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DYNAMICS OF TUPLES OF MATRICES

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Abstract. In this article we answer a question raised by N. Feldman in 2008 concerning the dynamics of tuples of operators on $\mathbb{R}^n$. In particular, we prove that for every positive integer $n \geq 2$ there exist n-tuples $(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices over $\mathbb{R}$ such that $(A_1, A_2, \ldots, A_n)$ is hypercyclic. We also establish related results for tuples of 2×2 matrices over $\mathbb{R}$ or $\mathbb{C}$ being in Jordan form.

1. Introduction

Following the recent work of Feldman in [4] an n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \ldots, T_n$ acting on a locally convex space X. The tuple $(T_1, T_2, \ldots, T_n)$ is hypercyclic if there exists a vector $x \in X$ such that the set

$$\{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} x : k_1, k_2, \ldots, k_n \geq 0\}$$

is dense in X. Such a vector x is called hypercyclic for $(T_1, T_2, \ldots, T_n)$ and the set of hypercyclic vectors for $(T_1, T_2, \ldots, T_n)$ will be denoted by $HC((T_1, T_2, \ldots, T_n))$. The above definition generalizes the notion of hypercyclicity to tuples of operators. For an account of results, comments and an extensive bibliography on hypercyclicity we refer to [1], [5], [6] and [7]. For results concerning the dynamics of tuples of operators see [2], [3], [4] and [9].

In [4] Feldman showed, among other things, that in $\mathbb{C}^n$ there exist diagonalizable $(n+1)$-tuples of matrices having dense orbits. In addition he proved that there is no n-tuple of diagonalizable matrices on $\mathbb{R}^n$ or $\mathbb{C}^n$ that has a somewhere dense orbit. Therefore the following question arose naturally.

Question (Feldman [4])**.** *Are there non-diagonalizable n-tuples on $\mathbb{R}^k$ that have somewhere dense orbits?*

We give a positive answer to this question in a very strong form, as the next theorem shows.

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Theorem 1.1. *For every positive integer $n \geq 2$ there exist n-tuples $(A_1, \ldots, A_n)$ of $n \times n$ non-(simultaneously) diagonalizable matrices over $\mathbb{R}$ such that $(A_1, \ldots, A_n)$ is hypercyclic.*

Restricting ourselves to tuples of 2×2 matrices in Jordan form either on $\mathbb{R}^2$ or $\mathbb{C}^2$, we prove the following.

Theorem 1.2. *There exist 2×2 matrices $A_j, j = 1, 2, 3, 4$, in Jordan form over $\mathbb{R}$ such that (A_1, A_2, A_3, A_4) is hypercyclic. In particular*

$$HC((A_1, A_2, A_3, A_4)) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\}.$$

Theorem 1.3. *There exist 2×2 matrices $A_j, j = 1, 2, \ldots, 8$, in Jordan form over $\mathbb{C}$ such that $(A_1, A_2, \ldots, A_8)$ is hypercyclic.*

2. Products of 2×2 matrices

Lemma 2.1. *Let m be a positive integer and for each $j = 1, 2, \ldots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, i.e. $A_j = \begin{pmatrix} a_j & 1 \\ 0 & a_j \end{pmatrix}$ for $a_1, a_2, \ldots, a_m \in \mathbb{F}$. Then $(A_1, A_2, \ldots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic if and only if the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$ (respectively $\mathbb{R}^2$).

Proof. We prove the above in the case $\mathbb{F} = \mathbb{C}$, since the other case is similar. Observe that

$${A_j}^l = \begin{pmatrix} {a_j}^l & l{a_j}^{l-1} \\ 0 & {a_j}^l \end{pmatrix}$$

for $l \in \mathbb{N}$. As a result we have

$${A_1}^{k_1} {A_2}^{k_2} \ldots {A_m}^{k_m} = \begin{pmatrix} \prod_{j=1}^m {a_j}^{k_j} & \prod_{j=1}^m {a_j}^{k_j} \sum_{s=1}^m \frac{k_s}{a_s} \\ 0 & \prod_{j=1}^m {a_j}^{k_j} \end{pmatrix}.$$

Assume that $(A_1, A_2, \ldots, A_m)$ is hypercyclic and let $\binom{z_1}{z_2} \in \mathbb{C}^2$ be a hypercyclic vector for $(A_1, A_2, \ldots, A_m)$. Then the sequence

$$\left\{ {A_1}^{k_1} {A_2}^{k_2} \ldots {A_m}^{k_m} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$
$$= \left\{ \begin{pmatrix} z_1 \prod_{j=1}^m {a_j}^{k_j} + z_2 \prod_{j=1}^m {a_j}^{k_j} \sum_{s=1}^m \frac{k_s}{a_s} \\ z_2 \prod_{j=1}^m {a_j}^{k_j} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. This implies that $z_2 \neq 0$. Dividing the element in the first row by that in the second, it can easily be shown that the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. The converse can easily be shown. ∎

Remark 2.2. Let m be a positive integer and for each $j = 1, 2, \ldots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. By the proof of Lemma 2.1 it is immediate that whenever $(A_1, A_2, \ldots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic, one can completely describe the set of hypercyclic vectors as

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 : z_2 \neq 0 \right\} \quad \left(\text{respectively } \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\} \right).$$

2.1. **The real case.** The one-dimensional version of Kronecker's theorem stated below (see for example [8, Theorem 438, p. 375]) will be used repeatedly throughout this work.

Theorem 2.3. *If x is a positive irrational number, then the sequence $\{kx - s : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Remark 2.4. If x is a positive irrational number, then the sequence $\{s - kx : k, s \in \mathbb{N}\}$ is also dense in $\mathbb{R}$. Likewise, if x is a negative irrational number, then the sequence $\{s + kx : k, s \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

We shall need the following well-known result; see for example [4].

Theorem 2.5. *If $a, b > 1$ and $\frac{\ln a}{\ln b}$ is irrational, then the sequence $\{\frac{a^n}{b^m} : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$.*

Lemma 2.6. *Let $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $b > 1$ and $\frac{\ln |a|}{\ln b}$ is irrational. Then the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Proof. Since $\frac{\ln |a|}{\ln b}$ is irrational it follows that $\ln b / \ln \frac{1}{a^2}$ is irrational as well. Applying Theorem 2.5 we conclude that the sequence $\{a^{2n} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$. On the other hand the fact that a is negative implies that the sequence $\{a^{2n+1} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^-$. This completes the proof of the lemma. □

Proposition 2.7. *There exist $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \frac{k_3}{a_3} + \frac{k_4}{a_4} \\ {a_1}^{k_1} {a_2}^{k_2} {a_3}^{k_3} {a_4}^{k_4} \end{pmatrix} : k_1, k_2, k_3, k_4 \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$.

Proof. By the lemma above fix $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $a + \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - x_2| < \epsilon$. Note that $a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s$ for every $k, s \in \mathbb{N}$. Note also that $a + \frac{1}{a} < 0$. Hence, by Remark 2.4, the sequence

$$\left\{ s + k \left(a + \frac{1}{a} \right) : k, s \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$; i.e. there exist $k, s \in \mathbb{N}$ such that

$$\left| s + k \left(a + \frac{1}{a} \right) - \left(x_1 - \frac{n}{a} - \frac{m}{b} \right) \right| < \epsilon,$$

i.e.

$$\left| \frac{n}{a} + \frac{m}{b} + k \left(a + \frac{1}{a} \right) + s - x_1 \right| < \epsilon.$$

Hence, setting $a_1 = a, a_2 = b, a_3 = \frac{1}{a}, a_4 = 1$ we prove the result. □

Proof of Theorem 1.2. This is an immediate consequence of Lemma 2.1, Proposition 2.7 and Remark 2.2.

Example 2.8. One may construct many concrete examples of four 2×2 matrices, in Jordan form over $\mathbb{R}$, being hypercyclic. For example, fix $a,b\in\mathbb{R}$ such that $-1<a<0$, $b>1$ and both $a+\frac{1}{a}$, $\frac{\ln|a|}{\ln b}$ are irrational. From the above we conclude that

$$\left(\begin{pmatrix} a & 1\\ 0 & a\end{pmatrix},\begin{pmatrix} b & 1\\ 0 & b\end{pmatrix},\begin{pmatrix} \frac{1}{a} & 1\\ 0 & \frac{1}{a}\end{pmatrix},\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\right)$$

is hypercyclic.

We shall now prove Theorem 1.1 for $n=2$; see Proposition 2.10 (ii). For this we need the following result due to Feldman; see Corollary 3.2 in [4].

Proposition 2.9 (Feldman)**.** *Let $\mathbb{D}$ denote the open unit disk centered at* 0 *in the complex plane. If $b\in\mathbb{D}\setminus\{0\}$, then there exists a dense set $\Delta\subset\mathbb{C}\setminus\mathbb{D}$ such that for every $a\in\Delta$ the sequence $\{a^nb^m : n,m\in\mathbb{N}\}$ is dense in $\mathbb{C}$.*

Proposition 2.10. *(i) Every pair (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ with A_j, $j=1,2$, being either diagonal or in Jordan form is not hypercyclic.*

(ii) There exist pairs (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ such that A_1 is diagonal, A_2 is antisymmetric (rotation matrix) and (A_1,A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1,A_2); i.e.

$$HC((A_1,A_2))=\mathbb{R}^2\setminus\{(0,0)\}.$$

(iii) There exist pairs (A_1,A_2) of 2×2 matrices over $\mathbb{R}$ such that both A_1 and A_2 are antisymmetric and (A_1,A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1,A_2), i.e.

$$HC((A_1,A_2))=\mathbb{R}^2\setminus\{(0,0)\}.$$

Proof. Let us prove assertion (i). The case of A_1,A_2 both diagonal is covered by Feldman; see [4].

Assume that A_1 is diagonal and A_2 is in Jordan form; i.e.

$$A_1=\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix},\quad A_2=\begin{pmatrix} b & 1\\ 0 & b\end{pmatrix}\quad \text{for } a,b\in\mathbb{R}.$$

Suppose that (A_1,A_2) is hypercyclic and let $\binom{x_1}{x_2}\in\mathbb{R}^2$ be a hypercyclic vector for (A_1,A_2). Then the sequence

$$\left\{{A_1}^n{A_2}^m\begin{pmatrix} x_1\\ x_2\end{pmatrix} : n,m\in\mathbb{N}\right\}=\left\{\begin{pmatrix} a^nb^mx_1+ma^nb^{m-1}x_2\\ a^nb^mx_2\end{pmatrix} : n,m\in\mathbb{N}\right\}$$

is dense in $\mathbb{R}^2$. Therefore b cannot be zero. Observe that x_2 cannot be zero either. Take any $y_1\in\mathbb{R}$ and $y_2\in\mathbb{R}\setminus\{0\}$. Then there exist sequences of positive integers $\{n_k\}$, $\{m_k\}$ such that $m_k\to+\infty$ and

$$\begin{aligned} &a^{n_k}b^{m_k}x_1+m_ka^{n_k}b^{m_k-1}x_2\to y_1,\\ &a^{n_k}b^{m_k}x_2\to y_2\end{aligned}$$

as $k\to+\infty$. Since $b\neq 0$, $y_2\neq 0$ and $x_2\neq 0$ we get that

$$a^{n_k}b^{m_k}x_1\to\frac{y_2x_1}{x_2}\quad\text{and}\quad |m_ka^{n_k}b^{m_k-1}x_2|=\frac{m_k}{|b|}|a^{n_k}b^{m_k}x_2|\to+\infty$$

as $k \to +\infty$. From the last, it clearly follows that

$$|a^{n_k} b^{m_k} x_1 + m_k a^{n_k} b^{m_k - 1} x_2| \to +\infty,$$

which is a contradiction.

Assume now that both A_1, A_2 are in Jordan form; i.e.

$$A_1 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},$$

for $a, b \in \mathbb{R}$ and (A_1, A_2) is hypercyclic. Lemma 2.1 implies that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ a^n b^m \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Observe that neither $|a|$ nor $|b|$ is equal to 1. By taking the absolute value in the second coordinate and then applying the logarithmic function, we find that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ n \ln|a| + m \ln|b| \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence the sequence

$$\left\{ \begin{pmatrix} n\frac{\ln|a|}{a} + m\frac{\ln|a|}{b} \\ n\frac{\ln|a|}{a} + m\frac{\ln|b|}{a} \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Subtracting the second coordinate from the first one, we conclude that the sequence

$$\left\{ m \left(\frac{\ln|a|}{b} - \frac{\ln|b|}{a} \right) : m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$, which is absurd. We proceed with the proof of assertion (ii). By Proposition 2.9 there exist $a \in \mathbb{R} \setminus \mathbb{Q}$ and $b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Write $b = |b| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b|\cos\theta & -|b|\sin\theta \\ |b|\sin\theta & |b|\cos\theta \end{pmatrix}.$$

Then we have

$$A_1{}^n A_2{}^m = \begin{pmatrix} a^n |b|^m \cos m\theta & -a^n |b|^m \sin m\theta \\ a^n |b|^m \sin m\theta & a^n |b|^m \cos m\theta \end{pmatrix}.$$

Applying in the above relation the vector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and taking into account that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$, we conclude that the sequence

$$\left\{ A_1{}^n A_2{}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n, m \in \mathbb{N} \right\} = \left\{ \begin{pmatrix} a^n |b|^m \cos m\theta \\ a^n |b|^m \sin m\theta \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence (A_1, A_2) is hypercyclic. It is now easy to show that every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1, A_2).

In order to prove the last assertion we follow a similar line of reasoning as above. That is, by Proposition 2.9 there exist $a, b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Write $a = |a| e^{i\phi}$, $b = |b| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a|\cos\phi & -|a|\sin\phi \\ |a|\sin\phi & |a|\cos\phi \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b|\cos\theta & -|b|\sin\theta \\ |b|\sin\theta & |b|\cos\theta \end{pmatrix}.$$

A direct computation gives that $\left\{ {A_1}^n {A_2}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n, m \in \mathbb{N} \right\}$ is equal to

$$\left\{ \begin{pmatrix} |a|^n |b|^m \cos(n\phi + m\theta) \\ |a|^n |b|^m \sin(n\phi + m\theta) \end{pmatrix} : n, m \in \mathbb{N} \right\},$$

and by the choice of a, b we conclude that the vector $\binom{1}{0}$ is hypercyclic for (A_1, A_2). This completes the proof of the proposition. □

Question 2.11. What is the minimum number of 2×2 matrices over $\mathbb{R}$ in Jordan form so that their tuple forms a hypercyclic operator?

2.2. **The complex case.** In what follows we will be writing $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of a complex number z respectively.

Proposition 2.12. *There exist $a_j \in \mathbb{C}$, $j = 1, 2, \ldots, 8$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_8}{a_8} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_8}^{k_8} \end{pmatrix} : k_1, k_2, \ldots, k_8 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

Proof. The proof is in the same spirit as the proof of Proposition 2.7. Fix $a, b \in \mathbb{C}$ such that $-1 < a < 0$, $a + \frac{1}{a}, a - \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$ (see Proposition 2.9). Let $z_1, z_2 \in \mathbb{C}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - z_2| < \epsilon$. Note that

$$a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s (ia)^\xi \left(\frac{1}{ia}\right)^\xi (4i)^\rho \left(-\frac{1}{4}\right)^\rho$$

for every $k, s, \xi \in \mathbb{N}$ and $\rho \in 4\mathbb{N}$. Note that $a + \frac{1}{a} < 0$ and $a - \frac{1}{a} > 0$. Hence, by Theorem 2.3, the sequence

$$\left\{ \xi \left(a - \frac{1}{a}\right) - \left(\frac{\rho}{4}\right) : \xi \in \mathbb{N}, \rho \in 4\mathbb{N} \right\}$$

is dense in $\mathbb{R}$. As a result, there exist $\xi \in \mathbb{N}$ and $\rho \in 4\mathbb{N}$ such that

$$\left| \Im \left(i\xi \left(a - \frac{1}{a}\right) - i\left(\frac{\rho}{4}\right) \right) - \Im \left(z_1 - \frac{n}{a} - \frac{m}{b} \right) \right| < \epsilon;$$

i.e. we have that

$$\left| \Im \left(\frac{n}{a} + \frac{m}{b} + i\xi \left(a - \frac{1}{a}\right) - i\left(\frac{\rho}{4}\right) \right) - \Im(z_1) \right| < \epsilon.$$

By Remark 2.4, the sequence

$$\left\{ k \left(a + \frac{1}{a}\right) + s : k, s \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$. Hence, there exist $k, s \in \mathbb{N}$ such that

$$\left| k \left(a + \frac{1}{a}\right) + s - \left(4\rho + \Re \left(z_1 - \frac{n}{a} - \frac{m}{b} \right) \right) \right| < \epsilon;$$

i.e. we have that

$$\left| \Re \left(\frac{n}{a} + \frac{m}{b} + k \left(a + \frac{1}{a}\right) - 4\rho + s \right) - \Re(z_1) \right| < \epsilon.$$

But this means that the real and imaginary parts of the complex number

$$\frac{n}{a}+\frac{m}{b}+k\left(a+\frac{1}{a}\right)+s+i\xi\left(a-\frac{1}{a}\right)-i\frac{\rho}{4}-4\rho$$

are within ϵ of the real and imaginary parts of z_1. Hence, setting $a_1=a, a_2=b, a_3=\frac{1}{a}, a_4=1, a_5=ia, a_6=\frac{1}{ia}, a_7=4i, a_8=-\frac{1}{4}$, we prove the result. □

Proof of Theorem 1.3. By Proposition 2.12, Lemma 2.1 and Remark 2.2 the assertion follows.

Example 2.13. Fix $a,b\in\mathbb{C}$ such that $-1<a<0$, $a+\frac{1}{a}, a-\frac{1}{a}\in\mathbb{R}\setminus\mathbb{Q}$ and $\{a^nb^m : n,m\in\mathbb{N}\}$ is dense in $\mathbb{C}$. From the above it is evident that the 8-tuple of 2×2 matrices in Jordan form over $\mathbb{C}$ given by

$$\begin{pmatrix} a & 1\\ 0 & a\end{pmatrix},\begin{pmatrix} b & 1\\ 0 & b\end{pmatrix},\begin{pmatrix} \frac{1}{a} & 1\\ 0 & \frac{1}{a}\end{pmatrix},\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},$$
$$\begin{pmatrix} ia & 1\\ 0 & ia\end{pmatrix},\begin{pmatrix} \frac{1}{ia} & 1\\ 0 & \frac{1}{ia}\end{pmatrix},\begin{pmatrix} 4i & 1\\ 0 & 4i\end{pmatrix},\begin{pmatrix} -\frac{1}{4} & 1\\ 0 & -\frac{1}{4}\end{pmatrix}$$

is hypercyclic.

Question 2.14. What is the minimum number of 2×2 matrices over $\mathbb{C}$ in Jordan form so that their tuple forms a hypercyclic operator?

3. Products of 3×3 matrices

In this section we start with the following special case of Corollary 3.5 in [4], due to Feldman, which will be of use to us in the following.

Proposition 3.1 (Feldman)**.** *If $b_1,b_2\in\mathbb{D}\setminus\{0\}$, then there exists a dense set $\Delta\subset\mathbb{C}\setminus\mathbb{D}$ such that for every $a_1,a_2\in\Delta$ the sequence*

$$\left\{\begin{pmatrix} {a_1}^n{b_1}^m\\ {a_2}^n{b_2}^l\end{pmatrix} : n,m,l\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}^2$.

In order to handle products of 3×3 matrices, we establish the following:

Corollary 3.2. *There exist $a\in\mathbb{C}$ and $b,c,d\in\mathbb{R}$ such that the sequence*

$$\left\{\begin{pmatrix} a^nb^m\\ c^nd^l\end{pmatrix} : n,m,l\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}\times\mathbb{R}$.

Proof. Fix two real numbers b_1,b_2 with $b_1,b_2\in(0,1)$. By Proposition 3.1 there exist $a_1,a_2\in\mathbb{C}\setminus\mathbb{D}$ such that the sequence

$$\left\{\begin{pmatrix} {a_1}^n{b_1}^m\\ {a_2}^n{b_2}^l\end{pmatrix} : n,m,l\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}^2$. Define $a=a_1$, $b=b_1$, $c=|a_2|$ and $d=-\sqrt{b_2}$. Observe that the sequence

$$\left\{\begin{pmatrix} a^nb^m\\ c^n{b_2}^l\end{pmatrix} : n,m,l\in\mathbb{N}\right\}$$

is dense in $\mathbb{C}\times[0,+\infty)$. Take $z\in\mathbb{C}$ and $x\in\mathbb{R}$.

Case I. $x \geq 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}, \{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to x.$$

Since ${b_2}^{l_k} = d^{2l_k}$ we get $c^{n_k} d^{2l_k} \to x$.

Case II. $x < 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}, \{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to \frac{x}{d}.$$

The last implies that $c^{n_k} d^{2l_k+1} \to x$. This completes the proof of the corollary. □

The main result of this section is to prove Theorem 1.1 for $n = 3$. This is stated and proved below.

Proposition 3.3. *There exist* 3 *tuples* (A_1, A_2, A_3) *of* 3×3 *matrices over* $\mathbb{R}$ *such that* (A_1, A_2, A_3) *is hypercyclic.*

Proof. By Corollary 3.2 there exist $a \in \mathbb{C}$ and $b, c, d \in \mathbb{R}$ such that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times \mathbb{R}$. Write $a = |a| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a| \cos\theta & -|a| \sin\theta & 0 \\ |a| \sin\theta & |a| \cos\theta & 0 \\ 0 & 0 & c \end{pmatrix}, \; A_2 = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Then we have

$${A_1}^n {A_2}^m {A_3}^l = \begin{pmatrix} |a|^n b^m \cos n\theta & -|a|^n b^m \sin n\theta & 0 \\ |a|^n b^m \sin n\theta & |a|^n b^m \cos n\theta & 0 \\ 0 & 0 & c^n d^l \end{pmatrix},$$

which in turn gives

$${A_1}^n {A_2}^m {A_3}^l \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} |a|^n b^m \cos n\theta \\ |a|^n b^m \sin n\theta \\ c^n d^l \end{pmatrix}.$$

The last and the choice of a, b, c, d imply that (A_1, A_2, A_3) is hypercyclic with $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ being a hypercyclic vector for (A_1, A_2, A_3). □

4. Proof of Theorem 1.1

By Proposition 2.10, there exist 2×2 matrices B_1 and B_2 such that (B_1, B_2) is hypercyclic.

Case I. $n = 2k$ for some positive integer k. For $k = 1$ the result follows by Proposition 2.10. Assume that $k > 1$. Each A_j will be constructed by blocks of 2×2 matrices. Let I_2 be the 2×2 identity matrix. We will be using the notation $diag(D_1, D_2, \ldots, D_n)$ to denote the diagonal matrix with diagonal entries the block matrices $D_1, D_2, \ldots, D_n$. Define $A_1 = diag(B_1, I_2, \ldots, I_2)$, $A_2 = diag(B_2, I_2, \ldots, I_2)$, $A_3 = diag(I_2, B_1, I_2, \ldots, I_2)$, $A_4 = diag(I_2, B_2, I_2, \ldots, I_2)$ and so on up to $A_{n-1} = diag(I_2, \ldots, I_2, B_1)$, $A_n = diag(I_2, \ldots, I_2, B_2)$.

It is now easy to check that $(A_1, A_2, \ldots, A_n)$ is hypercyclic and furthermore that the set $HC((A_1, A_2, \ldots, A_n))$ is

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{2j-1}^2 + x_{2j}^2 \neq 0, \forall j = 1, 2, \ldots, k\}.$$

Case II. $n = 2k + 1$ for some positive integer k. If $k = 1$ the result follows by Proposition 3.3. Suppose $k > 1$. For simplicity we treat the case $k = 2$, since the general case follows by similar arguments. By Proposition 3.3 there exist C_1, C_2, C_3, 3×3 matrices such that (C_1, C_2, C_3) is hypercyclic. Let I_3 be the 3×3 identity matrix. Define $A_1 = diag(B_1, I_3)$, $A_2 = diag(B_2, I_3)$, $A_3 = diag(I_2, C_1)$, $A_4 = diag(I_2, C_2)$ and $A_5 = diag(I_2, C_3)$.

It can easily be shown that $(A_1, A_2, \ldots, A_5)$ is hypercyclic. The details are left to the reader.

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