations. But before we can show the next proposition we have to recall the definition of simple representations.

Definition 7.2. The representation $S(j) := (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ with $V_i = 0$ for all $i \in Q_0$ with $i \neq j$, $V_j = K$ and $V_\alpha = 0$ for all $\alpha \in Q_1$ is called *the simple representation according to the point* $j \in Q_0$.

Proposition 7.3. *The functors S_j^- and S_j^+, $j \in Q_0$, preserve the indecomposability of non simple representations and the number of parameters on which families of representations depend.*

Proof. The assertion follows from [2], Theorem 1.1 and Lemma 1.1. If (V_i, V_α) was indecomposable and there was a decomposition $(W_i^{(1)}, W_\alpha^{(1)}) \oplus (W_i^{(2)}, W_\alpha^{(2)})$ of $S_j^-(V_i, V_\alpha)$, then one could decompose $(V_i, V_\alpha) = S_j^+ \circ S_j^-(V_i, V_\alpha)$ into $S_j^+(W_i^{(1)}, W_\alpha^{(1)}) \oplus S_j^+(W_i^{(2)}, W_\alpha^{(2)})$ (and analogously for S_j^+).

The second part follows from the fact that every parameter in a family of representations gives rise to an indeterminant in a representing matrix in the representations, and this indeterminant is preserved under the reflections, since it is preserved under taking kernels and cokernels. □

From now on let Q be a quiver without oriented cycles. If one chooses a sequence $\left\{ \begin{array}{c} S_{j_{|Q_0|}}^- \circ \cdots \circ S_{j_1}^- \\ S_{j_{|Q_0|}}^+ \circ \cdots \circ S_{j_1}^+ \end{array} \right\}$ of possible reflections which contains every vertex $j \in Q_0$ exactly once, one obtains the Coxeter functors $\left\{ \begin{array}{c} C^- \\ C^+ \end{array} \right\}$. (The Coxeter functors are independent of the chosen order of the points in the sequence, see [2].) Clearly, these also preserve the indecomposability of representations, if no simple representation occurs in the sequence of representations, and the numbers of parameters for a family of representations. Note that the underlying quiver $S_{j_{|Q_0|}} \circ \cdots \circ S_{j_1}(Q)$ is again Q, since every arrow is reversed exactly twice.

The Coxeter functors $\left\{ \begin{array}{c} C^- \\ C^+ \end{array} \right\}$ give exactly the construction of $\left\{ \begin{array}{c} \tau^- \\ \tau \end{array} \right\}$, where τ is the Auslander-Reiten translate, in the Auslander-Reiten quiver of Q (which contains all isomorphism classes of indecomposable representations), provided the underlying quiver does not contain any cycle of odd length. This was shown in 1976 by Brenner and Butler (see [1]). Unfortunately, there was a mistake in their proof[2], but Gabriel showed in [7], that

$$\tau \xrightarrow{\sim} C^+ \circ T = T \circ C^+,$$

where

$$T : \operatorname{rep} Q \to \operatorname{rep} Q$$

is given by

$$T((V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}) := (V_i, -V_\alpha)_{i \in Q_0, \alpha \in Q_1}.$$

And if the quiver does not contain any cycle of odd length — especially for a star —, we always have isomorphisms

$$(V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1} \cong (V_i, -V_\alpha)_{i \in Q_0, \alpha \in Q_1}.$$

[2]They claimed that the Coxeter functors gave *always* the construction of τ and τ^-.

7.2 Reflections for dimension vectors

For a quiver Q without loops we have the corresponding *Weyl group* $W = W(Q)$. That is the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the reflections r_i in the points $i \in Q_0$ which are defined in the following way: Let $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$, then $r_i(\mathbf{d}) = (r_i(\mathbf{d})_j)_{j\in Q_0}$ with

$$r_i(\mathbf{d})_j = d_j \text{ for } j \neq i, \text{ and } r_i(\mathbf{d})_i = -d_i + \sum_{j\in \mathrm{adj}(i)} d_j,$$

where

$$\mathrm{adj}(i) = \{j \in Q_0 \mid \exists \alpha \in Q_1 \text{ with } s(\alpha) = i \text{ and } t(\alpha) = j \text{ or } s(\alpha) = j \text{ and } t(\alpha) = i\}.$$

(So reflecting in point $i \in Q_0$ means leaving the entry in a point $j \neq i$ as it was and taking the sum over all entries of the neighbours of point i and subtracting the original entry for the point i.)

Remark 7.4. *If* $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}_0^{Q_0}$ *is the dimension vector of a non simple representation* $(V_i, V_\alpha)_{i\in Q_0,\alpha\in Q_1}$ *and* $i \in Q_0$ *is an accessible point for the reflection* $\left\{ \begin{array}{c} S_i^+ \\ S_i^- \end{array} \right\}$, *then* $r_i(\mathbf{d})$ *is the dimension vector of the representation* $\left\{ \begin{array}{c} S_i^+(V_i, V_\alpha)_{i\in Q_0,\alpha\in Q_1} \\ S_i^-(V_i, V_\alpha)_{i\in Q_0,\alpha\in Q_1} \end{array} \right\}$.

But note that reflections on representations are only defined for accessible points whereas reflections on the dimension vectors can always be done (and sometimes lead to negative entries).

8 Families of indecomposable representations and a Theorem of Kac

From now on let K be an algebraically closed field.

8.1 The number of parameters for families of indecomposable representations

The main aim of this work is to characterise the s-tame vectors as follows:

They have a one parameter family of indecomposable subspace representations, and for every s-decomposition there is no indecomposable n-parameter family of indecomposable subspace representations for either of the summands with $n \geq 2$.

A family of representations R_p, $p \in I$, for a dimension vector depends on n parameters if there are "normal forms" $N(\lambda_1, \dots, \lambda_n)$ of the representations — depending on $(\lambda_1, \dots, \lambda_n) \in K^n$ —, such that for every $(\lambda_1, \dots, \lambda_n) \in K^n$ there exists $p \in I$ with $N(\lambda_1, \dots, \lambda_n) \cong R_p$ and $N(\lambda_1, \dots, \lambda_n) \ncong N(\nu_1, \dots, \nu_n)$ if $(\lambda_1, \dots, \lambda_n) \neq (\nu_1, \dots, \nu_n)$.

For a dimension vector $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ the number $\mu_{\mathbf{d}}(Q)$ is the maximal number of parameters on which a family of indecomposable representations of Q with dimension vector $\mathbf{d}$ can depend.

8.2 Root systems for quivers and a Theorem of Kac

By Kac ([9], Theorems 1 and 3) we know that for a quiver Q there is an indecomposable representation of dimension $0 \neq \mathbf{d} \in \mathbb{Z}^{Q_0}$ over K if and only if $\mathbf{d}$ is an element of the positive root system Δ^+ for the quiver Q. (In this text roots will always be *positive* roots.) So let us first recall how the positive root system Δ^+ looks like.

Let $\langle -, - \rangle$ denote the *Euler form* of Q which is a bilinear form on $\mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$ and given by the following formula:

$$\langle \mathbf{d}, \mathbf{d}' \rangle = \sum_{i \in Q_0} d_i d'_i - \sum_{\alpha \in Q_1} d_{s(\alpha)} d'_{t(\alpha)}$$

for $\mathbf{d} = (d_i)_{i \in Q_0}, \mathbf{d}' = (d'_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$.

The *symmetric Euler form* $(-, -) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is given by

$$(\mathbf{d}, \mathbf{d}') = \langle \mathbf{d}, \mathbf{d}' \rangle + \langle \mathbf{d}', \mathbf{d} \rangle$$

for $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^{Q_0}$. Note that $(\mathbf{d}, \mathbf{d}) = 2 \cdot \langle \mathbf{d}, \mathbf{d} \rangle = 2 \cdot q(\mathbf{d})$ for all $\mathbf{d} \in \mathbb{Z}^{Q_0}$, where q is the Tits form.

Let $\mathbf{e_i}$ denote the dimension vectors of the simple representations of Q in the points $i \in Q_0$ and $\Pi := \{\mathbf{e_i} \mid i \in Q_0\}$.

Kac has shown ([9], Proposition 1.1) that Δ^+ decomposes into two disjoint parts, the set Δ^+_{re} of positive real roots and the set Δ^+_{im} of imaginary roots which can be described as follows:

$$\Delta^+_{\text{re}} = \bigcup_{w \in W} (w(\Pi) \cap (\mathbb{N}_0^{Q_0} \backslash \{0\})) \text{ and } \Delta^+_{\text{im}} = \bigcup_{w \in W} w(M),$$

where M denotes the so called *fundamental set*

$$\left\{\mathbf{d} \in \mathbb{N}_0^{Q_0} \backslash \{0\} \mid (\mathbf{d}, \mathbf{e_i}) \leq 0 \text{ for all } i \in Q_0,\ \mathbf{d} \text{ has connected support}\right\}$$

and W is the Weyl group for the quiver (see Chapter 7).

Moreover we can describe whether a root is real or imaginary (see [9], Lemma 2.1):

Lemma 8.1 (Kac, 1980). *Let* $\mathbf{d} \in \Delta^+$. *Then:*

- $\mathbf{d} \in \Delta^+_{re} \Leftrightarrow q(\mathbf{d}) = 1$
- $\mathbf{d} \in \Delta^+_{im} \Leftrightarrow q(\mathbf{d}) \leq 0$

As already mentioned in the introduction, the maximal number $\mu_{\mathbf{d}}(Q)$ of parameters for families of indecomposable representations for a dimension vector $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ is closely related to the Tits form of $\mathbf{d}$.

Kac has shown the following Theorem in [11], §1.10:

Theorem 8.2 (Kac, 1982). *Let K be an algebraically closed field and $Q = (Q_0, Q_1, s, t)$ a quiver. Then:*

- *There exists an indecomposable representation of dimension $0 \neq \mathbf{d} \in \mathbb{N}_0^{Q_0}$ if and only if $d \in \Delta^+$.*
- *There exists a unique indecomposable representation (up to isomorphism) of dimension $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ if and only if $d \in \Delta^+_{\text{re}}$.*
- *If $d \in \Delta^+_{\text{im}}$, then $\mu_{\mathbf{d}}(Q) = 1 - q(\mathbf{d})$.*

8.3 Existence of families of indecomposable representations for the s-tame and s-hypercritical dimension vectors

In the next lemma we show that reflecting tuples of compositions within the arms just means changing the order of the entries along the arms. This is a very useful property in order to show that an s-vector is indeed a root. For reflections within the arms we can restrict ourselves to a quiver Q of type $\mathbb{A}$, since nothing happens with the central point.

Lemma 8.3 (Reflecting tuples of compositions within their arms). *Let $(a_1, \dots, a_p) \in \mathbb{N}_0^p$ be a composition. Then*

$$\sigma^{-1} \circ r_i \circ \sigma(a_1, \dots, a_p) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_p)$$

for all $i = 1, \dots, p-1$, where σ is the summation map from Proposition 3.2.

Proof. We have $\sigma((a_1, \dots, a_p)) = (a_1, a_1 + a_2, \dots, \sum_{j=1}^{p} a_j)$. Reflecting in position i means changing the dimension $\sum_{j=1}^{i} a_j$ in position i to the dimension $-\sum_{j=1}^{i} a_j + \sum_{j=1}^{i-1} a_j + \sum_{j=1}^{i+1} a_j = \sum_{j=1}^{i-1} a_j + a_{i+1}$ and leaving the others as before. Therefore

$$\sigma^{-1} \circ r_i \circ \sigma((a_1, \dots, a_p)) = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_p).$$

□

Now we are going to prove that all s-hypercritical and all s-tame dimension vectors are indeed roots.

If the underlying quiver of a dimension vector is of hyperbolic type, then verifying that a dimension vector is an imaginary root is very easy. Namely, every non zero dimension vector with non positive Tits form is then a root.

The following result was also shown by Kac in [9], Lemma 2.1.

Lemma 8.4 (Kac, 1980). *If the quiver Q is of hyperbolic type, then the set of all imaginary roots is*

$$\left\{ \mathbf{d} \in \mathbb{Z}^{Q_0} \backslash \{0\} \mid q(\mathbf{d}) \leq 0 \right\},$$

where q is the Tits form of Q.

Now we can show the following:

Proposition 8.5. *If $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ is an s-tame vector, then it is also an imaginary root for Q.*

Proof. By Lemma 8.3 it suffices to show the property for all s-tame tuples of compositions which are ordered increasingly along their arms. The Tits form is preserved under the action of the Weyl group (see [9], Lemma 1.9), and so property (1) for s-tame tuples of compositions and Lemma 8.4 show that it is enough to show the following: every s-tame tuple of compositions which is ordered increasingly along its arms can be reflected to a tuple of compositions equivalent to an s-tame one with underlying quiver of hyperbolic type.

The cases in which the underlying quivers are *not* of hyperbolic type are the 13 cases connected with dotted lines in Figures 1 – 4.

The different cases for $k = 4$ (6., 7., and 8.) are just relabellings of the arms and therefore show the same behaviour for the reflections. For $k = 3$, the cases 40. and 41. and the cases 42. and 43. are also relabellings of the arms.

So we have to consider the following ten cases:

- $k = 4$ (no. 6.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) = ((1,3),(1,3),(2,2),(1,1,1,1))$: This can be reflected by applying C^- to $((2,1),(2,1),(1,2),(0,1,1,1))$ which is a root.
- $k = 3$ (no. 40.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,3),(1,2,2),(1,1,1,1,1))$: This can be reflected by applying C^- to $((2,1,1),(1,1,2),(0,1,1,1,1))$ which is a root.
- $k = 3$ (no. 42.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((1,1,4),(2,2,2),(1,1,1,1,1,1))$: This can be reflected by applying C^- to $((3,1,1),(1,2,2),(0,1,1,1,1,1))$ which is a root (by the previous step).
- $k = 3$ (no. 27.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,4),(1,1,2,2),(1,1,1,1,1,1))$: This can be reflected by applying C^- to $((3,2),(1,1,1,2),(0,1,1,1,1,1))$ which is a root (no. 8. in Proposition 5.4).
- $k = 3$ (no. 28.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,5),(1,2,2,2),(1,1,1,1,1,1,1))$: This can be reflected by applying C^- to $((4,2),(1,1,2,2),(0,1,1,1,1,1,1))$ which is a root (by the previous step).
- $k = 3$ (no. 29.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((2,6),(2,2,2,2),(1,1,1,1,1,1,1,1))$: This can be reflected by applying C^- to $((5,2),(1,2,2,2),(0,1,1,1,1,1,1,1))$ which is a root (by the previous step).
- $k = 3$ (no. 30.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,3),(1,1,1,3),(1,1,1,1,1,1))$: This can be reflected by applying C^- to $((2,3),(2,1,1,1),(0,1,1,1,1,1))$ which is a root (no. 8. in Proposition 5.4).
- $k = 3$ (no. 21.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,5),(2,3,3),(1,1,1,1,1,1,1,1))$: This can be reflected by applying C^- to $((4,3),(2,2,3),(0,1,1,1,1,1,1,1))$ which is a root (no. 2. in Proposition 5.4).
- $k = 3$ (no. 22.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((3,6),(3,3,3),(1,1,1,1,1,1,1,1,1))$: This can be reflected by applying C^- to $((5,3),(2,3,3),(0,1,1,1,1,1,1,1,1))$ which is a root (by the previous step).
- $k = 3$ (no. 23.): $(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) = ((4,4),(2,2,4),(1,1,1,1,1,1,1,1))$: This can be reflected by applying C^- to $((3,4),(3,2,2),(0,1,1,1,1,1,1,1))$ which is a root (no. 2. in Proposition 5.4).

□

Before we get the same result for the s-hypercritical vectors, we are going to prove another lemma which shows that for reduced dimension vectors $\mathbf{d}$ of stars we have to check the conditions

$$(\mathbf{d}, \mathbf{e_i}) \leq 0$$

for $\mathbf{d}$ being an element in the fundamental set only for the central points.

Lemma 8.6. *Let* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *be a reduced tuple of compositions and* $c \in Q_0$ *denote the central point of the underlying quiver. Then*

$$(\sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}), \mathbf{e}_\ell) \leq 0$$

for all $\ell \in Q_0 \backslash \{c\}$.

Proof. Since $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ is reduced, we have

$$a_{i,j} \leq a_{i,j+1}$$

for all $i = 1, \ldots, k$ and all $j = 1, \ldots, p_i - 1$.

Suppose $\ell = (i, j) \in Q_0$ for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, p_i - 1\}$. We have

$$\langle \sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}), \mathbf{e}_\ell \rangle = \sum_{t=1}^{j} a_{i,t} \cdot 1 - \sum_{t=1}^{j-1} a_{i,t} \cdot 1 = a_{i,j}$$

and

$$\langle \mathbf{e}_\ell, \sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}) \rangle = 1 \cdot \sum_{t=1}^{j} a_{i,t} - 1 \cdot \sum_{t=1}^{j+1} a_{i,t} = -a_{i,j+1},$$

so

$$(\sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}), \mathbf{e}_\ell) = \langle \sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}), \mathbf{e}_\ell \rangle + \langle \mathbf{e}_\ell, \sigma(\mathbf{a_1}, \ldots, \mathbf{a_k}) \rangle = a_{i,j} - a_{i,j+1} \leq 0.$$

□

Proposition 8.7. *If* $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ *is an s-hypercritical vector, then it is also an imaginary root for* Q.

Proof. As before we can restrict ourselves to the cases in which the tuples of compositions are ordered increasingly along their arms (by using Lemma 8.3) and the underlying quiver is *not* of hyperbolic type (by using Lemma 8.4).

We are going to show now that each of these tuples of compositions is already contained in the fundamental set.

The cases to deal with are the following:

$k=4$, cases 2. – 8. and $k=3$, cases 2., 5., 6., 8., 9. and 10., where cases 3., 4. and 5. for $k=4$ are just relabellings of the arms.

By Lemma 8.6 we only have to check the properties for the central points in the quivers.

- $k=4$ (no. 2)

$$\left(\begin{array}{cccccccccc} & 2 & & & & & 0 & & \\ 4 & 2 & & & & 1 & 0 & & \\ & 2 & & & , & & 0 & & \\ & 3 & 2 & 1 & & & 0 & 0 & 0 \end{array}\right) = 2\cdot 4-2-2-2-3=-1$$

- $k=4$ (no. 3)

$$\left(\begin{array}{cccccccccc} & 1 & & & & & 0 & & \\ 4 & 2 & & & & 1 & 0 & & \\ & 2 & & & , & & 0 & & \\ & 3 & 2 & 1 & & & 0 & 0 & 0 \end{array}\right) = 2\cdot 4-1-2-2-3=0$$

- $k=4$ (no. 6)

$$\left(\begin{array}{ccccccc} & 1 & & & & 0 & \\ 3 & 1 & & & 1 & 0 & \\ & 2 & 1 & , & & 0 & 0 \\ & 2 & 1 & & & 0 & 0 \end{array}\right) = 2\cdot 3-1-1-2-2=0$$

- $k=4$ (no. 7)

$$\left(\begin{array}{ccccccc} & 1 & & & & 0 & \\ 3 & 2 & 1 & & 1 & 0 & 0 \\ & 2 & 1 & , & & 0 & 0 \\ & 2 & 1 & & & 0 & 0 \end{array}\right) = 2\cdot 3-1-2-2-2=-1$$

- $k=4$ (no. 8)

$$\left(\begin{array}{ccccccc} & 2 & 1 & & & 0 & 0 \\ 3 & 2 & 1 & & 1 & 0 & 0 \\ & 2 & 1 & , & & 0 & 0 \\ & 2 & 1 & & & 0 & 0 \end{array}\right) = 2\cdot 3-2-2-2-2=-2$$

- $k=3$ (no. 2)

$$\left(\begin{array}{ccccccccccccccccc} & 4 & & & & & & & & & 0 & & & & & & \\ 8 & 5 & 2 & & & & & & , & 1 & 0 & 0 & & & & & \\ & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = 2\cdot 8-4-5-7=0$$

- $k = 3$ (no. 5)

$$\left(\begin{array}{ccccccccccccc} & 3 & & & & & & & 0 & & & & \\ 6 & 4 & 2 & 1 & & & , & 1 & 0 & 0 & 0 & & \\ & 5 & 4 & 3 & 2 & 1 & & & 0 & 0 & 0 & 0 & 0 \end{array} \right) = 2 \cdot 6 - 3 - 4 - 5 = 0$$

- $k = 3$ (no. 6)

$$\left(\begin{array}{ccccccccccc} & 2 & & & & & & 0 & & & \\ 5 & 4 & 3 & 2 & 1 & , & 1 & 0 & 0 & 0 & 0 \\ & 4 & 3 & 2 & 1 & & & 0 & 0 & 0 & 0 \end{array} \right) = 2 \cdot 5 - 2 - 4 - 4 = 0$$

- $k = 3$ (no. 8)

$$\left(\begin{array}{ccccccccccc} & 3 & 1 & & & & & 0 & 0 & & \\ 5 & 3 & 1 & & & , & 1 & 0 & 0 & & \\ & 4 & 3 & 2 & 1 & & & 0 & 0 & 0 & 0 \end{array} \right) = 2 \cdot 5 - 3 - 3 - 4 = 0$$

- $k = 3$ (no. 9)

$$\left(\begin{array}{ccccccccc} & 2 & 1 & & & & 0 & 0 & \\ 4 & 3 & 2 & 1 & , & 1 & 0 & 0 & 0 \\ & 3 & 2 & 1 & & & 0 & 0 & 0 \end{array} \right) = 2 \cdot 4 - 2 - 3 - 3 = 0$$

- $k = 3$ (no. 10)

$$\left(\begin{array}{ccccccccc} & 3 & 2 & 1 & & & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & , & 1 & 0 & 0 & 0 \\ & 3 & 2 & 1 & & & 0 & 0 & 0 \end{array} \right) = 2 \cdot 4 - 3 - 3 - 3 = -1$$

□

By applying Kac's Theorem we immediately get the following Corollary:

Corollary 8.8. *For every s-tame vector there is a one parameter family of indecomposable representations, but no n-parameter family of indecomposable representations with $n \geq 2$. For every s-hypercritical vector there is an n-parameter family of indecomposable representations with $n \geq 2$.*

Since (by Lemma 2.1) all indecomposable representations of s-vectors are already subspace representations, we can even replace "indecomposable representations" by "indecomposable subspace representations".

Remark 8.9. *This only shows the existence of the families of indecomposable (subspace) representations, but gives no constructive procedure since the reflections on the dimension vectors used here are sometimes non accessible reflections for the representations.*[3] *Constructing the families as reflections of representations only works with accessible reflections as used by Bernstein, Gel'fand, and Ponomarev in [2]. This is done in the Chapters 11.1 and 11.2.*

[3] A reflection r_i, $i \in Q_0$, is called *accessible*, if i is an accessible point for the given orientation of the quiver Q.

9 Characterisation of the s-tame and the s-hypercritical dimension vectors

9.1 Characterisation of the s-tame dimension vectors

The aim of this section is to give a characterisation of the s-tame dimension vectors.

Remember that (by Theorem 8.2) for a root $\mathbf{d}$ there is a unique isomorphism class of indecomposable representations if and only if $q(\mathbf{d}) = 1$ and that there is no indecomposable representation if $\mathbf{d}$ is not a root. Also, all roots have Tits form ≤ 1.

We have the following Theorem:

Theorem 9.1. *For a tuple* $(\mathbf{a_1}, \dots, \mathbf{a_k})$ *of compositions of a number the following assertions are equivalent:*

1. $(\mathbf{a_1}, \dots, \mathbf{a_k})$ *is s-tame.*
2. $(\mathbf{a_1^{red}}, \dots, \mathbf{a_k^{red}})$ *is contained in* **List 2** *(see Proposition 5.4).*
3. *There is a one parameter family of indecomposable subspace representations for* $(\mathbf{a_1}, \dots, \mathbf{a_k})$, *and for every s-decomposition* $(\mathbf{a_1}, \dots, \mathbf{a_k}) = (\mathbf{b_1^{(1)}}, \dots, \mathbf{b_k^{(1)}}) + (\mathbf{b_1^{(2)}}, \dots, \mathbf{b_k^{(2)}})$ *there is no n-parameter family of indecomposable subspace representations with* $n \geq 2$ *for* $(\mathbf{b_1^{(1)}}, \dots, \mathbf{b_k^{(1)}})$ *or* $(\mathbf{b_1^{(2)}}, \dots, \mathbf{b_k^{(2)}})$.

Proof. The equivalence of the conditions 1. and 2. is the content of Proposition 5.4.

One of the consequences of Kac's Theorem is that for every imaginary root the maximal number of parameters needed for families of indecomposable representations can be calculated by the Tits form: If $\mathbf{d}$ is an imaginary root, then $\mu_{\mathbf{d}} = 1 - q(\mathbf{d})$.

Let $(\mathbf{a_1}, \dots, \mathbf{a_k})$ be s-tame. Then we have a one parameter family of indecomposable subspace representations for $(\mathbf{a_1}, \dots, \mathbf{a_k})$, but no n-parameter family of indecomposable subspace representations with $n \geq 2$ (by Corollary 8.8). All smaller tuples of compositions are (by definition) also s-tame or have Tits form > 0. For the s-tame ones there is no n-parameter family of indecomposable subspace representations with $n \geq 2$ (also by Corollary 8.8), and if the Tits form is bigger than zero, there are no families of indecomposable subspace representations at all. Therefore both conditions in 3. are fulfilled.

Let now $(\mathbf{a_1}, \dots, \mathbf{a_k})$ be a tuple of compositions fulfilling both conditions in 3. Then it cannot be bigger than any s-hypercritical tuple of compositions (or an s-hypercritical tuple of compositions itself), because we had an n-parameter family of indecomposable subspace representations for a smaller tuple of compositions (or the tuple of compositions itself), otherwise , since for all s-hypercritical tuples of compositions there is an n-parameter family of indecomposable representations with $n \geq 2$ (by Corollary 8.8). Therefore, we have

$$\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) \geq 0$$

for all $(\mathbf{b_1}, \ldots, \mathbf{b_k}) \leq (\mathbf{a_1}, \ldots, \mathbf{a_k})$. And the condition on the existence of a one parameter family of indecomposable subspace representations for $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ gives that $\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) = 0$. □

9.2 Characterisation of the s-hypercritical dimension vectors

Not only the s-tame dimension vectors, but also the s-hypercritical dimension vectors can be characterised by the existence and non existence of families of representations depending on a certain amount of parameters.

We can show the following proposition:

Proposition 9.2. *For a tuple* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *of compositions of a number the following assertions are equivalent:*

1. $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *is s-hypercritical.*
2. $(\mathbf{a_1^{red}}, \ldots, \mathbf{a_k^{red}})$ *is contained in* **List 1** *(see Proposition 5.3).*
3. *There is an n-parameter family of subspace representations for* $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ *with* $n \geq 2$*, but for every proper s-decomposition* $(\mathbf{a_1}, \ldots, \mathbf{a_k}) = (\mathbf{b_1^{(1)}}, \ldots, \mathbf{b_k^{(1)}}) + (\mathbf{b_1^{(2)}}, \ldots, \mathbf{b_k^{(2)}})$ *there is no n-parameter family of indecomposable subspace representations with* $n \geq 2$ *for* $(\mathbf{b_1^{(1)}}, \ldots, \mathbf{b_k^{(1)}})$ *or* $(\mathbf{b_1^{(2)}}, \ldots, \mathbf{b_k^{(2)}})$*.*

Proof. The equivalence of the conditions 1. and 2. was shown in Proposition 5.3.

Let $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ be s-hypercritical. By Corollary 8.8 we have an n-parameter family of indecomposable subspace representations for $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ with $n \geq 2$. All smaller tuples of compositions must be either s-tame or have Tits form > 0. For the s-tame ones we already know (again by Corollary 8.8) that there is no n-parameter family of indecomposable subspace representations with $n \geq 2$, and for a tuple of compositions with positive Tits form there are no families of indecomposable representations at all. Therefore both conditions in 3. are fulfilled.

Let now $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ a tuple of compositions which fulfils both conditions in 3. The existence of an n-parameter family of indecomposable subspace representations shows that $\overline{q}(\mathbf{a_1}, \ldots, \mathbf{a_k}) < 0$. But for all smaller tuples of compositions there is never an n-parameter family of indecomposable subspace representations with $n \geq 2$. By the characterisation of the s-tame tuples of compositions (Theorem 9.1) we know that all smaller tuples of compositions are s-tame — thus have Tits form zero — or s-finite. If a tuple $(\mathbf{b_1}, \ldots, \mathbf{b_k})$ of compositions is s-finite, then in particular $\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) > 0$ (see [13]). Therefore $\overline{q}(\mathbf{b_1}, \ldots, \mathbf{b_k}) \geq 0$ for all $(\mathbf{b_1}, \ldots, \mathbf{b_k}) < (\mathbf{a_1}, \ldots, \mathbf{a_k})$. So $(\mathbf{a_1}, \ldots, \mathbf{a_k})$ is s-hypercritical. □

10 s-tame $\neq$ tame

If we skip the properties of the representations to be *subspace* representations, we can also define tameness for arbitrary dimension vectors of stars with subspace orientation.

Definition 10.1. A dimension vector $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}_0^{Q_0}$ is called *tame*, if there is a one parameter family of representations for this dimension vector, but there is no (arbitrary) decomposition into a sum of dimension vectors with an n-parameter family of indecomposable representations and $n \geq 2$ for one of the summands.

We will see that not all s-tame dimension vectors are tame. This is a consequence of the fact, that s-decompositions of dimension vectors are just special decompositions and therefore the partial order of the s-vectors differs from the partial order of usual dimension vectors.

10.1 An example: not all s-tame dimension vectors are tame

The following tuple of compositions of 4 is s-tame, but not tame:

$$((3,1),(2,2),(2,2),(1,1,2))$$

There is a two parameter family of representations for this tuple of compositions, which one can construct by decomposing the corresponding dimension vector

$$\begin{array}{ccc} & 3 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{array}$$

into the sum

$$\begin{array}{ccc} & 2 & \\ 2 & 4 & 2 \\ & 2 & \\ & 1 & \end{array} \oplus \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \\ & 0 & \end{array}.$$

For the dimension vector on the left hand side there is an indecomposable two parameter family of representations. (In fact, it is s-hypercritical.) But note that the family of representations constructed in this way is *not* a family of *subspace* representations, because there is no subspace representation for the dimension vector on the right hand side in this decomposition.

11 Construction methods for families of indecomposable representations

In Chapter 8 we have shown the existence of one parameter families of indecomposable subspace representations for the s-tame dimension vectors and the existence of n-parameter families of indecomposable subspace representations with $n \geq 2$ for the s-hypercritical dimension vectors. Another question arising from this is, how the families actually look like.

We construct a smaller quiver from the original one by deleting one point and one arrow at the end of one of the arms. If the representations for the smaller quiver are known, then the task is to find the right embedding(s) for the vector space(s) at the deleted point(s) in order to construct a family of indecomposable representations for the original quiver.

Since every representation (V_i, V_α) is isomorphic to one of type (K^{d_i}, f_α), it suffices to find representing matrices for the embeddings with respect to the standard bases of the K^{d_i}'s which contain sufficiently many indeterminants.

If not mentioned, in this chapter there are no non zero homomorphisms and only trivial extensions between the different representations needed for the construction. Furthermore, all representations for the restricted quivers have trivial endomorphism rings, i. e. all homomorphisms from the representations to themselves are given by scalar multiplications.

Since there are no non trivial extensions between the indecomposable representations for the decompositions of the smaller dimension vectors, we can even see how the restrictions of maps between the vector spaces at the sources of the restricted quivers look like.

We use the following Lemma, which is a special case of a Lemma by Happel and Ringel (see [8], Lemma 4.1).

Lemma 11.1. *Let $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $W = (W_i, W_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be non isomorphic indecomposable representations of a quiver $Q = (Q_0, Q_1, s, t)$ with $\operatorname{Ext}^1(W, V) = 0$. If $f \in \operatorname{Hom}(V, W)$, then f is either injective or surjective.*

As a consequence we get that the dimension of the restriction of the homomorphism space between two non isomorphic indecomposable representations of a quiver to two non zero vector spaces is at least 1, provided there are non zero homomorphisms, but no non trivial extensions between the representations.

Let $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}, W = (W_i, W_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \operatorname{rep} Q$ and $j \in Q_0$. Then set

$$\operatorname{Hom}(V, W)|_j := \{f_j : V_j \to W_j \mid \exists f = (f_i)_{i \in Q} \in \operatorname{Hom}(V, W)\}.$$

Corollary 11.2. *Let $V = (V_i, V_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $W = (W_i, W_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be non isomorphic indecomposable representations of a quiver $Q = (Q_0, Q_1, s, t)$ with $\dim_K \operatorname{Hom}(V, W) = 1$,*

$\dim_K \operatorname{Ext}^1(W, V) = 0$, $V_j \neq 0$ *and* $W_j \neq 0$ *for some* $j \in Q_0$. *Then*

$$\dim_K \operatorname{Hom}(V, W)|_j = 1.$$

Proof. Suppose there is a non zero homomorphism $(f_i)_{i \in Q_0} = f : V \to W$. By Lemma 11.1 we have the following two cases:

Case 1. f is injective. Therefore, also $f_j : V_j \to W_j$ is injective. But then $f_j(V_j) \neq 0$, since $V_j \cong f_j(V_j)$ and $V_j \neq 0$.

Case 2. f is surjective. Therefore, also $f_j : V_j \to W_j$ is surjective. But then $f_j(V_j) = W_j \neq 0$.

Therefore, $\dim_K \operatorname{Hom}(V, W)|_j \geq 1$. But we also have $\dim_K \operatorname{Hom}(V, W)|_j \leq 1$, since $\dim_K \operatorname{Hom}(V, W) = 1$. □

11.1 Construction methods for n-parameter families of indecomposable representations with $n \geq 2$

Proposition 11.3.

Ⓐ *Let* (V_i, V_α) *be a representation with* K^3 *at a source. If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}} K^3 ,$$

we get a two parameter family of indecomposable representations.

Ⓑ *Let* $(V_i^{(1)}(\lambda_1, \ldots, \lambda_{n_1}), V_\alpha^{(1)}(\lambda_1, \ldots, \lambda_{n_1}))_{(\lambda_1, \ldots, \lambda_{n_1}) \in K^{n_1}}$ *be an* n_1*-parameter family of representations and* $(V_i^{(2)}(\mu_1, \ldots, \mu_{n_2}), V_\alpha^{(2)}(\mu_1, \ldots, \mu_{n_2}))_{(\mu_1, \ldots, \mu_{n_2}) \in K^{n_2}}$ *be an* n_2*-parameter family of representations, each of which has a* K *at a source. If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \hline 1 \end{pmatrix}} K \oplus K ,$$

we get an $(n_1 + n_2)$*-parameter family of indecomposable representations.*

Ⓒ *Let* $(V_i(\lambda_1, \ldots, \lambda_n), V_\alpha(\lambda_1, \ldots, \lambda_n))_{(\lambda_1, \ldots, \lambda_n) \in K^n}$ *be an* n*-parameter family of representations with* K^2 *at a source. If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}} K^2 ,$$

we get an $(n+1)$-parameter family of indecomposable representations.

(D) *Let $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, \ldots, n_1$ with $n_1 \in \mathbb{N}_0$, be representations, each of which has a K at the same source, and $(W_i^{(j)}, W_\alpha^{(j)})$, $j = 1, \ldots, n_2$ with $n_2 \in \mathbb{N}\backslash\{1\}$, be representations, each of which has a K^2 at the source. If we choose the embedding*

$$K \xrightarrow{\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda_1 \\ \hline \vdots \\ \hline 1 \\ \lambda_{n_2} \end{array}\right)} K \oplus \cdots \oplus K \oplus K^2 \oplus \cdots \oplus K^2 ,$$

we get an n_2-parameter family of indecomposable representations.

Moreover, all representations of the constructed families have trivial endomorphism rings.

Proof. We have to show the following in each of the cases: If

$$\begin{array}{ccc} K^{a_1} & \xrightarrow{M} & K^{a_2} \\ {\scriptstyle B_1}\downarrow & & \downarrow{\scriptstyle B_2} \\ K^{a_1} & \xrightarrow[M']{} & K^{a_2} \end{array}$$

commutes for the allowed base change matrices A and B, then $A = 0$ and $B = 0$, or $M = M'$ and $A = b \cdot I_{a_1}$ and $B = b \cdot I_{a_2}$.

(A) We have to consider the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}} & K^3 \\ {\scriptstyle (a)}\downarrow & & \downarrow{\begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}} \\ K & \xrightarrow[\begin{pmatrix} 1 \\ \mu_1 \\ \mu_2 \end{pmatrix}]{} & K^3 \end{array} .$$

Then

$$\begin{pmatrix} a \\ a\mu_1 \\ a\mu_2 \end{pmatrix} = \begin{pmatrix} b \\ b\lambda_1 \\ b\lambda_2 \end{pmatrix},$$

and

$$\begin{cases} a = b = 0 \text{ or} \\ a = b \neq 0 \text{ and } \lambda_1 = \mu_1 \text{ and } \lambda_2 = \mu_2. \end{cases}$$

So we get a two parameter family of indecomposable representations with trivial endomorphism rings.

(B) We have to consider the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\left(\frac{1}{1}\right)} & K \oplus K \\ \Big\downarrow{\scriptstyle (a)} & & \Big\downarrow{\scriptstyle \left(\begin{array}{c|c} b_1 & 0 \\ \hline 0 & b_2 \end{array}\right)} \\ K & \xrightarrow[\left(\frac{1}{1}\right)]{} & K \oplus K \end{array} .$$

Then

$$\begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and

$$a = b_1 = b_2.$$

So we get an indecomposable representation for each of the $((\lambda_1, \dots, \lambda_{n_1}), (\mu_1, \dots, \mu_{n_2}))$, therefore an (n_1+n_2)-parameter family of indecomposable representations with trivial endomorphism rings.

(C) We have to consider the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}} & K^2 \\ \Big\downarrow{\scriptstyle (a)} & & \Big\downarrow{\scriptstyle \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}} \\ K & \xrightarrow[\begin{pmatrix} 1 \\ \mu \end{pmatrix}]{} & K^2 \end{array} .$$

Then

$$\begin{pmatrix} a \\ a\mu \end{pmatrix} = \begin{pmatrix} b \\ b\lambda \end{pmatrix},$$

and

$$\begin{cases} a = b = 0 \text{ or} \\ a = b \neq 0 \text{ and } \lambda = \mu. \end{cases}$$

So we get an $(n+1)$-parameter family of indecomposable representations with trivial endomorphism rings.

(D) We have to consider the following commutative diagram:

$$\begin{array}{ccc}
K & \xrightarrow{\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda_1 \\ \hline \vdots \\ \hline 1 \\ \lambda_{n_2} \end{array}\right)} & K \oplus \cdots \oplus K \oplus K^2 \oplus \cdots \oplus K^2 \,, \\
\Big\downarrow {\scriptstyle (a)} & & \Big\downarrow {\scriptstyle B} \\
K & \xrightarrow[\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \mu_1 \\ \hline \vdots \\ \hline 1 \\ \mu_{n_2} \end{array}\right)]{} & K \oplus \cdots \oplus K \oplus K^2 \oplus \cdots \oplus K^2
\end{array}$$

$$
\text{where } B := \left(\begin{array}{c|c|c|cc|c|cc}
b_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \hline
0 & \ddots & \ddots & & & & & \vdots \\ \hline
\vdots & \ddots & b_{n_1} & \ddots & & & & \vdots \\ \hline
\vdots & & \ddots & b_{n_1+1} & \ddots & & & \vdots \\
\vdots & & & \ddots & b_{n_1+1} & \ddots & & \vdots \\ \hline
\vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \hline
\vdots & & & & & \ddots & b_{n_1+n_2} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_{n_1+n_2}
\end{array}\right).
$$

Then

$$
\left(\begin{array}{c}
a \\ \hline \vdots \\ \hline a \\ \hline a \\ a\mu_1 \\ \hline \vdots \\ \hline a \\ a\mu_{n_2}
\end{array}\right)
=
\left(\begin{array}{c}
b_1 \\ \hline \vdots \\ \hline b_{n_1} \\ \hline b_{n_1+1} \\ b_{n_1+1}\lambda_1 \\ \hline \vdots \\ \hline b_{n_1+n_2} \\ b_{n_1+n_2}\lambda_{n_2}
\end{array}\right),
$$

and

$$
\begin{cases}
a = b_1 = \cdots = b_{n_1+n_2} = 0 \text{ or} \\
a = b_1 = \cdots = b_{n_1+n_2} \neq 0 \text{ and } \lambda_i = \mu_i \text{ for all } i = 1, \ldots, n_2.
\end{cases}
$$

So we get an n_2-parameter family of indecomposable representations with trivial endomorphism rings.

□

11.2 Construction methods for one parameter families of indecomposable representations

The following shows how one can construct one parameter families of subspace representations for the s-tame dimension vectors.

Proposition 11.4.

(A) *Let* $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, \ldots, n$ *with* $n \in \mathbb{N}_0$, *be representations, each of which has a* K *at the same source, and* (W_i, W_α) *a representation which has a* K^2 *at the source. If* *we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda \end{pmatrix}} K \oplus \cdots \oplus K \oplus K^2 \,,$$

we get a one parameter family of indecomposable representations.

(B) *Let* $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, \dots, n$ *with* $n \in \mathbb{N}_0$, *be representations, each of which has a* K *at the same source, and* $(W_i^{(1)}, W_\alpha^{(1)})$ *and* $(W_i^{(2)}, W_\alpha^{(2)})$ *two representations, each of which has a* K^2 *at the source.*
Let $\dim \operatorname{Hom}((W_i^{(1)}, W_\alpha^{(1)}), (W_i^{(2)}, W_\alpha^{(2)})) = 1$. *If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda \\ \hline 0 \\ 1 \end{pmatrix}} K \oplus \cdots \oplus K \oplus K^2 \oplus K^2 \,,$$

we get a one parameter family of indecomposable representations.

(C) *Let* $(V_i^{(1)}, V_\alpha^{(1)})$ *and* $(V_i^{(2)}, V_\alpha^{(2)})$ *be representations, each of which has a* K *at the same source, and* (W_i, W_α) *a representation which has* K^2 *at the source.*
Let $\dim \operatorname{Hom}((V_i^{(1)}, V_\alpha^{(1)}), (V_i^{(2)}, V_\alpha^{(2)})) = 1$. *If we choose the embedding*

$$K^2 \xrightarrow{\begin{pmatrix} 1 & \lambda \\ \hline 0 & 1 \\ \hline 1 & 0 \\ 1 & 1 \end{pmatrix}} K \oplus K \oplus K^2 \,,$$

we get a one parameter family of indecomposable representations.

(D) *Let* $(V_i^{(1)}, V_\alpha^{(1)})$ *and* $(V_i^{(2)}, V_\alpha^{(2)})$ *be representations, each of which has a* K *at the same source, and* (W_i, W_α) *a representation which has a* K^2 *at the source.*

Let $\dim \operatorname{Hom}((V_i^{(1)}, V_\alpha^{(1)}), (W_i, W_\alpha)) = 1$. *If we choose the embedding*

$$K^2 \xrightarrow{\begin{pmatrix} 1 & \lambda \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 1 \end{pmatrix}} K \oplus K^2 \oplus K \ ,$$

we get a one parameter family of indecomposable representations.

(E) *Let* $(V_i^{(1)}, V_\alpha^{(1)})$ *and* $(V_i^{(2)}, V_\alpha^{(2)})$ *be representations, each of which has a* K *at the same source, and* (W_i, W_α) *a representation which has a* K^2 *at the source.*
Let $\dim \operatorname{Hom}((W_i, W_\alpha), (V_i^{(1)}, V_\alpha^{(1)})) = 1$. *If we choose the embedding*

$$K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \hline 1 & 1 \\ \hline 1 & \lambda \end{pmatrix}} K^2 \oplus K \oplus K \ ,$$

we get a one parameter family of indecomposable representations.

(F) *Let* $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, \dots, 8$, *be representations, each of which has a* K *at the same source.*
Let $\dim \operatorname{Hom}((V_i^{(j)}, V_\alpha^{(j)}), (V_i^{(j+1)}, V_\alpha^{(j+1)})) = 1$, $j = 1, 3, 5, 7$. *If we choose the embedding*

$$K^2 \xrightarrow{\begin{pmatrix} 1 & \lambda \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \end{pmatrix}} K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \ ,$$

we get a one parameter family of indecomposable representations.

(G) *Let* $(V_i^{(1)}, V_\alpha^{(1)})$ *and* $(V_i^{(2)}, V_\alpha^{(2)})$ *be representations, each of which has a* K *at the same source, and* $(W_i(\lambda), W_\alpha(\lambda))_{\lambda \in K}$ *a one parameter family of representations which has* K *at the source. If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix} 1 \\ \hline 1 \\ \hline 1 \end{pmatrix}} K \oplus K \oplus K \ ,$$

we get a one parameter family of indecomposable representations.

Moreover, all representations of the constructed families have trivial endomorphism rings.

Proof. As in the last section we have to show the following in each case:

If

$$\begin{array}{ccc} K^{a_1} & \xrightarrow{M} & K^{a_2} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ K^{a_1} & \xrightarrow[M']{} & K^{a_2} \end{array}$$

commutes for the allowed base change matrices A and B, then $A = 0$ and $B = 0$, or $M = M'$ and $A = b \cdot I_{a_1}$ and $B = b \cdot I_{a_2}$. But in contrast to the last section we have to pay attention to the possible homomorphisms between the indecomposable representations for the restricted quivers when constructing the possible base change matrices. By Corollary 11.2 we also know the possible restrictions of the homomorphisms to the vector spaces at the sources of the quivers.

(A) We have to consider the following commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda \end{array}\right)} & K \oplus \cdots \oplus K \oplus K^2 \\ {\scriptstyle (a)}\Big\downarrow & & \Big\downarrow \left(\begin{array}{c|c|c|cc} b_1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \ddots & \ddots & & \vdots \\ \hline \vdots & \ddots & b_n & \ddots & \vdots \\ \hline \vdots & & \ddots & b & 0 \\ 0 & \cdots & \cdots & 0 & b \end{array}\right) \\ K & \xrightarrow[\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \mu \end{array}\right)]{} & K \oplus \cdots \oplus K \oplus K^2 \end{array} .$$

Then

$$\left(\begin{array}{c} a \\ \hline \vdots \\ \hline a \\ \hline a \\ a\mu \end{array}\right) = \left(\begin{array}{c} b_1 \\ \hline \vdots \\ \hline b_n \\ \hline b \\ b\lambda \end{array}\right),$$

and

$$\begin{cases} a = b = b_j = 0 \text{ for all } j = 1, \dots n \text{ or} \\ a = b = b_j \neq 0 \text{ for all } j = 1, \dots, n \text{ and } \lambda = \mu. \end{cases}$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(B) Since $\dim \operatorname{Hom}((W_i^{(1)}, W_\alpha^{(1)}), (W_i^{(2)}, W_\alpha^{(2)})) = 1$, we have to consider the following commutative diagrams:

$$\begin{array}{ccc} K & \xrightarrow{\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \lambda \\ \hline 0 \\ 1 \end{array}\right)} & K \oplus \cdots \oplus K \oplus K^2 \oplus K^2\,, \\ {\scriptstyle (a)}\Big\downarrow & & \Big\downarrow{\scriptstyle B} \\ K & \xrightarrow[\left(\begin{array}{c} 1 \\ \hline \vdots \\ \hline 1 \\ \hline 1 \\ \mu \\ \hline 0 \\ 1 \end{array}\right)]{} & K \oplus \cdots \oplus K \oplus K^2 \oplus K^2 \end{array}$$

where

$$B = \left(\begin{array}{c|c|c|cc|cc}
b_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hline
0 & \ddots & \ddots & & & & \vdots \\ \hline
\vdots & \ddots & b_n & \ddots & & & \vdots \\ \hline
\vdots & & 0 & c_1 & \ddots & & \vdots \\
\vdots & & \vdots & 0 & c_1 & \ddots & \vdots \\ \hline
\vdots & & \vdots & d & 0 & c_2 & 0 \\
0 & \cdots & 0 & 0 & d & 0 & c_2
\end{array}\right).$$

Then

$$\left(\begin{array}{c} a \\ \hline \vdots \\ \hline a \\ \hline a \\ a\mu \\ \hline 0 \\ a \end{array}\right) = \left(\begin{array}{c} b_1 \\ \hline \vdots \\ \hline b_n \\ \hline c_1 \\ c_1\lambda \\ \hline d \\ d\lambda + c_2 \end{array}\right),$$

and

$$\begin{cases} a = b = b_j = c_1 = c_2 = d = 0 \text{ for all } j = 1, \ldots n \text{ or} \\ a = b = b_j = c_1 = c_2 \neq 0 \text{ for all } j = 1, \ldots, n,\ d = 0 \text{ and } \lambda = \mu \end{cases}.$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(C) Since $\dim \operatorname{Hom}((V_i^{(1)}, V_\alpha^{(1)}), (V_i^{(2)}, V_\alpha^{(2)})) = 1$, we have to consider the following diagram:

$$\begin{array}{ccc}
K^2 & \xrightarrow{\left(\begin{array}{cc} 1 & \lambda \\ \hline 0 & 1 \\ \hline 1 & 0 \\ 1 & 1 \end{array}\right)} & K \oplus K \oplus K^2 \\
\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right) \Big\downarrow & & \Big\downarrow \left(\begin{array}{c|c|cc} b_1 & 0 & 0 & 0 \\ \hline c & b_2 & 0 & 0 \\ \hline 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_3 \end{array}\right) \\
K^2 & \xrightarrow[\left(\begin{array}{cc} 1 & \mu \\ \hline 0 & 1 \\ \hline 1 & 0 \\ 1 & 1 \end{array}\right)]{} & K \oplus K \oplus K^2
\end{array}.$$

Then

$$\left(\begin{array}{cc} a_{11}+a_{21}\mu & a_{12}+a_{22}\mu \\ \hline a_{21} & a_{22} \\ \hline a_{11} & a_{12} \\ a_{11}+a_{21} & a_{12}+a_{22} \end{array}\right) = \left(\begin{array}{cc} b_1 & b_1\lambda \\ \hline c & b_2 \\ \hline b_3 & 0 \\ b_3 & b_3 \end{array}\right),$$

and

$$\begin{cases} a_{11}=a_{22}=b_1=b_2=b_3=a_{12}=a_{21}=c=0 \text{ or} \\ a_{11}=a_{22}=b_1=b_2=b_3\neq 0,\ a_{12}=a_{21}=c=0 \text{ and } \lambda=\mu \end{cases}.$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(D) Since $\dim \operatorname{Hom}((V_i^{(1)}, V_\alpha^{(1)}), (W_i, W_\alpha)) = 1$, we have to consider the following commutative diagram

$$\begin{array}{ccc} K^2 & \xrightarrow{\left(\begin{array}{cc} 1 & \lambda \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 1 \end{array}\right)} & K\oplus K^2\oplus K \\ \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right)\Big\downarrow & & \Big\downarrow\left(\begin{array}{c|cc|c} b_1 & 0 & 0 & 0 \\ \hline c & b_2 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ \hline 0 & 0 & 0 & b_3 \end{array}\right) \\ K^2 & \xrightarrow[\left(\begin{array}{cc} 1 & \mu \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 1 \end{array}\right)]{} & K\oplus K^2\oplus K \end{array}.$$

Then

$$\left(\begin{array}{cc} a_{11}+a_{21}\mu & a_{12}+a_{22}\mu \\ \hline a_{11} & a_{12} \\ a_{21} & a_{22} \\ \hline a_{21} & a_{22} \end{array}\right) = \left(\begin{array}{cc} b_1 & b_1\lambda \\ \hline c+b_2 & c\lambda \\ c\lambda & b_2 \\ \hline 0 & b_3 \end{array}\right),$$

and

$$\begin{cases} a_{11}=a_{22}=b_1=b_2=b_3=a_{12}=a_{21}=c=0 \text{ or} \\ a_{11}=a_{22}=b_1=b_2=b_3\neq 0,\ a_{12}=a_{21}=c=0 \text{ and } \lambda=\mu \end{cases}.$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(E) Since $\dim \operatorname{Hom}((W_i, W_\alpha), (V_i^{(1)}, V_\alpha^{(1)})) = 1$, we have to consider the following commutative diagram:

$$\begin{array}{ccc}
K^2 & \xrightarrow{\left(\begin{array}{cc}1&0\\0&1\\\hline 1&1\\\hline 1&\lambda\end{array}\right)} & K^2\oplus K\oplus K \\
{\scriptstyle\left(\begin{array}{cc}a_{11}&a_{12}\\a_{21}&a_{22}\end{array}\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle\left(\begin{array}{cc|c|c}b_1&0&0&0\\0&b_1&0&0\\\hline c&0&b_2&0\\\hline 0&0&0&b_3\end{array}\right)} \\
K^2 & \xrightarrow[\left(\begin{array}{cc}1&0\\0&1\\\hline 1&1\\\hline 1&\mu\end{array}\right)]{} & K^2\oplus K\oplus K
\end{array}.$$

Then

$$\left(\begin{array}{cc}a_{11} & a_{12}\\ a_{21} & a_{22}\\\hline a_{11}+a_{21} & a_{12}+a_{22}\\\hline a_{11}+a_{21}\mu & a_{12}+a_{22}\mu\end{array}\right) = \left(\begin{array}{cc}b_1 & 0\\\hline 0 & b_1\\\hline c+b_2 & b_2\\\hline b_3 & b_3\lambda\end{array}\right),$$

and

$$\begin{cases} a_{11}=a_{22}=b_1=b_2=b_3=a_{12}=a_{21}=c=0 \text{ or} \\ a_{11}=a_{22}=b_1=b_2=b_3\neq 0,\ a_{12}=a_{21}=c=0 \text{ and } \lambda=\mu \end{cases}.$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(F) Since $\dim \operatorname{Hom}((V_i^{(j)}, V_\alpha^{(j)}), (V_i^{(j+1)}, V_\alpha^{(j+1)})) = 1$ for $j = 1, 3, 5, 7$, we have to consider the following commutative diagram:

$$
\begin{array}{ccc}
K^2 & \xrightarrow{\left(\begin{array}{cc} 1 & \lambda \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \end{array}\right)} & K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \ , \\
{\scriptstyle A_F := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle B_F} \\
K^2 & \xrightarrow[\left(\begin{array}{cc} 1 & \mu \\ \hline 0 & 1 \\ \hline 1 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline 0 & 1 \\ \hline 0 & 1 \\ \hline 1 & 0 \end{array}\right)]{} & K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K
\end{array}
$$

where

$$
B_F := \left(\begin{array}{c|c|c|c|c|c|c|c}
b_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
b_{21} & b_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & b_{33} & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & b_{43} & b_{44} & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & b_{55} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & b_{65} & b_{66} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & b_{77} & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & b_{87} & b_{88}
\end{array}\right).
$$

Then

$$
\left(\begin{array}{cc}
a_{11} + a_{21}\mu & a_{12} + a_{22}\mu \\ \hline
a_{21} & a_{22} \\ \hline
a_{11} + a_{21} & a_{12} + a_{22} \\ \hline
a_{21} & a_{22} \\ \hline
a_{11} & a_{12} \\ \hline
a_{21} & a_{22} \\ \hline
a_{21} & a_{22} \\ \hline
a_{11} & a_{12}
\end{array}\right)
=
\left(\begin{array}{cc}
b_{11} & b_{11}\lambda \\ \hline
b_{21} & b_{21}\lambda + b_{22} \\ \hline
b_{33} & b_{33} \\ \hline
b_{43} & b_{43} + b_{44} \\ \hline
b_{55} & 0 \\ \hline
b_{65} & b_{66} \\ \hline
0 & b_{77} \\ \hline
b_{88} & b_{87}
\end{array}\right),
$$

and

$$\begin{cases} a_{11} = a_{22} = b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = b_{88} \\ = a_{12} = a_{21} = b_{21} = b_{43} = b_{65} = b_{87} = 0 \text{ or} \\ a_{11} = a_{22} = b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = b_{88} \neq 0, \\ a_{12} = a_{21} = b_{21} = b_{43} = b_{65} = b_{87} = 0 \text{ and } \lambda = \mu \end{cases} .$$

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

(G) We have to consider the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\begin{pmatrix}1\\ \hline 1\\ \hline 1\end{pmatrix}} & K \oplus K \oplus K \\ \Big\downarrow{\scriptstyle (a)} & & \Big\downarrow{\left(\begin{array}{c|c|c} b_1 & 0 & 0 \\ \hline 0 & b_2 & 0 \\ \hline 0 & 0 & b_3 \end{array}\right)} \\ K & \xrightarrow[\begin{pmatrix}1\\ \hline 1\\ \hline 1\end{pmatrix}]{} & K \oplus K \oplus K \end{array} .$$

Then

$$\begin{pmatrix} a \\ \hline a \\ \hline a \end{pmatrix} = \begin{pmatrix} b_1 \\ \hline b_2 \\ \hline b_3 \end{pmatrix},$$

and $a = b_1 = b_2 = b_3$.

So we get a one parameter family of indecomposable representations with trivial endomorphism rings.

$\square$

11.3 Construction methods for indecomposable representations

Proposition 11.5.

(α) *Let $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, 2, 3$, be representations each of which has a K at the same source. If we choose the embedding*

$$K \xrightarrow{\begin{pmatrix}1\\ \hline 1\\ \hline 1\end{pmatrix}} K \oplus K \oplus K ,$$

we get an indecomposable representation.

(β) *Let* (V_i, V_α) *be a representation which has a* K^2 *at a source and* (W_i, W_α) *a representation which has a* K *at the source.*
Let $\dim \operatorname{Hom}((V_i, V_\alpha), (W_i, W_\alpha)) = 1$. *If we choose the embedding*

$$K \xrightarrow{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right)} K^2 \oplus K \ ,$$

we get an indecomposable representation.

Moreover, all representations have trivial endomorphism rings.

Proof. (α) We have to consider the following commutative diagram:

$$\begin{array}{ccc} K & \xrightarrow{\left(\begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array}\right)} & K \oplus K \oplus K \\ \Big\downarrow (a) & & \Big\downarrow \left(\begin{array}{c|c|c} b_1 & 0 & 0 \\ \hline 0 & b_2 & 0 \\ \hline 0 & 0 & b_3 \end{array}\right) \\ K & \xrightarrow[\left(\begin{array}{c} 1 \\ \hline 1 \\ \hline 1 \end{array}\right)]{} & K \oplus K \oplus K \end{array} .$$

Then

$$\left(\begin{array}{c} a \\ \hline a \\ \hline a \end{array}\right) = \left(\begin{array}{c} b_1 \\ \hline b_2 \\ \hline b_3 \end{array}\right),$$

and

$$a = b_1 = b_2 = b_3.$$

So we get an indecomposable representation with trivial endomorphism ring.

(β) Since $\dim \operatorname{Hom}((V_i, V_\alpha), (W_i, W_\alpha)) = 1$, we have to consider the following commutative diagram:

$$\begin{array}{ccc}
K^2 & \xrightarrow{\left(\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline 1 & 1\end{array}\right)} & K^2 \oplus K \oplus K \\
\left(\begin{array}{cc}a_{11} & a_{12}\\ a_{21} & a_{22}\end{array}\right)\Big\downarrow & & \Big\downarrow \left(\begin{array}{cc|c} b_1 & 0 & 0\\ 0 & b_1 & 0\\ \hline c & 0 & b_2\end{array}\right) \\
K^2 & \xrightarrow[\left(\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline 1 & 1\end{array}\right)]{} & K^2 \oplus K \oplus K
\end{array} .$$

Then

$$\left(\begin{array}{cc} a_{11} & a_{12}\\ a_{21} & a_{22}\\ \hline a_{11}+a_{21} & a_{12}+a_{22}\end{array}\right) = \left(\begin{array}{cc} b_1 & 0\\ 0 & b_1\\ \hline c+b_2 & b_2\end{array}\right),$$

and

$$a_{11} = a_{22} = b_1 = b_2 = b_3 = 0 \text{ and } a_{12} = a_{21} = c = 0.$$

So we get an indecomposable representation with trivial endomorphism ring.

□

11.4 Another construction method for one parameter families of indecomposable representations

(γ)

In the next proposition we also restrict the quiver, but instead of choosing an embedding for the vector space at the deleted point, we choose a projection to this vector space. If we can show that such a representation is indecomposable, we can change the orientation by applying the reflection functor for the point we deleted in the restricted quiver and get an indecomposable (subspace) representation for the original quiver.[4] (This is a onsequence of Lemma 7.3.)

Proposition 11.6. *Let* $(V_i^{(j)}, V_\alpha^{(j)})$, $j = 1, \dots, 8$, *be representations, each of which has a* K *at a source.*
Let $\dim \operatorname{Hom}((V_i^{(j)}, V_\alpha^{(j)}), (V_i^{(j-1)}, V_\alpha^{(j-1)})) = 1$, $j = 2, 4, 6, 8$. *If we choose the projection*

$$K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \oplus K \xrightarrow{\left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1\\ \lambda & 1 & 1 & 1 & 0 & 1 & 1 & 0\end{array}\right)} K^2 ,$$

[4] Of course, the dimension of the vector space at the "deleted" point is changed.

we get a one parameter family of indecomposable representations.

Proof. We have to consider the following commutative diagram:

$$\begin{array}{ccc} K\oplus K\oplus K\oplus K\oplus K\oplus K\oplus K\oplus K & \xrightarrow{\left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \lambda & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right)} & K^2 \\ \downarrow{\scriptstyle A'} & & \downarrow{\scriptstyle B'} \\ K\oplus K\oplus K\oplus K\oplus K\oplus K\oplus K\oplus K & \xrightarrow[\left(\begin{array}{c|c|c|c|c|c|c|c} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \mu & 1 & 1 & 1 & 0 & 1 & 1 & 0 \end{array}\right)]{} & K^2 \end{array},$$

where $A' := A_F^T$ and $B' := B_F^T$ from $\textcircled{F}$ of Proposition 11.4. This gives the same equations as the commutative diagram in case $\textcircled{F}$, such that

$$\begin{cases} a_{11} = a_{22} = b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = b_{88} \\ = a_{12} = a_{21} = b_{21} = b_{43} = b_{65} = b_{87} = 0 \text{ or} \\ a_{11} = a_{22} = b_{11} = b_{22} = b_{33} = b_{44} = b_{55} = b_{66} = b_{77} = b_{88} \neq 0, \\ a_{12} = a_{21} = b_{21} = b_{43} = b_{65} = b_{87} = 0 \text{ and } \lambda = \mu \end{cases}.$$

So we get a one parameter family of indecomposable representations. ☐

12 Orbits for the dimension vectors

The representations for the s-hypercritical and the s-tame dimension vectors can be sorted into orbits under the Auslander-Reiten translate. Accordingly, we get a decomposition of the set of all s-hypercritical dimension vectors and a decomposition of the set of all s-tame dimension vectors with respect to these orbits.

The orbits for the s-tame tuples of compositions are given by letters for the corresponding tame quivers (see Figures 1, 2, 3, and 4 in Chapter 6): $\tilde{\mathbb{D}}_4$ – A, $\tilde{\mathbb{E}}_6$ – B, $\tilde{\mathbb{E}}_7$ – C, $\tilde{\mathbb{E}}_8$ – D, and by numbers, which are used to distinguish the different orbits. The orbits for the s-hypercritical tuples of compositions are given by the letter H and by numbers, which are also used to distinguish the different orbits.

The tables can be used as follows: Choose an s-tame (or an s-hypercritical) dimension vector. First of all one has to find out in which of the τ-orbits the s-tame (s-hypercritical) dimension vector is contained (by the tables from this chapter). In the tables of the next chapter one can see how one can construct a family of indecomposable representations for one of the dimension vectors in the orbit by the methods from the previous chapter. And finally, one applies (sometimes repeatedly) the Coxeter transformations C^+ or C^- to the family of indecomposable subspace representations in order to get a family of indecomposable subspace representations for the chosen dimension vector (see also tables from this chapter). (The indecomposablity and the correct number of parameters for this family of representations are guaranteed by Proposition 7.3.)

12.1 Orbits for the s-hypercritical dimension vectors

No.	dim. vector	orbit	apply ...to constructed family
$k=5$:			
	$((1,1),(1,1),(1,1),(1,1),(1,1))$	H36	–
$k=4$:			
1.	$((2,2),(2,2),(2,2),(1,1,2))$	H1	–
	$((2,2),(2,2),(2,2),(1,2,1))$	H2	–
	$((2,2),(2,2),(2,2),(2,1,1))$	H1	C^-
2.	$((2,2),(2,2),(2,2),(1,1,1,1))$	H3	–
3.	$((1,3),(2,2),(2,2),(1,1,1,1))$	H4	–
	$((3,1),(2,2),(2,2),(1,1,1,1))$	H4	C^-
4.	see 3.		
5.	see 3.		
6.	$((1,2),(1,2),(1,1,1),(1,1,1))$	H5	–
	$((1,2),(2,1),(1,1,1),(1,1,1))$	H6	–
	$((2,1),(1,2),(1,1,1),(1,1,1))$:		

No.	dim. vector	orbit	apply ...to constructed family
	see 6.2 (with arms 1 and 2 exchanged) $((2,1),(2,1),(1,1,1),(1,1,1))$	H5	C^-
7.	$((1,2),(1,1,1),(1,1,1),(1,1,1))$	H7	–
	$((2,1),(1,1,1),(1,1,1),(1,1,1))$	H37	–
8.	$((1,1,1),(1,1,1),(1,1,1),(1,1,1))$	H8	–
$k=3$:			
1.	$((6,6),(4,4,4),(1,1,2,2,2,2,2))$	H9	–
	$((6,6),(4,4,4),(1,2,1,2,2,2,2))$	H10	–
	$((6,6),(4,4,4),(1,2,2,1,2,2,2))$	H11	–
	$((6,6),(4,4,4),(1,2,2,2,1,2,2))$	H12	–
	$((6,6),(4,4,4),(1,2,2,2,2,1,2))$	H13	–
	$((6,6),(4,4,4),(1,2,2,2,2,2,1))$	H14	–
	$((6,6),(4,4,4),(2,1,1,2,2,2,2))$	H9	C^-
	$((6,6),(4,4,4),(2,1,2,1,2,2,2))$	H10	C^-
	$((6,6),(4,4,4),(2,1,2,2,1,2,2))$	H11	C^-
	$((6,6),(4,4,4),(2,1,2,2,2,1,2))$	H12	C^-
	$((6,6),(4,4,4),(2,1,2,2,2,2,1))$	H13	C^-
	$((6,6),(4,4,4),(2,2,1,1,2,2,2))$	H9	$(C^-)^2$
	$((6,6),(4,4,4),(2,2,1,2,1,2,2))$	H10	$(C^-)^2$
	$((6,6),(4,4,4),(2,2,1,2,2,1,2))$	H11	$(C^-)^2$
	$((6,6),(4,4,4),(2,2,1,2,2,2,1))$	H12	$(C^-)^2$
	$((6,6),(4,4,4),(2,2,2,1,1,2,2))$	H9	$(C^-)^3$
	$((6,6),(4,4,4),(2,2,2,1,2,1,2))$	H10	$(C^-)^3$
	$((6,6),(4,4,4),(2,2,2,1,2,2,1))$	H11	$(C^-)^3$
	$((6,6),(4,4,4),(2,2,2,2,1,1,2))$	H9	$(C^-)^4$
	$((6,6),(4,4,4),(2,2,2,2,1,2,1))$	H10	$(C^-)^4$
	$((6,6),(4,4,4),(2,2,2,2,2,1,1))$	H9	$(C^-)^5$
2.	$((4,4),(2,3,3),(1,1,1,1,1,1,1,1))$	H15	–
	$((4,4),(3,2,3),(1,1,1,1,1,1,1,1))$	H15	C^-
	$((4,4),(3,3,2),(1,1,1,1,1,1,1,1))$	H15	$(C^-)^2$
3.	$((4,4),(2,2,2,2),(1,1,2,2,2))$	H16	–
	$((4,4),(2,2,2,2),(1,2,1,2,2))$	H17	–
	$((4,4),(2,2,2,2),(1,2,2,1,2))$	H18	–
	$((4,4),(2,2,2,2),(1,2,2,2,1))$	H19	–
	$((4,4),(2,2,2,2),(2,1,1,2,2))$	H16	C^-
	$((4,4),(2,2,2,2),(2,1,2,1,2))$	H17	C^-
	$((4,4),(2,2,2,2),(2,1,2,2,1))$	H18	C^-
	$((4,4),(2,2,2,2),(2,2,1,1,2))$	H16	$(C^-)^2$
	$((4,4),(2,2,2,2),(2,2,1,2,1))$	H17	$(C^-)^2$
	$((4,4),(2,2,2,2),(2,2,2,1,1))$	H16	$(C^-)^3$

No.	dim. vector	orbit	apply ...to constructed family
4.	$((5,5),(1,3,3,3),(2,2,2,2,2))$	H20	–
	$((5,5),(3,1,3,3),(2,2,2,2,2))$	H20	C^-
	$((5,5),(3,3,1,3),(2,2,2,2,2))$	H20	$(C^-)^2$
	$((5,5),(3,3,3,1),(2,2,2,2,2))$	H20	$(C^-)^3$
5.	$((3,3),(1,1,2,2),(1,1,1,1,1,1))$	H21	–
	$((3,3),(1,2,1,2),(1,1,1,1,1,1))$	H22	–
	$((3,3),(1,2,2,1),(1,1,1,1,1,1))$	H23	–
	$((3,3),(2,1,1,2),(1,1,1,1,1,1))$	H21	C^-
	$((3,3),(2,1,2,1),(1,1,1,1,1,1))$	H22	C^-
	$((3,3),(2,2,1,1),(1,1,1,1,1,1))$	H21	$(C^-)^2$
6.	$((2,3),(1,1,1,1,1),(1,1,1,1,1))$	H24	–
	$((3,2),(1,1,1,1,1),(1,1,1,1,1))$	H24	C^-
7.	$((2,2,2),(2,2,2),(1,1,2,2))$	H25	–
	$((2,2,2),(2,2,2),(1,2,1,2))$	H26	–
	$((2,2,2),(2,2,2),(1,2,2,1))$	H27	–
	$((2,2,2),(2,2,2),(2,1,1,2))$	H25	C^-
	$((2,2,2),(2,2,2),(2,1,2,1))$	H26	C^-
	$((2,2,2),(2,2,2),(2,2,1,1))$	H25	$(C^-)^2$
8.	$((1,2,2),(1,2,2),(1,1,1,1,1))$	H28	–
	$((1,2,2),(2,1,2),(1,1,1,1,1))$	H29	–
	$((1,2,2),(2,2,1),(1,1,1,1,1))$: see 8.7 (with arms 1 and 2 exchanged)		
	$((2,1,2),(1,2,2),(1,1,1,1,1))$	H30	–
	$((2,1,2),(2,1,2),(1,1,1,1,1))$	H31	–
	$((2,1,2),(2,2,1),(1,1,1,1,1))$	H29	C^-
	$((2,2,1),(1,2,2),(1,1,1,1,1))$	H32	–
	$((2,2,1),(2,1,2),(1,1,1,1,1))$	H30	C^-
	$((2,2,1),(2,2,1),(1,1,1,1,1))$	H28	$(C^-)^2$
9.	$((1,1,2),(1,1,1,1),(1,1,1,1))$	H33	–
	$((1,2,1),(1,1,1,1),(1,1,1,1))$	H34	–
	$((2,1,1),(1,1,1,1),(1,1,1,1))$	H33	C^-
10.	$((1,1,1,1),(1,1,1,1),(1,1,1,1))$	H35	–

12.2 Orbits for the s-tame dimension vectors

No.	dim. vector	orbit	apply ...to constructed family
$k=4$:			
1.	$((m,m),(m,m),(m,m),(m,m))$, $m \in \mathbb{N}$	known	–

No.	dim. vector	orbit	apply ...to constructed family
2.	$((1,2),(1,2),(1,2),(1,1,1))$	A1	C^-
	$((1,2),(1,2),(2,1),(1,1,1))$	A2	–
	$((1,2),(2,1),(1,2),(1,1,1))$: see 2.2 (with arms 2 and 3 exchanged)		
	$((1,2),(2,1),(2,1),(1,1,1))$: see 2.7 (with arms 1 and 3 exchanged)		
	$((2,1),(1,2),(1,2),(1,1,1))$: see 2.2 (with arms 1 and 3 exchanged)		
	$((2,1),(1,2),(2,1),(1,1,1))$: see 2.7 (with arms 2 and 3 exchanged)		
	$((2,1),(2,1),(1,2),(1,1,1))$	A2	C^-
	$((2,1),(2,1),(2,1),(1,1,1))$	A1	C^-
3.	$((1,3),(2,2),(2,2),(1,1,2))$	A8	–
	$((1,3),(2,2),(2,2),(1,2,1))$	A3	–
	$((1,3),(2,2),(2,2),(2,1,1))$	A4	–
	$((3,1),(2,2),(2,2),(1,1,2))$	A3	C^-
	$((3,1),(2,2),(2,2),(1,2,1))$	A4	C^-
	$((3,1),(2,2),(2,2),(2,1,1))$	A9	C^-
4.	see 3. (with arms 1 and 2 exchanged)		
5.	see 3. (with arms 1 and 3 exchanged)		
6.	$((1,3),(1,3),(2,2),(1,1,1,1))$	A5	–
	$((1,3),(3,1),(2,2),(1,1,1,1))$	A6	–
	$((3,1),(1,3),(2,2),(1,1,1,1))$: see 6.2 (with arms 1 and 2 exchanged)		
	$((3,1),(3,1),(2,2),(1,1,1,1))$	A7	–
7.	see 6 (with arms 1 and 2 exchanged)		
8.	see 6. (with arms 1 and 3 exchanged)		
$k=3$			
1.	$((3m,3m),(2m,2m,2m),(m,m,m,m,m,m))$, $m \in \mathbb{N}$	known	
2.	$((3,4),(2,2,3),(1,1,1,1,1,1,1))$	D1	C^-
	$((3,4),(2,3,2),(1,1,1,1,1,1,1))$	D2	–
	$((3,4),(3,2,2),(1,1,1,1,1,1,1))$	D2	$(C^+)^2$
	$((4,3),(2,2,3),(1,1,1,1,1,1,1))$	D2	C^+
	$((4,3),(2,3,2),(1,1,1,1,1,1,1))$	D2	$(C^+)^3$
	$((4,3),(3,2,2),(1,1,1,1,1,1,1))$	D1	$(C^+)^7$

No.	dim. vector	orbit	apply ...to constructed family
3.	$((4,4),(2,3,3),(1,1,1,1,1,1,2))$	D1	$(C^+)^8$
	$((4,4),(2,3,3),(1,1,1,1,1,2,1))$	D2	$(C^-)^2$
	$((4,4),(2,3,3),(1,1,1,1,2,1,1))$	D3	$(C^-)^5$
	$((4,4),(2,3,3),(1,1,1,2,1,1,1))$	D6	$(C^-)^2$
	$((4,4),(2,3,3),(1,1,2,1,1,1,1))$	D5	$(C^-)^2$
	$((4,4),(2,3,3),(1,2,1,1,1,1,1))$	D4	$(C^-)^2$
	$((4,4),(2,3,3),(2,1,1,1,1,1,1))$	D3	$(C^-)^2$
	$((4,4),(3,2,3),(1,1,1,1,1,1,2))$	D2	C^-
	$((4,4),(3,2,3),(1,1,1,1,1,2,1))$	D3	C^-
	$((4,4),(3,2,3),(1,1,1,1,2,1,1))$	D6	C^-
	$((4,4),(3,2,3),(1,1,1,2,1,1,1))$	D5	C^-
	$((4,4),(3,2,3),(1,1,2,1,1,1,1))$	D4	C^-
	$((4,4),(3,2,3),(1,2,1,1,1,1,1))$	D3	C^-
	$((4,4),(3,2,3),(2,1,1,1,1,1,1))$	D2	$(C^+)^4$
	$((4,4),(3,3,2),(1,1,1,1,1,1,2))$	D3	$(C^-)^3$
	$((4,4),(3,3,2),(1,1,1,1,1,2,1))$	D6	–
	$((4,4),(3,3,2),(1,1,1,1,2,1,1))$	D5	–
	$((4,4),(3,3,2),(1,1,1,2,1,1,1))$	D4	–
	$((4,4),(3,3,2),(1,1,2,1,1,1,1))$	D3	–
	$((4,4),(3,3,2),(1,2,1,1,1,1,1))$	D2	$(C^+)^5$
	$((4,4),(3,3,2),(2,1,1,1,1,1,1))$	D1	$(C^+)^8$
4.	$((4,5),(3,3,3),(1,1,1,1,1,2,2))$	D1	$(C^-)^2$
	$((4,5),(3,3,3),(1,1,1,1,2,1,2))$	D2	$(C^-)^3$
	$((4,5),(3,3,3),(1,1,1,1,2,2,1))$	D4	$(C^-)^5$
	$((4,5),(3,3,3),(1,1,1,2,1,1,2))$	D3	$(C^-)^6$
	$((4,5),(3,3,3),(1,1,1,2,1,2,1))$	D7	C^-
	$((4,5),(3,3,3),(1,1,1,2,2,1,1))$	D15	C^-
	$((4,5),(3,3,3),(1,1,2,1,1,1,2))$	D6	$(C^-)^3$
	$((4,5),(3,3,3),(1,1,2,1,1,2,1))$	D9	C^-
	$((4,5),(3,3,3),(1,1,2,1,2,1,1))$	D14	C^-
	$((4,5),(3,3,3),(1,1,2,2,1,1,1))$	D13	C^-
	$((4,5),(3,3,3),(1,2,1,1,1,1,2))$	D5	$(C^-)^3$
	$((4,5),(3,3,3),(1,2,1,1,1,2,1))$	D8	C^-
	$((4,5),(3,3,3),(1,2,1,1,2,1,1))$	D12	C^-
	$((4,5),(3,3,3),(1,2,1,2,1,1,1))$	D11	C^-
	$((4,5),(3,3,3),(1,2,2,1,1,1,1))$	D10	C^-
	$((4,5),(3,3,3),(2,1,1,1,1,1,2))$	D4	$(C^-)^3$
	$((4,5),(3,3,3),(2,1,1,1,1,2,1))$	D7	C^+
	$((4,5),(3,3,3),(2,1,1,1,2,1,1))$	D9	C^+
	$((4,5),(3,3,3),(2,1,1,2,1,1,1))$	D8	C^+
	$((4,5),(3,3,3),(2,1,2,1,1,1,1))$	D7	$(C^+)^3$

No.	dim. vector	orbit	apply ... to constructed family
	$((4,5),(3,3,3),(2,2,1,1,1,1,1))$	D6	$(C^+)^2$
	$((5,4),(3,3,3),(1,1,1,1,1,2,2))$	D4	$(C^-)^4$
	$((5,4),(3,3,3),(1,1,1,1,2,1,2))$	D7	–
	$((5,4),(3,3,3),(1,1,1,1,2,2,1))$	D15	–
	$((5,4),(3,3,3),(1,1,1,2,1,1,2))$	D9	–
	$((5,4),(3,3,3),(1,1,1,2,1,2,1))$	D14	–
	$((5,4),(3,3,3),(1,1,1,2,2,1,1))$	D13	–
	$((5,4),(3,3,3),(1,1,2,1,1,1,2))$	D8	–
	$((5,4),(3,3,3),(1,1,2,1,1,2,1))$	D12	–
	$((5,4),(3,3,3),(1,1,2,1,2,1,1))$	D11	–
	$((5,4),(3,3,3),(1,1,2,2,1,1,1))$	D10	–
	$((5,4),(3,3,3),(1,2,1,1,1,1,2))$	D7	$(C^+)^2$
	$((5,4),(3,3,3),(1,2,1,1,1,2,1))$	D9	$(C^+)^2$
	$((5,4),(3,3,3),(1,2,1,1,2,1,1))$	D8	C^+
	$((5,4),(3,3,3),(1,2,1,2,1,1,1))$	D7	$(C^+)^4$
	$((5,4),(3,3,3),(1,2,2,1,1,1,1))$	D6	$(C^+)^3$
	$((5,4),(3,3,3),(2,1,1,1,1,1,2))$	D6	C^+
	$((5,4),(3,3,3),(2,1,1,1,1,2,1))$	D5	C^+
	$((5,4),(3,3,3),(2,1,1,1,2,1,1))$	D4	C^+
	$((5,4),(3,3,3),(2,1,1,2,1,1,1))$	D3	C^+
	$((5,4),(3,3,3),(2,1,2,1,1,1,1))$	D2	$(C^+)^6$
	$((5,4),(3,3,3),(2,2,1,1,1,1,1))$	D1	$(C^+)^{10}$
5.	$((5,5),(3,3,4),(1,1,1,1,2,2,2))$	D1	$(C^-)^4$
	$((5,5),(3,3,4),(1,1,1,2,1,2,2))$	D2	$(C^-)^4$
	$((5,5),(3,3,4),(1,1,1,2,2,1,2))$	D4	$(C^-)^6$
	$((5,5),(3,3,4),(1,1,1,2,2,2,1))$	D10	$(C^-)^5$
	$((5,5),(3,3,4),(1,1,2,1,1,2,2))$	D3	$(C^-)^7$
	$((5,5),(3,3,4),(1,1,2,1,2,1,2))$	D7	$(C^-)^2$
	$((5,5),(3,3,4),(1,1,2,1,2,2,1))$	D18	C^-
	$((5,5),(3,3,4),(1,1,2,2,1,1,2))$	D15	$(C^-)^2$
	$((5,5),(3,3,4),(1,1,2,2,1,2,1))$	D30	C^-
	$((5,5),(3,3,4),(1,1,2,2,2,1,1))$	D29	C^-
	$((5,5),(3,3,4),(1,2,1,1,1,2,2))$	D6	$(C^-)^4$
	$((5,5),(3,3,4),(1,2,1,1,2,1,2))$	D9	$(C^-)^2$
	$((5,5),(3,3,4),(1,2,1,1,2,2,1))$	D17	$(C^+)^2$
	$((5,5),(3,3,4),(1,2,1,2,1,1,2))$	D14	$(C^-)^2$
	$((5,5),(3,3,4),(1,2,1,2,1,2,1))$	D28	C^-
	$((5,5),(3,3,4),(1,2,1,2,2,1,1))$	D27	C^-
	$((5,5),(3,3,4),(1,2,2,1,1,1,2))$	D13	$(C^-)^3$
	$((5,5),(3,3,4),(1,2,2,1,1,2,1))$	D26	C^-
	$((5,5),(3,3,4),(1,2,2,1,2,1,1))$	D25	C^-

No.	dim. vector	orbit	apply ...to constructed family
	$((5,5),(3,3,4),(1,2,2,2,1,1,1))$	D24	C^-
	$((5,5),(3,3,4),(2,1,1,1,1,2,2))$	D5	$(C^-)^4$
	$((5,5),(3,3,4),(2,1,1,1,2,1,2))$	D8	$(C^-)^2$
	$((5,5),(3,3,4),(2,1,1,1,2,2,1))$	D16	$(C^+)^2$
	$((5,5),(3,3,4),(2,1,1,2,1,1,2))$	D12	$(C^-)^2$
	$((5,5),(3,3,4),(2,1,1,2,1,2,1))$	D23	C^+
	$((5,5),(3,3,4),(2,1,1,2,2,1,1))$	D22	C^+
	$((5,5),(3,3,4),(2,1,2,1,1,1,2))$	D11	$(C^-)^2$
	$((5,5),(3,3,4),(2,1,2,1,1,2,1))$	D21	$(C^+)^3$
	$((5,5),(3,3,4),(2,1,2,1,2,1,1))$	D20	$(C^+)^5$
	$((5,5),(3,3,4),(2,1,2,2,1,1,1))$	D19	$(C^+)^4$
	$((5,5),(3,3,4),(2,2,1,1,1,1,2))$	D10	$(C^-)^2$
	$((5,5),(3,3,4),(2,2,1,1,1,2,1))$	D18	$(C^+)^2$
	$((5,5),(3,3,4),(2,2,1,1,2,1,1))$	D17	$(C^+)^5$
	$((5,5),(3,3,4),(2,2,1,2,1,1,1))$	D16	$(C^+)^5$
	$((5,5),(3,3,4),(2,2,2,1,1,1,1))$	D15	$(C^+)^3$
	$((5,5),(3,4,3),(1,1,1,1,2,2,2))$	D5	$(C^-)^5$
	$((5,5),(3,4,3),(1,1,1,2,1,2,2))$	D8	$(C^-)^3$
	$((5,5),(3,4,3),(1,1,1,2,2,1,2))$	D16	C^+
	$((5,5),(3,4,3),(1,1,1,2,2,2,1))$	D40	–
	$((5,5),(3,4,3),(1,1,2,1,1,2,2))$	D12	$(C^-)^3$
	$((5,5),(3,4,3),(1,1,2,1,2,1,2))$	D23	–
	$((5,5),(3,4,3),(1,1,2,1,2,2,1))$	D39	–
	$((5,5),(3,4,3),(1,1,2,2,1,1,2))$	D22	–
	$((5,5),(3,4,3),(1,1,2,2,1,2,1))$	D38	–
	$((5,5),(3,4,3),(1,1,2,2,2,1,1))$	D37	–
	$((5,5),(3,4,3),(1,2,1,1,1,2,2))$	D11	$(C^-)^3$
	$((5,5),(3,4,3),(1,2,1,1,2,1,2))$	D21	$(C^+)^3$
	$((5,5),(3,4,3),(1,2,1,1,2,2,1))$	D36	–
	$((5,5),(3,4,3),(1,2,1,2,1,1,2))$	D20	$(C^+)^4$
	$((5,5),(3,4,3),(1,2,1,2,1,2,1))$	D35	–
	$((5,5),(3,4,3),(1,2,1,2,2,1,1))$	D34	$(C^+)^5$
	$((5,5),(3,4,3),(1,2,2,1,1,1,2))$	D19	$(C^+)^3$
	$((5,5),(3,4,3),(1,2,2,1,1,2,1))$	D33	$(C^+)^3$
	$((5,5),(3,4,3),(1,2,2,1,2,1,1))$	D32	$(C^+)^5$
	$((5,5),(3,4,3),(1,2,2,2,1,1,1))$	D31	$(C^+)^4$
	$((5,5),(3,4,3),(2,1,1,1,1,2,2))$	D10	$(C^-)^3$
	$((5,5),(3,4,3),(2,1,1,1,2,1,2))$	D18	C^+
	$((5,5),(3,4,3),(2,1,1,1,2,2,1))$	D30	C^+
	$((5,5),(3,4,3),(2,1,1,2,1,1,2))$	D17	$(C^+)^4$
	$((5,5),(3,4,3),(2,1,1,2,1,2,1))$	D28	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((5,5),(3,4,3),(2,1,1,2,2,1,1))$	D26	C^+
	$((5,5),(3,4,3),(2,1,2,1,1,1,2))$	D16	$(C^+)^4$
	$((5,5),(3,4,3),(2,1,2,1,1,2,1))$	D23	$(C^+)^3$
	$((5,5),(3,4,3),(2,1,2,1,2,1,1))$	D21	$(C^+)^5$
	$((5,5),(3,4,3),(2,1,2,2,1,1,1))$	D18	$(C^+)^4$
	$((5,5),(3,4,3),(2,2,1,1,1,1,2))$	D15	$(C^+)^2$
	$((5,5),(3,4,3),(2,2,1,1,1,2,1))$	D14	$(C^+)^2$
	$((5,5),(3,4,3),(2,2,1,1,2,1,1))$	D12	$(C^+)^2$
	$((5,5),(3,4,3),(2,2,1,2,1,1,1))$	D9	$(C^+)^4$
	$((5,5),(3,4,3),(2,2,2,1,1,1,1))$	D5	$(C^+)^3$
	$((5,5),(4,3,3),(1,1,1,1,2,2,2))$	D10	$(C^-)^4$
	$((5,5),(4,3,3),(1,1,1,2,1,2,2))$	D18	–
	$((5,5),(4,3,3),(1,1,1,2,2,1,2))$	D30	–
	$((5,5),(4,3,3),(1,1,1,2,2,2,1))$	D29	–
	$((5,5),(4,3,3),(1,1,2,1,1,2,2))$	D17	$(C^+)^3$
	$((5,5),(4,3,3),(1,1,2,1,2,1,2))$	D28	–
	$((5,5),(4,3,3),(1,1,2,1,2,2,1))$	D27	–
	$((5,5),(4,3,3),(1,1,2,2,1,1,2))$	D26	–
	$((5,5),(4,3,3),(1,1,2,2,1,2,1))$	D25	–
	$((5,5),(4,3,3),(1,1,2,2,2,1,1))$	D24	–
	$((5,5),(4,3,3),(1,2,1,1,1,2,2))$	D16	$(C^+)^3$
	$((5,5),(4,3,3),(1,2,1,1,2,1,2))$	D23	$(C^+)^2$
	$((5,5),(4,3,3),(1,2,1,1,2,2,1))$	D22	$(C^+)^2$
	$((5,5),(4,3,3),(1,2,1,2,1,1,2))$	D21	$(C^+)^4$
	$((5,5),(4,3,3),(1,2,1,2,1,2,1))$	D20	$(C^+)^6$
	$((5,5),(4,3,3),(1,2,1,2,2,1,1))$	D19	$(C^+)^5$
	$((5,5),(4,3,3),(1,2,2,1,1,1,2))$	D18	$(C^+)^3$
	$((5,5),(4,3,3),(1,2,2,1,1,2,1))$	D17	$(C^+)^6$
	$((5,5),(4,3,3),(1,2,2,1,2,1,1))$	D16	$(C^+)^6$
	$((5,5),(4,3,3),(1,2,2,2,1,1,1))$	D15	$(C^+)^4$
	$((5,5),(4,3,3),(2,1,1,1,1,2,2))$	D15	C^+
	$((5,5),(4,3,3),(2,1,1,1,2,1,2))$	D14	C^+
	$((5,5),(4,3,3),(2,1,1,1,2,2,1))$	D13	C^+
	$((5,5),(4,3,3),(2,1,1,2,1,1,2))$	D12	C^+
	$((5,5),(4,3,3),(2,1,1,2,1,2,1))$	D11	C^+
	$((5,5),(4,3,3),(2,1,1,2,2,1,1))$	D10	C^+
	$((5,5),(4,3,3),(2,1,2,1,1,1,2))$	D9	$(C^+)^3$
	$((5,5),(4,3,3),(2,1,2,1,1,2,1))$	D8	$(C^+)^3$
	$((5,5),(4,3,3),(2,1,2,1,2,1,1))$	D7	$(C^+)^5$
	$((5,5),(4,3,3),(2,1,2,2,1,1,1))$	D6	$(C^+)^5$
	$((5,5),(4,3,3),(2,2,1,1,1,1,2))$	D5	$(C^+)^3$

No.	dim. vector	orbit	apply ...to constructed family
	$((5,5),(4,3,3),(2,2,1,1,1,2,1))$	D4	$(C^+)^2$
	$((5,5),(4,3,3),(2,2,1,1,2,1,1))$	D3	$(C^+)^2$
	$((5,5),(4,3,3),(2,2,1,2,1,1,1))$	D2	$(C^+)^7$
	$((5,5),(4,3,3),(2,2,2,1,1,1,1))$	D1	$(C^+)^{11}$
6.	$((5,6),(3,4,4),(1,1,1,2,2,2,2))$	D1	$(C^-)^5$
	$((5,6),(3,4,4),(1,1,2,1,2,2,2))$	D2	$(C^-)^5$
	$((5,6),(3,4,4),(1,1,2,2,1,2,2))$	D4	$(C^-)^7$
	$((5,6),(3,4,4),(1,1,2,2,2,1,2))$	D10	$(C^-)^6$
	$((5,6),(3,4,4),(1,1,2,2,2,2,1))$	D31	C^-
	$((5,6),(3,4,4),(1,2,1,1,2,2,2))$	D3	$(C^-)^8$
	$((5,6),(3,4,4),(1,2,1,2,1,2,2))$	D7	$(C^-)^3$
	$((5,6),(3,4,4),(1,2,1,2,2,1,2))$	D18	$(C^-)^2$
	$((5,6),(3,4,4),(1,2,1,2,2,2,1))$	D42	C^-
	$((5,6),(3,4,4),(1,2,2,1,1,2,2))$	D15	$(C^-)^3$
	$((5,6),(3,4,4),(1,2,2,1,2,1,2))$	D30	$(C^-)^2$
	$((5,6),(3,4,4),(1,2,2,1,2,2,1))$	D66	C^-
	$((5,6),(3,4,4),(1,2,2,2,1,1,2))$	D29	$(C^-)^2$
	$((5,6),(3,4,4),(1,2,2,2,1,2,1))$	D65	C^-
	$((5,6),(3,4,4),(1,2,2,2,2,1,1))$	D64	C^-
	$((5,6),(3,4,4),(2,1,1,1,2,2,2))$	D6	$(C^-)^5$
	$((5,6),(3,4,4),(2,1,1,2,1,2,2))$	D9	$(C^-)^3$
	$((5,6),(3,4,4),(2,1,1,2,2,1,2))$	D17	C^+
	$((5,6),(3,4,4),(2,1,1,2,2,2,1))$	D41	C^+
	$((5,6),(3,4,4),(2,1,2,1,1,2,2))$	D14	$(C^-)^3$
	$((5,6),(3,4,4),(2,1,2,1,2,1,2))$	D28	$(C^-)^2$
	$((5,6),(3,4,4),(2,1,2,1,2,2,1))$	D63	C^-
	$((5,6),(3,4,4),(2,1,2,2,1,1,2))$	D27	$(C^-)^2$
	$((5,6),(3,4,4),(2,1,2,2,1,2,1))$	D62	C^-
	$((5,6),(3,4,4),(2,1,2,2,2,1,1))$	D61	C^-
	$((5,6),(3,4,4),(2,2,1,1,1,2,2))$	D13	$(C^-)^3$
	$((5,6),(3,4,4),(2,2,1,1,2,1,2))$	D26	$(C^-)^2$
	$((5,6),(3,4,4),(2,2,1,1,2,2,1))$	D60	$(C^+)^2$
	$((5,6),(3,4,4),(2,2,1,2,1,1,2))$	D25	$(C^-)^2$
	$((5,6),(3,4,4),(2,2,1,2,1,2,1))$	D59	C^-
	$((5,6),(3,4,4),(2,2,1,2,2,1,1))$	D58	C^-
	$((5,6),(3,4,4),(2,2,2,1,1,1,2))$	D24	$(C^-)^2$
	$((5,6),(3,4,4),(2,2,2,1,1,2,1))$	D57	$(C^+)^3$
	$((5,6),(3,4,4),(2,2,2,1,2,1,1))$	D56	$(C^+)^5$
	$((5,6),(3,4,4),(2,2,2,2,1,1,1))$	D55	$(C^+)^4$
	$((5,6),(4,3,4),(1,1,1,2,2,2,2))$	D5	$(C^-)^6$
	$((5,6),(4,3,4),(1,1,2,1,2,2,2))$	D8	$(C^-)^4$

No.	dim. vector	orbit	apply ...to constructed family
	$((5,6),(4,3,4),(1,1,2,2,1,2,2))$	D16	–
	$((5,6),(4,3,4),(1,1,2,2,2,1,2))$	D40	C^-
	$((5,6),(4,3,4),(1,1,2,2,2,2,1))$	D54	–
	$((5,6),(4,3,4),(1,2,1,1,2,2,2))$	D12	$(C^-)^4$
	$((5,6),(4,3,4),(1,2,1,2,1,2,2))$	D23	C^-
	$((5,6),(4,3,4),(1,2,1,2,2,1,2))$	D39	C^-
	$((5,6),(4,3,4),(1,2,1,2,2,2,1))$	D53	C^-
	$((5,6),(4,3,4),(1,2,2,1,1,2,2))$	D22	C^-
	$((5,6),(4,3,4),(1,2,2,1,2,1,2))$	D38	C^-
	$((5,6),(4,3,4),(1,2,2,1,2,2,1))$	D52	C^-
	$((5,6),(4,3,4),(1,2,2,2,1,1,2))$	D37	C^-
	$((5,6),(4,3,4),(1,2,2,2,1,2,1))$	D51	C^-
	$((5,6),(4,3,4),(1,2,2,2,2,1,1))$	D50	C^-
	$((5,6),(4,3,4),(2,1,1,1,2,2,2))$	D11	$(C^-)^4$
	$((5,6),(4,3,4),(2,1,1,2,1,2,2))$	D21	C^+
	$((5,6),(4,3,4),(2,1,1,2,2,1,2))$	D36	C^-
	$((5,6),(4,3,4),(2,1,1,2,2,2,1))$	D49	C^+
	$((5,6),(4,3,4),(2,1,2,1,1,2,2))$	D20	$(C^+)^3$
	$((5,6),(4,3,4),(2,1,2,1,2,1,2))$	D35	C^-
	$((5,6),(4,3,4),(2,1,2,1,2,2,1))$	D48	C^-
	$((5,6),(4,3,4),(2,1,2,2,1,1,2))$	D34	$(C^+)^4$
	$((5,6),(4,3,4),(2,1,2,2,1,2,1))$	D47	C^-
	$((5,6),(4,3,4),(2,1,2,2,2,1,1))$	D46	$(C^+)^5$
	$((5,6),(4,3,4),(2,2,1,1,1,2,2))$	D19	$(C^+)^2$
	$((5,6),(4,3,4),(2,2,1,1,2,1,2))$	D33	$(C^+)^2$
	$((5,6),(4,3,4),(2,2,1,1,2,2,1))$	D45	$(C^+)^2$
	$((5,6),(4,3,4),(2,2,1,2,1,1,2))$	D32	$(C^+)^4$
	$((5,6),(4,3,4),(2,2,1,2,1,2,1))$	D44	C^-
	$((5,6),(4,3,4),(2,2,1,2,2,1,1))$	D43	$(C^+)^5$
	$((5,6),(4,3,4),(2,2,2,1,1,1,2))$	D31	$(C^+)^3$
	$((5,6),(4,3,4),(2,2,2,1,1,2,1))$	D42	$(C^+)^3$
	$((5,6),(4,3,4),(2,2,2,1,2,1,1))$	D41	$(C^+)^5$
	$((5,6),(4,3,4),(2,2,2,2,1,1,1))$	D40	$(C^+)^4$
	$((5,6),(4,4,3),(1,1,1,2,2,2,2))$	D19	–
	$((5,6),(4,4,3),(1,1,2,1,2,2,2))$	D33	–
	$((5,6),(4,4,3),(1,1,2,2,1,2,2))$	D45	–
	$((5,6),(4,4,3),(1,1,2,2,2,1,2))$	D70	–
	$((5,6),(4,4,3),(1,1,2,2,2,2,1))$	D80	–
	$((5,6),(4,4,3),(1,2,1,1,2,2,2))$	D32	$(C^+)^2$
	$((5,6),(4,4,3),(1,2,1,2,1,2,2))$	D44	$(C^-)^3$
	$((5,6),(4,4,3),(1,2,1,2,2,1,2))$	D69	–

No.	dim. vector	orbit	apply ... to constructed family
	$((5,6),(4,4,3),(1,2,1,2,2,2,1))$	D79	–
	$((5,6),(4,4,3),(1,2,2,1,1,2,2))$	D43	$(C^{+})^{3}$
	$((5,6),(4,4,3),(1,2,2,1,2,1,2))$	D68	–
	$((5,6),(4,4,3),(1,2,2,1,2,2,1))$	D78	–
	$((5,6),(4,4,3),(1,2,2,2,1,1,2))$	D67	$(C^{+})^{4}$
	$((5,6),(4,4,3),(1,2,2,2,1,2,1))$	D77	–
	$((5,6),(4,4,3),(1,2,2,2,2,1,1))$	D76	$(C^{+})^{5}$
	$((5,6),(4,4,3),(2,1,1,1,2,2,2))$	D31	C^{+}
	$((5,6),(4,4,3),(2,1,1,2,1,2,2))$	D42	C^{+}
	$((5,6),(4,4,3),(2,1,1,2,2,1,2))$	D66	C^{+}
	$((5,6),(4,4,3),(2,1,1,2,2,2,1))$	D65	C^{+}
	$((5,6),(4,4,3),(2,1,2,1,1,2,2))$	D41	$(C^{+})^{3}$
	$((5,6),(4,4,3),(2,1,2,1,2,1,2))$	D63	C^{+}
	$((5,6),(4,4,3),(2,1,2,1,2,2,1))$	D62	C^{+}
	$((5,6),(4,4,3),(2,1,2,2,1,1,2))$	D60	$(C^{+})^{4}$
	$((5,6),(4,4,3),(2,1,2,2,1,2,1))$	D59	C^{+}
	$((5,6),(4,4,3),(2,1,2,2,2,1,1))$	D57	$(C^{+})^{5}$
	$((5,6),(4,4,3),(2,2,1,1,1,2,2))$	D40	$(C^{+})^{2}$
	$((5,6),(4,4,3),(2,2,1,1,2,1,2))$	D39	$(C^{+})^{2}$
	$((5,6),(4,4,3),(2,2,1,1,2,2,1))$	D38	$(C^{+})^{2}$
	$((5,6),(4,4,3),(2,2,1,2,1,1,2))$	D36	$(C^{+})^{2}$
	$((5,6),(4,4,3),(2,2,1,2,1,2,1))$	D35	$(C^{+})^{2}$
	$((5,6),(4,4,3),(2,2,1,2,2,1,1))$	D33	$(C^{+})^{5}$
	$((5,6),(4,4,3),(2,2,2,1,1,1,2))$	D30	$(C^{+})^{3}$
	$((5,6),(4,4,3),(2,2,2,1,1,2,1))$	D28	$(C^{+})^{3}$
	$((5,6),(4,4,3),(2,2,2,1,2,1,1))$	D23	$(C^{+})^{5}$
	$((5,6),(4,4,3),(2,2,2,2,1,1,1))$	D14	$(C^{+})^{4}$
	$((6,5),(3,4,4),(1,1,1,2,2,2,2))$	D11	$(C^{-})^{5}$
	$((6,5),(3,4,4),(1,1,2,1,2,2,2))$	D21	–
	$((6,5),(3,4,4),(1,1,2,2,1,2,2))$	D36	$(C^{-})^{2}$
	$((6,5),(3,4,4),(1,1,2,2,2,1,2))$	D49	–
	$((6,5),(3,4,4),(1,1,2,2,2,2,1))$	D75	–
	$((6,5),(3,4,4),(1,2,1,1,2,2,2))$	D20	$(C^{+})^{2}$
	$((6,5),(3,4,4),(1,2,1,2,1,2,2))$	D35	$(C^{-})^{2}$
	$((6,5),(3,4,4),(1,2,1,2,2,1,2))$	D48	$(C^{-})^{2}$
	$((6,5),(3,4,4),(1,2,1,2,2,2,1))$	D74	–
	$((6,5),(3,4,4),(1,2,2,1,1,2,2))$	D34	$(C^{+})^{3}$
	$((6,5),(3,4,4),(1,2,2,1,2,1,2))$	D47	$(C^{-})^{2}$
	$((6,5),(3,4,4),(1,2,2,1,2,2,1))$	D73	–
	$((6,5),(3,4,4),(1,2,2,2,1,1,2))$	D46	$(C^{+})^{4}$
	$((6,5),(3,4,4),(1,2,2,2,1,2,1))$	D72	–

No.	dim. vector	orbit	apply ... to constructed family
	$((6,5),(3,4,4),(1,2,2,2,2,1,1))$	D71	$(C^+)^5$
	$((6,5),(3,4,4),(2,1,1,1,2,2,2))$	D19	C^+
	$((6,5),(3,4,4),(2,1,1,2,1,2,2))$	D33	C^+
	$((6,5),(3,4,4),(2,1,1,2,2,1,2))$	D45	C^+
	$((6,5),(3,4,4),(2,1,1,2,2,2,1))$	D70	C^+
	$((6,5),(3,4,4),(2,1,2,1,1,2,2))$	D32	$(C^+)^3$
	$((6,5),(3,4,4),(2,1,2,1,2,1,2))$	D44	$(C^-)^2$
	$((6,5),(3,4,4),(2,1,2,1,2,2,1))$	D69	C^+
	$((6,5),(3,4,4),(2,1,2,2,1,1,2))$	D43	$(C^+)^4$
	$((6,5),(3,4,4),(2,1,2,2,1,2,1))$	D68	C^+
	$((6,5),(3,4,4),(2,1,2,2,2,1,1))$	D67	$(C^+)^5$
	$((6,5),(3,4,4),(2,2,1,1,1,2,2))$	D31	$(C^+)^2$
	$((6,5),(3,4,4),(2,2,1,1,2,1,2))$	D42	$(C^+)^2$
	$((6,5),(3,4,4),(2,2,1,1,2,2,1))$	D66	$(C^+)^2$
	$((6,5),(3,4,4),(2,2,1,2,1,1,2))$	D41	$(C^+)^4$
	$((6,5),(3,4,4),(2,2,1,2,1,2,1))$	D63	$(C^+)^2$
	$((6,5),(3,4,4),(2,2,1,2,2,1,1))$	D60	$(C^+)^5$
	$((6,5),(3,4,4),(2,2,2,1,1,1,2))$	D40	$(C^+)^3$
	$((6,5),(3,4,4),(2,2,2,1,1,2,1))$	D39	$(C^+)^3$
	$((6,5),(3,4,4),(2,2,2,1,2,1,1))$	D36	$(C^+)^3$
	$((6,5),(3,4,4),(2,2,2,2,1,1,1))$	D30	$(C^+)^4$
	$((6,5),(4,3,4),(1,1,1,2,2,2,2))$	D31	–
	$((6,5),(4,3,4),(1,1,2,1,2,2,2))$	D42	–
	$((6,5),(4,3,4),(1,1,2,2,1,2,2))$	D66	–
	$((6,5),(4,3,4),(1,1,2,2,2,1,2))$	D65	–
	$((6,5),(4,3,4),(1,1,2,2,2,2,1))$	D64	–
	$((6,5),(4,3,4),(1,2,1,1,2,2,2))$	D41	$(C^+)^2$
	$((6,5),(4,3,4),(1,2,1,2,1,2,2))$	D63	–
	$((6,5),(4,3,4),(1,2,1,2,2,1,2))$	D62	–
	$((6,5),(4,3,4),(1,2,1,2,2,2,1))$	D61	–
	$((6,5),(4,3,4),(1,2,2,1,1,2,2))$	D60	$(C^+)^3$
	$((6,5),(4,3,4),(1,2,2,1,2,1,2))$	D59	–
	$((6,5),(4,3,4),(1,2,2,1,2,2,1))$	D58	–
	$((6,5),(4,3,4),(1,2,2,2,1,1,2))$	D57	$(C^+)^4$
	$((6,5),(4,3,4),(1,2,2,2,1,2,1))$	D56	$(C^+)^6$
	$((6,5),(4,3,4),(1,2,2,2,2,1,1))$	D55	$(C^+)^5$
	$((6,5),(4,3,4),(2,1,1,1,2,2,2))$	D40	C^+
	$((6,5),(4,3,4),(2,1,1,2,1,2,2))$	D39	C^+
	$((6,5),(4,3,4),(2,1,1,2,2,1,2))$	D38	C^+
	$((6,5),(4,3,4),(2,1,1,2,2,2,1))$	D37	C^+
	$((6,5),(4,3,4),(2,1,2,1,1,2,2))$	D36	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((6,5),(4,3,4),(2,1,2,1,2,1,2))$	D35	C^+
	$((6,5),(4,3,4),(2,1,2,1,2,2,1))$	D34	$(C^+)^6$
	$((6,5),(4,3,4),(2,1,2,2,1,1,2))$	D33	$(C^+)^4$
	$((6,5),(4,3,4),(2,1,2,2,1,2,1))$	D32	$(C^+)^6$
	$((6,5),(4,3,4),(2,1,2,2,2,1,1))$	D31	$(C^+)^5$
	$((6,5),(4,3,4),(2,2,1,1,1,2,2))$	D30	$(C^+)^3$
	$((6,5),(4,3,4),(2,2,1,1,2,1,2))$	D28	$(C^+)^2$
	$((6,5),(4,3,4),(2,2,1,1,2,2,1))$	D26	$(C^+)^2$
	$((6,5),(4,3,4),(2,2,1,2,1,1,2))$	D23	$(C^+)^4$
	$((6,5),(4,3,4),(2,2,1,2,1,2,1))$	D21	$(C^+)^6$
	$((6,5),(4,3,4),(2,2,1,2,2,1,1))$	D18	$(C^+)^5$
	$((6,5),(4,3,4),(2,2,2,1,1,1,2))$	D14	$(C^+)^3$
	$((6,5),(4,3,4),(2,2,2,1,1,2,1))$	D12	$(C^+)^3$
	$((6,5),(4,3,4),(2,2,2,1,2,1,1))$	D9	$(C^+)^5$
	$((6,5),(4,3,4),(2,2,2,2,1,1,1))$	D5	$(C^+)^4$
	$((6,5),(4,4,3),(1,1,1,2,2,2,2))$	D54	C^+
	$((6,5),(4,4,3),(1,1,2,1,2,2,2))$	D53	–
	$((6,5),(4,4,3),(1,1,2,2,1,2,2))$	D52	–
	$((6,5),(4,4,3),(1,1,2,2,2,1,2))$	D51	–
	$((6,5),(4,4,3),(1,1,2,2,2,2,1))$	D50	–
	$((6,5),(4,4,3),(1,2,1,1,2,2,2))$	D49	$(C^+)^2$
	$((6,5),(4,4,3),(1,2,1,2,1,2,2))$	D48	–
	$((6,5),(4,4,3),(1,2,1,2,2,1,2))$	D47	–
	$((6,5),(4,4,3),(1,2,1,2,2,2,1))$	D46	$(C^+)^6$
	$((6,5),(4,4,3),(1,2,2,1,1,2,2))$	D45	$(C^+)^3$
	$((6,5),(4,4,3),(1,2,2,1,2,1,2))$	D44	–
	$((6,5),(4,4,3),(1,2,2,1,2,2,1))$	D43	$(C^+)^6$
	$((6,5),(4,4,3),(1,2,2,2,1,1,2))$	D42	$(C^+)^4$
	$((6,5),(4,4,3),(1,2,2,2,1,2,1))$	D41	$(C^+)^6$
	$((6,5),(4,4,3),(1,2,2,2,2,1,1))$	D40	$(C^+)^5$
	$((6,5),(4,4,3),(2,1,1,1,2,2,2))$	D29	C^+
	$((6,5),(4,4,3),(2,1,1,2,1,2,2))$	D27	C^+
	$((6,5),(4,4,3),(2,1,1,2,2,1,2))$	D25	C^+
	$((6,5),(4,4,3),(2,1,1,2,2,2,1))$	D24	C^+
	$((6,5),(4,4,3),(2,1,2,1,1,2,2))$	D22	$(C^+)^3$
	$((6,5),(4,4,3),(2,1,2,1,2,1,2))$	D20	$(C^+)^7$
	$((6,5),(4,4,3),(2,1,2,1,2,2,1))$	D19	$(C^+)^6$
	$((6,5),(4,4,3),(2,1,2,2,1,1,2))$	D17	$(C^+)^7$
	$((6,5),(4,4,3),(2,1,2,2,1,2,1))$	D16	$(C^+)^7$
	$((6,5),(4,4,3),(2,1,2,2,2,1,1))$	D15	$(C^+)^5$
	$((6,5),(4,4,3),(2,2,1,1,1,2,2))$	D13	$(C^+)^2$

No.	dim. vector	orbit	apply …to constructed family
	$((6,5),(4,4,3),(2,2,1,1,2,1,2))$	D11	$(C^+)^2$
	$((6,5),(4,4,3),(2,2,1,1,2,2,1))$	D10	$(C^+)^2$
	$((6,5),(4,4,3),(2,2,1,2,1,1,2))$	D8	$(C^+)^4$
	$((6,5),(4,4,3),(2,2,1,2,1,2,1))$	D7	$(C^+)^6$
	$((6,5),(4,4,3),(2,2,1,2,2,1,1))$	D6	$(C^+)^5$
	$((6,5),(4,4,3),(2,2,2,1,1,1,2))$	D4	$(C^+)^3$
	$((6,5),(4,4,3),(2,2,2,1,1,2,1))$	D3	$(C^+)^3$
	$((6,5),(4,4,3),(2,2,2,1,2,1,1))$	D2	$(C^+)^8$
	$((6,5),(4,4,3),(2,2,2,2,1,1,1))$	D1	$(C^+)^{12}$
7.	$((2m,2m),(m,m,m,m),(m,m,m,m))$, $m \in \mathbb{N}$	known	
8.	$((2,3),(1,1,1,2),(1,1,1,1,1))$	C1	C^-
	$((2,3),(1,1,2,1),(1,1,1,1,1))$	C2	C^-
	$((2,3),(1,2,1,1),(1,1,1,1,1))$	C3	C^-
	$((2,3),(2,1,1,1),(1,1,1,1,1))$	C2	C^+
	$((3,2),(1,1,1,2),(1,1,1,1,1))$	C2	–
	$((3,2),(1,1,2,1),(1,1,1,1,1))$	C3	–
	$((3,2),(1,2,1,1),(1,1,1,1,1))$	C2	$(C^+)^2$
	$((3,2),(2,1,1,1),(1,1,1,1,1))$	C1	$(C^+)^5$
9.	$((3,3),(1,1,2,2),(1,1,1,1,2))$	C1	$(C^-)^2$
	$((3,3),(1,1,2,2),(1,1,1,2,1))$	C3	$(C^-)^3$
	$((3,3),(1,1,2,2),(1,1,2,1,1))$	C6	$(C^-)^2$
	$((3,3),(1,1,2,2),(1,2,1,1,1))$	C5	$(C^-)^2$
	$((3,3),(1,1,2,2),(2,1,1,1,1))$	C4	$(C^-)^2$
	$((3,3),(1,2,1,2),(1,1,1,1,2))$	C2	$(C^-)^2$
	$((3,3),(1,2,1,2),(1,1,1,2,1))$	C7	C^-
	$((3,3),(1,2,1,2),(1,1,2,1,1))$	C9	–
	$((3,3),(1,2,1,2),(1,2,1,1,1))$	C8	C^-
	$((3,3),(1,2,1,2),(2,1,1,1,1))$	C7	C^-
	$((3,3),(1,2,2,1),(1,1,1,1,2))$	C4	$(C^-)^2$
	$((3,3),(1,2,2,1),(1,1,1,2,1))$	C12	–
	$((3,3),(1,2,2,1),(1,1,2,1,1))$	C11	–
	$((3,3),(1,2,2,1),(1,2,1,1,1))$	C10	$(C^+)^3$
	$((3,3),(1,2,2,1),(2,1,1,1,1))$	C6	C^+
	$((3,3),(2,1,1,2),(1,1,1,1,2))$	C3	$(C^-)^2$
	$((3,3),(2,1,1,2),(1,1,1,2,1))$	C6	C^-
	$((3,3),(2,1,1,2),(1,1,2,1,1))$	C5	C^-
	$((3,3),(2,1,1,2),(1,2,1,1,1))$	C4	C^-
	$((3,3),(2,1,1,2),(2,1,1,1,1))$	C3	C^+
	$((3,3),(2,1,2,1),(1,1,1,1,2))$	C7	–
	$((3,3),(2,1,2,1),(1,1,1,2,1))$	C9	C^+

No.	dim. vector	orbit	apply ... to constructed family
	$((3,3),(2,1,2,1),(1,1,2,1,1))$	C8	–
	$((3,3),(2,1,2,1),(1,2,1,1,1))$	C7	$(C^+)^2$
	$((3,3),(2,1,2,1),(2,1,1,1,1))$	C2	$(C^+)^3$
	$((3,3),(2,2,1,1),(1,1,1,1,2))$	C6	–
	$((3,3),(2,2,1,1),(1,1,1,2,1))$	C5	–
	$((3,3),(2,2,1,1),(1,1,2,1,1))$	C4	–
	$((3,3),(2,2,1,1),(1,2,1,1,1))$	C3	$(C^+)^2$
	$((3,3),(2,2,1,1),(2,1,1,1,1))$	C1	$(C^+)^6$
10.	$((3,4),(1,2,2,2),(1,1,1,2,2))$	C1	$(C^-)^3$
	$((3,4),(1,2,2,2),(1,1,2,1,2))$	C3	$(C^-)^4$
	$((3,4),(1,2,2,2),(1,1,2,2,1))$	C10	–
	$((3,4),(1,2,2,2),(1,2,1,1,2))$	C6	$(C^-)^3$
	$((3,4),(1,2,2,2),(1,2,1,2,1))$	C14	C^-
	$((3,4),(1,2,2,2),(1,2,2,1,1))$	C32	C^-
	$((3,4),(1,2,2,2),(2,1,1,1,2))$	C5	$(C^-)^3$
	$((3,4),(1,2,2,2),(2,1,1,2,1))$	C13	C^+
	$((3,4),(1,2,2,2),(2,1,2,1,1))$	C31	$(C^+)^3$
	$((3,4),(1,2,2,2),(2,2,1,1,1))$	C30	$(C^+)^2$
	$((3,4),(2,1,2,2),(1,1,1,2,2))$	C2	$(C^-)^3$
	$((3,4),(2,1,2,2),(1,1,2,1,2))$	C7	$(C^-)^2$
	$((3,4),(2,1,2,2),(1,1,2,2,1))$	C20	C^-
	$((3,4),(2,1,2,2),(1,2,1,1,2))$	C9	C^-
	$((3,4),(2,1,2,2),(1,2,1,2,1))$	C19	C^-
	$((3,4),(2,1,2,2),(1,2,2,1,1))$	C33	C^-
	$((3,4),(2,1,2,2),(2,1,1,1,2))$	C8	$(C^-)^2$
	$((3,4),(2,1,2,2),(2,1,1,2,1))$	C18	C^+
	$((3,4),(2,1,2,2),(2,1,2,1,1))$	C29	$(C^+)^3$
	$((3,4),(2,1,2,2),(2,2,1,1,1))$	C26	$(C^+)^3$
	$((3,4),(2,2,1,2),(1,1,1,2,2))$	C4	$(C^-)^4$
	$((3,4),(2,2,1,2),(1,1,2,1,2))$	C12	C^-
	$((3,4),(2,2,1,2),(1,1,2,2,1))$	C17	C^-
	$((3,4),(2,2,1,2),(1,2,1,1,2))$	C11	C^-
	$((3,4),(2,2,1,2),(1,2,1,2,1))$	C16	C^-
	$((3,4),(2,2,1,2),(1,2,2,1,1))$	C15	C^-
	$((3,4),(2,2,1,2),(2,1,1,1,2))$	C10	$(C^+)^2$
	$((3,4),(2,2,1,2),(2,1,1,2,1))$	C14	C^+
	$((3,4),(2,2,1,2),(2,1,2,1,1))$	C13	$(C^+)^3$
	$((3,4),(2,2,1,2),(2,2,1,1,1))$	C12	$(C^+)^2$
	$((3,4),(2,2,2,1),(1,1,1,2,2))$	C26	C^+
	$((3,4),(2,2,2,1),(1,1,2,1,2))$	C25	–
	$((3,4),(2,2,2,1),(1,1,2,2,1))$	C24	–

No.	dim. vector	orbit	apply ...to constructed family
	((3,4),(2,2,2,1),(1,2,1,1,2))	C23	$(C^+)^2$
	((3,4),(2,2,2,1),(1,2,1,2,1))	C22	$(C^+)^3$
	((3,4),(2,2,2,1),(1,2,2,1,1))	C21	$(C^+)^3$
	((3,4),(2,2,2,1),(2,1,1,1,2))	C20	C^+
	((3,4),(2,2,2,1),(2,1,1,2,1))	C19	C^+
	((3,4),(2,2,2,1),(2,1,2,1,1))	C18	$(C^+)^3$
	((3,4),(2,2,2,1),(2,2,1,1,1))	C9	$(C^+)^3$
	((4,3),(1,2,2,2),(1,1,1,2,2))	C8	$(C^-)^3$
	((4,3),(1,2,2,2),(1,1,2,1,2))	C18	–
	((4,3),(1,2,2,2),(1,1,2,2,1))	C42	–
	((4,3),(1,2,2,2),(1,2,1,1,2))	C29	$(C^+)^2$
	((4,3),(1,2,2,2),(1,2,1,2,1))	C28	–
	((4,3),(1,2,2,2),(1,2,2,1,1))	C27	$(C^+)^3$
	((4,3),(1,2,2,2),(2,1,1,1,2))	C26	$(C^+)^2$
	((4,3),(1,2,2,2),(2,1,1,2,1))	C25	C^+
	((4,3),(1,2,2,2),(2,1,2,1,1))	C23	$(C^+)^3$
	((4,3),(1,2,2,2),(2,2,1,1,1))	C20	$(C^+)^2$
	((4,3),(2,1,2,2),(1,1,1,2,2))	C10	C^+
	((4,3),(2,1,2,2),(1,1,2,1,2))	C14	–
	((4,3),(2,1,2,2),(1,1,2,2,1))	C32	–
	((4,3),(2,1,2,2),(1,2,1,1,2))	C13	$(C^+)^2$
	((4,3),(2,1,2,2),(1,2,1,2,1))	C31	$(C^+)^4$
	((4,3),(2,1,2,2),(1,2,2,1,1))	C30	$(C^+)^3$
	((4,3),(2,1,2,2),(2,1,1,1,2))	C12	C^+
	((4,3),(2,1,2,2),(2,1,1,2,1))	C11	C^+
	((4,3),(2,1,2,2),(2,1,2,1,1))	C10	$(C^+)^4$
	((4,3),(2,1,2,2),(2,2,1,1,1))	C6	$(C^+)^2$
	((4,3),(2,2,1,2),(1,1,1,2,2))	C20	–
	((4,3),(2,2,1,2),(1,1,2,1,2))	C19	–
	((4,3),(2,2,1,2),(1,1,2,2,1))	C33	–
	((4,3),(2,2,1,2),(1,2,1,1,2))	C18	$(C^+)^2$
	((4,3),(2,2,1,2),(1,2,1,2,1))	C29	$(C^+)^4$
	((4,3),(2,2,1,2),(1,2,2,1,1))	C26	$(C^+)^4$
	((4,3),(2,2,1,2),(2,1,1,1,2))	C9	$(C^+)^2$
	((4,3),(2,2,1,2),(2,1,1,2,1))	C8	C^+
	((4,3),(2,2,1,2),(2,1,2,1,1))	C7	$(C^+)^3$
	((4,3),(2,2,1,2),(2,2,1,1,1))	C2	$(C^+)^4$
	((4,3),(2,2,2,1),(1,1,1,2,2))	C17	–
	((4,3),(2,2,2,1),(1,1,2,1,2))	C16	–
	((4,3),(2,2,2,1),(1,1,2,2,1))	C15	–
	((4,3),(2,2,2,1),(1,2,1,1,2))	C14	$(C^+)^2$

No.	dim. vector	orbit	apply ... to constructed family
	$((4,3),(2,2,2,1),(1,2,1,2,1))$	C13	$(C^+)^4$
	$((4,3),(2,2,2,1),(1,2,2,1,1))$	C12	$(C^+)^3$
	$((4,3),(2,2,2,1),(2,1,1,1,2))$	C5	C^+
	$((4,3),(2,2,2,1),(2,1,1,2,1))$	C4	C^+
	$((4,3),(2,2,2,1),(2,1,2,1,1))$	C3	$(C^+)^3$
	$((4,3),(2,2,2,1),(2,2,1,1,1))$	C1	$(C^+)^7$
11.	$((4,4),(1,2,2,3),(1,1,2,2,2))$	C2	$(C^-)^4$
	$((4,4),(1,2,2,3),(1,2,1,2,2))$	C7	$(C^-)^3$
	$((4,4),(1,2,2,3),(1,2,2,1,2))$	C20	$(C^-)^2$
	$((4,4),(1,2,2,3),(1,2,2,2,1))$	C46	C^-
	$((4,4),(1,2,2,3),(2,1,1,2,2))$	C9	$(C^-)^2$
	$((4,4),(1,2,2,3),(2,1,2,1,2))$	C19	$(C^-)^2$
	$((4,4),(1,2,2,3),(2,1,2,2,1))$	C45	–
	$((4,4),(1,2,2,3),(2,2,1,1,2))$	C33	$(C^-)^2$
	$((4,4),(1,2,2,3),(2,2,1,2,1))$	C44	–
	$((4,4),(1,2,2,3),(2,2,2,1,1))$	C43	–
	$((4,4),(1,2,3,2),(1,1,2,2,2))$	C11	$(C^-)^3$
	$((4,4),(1,2,3,2),(1,2,1,2,2))$	C16	$(C^-)^3$
	$((4,4),(1,2,3,2),(1,2,2,1,2))$	C40	–
	$((4,4),(1,2,3,2),(1,2,2,2,1))$	C61	–
	$((4,4),(1,2,3,2),(2,1,1,2,2))$	C15	$(C^-)^3$
	$((4,4),(1,2,3,2),(2,1,2,1,2))$	C39	–
	$((4,4),(1,2,3,2),(2,1,2,2,1))$	C60	–
	$((4,4),(1,2,3,2),(2,2,1,1,2))$	C38	$(C^+)^2$
	$((4,4),(1,2,3,2),(2,2,1,2,1))$	C59	–
	$((4,4),(1,2,3,2),(2,2,2,1,1))$	C58	$(C^+)^3$
	$((4,4),(1,3,2,2),(1,1,2,2,2))$	C21	–
	$((4,4),(1,3,2,2),(1,2,1,2,2))$	C35	–
	$((4,4),(1,3,2,2),(1,2,2,1,2))$	C56	–
	$((4,4),(1,3,2,2),(1,2,2,2,1))$	C70	–
	$((4,4),(1,3,2,2),(2,1,1,2,2))$	C34	C^+
	$((4,4),(1,3,2,2),(2,1,2,1,2))$	C55	–
	$((4,4),(1,3,2,2),(2,1,2,2,1))$	C69	–
	$((4,4),(1,3,2,2),(2,2,1,1,2))$	C54	$(C^+)^2$
	$((4,4),(1,3,2,2),(2,2,1,2,1))$	C68	–
	$((4,4),(1,3,2,2),(2,2,2,1,1))$	C67	$(C^+)^3$
	$((4,4),(2,1,2,3),(1,1,2,2,2))$	C4	$(C^-)^5$
	$((4,4),(2,1,2,3),(1,2,1,2,2))$	C12	$(C^-)^2$
	$((4,4),(2,1,2,3),(1,2,2,1,2))$	C17	$(C^-)^2$
	$((4,4),(2,1,2,3),(1,2,2,2,1))$	C41	C^-
	$((4,4),(2,1,2,3),(2,1,1,2,2))$	C11	$(C^-)^2$

No.	dim. vector	orbit	apply ...to constructed family
	$((4,4),(2,1,2,3),(2,1,2,1,2))$	C16	$(C^-)^2$
	$((4,4),(2,1,2,3),(2,1,2,2,1))$	C40	C^+
	$((4,4),(2,1,2,3),(2,2,1,1,2))$	C15	$(C^-)^2$
	$((4,4),(2,1,2,3),(2,2,1,2,1))$	C39	C^+
	$((4,4),(2,1,2,3),(2,2,2,1,1))$	C38	$(C^+)^3$
	$((4,4),(2,1,3,2),(1,1,2,2,2))$	C23	–
	$((4,4),(2,1,3,2),(1,2,1,2,2))$	C22	C^+
	$((4,4),(2,1,3,2),(1,2,2,1,2))$	C36	C^+
	$((4,4),(2,1,3,2),(1,2,2,2,1))$	C57	C^+
	$((4,4),(2,1,3,2),(2,1,1,2,2))$	C21	C^+
	$((4,4),(2,1,3,2),(2,1,2,1,2))$	C35	C^+
	$((4,4),(2,1,3,2),(2,1,2,2,1))$	C56	C^+
	$((4,4),(2,1,3,2),(2,2,1,1,2))$	C34	$(C^+)^2$
	$((4,4),(2,1,3,2),(2,2,1,2,1))$	C55	C^+
	$((4,4),(2,1,3,2),(2,2,2,1,1))$	C54	$(C^+)^3$
	$((4,4),(2,2,1,3),(1,1,2,2,2))$	C26	–
	$((4,4),(2,2,1,3),(1,2,1,2,2))$	C25	C^-
	$((4,4),(2,2,1,3),(1,2,2,1,2))$	C24	C^-
	$((4,4),(2,2,1,3),(1,2,2,2,1))$	C37	C^-
	$((4,4),(2,2,1,3),(2,1,1,2,2))$	C23	C^+
	$((4,4),(2,2,1,3),(2,1,2,1,2))$	C22	$(C^+)^2$
	$((4,4),(2,2,1,3),(2,1,2,2,1))$	C36	$(C^+)^2$
	$((4,4),(2,2,1,3),(2,2,1,1,2))$	C21	$(C^+)^2$
	$((4,4),(2,2,1,3),(2,2,1,2,1))$	C35	$(C^+)^2$
	$((4,4),(2,2,1,3),(2,2,2,1,1))$	C34	$(C^+)^3$
	$((4,4),(2,2,3,1),(1,1,2,2,2))$	C53	–
	$((4,4),(2,2,3,1),(1,2,1,2,2))$	C52	–
	$((4,4),(2,2,3,1),(1,2,2,1,2))$	C51	$(C^+)^3$
	$((4,4),(2,2,3,1),(1,2,2,2,1))$	C50	$(C^+)^3$
	$((4,4),(2,2,3,1),(2,1,1,2,2))$	C49	C^+
	$((4,4),(2,2,3,1),(2,1,2,1,2))$	C48	$(C^+)^3$
	$((4,4),(2,2,3,1),(2,1,2,2,1))$	C47	$(C^+)^3$
	$((4,4),(2,2,3,1),(2,2,1,1,2))$	C46	$(C^+)^2$
	$((4,4),(2,2,3,1),(2,2,1,2,1))$	C45	$(C^+)^3$
	$((4,4),(2,2,3,1),(2,2,2,1,1))$	C42	$(C^+)^3$
	$((4,4),(2,3,1,2),(1,1,2,2,2))$	C49	–
	$((4,4),(2,3,1,2),(1,2,1,2,2))$	C48	$(C^+)^2$
	$((4,4),(2,3,1,2),(1,2,2,1,2))$	C47	$(C^+)^2$
	$((4,4),(2,3,1,2),(1,2,2,2,1))$	C66	$(C^+)^2$
	$((4,4),(2,3,1,2),(2,1,1,2,2))$	C46	C^+
	$((4,4),(2,3,1,2),(2,1,2,1,2))$	C45	$(C^+)^2$

No.	dim. vector	orbit	apply ...to constructed family
	$((4,4),(2,3,1,2),(2,1,2,2,1))$	C44	$(C^+)^2$
	$((4,4),(2,3,1,2),(2,2,1,1,2))$	C42	$(C^+)^2$
	$((4,4),(2,3,1,2),(2,2,1,2,1))$	C28	$(C^+)^2$
	$((4,4),(2,3,1,2),(2,2,2,1,1))$	C25	$(C^+)^3$
	$((4,4),(2,3,2,1),(1,1,2,2,2))$	C65	–
	$((4,4),(2,3,2,1),(1,2,1,2,2))$	C64	$(C^+)^3$
	$((4,4),(2,3,2,1),(1,2,2,1,2))$	C63	$(C^+)^3$
	$((4,4),(2,3,2,1),(1,2,2,2,1))$	C62	$(C^+)^3$
	$((4,4),(2,3,2,1),(2,1,1,2,2))$	C41	C^+
	$((4,4),(2,3,2,1),(2,1,2,1,2))$	C40	$(C^+)^3$
	$((4,4),(2,3,2,1),(2,1,2,2,1))$	C39	$(C^+)^3$
	$((4,4),(2,3,2,1),(2,2,1,1,2))$	C32	$(C^+)^2$
	$((4,4),(2,3,2,1),(2,2,1,2,1))$	C31	$(C^+)^6$
	$((4,4),(2,3,2,1),(2,2,2,1,1))$	C11	$(C^+)^3$
	$((4,4),(3,1,2,2),(1,1,2,2,2))$	C46	–
	$((4,4),(3,1,2,2),(1,2,1,2,2))$	C45	C^+
	$((4,4),(3,1,2,2),(1,2,2,1,2))$	C44	C^+
	$((4,4),(3,1,2,2),(1,2,2,2,1))$	C43	C^+
	$((4,4),(3,1,2,2),(2,1,1,2,2))$	C42	C^+
	$((4,4),(3,1,2,2),(2,1,2,1,2))$	C28	C^+
	$((4,4),(3,1,2,2),(2,1,2,2,1))$	C27	$(C^+)^4$
	$((4,4),(3,1,2,2),(2,2,1,1,2))$	C25	$(C^+)^2$
	$((4,4),(3,1,2,2),(2,2,1,2,1))$	C23	$(C^+)^4$
	$((4,4),(3,1,2,2),(2,2,2,1,1))$	C20	C^-
	$((4,4),(3,2,1,2),(1,1,2,2,2))$	C41	–
	$((4,4),(3,2,1,2),(1,2,1,2,2))$	C40	$(C^+)^2$
	$((4,4),(3,2,1,2),(1,2,2,1,2))$	C39	$(C^+)^2$
	$((4,4),(3,2,1,2),(1,2,2,2,1))$	C38	$(C^+)^4$
	$((4,4),(3,2,1,2),(2,1,1,2,2))$	C32	C^+
	$((4,4),(3,2,1,2),(2,1,2,1,2))$	C31	$(C^+)^5$
	$((4,4),(3,2,1,2),(2,1,2,2,1))$	C30	$(C^+)^4$
	$((4,4),(3,2,1,2),(2,2,1,1,2))$	C11	$(C^+)^2$
	$((4,4),(3,2,1,2),(2,2,1,2,1))$	C10	$(C^+)^5$
	$((4,4),(3,2,1,2),(2,2,2,1,1))$	C6	$(C^+)^3$
	$((4,4),(3,2,2,1),(1,1,2,2,2))$	C37	–
	$((4,4),(3,2,2,1),(1,2,1,2,2))$	C36	$(C^+)^3$
	$((4,4),(3,2,2,1),(1,2,2,1,2))$	C35	$(C^+)^3$
	$((4,4),(3,2,2,1),(1,2,2,2,1))$	C34	$(C^+)^4$
	$((4,4),(3,2,2,1),(2,1,1,2,2))$	C33	C^+
	$((4,4),(3,2,2,1),(2,1,2,1,2))$	C29	$(C^+)^5$
	$((4,4),(3,2,2,1),(2,1,2,2,1))$	C26	$(C^+)^6$

No.	dim. vector	orbit	apply ...to constructed family
	$((4,4),(3,2,2,1),(2,2,1,1,2))$	C8	$(C^+)^2$
	$((4,4),(3,2,2,1),(2,2,1,2,1))$	C7	$(C^+)^4$
	$((4,4),(3,2,2,1),(2,2,2,1,1))$	C2	$(C^+)^5$
12.	$((4,5),(1,2,3,3),(1,2,2,2,2))$	C4	$(C^-)^6$
	$((4,5),(1,2,3,3),(2,1,2,2,2))$	C12	$(C^-)^3$
	$((4,5),(1,2,3,3),(2,2,1,2,2))$	C17	$(C^-)^3$
	$((4,5),(1,2,3,3),(2,2,2,1,2))$	C41	$(C^-)^2$
	$((4,5),(1,2,3,3),(2,2,2,2,1))$	C85	–
	$((4,5),(1,3,2,3),(1,2,2,2,2))$	C23	C^-
	$((4,5),(1,3,2,3),(2,1,2,2,2))$	C22	–
	$((4,5),(1,3,2,3),(2,2,1,2,2))$	C36	–
	$((4,5),(1,3,2,3),(2,2,2,1,2))$	C57	–
	$((4,5),(1,3,2,3),(2,2,2,2,1))$	C84	–
	$((4,5),(1,3,3,2),(1,2,2,2,2))$	C63	–
	$((4,5),(1,3,3,2),(2,1,2,2,2))$	C62	–
	$((4,5),(1,3,3,2),(2,2,1,2,2))$	C77	–
	$((4,5),(1,3,3,2),(2,2,2,1,2))$	C72	–
	$((4,5),(1,3,3,2),(2,2,2,2,1))$	C83	–
	$((4,5),(2,1,3,3),(1,2,2,2,2))$	C26	C^-
	$((4,5),(2,1,3,3),(2,1,2,2,2))$	C25	$(C^-)^2$
	$((4,5),(2,1,3,3),(2,2,1,2,2))$	C24	$(C^-)^2$
	$((4,5),(2,1,3,3),(2,2,2,1,2))$	C37	$(C^-)^2$
	$((4,5),(2,1,3,3),(2,2,2,2,1))$	C82	C^+
	$((4,5),(2,3,1,3),(1,2,2,2,2))$	C53	C^-
	$((4,5),(2,3,1,3),(2,1,2,2,2))$	C52	C^-
	$((4,5),(2,3,1,3),(2,2,1,2,2))$	C51	$(C^+)^2$
	$((4,5),(2,3,1,3),(2,2,2,1,2))$	C50	$(C^+)^2$
	$((4,5),(2,3,1,3),(2,2,2,2,1))$	C81	$(C^+)^2$
	$((4,5),(2,3,3,1),(1,2,2,2,2))$	C94	–
	$((4,5),(2,3,3,1),(2,1,2,2,2))$	C98	C^+
	$((4,5),(2,3,3,1),(2,2,1,2,2))$	C100	$(C^+)^2$
	$((4,5),(2,3,3,1),(2,2,2,1,2))$	C96	$(C^+)^3$
	$((4,5),(2,3,3,1),(2,2,2,2,1))$	C80	$(C^+)^3$
	$((4,5),(3,1,2,3),(1,2,2,2,2))$	C49	C^-
	$((4,5),(3,1,2,3),(2,1,2,2,2))$	C48	C^+
	$((4,5),(3,1,2,3),(2,2,1,2,2))$	C47	C^+
	$((4,5),(3,1,2,3),(2,2,2,1,2))$	C66	C^+
	$((4,5),(3,1,2,3),(2,2,2,2,1))$	C79	C^+
	$((4,5),(3,1,3,2),(1,2,2,2,2))$	C93	C^+
	$((4,5),(3,1,3,2),(2,1,2,2,2))$	C89	C^+
	$((4,5),(3,1,3,2),(2,2,1,2,2))$	C76	C^+

No.	dim. vector	orbit	apply ... to constructed family
	((4,5),(3,1,3,2),(2,2,2,1,2))	C71	C^+
	((4,5),(3,1,3,2),(2,2,2,2,1))	C78	C^+
	((4,5),(3,2,1,3),(1,2,2,2,2))	C65	C^-
	((4,5),(3,2,1,3),(2,1,2,2,2))	C64	$(C^+)^2$
	((4,5),(3,2,1,3),(2,2,1,2,2))	C63	$(C^+)^2$
	((4,5),(3,2,1,3),(2,2,2,1,2))	C62	$(C^+)^2$
	((4,5),(3,2,1,3),(2,2,2,2,1))	C77	$(C^+)^2$
	((4,5),(3,2,3,1),(1,2,2,2,2))	C92	–
	((4,5),(3,2,3,1),(2,1,2,2,2))	C87	C^+
	((4,5),(3,2,3,1),(2,2,1,2,2))	C93	$(C^+)^3$
	((4,5),(3,2,3,1),(2,2,2,1,2))	C89	$(C^+)^3$
	((4,5),(3,2,3,1),(2,2,2,2,1))	C76	$(C^+)^3$
	((4,5),(3,3,1,2),(1,2,2,2,2))	C91	$(C^+)^2$
	((4,5),(3,3,1,2),(2,1,2,2,2))	C85	$(C^+)^2$
	((4,5),(3,3,1,2),(2,2,1,2,2))	C61	$(C^+)^2$
	((4,5),(3,3,1,2),(2,2,2,1,2))	C60	$(C^+)^2$
	((4,5),(3,3,1,2),(2,2,2,2,1))	C59	$(C^+)^2$
	((4,5),(3,3,2,1),(1,2,2,2,2))	C90	–
	((4,5),(3,3,2,1),(2,1,2,2,2))	C82	$(C^+)^3$
	((4,5),(3,3,2,1),(2,2,1,2,2))	C57	$(C^+)^3$
	((4,5),(3,3,2,1),(2,2,2,1,2))	C56	$(C^+)^3$
	((4,5),(3,3,2,1),(2,2,2,2,1))	C55	$(C^+)^3$
	((5,4),(1,2,3,3),(1,2,2,2,2))	C48	–
	((5,4),(1,2,3,3),(2,1,2,2,2))	C47	–
	((5,4),(1,2,3,3),(2,2,1,2,2))	C66	–
	((5,4),(1,2,3,3),(2,2,2,1,2))	C79	–
	((5,4),(1,2,3,3),(2,2,2,2,1))	C75	–
	((5,4),(1,3,2,3),(1,2,2,2,2))	C89	–
	((5,4),(1,3,2,3),(2,1,2,2,2))	C76	–
	((5,4),(1,3,2,3),(2,2,1,2,2))	C71	–
	((5,4),(1,3,2,3),(2,2,2,1,2))	C78	–
	((5,4),(1,3,2,3),(2,2,2,2,1))	C74	–
	((5,4),(1,3,3,2),(1,2,2,2,2))	C88	–
	((5,4),(1,3,3,2),(2,1,2,2,2))	C97	–
	((5,4),(1,3,3,2),(2,2,1,2,2))	C99	–
	((5,4),(1,3,3,2),(2,2,2,1,2))	C95	–
	((5,4),(1,3,3,2),(2,2,2,2,1))	C73	–
	((5,4),(2,1,3,3),(1,2,2,2,2))	C64	C^+
	((5,4),(2,1,3,3),(2,1,2,2,2))	C63	C^+
	((5,4),(2,1,3,3),(2,2,1,2,2))	C62	C^+
	((5,4),(2,1,3,3),(2,2,2,1,2))	C77	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((5,4),(2,1,3,3),(2,2,2,2,1))$	C72	C^+
	$((5,4),(2,3,1,3),(1,2,2,2,2))$	C87	–
	$((5,4),(2,3,1,3),(2,1,2,2,2))$	C93	$(C^+)^2$
	$((5,4),(2,3,1,3),(2,2,1,2,2))$	C89	$(C^+)^2$
	$((5,4),(2,3,1,3),(2,2,2,1,2))$	C76	$(C^+)^2$
	$((5,4),(2,3,1,3),(2,2,2,2,1))$	C71	$(C^+)^2$
	$((5,4),(2,3,3,1),(1,2,2,2,2))$	C86	–
	$((5,4),(2,3,3,1),(2,1,2,2,2))$	C91	$(C^+)^3$
	$((5,4),(2,3,3,1),(2,2,1,2,2))$	C85	$(C^+)^3$
	$((5,4),(2,3,3,1),(2,2,2,1,2))$	C61	$(C^+)^3$
	$((5,4),(2,3,3,1),(2,2,2,2,1))$	C60	$(C^+)^3$
	$((5,4),(3,1,2,3),(1,2,2,2,2))$	C85	C^+
	$((5,4),(3,1,2,3),(2,1,2,2,2))$	C61	C^+
	$((5,4),(3,1,2,3),(2,2,1,2,2))$	C60	C^+
	$((5,4),(3,1,2,3),(2,2,2,1,2))$	C59	C^+
	$((5,4),(3,1,2,3),(2,2,2,2,1))$	C58	$(C^+)^2$
	$((5,4),(3,1,3,2),(1,2,2,2,2))$	C84	C^+
	$((5,4),(3,1,3,2),(2,1,2,2,2))$	C70	C^+
	$((5,4),(3,1,3,2),(2,2,1,2,2))$	C69	C^+
	$((5,4),(3,1,3,2),(2,2,2,1,2))$	C68	C^+
	$((5,4),(3,1,3,2),(2,2,2,2,1))$	C67	$(C^+)^2$
	$((5,4),(3,2,1,3),(1,2,2,2,2))$	C82	$(C^+)^2$
	$((5,4),(3,2,1,3),(2,1,2,2,2))$	C57	$(C^+)^2$
	$((5,4),(3,2,1,3),(2,2,1,2,2))$	C56	$(C^+)^2$
	$((5,4),(3,2,1,3),(2,2,2,1,2))$	C55	$(C^+)^2$
	$((5,4),(3,2,1,3),(2,2,2,2,1))$	C54	$(C^+)^4$
	$((5,4),(3,2,3,1),(1,2,2,2,2))$	C81	$(C^+)^2$
	$((5,4),(3,2,3,1),(2,1,2,2,2))$	C66	$(C^+)^3$
	$((5,4),(3,2,3,1),(2,2,1,2,2))$	C44	$(C^+)^3$
	$((5,4),(3,2,3,1),(2,2,2,1,2))$	C28	$(C^+)^3$
	$((5,4),(3,2,3,1),(2,2,2,2,1))$	C25	$(C^+)^4$
	$((5,4),(3,3,1,2),(1,2,2,2,2))$	C79	$(C^+)^2$
	$((5,4),(3,3,1,2),(2,1,2,2,2))$	C43	$(C^+)^2$
	$((5,4),(3,3,1,2),(2,2,1,2,2))$	C27	$(C^+)^4$
	$((5,4),(3,3,1,2),(2,2,2,1,2))$	C23	$(C^+)^5$
	$((5,4),(3,3,1,2),(2,2,2,2,1))$	C20	$(C^+)^4$
	$((5,4),(3,3,2,1),(1,2,2,2,2))$	C77	$(C^+)^3$
	$((5,4),(3,3,2,1),(2,1,2,2,2))$	C38	$(C^+)^4$
	$((5,4),(3,3,2,1),(2,2,1,2,2))$	C30	$(C^+)^5$
	$((5,4),(3,3,2,1),(2,2,2,1,2))$	C10	$(C^+)^6$

No.	dim. vector	orbit	apply ...to constructed family
	$((5,4),(3,3,2,1),(2,2,2,2,1))$	C6	$(C^+)^4$
13.	$((m,m,m),(m,m,m),(m,m,m))$, $m \in \mathbb{N}$	known	
14.	$((1,1,2),(1,1,2),(1,1,1,1))$	B1	C^-
	$((1,1,2),(1,2,1),(1,1,1,1))$	B2	$(C^-)^2$
	$((1,1,2),(2,1,1),(1,1,1,1))$	B3	C^-
	$((1,2,1),(1,1,2),(1,1,1,1))$	B2	$(C^-)^2$
	$((1,2,1),(1,2,1),(1,1,1,1))$	B4	–
	$((1,2,1),(2,1,1),(1,1,1,1))$	B2	–
	$((2,1,1),(1,1,2),(1,1,1,1))$	B2	C^-
	$((2,1,1),(1,2,1),(1,1,1,1))$	B3	–
	$((2,1,1),(2,1,1),(1,1,1,1))$	B1	$(C^+)^4$
15.	$((1,2,2),(1,2,2),(1,1,1,2))$	B1	$(C^-)^2$
	$((1,2,2),(1,2,2),(1,1,2,1))$	B4	$(C^-)^2$
	$((1,2,2),(1,2,2),(1,2,1,1))$	B6	C^-
	$((1,2,2),(1,2,2),(2,1,1,1))$	B5	$(C^-)^2$
	$((1,2,2),(2,1,2),(1,1,1,2))$	B2	$(C^-)^3$
	$((1,2,2),(2,1,2),(1,1,2,1))$	B10	C^-
	$((1,2,2),(2,1,2),(1,2,1,1))$	B16	C^-
	$((1,2,2),(2,1,2),(2,1,1,1))$	B13	$(C^+)^3$
	$((1,2,2),(2,2,1),(1,1,1,2))$	B7	C^+
	$((1,2,2),(2,2,1),(1,1,2,1))$	B15	–
	$((1,2,2),(2,2,1),(1,2,1,1))$	B14	$(C^+)^2$
	$((1,2,2),(2,2,1),(2,1,1,1))$	B9	C^+
	$((2,1,2),(1,2,2),(1,1,1,2))$	B3	$(C^-)^3$
	$((2,1,2),(1,2,2),(1,1,2,1))$	B9	C^-
	$((2,1,2),(1,2,2),(1,2,1,1))$	B8	C^-
	$((2,1,2),(1,2,2),(2,1,1,1))$	B7	C^+
	$((2,1,2),(2,1,2),(1,1,1,2))$	B4	C^-
	$((2,1,2),(2,1,2),(1,1,2,1))$	B6	–
	$((2,1,2),(2,1,2),(1,2,1,1))$	B5	C^-
	$((2,1,2),(2,1,2),(2,1,1,1))$	B4	C^+
	$((2,1,2),(2,2,1),(1,1,1,2))$	B10	–
	$((2,1,2),(2,2,1),(1,1,2,1))$	B16	–
	$((2,1,2),(2,2,1),(1,2,1,1))$	B13	$(C^+)^3$
	$((2,1,2),(2,2,1),(2,1,1,1))$	B2	C^+
	$((2,2,1),(1,2,2),(1,1,1,2))$	B13	C^+
	$((2,2,1),(1,2,2),(1,1,2,1))$	B12	–
	$((2,2,1),(1,2,2),(1,2,1,1))$	B11	$(C^+)^2$
	$((2,2,1),(1,2,2),(2,1,1,1))$	B10	C^+
	$((2,2,1),(2,1,2),(1,1,1,2))$	B9	–

No.	dim. vector	orbit	apply ...to constructed family
	$((2,2,1),(2,1,2),(1,1,2,1))$	B8	–
	$((2,2,1),(2,1,2),(1,2,1,1))$	B7	$(C^+)^3$
	$((2,2,1),(2,1,2),(2,1,1,1))$	B3	C^+
	$((2,2,1),(2,2,1),(1,1,1,2))$	B6	C^+
	$((2,2,1),(2,2,1),(1,1,2,1))$	B5	–
	$((2,2,1),(2,2,1),(1,2,1,1))$	B4	$(C^+)^2$
	$((2,2,1),(2,2,1),(2,1,1,1))$	B1	$(C^+)^5$
16.	$((5,7),(4,4,4),(1,1,2,2,2,2,2))$	D11	$(C^-)^6$
	$((5,7),(4,4,4),(1,2,1,2,2,2,2))$	D21	C^-
	$((5,7),(4,4,4),(1,2,2,1,2,2,2))$	D36	$(C^-)^3$
	$((5,7),(4,4,4),(1,2,2,2,1,2,2))$	D49	C^-
	$((5,7),(4,4,4),(1,2,2,2,2,1,2))$	D75	C^-
	$((5,7),(4,4,4),(1,2,2,2,2,2,1))$	D86	C^-
	$((5,7),(4,4,4),(2,1,1,2,2,2,2))$	D20	C^+
	$((5,7),(4,4,4),(2,1,2,1,2,2,2))$	D35	$(C^-)^3$
	$((5,7),(4,4,4),(2,1,2,2,1,2,2))$	D48	$(C^-)^3$
	$((5,7),(4,4,4),(2,1,2,2,2,1,2))$	D74	C^-
	$((5,7),(4,4,4),(2,1,2,2,2,2,1))$	D85	C^-
	$((5,7),(4,4,4),(2,2,1,1,2,2,2))$	D34	C^+
	$((5,7),(4,4,4),(2,2,1,2,1,2,2))$	D47	$(C^-)^3$
	$((5,7),(4,4,4),(2,2,1,2,2,1,2))$	D73	C^-
	$((5,7),(4,4,4),(2,2,1,2,2,2,1))$	D84	C^-
	$((5,7),(4,4,4),(2,2,2,1,1,2,2))$	D46	$(C^+)^3$
	$((5,7),(4,4,4),(2,2,2,1,2,1,2))$	D72	C^-
	$((5,7),(4,4,4),(2,2,2,1,2,2,1))$	D83	C^-
	$((5,7),(4,4,4),(2,2,2,2,1,1,2))$	D71	$(C^+)^4$
	$((5,7),(4,4,4),(2,2,2,2,1,2,1))$	D82	C^-
	$((5,7),(4,4,4),(2,2,2,2,2,1,1))$	D81	C^-
	$((7,5),(4,4,4),(1,1,2,2,2,2,2))$	D86	–
	$((7,5),(4,4,4),(1,2,1,2,2,2,2))$	D85	–
	$((7,5),(4,4,4),(1,2,2,1,2,2,2))$	D84	–
	$((7,5),(4,4,4),(1,2,2,2,1,2,2))$	D83	–
	$((7,5),(4,4,4),(1,2,2,2,2,1,2))$	D82	–
	$((7,5),(4,4,4),(1,2,2,2,2,2,1))$	D81	–
	$((7,5),(4,4,4),(2,1,1,2,2,2,2))$	D80	C^+
	$((7,5),(4,4,4),(2,1,2,1,2,2,2))$	D79	C^+
	$((7,5),(4,4,4),(2,1,2,2,1,2,2))$	D78	C^+
	$((7,5),(4,4,4),(2,1,2,2,2,1,2))$	D77	C^+
	$((7,5),(4,4,4),(2,1,2,2,2,2,1))$	D76	$(C^+)^6$
	$((7,5),(4,4,4),(2,2,1,1,2,2,2))$	D65	$(C^+)^3$
	$((7,5),(4,4,4),(2,2,1,2,1,2,2))$	D62	$(C^+)^2$

No.	dim. vector	orbit	apply ... to constructed family
	$((7,5),(4,4,4),(2,2,1,2,2,1,2))$	D59	$(C^+)^2$
	$((7,5),(4,4,4),(2,2,1,2,2,2,1))$	D57	$(C^+)^6$
	$((7,5),(4,4,4),(2,2,2,1,1,2,2))$	D38	$(C^+)^3$
	$((7,5),(4,4,4),(2,2,2,1,2,1,2))$	D35	$(C^+)^3$
	$((7,5),(4,4,4),(2,2,2,1,2,2,1))$	D33	$(C^+)^6$
	$((7,5),(4,4,4),(2,2,2,2,1,1,2))$	D28	$(C^+)^4$
	$((7,5),(4,4,4),(2,2,2,2,1,2,1))$	D23	$(C^+)^6$
	$((7,5),(4,4,4),(2,2,2,2,2,1,1))$	D14	$(C^+)^5$
17.	$((6,6),(3,4,5),(1,1,2,2,2,2,2))$	D5	$(C^-)^7$
	$((6,6),(3,4,5),(1,2,1,2,2,2,2))$	D8	$(C^-)^5$
	$((6,6),(3,4,5),(1,2,2,1,2,2,2))$	D16	C^-
	$((6,6),(3,4,5),(1,2,2,2,1,2,2))$	D40	$(C^-)^2$
	$((6,6),(3,4,5),(1,2,2,2,2,1,2))$	D54	C^-
	$((6,6),(3,4,5),(1,2,2,2,2,2,1))$	D98	C^-
	$((6,6),(3,4,5),(2,1,1,2,2,2,2))$	D12	$(C^-)^5$
	$((6,6),(3,4,5),(2,1,2,1,2,2,2))$	D23	$(C^-)^2$
	$((6,6),(3,4,5),(2,1,2,2,1,2,2))$	D39	$(C^-)^2$
	$((6,6),(3,4,5),(2,1,2,2,2,1,2))$	D53	$(C^-)^2$
	$((6,6),(3,4,5),(2,1,2,2,2,2,1))$	D97	C^-
	$((6,6),(3,4,5),(2,2,1,1,2,2,2))$	D22	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,1,2,1,2,2))$	D38	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,1,2,2,1,2))$	D52	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,1,2,2,2,1))$	D96	C^-
	$((6,6),(3,4,5),(2,2,2,1,1,2,2))$	D37	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,2,1,2,1,2))$	D51	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,2,1,2,2,1))$	D95	C^-
	$((6,6),(3,4,5),(2,2,2,2,1,1,2))$	D50	$(C^-)^2$
	$((6,6),(3,4,5),(2,2,2,2,1,2,1))$	D94	C^-
	$((6,6),(3,4,5),(2,2,2,2,2,1,1))$	D93	–
	$((6,6),(3,5,4),(1,1,2,2,2,2,2))$	D32	–
	$((6,6),(3,5,4),(1,2,1,2,2,2,2))$	D44	$(C^-)^5$
	$((6,6),(3,5,4),(1,2,2,1,2,2,2))$	D69	$(C^-)^2$
	$((6,6),(3,5,4),(1,2,2,2,1,2,2))$	D79	$(C^-)^2$
	$((6,6),(3,5,4),(1,2,2,2,2,1,2))$	D91	$(C^-)^2$
	$((6,6),(3,5,4),(1,2,2,2,2,2,1))$	D104	–
	$((6,6),(3,5,4),(2,1,1,2,2,2,2))$	D43	C^+
	$((6,6),(3,5,4),(2,1,2,1,2,2,2))$	D68	$(C^-)^2$
	$((6,6),(3,5,4),(2,1,2,2,1,2,2))$	D78	$(C^-)^2$
	$((6,6),(3,5,4),(2,1,2,2,2,1,2))$	D90	$(C^-)^2$
	$((6,6),(3,5,4),(2,1,2,2,2,2,1))$	D103	C^+
	$((6,6),(3,5,4),(2,2,1,1,2,2,2))$	D67	$(C^+)^2$

No.	dim. vector	orbit	apply ... to constructed family
	$((6,6),(3,5,4),(2,2,1,2,1,2,2))$	D77	$(C^-)^2$
	$((6,6),(3,5,4),(2,2,1,2,2,1,2))$	D89	$(C^-)^2$
	$((6,6),(3,5,4),(2,2,1,2,2,2,1))$	D102	$(C^-)^2$
	$((6,6),(3,5,4),(2,2,2,1,1,2,2))$	D76	$(C^+)^3$
	$((6,6),(3,5,4),(2,2,2,1,2,1,2))$	D88	$(C^-)^2$
	$((6,6),(3,5,4),(2,2,2,1,2,2,1))$	D101	$(C^+)^3$
	$((6,6),(3,5,4),(2,2,2,2,1,1,2))$	D87	$(C^+)^4$
	$((6,6),(3,5,4),(2,2,2,2,1,2,1))$	D100	$(C^+)^4$
	$((6,6),(3,5,4),(2,2,2,2,2,1,1))$	D99	$(C^+)^5$
	$((6,6),(4,3,5),(1,1,2,2,2,2,2))$	D19	C^-
	$((6,6),(4,3,5),(1,2,1,2,2,2,2))$	D33	C^-
	$((6,6),(4,3,5),(1,2,2,1,2,2,2))$	D45	C^-
	$((6,6),(4,3,5),(1,2,2,2,1,2,2))$	D70	C^-
	$((6,6),(4,3,5),(1,2,2,2,2,1,2))$	D80	C^-
	$((6,6),(4,3,5),(1,2,2,2,2,2,1))$	D92	C^-
	$((6,6),(4,3,5),(2,1,1,2,2,2,2))$	D32	C^+
	$((6,6),(4,3,5),(2,1,2,1,2,2,2))$	D44	$(C^-)^4$
	$((6,6),(4,3,5),(2,1,2,2,1,2,2))$	D69	C^-
	$((6,6),(4,3,5),(2,1,2,2,2,1,2))$	D79	C^-
	$((6,6),(4,3,5),(2,1,2,2,2,2,1))$	D91	C^-
	$((6,6),(4,3,5),(2,2,1,1,2,2,2))$	D43	$(C^+)^2$
	$((6,6),(4,3,5),(2,2,1,2,1,2,2))$	D68	C^-
	$((6,6),(4,3,5),(2,2,1,2,2,1,2))$	D78	C^-
	$((6,6),(4,3,5),(2,2,1,2,2,2,1))$	D90	C^-
	$((6,6),(4,3,5),(2,2,2,1,1,2,2))$	D67	$(C^+)^3$
	$((6,6),(4,3,5),(2,2,2,1,2,1,2))$	D77	C^-
	$((6,6),(4,3,5),(2,2,2,1,2,2,1))$	D89	C^-
	$((6,6),(4,3,5),(2,2,2,2,1,1,2))$	D76	$(C^+)^3$
	$((6,6),(4,3,5),(2,2,2,2,1,2,1))$	D88	C^-
	$((6,6),(4,3,5),(2,2,2,2,2,1,1))$	D87	$(C^+)^5$
	$((6,6),(4,5,3),(1,1,2,2,2,2,2))$	D110	–
	$((6,6),(4,5,3),(1,2,1,2,2,2,2))$	D109	–
	$((6,6),(4,5,3),(1,2,2,1,2,2,2))$	D108	–
	$((6,6),(4,5,3),(1,2,2,2,1,2,2))$	D107	–
	$((6,6),(4,5,3),(1,2,2,2,2,1,2))$	D106	–
	$((6,6),(4,5,3),(1,2,2,2,2,2,1))$	D105	–
	$((6,6),(4,5,3),(2,1,1,2,2,2,2))$	D98	C^+
	$((6,6),(4,5,3),(2,1,2,1,2,2,2))$	D97	C^+
	$((6,6),(4,5,3),(2,1,2,2,1,2,2))$	D96	C^+
	$((6,6),(4,5,3),(2,1,2,2,2,1,2))$	D95	C^+
	$((6,6),(4,5,3),(2,1,2,2,2,2,1))$	D94	C^+

No.	dim. vector	orbit	apply ...to constructed family
	((6,6),(4,5,3),(2,2,1,1,2,2,2))	D75	$(C^+)^2$
	((6,6),(4,5,3),(2,2,1,2,1,2,2))	D74	$(C^+)^2$
	((6,6),(4,5,3),(2,2,1,2,2,1,2))	D73	$(C^+)^2$
	((6,6),(4,5,3),(2,2,1,2,2,2,1))	D72	$(C^+)^2$
	((6,6),(4,5,3),(2,2,2,1,1,2,2))	D70	$(C^+)^3$
	((6,6),(4,5,3),(2,2,2,1,2,1,2))	D69	$(C^+)^3$
	((6,6),(4,5,3),(2,2,2,1,2,2,1))	D68	$(C^+)^3$
	((6,6),(4,5,3),(2,2,2,2,1,1,2))	D66	$(C^+)^4$
	((6,6),(4,5,3),(2,2,2,2,1,2,1))	D63	$(C^+)^4$
	((6,6),(4,5,3),(2,2,2,2,2,1,1))	D39	$(C^+)^5$
	((6,6),(5,3,4),(1,1,2,2,2,2,2))	D98	–
	((6,6),(5,3,4),(1,2,1,2,2,2,2))	D97	–
	((6,6),(5,3,4),(1,2,2,1,2,2,2))	D96	–
	((6,6),(5,3,4),(1,2,2,2,1,2,2))	D95	–
	((6,6),(5,3,4),(1,2,2,2,2,1,2))	D94	–
	((6,6),(5,3,4),(1,2,2,2,2,2,1))	D93	–
	((6,6),(5,3,4),(2,1,1,2,2,2,2))	D75	C^+
	((6,6),(5,3,4),(2,1,2,1,2,2,2))	D74	C^+
	((6,6),(5,3,4),(2,1,2,2,1,2,2))	D73	C^+
	((6,6),(5,3,4),(2,1,2,2,2,1,2))	D72	C^+
	((6,6),(5,3,4),(2,1,2,2,2,2,1))	D71	$(C^+)^6$
	((6,6),(5,3,4),(2,2,1,1,2,2,2))	D70	$(C^+)^2$
	((6,6),(5,3,4),(2,2,1,2,1,2,2))	D69	$(C^+)^2$
	((6,6),(5,3,4),(2,2,1,2,2,1,2))	D68	$(C^+)^2$
	((6,6),(5,3,4),(2,2,1,2,2,2,1))	D67	$(C^+)^6$
	((6,6),(5,3,4),(2,2,2,1,1,2,2))	D66	$(C^+)^3$
	((6,6),(5,3,4),(2,2,2,1,2,1,2))	D63	$(C^+)^3$
	((6,6),(5,3,4),(2,2,2,1,2,2,1))	D60	$(C^+)^6$
	((6,6),(5,3,4),(2,2,2,2,1,1,2))	D39	$(C^+)^4$
	((6,6),(5,3,4),(2,2,2,2,1,2,1))	D36	$(C^+)^4$
	((6,6),(5,3,4),(2,2,2,2,2,1,1))	D30	$(C^+)^5$
	((6,6),(5,4,3),(1,1,2,2,2,2,2))	D92	–
	((6,6),(5,4,3),(1,2,1,2,2,2,2))	D91	–
	((6,6),(5,4,3),(1,2,2,1,2,2,2))	D90	–
	((6,6),(5,4,3),(1,2,2,2,1,2,2))	D89	–
	((6,6),(5,4,3),(1,2,2,2,2,1,2))	D88	–
	((6,6),(5,4,3),(1,2,2,2,2,2,1))	D87	$(C^+)^6$
	((6,6),(5,4,3),(2,1,1,2,2,2,2))	D64	C^+
	((6,6),(5,4,3),(2,1,2,1,2,2,2))	D61	C^+
	((6,6),(5,4,3),(2,1,2,2,1,2,2))	D58	C^+
	((6,6),(5,4,3),(2,1,2,2,2,1,2))	D56	$(C^+)^7$

No.	dim. vector	orbit	apply ...to constructed family
	$((6,6),(5,4,3),(2,1,2,2,2,2,1))$	D55	$(C^+)^6$
	$((6,6),(5,4,3),(2,2,1,1,2,2,2))$	D37	$(C^+)^2$
	$((6,6),(5,4,3),(2,2,1,2,1,2,2))$	D34	$(C^+)^7$
	$((6,6),(5,4,3),(2,2,1,2,2,1,2))$	D32	$(C^+)^7$
	$((6,6),(5,4,3),(2,2,1,2,2,2,1))$	D31	$(C^+)^6$
	$((6,6),(5,4,3),(2,2,2,1,1,2,2))$	D26	$(C^+)^3$
	$((6,6),(5,4,3),(2,2,2,1,2,1,2))$	D21	$(C^+)^7$
	$((6,6),(5,4,3),(2,2,2,1,2,2,1))$	D18	$(C^+)^6$
	$((6,6),(5,4,3),(2,2,2,2,1,1,2))$	D12	$(C^+)^4$
	$((6,6),(5,4,3),(2,2,2,2,1,2,1))$	D9	$(C^+)^6$
	$((6,6),(5,4,3),(2,2,2,2,2,1,1))$	D5	$(C^+)^5$
18.	$((6,7),(3,5,5),(1,2,2,2,2,2,2))$	D19	$(C^-)^2$
	$((6,7),(3,5,5),(2,1,2,2,2,2,2))$	D33	$(C^-)^2$
	$((6,7),(3,5,5),(2,2,1,2,2,2,2))$	D45	$(C^-)^2$
	$((6,7),(3,5,5),(2,2,2,1,2,2,2))$	D70	$(C^-)^2$
	$((6,7),(3,5,5),(2,2,2,2,1,2,2))$	D80	$(C^-)^3$
	$((6,7),(3,5,5),(2,2,2,2,2,1,2))$	D92	$(C^-)^3$
	$((6,7),(3,5,5),(2,2,2,2,2,2,1))$	D112	C^-
	$((6,7),(5,3,5),(1,2,2,2,2,2,2))$	D110	C^-
	$((6,7),(5,3,5),(2,1,2,2,2,2,2))$	D109	C^-
	$((6,7),(5,3,5),(2,2,1,2,2,2,2))$	D108	C^-
	$((6,7),(5,3,5),(2,2,2,1,2,2,2))$	D107	C^-
	$((6,7),(5,3,5),(2,2,2,2,1,2,2))$	D106	C^-
	$((6,7),(5,3,5),(2,2,2,2,2,1,2))$	D105	C^-
	$((6,7),(5,3,5),(2,2,2,2,2,2,1))$	D111	C^-
	$((6,7),(5,5,3),(1,2,2,2,2,2,2))$	D114	–
	$((6,7),(5,5,3),(2,1,2,2,2,2,2))$	D112	C^+
	$((6,7),(5,5,3),(2,2,1,2,2,2,2))$	D104	$(C^+)^2$
	$((6,7),(5,5,3),(2,2,2,1,2,2,2))$	D103	$(C^+)^3$
	$((6,7),(5,5,3),(2,2,2,2,1,2,2))$	D102	$(C^+)^4$
	$((6,7),(5,5,3),(2,2,2,2,2,1,2))$	D101	$(C^+)^5$
	$((6,7),(5,5,3),(2,2,2,2,2,2,1))$	D100	$(C^+)^6$
	$((7,6),(3,5,5),(1,2,2,2,2,2,2))$	D109	$(C^-)^2$
	$((7,6),(3,5,5),(2,1,2,2,2,2,2))$	D108	$(C^-)^2$
	$((7,6),(3,5,5),(2,2,1,2,2,2,2))$	D107	$(C^-)^2$
	$((7,6),(3,5,5),(2,2,2,1,2,2,2))$	D106	$(C^-)^2$
	$((7,6),(3,5,5),(2,2,2,2,1,2,2))$	D105	$(C^-)^2$
	$((7,6),(3,5,5),(2,2,2,2,2,1,2))$	D111	$(C^-)^2$
	$((7,6),(3,5,5),(2,2,2,2,2,2,1))$	D113	–
	$((7,6),(5,3,5),(1,2,2,2,2,2,2))$	D112	–
	$((7,6),(5,3,5),(2,1,2,2,2,2,2))$	D104	C^+

No.	dim. vector	orbit	apply ... to constructed family
	((7,6),(5,3,5),(2,2,1,2,2,2,2))	D103	$(C^+)^2$
	((7,6),(5,3,5),(2,2,2,1,2,2,2))	D102	$(C^+)^3$
	((7,6),(5,3,5),(2,2,2,2,1,2,2))	D101	$(C^+)^4$
	((7,6),(5,3,5),(2,2,2,2,2,1,2))	D100	$(C^+)^5$
	((7,6),(5,3,5),(2,2,2,2,2,2,1))	D99	$(C^+)^6$
	((7,6),(5,5,3),(1,2,2,2,2,2,2))	D111	–
	((7,6),(5,5,3),(2,1,2,2,2,2,2))	D94	C^+
	((7,6),(5,5,3),(2,2,1,2,2,2,2))	D71	$(C^+)^7$
	((7,6),(5,5,3),(2,2,2,1,2,2,2))	D67	$(C^+)^7$
	((7,6),(5,5,3),(2,2,2,2,1,2,2))	D60	$(C^+)^7$
	((7,6),(5,5,3),(2,2,2,2,2,1,2))	D36	$(C^+)^5$
	((7,6),(5,5,3),(2,2,2,2,2,2,1))	D30	$(C^+)^6$
19.	((7,7),(3,5,6),(2,2,2,2,2,2,2))	D110	$(C^-)^2$
	((7,7),(3,6,5),(2,2,2,2,2,2,2))	D114	$(C^-)^2$
	((7,7),(5,3,6),(2,2,2,2,2,2,2))	D114	C^-
	((7,7),(5,6,3),(2,2,2,2,2,2,2))	D113	$(C^+)^2$
	((7,7),(6,3,5),(2,2,2,2,2,2,2))	D113	C^+
	((7,7),(6,5,3),(2,2,2,2,2,2,2))	D99	$(C^+)^7$
20.	((6,6),(4,4,4),(1,1,1,2,2,2,3))	D6	$(C^-)^6$
	((6,6),(4,4,4),(1,1,1,2,2,3,2))	D13	$(C^-)^5$
	((6,6),(4,4,4),(1,1,1,2,3,2,2))	D24	$(C^-)^5$
	((6,6),(4,4,4),(1,1,1,3,2,2,2))	D55	–
	((6,6),(4,4,4),(1,1,2,1,2,2,3))	D9	$(C^-)^4$
	((6,6),(4,4,4),(1,1,2,1,2,3,2))	D26	$(C^-)^4$
	((6,6),(4,4,4),(1,1,2,1,3,2,2))	D57	–
	((6,6),(4,4,4),(1,1,2,2,1,2,3))	D17	–
	((6,6),(4,4,4),(1,1,2,2,1,3,2))	D60	–
	((6,6),(4,4,4),(1,1,2,2,2,1,3))	D41	–
	((6,6),(4,4,4),(1,1,2,2,2,3,1))	D154	–
	((6,6),(4,4,4),(1,1,2,2,3,1,2))	D130	–
	((6,6),(4,4,4),(1,1,2,2,3,2,1))	D129	–
	((6,6),(4,4,4),(1,1,2,3,1,2,2))	D123	–
	((6,6),(4,4,4),(1,1,2,3,2,1,2))	D122	–
	((6,6),(4,4,4),(1,1,2,3,2,2,1))	D139	–
	((6,6),(4,4,4),(1,1,3,1,2,2,2))	D118	–
	((6,6),(4,4,4),(1,1,3,2,1,2,2))	D117	–
	((6,6),(4,4,4),(1,1,3,2,2,1,2))	D135	–
	((6,6),(4,4,4),(1,1,3,2,2,2,1))	D146	–
	((6,6),(4,4,4),(1,2,1,1,2,2,3))	D14	$(C^-)^4$
	((6,6),(4,4,4),(1,2,1,1,2,3,2))	D25	$(C^-)^4$
	((6,6),(4,4,4),(1,2,1,1,3,2,2))	D56	$(C^+)^2$

No.	dim. vector	orbit	apply ...to constructed family
	$((6,6),(4,4,4),(1,2,1,2,1,2,3))$	D28	$(C^-)^3$
	$((6,6),(4,4,4),(1,2,1,2,1,3,2))$	D59	$(C^-)^3$
	$((6,6),(4,4,4),(1,2,1,2,2,1,3))$	D63	$(C^-)^2$
	$((6,6),(4,4,4),(1,2,1,2,2,3,1))$	D153	–
	$((6,6),(4,4,4),(1,2,1,2,3,1,2))$	D128	–
	$((6,6),(4,4,4),(1,2,1,2,3,2,1))$	D127	–
	$((6,6),(4,4,4),(1,2,1,3,1,2,2))$	D121	–
	$((6,6),(4,4,4),(1,2,1,3,2,1,2))$	D120	–
	$((6,6),(4,4,4),(1,2,1,3,2,2,1))$	D138	–
	$((6,6),(4,4,4),(1,2,2,1,1,2,3))$	D27	$(C^-)^3$
	$((6,6),(4,4,4),(1,2,2,1,1,3,2))$	D58	$(C^-)^3$
	$((6,6),(4,4,4),(1,2,2,1,2,1,3))$	D62	$(C^-)^2$
	$((6,6),(4,4,4),(1,2,2,1,2,3,1))$	D152	–
	$((6,6),(4,4,4),(1,2,2,1,3,1,2))$	D126	–
	$((6,6),(4,4,4),(1,2,2,1,3,2,1))$	D125	–
	$((6,6),(4,4,4),(1,2,2,2,1,1,3))$	D61	$(C^-)^2$
	$((6,6),(4,4,4),(1,2,2,2,1,3,1))$	D151	–
	$((6,6),(4,4,4),(1,2,2,2,3,1,1))$	D124	–
	$((6,6),(4,4,4),(1,2,2,3,1,1,2))$	D119	–
	$((6,6),(4,4,4),(1,2,2,3,1,2,1))$	D137	–
	$((6,6),(4,4,4),(1,2,2,3,2,1,1))$	D136	–
	$((6,6),(4,4,4),(1,2,3,1,1,2,2))$	D116	–
	$((6,6),(4,4,4),(1,2,3,1,2,1,2))$	D134	–
	$((6,6),(4,4,4),(1,2,3,1,2,2,1))$	D145	–
	$((6,6),(4,4,4),(1,2,3,2,1,1,2))$	D133	–
	$((6,6),(4,4,4),(1,2,3,2,1,2,1))$	D144	–
	$((6,6),(4,4,4),(1,2,3,2,2,1,1))$	D143	–
	$((6,6),(4,4,4),(1,3,1,1,2,2,2))$	D115	–
	$((6,6),(4,4,4),(1,3,1,2,1,2,2))$	D132	–
	$((6,6),(4,4,4),(1,3,1,2,2,1,2))$	D142	–
	$((6,6),(4,4,4),(1,3,1,2,2,2,1))$	D150	–
	$((6,6),(4,4,4),(1,3,2,1,1,2,2))$	D131	–
	$((6,6),(4,4,4),(1,3,2,1,2,1,2))$	D141	–
	$((6,6),(4,4,4),(1,3,2,1,2,2,1))$	D149	–
	$((6,6),(4,4,4),(1,3,2,2,1,1,2))$	D140	–
	$((6,6),(4,4,4),(1,3,2,2,1,2,1))$	D148	–
	$((6,6),(4,4,4),(1,3,2,2,2,1,1))$	D147	–
	$((6,6),(4,4,4),(2,1,1,1,2,2,3))$	D13	$(C^-)^4$
	$((6,6),(4,4,4),(2,1,1,1,2,3,2))$	D24	$(C^-)^4$
	$((6,6),(4,4,4),(2,1,1,1,3,2,2))$	D55	C^+
	$((6,6),(4,4,4),(2,1,1,2,1,2,3))$	D26	$(C^-)^3$

No.	dim. vector	orbit	apply ... to constructed family
	$((6,6),(4,4,4),(2,1,1,2,1,3,2))$	D57	C^+
	$((6,6),(4,4,4),(2,1,1,2,2,1,3))$	D60	C^+
	$((6,6),(4,4,4),(2,1,1,2,2,3,1))$	D130	C^+
	$((6,6),(4,4,4),(2,1,1,2,3,1,2))$	D123	C^+
	$((6,6),(4,4,4),(2,1,1,2,3,2,1))$	D122	C^+
	$((6,6),(4,4,4),(2,1,1,3,1,2,2))$	D118	C^+
	$((6,6),(4,4,4),(2,1,1,3,2,1,2))$	D117	C^+
	$((6,6),(4,4,4),(2,1,1,3,2,2,1))$	D135	C^+
	$((6,6),(4,4,4),(2,1,2,1,1,2,3))$	D25	$(C^-)^3$
	$((6,6),(4,4,4),(2,1,2,1,1,3,2))$	D56	$(C^+)^3$
	$((6,6),(4,4,4),(2,1,2,1,2,1,3))$	D59	$(C^-)^2$
	$((6,6),(4,4,4),(2,1,2,1,2,3,1))$	D128	C^+
	$((6,6),(4,4,4),(2,1,2,1,3,1,2))$	D121	C^+
	$((6,6),(4,4,4),(2,1,2,1,3,2,1))$	D120	C^+
	$((6,6),(4,4,4),(2,1,2,2,1,1,3))$	D58	$(C^-)^2$
	$((6,6),(4,4,4),(2,1,2,2,1,3,1))$	D126	C^+
	$((6,6),(4,4,4),(2,1,2,2,3,1,1))$	D119	C^+
	$((6,6),(4,4,4),(2,1,2,3,1,1,2))$	D116	C^+
	$((6,6),(4,4,4),(2,1,2,3,1,2,1))$	D134	C^+
	$((6,6),(4,4,4),(2,1,2,3,2,1,1))$	D133	C^+
	$((6,6),(4,4,4),(2,1,3,1,1,2,2))$	D115	C^+
	$((6,6),(4,4,4),(2,1,3,1,2,1,2))$	D132	C^+
	$((6,6),(4,4,4),(2,1,3,1,2,2,1))$	D142	C^+
	$((6,6),(4,4,4),(2,1,3,2,1,1,2))$	D131	C^+
	$((6,6),(4,4,4),(2,1,3,2,1,2,1))$	D141	C^+
	$((6,6),(4,4,4),(2,1,3,2,2,1,1))$	D140	C^+
	$((6,6),(4,4,4),(2,2,1,1,1,2,3))$	D24	$(C^-)^3$
	$((6,6),(4,4,4),(2,2,1,1,1,3,2))$	D55	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,1,2,1,3))$	D57	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,1,2,3,1))$	D123	C^+
	$((6,6),(4,4,4),(2,2,1,1,3,1,2))$	D118	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,1,3,2,1))$	D117	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,2,1,1,3))$	D56	$(C^+)^4$
	$((6,6),(4,4,4),(2,2,1,2,1,3,1))$	D121	C^+
	$((6,6),(4,4,4),(2,2,1,2,3,1,1))$	D116	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,3,1,1,2))$	D115	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,3,1,2,1))$	D132	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,1,3,2,1,1))$	D131	$(C^+)^2$
	$((6,6),(4,4,4),(2,2,2,1,1,1,3))$	D55	$(C^+)^3$
	$((6,6),(4,4,4),(2,2,2,1,1,3,1))$	D118	$(C^+)^3$
	$((6,6),(4,4,4),(2,2,2,1,3,1,1))$	D115	$(C^+)^2$

No.	dim. vector	orbit	apply ...to constructed family
	$((6,6),(4,4,4),(2,2,2,3,1,1,1))$	D54	$(C^+)^5$
	$((6,6),(4,4,4),(2,2,3,1,1,1,2))$	D54	$(C^+)^4$
	$((6,6),(4,4,4),(2,2,3,1,1,2,1))$	D53	$(C^+)^3$
	$((6,6),(4,4,4),(2,2,3,1,2,1,1))$	D49	$(C^+)^5$
	$((6,6),(4,4,4),(2,2,3,2,1,1,1))$	D29	$(C^+)^4$
	$((6,6),(4,4,4),(2,3,1,1,1,2,2))$	D54	$(C^+)^3$
	$((6,6),(4,4,4),(2,3,1,1,2,1,2))$	D53	$(C^+)^2$
	$((6,6),(4,4,4),(2,3,1,1,2,2,1))$	D52	$(C^+)^2$
	$((6,6),(4,4,4),(2,3,1,2,1,1,2))$	D49	$(C^+)^4$
	$((6,6),(4,4,4),(2,3,1,2,1,2,1))$	D48	$(C^+)^2$
	$((6,6),(4,4,4),(2,3,1,2,2,1,1))$	D45	$(C^+)^5$
	$((6,6),(4,4,4),(2,3,2,1,1,1,2))$	D29	$(C^+)^3$
	$((6,6),(4,4,4),(2,3,2,1,1,2,1))$	D27	$(C^+)^3$
	$((6,6),(4,4,4),(2,3,2,1,2,1,1))$	D22	$(C^+)^5$
	$((6,6),(4,4,4),(2,3,2,2,1,1,1))$	D13	$(C^+)^4$
	$((6,6),(4,4,4),(3,1,1,1,2,2,2))$	D54	$(C^+)^2$
	$((6,6),(4,4,4),(3,1,1,2,1,2,2))$	D53	C^+
	$((6,6),(4,4,4),(3,1,1,2,2,1,2))$	D52	C^+
	$((6,6),(4,4,4),(3,1,1,2,2,2,1))$	D51	C^+
	$((6,6),(4,4,4),(3,1,2,1,1,2,2))$	D49	$(C^+)^3$
	$((6,6),(4,4,4),(3,1,2,1,2,1,2))$	D48	C^+
	$((6,6),(4,4,4),(3,1,2,1,2,2,1))$	D47	C^+
	$((6,6),(4,4,4),(3,1,2,2,1,1,2))$	D45	$(C^+)^4$
	$((6,6),(4,4,4),(3,1,2,2,1,2,1))$	D44	C^+
	$((6,6),(4,4,4),(3,1,2,2,2,1,1))$	D42	$(C^+)^4$
	$((6,6),(4,4,4),(3,2,1,1,1,2,2))$	D29	$(C^+)^2$
	$((6,6),(4,4,4),(3,2,1,1,2,1,2))$	D27	$(C^+)^2$
	$((6,6),(4,4,4),(3,2,1,1,2,2,1))$	D25	$(C^+)^2$
	$((6,6),(4,4,4),(3,2,1,2,1,1,2))$	D22	$(C^+)^4$
	$((6,6),(4,4,4),(3,2,1,2,1,2,1))$	D20	$(C^+)^8$
	$((6,6),(4,4,4),(3,2,1,2,2,1,1))$	D17	$(C^+)^8$
	$((6,6),(4,4,4),(3,2,2,1,1,1,2))$	D13	$(C^+)^3$
	$((6,6),(4,4,4),(3,2,2,1,1,2,1))$	D11	$(C^+)^3$
	$((6,6),(4,4,4),(3,2,2,1,2,1,1))$	D8	$(C^+)^5$
	$((6,6),(4,4,4),(3,2,2,2,1,1,1))$	D4	$(C^+)^4$
21.	$((3,5),(2,3,3),(1,1,1,1,1,1,1,1))$	D159	C^-
	$((3,5),(3,2,3),(1,1,1,1,1,1,1,1))$	D160	C^-
	$((3,5),(3,3,2),(1,1,1,1,1,1,1,1))$	D156	–
	$((5,3),(2,3,3),(1,1,1,1,1,1,1,1))$	D156	C^+
	$((5,3),(3,2,3),(1,1,1,1,1,1,1,1))$	D161	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((5,3),(3,3,2),(1,1,1,1,1,1,1,1))$	D162	C^+
22.	$((3,6),(3,3,3),(1,1,1,1,1,1,1,1,1))$	D163	C^-
	$((6,3),(3,3,3),(1,1,1,1,1,1,1,1,1))$	D164	C^+
23.	$((4,4),(2,2,4),(1,1,1,1,1,1,1,1))$	D157	C^-
	$((4,4),(2,4,2),(1,1,1,1,1,1,1,1))$	D155	–
	$((4,4),(4,2,2),(1,1,1,1,1,1,1,1))$	D158	C^+
24.	$((3,5),(2,2,2,2),(1,1,2,2,2))$	C8	$(C^-)^4$
	$((3,5),(2,2,2,2),(1,2,1,2,2))$	C18	C^-
	$((3,5),(2,2,2,2),(1,2,2,1,2))$	C42	C^-
	$((3,5),(2,2,2,2),(1,2,2,2,1))$	C132	C^-
	$((3,5),(2,2,2,2),(2,1,1,2,2))$	C29	C^+
	$((3,5),(2,2,2,2),(2,1,2,1,2))$	C28	C^-
	$((3,5),(2,2,2,2),(2,1,2,2,1))$	C131	C^-
	$((3,5),(2,2,2,2),(2,2,1,1,2))$	C27	$(C^+)^2$
	$((3,5),(2,2,2,2),(2,2,1,2,1))$	C130	C^-
	$((3,5),(2,2,2,2),(2,2,2,1,1))$	C129	C^-
	$((5,3),(2,2,2,2),(1,1,2,2,2))$	C132	–
	$((5,3),(2,2,2,2),(1,2,1,2,2))$	C131	–
	$((5,3),(2,2,2,2),(1,2,2,1,2))$	C130	–
	$((5,3),(2,2,2,2),(1,2,2,2,1))$	C129	–
	$((5,3),(2,2,2,2),(2,1,1,2,2))$	C24	C^+
	$((5,3),(2,2,2,2),(2,1,2,1,2))$	C22	$(C^+)^4$
	$((5,3),(2,2,2,2),(2,1,2,2,1))$	C21	$(C^+)^4$
	$((5,3),(2,2,2,2),(2,2,1,1,2))$	C19	$(C^+)^2$
	$((5,3),(2,2,2,2),(2,2,1,2,1))$	C18	$(C^+)^4$
	$((5,3),(2,2,2,2),(2,2,2,1,1))$	C9	$(C^+)^4$
25.	$((4,4),(2,2,2,2),(1,1,1,2,3))$	C5	$(C^-)^4$
	$((4,4),(2,2,2,2),(1,1,1,3,2))$	C30	–
	$((4,4),(2,2,2,2),(1,1,2,1,3))$	C13	–
	$((4,4),(2,2,2,2),(1,1,2,3,1))$	C140	–
	$((4,4),(2,2,2,2),(1,1,3,1,2))$	C134	–
	$((4,4),(2,2,2,2),(1,1,3,2,1))$	C139	–
	$((4,4),(2,2,2,2),(1,2,1,1,3))$	C31	$(C^+)^2$
	$((4,4),(2,2,2,2),(1,2,1,3,1))$	C138	–
	$((4,4),(2,2,2,2),(1,2,3,1,1))$	C137	–
	$((4,4),(2,2,2,2),(1,3,1,1,2))$	C133	–
	$((4,4),(2,2,2,2),(1,3,1,2,1))$	C136	–
	$((4,4),(2,2,2,2),(1,3,2,1,1))$	C135	–
	$((4,4),(2,2,2,2),(2,1,1,1,3))$	C30	C^+
	$((4,4),(2,2,2,2),(2,1,1,3,1))$	C134	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((4,4),(2,2,2,2),(2,1,3,1,1))$	C133	C^+
	$((4,4),(2,2,2,2),(2,3,1,1,1))$	C17	$(C^+)^2$
	$((4,4),(2,2,2,2),(3,1,1,1,2))$	C17	C^+
	$((4,4),(2,2,2,2),(3,1,1,2,1))$	C16	C^+
	$((4,4),(2,2,2,2),(3,1,2,1,1))$	C14	$(C^+)^3$
	$((4,4),(2,2,2,2),(3,2,1,1,1))$	C5	$(C^+)^2$
26.	$((5,5),(1,2,3,4),(2,2,2,2,2))$	C49	$(C^-)^2$
	$((5,5),(1,2,4,3),(2,2,2,2,2))$	C91	–
	$((5,5),(1,3,2,4),(2,2,2,2,2))$	C93	–
	$((5,5),(1,3,4,2),(2,2,2,2,2))$	C106	–
	$((5,5),(1,4,2,3),(2,2,2,2,2))$	C102	–
	$((5,5),(1,4,3,2),(2,2,2,2,2))$	C105	–
	$((5,5),(2,1,3,4),(2,2,2,2,2))$	C65	$(C^-)^2$
	$((5,5),(2,1,4,3),(2,2,2,2,2))$	C90	$(C^-)^2$
	$((5,5),(2,3,1,4),(2,2,2,2,2))$	C92	$(C^-)^2$
	$((5,5),(2,3,4,1),(2,2,2,2,2))$	C104	–
	$((5,5),(2,4,1,3),(2,2,2,2,2))$	C101	–
	$((5,5),(2,4,3,1),(2,2,2,2,2))$	C103	–
	$((5,5),(3,1,2,4),(2,2,2,2,2))$	C91	C^+
	$((5,5),(3,1,4,2),(2,2,2,2,2))$	C102	C^+
	$((5,5),(3,2,1,4),(2,2,2,2,2))$	C90	C^-
	$((5,5),(3,2,4,1),(2,2,2,2,2))$	C101	C^+
	$((5,5),(3,4,1,2),(2,2,2,2,2))$	C75	$(C^+)^2$
	$((5,5),(3,4,2,1),(2,2,2,2,2))$	C72	$(C^+)^3$
	$((5,5),(4,1,2,3),(2,2,2,2,2))$	C75	C^+
	$((5,5),(4,1,3,2),(2,2,2,2,2))$	C74	C^+
	$((5,5),(4,2,1,3),(2,2,2,2,2))$	C72	$(C^+)^2$
	$((5,5),(4,2,3,1),(2,2,2,2,2))$	C71	$(C^+)^3$
	$((5,5),(4,3,1,2),(2,2,2,2,2))$	C58	$(C^+)^5$
	$((5,5),(4,3,2,1),(2,2,2,2,2))$	C54	$(C^+)^5$
27.	$((2,4),(1,1,2,2),(1,1,1,1,1,1))$	C156	C^-
	$((2,4),(1,2,1,2),(1,1,1,1,1,1))$	C155	C^-
	$((2,4),(1,2,2,1),(1,1,1,1,1,1))$	C145	–
	$((2,4),(2,1,1,2),(1,1,1,1,1,1))$	C154	C^-
	$((2,4),(2,1,2,1),(1,1,1,1,1,1))$	C144	C^-
	$((2,4),(2,2,1,1),(1,1,1,1,1,1))$	C143	C^-
	$((4,2),(1,1,2,2),(1,1,1,1,1,1))$	C145	–
	$((4,2),(1,2,1,2),(1,1,1,1,1,1))$	C144	–
	$((4,2),(1,2,2,1),(1,1,1,1,1,1))$	C143	–
	$((4,2),(2,1,1,2),(1,1,1,1,1,1))$	C153	C^+
	$((4,2),(2,1,2,1),(1,1,1,1,1,1))$	C152	C^+

No.	dim. vector	orbit	apply ...to constructed family
	$((4,2),(2,2,1,1),(1,1,1,1,1,1))$	C151	C^+
28.	$((2,5),(1,2,2,2),(1,1,1,1,1,1,1))$	C162	C^-
	$((2,5),(2,1,2,2),(1,1,1,1,1,1,1))$	C161	C^-
	$((2,5),(2,2,1,2),(1,1,1,1,1,1,1))$	C160	C^-
	$((2,5),(2,2,2,1),(1,1,1,1,1,1,1))$	C146	C^-
	$((5,2),(1,2,2,2),(1,1,1,1,1,1,1))$	C146	–
	$((5,2),(2,1,2,2),(1,1,1,1,1,1,1))$	C159	C^+
	$((5,2),(2,2,1,2),(1,1,1,1,1,1,1))$	C158	C^+
	$((5,2),(2,2,2,1),(1,1,1,1,1,1,1))$	C157	C^+
29.	$((2,6),(2,2,2,2),(1,1,1,1,1,1,1,1))$	C147	–
	$((6,2),(2,2,2,2),(1,1,1,1,1,1,1,1))$	C148	–
30.	$((3,3),(1,1,1,3),(1,1,1,1,1,1))$	C150	C^-
	$((3,3),(1,1,3,1),(1,1,1,1,1,1))$	C142	–
	$((3,3),(1,3,1,1),(1,1,1,1,1,1))$	C141	–
	$((3,3),(3,1,1,1),(1,1,1,1,1,1))$	C149	C^+
31.	$((1,2,3),(2,2,2),(1,1,2,2))$	B3	$(C^-)^4$
	$((1,2,3),(2,2,2),(1,2,1,2))$	B9	$(C^-)^2$
	$((1,2,3),(2,2,2),(1,2,2,1))$	B22	C^-
	$((1,2,3),(2,2,2),(2,1,1,2))$	B8	$(C^-)^2$
	$((1,2,3),(2,2,2),(2,1,2,1))$	B21	C^-
	$((1,2,3),(2,2,2),(2,2,1,1))$	B20	C^-
	$((1,3,2),(2,2,2),(1,1,2,2))$	B11	–
	$((1,3,2),(2,2,2),(1,2,1,2))$	B18	$(C^-)^2$
	$((1,3,2),(2,2,2),(1,2,2,1))$	B25	–
	$((1,3,2),(2,2,2),(2,1,1,2))$	B17	C^+
	$((1,3,2),(2,2,2),(2,1,2,1))$	B24	–
	$((1,3,2),(2,2,2),(2,2,1,1))$	B23	$(C^+)^2$
	$((2,1,3),(2,2,2),(1,1,2,2))$	B13	–
	$((2,1,3),(2,2,2),(1,2,1,2))$	B12	C^-
	$((2,1,3),(2,2,2),(1,2,2,1))$	B19	C^-
	$((2,1,3),(2,2,2),(2,1,1,2))$	B11	C^+
	$((2,1,3),(2,2,2),(2,1,2,1))$	B18	C^-
	$((2,1,3),(2,2,2),(2,2,1,1))$	B17	$(C^+)^2$
	$((2,3,1),(2,2,2),(1,1,2,2))$	B28	–
	$((2,3,1),(2,2,2),(1,2,1,2))$	B27	$(C^+)^2$
	$((2,3,1),(2,2,2),(1,2,2,1))$	B26	$(C^+)^2$
	$((2,3,1),(2,2,2),(2,1,1,2))$	B22	C^+
	$((2,3,1),(2,2,2),(2,1,2,1))$	B21	C^+
	$((2,3,1),(2,2,2),(2,2,1,1))$	B15	$(C^+)^2$
	$((3,1,2),(2,2,2),(1,1,2,2))$	B22	–

No.	dim. vector	orbit	apply ...to constructed family
	$((3,1,2),(2,2,2),(1,2,1,2))$	B21	–
	$((3,1,2),(2,2,2),(1,2,2,1))$	B20	–
	$((3,1,2),(2,2,2),(2,1,1,2))$	B15	C^+
	$((3,1,2),(2,2,2),(2,1,2,1))$	B14	$(C^+)^3$
	$((3,1,2),(2,2,2),(2,2,1,1))$	B9	$(C^+)^2$
	$((3,2,1),(2,2,2),(1,1,2,2))$	B19	–
	$((3,2,1),(2,2,2),(1,2,1,2))$	B18	–
	$((3,2,1),(2,2,2),(1,2,2,1))$	B17	$(C^+)^3$
	$((3,2,1),(2,2,2),(2,1,1,2))$	B16	C^+
	$((3,2,1),(2,2,2),(2,1,2,1))$	B13	$(C^+)^4$
	$((3,2,1),(2,2,2),(2,2,1,1))$	B2	$(C^+)^2$
32.	see 31. (with arms 1 and 2 exchanged)		
33.	$((1,3,3),(2,2,3),(1,2,2,2))$	B13	C^-
	$((1,3,3),(2,2,3),(2,1,2,2))$	B12	$(C^-)^2$
	$((1,3,3),(2,2,3),(2,2,1,2))$	B19	$(C^-)^2$
	$((1,3,3),(2,2,3),(2,2,2,1))$	B29	–
	$((1,3,3),(2,3,2),(1,2,2,2))$	B27	–
	$((1,3,3),(2,3,2),(2,1,2,2))$	B26	–
	$((1,3,3),(2,3,2),(2,2,1,2))$	B38	$(C^-)^2$
	$((1,3,3),(2,3,2),(2,2,2,1))$	B37	–
	$((1,3,3),(3,2,2),(1,2,2,2))$	B32	–
	$((1,3,3),(3,2,2),(2,1,2,2))$	B31	–
	$((1,3,3),(3,2,2),(2,2,1,2))$	B36	–
	$((1,3,3),(3,2,2),(2,2,2,1))$	B35	–
	$((3,1,3),(2,2,3),(1,2,2,2))$	B28	C^-
	$((3,1,3),(2,2,3),(2,1,2,2))$	B27	C^+
	$((3,1,3),(2,2,3),(2,2,1,2))$	B26	C^+
	$((3,1,3),(2,2,3),(2,2,2,1))$	B38	C^-
	$((3,1,3),(2,3,2),(1,2,2,2))$	B33	C^+
	$((3,1,3),(2,3,2),(2,1,2,2))$	B32	C^+
	$((3,1,3),(2,3,2),(2,2,1,2))$	B31	C^+
	$((3,1,3),(2,3,2),(2,2,2,1))$	B36	C^+
	$((3,1,3),(3,2,2),(1,2,2,2))$	B29	C^+
	$((3,1,3),(3,2,2),(2,1,2,2))$	B25	C^+
	$((3,1,3),(3,2,2),(2,2,1,2))$	B24	C^+
	$((3,1,3),(3,2,2),(2,2,2,1))$	B23	$(C^+)^3$
	$((3,3,1),(2,2,3),(1,2,2,2))$	B34	$(C^+)^2$
	$((3,3,1),(2,2,3),(2,1,2,2))$	B33	$(C^+)^2$
	$((3,3,1),(2,2,3),(2,2,1,2))$	B32	$(C^+)^2$
	$((3,3,1),(2,2,3),(2,2,2,1))$	B31	$(C^+)^2$
	$((3,3,1),(2,3,2),(1,2,2,2))$	B30	$(C^+)^2$

No.	dim. vector	orbit	apply ... to constructed family
	$((3,3,1),(2,3,2),(2,1,2,2))$	B29	$(C^+)^2$
	$((3,3,1),(2,3,2),(2,2,1,2))$	B25	$(C^+)^2$
	$((3,3,1),(2,3,2),(2,2,2,1))$	B24	$(C^+)^2$
	$((3,3,1),(3,2,2),(1,2,2,2))$	B38	–
	$((3,3,1),(3,2,2),(2,1,2,2))$	B20	C^+
	$((3,3,1),(3,2,2),(2,2,1,2))$	B14	$(C^+)^4$
	$((3,3,1),(3,2,2),(2,2,2,1))$	B9	$(C^+)^3$
34.	see 33. (with arms 1 and 2 exchanged)		
35.	$((1,3,4),(2,3,3),(2,2,2,2))$	B28	$(C^-)^2$
	$((1,3,4),(3,2,3),(2,2,2,2))$	B33	–
	$((1,3,4),(3,3,2),(2,2,2,2))$	B40	–
	$((1,4,3),(2,3,3),(2,2,2,2))$	B30	–
	$((1,4,3),(3,2,3),(2,2,2,2))$	B39	–
	$((1,4,3),(3,3,2),(2,2,2,2))$	B41	–
	$((3,1,4),(2,3,3),(2,2,2,2))$	B34	C^+
	$((3,1,4),(3,2,3),(2,2,2,2))$	B30	C^+
	$((3,1,4),(3,3,2),(2,2,2,2))$	B39	C^+
	$((3,4,1),(2,3,3),(2,2,2,2))$	B42	$(C^+)^2$
	$((3,4,1),(3,2,3),(2,2,2,2))$	B40	$(C^+)^2$
	$((3,4,1),(3,3,2),(2,2,2,2))$	B37	$(C^+)^2$
	$((4,1,3),(2,3,3),(2,2,2,2))$	B40	C^+
	$((4,1,3),(3,2,3),(2,2,2,2))$	B37	C^+
	$((4,1,3),(3,3,2),(2,2,2,2))$	B35	C^+
	$((4,3,1),(2,3,3),(2,2,2,2))$	B39	$(C^+)^2$
	$((4,3,1),(3,2,3),(2,2,2,2))$	B36	$(C^+)^2$
	$((4,3,1),(3,3,2),(2,2,2,2))$	B23	$(C^+)^4$
36.	see 35. (with arms 1 and 2 exchanged)		
37.	$((1,4,4),(3,3,3),(2,2,2,3))$	B34	–
	$((1,4,4),(3,3,3),(2,2,3,2))$	B42	–
	$((1,4,4),(3,3,3),(2,3,2,2))$	B44	–
	$((1,4,4),(3,3,3),(3,2,2,2))$	B43	–
	$((4,1,4),(3,3,3),(2,2,2,3))$	B42	C^+
	$((4,1,4),(3,3,3),(2,2,3,2))$	B44	C^+
	$((4,1,4),(3,3,3),(2,3,2,2))$	B43	C^+
	$((4,1,4),(3,3,3),(3,2,2,2))$	B41	C^+
	$((4,4,1),(3,3,3),(2,2,2,3))$	B44	$(C^+)^2$
	$((4,4,1),(3,3,3),(2,2,3,2))$	B43	$(C^+)^2$
	$((4,4,1),(3,3,3),(2,3,2,2))$	B41	$(C^+)^2$
	$((4,4,1),(3,3,3),(3,2,2,2))$	B35	C^+
38.	see 37. (with arms 1 and 2 exchanged)		

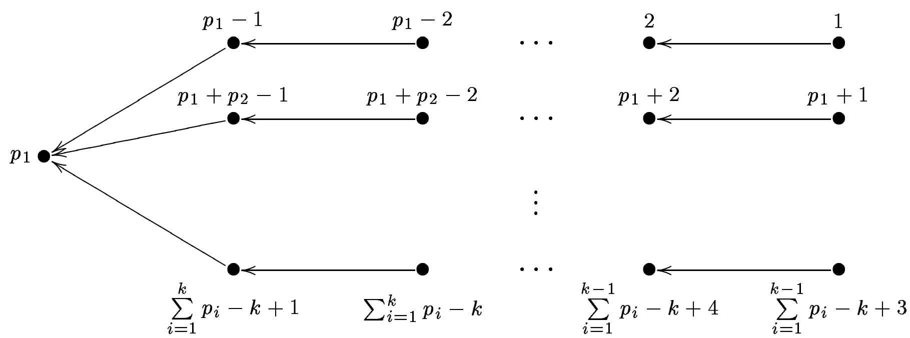
No.	dim. vector	orbit	apply ...to constructed family
39.	((2,2,2),(2,2,2),(1,1,1,3))	B5	$(C^-)^3$
	((2,2,2),(2,2,2),(1,1,3,1))	B45	–
	((2,2,2),(2,2,2),(1,3,1,1))	B46	–
	((2,2,2),(2,2,2),(3,1,1,1))	B6	$(C^+)^2$
40.	((1,1,3),(1,2,2),(1,1,1,1,1))	B52	C^-
	((1,1,3),(2,1,2),(1,1,1,1,1))	B53	C^-
	((1,1,3),(2,2,1),(1,1,1,1,1))	B47	–
	((1,3,1),(1,2,2),(1,1,1,1,1))	B47	–
	((1,3,1),(2,1,2),(1,1,1,1,1))	B49	–
	((1,3,1),(2,2,1),(1,1,1,1,1))	B50	–
	((3,1,1),(1,2,2),(1,1,1,1,1))	B47	C^+
	((3,1,1),(2,1,2),(1,1,1,1,1))	B54	C^+
	((3,1,1),(2,2,1),(1,1,1,1,1))	B55	C^+
41.	see 40. (with arms 1 and 2 exchanged)		
42.	((1,1,4),(2,2,2),(1,1,1,1,1,1))	B56	C^-
	((1,4,1),(2,2,2),(1,1,1,1,1,1))	B51	–
	((4,1,1),(2,2,2),(1,1,1,1,1,1))	B57	C^+
43.	see 42. (with arms 1 and 2 exchanged)		
44.	((4,6),(1,3,3,3),(2,2,2,2,2))	C64	–
	((4,6),(3,1,3,3),(2,2,2,2,2))	C87	C^-
	((4,6),(3,3,1,3),(2,2,2,2,2))	C86	C^-
	((4,6),(3,3,3,1),(2,2,2,2,2))	C108	–
	((6,4),(1,3,3,3),(2,2,2,2,2))	C107	–
	((6,4),(3,1,3,3),(2,2,2,2,2))	C83	C^+
	((6,4),(3,3,1,3),(2,2,2,2,2))	C78	$(C^+)^2$
	((6,4),(3,3,3,1),(2,2,2,2,2))	C59	$(C^+)^3$
45.	((5,5),(1,3,3,3),(1,2,2,2,3))	C25	$(C^-)^3$
	((5,5),(1,3,3,3),(1,2,2,3,2))	C51	–
	((5,5),(1,3,3,3),(1,2,3,2,2))	C96	–
	((5,5),(1,3,3,3),(1,3,2,2,2))	C110	–
	((5,5),(1,3,3,3),(2,1,2,2,3))	C24	$(C^-)^3$
	((5,5),(1,3,3,3),(2,1,2,3,2))	C50	–
	((5,5),(1,3,3,3),(2,1,3,2,2))	C80	–
	((5,5),(1,3,3,3),(2,2,1,2,3))	C37	$(C^-)^3$
	((5,5),(1,3,3,3),(2,2,1,3,2))	C81	–
	((5,5),(1,3,3,3),(2,2,2,1,3))	C82	–
	((5,5),(1,3,3,3),(2,2,2,3,1))	C128	–
	((5,5),(1,3,3,3),(2,2,3,1,2))	C127	–
	((5,5),(1,3,3,3),(2,2,3,2,1))	C126	–
	((5,5),(1,3,3,3),(2,3,1,2,2))	C125	–

No.	dim. vector	orbit	apply ...to constructed family
	$((5,5),(1,3,3,3),(2,3,2,1,2))$	C124	–
	$((5,5),(1,3,3,3),(2,3,2,2,1))$	C123	–
	$((5,5),(1,3,3,3),(3,1,2,2,2))$	C109	–
	$((5,5),(1,3,3,3),(3,2,1,2,2))$	C122	–
	$((5,5),(1,3,3,3),(3,2,2,1,2))$	C121	–
	$((5,5),(1,3,3,3),(3,2,2,2,1))$	C120	–
	$((5,5),(3,1,3,3),(1,2,2,2,3))$	C52	$(C^-)^2$
	$((5,5),(3,1,3,3),(1,2,2,3,2))$	C100	–
	$((5,5),(3,1,3,3),(1,2,3,2,2))$	C111	–
	$((5,5),(3,1,3,3),(1,3,2,2,2))$	C112	–
	$((5,5),(3,1,3,3),(2,1,2,2,3))$	C51	$(C^-)^2$
	$((5,5),(3,1,3,3),(2,1,2,3,2))$	C96	C^+
	$((5,5),(3,1,3,3),(2,1,3,2,2))$	C110	C^+
	$((5,5),(3,1,3,3),(2,2,1,2,3))$	C50	C^+
	$((5,5),(3,1,3,3),(2,2,1,3,2))$	C80	C^+
	$((5,5),(3,1,3,3),(2,2,2,1,3))$	C81	C^+
	$((5,5),(3,1,3,3),(2,2,2,3,1))$	C127	C^+
	$((5,5),(3,1,3,3),(2,2,3,1,2))$	C125	C^+
	$((5,5),(3,1,3,3),(2,2,3,2,1))$	C124	C^+
	$((5,5),(3,1,3,3),(2,3,1,2,2))$	C109	C^+
	$((5,5),(3,1,3,3),(2,3,2,1,2))$	C122	C^+
	$((5,5),(3,1,3,3),(2,3,2,2,1))$	C121	C^+
	$((5,5),(3,1,3,3),(3,1,2,2,2))$	C88	C^+
	$((5,5),(3,1,3,3),(3,2,1,2,2))$	C97	C^+
	$((5,5),(3,1,3,3),(3,2,2,1,2))$	C99	C^+
	$((5,5),(3,1,3,3),(3,2,2,2,1))$	C95	C^+
	$((5,5),(3,3,1,3),(1,2,2,2,3))$	C98	–
	$((5,5),(3,3,1,3),(1,2,2,3,2))$	C115	–
	$((5,5),(3,3,1,3),(1,2,3,2,2))$	C114	–
	$((5,5),(3,3,1,3),(1,3,2,2,2))$	C113	–
	$((5,5),(3,3,1,3),(2,1,2,2,3))$	C100	C^+
	$((5,5),(3,3,1,3),(2,1,2,3,2))$	C111	C^+
	$((5,5),(3,3,1,3),(2,1,3,2,2))$	C112	C^+
	$((5,5),(3,3,1,3),(2,2,1,2,3))$	C96	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,2,1,3,2))$	C110	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,2,2,1,3))$	C80	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,2,2,3,1))$	C125	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,2,3,1,2))$	C109	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,2,3,2,1))$	C122	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,3,1,2,2))$	C88	$(C^+)^2$
	$((5,5),(3,3,1,3),(2,3,2,1,2))$	C97	$(C^+)^2$

No.	dim. vector	orbit	apply ...to constructed family
	((5,5),(3,3,1,3),(2,3,2,2,1))	C99	$(C^+)^2$
	((5,5),(3,3,1,3),(3,1,2,2,2))	C84	$(C^+)^2$
	((5,5),(3,3,1,3),(3,2,1,2,2))	C70	$(C^+)^2$
	((5,5),(3,3,1,3),(3,2,2,1,2))	C69	$(C^+)^2$
	((5,5),(3,3,1,3),(3,2,2,2,1))	C68	$(C^+)^2$
	((5,5),(3,3,3,1),(1,2,2,2,3))	C119	–
	((5,5),(3,3,3,1),(1,2,2,3,2))	C118	–
	((5,5),(3,3,3,1),(1,2,3,2,2))	C117	–
	((5,5),(3,3,3,1),(1,3,2,2,2))	C116	–
	((5,5),(3,3,3,1),(2,1,2,2,3))	C115	C^+
	((5,5),(3,3,3,1),(2,1,2,3,2))	C114	C^+
	((5,5),(3,3,3,1),(2,1,3,2,2))	C113	C^+
	((5,5),(3,3,3,1),(2,2,1,2,3))	C111	$(C^+)^2$
	((5,5),(3,3,3,1),(2,2,1,3,2))	C112	$(C^+)^2$
	((5,5),(3,3,3,1),(2,2,2,1,3))	C110	$(C^+)^3$
	((5,5),(3,3,3,1),(2,2,2,3,1))	C109	$(C^+)^3$
	((5,5),(3,3,3,1),(2,2,3,1,2))	C88	$(C^+)^3$
	((5,5),(3,3,3,1),(2,2,3,2,1))	C97	$(C^+)^3$
	((5,5),(3,3,3,1),(2,3,1,2,2))	C84	$(C^+)^3$
	((5,5),(3,3,3,1),(2,3,2,1,2))	C70	$(C^+)^3$
	((5,5),(3,3,3,1),(2,3,2,2,1))	C69	$(C^+)^3$
	((5,5),(3,3,3,1),(3,1,2,2,2))	C79	$(C^+)^3$
	((5,5),(3,3,3,1),(3,2,1,2,2))	C43	$(C^+)^3$
	((5,5),(3,3,3,1),(3,2,2,1,2))	C27	$(C^+)^6$
	((5,5),(3,3,3,1),(3,2,2,2,1))	C23	$(C^+)^5$

13 Constructing families of indecomposable subspace representations

For the constructions of the families, the points of the quivers are always labelled in the following ways:



The *projective representation according to a point* $i \in Q_0$ is denoted by $P(i)$ and the *injective representation according to a point* $i \in Q_0$ by $I(i)$. The indecomposable projective and injective representations can be calculated easily by applying reflection functors to simple representations of a reflected quiver. A characterisation of the projective and the injective representations can be found in [5] (Proposition 2.4). τ is again the Auslander-Reiten translate.

Definition 13.1. An indecomposable representation R of a quiver Q is called *preprojective*, if $C^r(R) = P(i)$ for some $r \geq 0$ and some $i \in Q_0$, and R is called *preinjective*, if $C^{-r}(R) = I(i)$ for some $r \geq 0$ and some $i \in Q_0$, where C is the Coxeter transformation. If an indecomposable representation R is neither preprojective nor preinjective, it is called *regular*.

In the tables for each orbit one tuple of compositions is given and then the quiver to which we restrict ourselves in order to find the representations into which we have to embed a vector space by the methods given in Chapter 11. The next column shows the realisations of the representation for the restricted quiver. (The representations are given as τ-shifts of projective or injective representations or by their dimension vectors if they are regular. In the latter case there is either a unique indecomposable representation with this dimension vector or a whole one parameter family of representations with this dimension vector.) In the s-tame cases, the dimensions of the homomorphism spaces for the summands are given, which — in case there is a family of representations — are independent of the chosen representation in the family. And the last column shows the construction method according to Chapter 11.

Dlab and Ringel have shown in [5] (Proposition 2.1) that the endomorphism rings of representations are preserved under the reflection functors for non simple representations:

Proposition 13.2 (Dlab/Ringel, 1976).

- *Let $i \in Q_0$ be a sink and R be a representation of Q. Then there is a canonical monomorphism*
$$\mu : S_i^- S_i^+ R \to R,$$
and if R is indecomposable, then either $R = S(i)$ for some $i \in Q_0$ and $S_i^+ R = 0$ or $\operatorname{End}(S_i^+ R) \cong \operatorname{End}(R)$.

- *Let $i \in Q_0$ be a source and R be a representation of Q. Then there is a canonical epimorphism*
$$\varepsilon : R \to S_i^+ S_i^- R,$$
and if R is indecomposable, then either $R = S(i)$ for some $i \in Q_0$ and $S_i^- R = 0$ or $\operatorname{End}(S_i^- R) \cong \operatorname{End}(R)$.

From the construction of the indecomposable projective and indecomposable injective representations we get the following Corollary (since all simple representations have trivial endomorphism rings):

Corollary 13.3. *If R is an indecomposable projective or an indecomposable injective representation, then* $\operatorname{End}(R) \cong K$.

Remember that for the tame quivers the Auslander-Reiten quivers consist of three parts: the preprojective component, the regular component and the preinjective component. The regular component is (in these cases) always a stable separating tubular family. If an indecomposable representation R of a tame quiver is regular and belongs to a tube of rank r in a layer $\leq r$, then also $\operatorname{End}(R) \cong K$. Also, an indecomposable regular representation of a tame quiver belongs to a tube of rank r in a layer $\leq r$ if and only if its dimension vector is componentwise smaller than or equal to the critical dimension vector for the quiver.

The representations for the restricted quivers in the lists of this chapter always have trivial endomorphism rings: One can check that in each case exactly one of the following conditions holds:

1. The representation R is preprojective, i. e. $R = \tau^r P(i)$ for some $r \geq 0$ and some $i \in Q_0$, and
$$\operatorname{End}(R) = \operatorname{End}(\tau^r P(i)) \cong \operatorname{End}(P(i)) \cong K.$$
2. The representation R is preinjective, i. e. $R = \tau^{-r} I(i)$ for some $r \geq 0$ and some $i \in Q_0$, and
$$\operatorname{End}(R) = \operatorname{End}(\tau^{-r} I(i)) \cong \operatorname{End}(I(i)) \cong K.$$
3. The representation is regular, has underlying quiver of tame type and is contained in a tube of rank $r \in \mathbb{N}$, but in a layer of height $\leq r$. All of these have trivial endomorphism rings.

4. The representation is constructed by one of the construction methods of Chapter 11 which give families of representations with trivial endomorphism rings.

All (isomorphism classes of the) indecomposable representations for the tame quivers are known (see [5]), so the corresponding construction method for tuples of compositions of tame quivers is just labelled by “known”.

13.1 Constructing n-parameter families of indecomposable subspace representations for the s-hypercritical vectors with $n \geq 2$

H:

No.	dim. vector	quiver	construction	constr. method
1	$((2,2),(2,2),(2,2),(1,1,2))$	$\tilde{\mathbb{D}}_4$	$((1,1),(1,1),(1,1),(1,1)) \oplus ((1,1),(1,1),(1,1),(1,1))$	(B)
2	$((2,2),(2,2),(2,2),(1,2,1))$	$\tilde{\mathbb{D}}_4$	$\tau^4 I(5)$	(A)
3	$((2,2),(2,2),(2,2),(1,1,1,1))$	$T_{2,2,2,3}$	$((2,2),(2,2),(2,2),(2,1,1))$ (see orbit 1 in this list)	(A)
4	$((1,3),(2,2),(2,2),(1,1,1,1))$	$\mathbb{D}_6$	$\tau^{-2}P(2) \oplus \tau^{-1}P(5)$	(D)
5	$((1,2),(1,2),(1,1,1),(1,1,1))$	$\mathbb{E}_6$	$\tau^{-2}P(2)$	(A)
6	$((1,2),(2,1),(1,1,1),(1,1,1))$	$\mathbb{E}_6$	$\tau^{-3}P(2)$	(A)
7	$((1,2),(1,1,1),(1,1,1),(1,1,1))$	$\tilde{\mathbb{E}}_6$	$((1,1,1),(1,1,1),(1,1,1))$	(A)
8	$((1,1,1),(1,1,1),(1,1,1),(1,1,1))$	$T_{2,3,3,3}$	$((2,1),(1,1,1),(1,1,1),(1,1,1))$ (see orbit 7 in this list)	(C)
9	$((6,6),(4,4,4),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1)) \oplus ((3,3),(2,2,2),(1,1,1,1,1,1))$	(B)
10	$((6,6),(4,4,4),(1,2,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{60} I(5)$	(A)
11	$((6,6),(4,4,4),(1,2,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{30} I(6)$	(A)
12	$((6,6),(4,4,4),(1,2,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20} I(7)$	(A)
13	$((6,6),(4,4,4),(1,2,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15} I(8)$	(A)
14	$((6,6),(4,4,4),(1,2,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{12} I(9)$	(A)
15	$((4,4),(2,3,3),(1,1,1,1,1,1,1,1))$	$T_{2,3,7}$	$((4,4),(2,3,3),(2,1,1,1,1,1,1))$ (see orbit 4 in this list)	(C)
16	$((4,4),(2,2,2,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$((2,2),(1,1,1,1),(1,1,1,1)) \oplus ((2,2),(1,1,1,1),(1,1,1,1))$	(B)
17	$((4,4),(2,2,2,2),(1,2,1,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{24} I(6)$	(A)

No.	dim. vector	quiver	construction	constr. method
18	$((4,4),(2,2,2,2),(1,2,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(7)$	(A)
19	$((4,4),(2,2,2,2),(1,2,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(8)$	(A)
20	$((5,5),(1,3,3,3),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(2)\oplus\tau^{-7}P(7)$	(D)
21	$((3,3),(1,1,2,2),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1))$	(C)
22	$((3,3),(1,2,1,2),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(3)$	(A)
23	$((3,3),(1,2,2,1),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(8)$	(A)
24	$((2,3),(1,1,1,1,1),(1,1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(3)$	(C)
25	$((2,2,2),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$((1,1,1),(1,1,1),(1,1,1))\quad\oplus$ $((1,1,1),(1,1,1),(1,1,1))$	(B)
26	$((2,2,2),(2,2,2),(1,2,1,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{12}I(6)$	(A)
27	$((2,2,2),(2,2,2),(1,2,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{6}I(7)$	(A)
28	$((1,2,2),(1,2,2),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-6}P(8)$	(A)
29	$((1,2,2),(2,1,2),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-8}P(8)$	(A)
30	$((1,2,2),(2,2,1),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-10}P(2)$	(A)
31	$((2,1,2),(2,1,2),(1,1,1,1,1))$	$T_{3,3,4}$	$((2,1,2),(2,1,2),(2,1,1,1))$ (see case 15., **s-tame!!**)	(C) **!!!**
32	$((2,2,1),(1,2,2),(1,1,1,1,1))$		$\tau^{-10}P(2)$	(A)
33	$((1,1,2),(1,1,1,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$((2,2),(1,1,1,1),(1,1,1,1))$	(C)
34	$((1,2,1),(1,1,1,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(3)$	(A)
35	$((1,1,1,1),(1,1,1,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{2}I(3)$	(C)
36	$((1,1),(1,1),(1,1),(1,1),(1,1))$	$\mathbb{D}_4$	$\tau^{-1}P(2)$	(C)
37	$((2,1),(1,1,1),(1,1,1),(1,1,1))$	$T_{2,2,3,3}$	$((2,1),(2,1),(1,1,1),(1,1,1))$ (see case 6.)	(A)

13.2 Constructing one parameter families of indecomposable subspace representations for the s-tame vectors

A:

No.	dim. vector	quiver	construction	morphisms	constr. method
1	$((1,1),(1,1),(1,1),(0,1,1))$	$\tilde{\mathbb{D}}_4$	$((1,1),(1,1),(1,1),(1,1))$	–	known
2	$((1,2),(1,2),(2,1),(1,1,1))$	$\mathbb{D}_5$	$\tau^{-1}P(5)\oplus\tau^{-1}P(1)$	–	Ⓐ
3	$((1,3),(2,2),(2,2),(1,2,1))$	$\mathbb{D}_5$	$\tau^{-1}P(5)\oplus\tau^{-1}P(3)\oplus\tau^{-1}P(1)$	–	Ⓐ
4	$((1,3),(2,2),(2,2),(2,1,1))$	$\tilde{\mathbb{D}}_4$	$\tau^{2}I(1)\oplus\tau^{2}I(5)$	–	Ⓐ
5	$((1,3),(1,3),(2,2),(1,1,1,1))$	$\mathbb{D}_6$	$\tau^{-1}P(1)\oplus\tau^{-1}P(6)\oplus P(4)$	–	Ⓐ
6	$((1,3),(1,3),(2,2),(1,1,1,1))$	$\mathbb{D}_6$	$\tau^{-2}P(1)\oplus\tau^{-2}P(6)\oplus\tau^{-1}P(4)$	–	Ⓐ
7	$((3,1),(3,1),(2,2),(1,1,1,1))$	$T_{2,2,2,3}$	$\tau^{3}I(4)$	–	Ⓐ
8	$((1,3),(2,2),(2,2),(1,1,2))$	$\mathbb{D}_5$	$\tau^{-1}P(2)\oplus\tau^{-1}P(5)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	Ⓑ
9	$((1,2),(2,1),(2,1),(1,1,1))$	$\mathbb{D}_5$	$\tau^{-2}P(2)\oplus\tau^{-1}P(4)$		Ⓐ

B:

No.	dim. vector	quiver	construction	morphisms	constr. method
1	$((1,1,1),(1,1,1),(0,1,1,1))$	$\tilde{\mathbb{E}}_6$	$((1,1,1),(1,1,1),(1,1,1))$	–	known
2	$((1,2,1),(2,1,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{4}I(2)$	–	Ⓐ
3	$((2,1,1),(1,2,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{4}I(5)$	–	Ⓐ
4	$((1,2,1),(1,2,1),(1,1,1,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{9}I(6)$	–	Ⓐ
5	$((2,2,1),(2,2,1),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{3}I(3)$	–	Ⓐ
6	$((2,1,2),(2,1,2),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{10}I(6)$	–	Ⓐ
7	$((2,2,2),(2,1,3),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{-10}P(4)$	–	Ⓐ
8	$((2,2,1),(2,1,2),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{5}I(5)$	–	Ⓐ
9	$((2,2,1),(2,1,2),(1,1,1,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{10}I(1)$	–	Ⓐ
10	$((2,1,2),(2,2,1),(1,1,1,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{10}I(4)$	–	Ⓐ

No.	dim. vector	quiver	construction	morphisms	constr. method
11	$((1,3,2),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{-11}P(1)$	–	A
12	$((2,2,1),(1,2,2),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{11}I(4)$	–	A
13	$((2,1,3),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{-10}P(1)$	–	A
14	$((2,2,2),(1,3,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{-11}P(4)$	–	A
15	$((1,2,2),(2,2,1),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{11}I(1)$	–	A
16	$((2,1,2),(2,2,1),(1,1,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{5}I(2)$	–	A
17	$((3,2,2),(2,2,3),(1,1,2,3))$	$\tilde{\mathbb{E}}_6$	$\tau^{-12}P(1)$	–	A
18	$((3,2,1),(2,2,2),(1,2,1,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{4}I(1) \oplus \tau^{4}I(5)$	–	A
19	$((3,2,1),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{6}I(5)$	–	A
20	$((3,1,2),(2,2,2),(1,2,2,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{4}I(5) \oplus \tau^{4}I(6)$	–	A
21	$((3,1,2),(2,2,2),(1,2,1,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{6}I(1) \oplus \tau^{6}I(6)$	–	A
22	$((3,1,2),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{12}I(3)$	–	A
23	$((2,2,3),(2,3,2),(1,1,3,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{-13}P(1)$	–	A
24	$((1,3,2),(2,2,2),(2,1,2,1))$	$\mathbb{E}_7$	$\tau^{-4}P(3) \oplus \tau^{-5}P(1) \oplus \tau^{-5}P(5)$	–	A
25	$((1,3,2),(2,2,2),(1,2,2,1))$	$\mathbb{E}_7$	$\tau^{-4}P(3) \oplus \tau^{-5}P(2) \oplus \tau^{-5}P(5)$	–	A
26	$((1,3,3),(2,3,2),(2,1,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(2) \oplus \tau^{-4}P(4)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
27	$((1,3,3),(2,3,2),(1,2,2,2))$	$\mathbb{E}_7$	$\tau^{-3}P(7) \oplus \tau^{-4}P(2)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
28	$((2,3,1),(2,2,2),(1,1,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{13}I(1)$	–	A
29	$((1,3,3),(2,2,3),(2,2,2,1))$	$\mathbb{E}_7$	$\tau^{-4}P(4) \oplus \tau^{-3}P(5) \oplus \tau^{-4}P(7)$	–	A
30	$((1,4,3),(2,3,3),(2,2,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(2) \oplus \tau^{-4}P(4) \oplus \tau^{-3}P(5)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
31	$((1,3,3),(3,2,2),(2,1,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(4) \oplus \tau^{-5}P(2)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
32	$((1,3,3),(3,2,2),(1,2,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(4) \oplus \tau^{-4}P(1) \oplus \tau^{-3}P(6)$	–	A

No.	dim. vector	quiver	construction	morphisms	constr. method
33	$((1,3,4),(3,2,3),(2,2,2,2))$	$\mathbb{E}_7$	$\tau^{-3}P(3)\oplus\tau^{-4}P(2)\oplus\tau^{-3}P(6)$	–	$\textcircled{A}$
34	$((1,4,4),(3,3,3),(2,2,2,3))$	$\mathbb{E}_7$	$\tau^{-3}P(7)\oplus\tau^{-4}P(2)\oplus\tau^{-3}P(3)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
35	$((1,3,3),(3,2,2),(2,2,2,1))$	$\mathbb{E}_7$	$\tau^{-5}P(4)\oplus\tau^{-5}P(1)\oplus\tau^{-5}P(7)$	–	$\textcircled{A}$
36	$((1,3,3),(3,2,2),(2,2,1,2))$	$\mathbb{E}_7$	$\tau^{-5}P(2)\oplus\tau^{-5}P(7)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
37	$((1,3,3),(2,3,2),(2,2,2,1))$	$\mathbb{E}_7$	$\tau^{-4}P(3)\oplus\tau^{-5}P(2)\oplus\tau^{-4}P(6)$	–	$\textcircled{A}$
38	$((3,3,1),(3,2,2),(1,2,2,2))$	$\tilde{\mathbb{E}}_6$	$\tau^{3}I(3)\oplus\tau^{4}I(1)$	–	$\textcircled{A}$
39	$((1,4,3),(3,2,3),(2,2,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(4)\oplus\tau^{-5}P(2)\oplus\tau^{-3}P(5)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
40	$((1,3,4),(3,3,2),(2,2,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(4)\oplus\tau^{-4}P(1)\oplus\tau^{-4}P(7)$	–	$\textcircled{A}$
41	$((1,4,3),(3,3,2),(2,2,2,2))$	$\mathbb{E}_7$	$\tau^{-5}P(2)\oplus\tau^{-5}P(7)\oplus\tau^{-4}P(3)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
42	$((1,4,4),(3,3,3),(2,2,3,2))$	$\mathbb{E}_7$	$\tau^{-4}P(2)\oplus\tau^{-4}P(4)\oplus\tau^{-3}P(6)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
43	$((1,4,4),(3,3,3),(3,2,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(4)\oplus\tau^{-5}P(2)\oplus\tau^{-4}P(6)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
44	$((1,4,4),(3,3,3),(2,3,2,2))$	$\mathbb{E}_7$	$\tau^{-4}P(1)\oplus\tau^{-4}P(4)\oplus\tau^{-4}P(7)\oplus$ $\tau^{-3}P(5)$	–	$\textcircled{A}$
45	$((2,2,2),(2,2,2),(1,1,3,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{13}I(6)$	–	$\textcircled{A}$
46	$((2,2,2),(2,2,2),(1,3,1,1))$	$\tilde{\mathbb{E}}_6$	$\tau^{6}I(6)\oplus\tau^{3}I(7)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
47	$((1,1,3),(2,2,1),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-7}P(8)$	–	$\textcircled{A}$
48	$((1,3,1),(1,2,2),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-8}P(5)\oplus\tau^{-6}P(5)\oplus\tau^{-9}P(1)$	–	$\textcircled{A}$
49	$((1,3,1),(2,1,2),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-8}P(5)\oplus\tau^{-10}P(5)\oplus\tau^{-10}P(4)$	–	$\textcircled{A}$
50	$((1,3,1),(2,2,1),(1,1,1,1,1))$	$\mathbb{E}_8$	$\tau^{-10}P(3)\oplus\tau^{-11}P(8)\oplus\tau^{-10}P(5)$	–	$\textcircled{A}$
51	$((1,4,1),(2,2,2),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{5}I(5)\oplus\tau^{7}I(5)\oplus\tau^{9}I(5)\oplus\tau^{5}I(1)$	–	$\textcircled{A}$
52	$((2,1,1),(1,1,2),(0,1,1,1,1))$	$T_{3,3,4}$	$((2,1,1),(1,1,2),(1,1,1,1))$ (see case 14.)		

No.	dim. vector	quiver	construction	morphisms	constr. method
53	$((2,1,1),(1,2,1),(0,1,1,1,1))$	$T_{3,3,4}$	$((2,1,1),(1,2,1),(1,1,1,1))$ (see case 14.)		
54	$((1,1,2),(1,2,1),(1,1,1,1,0))$	$T_{3,3,4}$	$((1,1,2),(1,2,1),(1,1,1,1))$ (see case 14.)		
55	$((1,1,2),(2,1,1),(1,1,1,1,0))$	$T_{3,3,4}$	$((1,1,2),(2,1,1),(1,1,1,1))$ (see case 14.)		
56	$((3,1,1),(1,2,2),(0,1,1,1,1,1))$	$T_{3,3,5}$	$((3,1,1),(1,2,2),(1,1,1,1,1))$ (see case 33.)		
57	$((1,1,3),(2,2,1),(1,1,1,1,1,0))$	$T_{3,3,5}$	$((1,1,3),(2,2,1),(1,1,1,1,1))$ (see case 33.)		

C:

No.	dim. vector	quiver	construction	morphisms	constr. method
1	$((2,2),(1,1,1,1),(0,1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$((2,2),(1,1,1,1),(1,1,1,1))$	–	known
2	$((3,2),(1,1,1,2),(1,1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{15}I(3)$	–	Ⓐ
3	$((3,2),(1,1,2,1),(1,1,1,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(7)$	–	Ⓐ
4	$((3,3),(2,2,1,1),(1,1,2,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{4}I(2)$	–	Ⓐ
5	$((3,3),(2,2,1,1),(1,1,1,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(8)$	–	Ⓐ
6	$((3,3),(2,2,1,1),(1,1,1,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{9}I(4)$	–	Ⓐ
7	$((3,3),(2,1,2,1),(1,1,1,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{18}I(3)$	–	Ⓐ
8	$((3,3),(2,1,2,1),(1,1,2,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(5)$	–	Ⓐ
9	$((3,3),(1,2,1,2),(1,1,2,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{19}I(3)$	–	Ⓐ
10	$((3,4),(1,2,2,2),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{-20}P(6)$	–	Ⓐ
11	$((3,3),(1,2,2,1),(1,1,2,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{10}I(4)$	–	Ⓐ
12	$((3,3),(1,2,2,1),(1,1,1,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{20}I(6)$	–	Ⓐ
13	$((4,4),(2,2,2,2),(1,1,2,1,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{-21}P(6)$	–	Ⓐ

No.	dim. vector	quiver	construction	morphisms	constr. method
14	$((4,3),(2,1,2,2),(1,1,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{21}I(6)$	–	$\textcircled{A}$
15	$((4,3),(2,2,2,1),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{5}I(2)$	–	$\textcircled{A}$
16	$((4,3),(2,2,2,1),(1,1,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(8)$	–	$\textcircled{A}$
17	$((4,3),(2,2,2,1),(1,1,1,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{11}I(7)$	–	$\textcircled{A}$
18	$((4,3),(1,2,2,2),(1,1,2,1,2))$	$\mathbb{E}_8$	$\tau^{-7}P(1)\oplus\tau^{-7}P(7)$	–	$\textcircled{A}$
19	$((4,3),(2,2,1,2),(1,1,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{10}I(1)$	–	$\textcircled{A}$
20	$((4,3),(2,2,1,2),(1,1,1,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{21}I(3)$	–	$\textcircled{A}$
21	$((4,4),(1,3,2,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-23}P(3)$	–	$\textcircled{A}$
22	$((4,5),(1,3,2,3),(2,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4)\oplus\tau^{-6}P(1)\oplus\tau^{-5}P(5)$	–	$\textcircled{A}$
23	$((4,4),(2,1,3,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-22}P(3)$	–	$\textcircled{A}$
24	$((3,4),(2,2,2,1),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{11}I(1)$	–	$\textcircled{A}$
25	$((3,4),(2,2,2,1),(1,1,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{22}I(3)$	–	$\textcircled{A}$
26	$((4,4),(2,2,1,3),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-21}P(3)$	–	$\textcircled{A}$
27	$((4,5),(2,2,3,2),(1,1,2,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-13}P(1)$	–	$\textcircled{A}$
28	$((4,3),(1,2,2,2),(1,2,1,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-8}P(7)$	–	$\textcircled{A}$
29	$((5,4),(2,2,2,3),(1,1,2,2,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{-12}P(1)$	–	$\textcircled{A}$
30	$((4,4),(2,2,2,2),(1,1,1,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-22}P(6)$	–	$\textcircled{A}$
31	$((5,5),(2,2,3,3),(1,1,3,2,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{-9}P(8)$	–	$\textcircled{A}$
32	$((4,3),(2,1,2,2),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{11}I(4)$	–	$\textcircled{A}$
33	$((4,3),(2,2,1,2),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(5)$	–	$\textcircled{A}$
34	$((4,5),(3,2,2,2),(1,1,2,2,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{-24}P(3)$	–	$\textcircled{A}$
35	$((4,4),(1,3,2,2),(1,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3)\oplus\tau^{-8}P(3)\oplus\tau^{-7}P(7)$	–	$\textcircled{A}$

No.	dim. vector	quiver	construction	morphisms	constr. method
36	$((4,5),(1,3,2,3),(2,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3)\oplus\tau^{-7}P(8)\oplus\tau^{-5}P(5)$	–	$\textcircled{A}$
37	$((4,4),(3,2,2,1),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(5)$	–	$\textcircled{A}$
38	$((5,5),(3,2,2,3),(1,1,2,3,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{-13}P(4)$	–	$\textcircled{A}$
39	$((4,4),(1,2,3,2),(2,1,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(4)\oplus\tau^{-7}P(7)$	–	$\textcircled{A}$
40	$((4,4),(1,2,3,2),(1,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-7}P(1)\oplus\tau^{-7}P(8)$	–	$\textcircled{A}$
41	$((4,4),(3,2,1,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(4)$	–	$\textcircled{A}$
42	$((4,3),(1,2,2,2),(1,1,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{23}I(3)$	–	$\textcircled{A}$
43	$((4,4),(1,2,2,3),(2,2,2,1,1))$	$\mathbb{E}_8$	$\tau^{-9}P(2)\oplus\tau^{-8}P(6)$	–	$\textcircled{A}$
44	$((4,4),(1,2,2,3),(2,2,1,2,1))$	$\mathbb{E}_8$	$\tau^{-8}P(1)\oplus\tau^{-8}P(8)$	–	$\textcircled{A}$
45	$((4,4),(1,2,2,3),(2,1,2,2,1))$	$\mathbb{E}_8$	$\tau^{-7}P(4)\oplus\tau^{-7}P(7)$	–	$\textcircled{A}$
46	$((4,4),(3,1,2,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{24}I(3)$	–	$\textcircled{A}$
47	$((5,4),(1,2,3,3),(2,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(2)\oplus\tau^{-6}P(6)$	–	$\textcircled{A}$
48	$((5,4),(1,2,3,3),(1,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4)\oplus\tau^{-6}P(8)$	–	$\textcircled{A}$
49	$((4,4),(2,3,1,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{25}I(3)$	–	$\textcircled{A}$
50	$((5,5),(1,3,3,3),(2,1,2,3,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4)\oplus\tau^{-6}P(1)\oplus\tau^{-6}P(6)$	–	$\textcircled{A}$
51	$((5,5),(1,3,3,3),(1,2,2,3,2))$	$\mathbb{E}_8$	$\tau^{-6}P(2)\oplus\tau^{-6}P(4)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	$\textcircled{B}$
52	$((4,4),(2,2,3,1),(1,2,1,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{14}I(3)\oplus\tau^{12}I(6)$	–	$\textcircled{A}$
53	$((4,4),(2,2,3,1),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{26}I(3)$	–	$\textcircled{A}$
54	$((5,4),(2,2,2,3),(1,1,2,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-25}P(3)$	–	$\textcircled{A}$
55	$((4,4),(1,3,2,2),(2,1,2,1,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-9}P(1)\oplus\tau^{-7}P(5)$	–	$\textcircled{A}$
56	$((4,4),(1,3,2,2),(1,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3)\oplus\tau^{-8}P(8)\oplus\tau^{-7}P(5)$	–	$\textcircled{A}$
57	$((4,5),(1,3,2,3),(2,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(2)\oplus\tau^{-5}P(5)\oplus\tau^{-7}P(5)$	–	$\textcircled{A}$

No.	dim. vector	quiver	construction	morphisms	constr. method
58	$((5,5),(2,2,3,3),(1,1,3,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-14}P(4)$	–	A
59	$((4,4),(1,2,3,2),(2,2,1,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-9}P(8)$	–	A
60	$((4,4),(1,2,3,2),(2,1,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(2)\oplus\tau^{-7}P(6)$	–	A
61	$((4,4),(1,2,3,2),(1,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-8}P(2)\oplus\tau^{-7}P(6)$	–	A
62	$((4,5),(1,3,3,2),(2,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(8)\oplus\tau^{-7}P(7)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
63	$((4,5),(1,3,3,2),(1,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4)\oplus\tau^{-7}P(8)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
64	$((4,6),(1,3,3,3),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4)\oplus\tau^{-6}P(1)\oplus\tau^{-5}P(6)$	–	A
65	$((4,4),(2,3,2,1),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{13}I(4)$	–	A
66	$((5,4),(1,2,3,3),(2,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(2)\oplus\tau^{-6}P(6)$	–	A
67	$((4,5),(2,2,3,2),(1,1,3,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{-26}P(3)$	–	A
68	$((4,4),(1,3,2,2),(2,2,1,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(3)\oplus\tau^{-10}P(1)\oplus\tau^{-10}P(8)$	–	A
69	$((4,4),(1,3,2,2),(2,1,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(3)\oplus\tau^{-10}P(2)\oplus\tau^{-7}P(5)$	–	A
70	$((4,4),(1,3,2,2),(1,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-8}P(7)\oplus\tau^{-7}P(5)$	–	A
71	$((5,4),(1,3,2,3),(2,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(8)\oplus\tau^{-9}P(4)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
72	$((4,5),(1,3,3,2),(2,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3)\oplus\tau^{-9}P(2)\oplus\tau^{-7}P(5)$	–	A
73	$((5,4),(1,3,3,2),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-9}P(8)\oplus\tau^{-7}P(5)$	–	A
74	$((5,4),(1,3,2,3),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(3)\oplus\tau^{-10}P(2)\oplus\tau^{-8}P(6)$	–	A
75	$((5,4),(1,2,3,3),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(2)\oplus\tau^{-8}P(7)$	–	A
76	$((5,4),(1,3,2,3),(2,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(7)\oplus\tau^{-8}P(8)$	$\hom(\mathbf{d_1},\mathbf{d_2})=1$	B
77	$((4,5),(1,3,3,2),(2,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3)\oplus\tau^{-8}P(1)\oplus\tau^{-7}P(7)$	–	A
78	$((5,4),(1,3,2,3),(2,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4)\oplus\tau^{-9}P(1)\oplus\tau^{-8}P(6)$	–	A
79	$((5,4),(1,2,3,3),(2,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(4)\oplus\tau^{-8}P(8)$	–	A

No.	dim. vector	quiver	construction	morphisms	constr. method
80	$((5,5),(1,3,3,3),(2,1,3,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(2) \oplus \tau^{-7}P(7)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
81	$((5,5),(1,3,3,3),(2,2,1,3,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3) \oplus \tau^{-7}P(8) \oplus \tau^{-6}P(6)$	–	(A)
82	$((5,5),(1,3,3,3),(2,2,2,1,3))$	$\mathbb{E}_8$	$\tau^{-7}P(1) \oplus \tau^{-7}P(8) \oplus \tau^{-5}P(5)$	–	(A)
83	$((4,5),(1,3,3,2),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-9}P(8) \oplus \tau^{-7}P(5)$	–	(A)
84	$((4,5),(1,3,2,3),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-8}P(1) \oplus \tau^{-8}P(8) \oplus \tau^{-7}P(5)$	–	(A)
85	$((4,5),(1,2,3,3),(2,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-7}P(4) \oplus \tau^{-7}P(8)$	–	(A)
86	$((5,4),(2,3,3,1),(1,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(8) \oplus \tau^{8}I(6)$	–	(A)
87	$((5,4),(2,3,1,3),(1,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{13}I(3) \oplus \tau^{15}I(3)$	–	(A)
88	$((5,4),(1,3,3,2),(1,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3) \oplus \tau^{-8}P(8) \oplus \tau^{-7}P(6)$	–	(A)
89	$((5,4),(1,3,2,3),(1,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3) \oplus \tau^{-7}P(1) \oplus \tau^{-7}P(7)$	–	(A)
90	$((4,5),(3,3,2,1),(1,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(1) \oplus \tau^{7}I(7)$	–	(A)
91	$((5,5),(1,2,4,3),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(2) \oplus \tau^{-6}P(7)$	–	(A)
92	$((4,5),(3,2,3,1),(1,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{9}I(1) \oplus \tau^{10}I(3)$	–	(A)
93	$((5,5),(1,3,2,4),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3) \oplus \tau^{-7}P(2) \oplus \tau^{-5}P(5)$	–	(A)
94	$((4,5),(2,3,3,1),(1,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{14}I(3) \oplus \tau^{16}I(6)$	–	(A)
95	$((5,4),(1,3,3,2),(2,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-10}P(2) \oplus \tau^{-10}P(8)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
96	$((5,5),(1,3,3,3),(1,2,3,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(4) \oplus \tau^{-7}P(2)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
97	$((5,4),(1,3,3,2),(2,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-9}P(1) \oplus \tau^{-7}P(6)$	–	(A)
98	$((5,5),(3,3,1,3),(1,2,2,2,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{25}I(3) \oplus ((1,1),(1,0,0,1),(1,0,0,1))$	–	(A)
99	$((5,4),(1,3,3,2),(2,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-10}P(2)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)

No.	dim. vector	quiver	construction	morphisms	constr. method
100	$((5,5),(3,1,3,3),(1,2,2,3,2))$	$\tilde{\mathbb{E}}_7$	$((2,1),(1,0,1,1),(1,0,1,1)) \oplus ((1,2),(1,0,1,1),(1,1,1,0)) \oplus ((2,2),(1,1,1,1),(1,1,1,1))$	–	G
101	$((5,5),(2,4,1,3),(2,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{13}I(3) \oplus \tau^{7}I(3) \oplus \tau^{12}I(6)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	C
102	$((5,5),(1,4,2,3),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(8) \oplus \tau^{-9}P(4) \oplus \tau^{-7}P(5)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	B
103	$((5,5),(2,4,3,1),(2,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(6) \oplus \tau^{7}I(7) \oplus \tau^{6}I(4)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	C
104	$((5,5),(2,3,4,1),(2,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(7) \oplus \tau^{8}I(6) \oplus \tau^{10}I(3)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	E
105	$((5,5),(1,4,3,2),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-10}P(2) \oplus \tau^{-7}P(5)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	B
106	$((5,5),(1,3,4,2),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3) \oplus \tau^{-9}P(2) \oplus \tau^{-7}P(6)$	–	A
107	$((6,4),(1,3,3,3),(2,2,2,2,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-9}P(1) \oplus \tau^{-8}P(7)$	–	A
108	$((4,6),(3,3,3,1),(2,2,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{10}I(3) \oplus \tau^{7}I(1) \oplus \tau^{8}I(6)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	D
109	$((5,5),(1,3,3,3),(3,1,2,2,2))$	$\mathbb{E}_8$	$\tau^{-7}P(7) \oplus \tau^{-9}P(2)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	B
110	$((5,5),(1,3,3,3),(1,3,2,2,2))$	$\mathbb{E}_8$	$\tau^{-6}P(3) \oplus \tau^{-7}P(1) \oplus \tau^{-7}P(8)$	–	A
111	$((5,5),(3,1,3,3),(1,2,3,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{24}I(3) \oplus ((1,1),(0,0,1,1),(1,1,0,0))$	–	A
112	$((5,5),(3,1,3,3),(1,3,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(3) \oplus \tau^{15}I(3) \oplus ((0,1),(0,0,1,0),(1,0,0,0))$	–	A
113	$((5,5),(3,3,1,3),(1,3,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(3) \oplus \tau^{9}I(3) \oplus \tau^{7}I(1)$	–	A
114	$((5,5),(3,3,1,3),(1,2,3,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{10}I(1) \oplus \tau^{10}I(6)$	–	A
115	$((5,5),(3,3,1,3),(1,2,2,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{13}I(3) \oplus \tau^{18}I(6)$	–	A
116	$((5,5),(3,3,3,1),(1,3,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(3) \oplus \tau^{5}I(5) \oplus \tau^{5}I(1)$	–	A
117	$((5,5),(3,3,3,1),(1,2,3,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(5) \oplus \tau^{7}I(7)$	–	A
118	$((5,5),(3,3,3,1),(1,2,2,3,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{8}I(6) \oplus \tau^{8}I(5)$	–	A

No.	dim. vector	quiver	construction	morphisms	constr. method
119	$((5,5),(3,3,3,1),(1,2,2,2,3))$	$\tilde{\mathbb{E}}_7$	$\tau^{10}I(3) \oplus \tau^{11}I(7)$	–	Ⓐ
120	$((5,5),(1,3,3,3),(3,2,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(3) \oplus \tau^{-10}P(2) \oplus \tau^{-9}P(7)$	–	Ⓐ
121	$((5,5),(1,3,3,3),(3,2,2,1,2))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-9}P(1) \oplus \tau^{-9}P(7)$	–	Ⓐ
122	$((5,5),(1,3,3,3),(3,2,1,2,2))$	$\mathbb{E}_8$	$\tau^{-9}P(2) \oplus \tau^{-9}P(4)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	Ⓑ
123	$((5,5),(1,3,3,3),(2,3,2,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(4) \oplus \tau^{-9}P(8) \oplus \tau^{-8}P(6)$	–	Ⓐ
124	$((5,5),(1,3,3,3),(2,3,2,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3) \oplus \tau^{-9}P(2) \oplus \tau^{-8}P(6)$	–	Ⓐ
125	$((5,5),(1,3,3,3),(2,3,1,2,2))$	$\mathbb{E}_8$	$\tau^{-8}P(3) \oplus \tau^{-8}P(1) \oplus \tau^{-8}P(8)$	–	Ⓐ
126	$((5,5),(1,3,3,3),(2,2,3,2,1))$	$\mathbb{E}_8$	$\tau^{-9}P(2) \oplus \tau^{-8}P(7) \oplus \tau^{-7}P(5)$	–	Ⓐ
127	$((5,5),(1,3,3,3),(2,2,3,1,2))$	$\mathbb{E}_8$	$\tau^{-8}P(4) \oplus \tau^{-8}P(8) \oplus \tau^{-7}P(5)$	–	Ⓐ
128	$((5,5),(1,3,3,3),(2,2,2,3,1))$	$\mathbb{E}_8$	$\tau^{-8}P(1) \oplus \tau^{-8}P(8) \oplus \tau^{-7}P(6)$	–	Ⓐ
129	$((5,3),(2,2,2,2),(1,2,2,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{5}I(5) \oplus \tau^{5}I(7)$	–	Ⓐ
130	$((5,3),(2,2,2,2),(1,2,2,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(4) \oplus \tau^{6}I(1)$	–	Ⓐ
131	$((5,3),(2,2,2,2),(1,2,1,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{9}I(3) \oplus \tau^{8}I(7)$	–	Ⓐ
132	$((5,3),(2,2,2,2),(1,1,2,2,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(1)$	–	Ⓐ
133	$((4,4),(2,2,2,2),(1,3,1,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(6) \oplus \tau^{6}I(4)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	Ⓑ
134	$((4,4),(2,2,2,2),(1,1,3,1,2))$	$\tilde{\mathbb{E}}_7$	$\tau^{25}I(6)$	–	Ⓐ
135	$((4,4),(2,2,2,2),(1,3,2,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(4) \oplus \tau^{4}I(8)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	Ⓑ
136	$((4,4),(2,2,2,2),(1,3,1,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{6}I(4) \oplus \tau^{6}I(6) \oplus \tau^{8}I(6)$	–	Ⓐ
137	$((4,4),(2,2,2,2),(1,2,3,1,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{7}I(3) \oplus \tau^{6}I(5)$	–	Ⓐ
138	$((4,4),(2,2,2,2),(1,2,1,3,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{12}I(6) \oplus \tau^{14}I(6)$	–	Ⓐ
139	$((4,4),(2,2,2,2),(1,1,3,2,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{13}I(7)$	–	Ⓐ
140	$((4,4),(2,2,2,2),(1,1,2,3,1))$	$\tilde{\mathbb{E}}_7$	$\tau^{26}I(6)$	–	Ⓐ

No.	dim. vector	quiver	construction	morphisms	constr. method
141	$((3,3),(1,3,1,1),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^5 I(4) \oplus \tau^5 I(5) \oplus \tau^8 I(5)$	–	(A)
142	$((3,3),(1,1,3,1),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{16} I(3)$	–	(A)
143	$((4,2),(1,2,2,1),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^4 I(2) \oplus \tau^5 I(5)$	–	(A)
144	$((4,2),(1,2,1,2),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^5 I(5) \oplus \tau^7 I(8)$	–	(A)
145	$((4,2),(1,1,2,2),(1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10} I(1)$	–	(A)
146	$((2,5),(2,2,2,1),(1,1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^4 I(4) \oplus \tau^4 I(8) \oplus \tau^5 I(5)$	–	(α) in pt. 3 and then (A) in pt. 5
147	$((2,6),(2,2,2,2),(1,1,1,1,1,1,1,1))$	$\mathbb{A}_{11}$	$P(1) \oplus \tau^{-1}P(2) \oplus \tau^{-2}P(4) \oplus \tau^{-2}P(11) \oplus \tau^{-1}P(9) \oplus \tau^{-1}P(8) \oplus P(6) \oplus P(5)$	–	(F)
148	$((6,2),(2,2,2,2),(1,1,1,1,1,1,1,1))$	$\mathbb{A}_{11}$	$\tau^{-1}P(2) \oplus P(1) \oplus \tau^{-2}P(11) \oplus \tau^{-2}P(4) \oplus \tau^{-1}P(8) \oplus \tau^{-1}P(9) \oplus P(5) \oplus P(6)$		(γ)
149	$((3,2),(1,1,1,2),(1,1,1,1,1,0))$	$T_{2,4,5}$	$((3,2),(1,1,1,2),(1,1,1,1,1))$ (see case 8.)		
150	$((2,3),(2,1,1,1),(0,1,1,1,1,1))$	$T_{2,4,5}$	$((2,3),(2,1,1,1),(1,1,1,1,1))$ (see case 8.)		
151	$((2,3),(2,1,1,1),(1,1,1,1,1,0))$	$T_{2,4,5}$	$((2,3),(2,1,1,1),(1,1,1,1,1))$ (see case 8.)		
152	$((2,3),(1,2,1,1),(1,1,1,1,1,0))$	$T_{2,4,5}$	$((2,3),(1,2,1,1),(1,1,1,1,1))$ (see case 8.)		
153	$((2,3),(1,1,2,1),(1,1,1,1,1,0))$	$T_{2,4,5}$	$((2,3),(1,1,2,1),(1,1,1,1,1))$ (see case 8.)		
154	$((3,2),(1,2,1,1),(0,1,1,1,1,1))$	$T_{2,4,5}$	$((3,2),(1,2,1,1),(1,1,1,1,1))$ (see case 8.)		

No.	dim. vector	quiver	construction	morphisms	constr. method
155	$((3,2),(1,1,2,1),(0,1,1,1,1,1))$	$T_{2,4,5}$	$((3,2),(1,1,2,1),(1,1,1,1,1))$ (see case 8.)		
156	$((3,2),(1,1,1,2),(0,1,1,1,1,1))$	$T_{2,4,5}$	$((3,2),(1,1,1,2),(1,1,1,1,1))$ (see case 8.)		
157	$((2,4),(2,2,1,1),(1,1,1,1,1,1,0))$	$T_{2,4,6}$	$((2,4),(2,2,1,1),(1,1,1,1,1,1))$ (see case 27.)		
158	$((2,4),(2,1,2,1),(1,1,1,1,1,1,0))$	$T_{2,4,6}$	$((2,4),(2,1,2,1),(1,1,1,1,1,1))$ (see case 27.)		
159	$((2,4),(1,2,2,1),(1,1,1,1,1,1,0))$	$T_{2,4,6}$	$((2,4),(1,2,2,1),(1,1,1,1,1,1))$ (see case 27.)		
160	$((4,2),(1,2,2,1),(0,1,1,1,1,1,1))$	$T_{2,4,6}$	$((4,2),(1,2,2,1),(1,1,1,1,1,1))$ (see case 27.)		
161	$((4,2),(1,2,1,2),(0,1,1,1,1,1,1))$	$T_{2,4,6}$	$((4,2),(1,2,1,2),(1,1,1,1,1,1))$ (see case 27.)		
162	$((4,2),(1,1,2,2),(0,1,1,1,1,1,1))$	$T_{2,4,6}$	$((4,2),(1,1,2,2),(1,1,1,1,1,1))$ (see case 27.)		

D:

No.	dim. vector	quiver	construction	morphisms	constr. method
1	$((3,3),(2,2,2),(0,1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1))$	–	known
2	$((3,4),(2,3,2),(1,1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{36}I(5)$	–	$\textcircled{A}$
3	$((4,4),(3,3,2),(1,1,2,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{6}I(2)$	–	$\textcircled{A}$
4	$((4,4),(3,3,2),(1,1,1,2,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(9)$	–	$\textcircled{A}$
5	$((4,4),(3,3,2),(1,1,1,1,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(4)$	–	$\textcircled{A}$
6	$((4,4),(3,3,2),(1,1,1,1,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(7)$	–	$\textcircled{A}$
7	$((5,4),(3,3,3),(1,1,1,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{45}I(5)$	–	$\textcircled{A}$
8	$((5,4),(3,3,3),(1,1,2,1,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(7)$	–	$\textcircled{A}$

No.	dim. vector	quiver	construction	morphisms	constr. method
9	$((5,4),(3,3,3),(1,1,1,2,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{22}I(3)$	–	Ⓐ
10	$((5,4),(3,3,3),(1,1,2,2,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{7}I(2)$	–	Ⓐ
11	$((5,4),(3,3,3),(1,1,2,1,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(9)$	–	Ⓐ
12	$((5,4),(3,3,3),(1,1,2,1,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(4)$	–	Ⓐ
13	$((5,4),(3,3,3),(1,1,1,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{12}I(8)$	–	Ⓐ
14	$((5,4),(3,3,3),(1,1,1,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(1)$	–	Ⓐ
15	$((5,4),(3,3,3),(1,1,1,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{24}I(6)$	–	Ⓐ
16	$((5,6),(4,3,4),(1,1,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-50}P(5)$	–	Ⓐ
17	$((6,6),(4,4,4),(1,1,2,2,1,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-26}P(6)$	–	Ⓐ
18	$((5,5),(4,3,3),(1,1,1,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{50}I(5)$	–	Ⓐ
19	$((5,6),(4,4,3),(1,1,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-52}P(5)$	–	Ⓐ
20	$((7,6),(4,4,5),(1,1,2,2,2,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-20}P(1)$	–	Ⓐ
21	$((6,5),(3,4,4),(1,1,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-51}P(5)$	–	Ⓐ
22	$((5,5),(3,4,3),(1,1,2,2,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{26}I(6)$	–	Ⓐ
23	$((5,5),(3,4,3),(1,1,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{51}I(5)$	–	Ⓐ
24	$((5,5),(4,3,3),(1,1,2,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(2)$	–	Ⓐ
25	$((5,5),(4,3,3),(1,1,2,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(9)$	–	Ⓐ
26	$((5,5),(4,3,3),(1,1,2,2,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{12}I(4)$	–	Ⓐ
27	$((5,5),(4,3,3),(1,1,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{13}I(8)$	–	Ⓐ
28	$((5,5),(4,3,3),(1,1,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{16}I(1)$	–	Ⓐ
29	$((5,5),(4,3,3),(1,1,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{18}I(7)$	–	Ⓐ
30	$((5,5),(4,3,3),(1,1,1,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{25}I(3)$	–	Ⓐ

No.	dim. vector	quiver	construction	morphisms	constr. method
31	$((6,5),(4,3,4),(1,1,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1)) \oplus$ $((3,2),(2,1,2),(1,0,1,1,1,1))$	–	G
32	$((6,6),(3,5,4),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-29}P(3)$	–	A
33	$((5,6),(4,4,3),(1,1,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1)) \oplus$ $((2,3),(2,2,1),(1,1,0,1,1,1))$	–	G
34	$((6,7),(4,5,4),(1,1,2,2,2,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-21}P(1)$	–	A
35	$((5,5),(3,4,3),(1,2,1,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(7) \oplus \tau^{12}I(5)$	–	A
36	$((5,5),(3,4,3),(1,2,1,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{24}I(5) \oplus \tau^{15}I(6)$	–	A
37	$((5,5),(3,4,3),(1,1,2,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{13}I(4)$	–	A
38	$((5,5),(3,4,3),(1,1,2,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{17}I(1)$	–	A
39	$((5,5),(3,4,3),(1,1,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{26}I(3)$	–	A
40	$((5,5),(3,4,3),(1,1,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{54}I(5)$	–	A
41	$((6,6),(4,4,4),(1,1,2,2,2,1,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-55}P(5)$	–	A
42	$((6,5),(4,3,4),(1,1,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{55}I(5)$	–	A
43	$((6,7),(5,4,4),(1,1,2,2,2,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-30}P(3)$	–	A
44	$((6,5),(4,4,3),(1,2,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{7}I(4) \oplus \tau^{9}I(7)$	–	A
45	$((5,6),(4,4,3),(1,1,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{56}I(5)$	–	A
46	$((7,7),(5,4,5),(1,1,2,2,3,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-22}P(1)$	–	A
47	$((6,5),(4,4,3),(1,2,1,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(8) \oplus \tau^{10}I(3)$	–	A
48	$((6,5),(4,4,3),(1,2,1,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(7) \oplus \tau^{14}I(6)$	–	A
49	$((6,5),(3,4,4),(1,1,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{57}I(5)$	–	A
50	$((6,5),(4,4,3),(1,1,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(2)$	–	A

No.	dim. vector	quiver	construction	morphisms	constr. method
51	$((6,5),(4,4,3),(1,1,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(9)$	–	$\textcircled{A}$
52	$((6,5),(4,4,3),(1,1,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(8)$	–	$\textcircled{A}$
53	$((6,5),(4,4,3),(1,1,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{19}I(7)$	–	$\textcircled{A}$
54	$((5,6),(4,3,4),(1,1,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{58}I(5)$	–	$\textcircled{A}$
55	$((6,6),(4,4,4),(1,1,1,3,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-58}P(5)$	–	$\textcircled{A}$
56	$((7,7),(4,5,5),(1,1,3,2,2,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-21}P(7)$	–	$\textcircled{A}$
57	$((6,6),(4,4,4),(1,1,2,1,3,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-57}P(5)$	–	$\textcircled{A}$
58	$((6,5),(4,3,4),(1,2,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(5)\oplus\tau^{9}I(9)$	–	$\textcircled{A}$
59	$((6,5),(4,3,4),(1,2,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(5)\oplus\tau^{10}I(8)$	–	$\textcircled{A}$
60	$((6,6),(4,4,4),(1,1,2,2,1,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-56}P(5)$	–	$\textcircled{A}$
61	$((6,5),(4,3,4),(1,2,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(5)\oplus\tau^{12}I(8)$	–	$\textcircled{A}$
62	$((6,5),(4,3,4),(1,2,1,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{13}I(1)\oplus\tau^{15}I(5)$	–	$\textcircled{A}$
63	$((6,5),(4,3,4),(1,2,1,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20}I(5)\oplus\tau^{17}I(3)$	–	$\textcircled{A}$
64	$((6,5),(4,3,4),(1,1,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(4)$	–	$\textcircled{A}$
65	$((6,5),(4,3,4),(1,1,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{18}I(1)$	–	$\textcircled{A}$
66	$((6,5),(4,3,4),(1,1,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{27}I(3)$	–	$\textcircled{A}$
67	$((7,6),(4,4,5),(1,1,2,2,2,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-31}P(3)$	–	$\textcircled{A}$
68	$((5,6),(4,4,3),(1,2,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{13}I(3)\oplus\tau^{15}I(6)$	–	$\textcircled{A}$
69	$((5,6),(4,4,3),(1,2,1,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{18}I(5)\oplus\tau^{20}I(6)$	–	$\textcircled{A}$
70	$((5,6),(4,4,3),(1,1,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{28}I(3)$	–	$\textcircled{A}$
71	$((7,7),(4,5,5),(1,1,2,3,2,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-23}P(1)$	–	$\textcircled{A}$
72	$((6,5),(3,4,4),(1,2,2,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(4)\oplus\tau^{12}I(5)$	–	$\textcircled{A}$

No.	dim. vector	quiver	construction	morphisms	constr. method
73	$((6,5),(3,4,4),(1,2,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(3)\oplus\tau^{15}I(6)$	–	Ⓐ
74	$((6,5),(3,4,4),(1,2,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{17}I(3)\oplus\tau^{24}I(5)$	–	Ⓐ
75	$((6,5),(3,4,4),(1,1,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{29}I(3)$	–	Ⓐ
76	$((6,7),(4,5,4),(1,1,2,2,3,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-32}P(3)$	–	Ⓐ
77	$((5,6),(4,4,3),(1,2,2,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(1)\oplus\tau^{10}I(7)$	–	Ⓐ
78	$((5,6),(4,4,3),(1,2,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(3)\oplus\tau^{11}I(1)$	–	Ⓐ
79	$((5,6),(4,4,3),(1,2,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(7)\oplus\tau^{18}I(5)$	–	Ⓐ
80	$((5,6),(4,4,3),(1,1,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{19}I(1)$	–	Ⓐ
81	$((7,5),(4,4,4),(1,2,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{7}I(2)\oplus\tau^{9}I(6)$	–	Ⓐ
82	$((7,5),(4,4,4),(1,2,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(1)\oplus\tau^{9}I(8)$	–	Ⓐ
83	$((7,5),(4,4,4),(1,2,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(1)\oplus\tau^{10}I(7)$	–	Ⓐ
84	$((7,5),(4,4,4),(1,2,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(4)\oplus\tau^{15}I(5)$	–	Ⓐ
85	$((7,5),(4,4,4),(1,2,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(1)\oplus\tau^{15}I(5)$	–	Ⓐ
86	$((7,5),(4,4,4),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20}I(1)$	–	Ⓐ
87	$((7,7),(5,4,5),(1,1,2,3,2,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-33}P(3)$	–	Ⓐ
88	$((6,6),(5,4,3),(1,2,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{7}I(4)\oplus\tau^{8}I(8)$	–	Ⓐ
89	$((6,6),(5,4,3),(1,2,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(9)\oplus\tau^{10}I(6)$	–	Ⓐ
90	$((6,6),(5,4,3),(1,2,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(4)\oplus\tau^{10}I(6)$	–	Ⓐ
91	$((6,6),(5,4,3),(1,2,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(1)\oplus\tau^{14}I(6)$	–	Ⓐ
92	$((6,6),(5,4,3),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(4)$	–	Ⓐ
93	$((6,6),(3,4,5),(2,2,2,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{8}I(2)\oplus\tau^{10}I(5)$	–	Ⓐ
94	$((6,6),(5,3,4),(1,2,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(3)\oplus\tau^{10}I(8)$	–	Ⓐ

No.	dim. vector	quiver	construction	morphisms	constr. method
95	$((6,6),(5,3,4),(1,2,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(1) \oplus \tau^{12}I(3)$	–	Ⓐ
96	$((6,6),(5,3,4),(1,2,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(3) \oplus \tau^{15}I(6)$	–	Ⓐ
97	$((6,6),(5,3,4),(1,2,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20}I(3) \oplus \tau^{20}I(5)$	–	Ⓐ
98	$((6,6),(5,3,4),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{30}I(3)$	–	Ⓐ
99	$((7,7),(4,5,5),(1,1,3,2,2,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{-34}P(3)$	–	Ⓐ
100	$((7,7),(5,4,5),(1,2,1,3,2,2,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{-58}P(5) \oplus ((1,1),(1,0,1),(1,0,0,0,0,1))$	–	Ⓐ
101	$((6,7),(4,5,4),(1,2,2,1,3,2,2))$	$\tilde{\mathbb{E}}_8$	$((3,3),(2,2,2),(1,1,1,1,1,1)) \oplus ((2,3),(2,2,1),(1,1,0,1,1,1)) \oplus ((1,1),(0,1,1),(1,0,0,1,0,0))$	–	Ⓖ
102	$((7,6),(4,4,5),(1,2,2,2,1,3,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{52}I(5) \oplus ((2,1),(1,1,1),(1,0,0,0,1,1))$	–	Ⓐ
103	$((6,7),(5,4,4),(1,2,2,2,2,1,3))$	$\tilde{\mathbb{E}}_8$	$\tau^{28}I(3) \oplus ((1,1),(1,0,1),(1,0,0,0,0,1))$	–	Ⓐ
104	$((6,6),(3,5,4),(1,2,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{24}I(5) \oplus \tau^{21}I(6)$	–	Ⓐ
105	$((6,6),(4,5,3),(1,2,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(1) \oplus \tau^{9}I(8)$	–	Ⓐ
106	$((6,6),(4,5,3),(1,2,2,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(4) \oplus \tau^{11}I(6)$	–	Ⓐ
107	$((6,6),(4,5,3),(1,2,2,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(6) \oplus \tau^{14}I(7)$	–	Ⓐ
108	$((6,6),(4,5,3),(1,2,2,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{16}I(3) \oplus \tau^{15}I(6)$	–	Ⓐ
109	$((6,6),(4,5,3),(1,2,1,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{24}I(5) \oplus \tau^{20}I(6)$	–	Ⓐ
110	$((6,6),(4,5,3),(1,1,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{31}I(3)$	–	Ⓐ
111	$((7,6),(5,5,3),(1,2,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{9}I(7) \oplus \tau^{8}I(9)$	–	Ⓐ
112	$((7,6),(5,3,5),(1,2,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(3) \oplus \tau^{17}I(3)$	–	Ⓐ
113	$((7,6),(3,5,5),(2,2,2,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{14}I(3) \oplus \tau^{12}I(5) \oplus \tau^{15}I(6)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	Ⓒ

No.	dim. vector	quiver	construction	morphisms	constr. method
114	$((6,7),(5,5,3),(1,2,2,2,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{13}I(3) \oplus \tau^{17}I(7)$	–	(A)
115	$((6,6),(4,4,4),(1,3,1,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{30}I(5) \oplus \tau^{15}I(6)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
116	$((6,6),(4,4,4),(1,2,3,1,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{16}I(5) \oplus \tau^{15}I(7)$	–	(A)
117	$((6,6),(4,4,4),(1,1,3,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{31}I(6)$	–	(A)
118	$((6,6),(4,4,4),(1,1,3,1,2,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{61}I(5)$	–	(A)
119	$((6,6),(4,4,4),(1,2,2,3,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{12}I(4) \oplus \tau^{12}I(5)$	–	(A)
120	$((6,6),(4,4,4),(1,2,1,3,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{18}I(5) \oplus \tau^{22}I(3)$	–	(A)
121	$((6,6),(4,4,4),(1,2,1,3,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{30}I(5) \oplus \tau^{32}I(5)$	–	(A)
122	$((6,6),(4,4,4),(1,1,2,3,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{32}I(6)$	–	(A)
123	$((6,6),(4,4,4),(1,1,2,3,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{62}I(5)$	–	(A)
124	$((6,6),(4,4,4),(1,2,2,2,3,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(4) \oplus \tau^{12}I(6)$	–	(A)
125	$((6,6),(4,4,4),(1,2,2,1,3,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(6) \oplus \tau^{18}I(6)$	–	(A)
126	$((6,6),(4,4,4),(1,2,2,1,3,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20}I(3) \oplus \tau^{21}I(5)$	–	(A)
127	$((6,6),(4,4,4),(1,2,1,2,3,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{24}I(5) \oplus \tau^{20}I(3)$	–	(A)
128	$((6,6),(4,4,4),(1,2,1,2,3,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{30}I(5) \oplus \tau^{33}I(5)$	–	(A)
129	$((6,6),(4,4,4),(1,1,2,2,3,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{33}I(6)$	–	(A)
130	$((6,6),(4,4,4),(1,1,2,2,3,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{63}I(5)$	–	(A)
131	$((6,6),(4,4,4),(1,3,2,1,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15}I(6) \oplus \tau^{10}I(7)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
132	$((6,6),(4,4,4),(1,3,1,2,1,2,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{20}I(5) \oplus \tau^{15}I(6) \oplus \tau^{12}I(5)$	–	(A)
133	$((6,6),(4,4,4),(1,2,3,2,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10}I(8) \oplus \tau^{11}I(6)$	–	(A)
134	$((6,6),(4,4,4),(1,2,3,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{11}I(7) \oplus \tau^{15}I(6)$	–	(A)
135	$((6,6),(4,4,4),(1,1,3,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{21}I(7)$	–	(A)

No.	dim. vector	quiver	construction	morphisms	constr. method
136	$((6,6),(4,4,4),(1,2,2,3,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^8 I(2) \oplus \tau^{12} I(5)$	–	(A)
137	$((6,6),(4,4,4),(1,2,2,3,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10} I(9) \oplus \tau^{12} I(5)$	–	(A)
138	$((6,6),(4,4,4),(1,2,1,3,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{15} I(1) \oplus \tau^{18} I(5)$	–	(A)
139	$((6,6),(4,4,4),(1,1,2,3,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{22} I(7)$	–	(A)
140	$((6,6),(4,4,4),(1,3,2,2,1,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10} I(7) \oplus \tau^{7} I(4)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
141	$((6,6),(4,4,4),(1,3,2,1,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{10} I(7) \oplus \tau^{8} I(3) \oplus \tau^{15} I(5)$	–	(A)
142	$((6,6),(4,4,4),(1,3,1,2,2,1,2))$	$\tilde{\mathbb{E}}_8$	$\tau^{15} I(5) \oplus \tau^{11} I(1) \oplus \tau^{12} I(5)$	–	(A)
143	$((6,6),(4,4,4),(1,2,3,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^8 I(3) \oplus \tau^7 I(2)$	–	(A)
144	$((6,6),(4,4,4),(1,2,3,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^8 I(4) \oplus \tau^{10} I(7)$	–	(A)
145	$((6,6),(4,4,4),(1,2,3,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{11} I(3) \oplus \tau^{10} I(8)$	–	(A)
146	$((6,6),(4,4,4),(1,1,3,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{16} I(8)$	–	(A)
147	$((6,6),(4,4,4),(1,3,2,2,2,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^7 I(4) \oplus \tau^6 I(9)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(B)
148	$((6,6),(4,4,4),(1,3,2,2,1,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^7 I(4) \oplus \tau^7 I(1) \oplus \tau^{10} I(5)$	–	(A)
149	$((6,6),(4,4,4),(1,3,2,1,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^9 I(8) \oplus \tau^{10} I(5) \oplus \tau^8 I(3)$	–	(A)
150	$((6,6),(4,4,4),(1,3,1,2,2,2,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{10} I(5) \oplus \tau^{10} I(4) \oplus \tau^{12} I(5)$	–	(A)
151	$((6,6),(4,4,4),(1,2,2,2,1,3,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{15} I(1) \oplus \tau^{16} I(5)$	–	(A)
152	$((6,6),(4,4,4),(1,2,2,1,2,3,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{16} I(5) \oplus \tau^{24} I(6)$	–	(A)
153	$((6,6),(4,4,4),(1,2,1,2,2,3,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{30} I(5) \oplus \tau^{34} I(5)$	–	(A)
154	$((6,6),(4,4,4),(1,1,2,2,2,3,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{64} I(5)$	–	(A)
155	$((4,4),(2,4,2),(1,1,1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^9 I(8) \oplus \tau^6 I(5)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(A) and (β)
156	$((3,5),(3,2,3),(1,1,1,1,1,1,1,1))$	$\tilde{\mathbb{E}}_8$	$\tau^{11} I(1) \oplus \tau^6 I(5)$	$\hom(\mathbf{d_1},\mathbf{d_2}) = 1$	(A) and (β)

No.	dim. vector	quiver	construction	morphisms	constr. method
157	$((3,4),(3,2,2),(0,1,1,1,1,1,1,1))$	$T_{2,3,7}$	$((3,4),(3,2,2),(1,1,1,1,1,1,1))$ (see case 2.)		
158	$((4,3),(2,2,3),(1,1,1,1,1,1,1,0))$	$T_{2,3,7}$	$((4,3),(2,2,3),(1,1,1,1,1,1,1))$ (see case 2.)		
159	$((4,3),(2,2,3),(0,1,1,1,1,1,1,1))$	$T_{2,3,7}$	$((4,3),(2,2,3),(1,1,1,1,1,1,1))$ (see case 2.)		
160	$((4,3),(2,3,2),(0,1,1,1,1,1,1,1))$	$T_{2,3,7}$	$((4,3),(2,3,2),(1,1,1,1,1,1,1))$ (see case 2.)		
161	$((3,4),(2,3,2),(1,1,1,1,1,1,1,0))$	$T_{2,3,7}$	$((3,4),(2,3,2),(1,1,1,1,1,1,1))$ (see case 2.)		
162	$((3,4),(3,2,2),(1,1,1,1,1,1,1,0))$	$T_{2,3,7}$	$((3,4),(3,2,2),(1,1,1,1,1,1,1))$ (see case 2.)		
163	$((5,3),(2,3,3),(0,1,1,1,1,1,1,1))$	$T_{2,3,7}$	$((5,3),(2,3,3),(1,1,1,1,1,1,1))$ (see case 21.)		
164	$((3,5),(3,3,2),(1,1,1,1,1,1,1,0))$	$T_{2,3,7}$	$((3,5),(3,3,2),(1,1,1,1,1,1,1))$ (see case 21.)		

Appendix

By using Lemma 4.2 one can easily prove the results of the positiveness (resp. non negativeness) of the Tits form in the finite (resp. tame star) cases.

A Proof of the positiveness of the Tits form in the finite cases

$\mathbb{A}_n$: $k = 1$:

$$\overline{q}(\mathbf{a_1}) \geq \frac{1}{2}\left(\frac{n^2}{p_1} + (2-1)\cdot n^2\right) = \frac{(p_1+1)n^2}{2p_1} > 0$$

$k = 2$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}) \geq \frac{1}{2}\left(\frac{n^2}{p_1} + \frac{n^2}{p_2} + (2-2)\cdot n^2\right) = \frac{(p_1+p_2)n^2}{2p_1p_2} > 0$$

$\mathbb{D}_n$: $k = 3$, $p_1 = p_2 = 2$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{p_3} + (2-3)n^2\right) = \frac{n^2}{2p_3} > 0$$

$\mathbb{E}_6$: $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 3$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{3} + \frac{n^2}{3} + (2-3)n^2\right) = \frac{n^2}{12} > 0$$

$\mathbb{E}_7$: $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 4$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{3} + \frac{n^2}{4} + (2-3)n^2\right) = \frac{n^2}{24} > 0$$

$\mathbb{E}_8$: $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 4$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{3} + \frac{n^2}{5} + (2-3)n^2\right) = \frac{n^2}{60} > 0$$

B Proof of the non negativeness of the Tits form in the tame cases (only for stars)

$\tilde{\mathbb{D}}_4$: $k = 4$, $p_1 = p_2 = p_3 = p_4 = 2$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{2} + \frac{n^2}{2} + (2-4)\cdot n^2\right) = 0$$

$\tilde{\mathbb{E}}_6$: $k = 3$, $p_1 = p_2 = p_3 = 3$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{3} + \frac{n^2}{3} + \frac{n^2}{3} + (2-3)\cdot n^2\right) = 0$$

$\tilde{\mathbb{E}}_7$: $k = 3$, $p_1 = 2$, $p_2 = p_3 = 4$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{4} + (2-3)\cdot n^2\right) = 0$$

$\tilde{\mathbb{E}}_8$: $k = 3$, $p_1 = 2$, $p_2 = 3$, $p_3 = 6$:

$$\overline{q}(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}) \geq \frac{1}{2}\left(\frac{n^2}{2} + \frac{n^2}{3} + \frac{n^2}{6} + (2-3)\cdot n^2\right) = 0$$

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