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NORMAL INVARIANTS OF MANIFOLD PAIRS AND ASSEMBLY MAPS

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Abstract. This article constructs the structure set of a manifold pair, describes its normal invariants, and develops exact sequences and braids of exact sequences comparing the structure set and its normal invariants to those of the manifold and the submanifold.

1. Introduction.

Let X^n be a connected closed n-dimensional topological manifold with fundamental group $\pi_1 = \pi_1(X)$ and orientation character $w : \pi_1 \to \{\pm 1\}$. A fundamental problem of geometric topology is to describe all possible connected closed n-dimensional topological manifolds which are homotopy (simple homotopy) equivalent to X.

Let $h\colon M \to X$ be an orientation preserving simple homotopy equivalence, where M is a connected closed n-manifold in the category TOP of topological manifolds. Two such maps $f_i : M_i \to X (i = 1, 2)$ are said to be equivalent if there is an s-cobordism W between them together with a map from W to X extending the maps $f_i (i = 1, 2)$ on the boundary [11, page 542]. The set of equivalence classes forms the structure set $\mathcal{S}^s_n(X)$ and fits into the surgery exact sequence (see, for example, [10, page 278] and [11, page 559])

$$\cdots \to [\Sigma X, G/TOP] \overset{\sigma_{n+1}}{\to} L_{n+1}(\pi_1) \to \mathcal{S}^s_n(X) \to [X, G/TOP] \overset{\sigma_n}{\to} L_n(\pi_1).$$

The elements of the set $[X, G/TOP]$ are called normal invariants. To describe the set $\mathcal{S}^s_n(X)$ we must know the set of normal invariants, the surgery obstruction groups $L_n(\pi_1) = L^s_n(\pi_1)$, and the assembly map σ. There has been much progress computing L-groups (see [1], [2], [4], and [15]) and there are many results concerning normal invariants and assembly maps (see, for example, [3], [12], and [14]), but still the description obtained of $\mathcal{S}^s_n(X)$ is in general unsatisfactory. One significant reason for this is the difficulty of analyzing assembly maps. The goal of this paper is to obtain information about assembly maps by examining their relationship to submanifolds of X.

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Let $Y \subset X$ be a submanifold of codimension q in X. A homotopy equivalence $f : M \to X$ splits along the submanifold Y if by definition it is homotopy equivalent to a map g transversal to Y, such that for $N = g^{-1}(Y)$ the restrictions

$$g|_N : N \to Y, \quad g|_{(M\setminus N)} : M \setminus N \to X \setminus Y$$

are simple homotopy equivalences. By definition (see [14, page118]) $M\setminus N$ (similarly $X \setminus Y$) is the closure of the complement of a tubular neighborhood of N. According to [11, 7.2] there exists a group $LS_{n-q}(F)$ of obstructions to splitting which depends only on $n - q$ mod 4 and a pushout square

$$F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(U) & \to & \pi_1(X) \end{pmatrix} \tag{1}$$

of fundamental groups with orientations, where ∂U is the boundary of a tubular neighborhood U of Y in X. If $f : M \to X$ is a normal map then by [11, 7.2] there exists a group $LP_{n-q}(F)$ of obstructions to surgery on pairs of manifolds (X, Y) which depends as well only on $n - q$ mod 4 and the square F. The groups $LS_*(F)$ and $LP_*(F)$ are closely related to the surgery obstruction groups $L_*(\pi_1(X))$ and $L_*(\pi_1(Y))$, and are related to the classical surgery exact sequence by the commutative diagram

$$\begin{array}{ccccccc} \cdots \to & \mathcal{S}^s_n(X) & \to & [X, G/TOP] & \xrightarrow{\sigma} & L_n(\pi_1(X)) & \\ & \downarrow & & \downarrow v_\xi & & \downarrow = & \\ \cdots \to & LS_{n-q}(F) & \to & LP_{n-q}(F) & \xrightarrow{p_0} & L_n(\pi_1(X)) \to & \\ & \downarrow = & & \downarrow_{p_1} & & \downarrow & \\ \cdots \to & LS_{n-q}(F) & \to & L_{n-q}(\pi_1(Y)) & \to & L_n(\pi_1(X \setminus Y) \to \pi_1(X)) \to, & \end{array} \tag{2}$$

in which all rows are exact [11, 7.2] and [14, page 136].

The maps p_i in the diagram are the forgetful maps and the map v_ξ is defined in a natural way such that the composition $p_0 v_\xi$ is the assembly map σ in the classical surgery exact sequence. It is necessary to remark that the map $p_1 v_\xi$ in diagram (2) has the natural geometrical description. Each normal map $h \in [X, G/TOP]$ gives by restriction to the submanifold Y a normal map from $[Y, G/TOP]$. The obstruction to surgery of the restricted map is exactly $p_1 v_\xi(h) \in L_{n-q}(\pi_1(Y))$.

In [11, page 571] Ranicki introduced a set $\mathcal{S}_n(X, Y, \xi)$ of s-triangulations of a pair of manifolds (X, Y), where ξ denotes the normal bundle of Y in X. This set consists of concordance classes of Y-split maps $f : (M, N) \to (X, Y)$ where $N = f^{-1}(Y)$ and fits into an exact sequence

$$\cdots \to \mathcal{S}_n(X, Y, \xi) \to [X, G/TOP] \xrightarrow{v_\xi} LP_{n-q}(F) \to \dots \tag{3}$$

This sequence extends to the right and to the left and the set $\mathcal{S}_n(X, Y, \xi)$ has a group structure. In [11, Proposition 7.2.6] there is a description of properties of the groups $\mathcal{S}_n(X, Y, \xi)$, including an exact sequence which is compared with the surgery exact sequences for the manifolds X and Y.

In the current paper, we introduce the structure set $\mathcal{NS}_*(X, Y)$ and show that it fits into an exact sequence

$$\cdots \to L_{n-q+1}(\pi_1(Y)) \to \mathcal{NS}_n(X, Y) \to [X, G/TOP] \xrightarrow{p_1 v_\xi} L_{n-q}(\pi_1(Y)), \tag{4}$$

where, as usual, exactness at $L_{n-q+1}(\pi_1(Y))$ and $\mathcal{NS}_n(X,Y)$ is defined in terms of the group action $L_{n-q+1}(\pi_1(Y))$ on $\mathcal{NS}_n(X,Y)$. This is Theorem 1 of the paper. The definition of $\mathcal{NS}_*(X,Y)$ is a natural straightforward extension of mixed type of structure [14, p. 116] for a manifold with the boundary, in which Y plays the role of ∂X.

Letting $\mathbf{L}_\bullet$ denote the 1-connected cover of the algebraic L-theory spectrum of a point and letting $\mathbf{L}(\pi_1(Y))$ denote the algebraic L-theory spectrum of the manifold Y (see [3, page 28], [10], and [11, page 544]), we construct a map $\Omega^q(X_+ \wedge \mathbf{L}_\bullet) \to \mathbf{L}(\pi_1(Y))$ of spectra, identify the homotopy group $\pi_{n-q}(\mathbf{F})$ of its fiber $\mathbf{F}$ with the structure sets $\mathcal{NS}_n(X,Y)$, and show that there is an isomorphism

$$\begin{array}{cccc}
\to \pi_{n-q+1}\mathbf{L}(\pi_1(Y)) \to & \pi_{n-q}(\mathbf{F}) \to & \pi_{n-q}(\Omega^q(X_+ \wedge \mathbf{L}_\bullet)) & \to \pi_{n-q}\mathbf{L}(\pi_1(Y)) \\
\downarrow = & \downarrow\cong & \downarrow\cong & \downarrow = \\
\to L_{n-q+1}(\pi_1(Y))) \to & \mathcal{NS}_n(X,Y) \to & [X, G/TOP] & \to L_{n-q}(\pi_1(Y))
\end{array}$$

of the long exact sequence of homotopy groups of the map $\Omega^q(X_+ \wedge \mathbf{L}_\bullet) \to \mathbf{L}(\pi_1(Y))$, beginning at level $n-q$, with the exact sequence (4) above. This is Theorem 2 of the paper. As a trivial consequence of the theorem, one obtains that $\mathcal{NS}_n(X,Y)$ has a group structure making (4) an exact sequence of groups. Five other theorems are developed which provide exact rectangular and braid diagrams of homotopy groups of spectra, relating in various ways all of the groups introduced above.

The rest of the paper is organized as follows. In section 2, we recall general constructions and properties of algebraic L-theory spectra relating a manifold with a submanifold. In section 3, we prove Theorem 1 and Theorem 2 and in section 4, we provide our exact diagrams relating the groups introduced above.

2. Spectra and L-groups.

Let R be a ring with involution. According to [6, §7] and [14, page 253], there exists an Ω-spectrum $\mathbf{L}(R) = \{\mathbf{L}_{-k}(R) : k \in \mathbb{Z}\}$ such that $\mathbf{L}_{m+1}(R) \simeq \Omega\mathbf{L}_m(R)$ and $L_n(R) = \pi_n(\mathbf{L}(R)) = \pi_n(\mathbf{L}_0(R))$. For every morphism $i : R \to T$ of rings with involution, there exists a cofibration of spectra

$$\cdots \to \mathbf{L}(R) \to \mathbf{L}(T) \to \mathbf{L}(R \to T) \to \cdots \tag{5}$$

whose homotopy long exact sequence is the relative exact sequence of L-groups of the map i [6, §7]:

$$\begin{array}{ccccccc}
\cdots \to & L_n(R) & \to & L_n(T) & \to & L_n(R \to T) & \to \dots \\
& || & & || & & || & \\
\to & \pi_n(\mathbf{L}(R)) & \to & \pi_n(\mathbf{L}(T)) & \to & \pi_n(\mathbf{L}(R \to T)) & \to \dots
\end{array}$$

If π denotes a group equipped with a homomorphism $w : \pi \to \{\pm 1\}$, usually called the orientation homomorphism, and if $\mathbb{Z}\pi$ denotes the integral group ring supplied with the involution defined by the rule

$$\Sigma a_g g \mapsto \Sigma a_g w(g) g^{-1},\ a_g \in \mathbb{Z}, g \in \pi$$

then it is customary to denote the spectrum $\mathbf{L}(\mathbb{Z}\pi)$ by $\mathbf{L}(\pi)$. If it necessary to make the orientation homomorphism w explicit then we shall write $\mathbf{L}(\pi, w)$ instead of $\mathbf{L}(\pi)$.

Let

$$(D^q, S^{q-1}) \to (U, \partial U) \xrightarrow{p} Y$$

denote a disk bundle over a closed manifold Y. Since the transfer map of L-groups is defined on the spectra level (see [6, §7] and [14, page 253]), we have the commutative diagram

$$\begin{array}{ccc} \mathbf{L}(\pi_1(Y)) & \xrightarrow{p^\sharp} & \Omega^q\mathbf{L}(\pi_1(\partial U) \to \pi_1(U)) \\ & \searrow & \downarrow \delta \\ & & \Omega^{q-1}\mathbf{L}(\pi_1(\partial U)), \end{array} \tag{6}$$

where δ is the connecting map in the cofibration sequence (5) of spectra for the natural map of fundamental groups $\pi_1(\partial U) \to \pi_1(U)$.

Now let $Y \subset X$ be a codimension q submanifold of a connected closed manifold X. A tubular neighborhood U of Y in X with boundary ∂U is a disk bundle over the submanifold Y, and the square F in (1) of fundamental groups is defined. There is a homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \mathbf{L}(\pi_1(Y)) & \xrightarrow{p^\sharp} & \Omega^q\mathbf{L}(\pi_1(\partial U) \to \pi_1(U)) & \xrightarrow{\alpha} & \Omega^q\mathbf{L}(\pi_1(X\setminus Y) \to \pi_1(X)) \\ & \searrow & \downarrow \delta & & \downarrow \delta_1 \\ & & \Omega^{q-1}\mathbf{L}(\pi_1(\partial U)) & \xrightarrow{\beta} & \Omega^{q-1}\mathbf{L}(\pi_1(X\setminus Y)), \end{array} \tag{7}$$

where the left triangle is that in (6) and the horizontal maps of the right hand square are induced by the horizontal maps of F.

We define the spectrum

$$\mathbf{LS}(F) = \textit{homotopy cofiber}\left[\Omega(\alpha p^\sharp) : \Omega\mathbf{L}(\pi_1(Y)) \to \Omega^{q+1}\mathbf{L}(\pi_1(X\setminus Y) \to \pi_1(X))\right]$$

where α and $p^\sharp$ are as in (7). We define the spectrum

$$\mathbf{LP}(F) = \textit{homotopy cofiber}\left[\Omega(\beta\delta p^\sharp) : \Omega\mathbf{L}(\pi_1(Y)) \to \Omega^{q}\mathbf{L}(\pi_1(X\setminus Y))\right]$$

where β, δ, and $p^\sharp$ are as in (7). (See [5], [8], and [9] for the special case of submanifolds).

Recall that in the homotopy category of spectra the concepts pullback and pushout squares are equivalent. A homotopy commutative square of spectra is a pullback iff the fibres (and hence cofibres) of any two parallel maps are naturally homotopy equivalent.

Proposition 1. *The homotopy commutativity of (7) induces up to homotopy a map* $\mathbf{LS}(F) \to \mathbf{LP}(F)$ *of spectra such that the diagram*

$$\begin{array}{ccccc} \Omega\mathbf{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1}\mathbf{L}(\pi_1(X\setminus Y) \to \pi_1(X)) & \longrightarrow & \mathbf{LS}(F) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \Omega\mathbf{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q}\mathbf{L}(\pi_1(X\setminus Y)) & \longrightarrow & \mathbf{LP}(F) \end{array}$$

of spectra is homotopy commutative, the horizontal rows are cofibrations and the right square is pushout. Moreover if $LS_n(F)$ *and* $LP_n(F)$ *denote respectively the surgery obstruction groups to splitting and to surgery on on* (X, Y)*, defined by Ranicki in* [11, 7.2] *then there are isomorphisms*

$$\pi_n(\mathbf{LS}(F)) \cong LS_n(F),\ \pi_n(\mathbf{LP}(F)) \cong LP_n(F)$$

which are functorial in F.

Proof. The assertions for the diagram are routine (cf. [13]). The isomorphisms for the obstruction groups follow from the five-lemma and the definition of the LS_* and LP_* groups. □

Remark. Let $C = \pi_1(X \setminus Y)$ and $D = \pi_1(X)$. Then the homotopy long exact sequences of the maps in the pushout square of Proposition 1 fit together to form a braid of exact sequences

$$\begin{array}{ccccccccc}
 & & L_{n+q}(C) & \rightarrow & L_{n+q}(D) & \xrightarrow{\Theta} & LS_{n-1}(F) & & \\
 & \nearrow & & \searrow & \nearrow & \searrow & \nearrow & & \\
L_{n+q+1}(C \to D) & & & & LP_n(F) & & L_{n+q}(C \to D) & & \\
 & \searrow\nearrow & & & \searrow & \nearrow & \searrow & & \\
 & & LS_n(F) & \rightarrow & L_n(B) & \longrightarrow & L_{n+q-1}(C) & &
\end{array} \tag{8}$$

due to Wall [14, page 264].

3. Surgery exact sequence of a manifold pair.

Let $\mathbf{L}_\bullet$ denote the 1-connected cover of the surgery Ω-spectrum $\mathbf{L}(\mathbb{Z})$ such that $\mathbf{L}_{\bullet 0} \simeq G/TOP$. For any topological space X such that $\pi_1(X) = \pi$, there exists an algebraic surgery exact sequence (see [10, page 278], [11, page 559], and [14, page 116])

$$\cdots \to L_{m+1}(\pi) \to \mathbb{S}_{m+1}(X) \to H_m(X; \mathbf{L}_\bullet) \to L_m(\pi) \to \cdots \tag{9}$$

By definition, (9) is the homotopy long exact sequence of the cofibration

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbf{L}(\pi).$$

of spectra with homotopy fiber $\mathbf{S}(X)$ where

$$\begin{array}{rcl}
\mathbb{S}_{m+1}(X) & = & \pi_m(\mathbf{S}(X)) \\
H_m(X; \mathbf{L}_\bullet) & = & \pi_m(X_+ \wedge \mathbf{L}_\bullet) \\
L_m(\pi) & = & \pi_m(\mathbf{L}(\pi)).
\end{array}$$

If X is a closed n-dimensional topological manifold then the exact sequence (9) corresponds for $m \geq n$ to the surgery exact sequence, because $H_n(X; \mathbf{L}_\bullet) = [X, G/TOP]$ and $\mathbb{S}_{n+1}(X) = \mathcal{S}_n^s(X)$.

Let $Y^{n-q} \subset X^n$ be a submanifold of codimension q of the n-dimensional topological manifold X. Then the square F in (1) of fundamental groups is defined. From diagram (2) and the algebraic surgery exact sequence (9) (see [14, page 136]) we obtain the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to & \mathbb{S}_{m+1}(X) & \to & H_m(X; \mathbf{L}_\bullet) & \to & L_m(\pi_1(X)) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & LS_{m-q}(F) & \to & L_{m-q}(\pi_1(Y)) & \to & L_m(\pi_1(X \setminus Y) \to \pi_1(X)) & \to \cdots
\end{array} \tag{10}$$

Proposition 2. *There is a homotopy commutative square*

$$\begin{array}{ccc} \Omega^q(X_+ \wedge \mathbf{L}_\bullet) & \stackrel{\mathbf{A}}{\to} & \Omega^q\mathbf{L}(\pi_1(X)) \\ \downarrow & & \downarrow \\ \mathbf{L}(\pi_1(Y)) & \to & \Omega^q\mathbf{L}(\pi_1(X\setminus Y) \to \pi_1(X)) \end{array} \tag{11}$$

of spectra such that the commutative diagram in (10) is that obtained by canonically mapping the homotopy long exact sequence of the top map in (11) to that of the bottom map in (11).

Proof. Let U be a tubular neighborhood with the boundary ∂U of the submanifold Y and let $Z = X \setminus Y$. Define the commutative diagram

$$\begin{array}{ccc} H_{m-q}(Y;\mathbf{L}_\bullet) & \to & L_{m-q}(\pi_1(Y)) \\ \downarrow & & \downarrow \\ H_m(U,\partial U;\mathbf{L}_\bullet) & \to & L_m(\pi_1(\partial U) \to \pi_1(U)) \\ \downarrow & & \downarrow \\ H_m(X,Z;\mathbf{L}_\bullet) & \to & L_m(\pi_1(X\setminus Y) \to \pi_1(X)) \end{array} \tag{12}$$

such that the horizontal maps correspond to assembly maps, the top vertical maps are transfer maps (see [11, page 579]), and the bottom vertical maps are the canonical ones. The square F induces the homotopy commutative diagram of spectra [13]

$$\begin{array}{ccc} \partial U_+ \wedge \mathbf{L}_\bullet & \to & Z_+ \wedge \mathbf{L}_\bullet \\ \downarrow \mathbf{i} & & \downarrow \mathbf{j} \\ U_+ \wedge \mathbf{L}_\bullet & \to & X_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ \mathcal{C}of\,\mathbf{i} & \stackrel{\mathbf{r}}{\to} & \mathcal{C}of\,\mathbf{j}, \end{array} \tag{13}$$

whose vertical columns are cofibrations. For all m, we have the commutative diagram

$$\begin{array}{ccc} \pi_m(\mathcal{C}of\,\mathbf{i}) & \underset{\cong}{\stackrel{\pi_m(r)}{\to}} & \pi_m(\mathcal{C}of\,\mathbf{j}) \\ || & & || \\ H_m(U,\partial U;\mathbf{L}_\bullet) & \to & H_m(X,Z;\mathbf{L}_\bullet) \end{array} \tag{13$'$}$$

where the equalities are given by the definition of $H_m(\quad;\quad)$ and the bottom horizontal map is an isomorphism by excision. Hence the map $\pi_m(r)$ is an isomorphism and the map $\mathbf{r}$ is a weak homotopy equivalence. Now it follows from [11] that $\mathbf{r}$ is a homotopy equivalence. By [11, page 579], the upper square of diagram (12) is realized on the spectra level by the homotopy commutative diagram of spectra

$$\begin{array}{ccc} Y_+ \wedge \mathbf{L}_\bullet & \to & \mathbf{L}(\pi_1(Y)) \\ \sim\downarrow \mathbf{s} & & \downarrow \\ \Omega^q\mathcal{C}of\,\mathbf{i} & \to & \Omega^q\mathbf{L}(\pi_1(\partial U) \to \pi_1(U)), \end{array} \tag{14}$$

in which the left vertical map is a weak homotopy equivalence [11, page 579] and hence by [13] a homotopy equivalence. Because of the naturality of the assembly map, the lower square of diagram (12) is realized on the spectra level by the homotopy commutative diagram of spectra

$$\begin{array}{ccc} \mathcal{C}of\,\mathbf{i} & \to & \mathbf{L}(\pi_1(\partial U) \to (\pi_1(U)) \\ \sim\downarrow & & \downarrow \\ \mathcal{C}of\,\mathbf{j} & \to & \mathbf{L}(\pi_1(X\setminus Y) \to \pi_1(X)) \end{array} \tag{14$'$}$$

and by (13′), the left vertical map is a homotopy equivalence. From diagrams (14) and (14′), we obtain a homotopy commutative diagram

$$\begin{array}{ccc} Y_+ \wedge \mathbf{L}_\bullet & \to & \mathbf{L}(\pi_1(Y)) \\ \sim\downarrow & & \downarrow \\ \Omega^q \mathcal{C}of\mathbf{j} & \to & \Omega^q \mathbf{L}(\pi_1(X \setminus Y) \to \pi_1(X)), \end{array} \tag{15}$$

in which the left vertical map is a homotopy equivalence. Because of the naturality of the assembly map, there is a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \Omega^q X_+ \wedge \mathbf{L}_\bullet & \to & \Omega^q \mathbf{L}(\pi_1(X)) \\ \downarrow & & \downarrow \\ \Omega^q \mathcal{C}of\mathbf{j} & \to & \Omega^q \mathbf{L}(\pi_1(X \setminus Y) \to \pi_1(X)). \end{array} \tag{16}$$

By (15), the bottom horizontal map factors up to homotopy through $\mathbf{L}(\pi_1(Y))$. This establishes the proposition. □

Recall the definition of a normal map into a closed topological manifold X [11, p. 604]. Suppose X has dimension n. Then a topological normal map $(f,b) : M \to X$ consists of the following.

i) an n-dimensional manifold M with a normal topological block bundle

$$\nu_M = \nu_{M \subset S^{n+k}} : M \to BTOP(k),$$

$$\rho_M : S^{n+k} \to S^{n+k}/\overline{S^{n+k} - E(\nu_M)} = T(\nu_M)$$

ii) an n-dimensional manifold X with a topological block bundle

$$\nu_X : X \to BTOP(k),$$

$$\rho_X : S^{n+k} \to T(\nu_X)$$

iii) a degree one map $f : M \to X$

iv) a map of topological block bundles $b : \nu_M \to \nu_X$ covering f, such that

$$T(b)_*(\rho_M) = \rho_X \in \pi_{n+k}(T(\nu_X)).$$

The topological normal structure set of the manifold X is the set of concordance classes of normal maps $(f,b) : M \to X$. For $n \geq 5$ this set coincides with the set $[X, G/TOP]$ (see [11, pp. 553–559]).

Let $f : M \to X$ be a normal map which is transversal to a submanifold Y. Let $N = f^{-1}(Y)$. Suppose that the restriction $f|_N : N \to Y$ is a simple homotopy equivalence. Two such normal maps $f_i : M_i \to X$ $(i = 1,2)$ with $N_i = f_i^{-1}(Y)$ will be considered *equivalent* if there exists a normal cobordism $F : W \to X$ with the following properties.

i) $\partial W = M_1 \cup M_2$ and $F|_{M_i} = f_i$ $(i = 1,2)$.

ii) F is transversal to Y with $F^{-1}(Y) = V$ and $\partial V = N_1 \cup N_2$.

iii) the restriction $F|_V$ is an s-cobordism between $F|_{N_i} = f_i$ $(i = 1,2)$.

Denote the equivalence classes of such maps by $\mathcal{NS}_n(X,Y)$ where $n = \dim X$.

The set $\mathcal{NS}_n(X,Y)$ has a base point which is given by a representative $f : (M,N) \to (X,Y)$, where $f|_M : M \to X$ is a homeomorphism.

There exists a natural map

$$\phi : \mathcal{NS}_n(X,Y) \to [X, G/TOP].$$

defined by composition of the forgetful map

$$\mathcal{NS}_n(X,Y) \to \mathcal{T}^{TOP}(X)$$

with the map $\mathcal{T}^{TOP} \to [X, G/TOP]$, where $\mathcal{T}^{TOP}(X)$ is the set of concordance classes of t-triangulations $(f,b) : M \to X$ (see [11, pages 553–555]).

We define a map (action)

$$\lambda : L_{n-q+1}(\pi_1(Y)) \to \mathcal{NS}_n(X,Y)$$

in the following way. Let $f : (M,N) \to (X,Y)$ represent some class $a \in \mathcal{NS}_n(X,Y)$. For any element $x \in L_{n-q+1}(\pi_1(Y))$, there is an action of x on the restriction $f|_N : N \to Y$ such that if $f_1 = x(f|_N) : N_1 \to Y$ is the result of this action then f_1 is a simple homotopy equivalence (see [12, page 196], [14; 5.6, 6.5, and §10]).

The construction of this action in [14] produces a normal map of manifolds with boundaries $G : V \to Y \times I$ with $\partial V = N \cup N_1$, $G|_N = f|_N : N \to Y \times \{0\}$, and $G|_{N_1} = f_1 : N_1 \to Y \times \{1\}$. The surgery obstruction for the map G relative to boundary ∂V is $x \in L_{n-q+1}(\pi_1(Y))$.

Let E_ξ be a tubular neighborhood of the submanifold Y in X and ∂E_ξ its boundary. The map f is transversal to the submanifold Y and $F|_N : N \to Y$ is a simple homotopy equivalence. Hence $f|_N$ induces a simple homotopy equivalence of tubular neighborhoods and their boundaries. We denote the equivalence by

$$g_0 = f\,\big|_{(E_\nu, \partial E_\nu)} : (E_\nu, \partial E_\nu) \to (E_\xi, \partial E_\xi)$$

(see for example [14, p. 8] and [11, p. 579]) where E_ν is a tubular neighborhood of the submanifold N in M and ∂E_ν its boundary. The transfer map $p^\sharp$ of spectra in diagram (6) induces a transfer map of L-groups

$$p^! : L_{n-q+1}(\pi_1(Y)) \to L_{n+1}(\pi_1(\partial E_\xi) \to \pi_1(E_\xi)).$$

and a transfer map

$$p^\flat : \mathcal{S}_{n-q}(Y) \to \mathcal{S}_n(E_\xi, \partial E_\xi)$$

of structure sets.

The geometrical definition of the map $p^!(x)$ (see [11, pp. 562–565] and [14, 133]) gives a normal map of manifolds with boundaries $H : \Omega \to (E_\xi, \partial E_\xi) \times I$ such that H is transversal to $Y \times I$, $H^{-1}(Y \times I) = V$, and $H|_V = G$. The restriction of the map H to the bottom boundary is $g_0 = f\,\big|_{(E_\nu, \partial E_\nu)}$. The restriction of H to the top boundary is a simple homotopy equivalence of pairs

$$g_1 : (E_\eta, \partial E_\eta) \to (E_\xi, \partial E_\xi)$$

which is by definition the result of the action of $p^!(x)$ on the simple homotopy equivalence $(f\,\big|_{(E_\nu, \partial E_\nu)})$. The restriction $g_1|_{N_1}$ is f_1. In particular, the element

$$p^\flat(f_1 : N_1 \to Y) \in \mathcal{S}_n(E_\xi, \partial E_\xi)$$

is represented by a simple homotopy equivalence of pairs

$$g_1 : (E_\eta, \partial E_\eta) \to (E_\xi, \partial E_\xi)$$

which is transversal to the submanifold Y with $g_1^{-1}(Y) = N_1$. The cobordism Ω and the map $f : M \to X$ are extended to a normal cobordism $F : W \to X$ with $\partial W = M \cup M_1$ [7, page 45]. The restriction $F|_{M_1}$ defines an element $x(a) \in \mathcal{NS}_n(X, Y)$. Note that for $x \neq 1$ the map F does not define an equivalence between $f = F|_M$ and $F|_{M_1}$, in the sense above, because V is not an s-cobordism.

Lemma 1. *The definition of λ is well defined and is an action of the group $L_{n-q+1}(\pi_1(Y))$ on the set $\mathcal{NS}_n(X, Y)$.*

Proof. Let $G' : V' \to Y$ denote another normal cobordism constructed as above such that $\partial V' = N \cup N_1'$, $G'|_N = f|_N$, and $G'|_{N_1'} = f_1'$. Let $F' : W' \to X$ denote a normal cobordism extending G' and $f : M \to X$ such that $\partial W = M \cup M_1'$ [7, page 45]. The restriction $F|_{M_1'}$ is another representative of the element $x(a) \in \mathcal{NS}_n(X, Y)$. We shall prove that the representatives $F|_{M_1}$ and $F'|_{M_1'}$ are equivalent in $\mathcal{NS}_n(X, Y)$. From [14, 10.4], it follows that there is a normal cobordism map $\Psi : \Lambda \to Y$ whose bottom is $f|_N \times I : N \times I \to Y$, whose left side is $G : V \to Y$, whose right side is $G' : V' \to Y$, and whose top is a normal s-cobordism from $G|_{N_1} : N_1 \to Y$ to $G'|_{N_1'} : N_1' \to Y$. Let $W \cup_\tau M \times I \cup_{\tau'} W'$ denote the space got from the obvious attaching maps $\tau : M \times \{0\} \to W$ and $\tau' : M \times \{1\} \to W'$. Let

$$H = F \cup (f \times I) \cup F' : W \cup_\tau M \times I \cup_{\tau'} W' \to X.$$

Let

$$\rho : V \cup_{\tau|_{N \times \{0\}}} N \times I \cup_{\tau'|_{N \times \{1\}}} W' \to W \cup_\tau M \times I \cup_{\tau'} W'$$

denote the canonical embedding and use ρ to attach Λ to $W \cup_\tau M \times I \cup_{\tau'} W'$ to get the space

$$\tilde{\Lambda} := \Lambda \cup_\rho (W \cup_\tau M \times I \cup_{\tau'} W')$$

and a map

$$\Psi \cup_\rho (H) : \tilde{\Lambda} \to X.$$

By [7, p. 45] we can extend $\Psi \cup_\rho (H)$ to a normal cobordism $\Theta : \Omega \to X$ whose bottom is $f \times I : M \times I \to X$ and whose top is an equivalence in the sense defined above between $f_1 : M_1 \to X$ and $f_1' : M_1' \to X$. Using the same line of argument as in the definition of the map λ we can arrange that the restriction of the map Θ on the top boundary is transversal to the submanifold Y. The proof that we have a group action follows routinely from the fact that the action of $L_{n-q+1}(\pi_1(Y))$ on $\mathcal{S}_{n-q}(Y)$ is a group action [14, §10]. $\square$

There exists a natural geometric map

$$\alpha_1 : [X \times D^1, X \times S^0; G/TOP, *] \to L_{n-q+1}(\pi_1(Y))$$

got by composing the restriction map

$$[X \times D^1, X \times S^0; G/TOP, *] \to [Y \times D^1, Y \times S^0; G/TOP, *]$$

with the map

$$[Y \times D^1, Y \times S^0; G/TOP, *] \to L_{n-q+1}(\pi_1(Y)).$$

Similarly, one has a natural geometric map $\alpha : [X; G/TOP] \to L_{n-q+1}(\pi_1(Y))$.

Theorem 1. *The sequence*

$$\begin{aligned} \cdots \to [X \times D^1, X \times S^0; G/TOP, *] \xrightarrow{\alpha_1} L_{n-q+1}(\pi_1(Y)) \to \\ \xrightarrow{\lambda} \mathcal{NS}_n(X,Y) \xrightarrow{\phi} [X, G/TOP] \xrightarrow{\alpha} L_{n-q}(\pi_1(Y)) \end{aligned} \tag{17}$$

is exact.

Proof. Exactness at $[X, G/TOP]$. There is a natural map $\mathcal{NS}_n(X,Y) \to \mathcal{S}_{n-q}(Y)$ got by restricting to Y. The diagram

$$\begin{tikzcd} \mathcal{NS}_n(X,Y) \arrow[r, "\phi"] \arrow[d] & {[X, G/TOP]} \arrow[d] \arrow[dr, "\alpha"] & \\ \mathcal{S}_{n-q}(Y) \arrow[r] & {[Y, G/TOP]} \arrow[r, "\sigma"] & L_{n-q}(\pi_1(Y)) \end{tikzcd}$$

commutes and the bottom row is exact by [11, Proposition 7.1.4]. Thus $\alpha\phi = 0$. Let $f : M \to X$ denote a normal map defining a class in $[X, G/TOP]$ such that $\alpha(f) = 0$. Then $\sigma(f|_N) = 0$ and so $f|_N$ is normal cobordant, say by $G : V \to Y$, to a normal map $g : N_1 \to Y$ which is a simple homotopy equivalence. By [7, p. 45] we can extend $f \cup G : M \cup V \to X$ by a normal cobordism $F : W \to X$ such that $F|_{N_1} = g$ is a simple homotopy equivalence. Using the same line of argument as in the definition of the map λ we can arrange that the restriction of the map $F|_{M_1}$ on the top boundary is transversal to the submanifold Y. By definition the class $F|_{M_1}$ lies in the image of ϕ.

Exactness at $\mathcal{NS}(X,Y)$. If $x \in L_{n-q+1}(\pi_1(Y))$ and $f : (M,N) \to (X,Y)$ represents an element of $\mathcal{NS}(X,Y)$ then the normal map $x(f)$ is by definition normal cobordant to f and so f and $x(f)$ have the same image in $[X, G/TOP]$. This shows by definition that $\phi\lambda = 0$. Let

$$f_i : (M_i, N_i) \to (X,Y) \ (i = 1,2)$$

denote normal maps representing elements of $\mathcal{NS}_n(X,Y)$. If the f_i's represent the same element of $[X, G/TOP]$ then there is by definition a normal cobordism $F' : W' \to X$ from f_1 to f_2. Since $f_1|_{N_1}$ and $f_2|_{N_2}$ are obviously the same in $[Y, G/TOP]$, there is by [14, §10] an element $x \in L_{n-q+1}(\pi_1(Y))$ such that $x(f_1|_{N_1})$ is normal s-cobordant to $f_2|_{N_2}$. Let $F : W \to X$ be the normal cobordism constructed in the definition of $x(f_1)$. We want to show $x(f_1)$ is equivalent, in the sense defined prior to Lemma 1, to f_2. Let $\Psi : \Lambda \to Y$ be the normal cobordism corresponding to the picture given by [14, §10]. This corresponds to the normal cobordism $\Psi : \Lambda \to Y$ in the proof of Lemma 1. Now as in the proof of this lemma, there is a normal cobordism $\Theta : \Omega \to X$ extending Ψ. The top of this cobordism is a normal cobordism $\mathcal{J}$ from $x(f_1)$ to f_2 such that $\mathcal{J}|_{\mathcal{J}^{-1}(Y)}$ is a normal s-cobordism from $x(f_1|_{N_1})$ to $f_2|_{N_2}$. As in the definition of the map λ we can arrange that the map $\mathcal{J}$ is transversal to the submanifold Y.

Exactness at $L_{n-q+1}(\pi_1(Y))$. Let $x \in image(\alpha_1)$ and let $f : (M,N) \to (X,Y)$ represent an element of $\mathcal{NS}_n(X,Y)$. Let G denote the normal cobordism in the definition of $x(f)$. By definition, the bottom map in G is $f|_N$ and the top $x(f)|_{N_1}$. Since the diagram

$$\begin{tikzcd} {[X \times D^1, X \times S^0; G/TOP]} \arrow[d] \arrow[dr, "\alpha_1"] & & \\ {[Y \times D^1, Y \times S^0; G/TOP]} \arrow[r, "\sigma"] & L_{n-q+1}(\pi_1(Y)) \arrow[r] & \mathcal{S}_{n-q}(Y) \end{tikzcd}$$

commutes and the bottom row is exact by [11, Proposition 7.1.4] and [14, Proposition 10.8], it follows that G is an s-cobordism by [14, Proposition 10.4]. Thus the cobordism F in the definition of $x(f)$ is an equivalence, in the sense defined prior to Lemma 1, between f and $x(f)$. Let $x \in L_{n-q+1}(\pi_1(Y))$ and $f \in \mathcal{NS}_n(X,Y)$ such that f is equivalent to $x(f)$. Let $F : (W,V) \to (X,Y) \times I$ denote the normal map constructed in the definition of the action of x on $f : (M,N) \to (X,Y)$ and let $f_1 : (M_1,N_1) \to (X,Y)$ denote the top map of F. Let $G : V \to Y \times I$ denote the restriction of F to $F^{-1}(Y)$. Let $F' : (W',V') \to (X,Y)\times I$ denote a normal map defining an equivalence from $x(f)$ to f. Here the bottom map is $f_1 : (M_1,N_1) \to (X,Y)$ and the top map $f : (M,N) \to (X,Y)$. Let $G' : V' \to Y \times I$ denote the restriction of F' to $(F')^{-1}(Y)$. By definition $(F')^{-1}(Y) = V'$ is an s-cobordism. We identify the top boundary of W with the bottom boundary of W' and obtain a new normal map $H : (\Omega,\Lambda) \to (X,Y) \times I$ whose bottom and top maps coincide with $f : (M,N) \to (X,Y)$. Hence the normal map H lies in $[X \times D^1, Y \times S^0; G/TOP, *]$. The normal map $H|_\Lambda : \Lambda \to Y \times I$ is evidently the union of the normal maps given by G and G' and has the element $x \in L_{n-q+1}(\pi_1(Y))$ as obstruction to surgery relative boundary, because of the additivity of surgery obstructions. Thus $\alpha_1(H) = x$. $\square$

Let $\mathbf{F}$ denote the homotopy fiber of the map

$$\Omega^q(X_+ \wedge \mathbf{L}_\bullet) \to \mathbf{L}(\pi_1(Y))$$

in square (11). Let $\mathbf{NS}(X,Y)$ denote the q'th delooping of $\mathbf{F}$ and identify $\mathbf{F} = \Omega^q\mathbf{NS}(X,Y)$. Let

$$\mathbb{NS}_m(X,Y) = \pi_m(\mathbf{NS}(X,Y)),$$

and let

$$\cdots \to \mathbb{NS}_m(X,Y) \to H_m(X;\mathbf{L}_\bullet) \to L_{m-q}(\pi_1(Y)) \to \cdots \tag{18}$$

denote the long homotopy exact sequence of

$$\Omega^q(X_+ \wedge \mathbf{L}_\bullet) \to \mathbf{L}(\pi_1(Y)).$$

Theorem 2. *Use the group homomorphism*

$$L_{n-q+1}(\pi_1(Y)) \to \mathbb{NS}_n(X,Y)$$

in (18) to give $\mathbb{NS}_n(X,Y)$ a left action by multiplication of $L_{n-q+1}(\pi_1(Y))$. We assert that there is an isomorphism

$$\mathbb{NS}_n(X,Y) \to \mathcal{NS}_n(X,Y)$$

of $L_{n-q+1}(\pi_1(Y))$-actions making the diagram

$$\begin{array}{ccccccc}
\cdots \to & L_{n-q+1}(\pi_1(Y)) & \to & \mathbb{NS}_n(X,Y) & \to & H_n(X;\mathbf{L}_\bullet) & \to L_{n-q}(\pi_1(Y)) \\
 & \downarrow = & & \downarrow \cong & & \downarrow \cong & \downarrow = \\
\to & L_{n-q+1}(\pi_1(Y)) & \to & \mathcal{NS}_n(X,Y) & \to & [X, G/TOP] & \to L_{n-q}(\pi_1(Y))
\end{array}$$

commute. In particular there is a group structure on $\mathcal{NS}_n(X,Y)$ making (17) an exact sequence of groups.

Proof. By [14, pp. 116-117, 136] and [11, §7.2] there are commutative squares

$$\begin{array}{ccc} H_{n+1}(X;\mathbf{L}_\bullet) & \to & L_{n-q+1}(\pi_1(Y)) \\ \downarrow\cong & & \downarrow= \\ [X\times D^1, X\times S^0; G/TOP] & \to & L_{n-q+1}(\pi_1(Y)) \end{array}$$

and

$$\begin{array}{ccc} H_n(X;\mathbf{L}_\bullet) & \to & L_{n-q}(\pi_1(Y)) \\ \downarrow\cong & & \downarrow= \\ [X;G/TOP] & \to & L_{n-q}(\pi_1(Y)). \end{array}$$

It is now an elementary exercise using the above and exact sequences in (18) and (17) to construct the desired isomorphism of $L_{n-q+1}(\pi_1(Y))$-actions. The final assertion of the theorem is trivial. □

4. Algebraic properties of the groups $\mathcal{NS}_n(X,Y)$.

Define the spectrum $\Omega^q\mathbf{S}(X,Y,\xi)$ as the homotopy fiber of the natural map

$$\Omega^q\mathbf{S}(X)\to\mathbf{LS}(F).$$

Since $\mathbf{S}(X)$ has been defined as the homotopy fiber of $X_+\wedge\mathbf{L}_\bullet\to\mathbf{L}(\pi_1(X))$, it follows that $\Omega^q\mathbf{S}(X)$ is the fiber of the top map in the square (11). The spectrum $\mathbf{LS}(F)$ is the homotopy fiber of the bottom map in the square in (11) by [13, §8.32]. We obtain a cofibration sequence

$$\Omega^q\mathbf{S}(X,Y,\xi)\to\Omega^q\mathbf{S}(X)\to\mathbf{LS}(F).$$

By [11, §7] we have an isomorphism

$$\pi_m(\mathbf{S}(X,Y,\xi))=\mathbb{S}_{m+1}(X,Y,\xi),$$

and the groups $\mathbb{S}_m(X,Y,\xi)$ fit into an exact sequence [11, Proposition 7.2.6]

$$\cdots\to\mathbb{S}_{m+q+1}(X,Y,\xi)\to\mathbb{S}_{m+q+1}(X)\to LS_m(F)\to\cdots \tag{19}$$

We can extend by [13, §8.31] and [8, Lemma 1] the homotopy commutative square in (11) to a biinfinite homotopy commutative diagram of spectra

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega^q\mathbf{S}(X,Y,\xi) & \longrightarrow & \Omega^q\mathbf{NS}(X,Y) & \longrightarrow & \Omega^q\mathbf{L}(\pi_1(X\setminus Y)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \Omega^q\mathbf{S}(X) & \longrightarrow & \Omega^q(X_+\wedge\mathbf{L}_\bullet) & \longrightarrow & \Omega^q L(\pi_1(X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathbf{LS}(F) & \longrightarrow & \mathbf{L}(\pi_1(Y)) & \longrightarrow & \Omega^q\mathbf{L}(\pi_1(X\setminus Y)\to\pi_1(X)) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{20}$$

in which each row and column is a cofibration sequence.

Theorem 3. *Applying π_0 to (20) we obtain the biinfinite commutative complex of groups*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathbb{S}_{m+1}(X,Y,\xi) \arrow[r] \arrow[d] & \mathbb{NS}_m(X,Y) \arrow[r] \arrow[d] & L_m(\pi_1(X\setminus Y)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{S}_{m+1}(X) \arrow[r] \arrow[d] & H_m(X;\mathbf{L}_\bullet) \arrow[r] \arrow[d] & L_m(\pi_1(X)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & LS_{m-q}(F) \arrow[r] \arrow[d] & L_{m-q}(\pi_1(Y)) \arrow[r] \arrow[d] & L_m(\pi_1(X\setminus Y)\to\pi_1(X)) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{21}$$

in which all rows and columns are exact.

Proof. Since the complex in (20) is homotopy commutative, it follows that the complex in the theorem is commutative. Since all columns and rows in (20) are cofibration sequences, it follows that all rows and columns in the theorem are exact. $\square$

Lemma 2. *Let*

$$\begin{tikzcd}
 & \bullet \arrow[d] \arrow[dr] & \\
\bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[d] & \bullet \\
 & \bullet &
\end{tikzcd},$$

denote a homotopy commutative diagram of spectra in which the row and column are cofibrations. Then the cofibres of the sloping maps are naturally homotopy equivalent.

Proof. See [8]. $\square$

The biinfinite homotopy commutative diagram (20) contains a homotopy commutative diagram of spectra

$$\begin{tikzcd}
\Omega\mathbf{LS}(F) \arrow[r] \arrow[d] & \Omega\mathbf{L}(\pi_1(Y)) \arrow[r] \arrow[d] & \Omega^{q+1}\mathbf{L}(\pi_1(X\setminus Y)\to\pi_1(X)) \arrow[d] \\
\Omega^q\mathbf{S}(X,Y,\xi) \arrow[r] \arrow[d] & \Omega^q\mathbf{NS}(X,Y) \arrow[r] \arrow[d] & \Omega^q\mathbf{L}(\pi_1(X\setminus Y)) \arrow[d] \\
\Omega^q\mathbf{S}(X) \arrow[r] & \Omega^q(X_+\wedge\mathbf{L}_\bullet) \arrow[r] & \Omega^q\mathbf{L}(\pi_1(X))
\end{tikzcd} \tag{22}$$

in which all rows and columns are cofibrations. By Lemma 2, the cofiber of the map

$$\Omega^q\mathbf{S}(X,Y,\xi)\to\Omega^q(X_+\wedge\mathbf{L}_\bullet)$$

obtained from diagram (22) is naturally homotopy equivalent to the cofiber of the map

$$\Omega\mathbf{L}(\pi_1(Y)) \to \Omega^q\mathbf{L}(\pi_1(X\setminus Y))$$

obtained from diagram (22). But the last cofiber is the spectrum $\mathbf{LP}(F)$, by definition. Thus we obtain a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \Omega^q\mathbf{NS}(X,Y) & \to & \Omega^q\mathbf{L}(\pi_1(X\setminus Y)) \\ \downarrow & & \downarrow \\ \Omega^q(X_+\wedge\mathbf{L}_\bullet) & \to & \mathbf{LP}(F). \end{array} \tag{23}$$

Since the homotopy fiber of the top and bottom maps in (23) are the same, namely $\Omega^q\mathbf{S}(X,Y,\xi)$, it follows that (23) is a pullback and therefore a pushout homotopy commutative square of spectra.

Theorem 4. *Let* $\mathbb{NS}_{m+q} = \mathbb{NS}_{m+q}(X,Y), \mathbb{S}_{m+q}(\xi) = \mathbb{S}_{m+q}(X,Y,\xi)$. *Then the braid of exact sequences defined by the pushout square (23) is*

$$\begin{array}{ccccccccccc} \to & L_{m+1}(\pi_1(Y)) & & \longrightarrow & & L_{m+q}(\pi_1(X\setminus Y)) & & \to & & \mathbb{S}_{m+q}(\xi) & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\ & & & \mathbb{NS}_{m+q} & & & & LP_m(F) & & & & . \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\ \to & \mathbb{S}_{m+q+1}(\xi) & & \longrightarrow & & H_{m+q}(X;\mathbf{L}) & & \longrightarrow & & L_m(\pi_1(Y)) & \to \end{array} \tag{24}$$

□

The biinfinite diagram (20) contains the homotopy commutative subdiagram of spectra

$$\begin{array}{ccccc} \Omega^{q+1}\mathbf{L}(\pi_1(X)) & \longrightarrow & \Omega^q\mathbf{S}(X) & \longrightarrow & \Omega^q(X_+\wedge\mathbf{L}_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^{q+1}\mathbf{L}(\pi_1(X\setminus Y)\to\pi_1(X)) & \longrightarrow & \mathbf{LS}(F) & \longrightarrow & \mathbf{L}(\pi_1(Y)). \end{array}$$

Since each row in this diagram is a cofibration sequence, it follows that the right hand vertical map $\Omega^q(X_+\wedge\mathbf{L}_\bullet)\to\mathbf{L}(\pi_1(Y))$ in this diagram factors up to homotopy through the homotopy cofiber of the map $\Omega^{q+1}\mathbf{L}(\pi_1(X))\to\mathbf{LS}(F)$. But by Lemma 2 and the biinfinite diagram (20),

$$\begin{gathered} homotopy\ cofiber\left[\Omega^{q+1}(\mathbf{L}(\pi_1(X))\to\mathbf{LS}(F)\right]\sim \\ \sim homotopy\ cofiber\,[\Omega^q(\mathbf{S}(X,Y,\xi)\to\Omega^q(X_+\wedge\mathbf{L}_\bullet]\sim \\ \sim homotopy\ cofiber\,[\Omega(\mathbf{L}(\pi_1(Y))\to\Omega^q\mathbf{L}(\pi_1(X\setminus Y))] = \\ \text{(by definition)} = \mathbf{LP}(F). \end{gathered}$$

Thus we obtain a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \Omega^q(X_+\wedge\mathbf{L}_\bullet) & & \\ \downarrow & \searrow & \\ \mathbf{LP}(F) & \to & \mathbf{L}(\pi_1(Y)). \end{array}$$

However the definition of the map $\Omega^q(X_+\wedge\mathbf{L}_\bullet)\to\mathbf{L}(\pi_1(Y))$ is given under (16) in the proof of Proposition 2 and is the composition of 4 maps

$$\Omega^q(X_+\wedge\mathbf{L}_\bullet)\to\Omega^q\mathcal{C}of\mathbf{j}\xrightarrow{\sim}Y_+\wedge\mathbf{L}_\bullet\to\mathbf{L}(\pi_1(Y)).$$

Thus we can extend the diagram above to a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \Omega^q(X_+ \wedge \mathbf{L}_\bullet) & \to & Y_+ \wedge \mathbf{L}_\bullet \\ \downarrow & & \downarrow \\ \mathbf{LP}(F) & \to & \mathbf{L}(\pi_1(Y)). \end{array} \tag{25}$$

As in the proof of Theorem 3, we can associate to (25) (rather than [11]) a biinfinite homotopy commutative diagram and apply π_0 to this diagram to get the biinfinite commutative diagram of groups

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_m(X \setminus Y; \mathbf{L}_\bullet) & \longrightarrow & H_m(X; \mathbf{L}_\bullet) & \longrightarrow & H_{m-q}(Y; \mathbf{L}_\bullet) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & L_{m-q+1}(\pi_1(X \setminus Y)) & \longrightarrow & LP_{m-q}(F) & \longrightarrow & L_{m-q}(\pi_1(Y)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{S}_m(X \setminus Y) & \longrightarrow & \mathbb{S}_m(X, Y, \xi) & \longrightarrow & \mathbb{S}_{m-q}(Y) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots, & & \end{array} \tag{26}$$

whose rows and columns are exact. This is the diagram of [11, Proposition 7.2.6] and the homotopy commutative square (25) serves to realize the biinfinite commutative diagram (26) on the level of spectra.

Theorem 5. *As in the proof of Theorem 4, we can associate to the homotopy square (25) (rather than* [11]*) a biinfinite homotopy commutative diagram whose rows and columns are cofibration sequences and derive from the diagram a homotopy pushout square which in the current situation is*

$$\begin{array}{ccc} Y_+ \wedge \mathbf{L}_\bullet & \to & \Omega^{q-1}((X \setminus Y)_+ \wedge \mathbf{L}_\bullet) \\ \downarrow & & \downarrow \\ \mathbf{L}(\pi_1(Y)) & \to & \Omega^{q-1}\mathbf{SN}(X, Y). \end{array} \tag{27}$$

The braid of exact sequences defined by (27) is

$$\begin{array}{ccccccccccccc} \to & \mathbb{S}_{m-q+1}(Y) & & \longrightarrow & & H_{m-1}(X \setminus Y; \mathbf{L}_\bullet) & & \to & & H_{m-1}(X; \mathbf{L}) & & \to \\ & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & H_{m-q}(Y; \mathbf{L}_\bullet) & & & & \mathbb{NS}_{m-1}(X, Y) & & & & \\ & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \to & H_m(X; \mathbf{L}_\bullet) & & \longrightarrow & & L_{m-q}(\pi_1(Y)) & & \longrightarrow & & \mathbb{S}_{m-q}(Y) & & \to . \end{array}$$

□

Theorem 6. *As in the proof of Theorem 4, we can associate to the homotopy square (25) (rather than* [11]*) a biinfinite homotopy commutative diagram whose rows and columns are cofibration sequences and derive from the diagram a homotopy pushout square which in the current situation is*

$$\begin{array}{ccc} \mathbf{S}(X\setminus Y) & \to & \mathbf{S}(X,Y,\xi) \\ \downarrow & & \downarrow \\ Z\wedge \mathbf{L}_\bullet & \to & \mathbf{NS}(X,Y). \end{array} \tag{28}$$

The braid of exact sequences defined by (28) is

$$\begin{array}{ccccccccccc} \to L_m(\pi_1(X\setminus Y)) & & \to & & \mathbb{S}_m(X,Y,\xi) & & \to & & \mathbb{S}_{m-q}(Y) & & \to \\ \nearrow & & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & \mathbb{S}_m(X\setminus Y) & & & \mathbb{NS}_{m-1}(X,Y) & & & & \\ \searrow & & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \to \mathbb{S}_{m-q-1}(Y) & & \to & & H_{m-1}(X\setminus Y;\mathbf{L}_\bullet) & & \to & & L_{m-1}(\pi_1(X\setminus Y)) & & \to . \end{array}$$

□

Let $Y\subset X\subset W$ be a triple of closed topological manifolds such that n is the dimension of X, q is the codimension of Y in X, and q' is the codimension of X in W. We shall suppose that every submanifold is locally flat in the ambient manifold and satisfies the conditions on a manifold pair given in [12,page 570]. In this case we can describe relations between the groups $\mathbb{NS}_*(W,Y)$, $\mathbb{NS}_*(W,Y)$, and the sets of normal invariants of the manifolds X, W, and $W\setminus X$.

Theorem 7. *There exists a braid of exact sequences*

$$\begin{array}{ccccccccccc} \to L_{n-q-1}(\pi_1(Y)) & & \longrightarrow & & \mathbb{NS}_n(X,Y) & & \to & & H_{m-1}(W\setminus X;\mathbf{L}_\bullet) & & \to \\ \nearrow & & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & & \mathbb{NS}_m(W,Y) & & & H_n(X;\mathbf{L}_\bullet) & & & & \\ \searrow & & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \\ \to H_m(W\setminus X;\mathbf{L}_\bullet) & & \longrightarrow & & H_m(W;\mathbf{L}_\bullet) & & \longrightarrow & & L_{n-q}(\pi_1(Y)) & & \to \end{array} \tag{29}$$

where $m=n+q'$ is the dimension of the manifold W.

Proof. Consider the homotopy commutative square of spectra

$$\begin{array}{ccc} \Omega^{q'+q}(W_+\wedge \mathbf{L}_\bullet) & \to & \mathbf{L}(\pi_1(Y)) \\ \downarrow & & \downarrow= \\ \Omega^{q}(X_+\wedge \mathbf{L}_\bullet) & \to & \mathbf{L}(\pi_1(Y)), \end{array} \tag{30}$$

where the left vertical map is a realization on the spectra level of the composition of the map

$$H_m(W;\mathbf{L}_\bullet)\to H_m(W,W\setminus X;\mathbf{L}_\bullet)$$

in the homology long exact sequence of the pair $(W,W\setminus X)$ and the isomorphism

$$H_m(W,W\setminus X;\mathbf{L}_\bullet)\to H_{m-q'}(X;\mathbf{L}_\bullet).$$

This is similar to the situation in Proposition 2. The vertical maps in square (30) induce a map between the homotopy fibers of the horizontal maps [13] and we obtain a homotopy commutative square

$$\begin{array}{ccc} \Omega^{q+q'}\mathbf{NS}(W,Y) & \to & \Omega^{q+q'}(W_+\wedge \mathbf{L}_\bullet) \\ \downarrow & & \downarrow \\ \Omega^{q}\mathbf{NS}(X,Y) & \to & \Omega^{q}(X_+\wedge \mathbf{L}_\bullet). \end{array} \tag{31}$$

It is a pushout, since the cofibers of the horizontal maps are naturally homotopy equivalent to the spectra $\mathbf{L}(\pi_1(Y))$. The braid of exact sequences defined by (31) is given in diagram (29). □

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