MINIMAL FLOWS ON MULTIPUNCTURED SURFACES OF INFINITE TYPE

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A multipunctured surface is an open 2-manifold obtained from a closed 2-manifold M by removing a nonempty, closed and totally disconnected set F. The multipunctured surface $M\backslash F$ will be called of finite type if F is finite, and of infinite type otherwise. In this short note we study the behaviour of the orbits of a given minimal flow on a multipunctured surface $M\backslash F$ at infinity, that is, near the points of F. Any flow on $M\backslash F$ has an extension to a flow on M that fixes F pointwise [**1**, Satz 2.3]. In view of the work of C. Gutierrez [**5**], there is no loss of generality if we assume that everything is smooth.

Examples of minimal flows on multipunctured surfaces of finite type are given in [**4**] and [**7**], and from these one can obtain minimal flows on multipunctured surfaces of infinite type, by multiplying the infinitesimal generator with a suitable smooth function. The aim of the present note is to show that every minimal flow on a multipunctured surface of infinite type is obtained in this way; see Theorem 4 below.

In the finite type case, every point of F has to be a possibly degenerate saddle, and it follows from the Poincaré–Hopf Index Theorem that the number of orbits in $M\backslash F$ with empty positive (negative) limit set in $M\backslash F$ is equal to $|F|-\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. So if F is a finite subset of the torus T^2, then every minimal flow on $T^2\backslash F$ is constructed (modulo topological equivalence) from an irrational vector field by multiplication with a smooth function which vanishes exactly on F. Using the examples in [**4**], it is not hard to see that if $\chi(M) < 0$, then all possible cases of behaviour at infinity of minimal flows on multipunctured surfaces of finite type $M\backslash F$ can occur.

We turn now to the study of minimal flows on multipunctured surfaces of infinite type. We shall use the machinery of isolated invariant sets and isolating blocks developed by C. Conley and R. Easton in [**3**]. Recall that a compact invariant set A of a given smooth flow on a manifold M is called isolated if it is the maximal invariant set in some of its compact neighbourhoods. An isolating block N is a compact submanifold with boundary of M, of the same dimension, such that the boundary ∂N of N is the union of two submanifolds n^+, n^- of codimension 1 in M, with common boundary τ such that the flow is transverse into N on $n^+\backslash\tau$, transverse out of N on $n^-\backslash\tau$ and externally tangent to N on τ. It is proved in [**3**] that every neighbourhood of an isolated invariant set A contains an isolating block N having A in its interior, and A is the maximal invariant set in N.

If $M\backslash F$ is a multipunctured surface carrying a minimal flow, then F is an isolated invariant set with respect to the extended flow on M. Moreover, every point of F has arbitrarily small compact neighbourhoods in F which are isolated invariant sets, because F is totally disconnected. By the classification of the noncompact surfaces

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given in [6], we may always assume that F is contained in a disc in M. It follows that F is contained in the interior of an isolating block whose connected components are closed discs with finitely many holes.

In the sequel we always assume that $M \setminus F$ is a multipunctured surface of infinite type carrying a minimal flow. All our notation refers to the extended flow on M. We denote by tx the translation of the point x along its orbit $C(x)$ in time t, by $C^+(x)$ its positive semiorbit, and by $C^-(x)$ its negative semiorbit. The positive limit set of x is denoted by $L^+(x)$ and the negative limit set by $L^-(x)$. We recall that two flows on M are called topologically equivalent if there exists a homeomorphism $h: M \to M$ which sends orbits of one flow onto orbits of the other, preserving their orientations.

LEMMA 1. *For every $e \in F$, there exist points $x, y \in M \setminus F$ such that*

$$L^+(x) = L^-(y) = \{e\}.$$

Proof. Since F is closed and totally disconnected, the point e has a neighbourhood basis $\{V_k : k \in \mathbb{N}\}$ in M consisting of closed discs such that $\partial V_k \subset M \setminus F$ for all $k \in \mathbb{N}$. The set $F \cap V_k$ is a nonempty isolated invariant set and is not positively or negatively stable, because the flow on $M \setminus F$ is minimal. According to [2, Chapter VI, Theorem 1.1], there exist points $x_k, y_k \in M \setminus (F \cap V_k)$ such that $L^+(x_k)$ and $L^-(y_k)$ are subsets of $F \cap V_k$. Since F is totally disconnected, $L^+(x_k) = \{e_k\}$ and $L^-(y_k) = \{e'_k\}$ for some $e_k, e'_k \in F \cap V_k$. Obviously, the sequences $\{e_k : k \in \mathbb{N}\}$ and $\{e'_k : k \in \mathbb{N}\}$ converge to e. Let N be an isolating block for F in M, with $N \neq M$ and $\alpha^{\pm} = \{x \in \partial N : C^{\pm}(x) \subset N\}$. Then $\alpha^{\pm}$ are compact, and clearly we may assume that $x_k \in \alpha^+$ and $y_k \in \alpha^-$. So we may further assume that the sequences $\{x_k : k \in \mathbb{N}\}$ and $\{y_k : k \in \mathbb{N}\}$ have limits $x \in \alpha^+$ and $y \in \alpha^-$, respectively. We shall prove that $L^+(x) = \{e\}$. It is similarly proved that $L^-(y) = \{e\}$. We shall assume that $L^+(x) \neq \{e\}$, and arrive at a contradiction. Since $x \in \alpha^+$, there is then another point $e' \neq e$ in F such that $L^+(x) = \{e'\}$. From the continuity of the flow, it follows that there are times $T_k \to +\infty$ such that $T_k x_k \to e'$. Let V and W be disjoint compact neighbourhoods of e and e' respectively, whose boundaries are subsets of $M \setminus F$. We may assume that $e_k \in V$ and $T_k x_k \in W$ for all $k \in \mathbb{N}$. By connectedness of the orbits, there are $t_k \geqslant T_k$ such that $t_k x_k \in \partial W$ for every $k \in \mathbb{N}$. Since ∂W is compact, we may assume that the sequence $\{t_k x_k : k \in \mathbb{N}\}$ has a limit $z \in \partial W$. Then for every $t \in \mathbb{R}$, we have eventually $t_k \geqslant -t$ and therefore $t(t_k x_k) = (t + t_k) x_k \in C^+(x_k) \subset N$. It follows that $C(z) \subset N$, and at the same time $z \in M \setminus F$. This contradicts the minimality of the flow in $M \setminus F$.

THEOREM 2. *There exists a (possibly empty) finite set $K \subset F$ such that for every point $e \in F \setminus K$, there exist unique disjoint orbits $C(x)$ and $C(y)$ in $M \setminus F$ with $L^+(x) = L^-(y) = \{e\}$.*

Proof. Suppose that there is no such finite set K. By Lemma 1 and the compactness of F, there is then a sequence $\{e_k : k \in \mathbb{N}\}$ of mutually different points of F converging to a point $e \in F$, such that for every $k \in \mathbb{N}$ there exist points $x_k, x'_k \in M \setminus F$ with $C(x_k) \neq C(x'_k)$, and such that $L^+(x_k) = L^+(x'_k) = \{e_k\}$. Let N_0 be an isolating block for F contained in a disc neighbourhood of F in M. The connected component N of N_0 which contains e is also an isolating block. All the notation used below refers to N. We may assume that $e_k \in N$ and $x_k, x'_k \in \alpha^+$ for all $k \in \mathbb{N}$, where α^+ is defined as in the proof of Lemma 1. The proof of Lemma 1 shows that we may further assume that the sequences $\{x_k : k \in \mathbb{N}\}$ and $\{x'_k : k \in \mathbb{N}\}$ have limits x and x' in α^+, respectively, such that $L^+(x) = L^+(x') = \{e\}$. Let I be the connected component of n^+ which contains x.

If $x' \in I$, then x and x' are the endpoints of a subinterval J of I which is transverse to the flow. The set $\overline{C^+(x)} \cup \overline{C^+(x')} \cup J$ is a simple closed curve and bounds a disc D in M, by the Jordan–Schönflies Theorem, since N is contained in a disc neighbourhood of F. The disc D should be positively invariant, which contradicts the minimality of the flow in $M \backslash F$. Hence the point x' must be contained in another connected component $I' \neq I$ of n^+, and we may assume that $x_k \in I$ and $x'_k \in I'$ for all $k \in \mathbb{N}$. Let J_k be the subinterval of I with endpoints x_k and x, and J'_k the subinterval of I' with endpoints x'_k and x'. The sets $A = \overline{C^+(x)} \cup \overline{C^+(x')}$ and $A_k = \overline{C^+(x_k)} \cup \overline{C^+(x'_k)}$ are nonintersecting arcs whose interiors are contained in the interior of N. Hence the set $A \cup J_k \cup A_k \cup J'_k$ is a simple closed curve and bounds a positively invariant disc in M. This again contradicts the minimality of the flow in $M \backslash F$ and completes the proof.

On $M \backslash K$ we can now define a family of curves as follows. If $x \in M \backslash F$ and $L^+(x) = L^-(x) = M$, we set $A_x = C(x)$. If $e \in F \backslash K$ and $C(x), C(y)$ are, according to Theorem 2, the unique disjoint orbits in $M \backslash F$ such that $L^+(x) = L^-(y) = \{e\}$, we set $A_x = A_y = A_e = \overline{C(x)} \cup \overline{C(y)}$. Then the A_z, $z \in M \backslash K$, form a family of nonintersecting oriented curves filling $M \backslash K$.

LEMMA 3. *The family of curves* $\{A_z : z \in M \backslash K\}$ *is regular.*

Proof. Only the regularity at points $e \in F \backslash K$ needs proof. Let x, y be points in $M \backslash F$ such that $L^+(x) = L^-(y) = \{e\}$. By the continuity of the flow in $M \backslash F$, it suffices to show that given any compact neighbourhood U of $\overline{C^-(y)}$ and a compact neighbourhood W of y in $M \backslash K$, there is a neighbourhood V of e such that if $z \in V$, then there is a point $z' \in A_z$ such that $[z, z'] \subset U$ and $z' \in W$, where $[z, z']$ denotes the interval with endpoints z and z' on A_z, in the obvious orientations. Suppose that $\{z_k : k \in \mathbb{N}\}$ is a sequence in U converging to e. By the definition of the curves, we may assume that $z_k \in M \backslash F$ for all $k \in \mathbb{N}$. Let now $T_k = \sup\{t \geqslant 0 : [0, t] z_k \subset U\}$. Then $T_k z_k \in \partial U$, and since ∂U is compact, we may assume that the sequence $\{T_k z_k : k \in \mathbb{N}\}$ has a limit $z_0 \in \partial U$. It follows that $T_k \to +\infty$ and $C^-(z_0) \subset U$. So $L^-(z_0) \subset F \cap U$, and it can be proved in the same way as in Lemma 1 that $L^-(z_0) = \{e\}$. Hence $C(z_0) = C(y)$, from which the conclusion follows.

Combining now Lemma 3 with the results of [**5**] and [**8**], we have the following.

THEOREM 4. *Let* $M \backslash F$ *be a multipunctured surface of infinite type carrying a minimal flow. Then there are a finite set* $K \subset F$, *a smooth vector field* ξ *on* $M \backslash K$ *with minimal flow, and a smooth function* $f : M \backslash K \to [0, 1]$ *with* $f^{-1}(0) = F \backslash K$, *such that the flow on* M *is topologically equivalent to the extension of the flow of* $f \cdot \xi$ *to* M.

COROLLARY 5. *Every minimal flow on a multipunctured torus* $T^2 \backslash F$, *of finite or infinite type, is topologically equivalent to the restriction to* $T^2 \backslash F$ *of the flow of the product of an irrational vector field by a smooth function* $f : T^2 \to [0, 1]$ *such that* $f^{-1}(0) = F$.

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