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The Jacobson Radical for Analytic Crossed Products

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We characterise the Jacobson radical of an analytic crossed product $C_0(X)\times_\phi \mathbb{Z}_+$, answering a question first raised by Arveson and Josephson in 1969. In fact, we characterise the Jacobson radical of analytic crossed products $C_0(X)\times_\phi \mathbb{Z}_+^d$. This consists of all elements whose "Fourier coefficients" vanish on the recurrent points of the dynamical system (and the first one is zero). The multidimensional version requires a variation of the notion of recurrence, taking into account the various degrees of freedom.

Key Words: semicrossed products; analytic crossed products; Jacobson radical; recurrence; wandering sets.

There is a rich interplay between operator algebras and dynamical systems, going back to the founding work of Murray and von Neumann in the 1930's. Crossed product constructions continue to provide fundamental examples of von Neumann algebras and C*-algebras. Comparatively recently, Arveson [1] in 1967 introduced a nonselfadjoint crossed product construction, called the analytic crossed product or the semicrossed product, which has the remarkable property of capturing all of the information about the dynamical system.

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The construction starts with a dynamical system (X, ϕ), i.e., a locally compact Hausdorff space X and a continuous, proper surjection ϕ: $X \to X$. Regarding the elements of $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $\sum_{n \geqslant 0} U^n f_n$, define a multiplication by requiring $fU = U(f \circ \phi)$. The analytic crossed product, $C_0(X) \times_\phi \mathbb{Z}_+$, is a suitable completion of $\ell^1(\mathbb{Z}_+, C_0(X))$; we give a detailed discussion below. Then the property mentioned above is that, subject to a mild condition on periodic points, two analytic crossed product algebras are isomorphic as complex algebras if, and only if, the underlying dynamical systems are topologically conjugate; i.e., there is a homeomorphism between the spaces that intertwines the two actions. In this generality, the result is due to Hadwin and Hoover [9, 10]—see also [20], which gives an elegant direct proof of this if the maps ϕ are homeomorphisms and extends the result to analytic crossed products by finitely many distinct commuting homeomorphisms on X, i.e., by $\mathbb{Z}_+^d$.

Arveson's original work [1] was for weakly-closed operator algebras and Arveson and Josephson in [2] gave an extension to norm closed operator algebras, including a structure theorem for bounded isomorphisms between two such algebras. Motivated by this, they asked if the analytic crossed product algebras were always semisimple (which would imply that all isomorphisms are bounded), noting that the evidence suggested a negative answer. This question stimulated considerable work on the ideal structure of analytic crossed products.

Another stimulus is the close connections between the ideal structure of C*-crossed products and dynamical systems, such as the characterisation of primitive ideals of C*-crossed products in terms of orbit closures by Effros and Hahn [5]. In this connection, we should mention Lamoureux's development of a generalisation of the primitive ideal space for various nonselfadjoint operator algebras, including analytic crossed products [12, 13].

We state our main result for the case $d = 1$. Recall that a point $x \in X$ is *recurrent* for the dynamical system (X, ϕ) if for every neighbourhood V of x, there is $n \geqslant 1$ so that $\phi^n(x) \in V$. If X is a metric space, then this is equivalent to having a sequence (n_k) tending to infinity so that $\phi^{n_k}(x)$ converges to x. Let X_r denote the recurrent points of (X, ϕ). Denoting elements of the analytic crossed product by formal series $\sum_{n \geqslant 0} U^n f_n$ we prove:

THEOREM 1. *If X is a locally compact metrisable space, then*

$$\operatorname{Rad}(C_0(X) \times_\phi \mathbb{Z}_+) = \left\{ \sum_{n \geqslant 1} U^n f_n \in C_0(X) \times_\phi \mathbb{Z}_+ : f_n|_{X_r} = 0 \textit{ for all } n \right\}.$$

Important progress towards a characterisation has been made by a number of authors. In [16], Muhly gave two sufficient conditions, one for an analytic crossed product to be semisimple and another for the Jacobson radical to be nonzero. The sufficient condition for a nonzero Jacobson radical is that the dynamical system (X, ϕ) possess a *wandering set*, i.e., an open set $V \subset X$ so that $V, \phi^{-1}(V), \phi^{-2}(V), \ldots$ are pairwise disjoint. If there are no wandering open sets, then the recurrent points are dense, so it turns out that this sufficient condition is also necessary.

Peters in [18, 19] characterised the strong radical (namely, the intersection of the maximal (modular, two-sided) ideals) and the closure of the prime radical and described much of the ideal structure for analytic crossed products arising from *free* actions of $\mathbb{Z}^+$. He also gave a sufficient condition for semisimplicity and showed that this condition is necessary and sufficient for semisimplicity of the norm dense subalgebra of polynomials in the analytic crossed product.

Most recently, Mastrangelo *et al.* [15], using powerful coordinate methods and the crucial idea from [4], characterised the Jacobson radical for analytic subalgebras of groupoid C*-algebras. For those analytic crossed products that can be coordinatised in this way (those with a free action), their characterisation is the same as ours. The asymptotic centre of the dynamical system that is used in [15] is also important to our approach.

However, we are able to dispose of the assumption of freeness (and thus our dynamical systems can have fixed points or periodic points); in fact, our methods are applicable to irreversible dynamical systems having several degrees of freedom (that is, actions of $\mathbb{Z}_+^d$). In the multidimensional case the usual notions of recurrence and centre are not sufficient to describe the Jacobson radical, as we show by an example. Accordingly, we introduce appropriate modifications.

After discussing the basic properties of analytic crossed products and some of the radicals of Banach algebras, we develop the key lemma in Section 1. This lemma, which is based on the idea of [4, Lemma 1], relates (multi-) recurrent points in the dynamical system with elements not in the Jacobson radical. In Section 2, we give a characterisation of semisimplicity. The proof has three ingredients: the key lemma, a sufficient condition for an element to belong to the prime radical (a descendant of Muhly's condition mentioned earlier), and a basic fact from dynamical systems theory which is known in the one-dimensional case. Our main result, Theorem 18, is proved in the last section using a modification of the centre of a dynamical system.

0.1. *Definition of analytic crossed products.* Analytic crossed products or semicrossed products have been defined in various degrees of generality by several authors (see for example [9, 13, 18, 19, 20]), generalising the concept of the crossed product of a C*-algebra by a group of *-automorphisms. To fix our conventions, we present the definition in the form that we will use it. Let X be a locally compact Hausdorff space and $\Phi = \{\phi_{\mathbf{n}} : \mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d\}$ be a semigroup of continuous and proper surjections isomorphic (as a semigroup) to $\mathbb{Z}_+^d$.

An action of $\Sigma = \mathbb{Z}_+^d$ on $C_0(X)$ by isometric *-endomorphisms $\alpha_{\mathbf{n}}$ $(\mathbf{n} \in \Sigma)$ is obtained by defining $\alpha_{\mathbf{n}}(f) = f \circ \phi_{\mathbf{n}}$.

We write elements of the Banach space $\ell^1(\Sigma, C_0(X))$ as formal multi-series $A = \sum_{\mathbf{n} \in \Sigma} U_{\mathbf{n}} f_{\mathbf{n}}$ with the norm given by $\|A\|_1 = \sum \|f_{\mathbf{n}}\|_{C_0(X)}$. The multiplication on $\ell^1(\Sigma, C_0(X))$ is defined by setting

$$U_{\mathbf{n}} f U_{\mathbf{m}} g = U_{\mathbf{n}+\mathbf{m}}(\alpha_{\mathbf{m}}(f)\, g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\Sigma, C_0(X))$ is a Banach algebra.

We will represent $\ell^1(\Sigma, C_0(X))$ faithfully as a (concrete) operator algebra on Hilbert space, and define the analytic crossed product as the closure of the image.

Assuming we have a faithful action of $C_0(X)$ on a Hilbert space H_o, we can define a faithful contractive representation π of $\ell^1(\Sigma, C_0(X))$ on the Hilbert space $\mathscr{H} = H_o \otimes \ell^2(\Sigma)$ by defining $\pi(U_{\mathbf{n}} f)$ as

$$\pi(U_{\mathbf{n}} f)(\xi \otimes e_{\mathbf{k}}) = \alpha_{\mathbf{k}}(f)\, \xi \otimes e_{\mathbf{k}+\mathbf{n}}$$

To show that π is faithful, let $A = \sum_{\mathbf{n} \in \Sigma} U_{\mathbf{n}} f_{\mathbf{n}}$ be in $\ell^1(\mathbb{Z}_+^d, C_0(X))$ and $x, y \in H_o$ be unit vectors. Since π is clearly contractive, the series $\pi(A) = \sum_{\mathbf{n} \in \Sigma} \pi(U_{\mathbf{n}} f_{\mathbf{n}})$ converges absolutely. For $\mathbf{m} \in \Sigma$, we have

$$\begin{aligned}
\langle \pi(A)(x \otimes e_0), y \otimes e_{\mathbf{m}} \rangle &= \sum_{\mathbf{n}} \langle \pi(U_{\mathbf{n}} f_{\mathbf{n}})(x \otimes e_0), y \otimes e_{\mathbf{m}} \rangle \\
&= \sum_{\mathbf{n}} \langle f_{\mathbf{n}} x \otimes e_{\mathbf{n}}, y \otimes e_{\mathbf{m}} \rangle \\
&= \langle f_{\mathbf{m}} x \otimes e_{\mathbf{m}}, y \otimes e_{\mathbf{m}} \rangle = \langle f_{\mathbf{m}} x, y \rangle
\end{aligned}$$

as $x \otimes e_{\mathbf{n}}$ and $y \otimes e_{\mathbf{m}}$ are orthogonal for $\mathbf{n} \neq \mathbf{m}$. It follows that

$$\|\pi(A)\| \geqslant \|f_{\mathbf{m}}\|.$$

Hence if $\pi(A) = 0$ then $f_{\mathbf{m}} = 0$ for all $\mathbf{m}$, showing $A = 0$. Thus π is a monomorphism.

DEFINITION 2. The *analytic crossed product* $\mathscr{A} = C_0(X) \times_\phi \mathbb{Z}_+^d$ is the closure of the image of $\ell^1(\mathbb{Z}_+^d, C_0(X))$ in $\mathscr{B}(\mathscr{H})$ in the representation just defined.

This is a generalisation of the definition given in [19]. Note that $\mathscr{A}$ is in fact independent of the faithful action of $C_0(X)$ on H_o (up to isometric isomorphism).

For $A = \sum U_{\mathbf{n}} f_{\mathbf{n}} \in \ell^1(\Sigma, C_0(X))$ we call $f_{\mathbf{n}} \equiv E_{\mathbf{n}}(A)$ the $\mathbf{n}$*th Fourier coefficient* of A. We have shown above that the maps $E_{\mathbf{n}}: \ell^1(\Sigma, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $\mathscr{A}$, hence they extend to contractions $E_{\mathbf{n}}: \mathscr{A} \to C_0(X)$.

Moreover,

$$U_{\mathbf{m}} E_{\mathbf{m}}(A) = \frac{1}{(2\pi)^d} \int_{([-\pi, \pi])^d} \theta_{\mathbf{t}}(A) \exp(-i\mathbf{m}.\mathbf{t})\, d\mathbf{t},$$

where $\mathbf{m}.\mathbf{t} = m_1 t_1 + \ldots + m_d t_d$ and the automorphism $\theta_{\mathbf{t}}$ is defined first on the dense subalgebra $\ell^1(\Sigma, C_0(X))$ by

$$\theta_{\mathbf{t}}\left(\sum U_{\mathbf{n}} f_{\mathbf{n}}\right) = \sum U_{\mathbf{n}}(\exp(i\mathbf{t}.\mathbf{n})\ f_{\mathbf{n}})$$

and then extended to $\mathscr{A}$ by continuity.

Thus, by injectivity of the Fourier transform on $C(([-\pi, \pi])^d)$, if a continuous linear form η on $\mathscr{A}$ satisfies $\eta(E_{\mathbf{m}}(A)) = 0$ for all $\mathbf{m}$ then (the function $\mathbf{t} \to \eta(\theta_{\mathbf{t}}(A))$ vanishes and hence) $\eta(A) = 0$. The Hahn–Banach Theorem yields the following remark.

Remark. Any $A \in \mathscr{A}$ belongs to the closed linear span of the set $\{U_{\mathbf{m}} E_{\mathbf{m}}(A) : \mathbf{m} \in \Sigma\}$ of its "associated monomials".

In particular, $\mathscr{A}$ is the closure of the subalgebra $\mathscr{A}_0$ of trigonometric polynomials, i.e., finite sums of monomials.

As $\theta_{\mathbf{t}}$ is an automorphism of $\mathscr{A}$, we conclude that if $\mathscr{J} \subseteq \mathscr{A}$ is a closed automorphism invariant ideal (in particular, the Jacobson radical) then for all $B \in \mathscr{J}$ and $\mathbf{m} \in \Sigma$ we obtain $U_{\mathbf{m}} E_{\mathbf{m}}(B) \in \mathscr{J}$. Thus, an element $\sum U_{\mathbf{n}} f_{\mathbf{n}}$ is in $\mathscr{J}$ if and only if each monomial $U_{\mathbf{n}} f_{\mathbf{n}}$ is in $\mathscr{J}$; this was first observed (for $d = 1$) in [16, Proposition 2.1]. It now follows from the remark that any such ideal is the closure of the trigonometric polynomials it contains.

0.2 *Radicals in Banach algebras.* Recall that an ideal $\mathscr{J}$ of an algebra $\mathscr{A}$ is said to be *primitive* if it is the kernel of an (algebraically) irreducible representation. The intersection of all primitive ideals of $\mathscr{A}$ is the *Jacobson radical* of $\mathscr{A}$, denoted Rad $\mathscr{A}$.

An ideal $\mathscr{J}$ is *prime* if it cannot factor as the product of two distinct ideals, i.e., if $\mathscr{J}_1, \mathscr{J}_2$ are ideals of $\mathscr{A}$ such that $\mathscr{J}_1 \mathscr{J}_2 \subseteq \mathscr{J}$ then either $\mathscr{J}_1 \subseteq \mathscr{J}$ or $\mathscr{J}_2 \subseteq \mathscr{J}$. The intersection of all prime ideals is the *prime radical* of $\mathscr{A}$, denoted PRad $\mathscr{A}$. An algebra $\mathscr{A}$ is *semisimple* if Rad $\mathscr{A} = \{0\}$ and *semiprime* if PRad $\mathscr{A} = \{0\}$, or equivalently, if there are no (nonzero) nilpotent ideals.

As a primitive ideal is prime, PRad $\mathcal{A} \subseteq$ Rad $\mathcal{A}$. Thus a semisimple algebra is semiprime. If $\mathcal{A}$ is a Banach algebra, then the Jacobson radical is closed; indeed every primitive ideal is the kernel of some *continuous* representation of $\mathcal{A}$ on a Banach space. In fact an element $A \in \mathcal{A}$ is in Rad $\mathcal{A}$ if and only if the spectral radius of AB vanishes for all $B \in \mathcal{A}$.

The prime radical need not be closed; it is closed if and only if it is a nilpotent ideal (see [8] or [17, Theorem 4.4.11]). Thus for a general Banach algebra, PRad $\mathcal{A} \subseteq \overline{\text{PRad}\, \mathcal{A}} \subseteq$ Rad $\mathcal{A}$.

1. RECURRENCE AND MONOMIALS

Our main results will be proved for metrisable dynamical systems; hence we make the blanket assumption that X *will be a locally compact metrisable space*. As in the one-dimensional case, we say that a point $x \in X$ is *recurrent* for the dynamical system (X, Φ) if there exists a sequence $(\mathbf{n}_k)$ tending to infinity so that $\phi_{\mathbf{n}_k}(x) \to x$. We will need the following variant:

DEFINITION 3. Let $J \subseteq \{1, 2, ..., d\}$. Say $x \in X$ is *J-recurrent* if there exists a sequence $(\mathbf{n}_k)$ which is strictly increasing in the directions of J (that is, the jth entry of $\mathbf{n}_{k+1}$ is greater than the jth entry of $\mathbf{n}_k$ for every $j \in J$ and $k \in \mathbb{N}$) such that $\lim_k \phi_{\mathbf{n}_k}(x) = x$. Denote the set of all J-recurrent points by X_{Jr}.

We say that a point $x \in X$ is *strongly recurrent* if it is $\{1, 2, ..., d\}$-recurrent. Finally, Σ_J denotes $\{\mathbf{n} \in \mathbb{Z}_+^d : \mathbf{n}_j > 0 \text{ for all } j \in J\}$.

In the multidimensional case, the Jacobson radical cannot be characterised in terms of either the recurrent points (in the traditional sense) or the strongly recurrent points. To justify this, we give the following example.

EXAMPLE 4. Let $X = X_0 \cup X_1 \cup X_2$ where $X_i = \mathbb{R} \times \{i\}$. Consider the dynamical system $(X, (\phi_1, \phi_2))$, where ϕ_1 acts as translation by 1 on X_1 and as the identity on $X_0 \cup X_2$ while ϕ_2 acts as translation by 1 on X_2 and as the identity on $X_0 \cup X_1$. It is easy to see that the set of $\{1\}$-recurrent points is $X_0 \cup X_2$, the set of $\{2\}$-recurrent points is $X_0 \cup X_1$ and the set of strongly recurrent points is X_0.

Choose small neighbourhoods $V_1 \subseteq X_1$ and $V_2 \subseteq X_2$ of $(0, 1)$ and $(0, 2)$ respectively such that $\phi_1(V_1) \cap V_1 = \varnothing$ and $\phi_2(V_2) \cap V_2 = \varnothing$. Let $f \in C_0(X)$ be any function supported on $V_1 \cup V_2$ such that $f(0, 1) = f(0, 2) = 1$.

Then one can verify (as in the proof of Lemma 8 in the next section) that $U_1 U_2 f$ is in the prime radical. On the other hand, neither $U_1 f$ nor $U_2 f$ belong to the Jacobson radical (they are not even quasinilpotent).

Here, the associated semicrossed product has nonzero Jacobson radical, although every point is recurrent. Also, the monomial $U_1 f$ is not in the Jacobson radical, although f vanishes on the strongly recurrent points. The next lemma shows that for such a monomial to be in the Jacobson radical, f must vanish on the $\{1\}$-recurrent points.

The main result of this section is the following lemma, which is crucial to our analysis.

LEMMA 5. *Let $U_{\mathbf{q}} f \in Rad(C_0(X) \times_\phi \mathbb{Z}_+^d)$. If J contains the support of $\mathbf{q}$, then f vanishes on each J-recurrent point of (X, Φ).*

In order to prove this lemma, we need a basic property of recurrent points, adapted to our circumstances.

DEFINITION 6. Given a sequence $\bar{\mathbf{n}} = (\mathbf{n}_k) \subseteq \mathbb{Z}_+^d$ we define recursively the *family of indices associated to* $\bar{\mathbf{n}}$, denoted $\mathscr{S}(\bar{\mathbf{n}}) = (S_0, S_1, S_2, \ldots)$ as follows: $S_0 = \{\mathbf{0}\}$, $S_1 = \{\mathbf{n}_1\}$ and generally

$$S_{k+1} = \left\{ \mathbf{n}_{k+1} + \mathbf{m}_k + \mathbf{j} : \mathbf{j} \in \bigcup_{i=0}^{k} S_i \right\}$$

where $\mathbf{m}_0 = \mathbf{0}$ and $\mathbf{m}_k = \mathbf{n}_k + 2\mathbf{m}_{k-1}$.

The sets in $\mathscr{S}(\bar{\mathbf{n}})$ will be needed in the proof of Lemma 5: they are the indices of ϕ occurring in the simplification of the inductive sequence of products given by $P_1 = U_{\mathbf{n}_1} g$ and $P_k = P_{k-1}(U_{\mathbf{n}_k}(g/2^{k-1}))\, P_{k-1}$. We should also point out that $\bigcup_i S_i$ is an IP-set (see [6, Section 8.4]) and the next lemma is a variant on [6, Theorem 2.17].

Recall Σ_J denotes $\{(n_1, n_2, \ldots, n_d) : n_j \neq 0 \text{ for all } j \in J\}$. Let Δ_J be the subset of Σ_J with entries in the directions of J^c identically zero.

LEMMA 7. *Let x be in X, J be a subset of $\{1, 2, \ldots, d\}$. Suppose that $\lim_k \phi_{\mathbf{p}_k}(x) = x$, where $(\mathbf{p}_k)$ is a sequence whose restriction to J is strictly increasing while its restriction to J^c is constant.*

For each open neighbourhood V of x and each $k \in \mathbb{N}$, there is $\mathbf{n}_k \in \Delta_J$ and $x_k \in V$ with

$$\phi_{\mathbf{s}}(x_k) \in V \quad \textit{for all} \quad \mathbf{s} \in \bigcup_{i=0}^{k} S_i,$$

where $\mathscr{S}(\bar{\mathbf{n}}) = (S_0, \ldots)$ is the family of indices associated to the sequence $(\mathbf{n}_k)$.

Proof. We inductively find indices $\mathbf{n}_1, \mathbf{n}_2, \ldots,$ as above, open sets $V \supseteq V_1 \supseteq V_2 \supseteq \cdots$ and points $x_1, x_2, \ldots$ with $x_i \in V_i$ and $x_i = \phi_{\mathbf{k}_i}(x)$ for some index $\mathbf{k}_i$, so that

$$\phi_{\mathbf{s}}(V_i) \subseteq V \quad \text{for all} \quad \mathbf{s} \in S_i.$$

This will prove the lemma, for if $k \in \mathbb{N}$ and $\mathbf{s} \in S_i$ for some $i \leqslant k$ then, since $x_k \in V_k \subseteq V_i$ it will follow that $\phi_{\mathbf{s}}(x_k) \in \phi_{\mathbf{s}}(V_k) \subseteq \phi_{\mathbf{s}}(V_i) \subseteq V$.

Since $\lim_k \phi_{\mathbf{p}_k}(x) = x \in V$, there is $\mathbf{p}_{i_1}$ with $x_1 = \phi_{\mathbf{p}_{i_1}}(x) \in V$. Let $\mathbf{k}_1 = \mathbf{p}_{i_1}$. Using $\lim_k \phi_{\mathbf{p}_k}(x) = x \in V$ and the form of the $\mathbf{p}_k$, it follows that there is $\mathbf{n}_1 \in \Delta_J$ so that $\phi_{\mathbf{n}_1 + \mathbf{k}_1}(x) \in V$. Now

$$\phi_{\mathbf{n}_1}(x_1) = \phi_{\mathbf{n}_1}(\phi_{\mathbf{k}_1}(x)) = \phi_{\mathbf{n}_1 + \mathbf{k}_1}(x) \in V,$$

and so there is $V_1 \subseteq V$, an open neighbourhood of x_1, so that $\phi_{\mathbf{n}_1}(V_1) \subseteq V$. Since $S_1 = \{\mathbf{n}_1\}$, this establishes the base step.

For the inductive step, assume we have chosen indices $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_q$, open subsets of V, $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_q$ and points $x_1, x_2, \ldots, x_q$, with $x_i \in V_i$ and $x_i = \phi_{\mathbf{k}_i}(x)$, so that, for $i = 1, \ldots, q$, we have

$$\phi_{\mathbf{s}}(V_i) \subseteq V \quad \text{for all} \quad \mathbf{s} \in S_i. \tag{1}$$

Since $\lim_k \phi_{\mathbf{p}_k}(x_q) = \phi_{\mathbf{k}_q}(\lim_k \phi_{\mathbf{p}_k}(x)) = x_q \in V_q$, there is $\mathbf{k}_{q+1} = \mathbf{p}_{i_q}$ so that $x_{q+1} = \phi_{\mathbf{k}_{q+1}}(x_q) \in V_q$. Notice that $\mathbf{m}_q$ (as in Definition 6) is in Δ_J. It follows that there exists $\mathbf{n}_{q+1} \in \Delta_J$ such that $\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q + \mathbf{k}_{q+1}}(x_q) \in V_q$ and so $\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(x_{q+1}) \in V_q$. Hence there exists an open neighbourhood V_{q+1} of x_{q+1}, contained in V_q, so that

$$\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(V_{q+1}) \subseteq V_q. \tag{2}$$

It remains only to show that $\phi_{\mathbf{s}}(V_{q+1}) \subseteq V$ for all $\mathbf{s} \in S_{q+1}$. An element $\mathbf{s}$ in S_{q+1} is of the form $\mathbf{s} = \mathbf{n}_{q+1} + \mathbf{m}_q + \mathbf{j}$ for some $\mathbf{j} \in \bigcup_{i=0}^{q} S_i$. Assuming $\mathbf{j} \in S_i$ for some i, we have

$$\begin{aligned} \phi_{\mathbf{s}}(V_{q+1}) = \phi_{\mathbf{j}}(\phi_{\mathbf{n}_{q+1} + \mathbf{m}_q}(V_{q+1})) &\subseteq \phi_{\mathbf{j}}(V_q) && \text{by (2)} \\ \subseteq \phi_{\mathbf{j}}(V_i) &\subseteq V && \text{by (1)} \end{aligned}$$

completing the induction. ∎

Proof of Lemma 5. Assume that $f(x) \neq 0$ for some J-recurrent point x. We will find $B \in \mathscr{A}$ such that $BU_{\mathbf{q}} f$ has nonzero spectral radius. We may scale f so that there exists a relatively compact open neighbourhood V of x such that $|f(y)| \geqslant 1$ for all $y \in V$. Since $U_{\mathbf{q}} |f|^2 = (U_{\mathbf{q}} f) f^* \in \operatorname{Rad} \mathscr{A}$ when $U_{\mathbf{q}} f \in \operatorname{Rad} \mathscr{A}$, we may also assume that $f \geqslant 0$.

Since x is J-recurrent, there exists a sequence $(\mathbf{p}_k)$ which is strictly increasing in the directions of J such that $\lim_k \phi_{\mathbf{p}_k}(x) = x$. Deleting some initial segment, we may assume that $\phi_{\mathbf{p}_k}(x) \in V$ for all $k \in \mathbb{N}$.

If $(\mathbf{p}_k)$ has all entries going to infinity, then we may apply Lemma 7 with $J = \{1, 2, ..., d\}$, to find a strictly increasing sequence $(\mathbf{n}_k)$ such that $\mathbf{n}_k > \mathbf{q}$ for all k and points $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s}$ in $\bigcup_{i=0}^k S_i$.

If not, enlarging J and passing to a subsequence if necessary, we may assume that the restriction of $(\mathbf{p}_k)$ to J^c takes only finitely many values. Passing to another subsequence, we may further assume that this restriction is constant. Applying Lemma 7, we may find a strictly increasing sequence $(\mathbf{n}_k)$ in $\mathbb{Z}_+^d$ with $\mathbf{n}_k \in \Delta_J$ and points $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s}$ in $\bigcup_{i=0}^k S_i$. We may suppose that $\mathbf{n}_k - \mathbf{q} \in \Sigma_J$ for all k. Thus $U_{\mathbf{n}_k - \mathbf{q}}$ is an admissible term in the formal power series of an element of $C_0(X) \times_\phi \mathbb{Z}_+^d$.

Fix a nonnegative function $h \in C_0(X)$ such that $h(\phi_{\mathbf{q}}(y)) = 1$ for all $y \in V$ and consider

$$B = \sum_{k=1}^{\infty} U_{\mathbf{n}_k - \mathbf{q}} \frac{h}{2^{k-1}}.$$

This is an element of $\mathscr{A}$ since the series converges absolutely. To complete the proof, it suffices to show that the spectral radius of $A \equiv BU_{\mathbf{q}} f$ is strictly positive. Note that

$$A = \sum U_{\mathbf{n}_k} \frac{g}{2^{k-1}},$$

where g is $f.(h \circ \phi_{\mathbf{q}})$, a nonnegative function satisfying $g(y) \geqslant 1$ for all $y \in V$. Thus each Fourier coefficient $E_{\mathbf{n}}(A^m)$ of A^m is a finite sum of nonnegative functions, and hence its norm dominates the (supremum) norm of each summand. Since $\|A^{2^k-1}\| \geqslant \|E_{\mathbf{n}}(A^{2^k-1})\|$, it suffices to find $\varepsilon > 0$ such that for each k there exists $\mathbf{n}$ such that the norm of *some* summand of $E_{\mathbf{n}}(A^{2^k-1})$ exceeds ε^{2^k-1}.

If we let $P_1 = U_{\mathbf{n}_1} g$, then trivially P_1 is a term in A. In the next product, $A^3 = A(\sum U_{\mathbf{n}_k} \frac{g}{2^{k-1}})\, A = \sum A(U_{\mathbf{n}_k} \frac{g}{2^{k-1}})\, A$, we have the term

$$P_2 = U_{\mathbf{n}_1} g \left(U_{\mathbf{n}_2} \frac{g}{2} \right) U_{\mathbf{n}_1} g.$$

Generally, one term in the expansion of $A^{2^k-1} = A^{2^{k-1}-1} A A^{2^{k-1}-1}$ is

$$P_k = P_{k-1} \left(U_{\mathbf{n}_k} \frac{g}{2^{k-1}} \right) P_{k-1}.$$

CLAIM. *If* $\lambda_1 = 1$ *and* $\lambda_{k+1} = \lambda_k^2/2^k$, *then* $P_k = U_{\mathbf{m}_k}\lambda_k \prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}}$ *where* $\mathbf{m}_k$ *is as in the definition of* $\mathscr{S}(\bar{\mathbf{n}})$ *and the product is over all* $\mathbf{s}$ *in* $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$.

Proof of Claim. For $k = 1$, the claim holds trivially as $(S_0 \cup S_1) \setminus \{\mathbf{m}_1\} = \{0\}$. Assuming the claim is true for some k, we have

$$P_{k+1} = P_k \left(U_{\mathbf{n}_{k+1}} \frac{g}{2^k} \right) P_k$$
$$= U_{\mathbf{m}_k} \lambda_k \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right) \left(U_{\mathbf{n}_{k+1}} \frac{g}{2^k} \right) U_{\mathbf{m}_k} \lambda_k \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{t}} \right)$$

(where $\mathbf{s}, \mathbf{t}$ range over $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$)

$$= U_{\mathbf{m}_k} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right) U_{\mathbf{n}_{k+1}+\mathbf{m}_k} (g \circ \phi_{\mathbf{m}_k}) \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{s}} \right)$$
$$= U_{2\mathbf{m}_k+\mathbf{n}_{k+1}} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}+\mathbf{n}_{k+1}+\mathbf{m}_k} \right) (g \circ \phi_{\mathbf{m}_k}) \left(\prod_{\mathbf{t}} g \circ \phi_{\mathbf{t}} \right)$$
$$= U_{2\mathbf{m}_k+\mathbf{n}_{k+1}} \frac{\lambda_k^2}{2^k} \left(\prod_{\mathbf{s}'} g \circ \phi_{\mathbf{s}'} \right) \left(\prod_{\mathbf{t}'} g \circ \phi_{\mathbf{t}'} \right),$$

where $\mathbf{s}'$ ranges over $\{\mathbf{n}_{k+1} + \mathbf{m}_k + \mathbf{s}\}$, for $\mathbf{s} \in (\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$, and $\mathbf{t}'$ ranges over $(\bigcup_{i=0}^k S_i)$. Therefore

$$P_{k+1} = U_{\mathbf{m}_{k+1}} \lambda_{k+1} \left(\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}} \right)$$

for $\mathbf{s}$ in $(\bigcup_{i=0}^{k+1} S_i) \setminus \{\mathbf{m}_{k+1}\}$, proving the claim.

Recall that for each $k \in \mathbb{N}$ there exists $x_k \in V$ such that $\phi_{\mathbf{s}}(x_k) \in V$ for all $\mathbf{s} \in \bigcup_{i=0}^k S_i$. Since $g|_V \geqslant 1$, we have $\prod_{\mathbf{s}} g(\phi_{\mathbf{s}}(x_k)) \geqslant 1$ where $\mathbf{s}$ ranges over $(\bigcup_{i=0}^k S_i) \setminus \{\mathbf{m}_k\}$ and hence $\|\prod_{\mathbf{s}} g \circ \phi_{\mathbf{s}}\| \geqslant 1$. From the claim, it follows that $\|P_k\| \geqslant \lambda_k$ and so, by the earlier remarks,

$$\|A^{2^k-1}\| \geqslant \|E_{\mathbf{m}_k}(A^{2^k-1})\| \geqslant \|P_k\| \geqslant \lambda_k.$$

Thus the proof will be complete if we show that $\lambda_k \geqslant (\frac{1}{2})^{2^k-1}$ or equivalently $\log_2 \lambda_k^{-1} \leqslant 2^k - 1$ for all k. Setting $\mu_k = \log_2 \lambda_k^{-1}$, the recurrence relation for λ_k becomes $\mu_{k+1} = 2\mu_k + k$ and $\mu_1 = 0$, which has solution $\mu_k = 2^k - k - 1$. ∎

2. WANDERING SETS AND SEMISIMPLICITY

We characterise semisimplicity of analytic crossed products and show this is equivalent to being semiprime. Part of this characterisation is of course a special case of our main result, Theorem 18, but we will need the preliminary results in any case.

A *wandering open set* is an open set $V \subset X$ so that $\phi_{\mathbf{n}}^{-1}(V) \cap V = \varnothing$ whenever $\mathbf{n} \in \mathbb{Z}_+^d$ is nonzero. A *wandering point* is a point with a wandering neighbourhood.

We will need the following variant: let $J \subseteq \{1, ..., d\}$. An open set $V \subseteq X$ is said to be *wandering in the directions of* J, or J*-wandering*, if $\phi_{\mathbf{n}}^{-1}(V) \cap V = \varnothing$ whenever $\mathbf{n}$ is in Σ_J. It is easily seen that, if X_{Jw} denotes the set of all J-wandering points (those with a J-wandering neighbourhood), then X_{Jw} is open and its complement is invariant and contains the set X_{Jr} of J-recurrent points.

Note, however, that it is possible for a recurrent point (in the usual sense) to have a neighbourhood that is J-wandering (for some J). For example, if $X = \mathbb{R}^2$ and $\phi_1(x, y) = (x+1, y)$ while $\phi_2(x, y) = (x, 3y)$, then the origin is recurrent for the dynamical system $(X, (\phi_1, \phi_2))$, but it also has a $\{1\}$-wandering neighbourhood.

The idea of the following Lemma comes from [16, Theorem 4.2].

LEMMA 8. *Suppose $V \subseteq X$ is an open set which is J-wandering and $g \in C_0(X)$ is a nonzero function with support contained in V. If $\mathbf{e}_J$ denotes the characteristic function of J, then $B = U_{\mathbf{e}_J} g$ generates a nonzero ideal $\mathscr{A} B \mathscr{A}$ whose square is* 0.

Proof. Let $C \in \mathscr{A}$ be arbitrary and $h = E_{\mathbf{k}}(C)$. Then

$$BU_{\mathbf{k}} E_{\mathbf{k}}(C)\, B = U_{\mathbf{e}_J}\, gU_{\mathbf{k}} hU_{\mathbf{e}_J}\, g = U_{\mathbf{k}+2\mathbf{e}_J}(\alpha_{\mathbf{k}+\mathbf{e}_J}(g)\, \alpha_{\mathbf{e}_J}(h)\, g),$$

which is zero since g is supported on V and $\alpha_{\mathbf{k}+\mathbf{e}_J}(g)$ is supported on the disjoint set $\phi_{\mathbf{k}+\mathbf{e}_J}^{-1}(V)$. This shows that all Fourier coefficients of BCB will vanish, and hence $BCB = 0$. It follows that all products $(C_1 BC_2)(C_3 BC_4)$ vanish and hence $(\mathscr{A} B \mathscr{A})^2 = 0$. On the other hand, choosing functions $h_1 \in C_0(X)$ equal to 1 on $\phi_{\mathbf{e}_J}^{-1}(V)$ and h_2 equal to 1 on V, we find $E_{\mathbf{e}_J}(h_1 B h_2) = \alpha_{\mathbf{e}_J}(h_1)\, gh_2 = g \neq 0$, so the ideal $\mathscr{A} B \mathscr{A}$ is nonzero. ∎

The following proposition is known for the usual notions of recurrence and wandering in the case $d = 1$; see [6, Theorem 1.27].

PROPOSITION 9. *Suppose X is a locally compact metrisable space. If (X, Φ) has no nonempty J-wandering open sets, then the J-recurrent points are dense.*

Proof. Let $V \subseteq X$ be a relatively compact open set. We wish to find a J-recurrent point in V.

Since V is not J-wandering, there exists $\mathbf{n}_1 \in \Sigma_J$ such that $\phi_{\mathbf{n}_1}^{-1}(V) \cap V \neq \emptyset$. Hence there is a nonempty, relatively compact, open set V_1 with $\operatorname{diam}(V_1) < 1$ such that $\overline{V_1} \subseteq \phi_{\mathbf{n}_1}^{-1}(V) \cap V$.

Since V_1 contains no J-wandering subsets, a similar argument shows that there exists $\mathbf{n}_2$ such that $\phi_{\mathbf{n}_2}^{-1}(V_1) \cap V_1 \neq \emptyset$ and the jth entry of $\mathbf{n}_2$ is greater than that of $\mathbf{n}_1$ for every $j \in J$.

Inductively one obtains a sequence of open sets V_k and $\mathbf{n}_k$ strictly increasing in the directions of J with $\overline{V_k} \subseteq \phi_{\mathbf{n}_k}^{-1}(V_{k-1}) \cap V_{k-1}$ and $\operatorname{diam}(V_k) < 1/k$ all contained in the compact metrisable space $\overline{V_0}$. It follows from Cantor's theorem that the intersection $\bigcap_{n \geqslant 1} \overline{V_n}$ is a singleton, say x. Since $x \in \overline{V_k} \subseteq \phi_{\mathbf{n}_k}^{-1}(V_{k-1})$ we have $\phi_{\mathbf{n}_k}(x) \in V_{k-1}$ for all k and so $\phi_{\mathbf{n}_k}(x) \to x$; hence $x \in X_{Jr}$. ∎

THEOREM 10. *If X is a metrisable, locally compact space, then the following are equivalent:*

1. *the strongly recurrent points are dense in X,*
2. *$C_0(X) \times_\phi \mathbb{Z}_+^d$ is semisimple, and*
3. *$C_0(X) \times_\phi \mathbb{Z}_+^d$ is semiprime.*

Proof. If the strongly recurrent points are dense in X, then by Lemma 5 there are no nonzero monomials in the Jacobson radical of $C_0(X) \times_\phi \Sigma$. But we have already observed that an element A is in the Jacobson radical if and only if each monomial $U_\mathbf{n} E_\mathbf{n}(A)$ is. Thus $C_0(X) \times_\phi \Sigma$ is semisimple and hence semiprime.

Suppose that $C_0(X) \times_\phi \Sigma$ is semiprime. Then Lemma 8 shows that there are no nonempty J-wandering open sets for $J = \{1, 2, \ldots, d\}$. Thus, by Proposition 9, the strongly recurrent points are dense. ∎

3. CENTRES AND THE JACOBSON RADICAL

In order to describe the Jacobson radical of an analytic crossed product, we need to characterise the closure of the J-recurrent points, for a dynamical system (X, Φ) with X a locally compact metrisable space.

LEMMA 11. (i) *If $Y \subseteq X$ is a closed invariant set, the set Y_{Jr} of J-recurrent points for the dynamical system (Y, Φ) equals $X_{Jr} \cap Y$.*

(ii) *The set $\overline{X_{Jr}}$ is the largest closed invariant set $Y \subseteq X$ such that (Y, Φ) has no J-wandering points.*

Proof. (i) To see that $Y_{Jr} \subseteq X_{Jr}$, note that if $y \in Y_{Jr}$ then for every neighbourhood V of y (in X) the set $V \cap Y$ is a neighbourhood of y in the relative topology of Y, so there exists $\mathbf{n} \in \Sigma_J$ such that $\phi_{\mathbf{n}}(y) \in V \cap Y$. Thus $\phi_{\mathbf{n}}(y) \in V$ showing that $y \in X_{Jr}$. On the other hand if $y \in Y \cap X_{Jr}$ then for each relative neighbourhood $V \cap Y$ of y, since V is a neighbourhood of y in X there exists $\mathbf{n} \in \Sigma_J$ such that $\phi_{\mathbf{n}}(y) \in V$. Since $y \in Y$ and Y is invariant, $\phi_{\mathbf{n}}(y) \in V \cap Y$ establishing (i).

(ii) Given a closed invariant set $Y \subseteq X$, if $(Y, \Phi|_Y)$ has no J-wandering points, then Y_{Jr} is dense in Y by Proposition 9, and hence $Y \subseteq \overline{X_{Jr}}$. On the other hand, $(\overline{X_{Jr}}, \Phi)$ clearly has no J-wandering open sets. ∎

The set $\overline{X_{Jr}}$ is found by successively "peeling off" the J-wandering parts of the dynamical system. This construction and Lemma 13 generalise the well known concept of the centre of a dynamical system (X, ϕ) [7, 7.19].

If $V \subseteq X$ is the union of the J-wandering open subsets of X, then let $X_{J,1}$ be the closed invariant set $X \backslash V$. Consider the dynamical system $(X_{J,1}, \Phi_{J,1})$, where $\Phi_{J,1} \equiv \Phi|_{X_{J,1}}$. Let $X_{J,2}$ be the complement of the union of all J-wandering open sets of $(X_{J,1}, \Phi_{J,1})$. Again we have a closed invariant set, and we may form the dynamical subsystem $(X_{J,2}, \Phi_{J,2})$ where $\Phi_{J,2} \equiv \Phi|_{X_{J,2}}$. By transfinite recursion, we obtain a decreasing family $(X_{J,\gamma}, \Phi_{J,\gamma})$ of dynamical systems: indeed, if $(X_{J,\gamma}, \Phi_{J,\gamma})$ has been defined, we let $X_{J,\gamma+1} \subseteq X_{J,\gamma}$ be the set of points in $(X_{J,\gamma}, \Phi_{J,\gamma})$ having no J-wandering neighbourhood and we define $\Phi_{J,\gamma+1} = \Phi|_{X_{J,\gamma+1}}$; if β is a limit ordinal and the systems $(X_{J,\gamma}, \Phi_{J,\gamma})$ have been defined for all $\gamma < \beta$, then we set $X_{J,\beta} = \bigcap_{\gamma<\beta} X_{J,\gamma}$ and $\Phi_{J,\beta} = \Phi|_{X_{J,\beta}}$. (We write $X_{J,0} = X$ and $\Phi_{J,0} = \Phi$.) This process must stop, for the cardinality of the family $\{X_{J,\gamma}\}$ cannot exceed that of the power set of X.

Definition 12. By the above argument, there exists a least ordinal γ such that $X_{J,\gamma+1} = X_{J,\gamma}$. The set $X_{J,\gamma}$ is called *the strong J-centre* of the dynamical system, and γ is called *the depth* of the strong J-centre.

Lemma 13. *If X is metrisable, then the strong J-centre of the dynamical system is the closure of the J-recurrent points.*

Proof. As a J-recurrent point cannot be J-wandering, $X_{Jr} \subseteq X_{J,1}$. If $X_{Jr} \subseteq X_{J,\gamma}$ for some γ, then by Lemma 11 the set $(X_{J,\gamma})_{Jr}$ of J-recurrent points of the subsystem $(X_{J,\gamma}, \Phi_{J,\gamma})$ equals $X_{Jr} \cap X_{J,\gamma}$, so $(X_{J,\gamma})_{Jr} = X_{Jr}$; but $(X_{J,\gamma})_{Jr} \subseteq X_{J,\gamma+1}$, and so $X_{Jr} \subseteq X_{J,\gamma+1}$. Finally, if γ is a limit ordinal and we assume that $X_{Jr} \subseteq X_{J,\delta}$ for all $\delta < \gamma$ then $X_{Jr} \subseteq \bigcap_{\delta<\gamma} X_{J,\delta} = X_{J,\gamma}$. This shows that $X_{Jr} \subseteq \bigcap_\gamma X_{J,\gamma}$ and so $\overline{X_{Jr}} \subseteq \bigcap_\gamma X_{J,\gamma}$ since the sets $X_{J,\gamma}$ are closed.

But on the other hand, if γ_0 is the depth of the strong J-centre we have $\bigcap_\gamma X_{J,\gamma} = X_{J,\gamma_0}$, a closed invariant set. Since $X_{J,\gamma_0+1} = X_{J,\gamma_0}$, the dynamical system $(X_{J,\gamma_0}, \Phi_{J,\gamma_0})$ can have no J-wandering points. Thus it follows from Lemma 11 that $X_{J,\gamma_0} \subseteq \overline{X_{Jr}}$ and hence equality holds. ∎

Remark. If X is a locally compact (not necessarily metrisable) space and $\{\phi_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ is an action of an *equicontinuous* group of homeomorphisms (with respect to a uniformity compatible with the topology of X) then $X_{Jr} = X \setminus X_{Jw}$ (see [14, Proposition 4.15]).

LEMMA 14. *For any ordinal δ, any $f \in C_c(X^c_{J,\delta+1})$ (i.e., f has compact support disjoint from $X_{J,\delta+1}$) can be written as a* finite *sum $f = \sum f_k$ where each f_k has compact support contained in a set V_k such that $V_k \cap X_{J,\delta}$ is J-wandering set for $(X_{J,\delta}, \Phi_{J,\delta})$.*

Proof. If K is the support of f then $K \cap X_{J,\delta} \subseteq X_{J,\delta} \setminus X_{J,\delta+1}$; in other words the compact set $K \cap X_{J,\delta}$ consists of J-wandering points for $(X_{J,\delta}, \Phi_{J,\delta})$. This means that each $x \in K \cap X_{J,\delta}$ has an open neighbourhood V_x so that the (relatively open) set $V_x \cap X_{J,\delta}$ is J-wandering for $(X_{J,\delta}, \Phi_{J,\delta})$. Each $y \in K \setminus X_{J,\delta}$ has an open neighbourhood V_y such that $V_y \cap X_{J,\delta}$ is empty (and so J-wandering).

The family $\{V_x : x \in K\}$ is an open cover for K. Thus, there is a partition of unity for f, i.e., a finite subcover, $\{V_k : 1 \leqslant k \leqslant m\}$, and functions f_k, $1 \leqslant k \leqslant m$, with $\operatorname{supp}(f_k)$ a compact subset of V_k, so that $f = f_1 + \cdots + f_m$. ∎

DEFINITION 15. We denote by $\mathscr{R}_{J,\gamma}$ the closed ideal generated by all monomials of the form $U_{\mathbf{n}} f$ where $\mathbf{n}$ is in Σ_J and $f \in C_0(X)$ vanishes on the set $X_{J,\gamma}$ and by $\mathscr{S}_{J,\gamma}$ the set of all elements of the form Bf where $B \in \mathscr{R}_{J,\gamma}$ and f has compact support disjoint from $X_{J,\gamma}$.

Note that a monomial $U_{\mathbf{n}} f \in \mathscr{R}_{J,\gamma}$ may be written in the form $CU_{\mathbf{e}_J} f$ with $C \in \mathscr{A}$, since $\mathbf{n} \in \Sigma_J$.

Also observe that $\mathscr{S}_{J,\gamma}$ is dense in $\mathscr{R}_{J,\gamma}$. Indeed if $U_{\mathbf{n}} f \in \mathscr{R}_{J,\gamma}$, then f can be approximated by some $g \in C_c(X^c_{J,\gamma})$; now $U_{\mathbf{n}} g$ is in $\mathscr{S}_{J,\gamma}$ and approximates $U_{\mathbf{n}} f$.

PROPOSITION 16. *For each ordinal γ and each $J \subseteq \{1, 2, ..., d\}$, the set $\mathscr{S}_{J,\gamma}$ is contained in Rad $\mathscr{A}$. Hence $\mathscr{R}_{J,\gamma}$ is contained in Rad $\mathscr{A}$.*

If PRad $\mathscr{A}$ is closed, then $\mathscr{R}_{J,\gamma}$ is contained in PRad $\mathscr{A}$.

Proof. Since $\mathscr{S}_{J,\gamma}$ is dense in $\mathscr{R}_{J,\gamma}$, it suffices to prove that any $A = Bf \in \mathscr{S}_{J,\gamma}$ is contained in Rad $\mathscr{A}$.

Suppose $\gamma = 1$. By Lemma 14 we may write A as a finite sum $A = \sum_k Bf_k$ where each f_k is supported on a compact set that is J-wandering. Since $A_k \equiv Bf_k = DU_{\mathbf{e}_J} f_k$ for some $D \in \mathscr{A}$ as observed above, by Lemma 8 we have $(\mathscr{A} A_k \mathscr{A})^2 = 0$ and so $A_k \in \mathrm{PRad}\,\mathscr{A}$. Thus $A \in \mathrm{PRad}\,\mathscr{A} \subseteq \mathrm{Rad}\,\mathscr{A}$.

Suppose the result has been proved for all ordinals less than some γ.

Let γ be a limit ordinal. If $\operatorname{supp} f = K \subseteq X^c_{J,\gamma}$, we have $K \subseteq X^c_{J,\gamma} = \bigcup_{\delta<\gamma} X^c_{J,\delta}$; hence K can be covered by finitely many of the $X^c_{J,\delta}$, hence (since they are decreasing) by one of them. Thus f has compact support contained in some $X^c_{J,\delta}$ $(\delta < \gamma)$ and so $Bf \in \mathscr{S}_{J,\delta}$. Therefore $A = Bf \in \mathrm{Rad}\,\mathscr{A}$ by the induction hypothesis.

Now suppose that γ is a successor, $\gamma = \delta + 1$. By Lemma 14, we may write $f = \sum f_k$ where the support of f_k is compact and contained in an open set V_k such that $V_k \cap X_{J,\delta}$ is J-wandering for $(X_{J,\delta}, \Phi_{J,\delta})$, i.e.,

$$\phi_{\mathbf{n}}^{-1}(V_k \cap X_{J,\delta}) \cap (V_k \cap X_{J,\delta}) = \varnothing$$

when $\mathbf{n} \in \Sigma_J$. This can easily be seen to imply $\phi_{\mathbf{n}}^{-1}(V_k) \cap V_k \subseteq X^c_{J,\delta}$.

Let $C \in \mathscr{A}$ be arbitrary. Writing $A_k = DU_{\mathbf{e}_J} f_k$ as above, it follows as in the proof of Lemma 8 that for each k all Fourier coefficients of $A_k C A_k$ are supported in $V_k \cap \phi_{\mathbf{n}}^{-1}(V_k)$ (for some $\mathbf{n} \in \Sigma_J$) which is contained in $X^c_{J,\delta}$ by the previous paragraph.

Thus $A_k C A_k \in \mathscr{R}_{J,\delta}$. By the induction hypothesis, $A_k C A_k$ must be contained in $\mathrm{Rad}\,\mathscr{A}$. Thus $(A_k C)^2$ is quasinilpotent, hence so is $A_k C$ (by the spectral mapping theorem). Since $C \in \mathscr{A}$ is arbitrary, it follows that $A_k \in \mathrm{Rad}\,\mathscr{A}$ for each k, so that $A \in \mathrm{Rad}\,\mathscr{A}$.

Finally, we suppose that $\mathrm{PRad}\,\mathscr{A}$ is closed. Then the argument above can be repeated exactly up to the previous paragraph, changing $\mathrm{Rad}\,\mathscr{A}$ to $\mathrm{PRad}\,\mathscr{A}$. The previous paragraph can be replaced by the following argument.

Thus $A_k C A_k \in \mathscr{R}_{J,\delta}$. By the induction hypothesis, $A_k C A_k$ must be contained in $\mathrm{PRad}\,\mathscr{A}$. Thus all products $(C_1 A_k C_2)(C_3 A_k C_4)$ are in $\mathrm{PRad}\,\mathscr{A}$ and so the (possibly non-closed) ideal $\mathscr{J}_k$ generated by A_k satisfies $\mathscr{J}_k \mathscr{J}_k \subseteq \mathrm{PRad}\,\mathscr{A}$. For every prime ideal $\mathscr{P}$, we have $\mathscr{J}_k \mathscr{J}_k \subseteq \mathscr{P}$ and so $\mathscr{J}_k \subseteq \mathscr{P}$. Hence $\mathscr{J}_k \subseteq \mathrm{PRad}\,\mathscr{A}$, and therefore $A_k \in \mathrm{PRad}\,\mathscr{A}$ for each k, so that $A \in \mathrm{PRad}\,\mathscr{A}$. ∎

One cannot conclude that $\mathscr{R}_{J,\gamma} \subseteq \mathrm{PRad}\,\mathscr{A}$ in general, even for finite γ, as the following example shows. Thus the prime radical is not always closed. Note that Hudson has given examples of TAF algebras in which the prime radical is not closed [11, Example 4.9].

EXAMPLE 17. We use a continuous dynamical system $(X, \{\phi_t\}_{t\in\mathbb{R}})$ based on [3, Example 3.3.4, p. 20] and look at the discrete system given by the maps $\{\phi_t\}$ for $t \in \mathbb{Z}_+$. The space X is the closed unit disc in $\mathbb{R}^2$. For the continuous system, the trajectories consist of: (i) three fixed points, namely the origin O and the points $A(1,0)$ and $B(-1,0)$ on the unit circle, (ii) the two semicircles on the unit circle joining A and B and (iii) spiraling trajectories emanating at the origin and converging to the boundary.

Let $\phi = \phi_1$. The recurrent points for the (discrete) dynamical system (X, ϕ) are $X_r = \{A, B, O\}$ and the set of wandering points is the open unit disc except the origin. Hence $X_2 = X_r$ and so the depth of the dynamical system is 2.

Now choose small disjoint open neighbourhoods V_A, V_B, V_O around the fixed points and let $f \in C(X)$ be a nonnegative function which is 1 outside these open sets and vanishes only at A, B and O. Then the element $Uf \in \mathcal{A}$ is clearly not nilpotent, so $Uf \notin \operatorname{PRad} \mathcal{A}$. However $Uf \in \operatorname{Rad} \mathcal{A}$ by the next theorem.

THEOREM 18. *Let (X, Φ) be a dynamical system with X metrisable. The Jacobson radical, $\mathit{Rad}(C_0(X) \times_\phi \mathbb{Z}_+^d)$, is the closed ideal generated by all monomials $U_{\mathbf{n}} f$ $(\mathbf{n} \neq \mathbf{0})$ where f vanishes on the set X_{Jr} of J-recurrent points corresponding to the support J of* $\mathbf{n}$.

Moreover, PRad $\mathcal{A}$ = Rad $\mathcal{A}$ if and only if PRad $\mathcal{A}$ is closed.

Proof. Let $U_{\mathbf{n}} f$ be a monomial contained in $\operatorname{Rad} \mathcal{A}$ and let J be the support of $\mathbf{n}$. Then Lemma 5 shows that f must vanish on X_{Jr}.

On the other hand, let $U_{\mathbf{n}} f$ be as in the statement of the Theorem, so that f vanishes on X_{Jr} (where $J = \operatorname{supp} \mathbf{n}$). We will show that $U_{\mathbf{n}} f$ is in $\operatorname{Rad} \mathcal{A}$. It is enough to suppose that the support K of f is compact. Since K is contained in $(\overline{X_{Jr}})^c = \bigcup_\gamma X_{J,\gamma}^c$, it is contained in finitely many, hence one, $X_{J,\gamma}^c$. It follows by Proposition 16 that $U_{\mathbf{n}} f \in \operatorname{Rad} \mathcal{A}$.

In the final statement of the theorem, one direction is obvious. For the other, suppose $\operatorname{PRad} \mathcal{A}$ is closed. Then by the final statement of Proposition 16, we have $\mathcal{R}_{J,\gamma} \subseteq \operatorname{PRad} \mathcal{A}$. ∎

This theorem leaves open the possibility that the closure of the prime radical is always equal to the Jacobson radical.

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