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Surgery on a pair of transversal manifolds

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Abstract In the paper we construct the algebraic surgery theory of a topological space consisting of two manifolds which intersect transversally. Such a space is a basic example of stratified space. For such spaces, we define the notion of a normal map, an s-triangulation, and surgery obstruction groups, and show that these notions can be realized on the level of spectra. We construct commutative braids of exact sequences which relate the new notions to their classical counterparts. We examine the example of a transversal intersection of two real projective spaces and obtain explicit results regarding structure sets in this situation.

Keywords Surgery obstruction groups · Splitting obstruction groups · Stratified manifolds · Surgery exact sequence

Mathematics Subject Classification 57R67 · 19J25 · 57N99

1 Introduction

The surgery theory of Milnor–Novikov–Browder–Sullivan–Wall is an effective tool for classifying manifolds in anyone of the categories TOP, PL, or Diff, which are homotopy equivalent to a given manifold or Poincaré complex [12]. The generalization

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of this theory, which we develop, can be applied to classifying stratified spaces (see, for example [4,5,13]).

In the current paper we construct the algebraic surgery theory of a stratified space $\mathcal{X} = (X_1, X_2; X_1 \cap X_2 = Y)$ of two manifolds X_1, X_2 with transversal intersection Y. Although we consider only the case $\dim X_1 = \dim X_2$, our result can be extended to the situation that $\dim X_1$ and $\dim X_2$ are arbitrary. Under the assumption that $\dim X_1 = \dim X_2$, we define the notions of normal map, s-triangulation, t-triangulation, and surgery obstruction groups, and show that the notions can be realized on the level of spectra. We construct commutative braids of exact sequences which relate the new notions with their classical counterparts for manifold pairs $Y \subset X_1$ and $Y \subset X_2$. We consider the example of a transversal intersection of two real projective spaces and obtain explicit results regarding the structure sets in this situation. The main results are given in Theorems 3.2, 3.3, 3.4, 3.6, and 3.7.

The rest of the paper is organized as follows. In Sect. 2, we provide preliminary material regarding the algebraic surgery and splitting theory (see [2–4,6,10–12]).

In Sect. 3, we construct the algebraic surgery theory of a stratified space $\mathcal{X}$ of two manifolds with a transversal intersection Y.

In Sect. 4, we show that one can compute structure sets by treating the example of a transversal intersection of two real projective spaces.

2 Algebraic L-theory for a manifold pair

In this section we recall the algebraic surgery theory of manifold pairs (see [2,3,6,10–12]). Throughout the section, X denotes a compact or closed n-dimensional manifold, $n \geq 5$, and for a manifold pair $Y^{n-q} \subset X$, we shall suppose that Y is a locally flat topological submanifold equipped with the structure of a normal topological bundle (see [11, pages 562–563]).

Let X^n, M^n be closed n-dimensional manifolds. A simple homotopy equivalence $f : M \to X$ is called an *s-triangulation* of the manifold X. Two s-triangulations

$$f_i : M_i \to X, \ i = 1, 2$$

are equivalent if there exists a homeomorphism $g : M_1 \to M_2$ such that $f_2 g$ is homotopic to f_1. The set of equivalence classes of s-triangulations is denoted by $\mathcal{S}^{TOP}(X)$.

Let X^n, M^n be closed n-dimensional manifolds. Let ν_M be a stable normal bundle over M and ξ_X a stable bundle over X. A *t-triangulation* of X is a pair (f, b) of maps

$$f \colon M \to X, \ b \colon \nu_M \to \xi_X \tag{2.1}$$

where f is any degree one continuous map and b is a map of stable bundles over f. (A t-triangulation is often called a *normal map* in the literature). The set of concordance, i.e. normal cobordism, classes of normal maps is denoted by $\mathcal{T}^{TOP}(X)$. ($\mathcal{T}^{TOP}(X)$ is often called in the literature the set of normal invariants of X.)

Let $\pi = \pi_1(X)$ denote the fundamental group of X equipped with an orientation character $w \colon \pi \to \{\pm 1\}$. The structure sets $\mathcal{S}^{TOP}(X)$ and $\mathcal{T}^{TOP}(X)$ fit into the

Wall–Novikov–Sullivan surgery exact sequence (see [9] and [10, §10])

$$\cdots \to L_{n+1}(\pi) \to \mathcal{S}^{TOP}(X) \to [X, G/TOP] \to L_n(\pi) \tag{2.2}$$

where $[X, G/TOP]$ is isomorphic to the set of normal invariants $\mathcal{T}^{TOP}(X)$ and $L_n(\pi) = L_n(\pi, w)$, $i = 0, 1, 2, 3 \bmod 4$ are the surgery obstruction groups. Since the members of the structure set $\mathcal{S}^{TOP}$ are defined by simple homotopy equivalences, the surgery groups L_n above are often decorated by a superscript s, yielding L_n^s. The surgery exact sequence (2.2) is isomorphic to the homotopy long exact sequence in dimensions $\geq n$ of the cofibration of spectra (see [10, p. 276] and [11])

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi) \to \mathbb{S}(X) \tag{2.3}$$

where $\mathbf{L}_\bullet$ is the 1-connected cover of the spectrum $\mathbb{L}(1)$ with $\mathbf{L}_{\bullet 0} \simeq G/TOP$ and the surgery obstruction groups $L_i(\pi) = L_i(\pi, w)$, $i = 0, 1, 2, 3 \bmod 4$ are realized by the Ω-spectrum

$$\mathbb{L}(\pi, w) = \mathbb{L}(\pi) = \{\mathbb{L}_{-k}(\pi) : k \in \mathbb{Z}\}, \;\; L_n(\pi) = \pi_n(\mathbb{L}(\pi)).$$

In this case, we have

$$H_n(X; \mathbf{L}_\bullet) \cong [X, G/TOP] \cong \mathcal{T}^{TOP}(X)$$

and

$$\mathbb{S}_m(X) = \pi_m(\mathbb{S}(X)), \;\; \mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X).$$

Note that the cofibration (2.3) is defined functorially for any topological space X with $\pi_1(X) = \pi$.

Let $(X, \partial X)$, $(M, \partial M)$ denote compact n-dimensional manifolds with boundary. Let ν_M be a stable normal bundle over M and ξ_X a stable bundle over X. A *t-triangulation of a manifold with boundary* $(X, \partial X)$ is a pair $((f, b), (\partial f, \partial b))$ of pairs (f, b) and $(\partial f, \partial b)$ where the pair

$$f : M \to X, \;\; b : \nu_M \to \xi_X$$

satisfies f is any continuous map and b is a map of stable bundles over f and the pair

$$\partial f = f|_{\partial M} : \partial M \to \partial X, \;\; \partial b = b|_{(\nu_M|_{\partial M})} : \nu_M|_{\partial M} \to \xi_X|_{\partial X}$$

is a t-triangulation of ∂X (note that ∂X and ∂M are necessary closed manifolds).

Two t-triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \to (X, \partial X), \;\; i = 0, 1$$

are *concordant* (see [6,11]) if there exists a t-triangulation

$$((F, B), (\partial F, \partial B)) : (W, \partial W) \to (X \times I, \partial(X \times I))$$

of the manifold with boundary $(X \times I, \partial(X \times I))$ such that

$$\partial W = M_0 \cup_{\partial M_0} V \cup_{\partial M_1} M_1, \quad \partial V = \partial M_0 \cup \partial M_1,$$

$$F|_{M_i} = f_i, \; B|_{\nu_{M_i}} = b_i, \; i = 0, 1.$$

In this situation, we shall say that we have *a normal map of 4-ads* and write

$$\begin{aligned} &((F, B); (g, c), (f_0, b_0), (f_1, b_1)) : (W; V, M_0, M_1) \\ &\quad \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\}), \end{aligned} \tag{2.4}$$

where

$$(g, c) : V \to \partial X \times I$$

is the restriction of the normal map (F, B). The set of concordance classes is denoted by $\mathcal{T}^{TOP}(X, \partial X)$.

A t_∂*-triangulation of a manifold with boundary* $(X, \partial X)$ is a t-triangulation

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \to (X, \partial X)$$

of a manifold with boundary such that $f|_{\partial M} = \partial f$ is a homeomorphism $\partial M \to \partial X$. Two t_∂-triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \to (X, \partial X), \; i = 0, 1$$

are *concordant* (see [6,11]) if there exists a t-triangulation of the $4 - ad$ (2.4) such that

$$V = \partial M_0 \times I \text{ and } g = \partial f_0 \times I : V \to \partial X \times I.$$

Since ∂f_0 is a homeomorphism so is g. The set of concordance classes is denoted by $\mathcal{T}_\partial^{TOP}(X, \partial X)$.

An s*-triangulation of a manifold with boundary* $(X, \partial X)$ is a simple homotopy equivalence of manifolds with boundary (see [6,10,11])

$$(f, \partial f) : (M, \partial M) \to (X, \partial X).$$

This means that $\partial f = f|_{\partial M}$ and the maps f and ∂f are simple homotopy equivalences. Two s-triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \to (X, \partial X),\ \ i = 0, 1$$

are *concordant*, i.e. cobordant, if there exists a simple homotopy equivalence

$$(F; g, f_0, f_1)\colon (W; V, M_0, M_1) \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

of 4-ads with

$$\partial W = M_0 \cup_{\partial M_0} V \cup_{\partial M_1} M_1,\ \ \partial V = \partial M_0 \cup \partial M_1.$$

This means that $g = F|_V$, $f_i = F|_{M_i}$, and F, g, f_i are simple homotopy equivalences. The set of concordance classes is denoted by $\mathcal{S}^{TOP}(X, \partial X)$. Note that if $\partial X = \emptyset$ then $\mathcal{S}^{TOP}(X, \partial X) = \mathcal{S}^{TOP}(X)$ by the s-cobordism theorem [8].

An s_∂*-triangulation of a manifold with boundary* $(X, \partial X)$ is an s-triangulation

$$(f, \partial f) : (M, \partial M) \to (X, \partial X)$$

of a manifold with boundary such that $f|_{\partial M} = \partial f$ is a homeomorphism $\partial M \to \partial X$. Two s_∂-triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \to (X, \partial X),\ \ i = 0, 1$$

are *concordant* (see [6,11]) if there exists an s-triangulation of the $4 - ad$ (2.4) such that

$$V = \partial M_0 \times I \text{ and } g = \partial f_0 \times I : V \to \partial X \times I.$$

Since ∂f_0 is a homeomorphism, so is g. The set of concordance classes is denoted $\mathcal{S}^{TOP}_\partial(X, \partial X)$.

The natural inclusion $\partial X \to X$ induces a map of cofibrations in (2.3) such that we obtain a homotopy commutative diagram of spectra

$$\begin{array}{ccc} (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi_1(\partial X)) \\ \downarrow & & \downarrow \\ X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi_1(X)) \end{array}$$

which we can extend (see [9, Lemma 1]) to the following homotopy commutative diagram of spectra

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega\mathbb{S}(\partial X) & \longrightarrow & (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi_1(\partial X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega\mathbb{S}_\partial(X, \partial X) & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi_1(X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega\mathbb{S}(X, \partial X) & \longrightarrow & (X/\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}^{rel} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbb{S}(\partial X) & \longrightarrow & \Sigma(\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma\mathbb{L}(\pi_1(\partial X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{2.5}
$$

where $\mathbb{L}^{rel} = \mathbb{L}(\pi_1(\partial X) \to \pi_1(X))$ is the spectrum in [10] for relative L-groups and Σ is the suspension functor.

The homotopy long exact sequences of any three consecutive rows of (2.5) are the usual algebraic surgery exact sequences for the manifold with boundary $(X, \partial X)$. We shall use the following notation:

$$
\begin{aligned}
&\mathcal{T}_\partial^{TOP}(X, \partial X) = \pi_n(X_+ \wedge \mathbf{L}_\bullet), \quad \mathcal{T}^{TOP}(X, \partial X) = \pi_n\left((X/\partial X)_+ \wedge \mathbf{L}_\bullet\right),\\
&\mathcal{S}_\partial^{TOP}(X, \partial X) = \pi_{n+1}(\mathbb{S}_\partial(X, \partial X)), \quad \mathcal{S}^{TOP}(X, \partial X) = \pi_{n+1}(\mathbb{S}(X, \partial X)).
\end{aligned}
$$

Let X^n be a closed n-dimensional manifold and let Y^{n-q} be a closed submanifold of codimension q. A pair $(X, Y) = (X^n, Y^{n-q}, \xi)$ is called a *closed manifold pair* (see [2,3,6] and [11, page 570]), if Y is a locally flat submanifold of X and is equipped with a topological normal block bundle ξ.

A *t-triangulation of a closed manifold pair* (X, Y) [10, page 570]

$$
(f, g)\colon (M, N) \to (X, Y) \tag{2.6}
$$

is a t-triangulation (2.1) $f\colon M \to X$ such that $N = f^{-1}(Y)$ is a transversal preimage of Y, $g = f|_N$, and $(M, N) = (M, N, \nu)$ is a closed manifold pair with the block bundle ν induced from ξ. Let $Q = X \setminus Y$ be the compliment of the submanifold Y in X with the boundary $\partial(X \setminus Y) = \partial Q = S(\xi)$ and let $P = M \setminus N$ be the compliment of the submanifold N in M with the boundary $\partial(M \setminus N) = \partial P = S(\nu)$ [10]. It is easy to show that the t-triangulation (2.6) has the following properties:

(i) the restriction

$$g = f|_N \colon N \to Y$$

is a t-triangulation of the manifold Y;

(ii) the restriction

$$(h, \partial h) = (f|_P, \partial(f_P)) \colon (P, \partial P) \to (Q, \partial Q)$$

is a t-triangulation of the manifold with boundary $(Q, \partial Q)$;

(iii) the restriction ∂h coincides with the map

$$\partial g^! : S(\nu) \to S(\xi),$$

where the map

$$(g^!, \partial g^!) : (E(\nu), S(\nu)) \to (E(\xi), S(\xi)),$$

is induced by transfer and $f = g^! \cup_{\partial h} h$.

We have a natural isomorphism [11, §7.2]

$$\mathcal{T}^{TOP}(X, Y) \overset{\cong}{\to} \mathcal{T}^{TOP}(X)$$

and a natural map

$$\mathcal{T}^{TOP}(X) \to \mathcal{T}^{TOP}(Y)$$

which is given by restricting the normal map. This map is realized on the spectra level (see [1,2,6,11]) by the map of spectra

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^q(Y_+ \wedge \mathbf{L}_\bullet) \tag{2.7}$$

which fits into the cofibration

$$(X \setminus Y)_+ \wedge \mathbf{L}_\bullet \to X_+ \wedge \mathbf{L}_\bullet \to \Sigma^q(Y_+ \wedge \mathbf{L}_\bullet).$$

An *s-triangulation of the closed manifold pair* (X, Y) [11, page 571] is a t-triangulation of the pair (X, Y) such that

$$f \colon M \to X,\ g = f|_N \colon N \to Y,\ \text{and } h = f|_P \colon (P, S(\nu)) \to (Q, S(\xi)) \tag{2.8}$$

are simple homotopy equivalences. Two s-triangulations of closed manifold pairs

$$(f_i; g_i, h_i) \colon (M_i; N_i, M_i \setminus N_i) \to (X; Y, X \setminus Y),\ i = 0, 1$$

are *concordant*, if there exists a triple of simple homotopy equivalences

$$(F; G, H)\colon (W; V, W \setminus V) \to (X \times I; Y \times I, (X \setminus Y) \times I)$$

where

$$\partial W = M_0 \cup M_1, \quad \partial V = N_0 \cup N_1,$$

$$\partial(W \setminus V) = (M_0 \setminus N_0) \cup (M_1 \setminus N_1) \cup F^{-1}(\partial Y \times I),$$

$$F_{M_i} = f_i, \quad G_{N_i} = g_i, \quad H_{M_i \setminus N_i} = h_i.$$

Denote by $\mathcal{S}^{TOP}(X, Y)$ the set of concordance classes of s-triangulations of the manifold pair (X, Y, ξ).

A simple homotopy equivalence $f : M \to X$ splits along a submanifold Y if it is homotopy equivalent to an s-triangulation of the manifold pair (X, Y). By the definition, every s-triangulation $(f; g, h)$ of the manifold pair (X, Y) defines s-triangulations f, g, and h of the manifolds X, Y, and the manifold with boundary $(X \setminus Y, \partial(X \setminus Y))$, respectively.

Thus we obtain natural maps [11, §7.2]

$$\begin{aligned} &\mathcal{S}^{TOP}(X, Y) \to \mathcal{S}^{TOP}(X), \quad \mathcal{S}^{TOP}(X, Y) \to \mathcal{S}^{TOP}(Y), \quad \text{and} \\ &\mathcal{S}^{TOP}(X, Y) \to \mathcal{S}^{TOP}(X \setminus Y, \partial(X \setminus Y)). \end{aligned} \tag{2.9}$$

Let

$$F = \begin{pmatrix} \pi_1(S(\xi)) & \to \pi_1(X \setminus Y) \\ \downarrow & \quad\downarrow \\ \pi_1(E(\xi)) & \to \pi_1(X) \end{pmatrix} \tag{2.10}$$

be a push-out square of fundamental groups with orientations for a manifold pair (X, Y) where $S(\xi)$ is a boundary of a tubular neighborhood $E(\xi)$ of Y in X.

Denote by $LS_{n-q}(F)$, $n - q = 0, 1, 2, 3 \bmod 4$ the group of obstructions to splitting a simple homotopy equivalence $f : M \to X$ along the submanifold Y (see [2,3,6,10,11]), and let

$$\Theta^{TOP} : \mathcal{S}^{TOP}(X) \to LS_{n-q}(F). \tag{2.11}$$

be the splitting obstruction map.

Denote also by $LP_{n-q}(F)$, $n = 0, 1, 2, 3 \bmod 4$ the group of obstructions to performing surgeries on the closed manifold pair (X, Y) (see [2,3,6,11,12]), and let

$$\mathcal{T}^{TOP}(X) \to LP_{n-q}(F) \tag{2.12}$$

be the corresponding surgery obstruction map.

The structure set $\mathcal{S}(X, Y)$ and the obstruction groups $LS_*(F)$, $LP_*(F)$, are realized by the spectra $\mathbb{S}(X, Y)$, $\mathbb{L}S(F)$, and $\mathbb{L}P(F)$ (see [2,3,6,10,11]) whose homotopy groups are respectively

$$\pi_m(\mathbb{S}(X, Y)) = \mathcal{S}_m(X, Y),\ \pi_m(\mathbb{L}S(F)) = LS_m(F), \pi_m(\mathbb{L}P(F)) = LP_m(F),$$
$$\text{and } \pi_{n+1}(\mathbb{S}(X, Y)) = \mathcal{S}_{n+1}(X, Y) = \mathcal{S}^{TOP}(X, Y).$$

The spectra $\mathbb{L}S(F)$ and $\mathbb{L}P(F)$ fit into the homotopy commutative diagram

$$\begin{array}{ccccc} \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \longrightarrow & \mathbb{L}S(F) \\ \Vert & & \downarrow & & \downarrow \\ \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q}\mathbb{L}(\pi_1(X \setminus Y)) & \longrightarrow & \mathbb{L}P(F) \end{array} \tag{2.13}$$

in which the right square is a pullback, the left hand horizontal maps are suspensions of the transfer maps, and the homotopy fiber of the two vertical maps furthest to the right is $\Omega^{q+1}\mathbb{L}(\pi_1(X))$ (see [2,3,6,11]).

The homotopy long exact sequences of the maps in (2.13) form a braid of exact sequences

$$\begin{array}{ccccccccc} \to L_n(\pi_1(X \setminus Y)) & & \longrightarrow & & L_n(\pi_1(X)) & \to & LS_{n-q-1}(F) & & \longrightarrow \\ \nearrow & \searrow & & \nearrow & & \searrow & \nearrow & \searrow & \\ & & LP_{n-q}(F) & & & & L_n^{rel} & & \\ \searrow & \nearrow & & \searrow & & \nearrow & \searrow & \nearrow & \\ \to LS_{n-q}(F) & & \longrightarrow & & L_{n-q}(\pi_1(Y)) & \longrightarrow & L_{n-1}(\pi_1(X \setminus Y)) & & \to \end{array}, \tag{2.14}$$

as in [12, page 264], where L_n^{rel} are the relative groups $L_n(\pi_1(X \setminus Y) \to \pi_1(X))$.

The maps in (2.11) and (2.12) are realized on the spectra level by maps fitting into the homotopy commutative diagram of spectra (see [3,6], and [11, §7.2])

$$\begin{array}{ccccc} \Omega\mathbb{S}(X, Y) & \longrightarrow & \Omega\mathbb{S}(X) & \longrightarrow & \Sigma^q\mathbb{L}S(F) \\ \Vert & & \downarrow & & \downarrow \\ \Omega\mathbb{S}(X, Y) & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q\mathbb{L}P(F) \end{array} \tag{2.15}$$

in which the right square is a pullback. The homotopy cofiber of the two vertical maps furthest to the right is $\mathbb{L}(\pi_1(X))$.

The homotopy long exact sequences of the maps in the pullback square in (2.15) form a braid of exact sequences [11, Proposition 7.2.6]

$$\begin{array}{ccccccccccc}
\longrightarrow L_{n+1}(\pi_1(X)) & & \longrightarrow & & LS_{n-q}(F) & & \to & & \mathcal{S}_n(X,Y,\xi) \to & & \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & \mathcal{S}_{n+1}(X) & & & & LP_{n-q}(F) & & & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to \mathcal{S}_{n+1}(X,Y,\xi) & & \longrightarrow & & H_n(X;\mathbf{L}_\bullet) & & \longrightarrow & & L_n(\pi_1(X)) \to . & &
\end{array} \tag{2.16}$$

The spectrum $\mathbb{L}P(F)$ fits also into a homotopy commutative diagram (see [2,3,6])

$$\begin{array}{ccc}
X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q Y_+ \wedge \mathbf{L}_\bullet \\
{\scriptstyle\Theta}\downarrow & & \downarrow \\
\Sigma^q \mathbb{L}P(F) & \longrightarrow & \Sigma^q \mathbb{L}(\pi_1(Y))
\end{array} \tag{2.17}$$

which realizes the diagram in [11, Proposition 7.2.6, iii)]. The other relations between various structure sets and obstruction groups for a closed manifold pair (X, Y) are expressed by many braids of exact sequences, all of which are realized on the spectra level (see [2,3,6,11]).

3 Surgery on two manifolds with a transversal intersection

In this section we construct the algebraic surgery theory of a stratified space [13, sections 5, 7]

$$\mathcal{X} = (X_1, X_2; X_1 \cap X_2 = Y), \tag{3.1}$$

where X_1 and X_2 are n-dimensional manifolds with a transversal intersection Y of codimension q and $\dim Y = n - q \geq 5$. We shall consider only the case when $X = X_1 \cup_Y X_2$, X_1, X_2, and Y are connected manifolds, but all results will be true in the general case. Let $X = X_1 \cup_Y X_2$.

Let $\mathcal{N} = (N_1, N_2; N_1 \cap N_2 = K)$ be a stratified space consisting of two transversal manifolds N_1 and N_2 with intersection K such that $\dim N_1 = \dim N_2 = n$, $\dim K = n - q$, and $N = N_1 \cup N_2$. Let ξ_1 be a normal bundle of Y in X_1, and let ξ_2 be a normal bundle of Y in X_2. We have associated to these bundles spherical fibrations [10]

$$(D^q, S^{q-1}) \to (E(\xi_1), S(\xi_1)) \to Y,$$
$$(D^q, S^{q-1}) \to (E(\xi_2), S(\xi_2)) \to Y.$$

Let $f: K \to Y$ be a t-triangulation of the manifold Y. The transfer maps (see [11, §7] and [13, §7.2])

$$\tau_1: \mathcal{T}(Y) \to \mathcal{T}(E(\xi_1), S(\xi_1))$$

and

$$\tau_2: \mathcal{T}(Y) \to \mathcal{T}(E(\xi_2), S(\xi_2))$$

define t-triangulations

$$\begin{aligned}(\tau_1(f), \partial\tau_1(f)) &: (E_1, \partial E_1) \to (E(\xi_1), S(\xi_1)),\\ (\tau_2(f), \partial\tau_2(f)) &: (E_2, \partial E_2) \to (E(\xi_2), S(\xi_2))\end{aligned}$$

such that the restrictions

$$\begin{aligned}\partial\tau_1(f) &: \partial E_1 \to S(\xi_1)\\ \partial\tau_2(f) &: \partial E_2 \to S(\xi_2)\end{aligned}$$

are t-triangulations of closed manifolds $S(\xi_1)$ and $S(\xi_2)$, respectively.

Definition 3.1 Let

$$\mathcal{F} = (f_1, f_2, f): \mathcal{N} \to \mathcal{X}$$

denote a map $N \to X$ defined by t-triangulations

$$\begin{aligned} f_1 &: N_1 \to X_1,\\ f_2 &: N_2 \to X_2,\\ f &: K \to Y\end{aligned}$$

such that f_i $(i = 1, 2)$ is transversal to Y with $f_i^{-1}(Y) = K$, (Y_i, N) is a closed manifold pair with normal block bundle ν_i, and $f = f_1|_K = f_2|_K$. Such a map $\mathcal{F}$ is called *t-triangulation of the stratified space $\mathcal{X}$*, if the constituent maps

$$(f_i, f): (N_i, K, \nu_i) \to (X_i, Y, \xi_i),\ i = 1, 2$$

are t-triangulations of the manifold pairs (X_i, Y, ξ_i).

Consider two t-triangulations $(\mathcal{F}, \mathcal{N})$ and $(\mathcal{G}, \mathcal{M})$ of a stratified space $\mathcal{X}$ in (3.1), where $\mathcal{N} = (N_1, N_2; K)$, $\mathcal{F} = (f_1, f_2; f)$, $\mathcal{M} = (M_1, M_2; L)$, and $\mathcal{G} = (g_1, g_2; g)$.

Definition 3.2 Two t-triangulations $(\mathcal{F}, \mathcal{N})$ and $(\mathcal{G}, \mathcal{M})$ are *concordant* if the following conditions are satisfied:

(i) There exists a stratified space $\mathcal{V} = (V_1, V_2; S = V_1 \cap V_2)$ with boundary $\partial\mathcal{V} = \mathcal{N} \dot{\cup} \mathcal{M}$ such that

$$\partial V_1 = N_1 \dot{\cup} M_1,\ \partial V_2 = N_2 \dot{\cup} M_2, \partial S = K \dot{\cup} L.$$

(ii) There exists a stratified map

$$\Psi = (F_1, F_2; F)\colon \mathcal{V} \to \mathcal{X} \times I, I = [0, 1]$$

defined by t-triangulations

$$\begin{aligned} F_1\colon V_1 &\to X_1 \times I,\\ F_2\colon V_2 &\to X_2 \times I,\\ F = F_1|_S = F_2|_S\colon &: S \to Y \times I \end{aligned}$$

such that F_i is transversal to $Y \times I$ and $S = F_i^{-1}(Y \times I)$ $(i = 1, 2)$,

(iii) For $i = 1, 2$, the pairs (F_i, F) define concordances of the t-triangulations (f_i, f) and (g_i, g) of the closed manifold pair (X_i, Y).

Definitions 3.1 and 3.2 are a special case of the notion of t-triangulation of a stratified space considered in [11]. The set of concordance classes of t-triangulations of $\mathcal{X}$ is denoted by $\mathcal{T}^{TOP}(\mathcal{X})$ and, by Definitions 3.1 and 3.2, fits into commutative diagram of structure sets

$$\begin{array}{ccc} \mathcal{T}^{TOP}(\mathcal{X}) & \longrightarrow & \bigoplus_i \mathcal{T}^{TOP}(X_i \setminus Y, S(\xi_i)) \\ \downarrow & & \downarrow{\scriptstyle(\partial\oplus\partial)} \\ \mathcal{T}^{TOP}(Y) & \longrightarrow & \bigoplus_i \mathcal{T}^{TOP}(S(\xi_i)) \end{array} \tag{3.2}$$

in which the bottom map is the composition

$$\mathcal{T}^{TOP}(Y) \xrightarrow{(\tau_1 \top \tau_2)} \bigoplus_{i=1,2} \mathcal{T}^{TOP}(E(\xi_i), S(\xi_i)) \xrightarrow{(\partial\oplus\partial)} \bigoplus_{i=1,2} \mathcal{T}^{TOP}(S(\xi_i))$$

of the transfer maps and the restriction maps and the other maps in the diagram are restrictions of t-triangulations.

Define the spectrum $\mathbb{T}(S(\xi))$ by the pullback square of spectra

$$\begin{array}{ccc} \mathbb{T}(S(\xi)) & \longrightarrow & S(\xi_1)_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ S(\xi_2)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & * \end{array} \tag{3.3}$$

where

$$\pi_{n-1}(S(\xi_1)_+ \wedge \mathbf{L}_\bullet) = \mathcal{T}^{TOP}(S(\xi_1)),\ \ \pi_{n-1}(S(\xi_2)_+ \wedge \mathbf{L}_\bullet) = \mathcal{T}^{TOP}(S(\xi_2)).$$

The spectrum $\mathbb{T}(S(\xi))$ realizes the structure set $\mathcal{T}^{TOP}(S(\xi_1)) \oplus \mathcal{T}^{TOP}(S(\xi_2))$ and has homotopy groups

$$\pi_m(\mathbb{T}(S(\xi))) = H_m(S(\xi_1); L_\bullet) \oplus H_m(S(\xi_2); L_\bullet)$$

such that

$$\pi_{n-1}(\mathbb{T}(S(\xi))) = \mathcal{T}^{TOP}(S(\xi_1)) \oplus \mathcal{T}^{TOP}(S(\xi_2)).$$

Define the spectrum $\mathbb{T}(X \setminus Y, S(\xi))$ by the pullback square of spectra

$$\begin{array}{ccc} \mathbb{T}(X \setminus Y, S(\xi)) & \longrightarrow & ((X_1 \setminus Y)/S(\xi_1))_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ ((X_2 \setminus Y)/S(\xi_2))_+ \wedge \mathbf{L}_\bullet & \longrightarrow & * \end{array} \tag{3.4}$$

where

$$\pi_n((X_1 \setminus Y)/S(\xi_1))_+ \wedge \mathbf{L}_\bullet) = \mathcal{T}^{TOP}(X_1 \setminus Y, S(\xi_1))$$

and

$$\pi_n((X_2 \setminus Y)/S(\xi_2))_+ \wedge \mathbf{L}_\bullet) = \mathcal{T}^{TOP}(X_2 \setminus Y, S(\xi_2)).$$

The spectrum $\mathbb{T}(X \setminus Y, S(\xi))$ realizes the structure set

$$\mathcal{T}^{TOP}(X_1 \setminus Y, S(\xi_1)) \oplus \mathcal{T}^{TOP}(X_2 \setminus Y, S(\xi_2))$$

and has homotopy groups

$$\pi_m(\mathbb{T}(X \setminus Y, S(\xi))) = H_m(X_1 \setminus Y, S(\xi_1); \mathbf{L}_\bullet) \oplus H_m(X_2 \setminus Y, S(\xi_2); \mathbf{L}_\bullet).$$

The structure set $\mathcal{T}^{TOP}(\mathcal{X})$ is realized on the spectra level by a spectrum $\mathbb{T}(\mathcal{X})$ (see [11, §7], [12, §17A] and [13, §3,§7]) with $\pi_n(\mathbb{T}(\mathcal{X})) = \mathcal{T}_n(\mathcal{X}) = \mathcal{T}^{TOP}(\mathcal{X})$ and fits into the pullback square of spectra

$$\begin{array}{ccc} \mathbb{T}(\mathcal{X}) & \longrightarrow & \mathbb{T}(X \setminus Y, S(\xi)) \\ \downarrow & & \delta\downarrow \\ \Sigma^q(Y_+ \wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma\mathbb{T}(S(\xi)) \end{array} \tag{3.5}$$

which realizes the diagram of structure sets (3.2).

It follows immediately from Definitions 3.1 and 3.2 that we have a commutative diagram of structure sets

$$\begin{array}{ccc} \mathcal{T}^{TOP}(\mathcal{X}) & \longrightarrow & \mathcal{T}^{TOP}(X_1) \\ \downarrow & & \downarrow \\ \mathcal{T}^{TOP}(X_2) & \longrightarrow & \mathcal{T}^{TOP}(Y) \end{array} \tag{3.6}$$

in which all maps are restrictions of t-triangulations.

Theorem 3.1 *The diagram* (3.6) *is realized by a pullback square of spectra*

$$\begin{array}{ccc} \mathbb{T}(\mathcal{X}) & \longrightarrow & X_{1+} \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ X_{2+} \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q(Y_+ \wedge \mathbf{L}_\bullet) \end{array} \tag{3.7}$$

where the bottom horizontal map and the right vertical map are defined as in (2.7).

Proof Denote by $\mathbb{T}(X \setminus Y)$ a spectrum which realizes the structure set $\mathcal{T}^{TOP}(X_1 \setminus Y) \oplus \mathcal{T}^{TOP}(X_2 \setminus Y)$. This spectrum fits into the pullback square

$$\begin{array}{ccc} \mathbb{T}(X \setminus Y) & \longrightarrow & (X_1 \setminus Y)_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ (X_2 \setminus Y)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & * \end{array} \tag{3.8}$$

On spectra level, restricting a t-triangulation to the boundary provides a natural map Δ of square (3.4) to the suspended square (3.3) whose cofiber coincides with the suspended square (3.8). The map Δ in the left upper corner coincides with the map δ in (3.5). Now from pullback square (3.5) and the cofibrations (2.7) for manifolds pairs (X_i, Y) $(i = 1, 2)$, we obtain a map of the spectrum $\Sigma^q(Y_+ \wedge \mathbf{L}_\bullet)$ into each of the spectra in the suspended square (3.8) such that the diagram which has the form of a pyramid with top $\Sigma^q(Y_+ \wedge \mathbf{L}_\bullet)$ is homotopy commutative. Now the fibers of the maps from the top of the pyramid to the spectra in the bottom of the pyramid form a pullback square of spectra (3.7). This proves the theorem. □

Corollary 3.1 *The homotopy long exact sequences of the maps in the pullback square* (3.7) *form a braid of exact sequences*

$$\begin{array}{ccccccccc} \longrightarrow H_n(X_1 \setminus Y; \mathbf{L}_\bullet) & \longrightarrow & H_n(X_1; \mathbf{L}_\bullet) & & \to & & H_{n-1}(X_2 \setminus Y; \mathbf{L}_\bullet) \to \\ \nearrow \quad \searrow & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow \\ & \mathcal{T}_n(\mathcal{X}) & & & H_{n-q}(Y; \mathbf{L}_\bullet) & & \\ \searrow \quad \nearrow & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow \\ \to H_n(X_2 \setminus Y; \mathbf{L}_\bullet) & \longrightarrow & H_n(X_2; \mathbf{L}_\bullet) & & \longrightarrow & & H_{n-1}(X_1 \setminus Y; \mathbf{L}_\bullet) \to . \end{array}$$

Corollary 3.2 *Let $i = 1$, $j = 2$ or $i = 2$, $j = 1$. The diagram of spectra*

$$\begin{array}{ccccc} & & \Omega^{q-1}(X_j \setminus Y)_+ \wedge \mathbf{L}_\bullet & \xrightarrow{=} & \Omega^{q-1}(X_j \setminus Y)_+ \wedge \mathbf{L}_\bullet \\ & & \downarrow & & \downarrow \\ Y_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Omega^{q-1}\mathbb{T}(X \setminus Y) & \longrightarrow & \Omega^{q-1}\mathbb{T}(\mathcal{X}) \\ = \downarrow & & \downarrow & & \downarrow \\ Y_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Omega^{q-1}(X_i \setminus Y)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Omega^{q-1}(X_{i+} \wedge \mathbf{L}_\bullet) \end{array} \tag{3.9}$$

is homotopy commutative and the two bottom rows and two right columns are cofibration sequences. The homotopy long exact sequences of the maps in (3.9) form a braid of exact sequences

$$\begin{array}{ccccccccc} \longrightarrow H_n(X_j \setminus Y; \mathbf{L}_\bullet) & & \longrightarrow & & \mathcal{T}_n(\mathcal{X}) & & \rightarrow & & H_{n-q}(Y; \mathbf{L}_\bullet) \rightarrow \\ \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\ & & \mathcal{T}_n(X \setminus Y) & & & & H_n(X_i; \mathbf{L}_\bullet) & & \\ \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\ \rightarrow H_{n-q+1}(Y; \mathbf{L}_\bullet) & & \longrightarrow & & H_n(X_i \setminus Y; \mathbf{L}_\bullet) & & \longrightarrow & & H_{n-1}(X_j \setminus Y; \mathbf{L}_\bullet) \rightarrow \end{array} \tag{3.10}$$

where

$$\mathcal{T}_m(X \setminus Y) = \pi_m(\mathbb{T}(X \setminus Y)) = H_m(X_1 \setminus Y; \mathbf{L}_\bullet) \oplus H_m(X_2 \setminus Y; \mathbf{L}_\bullet).$$

Proof Consider the commutative diagram which forms of pyramid in the proof of Theorem 3.1. The left bottom square of (3.9) is the q-desuspended part of this diagram for $i = 2$, $j = 1$. Hence the right bottom square in (3.9) is a pullback and the result follows. Note that the two right bottom vertical maps in (3.9) geometrically correspond to restricting a t-triangulation of the manifold with boundary $X \setminus Y$ to a t-triangulation of the manifold with boundary $X_i \setminus Y$ and to restricting a t-triangulation of $\mathcal{X}$ to a t-triangulation of the manifold X_i, respectively. □

It is worth noting the braids of exact sequences obtained in Corollaries 3.1 and 3.2 together with the Atiyah–Hirzebruch spectral sequence for $H_n(?; \mathbf{L}_\bullet)$ [1] provide a tool for computing the structure sets $\mathcal{T}_i(\mathcal{X})$.

Using (3.1), we define now a spectrum $\mathbb{L}(\mathcal{X})$ for surgery obstruction groups of a stratified space $\mathcal{X}$ (see [13, §6]).

For $i = 1, 2$, let $\partial\tau_i$ denote the composition of the maps

$$L_m(\pi_1(Y)) \xrightarrow{\tau_i} L_{m+q}(\pi_1(S(\xi_i)) \to \pi_1(E(\xi_i))) \xrightarrow{\partial} L_{m+q-1}(\pi_1(S(\xi_i))) \tag{3.11}$$

where the first map is the transfer map and the second is the boundary map in the relative long exact sequence of L-groups for the map $\pi_1(S(\xi_i)) \to \pi_1(E(\xi_i))$ (see [2,3,10,11]). For future use below, we record the diagram of groups

$$\begin{array}{ccc} L_m(\pi_1(Y)) & \xrightarrow{(\partial\tau_1\ \partial\tau_2)} & L_{m+q-1}(\pi_1(S(\xi_1)) \oplus L_{m+q-1}(\pi_1(S(\xi_2)) \\ & & \downarrow{\scriptstyle i_{1*}\oplus i_{2*}} \\ & & L_{m+q-1}(\pi_1(X_1 \setminus Y)) \oplus L_{m+q-1}(\pi_1(X_2 \setminus Y)) \end{array} \tag{3.12}$$

where the first map is given by (3.11) and the second is induced by the inclusions $i_1 : S(\xi_1) \to (X_1 \setminus Y)$ and $i_2 : S(\xi_2) \to (X_2 \setminus Y)$. Note that all maps in (3.11) and (3.12) are realized on the spectra level.

Define the spectrum $\mathbb{L}(X \setminus Y)$ with homotopy groups

$$\pi_m(\mathbb{L}(X \setminus Y)) = L_m(\pi_1(X_1 \setminus Y)) \oplus L_m(\pi_1(X_2 \setminus Y))$$

by the pullback square

$$\begin{array}{ccc} \mathbb{L}(X \setminus Y) & \longrightarrow & \mathbb{L}(\pi_1(X_1 \setminus Y)) \\ \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(X_2 \setminus Y)) & \longrightarrow & *\,, \end{array}$$

and define the spectrum $\mathbb{L}(S(\xi))$ with homotopy groups

$$\pi_m(\mathbb{L}(S(\xi))) = L_m(\pi_1(S(\xi_1))) \oplus L_m(\pi_1(S(\xi_2)))$$

by the pullback square

$$\begin{array}{ccc} \mathbb{L}(S(\xi)) & \longrightarrow & \mathbb{L}(\pi_1(S(\xi_1))) \\ \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(S(\xi_2))) & \longrightarrow & *. \end{array}$$

The diagram (3.12) can be written on the spectra level in the form

$$\begin{array}{ccc} \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^q\mathbb{L}(S(\xi)) \\ & & \downarrow \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(X \setminus Y)). \end{array} \tag{3.13}$$

We denote the composition of the maps in (3.13) by $\mathbf{d}$. Let $\mathbb{L}(\mathcal{X})$ denote the homotopy fiber of the map $\mathbf{d}$. It fits into a cofibration sequence (see [13, §6])

$$\mathbb{L}(\mathcal{X}) \longrightarrow \mathbb{L}(\pi_1(Y)) \xrightarrow{\mathbf{d}} \Omega^{q-1}\mathbb{L}(X \setminus Y). \tag{3.14}$$

Let F_i $(i = 1, 2)$ denote the square of fundamental groups in the splitting problem for the closed manifold pair $Y \subset X_i$.

Theorem 3.2 *The spectrum* $\mathbb{L}(\mathcal{X})$ *fits into a pullback square*

$$\begin{array}{ccc} \mathbb{L}(\mathcal{X}) & \longrightarrow & \mathbb{L}P(F_1) \\ \downarrow & & \downarrow \\ \mathbb{L}P(F_2) & \longrightarrow & \mathbb{L}(\pi_1(Y)) \end{array} \tag{3.15}$$

of spectra and the homotopy long exact sequences of the maps in (3.15) *form a braid of exact sequences*

$$\begin{array}{ccccccccc} \longrightarrow & L_n(C_1) & & \longrightarrow & LP_{n-q}(F_1) & & \rightarrow & L_{n-1}(C_2)\rightarrow & \\ \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow \\ & & L_{n-q}(\mathcal{X}) & & & L_{n-q}(\pi_1(Y)) & & & \\ \searrow & & \nearrow & \searrow & & \nearrow & \searrow & & \nearrow \\ \rightarrow & L_n(C_2) & & \longrightarrow & LP_{n-q}(F_2) & & \longrightarrow & L_{n-1}(C_1)\rightarrow & \end{array} \tag{3.16}$$

where $C_i = \pi_1(X_i \setminus Y)$ $(i = 1, 2)$.

Proof The maps in diagram (3.13) define a map of the spectrum $\mathbb{L}(\pi_1(Y))$ into the pullback square

$$\begin{array}{ccc} \Omega^{q-1}\mathbb{L}(X \setminus Y) & \longrightarrow & \Omega^{q-1}\mathbb{L}(\pi_1(X_1 \setminus Y)) \\ \downarrow & & \downarrow \\ \Omega^{q-1}\mathbb{L}(\pi_1(X_2 \setminus Y)) & \longrightarrow & *, \end{array} \tag{3.17}$$

and we obtain a homotopy commutative diagram in the form of a pyramid whose base is the diagram in (3.17). By (2.13) and (3.14) the fibers of the maps from $\mathbb{L}(\pi_1(Y))$ to the spectra in the base of the pyramid form a pullback square of spectra (3.15). The homotopy long exact sequences of the maps in (3.15) form the braid exact sequences (3.16). □

Corollary 3.3 *Let* $i = 1, j = 2$ *or* $i = 2, j = 1$. *The diagram of spectra*

$$\begin{array}{ccccc} & & \Omega^{q-1}\mathbb{L}(\pi_1(X_j \setminus Y)) & \xrightarrow{=} & \Omega^{q-1}\mathbb{L}(\pi_1(X_j \setminus Y)) \\ & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q-1}\mathbb{L}(X \setminus Y) & \longrightarrow & \Omega^{-1}\mathbb{L}(\mathcal{X}) \\ {=}\downarrow & & \downarrow & & \downarrow \\ \mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q-1}\mathbb{L}(\pi_1(X_i \setminus Y)) & \longrightarrow & \Omega^{-1}\mathbb{L}P(F_i) \end{array} \tag{3.18}$$

is homotopy commutative and the two bottom rows and two right columns are cofibration sequences. The homotopy long exact sequences of the maps in (3.18) *form a braid of exact sequences*

$$\begin{array}{ccccccccc} \longrightarrow & L_n(C_j) & & \longrightarrow & & L_{n-q}(\mathcal{X}) & \to & L_{n-q}(\pi_1(Y)) \to & \\ & \nearrow & \searrow & & \nearrow & & \searrow & \nearrow \quad \searrow & \\ & & & L_n(C_1)\oplus L_n(C_2) & & & LP_{n-q}(F_i) & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & \searrow \quad \nearrow & \\ \to L_{n-q+1}(\pi_1(Y)) & & & \longrightarrow & & L_n(C_i) & \longrightarrow & L_{n-1}(C_j) & \to . \end{array} \tag{3.19}$$

Proof Similar to that of Corollary 3.2. □

Theorem 3.3 *Let* $i=1,\ j=2$ *or* $i=2,\ j=1$. *The diagram of spectra*

$$\begin{array}{ccc} \mathbb{L}(\mathcal{X}) & \longrightarrow & \mathbb{L}P(F_i) \\ \downarrow & & \downarrow \\ \Omega^q\mathbb{L}(\pi_1(X_j)) & \longrightarrow & \Omega^q\mathbb{L}(\pi_1(X_j\setminus Y)\to\pi_1(X_j)) \end{array} \tag{3.20}$$

is a pullback diagram. The homotopy long exact sequences of the horizontal maps in (3.20) *form a commutative diagram*

$$\begin{array}{ccccccccc} \longrightarrow & L_n(C_j) & \longrightarrow & L_{n-q}(\mathcal{X}) & \longrightarrow & LP_{n-q}(F_i) & \longrightarrow \\ & =\downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & L_n(C_j) & \longrightarrow & L_n(\pi_1(X_j)) & \longrightarrow & L_n(C_j\to\pi_1(X_j)) & \longrightarrow \end{array} \tag{3.21}$$

where $C_j=\pi_1(X_j\setminus Y)$.

Proof Consider the homotopy commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{L}(\mathcal{X}) & \longrightarrow & \mathbb{L}P(F_i) \\ \downarrow & & \downarrow \\ \mathbb{L}P(F_j) & \longrightarrow & \mathbb{L}(\pi_1(Y)) \\ \downarrow & & \downarrow \\ \Omega^q\mathbb{L}(\pi_1(X_j)) & \longrightarrow & \Omega^q\mathbb{L}(C_j\to\pi_1(X_j)). \end{array} \tag{3.22}$$

The upper square is (3.15) and the bottom square is the central square in(2.14) on the spectra level. The squares in (3.22) are pullback squares and the fibers of all horizontal maps are naturally homotopy equivalent to $\Omega^q\mathbb{L}(C_j)$. □

Definition 3.3 ([13, §7]) A t-triangulation

$$\mathcal{F} = (f_1, f_2, f): \mathcal{N} \to \mathcal{X}$$

of the stratified space $\mathcal{X}$ (see 3.1) is called an *s-triangulation*, if $\mathcal{F}$ is given by a simple homotopy equivalence $N \to X$ and the constituent maps

$$(f_i, f): (N_i, K, \nu_i) \to (X_i, Y, \xi_i),\ i = 1, 2$$

are s-triangulations of the closed manifold pairs (X_i, Y, ξ_i).

The definition of concordance of s-triangulations of a stratified space is similar to concordance of t-triangulations of stratified space in Definition 3.2.

Definition 3.4 Two s-triangulations $(\mathcal{F}, \mathcal{N})$ and $(\mathcal{G}, \mathcal{M})$ of $\mathcal{X}$ are *concordant*, if the following conditions are satisfied:

(i) There exists a concordance

$$\Psi = (F_1, F_2; F): \mathcal{V} = (V_1, V_2; S = V_1 \cap V_2) \to \mathcal{X} \times I, I = [0, 1]$$

of the t-triangulations $(\mathcal{F}, \mathcal{N})$ and $(\mathcal{G}, \mathcal{M})$ such that the map

$$\psi = \Psi_{V_1 \cup_S V_2}: V_1 \cup_S V_2 \to X \times I$$

is a simple homotopy equivalence.

(ii) For $i = 1, 2$, the pair (F_i, F) is a concordance of the s-triangulations (f_i, f) and (g_i, g) of the closed manifold pair (X_i, Y).

The set of concordance classes of s-triangulations is denoted by $\mathcal{S}^{TOP}(\mathcal{X})$ and is realized by a spectrum $\mathbb{S}(\mathcal{X})$ with homotopy groups $\pi_m(\mathbb{S}(\mathcal{X})) = \mathcal{S}_m(\mathcal{X})$ such that $\mathcal{S}^{TOP}(\mathcal{X}) = \mathcal{S}_{n+1}(\mathcal{X}),\ n = \dim X_i$ [11]. Letting $\mathbb{T}(\mathcal{X})$ be as in (3.5), we obtain a cofibration sequence of spectra [11]

$$\mathbb{T}(\mathcal{X}) \to \Sigma^q \mathbb{L}(\mathcal{X}) \to \mathbb{S}(\mathcal{X}) \tag{3.23}$$

whose homotopy long exact sequence is

$$\cdots \to \mathcal{S}_{n+1}(\mathcal{X}) \to \mathcal{T}_n(\mathcal{X}) \to L_{n-q}(\mathcal{X}) \to \mathcal{S}_n(\mathcal{X}) \to \ldots. \tag{3.24}$$

where n is the dimension of the ambient manifolds $X_i (i = 1, 2)$. Recall that $\mathcal{T}_n(\mathcal{X}) = \mathcal{T}^{TOP}(\mathcal{X})$ and that we are using the convention that the dimension of the surgery obstruction group of a t-triangulation of $\mathcal{X}$ is the dimension of the submanifold Y. It follows immediately from Definitions 3.3 and 3.4 that the spectrum $\mathbb{S}(\mathcal{X})$ fits into a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{S}(\mathcal{X}) & \longrightarrow & \mathbb{S}(X_1, Y) \\ \downarrow & & \downarrow \\ \mathbb{S}(X_2, Y) & \longrightarrow & \Sigma^q \mathbb{S}(Y). \end{array} \tag{3.25}$$

Theorem 3.4 *The square in (3.25) is a pullback square. The fiber of the vertical maps is $\mathbb{S}(X_1 \setminus Y)$ and the fiber of the horizontal maps is $\mathbb{S}(X_2 \setminus Y)$. The homotopy long exact sequences of the maps in (3.25) form a braid of exact sequences*

$$\begin{array}{ccccccccc}
\longrightarrow \mathcal{S}_m(X_1 \setminus Y) & \longrightarrow & \mathcal{S}_m(X_1, Y) & \rightarrow & \mathcal{S}_{m-1}(X_2 \setminus Y) \rightarrow \\
\nearrow \quad \searrow & & \nearrow \quad \searrow & & \nearrow \quad \searrow \\
& \mathcal{S}_m(\mathcal{X}) & & \mathcal{S}_{m-q}(Y) & \\
\searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow \\
\rightarrow \mathcal{S}_m(X_2 \setminus Y) & \longrightarrow & \mathcal{S}_m(X_2, Y) & \longrightarrow & \mathcal{S}_{m-1}(X_1 \setminus Y) \rightarrow .
\end{array} \tag{3.26}$$

Proof By (2.17), we get a homotopy commutative diagram

$$\begin{array}{ccccc}
X_{2+} \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q(Y_+ \wedge \mathbf{L}_\bullet) & \longleftarrow & X_{1+} \wedge \mathbf{L}_\bullet \\
\Theta_2 \downarrow & & \Sigma^q\Theta_0 \downarrow & & \Theta_1 \downarrow \\
\Sigma^q \mathbb{L}P(F_2) & \longrightarrow & \Sigma^q \mathbb{L}(\pi_1(Y)) & \longleftarrow & \Sigma^q \mathbb{L}P(F_1)
\end{array} \tag{3.27}$$

where the squares are defined by the closed manifold pairs (X_i, Y) and q denotes the codimension of Y in X_i. The vertical maps in (3.27) induce the map

$$\mathbb{T}(\mathcal{X}) \rightarrow \Sigma^q \mathbb{L}(\mathcal{X})$$

in the cofibration (3.23). Thus we obtain a map Λ of the square (3.7) to the q-suspended square (3.15). This map is given by a commutative diagram which has the form of a cube. The homotopy cofibers of the maps from the corners of the square (3.7) to the corresponding corners of the pullback square (3.15) form the pullback square (3.25). The theorem follows. □

Define the spectrum $\mathbb{S}(X \setminus Y)$ by the pullback square

$$\begin{array}{ccc}
\mathbb{S}(X \setminus Y) & \longrightarrow & \mathbb{S}(X_1 \setminus Y) \\
\downarrow & & \downarrow \\
\mathbb{S}(X_2 \setminus Y) & \longrightarrow & *.
\end{array}$$

The spectrum $\mathbb{S}(X \setminus Y)$ has the homotopy groups

$$\mathcal{S}_m(X \setminus Y) = \pi_m(\mathbb{S}(X \setminus Y)) = \mathcal{S}_m(X_1 \setminus Y) \oplus \mathcal{S}_m(X_2 \setminus Y).$$

Theorem 3.5 *The diagram of spectra*

$$\begin{array}{ccccc}
Y_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Omega^{q-1}\mathbb{T}(X\setminus Y) & \longrightarrow & \Omega^{q-1}\mathbb{T}(\mathcal{X}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q-1}\mathbb{L}(X\setminus Y) & \longrightarrow & \Omega^{-1}\mathbb{L}(\mathcal{X}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}(Z) & \longrightarrow & \Omega^{q-1}\mathbb{S}(X\setminus Y) & \longrightarrow & \Omega^{q-1}\mathbb{S}(\mathcal{X})
\end{array} \tag{3.28}$$

is homotopy commutative and the rows and columns are cofibrations. Let $C_i = \pi_1(X_i \setminus Y)$ for $i = 1, 2$. The homotopy long exact sequences of the maps in (3.28) form a commutative diagram

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{n-q}(Y;\mathbf{L}_\bullet) & \longrightarrow & \mathcal{T}_{n-1}(X\setminus Y) & \longrightarrow & \mathcal{T}_{n-1}(\mathcal{X}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & L_{n-q}(\pi_1(Y)) & \longrightarrow & \oplus_{i=1,2} L_{n-1}(C_i) & \longrightarrow & L_{n-q-1}(\mathcal{X}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{S}_{n-q}(Y) & \longrightarrow & \mathcal{S}_{n-q}(X\setminus Y) & \longrightarrow & \mathcal{S}_{n-1}(\mathcal{X}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array} \tag{3.29}$$

whose rows and columns are exact.

Proof The upper row of diagram (3.28) is the middle row of diagram (3.9), and the middle row of diagram (3.28) is the middle row of diagram (3.18). The upper vertical maps in (3.28) are the surgery obstruction maps on the spectra level. The theorem follows. □

Theorem 3.6 *Let $C_i = \pi_1(X_i \setminus Y)$ for $i = 1, 2$, and $i = 1$, $j = 2$ or $i = 2$, $j = 1$. The diagram of groups*

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathcal{T}_n(\mathcal{X}) & \longrightarrow & L_{n-q}(\mathcal{X}) & \longrightarrow & \mathcal{S}_n(\mathcal{X}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_n(X_i;\mathbf{L}_\bullet) & \longrightarrow & LP_{n-q}(F_i) & \longrightarrow & \mathcal{S}_n(X_i, Y) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{n-1}(X_j\setminus Y;\mathbf{L}_\bullet) & \longrightarrow & L_{n-1}(C_j) & \longrightarrow & \mathcal{S}_{n-1}(X_j\setminus Y) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array} \tag{3.30}$$

is commutative and its rows and columns are exact.

For $i = 1, 2$, the is diagram of groups

$$\begin{array}{ccccc} \cdots \to \mathcal{T}_n(\mathcal{X}) & \longrightarrow & L_{n-q}(\mathcal{X}) & \longrightarrow & \mathcal{S}_n(\mathcal{X}) \to \dots \\ \downarrow & & \downarrow & & \downarrow \\ \cdots \to H_n(X_i; \mathbf{L}_\bullet) & \longrightarrow & L_n(\pi_1(X_i)) & \longrightarrow & \mathcal{S}_n(X_i) \to \dots \end{array} \tag{3.31}$$

is commutative and its rows are exact. The diagrams (3.30) *and* (3.31) *are realized on the spectra level.*

Proof For $i = 1, 2$, consider the homotopy commutative square of spectra

$$\begin{array}{ccc} \mathbb{T}(X) & \longrightarrow & \Sigma^q \mathbb{L}(\mathcal{X}) \\ \downarrow & & \downarrow \\ X_{i+} \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{L} P(F_i). \end{array} \tag{3.32}$$

The horizontal maps are the surgery obstruction maps on the spectra level. The vertical maps are described in Theorems 3.1 and 3.2. The square extends to an infinite homotopy commutative bicomplex of spectra which is the realization of the diagram (3.30) on the spectra level. This proves the assertion of the theorem regarding the diagram (3.30). The assertion for diagram (3.31) is proved similarly. □

Theorem 3.7 *Let $i = 1, j = 2$ or $i = 2, j = 1$. The diagram of spectra*

$$\begin{array}{ccccc} & & \Omega^{q-1}\mathbb{S}(X_j \setminus Y) & \xrightarrow{=} & \Omega^{q-1}\mathbb{S}(X_j \setminus Y) \\ & & \downarrow & & \downarrow \\ \mathbb{S}(Y) & \longrightarrow & \Omega^{q-1}\mathbb{S}(X \setminus Y) & \longrightarrow & \Omega^{-1}\mathbb{S}(\mathcal{X}) \\ = \downarrow & & \downarrow & & \downarrow \\ \mathbb{S}(Y) & \longrightarrow & \Omega^{q-1}\mathbb{S}(X_i \setminus Y) & \longrightarrow & \Omega^{q-1}\mathbb{S}(X_i, Y) \end{array} \tag{3.33}$$

is homotopy commutative and its rows and columns are cofibration sequences. The homotopy long exact sequences of the maps in (3.33) *form a braid of exact sequences*

$$\begin{array}{ccccccccc} \longrightarrow S_n(X_j \setminus Y) & & \longrightarrow & & S_n(\mathcal{X}) & & \to & & S_{n-q}(Y) \quad \to \\ \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\ & & S_n(X \setminus Y) & & & & S_n(X_i, Y) & & \\ \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\ \to S_{n-q+1}(Y) & & \longrightarrow & & S_n(X_i \setminus Y) & & \longrightarrow & & S_{n-1}(X_j \setminus Y) \to . \end{array}$$

Proof On the spectra level, consider surgery obstruction maps from the spectra in diagram (3.9) to the corresponding spectra in diagram (3.18). The cofibers of these maps form the homotopy commutative diagram (3.33). □

4 An example

In this section we compute the structure sets of a stratified manifold consisting of two real projective spaces which intersect transversally. Commutative diagrams in Sect. 3 allow us to obtain very explicit results.

Consider a stratified manifold $\mathcal{X}$ as in (3.1) where $X_i = \mathbb{R}P^n$ ($i = 1, 2$) and $Y = X_1 \cap X_2 = \mathbb{R}P^{n-1}$. We shall suppose that $n \geq 6$. There are isomorphisms [7, pp. 48, 53] (see also [12, §14D])

$$\mathcal{T}^{TOP}(\mathbb{R}P^{2i+5}) = \mathcal{T}^{TOP}(\mathbb{R}P^{2i+4}) = \bigoplus_{j=1}^{i+2} \mathbb{Z}_2 \tag{4.1}$$

and

$$\mathcal{S}^{TOP}(\mathbb{R}P^n) = \bigoplus_{j=1}^{2k} \mathbb{Z}_2, \quad \bigoplus_{j=1}^{2k} \mathbb{Z}_2, \quad \bigoplus_{j=1}^{2k} \mathbb{Z}_2 \oplus \mathbb{Z}, \quad \bigoplus_{j=1}^{2k+1} \mathbb{Z}_2 \tag{4.2}$$

where in (4.2), k is the largest integer such that $4k + 1 \leq n$. The manifold with boundary

$$X_i \setminus Y = \mathbb{R}P^n \setminus \mathbb{R}P^{n-1} \quad (i = 1, 2)$$

is a disk D^n, hence

$$H_n(X_i \setminus Y; \mathbf{L}_\bullet) = L_n(1) = \mathbb{Z}, \ 0, \ \ \mathbb{Z}_2, \ \ 0 \tag{4.3}$$

for $n = 0, 1, 2, 3$ mod 4, respectively.

Theorem 4.1 *Let*

$$\mathcal{X} = (\mathbb{R}P^n, \mathbb{R}P^n; \mathbb{R}P^{n-1}), \quad n \geq 6.$$

Then

$$\mathcal{T}^{TOP}(\mathcal{X}) \cong \mathcal{T}^{TOP}(\mathbb{R}P^{4k+1}) = \bigoplus_{j=1}^{2k} \mathbb{Z}_2,$$

$$\mathcal{T}^{TOP}(\mathcal{X}) \cong \mathcal{T}^{TOP}(\mathbb{R}P^{4k+2}) \oplus \mathbb{Z}_2 = \bigoplus_{j=1}^{2k+2} \mathbb{Z}_2,$$

$$\mathcal{T}^{TOP}(\mathcal{X}) \cong \mathcal{T}^{TOP}(\mathbb{R}P^{4k+3}) = \bigoplus_{j=1}^{2k+1} \mathbb{Z}_2$$

$$\mathcal{T}^{TOP}(\mathcal{X}) \cong \mathcal{T}^{TOP}(\mathbb{R}P^{4k+4}) \oplus \mathbb{Z}_2 = \bigoplus_{j=1}^{2k+3} \mathbb{Z}_2,$$

for $n = 4k+1, 4k+2, 4k+3, 4k+4$*, respectively.*

Proof In dimensions $4k+1$, $4k+2$, and $4k+3$, the result follows from Corollary 3.1 and (4.1). In dimension $4k+4$, it is necessary additionally to use the fact that $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. □

Theorem 4.2 *Let*

$$\mathcal{X} = (\mathbb{R}P^n, \mathbb{R}P^n; \mathbb{R}P^{n-1}), \quad n \geq 6.$$

Then there are isomorphisms

$$\mathcal{S}^{TOP}(\mathcal{X}) \cong \mathcal{S}^{TOP}(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \cong \mathcal{S}^{TOP}(\mathbb{R}P^{n-1}). \tag{4.4}$$

Proof The braid (3.26) provides the isomorphisms in (4.4), since $\mathcal{S}_m(pt_*) = 0$ [10, page 276]. □

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