Wintersemester 2007-2008 A. Manoussos

Topologie I Feb 12, 2008

KLAUSUR

1. Let X be a topological space and $A \subseteq X$ be a closed set. Show that

$$A^o = (\,\overline{A^o}\,)^o.$$

2. Let X be a topological space and $A, B \subseteq X$ such that $\partial A \cap \partial B = \emptyset$. Show that $(\,A \cup B\,)^o = A^o \cup B^o$.

3. Let X, Y be two topological spaces and $f : X \to Y$ be a map. Show that f is continuous if and only if $\partial f^{-1}(B) \subseteq f^{-1}(\partial B)$ for every $B \subseteq Y$.

4. Let X, Y be two topological spaces and $f : X \to Y$ be a continuous, open surjection. Show that Y is Hausdorff if and only if the set

$$\{\,(x_1, x_2) \,:\, x_1, x_2 \in X \text{ such that } f(x_1) = f(x_2)\,\}$$

is a closed subset of $X \times X$.

5. Let (X, d) be a metric space and $A \subseteq X$. Let $diam(A) := \sup\{d(x, y) \,:\, x, y \in A\}$. Show that the following are equivalent:

(a) (X, d) is complete.

(b) Each decreasing sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ of closed subsets of X with $diam(F_n) \to 0$ has non-empty intersection (i.e. $\bigcap_{n=1}^{+\infty} F_n \neq \emptyset$).

6. Let $\{(X_i, \mathcal{T}_i) \,:\, i \in I\}$ be a family of topological spaces and $A_i \subseteq X_i$, for every $i \in I$. Show that

$$\prod \overline{A_i} = \overline{\prod A_i}.$$

7. Let X be a locally compact space and A be a compact subset of X. Show that for every open set $W \subseteq X$ such that $A \subseteq W$ there exists an open set $V \subseteq X$ such that $\overline{V}$ is compact and

$$A \subseteq V \subseteq \overline{V} \subseteq W.$$

8. Let (X, d) be a metric space and K be a non-empty compact subset of X. Let $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ be a sequence such that for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $m \geq n$ such that $x_m \in \bigcup_{x \in K} S(x, \epsilon)$, where $S(x, \epsilon)$ denotes the ball centered at $x \in K$ with radius $\epsilon > 0$. Show that there exist a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ and a point $x \in K$ such that $x_{n_k} \to x$.

9. Show that the closed unit interval $[0, 1]$ is not homeomorphic to the circle $S^1 := \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$ (in both $[0, 1]$ and S^1 we have the usual topologies).

10. Let X be a topological space and $x \in X$. Let

$$S(x) := \bigcap \{A \subseteq X \ : \ \text{such that } x \in A \text{ and } A \text{ is open and closed in } X\}.$$

Show that the set $S(x)$ contains the connected component $C(x)$ of x in X.

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