Münster J. of Math. **3** (2010), 209–212
Münster Journal of Mathematics

On the action of the group of isometries on a locally compact metric space

Antonios Manoussos

(Communicated by Katrin Tent)

Abstract. In this short note we give an answer to the following question. Let X be a locally compact metric space with group of isometries G. Let $\{g_i\}$ be a net in G for which g_ix converges to y, for some $x, y \in X$. What can we say about the convergence of $\{g_i\}$? We show that there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_y$, where C_x and C_y denote the pseudo-components of x and y respectively. Applying this we give short proofs of the van Dantzig-van der Waerden theorem (1928) and Gao-Kechris theorem (2003).

The main result and some applications

A few words about the notation we shall be using. In what follows, X will denote a locally compact metric space with group of isometries G. If we endow G with the topology of pointwise convergence then G is a topological group [2, Ch. X, §3.5 Cor.]. On G there is also the topology of uniform convergence on compact subsets which is the same as the compact-open topology. In the case of a group of isometries these topologies coincide with the topology of pointwise convergence, and the natural action of G on X with $(g, x) \mapsto g(x)$, $g \in G$, $x \in X$, is continuous [2, Ch. X, §2.4 Thm. 1 and §3.4 Cor. 1]. For $F \subset G$, let $K(F) := \{x \in X \mid \text{ the set } Fx \text{ has compact closure in } X\}$. The sets $K(F)$ are clopen [6, Lem. 3.1].

Lemma 1. *Let $\Gamma = \{g_i\}$ be a net in G and $x \in K(\Gamma)$ such that g_ix converges to y for some $y \in X$. Then a subnet of Γ converges to an isometry $f : K(\Gamma) \to X$ on $K(\Gamma)$.*

Proof. Let $g_i|_{K(\Gamma)}$ denote the restriction of g_i on $K(\Gamma)$. Arzela-Ascoli theorem implies that the set $\{g_i|_{K(\Gamma)} : K(\Gamma) \to X\}$ has compact closure in the set of

During this research the author was fully supported by SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" at the University of Bielefeld, Germany. He would also like to express his gratitude to Professor H. Abels for his support.

all continuous maps from $K(\Gamma)$ to X. Thus, there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : K(\Gamma) \to X$ such that $g_j \to f$ on $K(\Gamma)$. $\square$

In [4] S. Gao and A. S. Kechris introduced the concept of pseudo-components. These are the equivalence classes C_x of the following equivalence relation: $x \sim y$ if and only if x and y, as also y and x, can be connected by a finite sequence of intersecting open balls with compact closure. The pseudo-components are clopen [4, Prop. 5.3]. We call X pseudo-connected if it has only one pseudo-component. An immediate consequence of the definitions is that $gC_x = C_{gx}$ for every $g \in G$. Another notion, that will be used in the proofs, is the radius of compactness $\rho(x)$ of $x \in X$ [4]. Let $B_r(x)$ denote the open ball centered at x with radius $r > 0$. Then $\rho(x) := \sup\{r > 0 \mid B_r(x)$ has compact closure$\}$. If $\rho(x) = +\infty$ for some $x \in X$ then every ball has compact closure (i.e., X has the Heine-Borel property), hence $\rho(x) = +\infty$ for every $x \in X$. If $\rho(x)$ is finite for some $x \in X$ then the radius of compactness is a Lipschitz map [4, Prop. 5.1]. Note that ρ is G-invariant.

Lemma 2. *Let $x, y \in X$ and $\{g_i\}_I$ be a net in G with $g_i x \to y$. Then there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \mid i \geq i_0\}$.*

Proof. Since X is locally compact there exists an index i_0 such that the set $F(x)$ has compact closure, where $F := \{g_i \mid i \geq i_0\}$. We claim that for every $z \in C_x$ the set $F(z)$ also has compact closure, hence $C_x \subset K(F)$. The strategy is to start with an open ball $B_r(x)$ with radius $r < \rho(x)$ and prove that $F(z)$ has compact closure for every $z \in B_r(x)$. Then our claim follows from the definition of C_x. To prove the claim take a sequence $\{g_n z\} \subset F$. Since the closure of $F(x)$ is compact we may assume, upon passing to a subsequence, that $g_n x \to w$ for some w in the closure of $F(x)$. Assume that $\rho(x)$ is finite and take a positive number ε such that $r + \varepsilon < \rho(x)$. Then for n big enough

$$d(g_n z, w) \leq d(g_n z, g_n x) + d(g_n x, w) = d(z, x) + d(g_n x, w) < r + \varepsilon < \rho(x).$$

Recall that the radius of convergence is a continuous map, and since $g_n x \to w$ then $\rho(x) = \rho(w)$. So, the sequence $\{g_n z\}$ is contained eventually in a ball of w with compact closure, hence it has a convergence subsequence. The same also holds in the case where $\rho(x) = +\infty$. $\square$

Theorem 3. *Let X be a locally compact metric space with group of isometries G and let $\{g_i\}$ be a net in G for which $g_i x$ converges to y, for some $x, y \in X$. Then there exist a subnet $\{g_j\}$ of $\{g_i\}$ and an isometry $f : C_x \to X$ such that g_j converges to f pointwise on C_x and $f(C_x) = C_{f(x)}$*

Proof. By Lemma 2 there is an index $i_0 \in I$ such that $C_x \subset K(F)$, where $F := \{g_i \mid i \geq i_0\}$. Hence, by Lemma 1, there exists a subnet $\{g_j\}$ of $\{g_i\}$ which converges to an isometry $f : K(F) \to X$ on $K(F)$. Therefore, $g_j \to f$ on C_x. Let us show that $f(C_x) = C_{f(x)}$. Since $d(x, g_j^{-1} f(x)) = d(g_j x, f(x)) \to 0$ it follows that $g_j^{-1} f(x) \to x$. Hence, by repeating the previous procedure, there exist a subnet $\{g_k\}$ of $\{g_j\}$ and an isometry $h : C_{f(x)} \to X$ such that $g_k^{-1} \to h$

pointwise on $C_{f(x)}$ and $h(f(x)) = x$. Note that $g_k x \in C_{f(x)}$ eventually for every k, since $g_k x \to f(x)$ and $C_{f(x)}$ is clopen. Therefore, $g_k C_x = C_{g_k x} = C_{f(x)}$. Take a point $z \in C_x$. Then, $g_k z \to f(z)$ and since $C_{f(x)}$ is clopen then $f(z) \in C_{f(x)}$, so $f(C_x) \subset C_{f(x)}$. By repeating the same arguments as before, it follows that $hC_{f(x)} \subset C_x$. Take now a point $w \in C_{f(x)}$. Then $h(w) \in C_x$, hence $g_k^{-1}(w) \in C_x$ eventually for every k. So, $w = g_k g_k^{-1}(w) \to f(h(w)) \in f(C_x)$ from which follows that $C_{f(x)} \subset f(C_x)$. □

A few words about properness. A continuous action of a topological group H on a topological space Y is called proper (or Bourbaki proper) if the map $H \times Y \to Y \times Y$ with $(g, x) \mapsto (x, gx)$ for $g \in H$ and $x \in Y$, is proper, i.e., it is continuous, closed and the inverse image of a singleton is a compact set [1, Ch. III, §4.1 Def. 1]. In terms of nets, a continuous action is proper if and only if whenever we have two nets $\{g_i\}$ in H and $\{x_i\}$ in Y, for which both $\{x_i\}$ and $\{g_i x_i\}$ converge, then $\{g_i\}$ has a convergent subnet. For isometric actions, it is easy to see that a continuous action is proper if and only if whenever we have a net $\{g_i\}$ in H for which $\{g_i x\}$ converges for some $x \in Y$, then $\{g_i\}$ has a convergent subnet. If H is locally compact and Y is Hausdorff, then H acts properly on Y if and only if for every $x, y \in Y$ there exist neighborhoods U and V of x and y, respectively, such that the set $\{g \in H \mid gU \cap V \neq \varnothing\}$ has compact closure in H [1, Ch. III, §4.4 Prop. 7]. Observe that if H acts properly on a locally compact space Y then H is also locally compact.

A direct implication of Theorem 3 is the van Dantzig-van der Waerden Theorem [3]. The advantage of our proof, comparing to the proofs given in the original work of van Dantzig-van der Waerden or in [5, Thm. 4.7, pp. 46–49], is that it is considerably shorter.

Corollary 4. *(van Dantzig-van der Waerden theorem 1928) Let X be a connected locally compact metric space with group of isometries G. Then G acts properly on X and is locally compact.*

Another application of Theorem 3 is that we can rederive the results of Gao and Kechris in [4, Thm. 5.4 and Cor. 6.2].

Corollary 5. *(Gao-Kechris theorem 2003) Let X be a locally compact metric space with finitely many pseudo-components. Then the group of isometries G of X is locally compact. If X is pseudo-connected, then G acts properly on X.*

Proof. Let $C_1, C_2, \ldots, C_n$ denote the pseudo-components of X and take points $x_1 \in C_1, x_2 \in C_2, \ldots, x_n \in C_n$ and open balls $B_r(x_m) \subset C_m$, $m = 1, 2, \ldots, n$, $r > 0$ such that all $B_r(x_m)$ have compact closures. We will show that the set $V := \bigcap_{m=1}^{n} \{g \in G \mid gx_m \in B_r(x_m)\}$ is an open neighborhood of the identity in G with compact closure. Indeed, take a net $\{g_i\}$ in V. Since each $B_r(x_m)$ has compact closure there exist a subnet $\{g_j\}$ of $\{g_i\}$ and points $y_1 \in C_1, y_2 \in C_2, \ldots, y_n \in C_n$ such that $g_j x_m \to y_m$ for every $m = 1, 2, \ldots, n$. Theorem 3 implies that there exist a subnet $\{g_l\}$ of $\{g_j\}$ and isometries $f_m : C_m \to X$ such that $g_l \to f_m$ on C_m and $f_m(C_m) = C_m$ for all m. The last implies that $\{g_l\}$ converges to an isometry on X, hence V has compact closure.

If X is pseudo-connected the proof of the statement follows directly from Theorem 3. □

Remark 6. Note that in Corollary 5 we do not require that X is separable as in [4, Thm 5.4 and Cor. 6.2]. This is not a real improvement since if X has countably many pseudo-components then it is separable. Indeed, we define a relation on X by $x\mathcal{S}y$ if and only if there exist separable balls $B_r(x)$ and $B_l(y)$ with $y \in B_r(x)$ and $x \in B_l(y)$. Let $U(x)$ be the equivalence class of x in the transitive closure of the relation $\mathcal{S}$. Then, each $U(x)$ is a separable clopen subset of X [5, Lem. 3 in App. 2]. By construction $C_x \subset U(x)$, therefore X is separable.

Acknowledgements

We would like to thank the referee for an extremely careful reading of the manuscript. Her/his remarks and comments helped us to improve considerably the presentation of the paper.

References

[1] N. Bourbaki, *Elements of mathematics. General topology. Part 1*, Hermann, Paris, 1966. MR0205210 (34 #5044a)

[2] N. Bourbaki, *Elements of mathematics. General topology. Part 2*, Hermann, Paris, 1966. MR0205211 (34 #5044b)

[3] D. van Dantzig and B. L. van der Waerden, Über metrisch homogene Räume, Abh. Math. Seminar Hamburg **6** (1928), 367–376.

[4] S. Gao and A. S. Kechris, On the classification of Polish metric spaces up to isometry, Mem. Amer. Math. Soc. **161** (2003), no. 766, viii+78 pp. MR1950332 (2004b:03067)

[5] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, a division of John Wiley & Sons, New York, 1963. MR0152974 (27 #2945)

[6] A. Manoussos and P. Strantzalos, On the group of isometries on a locally compact metric space, J. Lie Theory **13** (2003), no. 1, 7–12. MR1958572 (2004a:54045)

Received January 17 2010; accepted March 16 2010

Antonios Manoussos
Fakultät für Mathematik, SFB 701, Universität Bielefeld
Postfach 100131, D-33501 Bielefeld, Germany
E-mail: `amanouss@math.uni-bielefeld.de`
URL: `http://www.math.uni-bielefeld.de/~amanouss`