A group of isometries with non-closed orbits

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A B S T R A C T

In this note we give an example of a one-dimensional manifold with two connected components and a complete metric whose group of isometries has an orbit which is not closed. This answers a question of S. Gao and A.S. Kechris.

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1. Preliminaries and the construction of the example

In [3, p. 35] S. Gao and A.S. Kechris asked the following question. Let (X, d) be a locally compact complete metric space with finitely many pseudo-components or connected components. Does its group of isometries have closed orbits? This is the case if X is connected since then the group of isometries acts properly by an old result of van Dantzig and van der Waerden [1] and hence all of its orbits are closed. The above question arose in the following context. Suppose a locally compact group with a countable base acts on a locally compact space with a countable base. Then the action has locally closed orbits (i.e. orbits which are open in their closures) if and only if there exists a Borel section for the action (see [4,2]) or, in other terminology, the corresponding orbit equivalence relation is smooth. For isometric actions it is easy to see that an orbit is locally closed if and only if it is closed. In this note we give a negative answer to the question of Gao and Kechris. Our space is a one-dimensional manifold with two connected components, one compact isometric to S^1, and one non-compact, the real line with a locally Euclidean metric. It has a complete metric whose group of isometries has non-closed dense orbits on the compact component. In the course of the construction we give an example of a 2-dimensional manifold with two connected components one compact and one non-compact and a complete metric whose group G of isometries also has non-closed dense orbits on the compact component. The difference is that G contains a subgroup of index 2 which is isomorphic to $\mathbb{R}$.

Let (Y, d_1) be a metric space. Later on Y will be a torus with a flat Riemannian metric. Let $Z = Y \cup (Y \times \mathbb{R})$. We fix two positive real numbers R and M. We endow Z with the following metric d depending on R and M.

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$$d(y_1, y_2) = d_1(y_1, y_2)$$
$$d\big((y_1, t_1), (y_2, t_2)\big) = d_1(y_1, y_2) + \min\big(|t_1 - t_2|, M\big)$$
$$d\big(y_1, (y_2, t_2)\big) = d\big((y_2, t_2), y_1\big) = d(y_1, y_2) + R,$$

for $y_1, y_2 \in Y$ and $t_1, t_2 \in \mathbb{R}$. It is easy to check that d is a metric on Z if $2R \geqslant M$. The metric space Z has the following properties:

1.1. (a) For a given point $(y, r) \in Y \times \mathbb{R}$ there is a unique point in Y which is closest to (y, r), namely y.

(b) Given a point $y \in Y$ the set of points in $Y \times \mathbb{R}$ which are closest to y is the line $\{y\} \times \mathbb{R}$.

(c) For every point $(y, r) \in Y \times \mathbb{R}$ and every $y' \in Y$ there is a unique point on the line $\{y'\} \times \mathbb{R}$ which is closest to (y, r), namely (y', r).

(d) Let g_Y be an isometry of Y and let $g_\mathbb{R}$ be an isometry of the Euclidean line $\mathbb{R}$. Define a map $g = g(g_Y, g_\mathbb{R}) : Z \to Z$ by $g|Y := g_Y$ and $g(y, r) = (g_Y(y), g_\mathbb{R}(r))$ for $(y, r) \in Y \times \mathbb{R}$. Then g is an isometry of Z.

(e) Every isometry of Z is of the form given in (d) if Y is compact.

Proof. (a) through (d) are easily checked. To prove (e) let g be an isometry of Z. Then $g(Y) = Y$ and $g(Y \times \mathbb{R}) = Y \times \mathbb{R}$, since Y is compact and $Y \times \mathbb{R}$ consists of non-compact components. Then $g_Y := g|Y$ is an isometry of Y. The map $g(g_Y, id)^{-1} \circ g$, where id denotes the identity map, is an isometry of Z which fixes Y, hence maps every line $\{y\} \times \mathbb{R}$ to itself, by (b). Let $h_y : \mathbb{R} \to \mathbb{R}$ be defined by $g(y, t) = (y, h_y(t))$. Then h_y is an isometry of the Euclidean line $\mathbb{R}$ for every $y \in Y$ and all the h_y's are the same, by (c), say $h_y = g_\mathbb{R}$. Thus $g = (g_Y, g_\mathbb{R})$. $\quad\square$

1.2. Let now Y be a 2-dimensional torus with a flat Riemannian metric. Y is also an abelian Lie group whose composition we write as multiplication. Every translation L_x of Y, $L_x(y) = x \cdot y$, is an isometry. Let $g(t)$, $t \in \mathbb{R}$, be a dense one parameter subgroup of Y. Let $H \subset Y \times \mathbb{R}$ be its graph, $H = \{(g(t), t); t \in \mathbb{R}\}$. Our example is $X = Y \cup H$ with the metric induced from $Z = Y \cup (Y \times \mathbb{R})$.

1.3. (a) If $g_\mathbb{R}$ is an isometry of the Euclidean line $\mathbb{R}$ then there is a unique isometry g of X such that $g(y, t) \in Y \times \{g_\mathbb{R}(t)\}$. If $g_\mathbb{R}$ is the translation by a, so $g_\mathbb{R} = L_a$ with $L_a(t) = t + a$, then g is the restriction of $g(L_{g(a)}, L_a)$ to X. If $g_\mathbb{R}$ is the reflection at the origin, $g_\mathbb{R} = -\mathbf{1}$, then g is the restriction of $g(inv, -\mathbf{1})$ to X, where $inv : Y \to Y$, $inv(y) = y^{-1}$. The reflection in $a \in \mathbb{R}$ is the composition $L_{-2a} \circ (-\mathbf{1}) = -\mathbf{1} \circ L_{2a}$.

(b) Every isometry of X is of the form in (a). It follows that the group of isometries of X has dense non-closed orbits on Y and the other component H is one orbit.

(c) H is locally isometric to the real line with the Euclidean metric, actually $d((g(t), t), (g(s), s)) = (1 + \|\overset{\bullet}{g}(0)\|)\, |t - s|$ for small $|t - s|$, where $\overset{\bullet}{g}(0)$ is the tangent of the one-parameter group $g(t)$, $t \in \mathbb{R}$, and $\|\cdot\|$ is the norm on the tangent space of Y at the identity element derived from the Riemannian tensor.

Proof. (c) follows from the definition of the metric d on $Y \times \mathbb{R}$. The maps given in (a) are isometries of Z and map X to X, hence are isometries of X. To prove the uniqueness claim in (a) it suffices to prove it for $g_\mathbb{R} = id$. But then g is the identity on the image of the one-parameter group $g(t)$, $t \in \mathbb{R}$, by 1.1(a) and hence on all of Y. Hence g has the form given by 1.1(d). To show (b) it suffices to show that every isometry h of H is of the form given in (a). This follows from (c). $\quad\square$

1.4. Remark. In our example the space has dimension 2 and the group of orientation preserving isometries is of index 2 in the group of all isometries and is isomorphic to $\mathbb{R}$. We can reduce the dimension of our space to 1 to obtain a group of isometries with closed orbits on the non-compact component, which is diffeomorphic and locally isometric to $\mathbb{R}$, and non-closed dense orbits on the compact component, which is isometric to S^1. The example is as follows. Take a one-dimensional subtorus Y_1 of Y containing the identity element of Y. Define $X_1 = Y_1 \cup H \subset Y \cup H$. Then the group of isometries of X_1 consists of those maps $g_a = g(L_{g(a)}, L_a)$ restricted to Y_1 with $g(a) \in Y_1$, and of the maps $g(inv \circ L_{g(2a)}, -\mathbf{1} \circ L_a)$ restricted to Y_1 with $g(2a) \in Y_1$. The proof follows from the proof of 1.3.

References

[1] D. van Dantzig, B.L. van der Waerden, Über metrisch homogene Räume, Abh. Math. Seminar Hamburg 6 (1928) 367–376.

[2] E.G. Effros, Transformation groups and C^*-algebras, Ann. of Math. (2) 81 (1965) 38–55.

[3] S. Gao, A.S. Kechris, On the classification of Polish metric spaces up to isometry, Mem. Amer. Math. Soc. 161 (766) (2003).

[4] J. Glimm, Locally compact transformation groups, Trans. Amer. Math. Soc. 101 (1961) 124–138.