The tame dimension vectors for trees

ICRA XIII

Angela Holtmann

The tame dimension vectors for trees

Thm.
Let $\mathbf{d}$ be a dimension vector of a tree.
Then $\mathbf{d}$ is tame if and only if
$q(\mathbf{d}) = 0$ and $q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' \leq \mathbf{d}$.

- *tree*: quiver without cycles, $Q = (Q_0, Q_1, s, t)$
- $\mathbf{d}' \leq \mathbf{d} :\Leftrightarrow d_i' \leq d_i$ for all $i \in Q_0$
- q: Tits form of Q, $q(\mathbf{d}) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}$
- $\mathbf{d}$ *tame* if
  - there is a one parameter family of indecomposable representations and
  - for all $\mathbf{d}' \leq \mathbf{d}$ the families of indecomposable representations depend on at most one parameter

Infinitely many tame dimension vectors

Note that there are *infinitely many* tame dimension vectors (if we consider *all* possible trees at a time):

Take for example one tame dimension vector $\mathbf{d_1} - 1$, e. g. a tame root of a Euclidean quiver, and glue it together with a root of quiver of shape $1 - \mathbf{d_2} - 1$ for which there is (up to isomorphism) only one indecomposable representation, e. g. a root of a Dynkin quiver.

$$\mathbf{d_1} - 1 - \mathbf{d_2} - 1$$

Then this is again tame.

Can continue in this way:

$$\mathbf{d_1} - 1 - \mathbf{d_2} - 1 - \mathbf{d_3} - 1$$

etc.

Combinatorially tame (and combinatorially finite) dimension vectors

Def.

- $\mathbf{d}$ is *combinatorially tame* if $q(\mathbf{d}) = 0$ and $q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' \leq \mathbf{d}$.
- $\mathbf{d}$ is *combinatorially finite* if $q(\mathbf{d}) = 1$ and $q(\mathbf{d}') \geq 1$ for all $\mathbf{d}' \leq \mathbf{d}$.

Do the same as for tame dimension vectors:

Take a *combinatorially tame* dimension vector $\mathbf{d_1} - 1$ and glue it together with *combinatorially finite* dimension vector of shape $1 - \mathbf{d_2} - 1$.

$$\mathbf{d_1} - 1 - \mathbf{d_2} - 1$$

Then this is again combinatorially tame.

Combinatorially tame dimension vectors

Have

$$\mathbf{d_1} - 1 - \mathbf{d_2} - 1,$$

again combinatorially tame.

Can continue in this way:

$$\mathbf{d_1} - 1 - \mathbf{d_2} - 1 - \mathbf{d_3} - 1$$

etc.

Building blocks for (combinatorially) tame dimension vectors

One can show:

Lemma
All (combinatorially) tame dimension vectors arise in this way.

Start with exactly one (combinatorially) tame dimension vector and glue it together with several (combinatorially) finite dimension vectors at vertices with dimension 1.
(several $= 0, 1, 2, 3, \ldots$)

Rmk. The (combinatorially) finite dimension vectors were (implicitly) classified by P. Magyar, J. Weyman and A. Zelevinsky in 1999.

Building blocks for (combinatorially) tame dimension vectors

Aim: Classify the corresponding building blocks!

Can classify the set of minimal dimension vectors which are *not* (combinatorially) tame (=: *(combinatorially) hypercritical*).

Thm.
Let $\mathbf{d}$ be a dimension vector of a tree.
Then $\mathbf{d}$ is hypercritical if and only if
$q(\mathbf{d}) < 0$ and $q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' < \mathbf{d}$, i. e. if it is combinatorially hypercritical.

This is the case if and only if $\mathbf{d}$ is contained in a finite *explicitly given* list.

Building blocks for combinatorially tame dimension vectors

Aim: Find all dimension vectors $\mathbf{d}$ which are *incomparable* with or *smaller* than the ones in the (combinatorially) hypercritical list which have Tits form $q(\mathbf{d}) = 0$.

Can show:

- It is *not* possible to have combinatorially tame building blocks with more than two branching vertices in the underlying tree.
- There are (basically) only finitely many dimension vectors with one or two branching vertices which are incomparable with or smaller than the combinatorially hypercritical ones and have Tits form zero. (And it is possible to list them *explicitly*.)
- All these dimension vectors are *roots*.

Building blocks for (combinatorially) tame dimension vectors

"(Basically) only finitely many dimension vectors..." – What does that mean?

Neighbouring vertices in the quiver shall have different dimensions.

So: Delete all double dimension entries in the dimension vectors!
($=$: *reduced* dimension vectors)

Rmk.
This does have *no* influence on the property whether the dimension vector is a root or not.
This does have *no* influence on the Tits form of the dimension vector.

Equivalence of tame and combinatorially tame dimension vectors

- Can restrict ourselves to building blocks of (combinatorially) tame dimension vectors.
- All combinatorially tame building blocks are tame.
  - Reduce the dimension vector. The reduced dimension vector is still combinatorially tame, therefore a root. So the original one is also a root.
  - The condition on the Tits forms for the smaller dimension vectors ($q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' \leq \mathbf{d}$) implies that there are no m-parameter families of indecomposable representations for them with $m \geq 2$. (Use Kac's Theorem.)

Equivalence of tame and combinatorially tame dimension vectors

- All tame building blocks are also combinatorially tame.
  - Clearly, they are not bigger than any hypercritical dimension vector, i. e. not bigger than any combinatorially hypercritical dimension vector (by the second Thm.).
  - So the Tits form of the dimension vector and all smaller ones has to be non-negative ($q(\mathbf{d}') \geq 0$ for all $\mathbf{d}' \leq \mathbf{d}$).
  - The existence of a one parameter family of indecomposable representations gives that the Tits form for the building block itself is exactly zero ($q(\mathbf{d}) = 0$). (Use Kac's Theorem.)