6. Let (X, d) be a metric space and A, B be two closed subsets of X which are disjoint, i.e. $A \cap B = \emptyset$. Show that there exist two disjoint open subsets of X, let say U and V such that $A \subseteq U$ and $B \subseteq V$.

Hint: Use exercise 5.

7. Let (X, d), (Y, p) be two metric spaces. A map $f : X \to Y$ is called *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $p(f(x), f(y)) < \epsilon$, for every $x, y \in X$. Show that if f is uniformly continuous and $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X then $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

8. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be a continuous function ($\mathbb{R}$ is endowed with the usual metric). Show that the map

$$p(x, y) = d(x, y) + |f(x) - f(y)|$$

for every $x, y \in X$, defines a metric on X. Moreover $d(x_n, x) \to 0$ if and only if $p(x_n, x) \to 0$. Show, also, that if f is uniformly continuous then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, p).

Hint: Use exercise 7.

9. Let (X, d) be a metric space and $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be two Cauchy sequences in X. Show that there exists a real number l such that $d(x_n, y_n) \to l$.

Hint: Use exercise 2.

10. (**Fredholm's second (linear) integral equation**) Let $G : [0, 1] \to \mathbb{R}$ and $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be two continuous functions and let $|\lambda| < \frac{1}{M}$ where

$$M = \sup\{|K(x, y)| \,:\, x, y \in [0, 1]\}.$$

Let $C[0, 1]$ be the space of all continuous functions $f : [0, 1] \to \mathbb{R}$ endowed with the supremum metric (i.e. $d(f, g) = \sup\{\, |f(x) - g(x)| \,:\, x \in X\,\}$). With this metric $C[0, 1]$ is complete. Show that the map $T : C[0, 1] \to C[0, 1]$ with

$$T(f) = \lambda \int_0^1 K(x, y) f(y) dy + G(x)$$

is a contraction (i.e. there exists $0 < c < 1$ such that $d(T(f), T(g)) \leq c \cdot d(f, g)$, for every $f, g \in C[0, 1]$), hence T has a fixed point in $C[0, 1]$.