11. (**Zariski topology**) Let R be a commutative ring with identity. Recall that a proper ideal $\mathfrak{p}$ of R (i.e. $\mathfrak{p} \subsetneqq R$) is called *prime* if $ab \in \mathfrak{p}$ implies that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. For example, let $R = \mathbb{Z}$; then every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z} := \{nm : m \in \mathbb{Z}\}$ with $n \in \mathbb{N} \cup \{0\}$, and it is a prime ideal of $\mathbb{Z}$ if and only if $n = 0$ or a prime number. Let

$$\mathrm{Spec}(R) := \{\mathfrak{p} : \mathfrak{p} \text{ is a prime ideal of } R\}.$$

We now define a topology on $\mathrm{Spec}(R)$ by declaring certain subsets of $\mathrm{Spec}(R)$ as closed.

For any ideal I of R, let

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) \ : \ I \subseteq \mathfrak{p}\}$$

Show that, in particular $\emptyset = V(R)$ and $\mathrm{Spec}(R) = V(\{0\})$. Let $\mathcal{I}$ be a family of ideals of R, and let $\sum\{I : I \in \mathcal{I}\}$ be the set of all finite sums $\sum_{j=1}^{n} a_j$ such that there are $I_1, \dots, I_n \in \mathcal{I}$ with $a_j \in I_j$ for every $j = 1, \dots, n$. *Show that* $\sum\{I : I \in \mathcal{I}\}$ is again an ideal of R and

$$\bigcap\{V(I) : I \in \mathcal{I}\} = V\left(\sum\{I : I \in \mathcal{I}\}\right).$$

Let I_1 and I_2 be ideals of R, and let I be the ideal of R generated by the set $\{ab : a \in I_1, b \in I_2\}$ (i.e. the smallest ideal of R that contains the set $\{ab : a \in I_1, b \in I_2\}$). *Show that*, I consists precisely of those elements of R that are of the form $\sum_{j=1}^{n} a_j b_j$ with $a_1, \dots, a_n \in I_1$ and $b_1, \dots, b_n \in I_2$. Moreover, *show that* $V(I_1) \cup V(I_2) = V(I)$.

The sets of the form $V(I)$, where I is an ideal of R, are thus the closed sets of a topology on $\mathrm{Spec}(R)$. This topology is called the *Zariski topology*.

12. Let $X = \{a, b, c\}$. Find all possible topologies on X.

Hint: There exist exactly 29 topologies on X! Describe them in a short way!!

13. (**The lattice of topologies**) Let $X \neq \emptyset$. Show that

(1) The intersection of any family of topologies on X is a topology on X. [*Note*: Intersect the topologies, not the sets which are elements of the topologies!]

(2) The union of two topologies on X need not be a topology on X (*Hint*: Use exercise 12). Show, also, that for any family of topologies on X, there is a smallest topology which contains all of them.

14. (G_δ **and** F_σ **sets**) A subset of a topological space X is called G_δ if it a countable intersection of open sets and is called F_σ if it is a countable union of closed sets. Show that

(1) The complement of a G_δ set is an F_σ set and vice versa.

(2) An F_σ set can be written as the union of an *increasing* sequence $F_1 \subseteq F_2 \subseteq \ldots$ of closed sets. Hence, by item (1), a G_δ set can be written as the intersection of a decreasing sequence of open sets.

(3) A closed set in a metric space (X, d) is a G_δ set (*Hint*: If A is a closed set let $A_n := \{x : d(x, A) < \frac{1}{n}\}$ and use exercise 3). Hence, an open set in a metric space is F_σ.

15. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on a non-empty set X with bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Show that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ (i.e. $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$) if and only if for every $B_1 \in \mathcal{B}_1$ and $x \in B_1$ there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.