41. Let X be a compact topological space and $\{V_n : n \in \mathbb{N}\}$ be a decreasing sequence (i.e. $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$) of *closed* neighborhoods of $x \in X$ such that $\bigcap_{n=1}^{+\infty} V_n = \{x\}$. Show that $\{V_n : n \in \mathbb{N}\}$ is a neighborhood base of x.

42. Let X be a compact topological space, V be an *open* subset of X and $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence (i.e. $F_{n+1} \subseteq F_n$ for every $n \in \mathbb{N}$) of closed subsets of X such that $\bigcap_{n=1}^{+\infty} F_n \subseteq V$. Show that there exists $n_0 \in \mathbb{N}$ such that $F_{n_0} \subseteq V$.

43. Let X be a locally compact space and A be a compact subset of X. Show that for every open set $W \subseteq X$ such that $A \subseteq W$ there exists an open set $V \subseteq X$ such that $\overline{V}$ is compact and

$$A \subseteq V \subseteq \overline{V} \subseteq W.$$

44. (σ-**compact spaces**) Let X be a locally compact, second countable space. Show that X is a σ-compact space, that is there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of X such that

$$\overline{V_n} \text{ is compact and } \overline{V_n} \subseteq V_{n+1}, \text{ for every } n \in \mathbb{N}, \text{ with } X = \bigcup_{n=1}^{+\infty} V_n.$$

Hint: Use Lindelöf's lemma.

45. Let X be a topological spaces endowed with two topologies $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Show that if $(X, \mathcal{T}_2)$ is connected then $(X, \mathcal{T}_1)$ is also connected.