1. Let (X, d) be a metric space. Fix a point $p \in X$. Show that the map

$$d_p(x, y) := \begin{cases} 0, & \text{if } x = y \\ d(x, p) + d(p, y), & \text{if } x \neq y \end{cases}$$

for every $x, y \in X$, is a metric on X.

2. Let (X, d) be a metric space. Show that

$$|\, d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$$

for every $x, y, z, w \in X$.

3. Let (X, d) be a metric space and $A \subset X$. Show that the following are equivalent:

(1) A is a closed subset of X.

(2) If $\{x_n\} \subset A$ and $x_n \to x$, then $x \in A$.

(3) $A = \{x \in X \,:\, d(x, A) = 0\}$, where $d(x, A) = \inf\{d(x, y) \,:\, y \in A\}$.

4. (**The p-adic metric on $\mathbb{Z}$**) Let $X = \mathbb{Z}$ and $p \in \mathbb{N}$ be a prime number. The fundamental theorem of arithmetic states that every positive integer larger than 1 can be written as a product of one or more primes in a unique way, i.e. unique except for the order. Let $\nu_p(x)$ denotes the exponent of p in the analysis of $|x|$ as a product of prime numbers. Show that the map

$$d(x, y) := \begin{cases} 0, & \text{if } x = y \\ p^{-\nu_p(x-y)}, & \text{if } x \neq y \end{cases}$$

for every $x, y \in X$, defines a metric on X. This metric is called the p*-adic metric* on $\mathbb{Z}$.

Hint: To prove the triangle inequality show the stronger property:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \leq d(x, z) + d(z, y).$$

5. Let (X, d) be a metric space and A, B be two closed subsets of X which are disjoint, i.e. $A \cap B = \emptyset$. Show that there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$.

Hint: Use exercise 3 and the fact that if D is a subset of X then the function $f : X \to \mathbb{R}$ with $f(x) = \inf\{d(x, y) \,:\, y \in D\}$ is continuous.