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Stability for Quadratic K_1

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dedicated to Hyman Bass on his 70'th birthday

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Abstract

The general quadratic group GQ_{2n} and its elementary subgroup EQ_{2n} are analogs in the theory of quadratic forms of the general linear group GL_n and its elementary subgroup E_n. This article proves that the

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stabilization map $GQ_{2n}/EQ_{2n} \longrightarrow GQ_{2(n+1)}/EQ_{2(n+1)}$ is an isomorphism whenever $n \geq \Lambda S + 1$ and ΛS denotes the Λ-stable rank of rings with antiinvolution. As a corollary, a result is obtained which has been anticipated since the late 1960's: Over rings of finite Bass-Serre dimension d, the stabilization map is an isomorphism whenever $n \geq d + 2$.

1 Introduction

Let R denote an associative ring with identity and antiinvolution. Let Λ denote a form parameter on R defined with respect to the antiinvolution. Let $GQ_{2n}(R,\Lambda)$ denote the general quadratic group of rank n over the form ring (R,Λ) and let $EQ_{2n}(R,\Lambda)$ denote its elementary subgroup. The role of the group GQ_{2n} and its elementary subgroup EQ_{2n} in the theory of nonsingular quadratic modules is analogous to that of the general linear group GL_n and its elementary subgroup E_n in the theory of finitely generated projective modules.

Let $KQ_{1,n}(R,\Lambda) = GQ_{2n}(R,\Lambda)/EQ_{2n}(R,\Lambda)$. There is a natural embedding $GQ_{2n}(R,\Lambda) \to GQ_{2(n+1)}(R,\Lambda)$ of groups which induces a map $KQ_{1,n}$ $(R,\Lambda) \to KQ_{1,n+1}(R,\Lambda)$ of coset spaces called the stabilization map. Our main result is the following.

Theorem 1.1 Suppose R satisfies the Λ-stable range condition ΛS_m where $m \geq 2$. If $n > m$ then

$$KQ_{1,n}(R,\Lambda)$$

is a group, the stabilization map

$$KQ_{1,n-1}(R,\Lambda) \to KQ_{1,n}(R,\Lambda)$$

is surjective, and the stabliziation map

$$KQ_{1,n}(R\Lambda) \to KQ_{1,n+1}(R,\Lambda)$$

is an isomorphism of groups.

Stability results of the kind above have been available in the literature for a long time, beginning with the papers [B1] and [B2] for surjective stability and the papers [V1] and [V2] for surjective and injective stability. These papers and later publications employed notions of stable range and dimension which are stronger than Λ-stable range. Their precise relation to Λ-stable range is given in the theorem below.

Several of the notions of stable range and dimension are defined for any associative ring with identity. They are Bass-Serre dimension BS, Krull dimension $K\dim$ of Rentschler and Gabriel, max Krull dimension $K\max$ of Stafford, dim(max spec) = length of longest chain of Jacobson prime ideals, and absolute stable range AS_m of level m. For an associative ring R with identity, antiinvolution, and form parameter Λ, let ΛUS_m denote the unitary stable range condition of level m on (R, Λ).

For any associative ring R with identity, let $\mathfrak{J}(0)$ denote its Jacobson radical.

Theorem 1.2 Suppose R is a ring with antiinvolution and form parameter Λ. Then R satisfies ΛS_m, whenever one of the following conditions is satisfied.

(1.2.1) R satisfies AS_m.
(1.2.2) R satisfies ΛUS_m.
(1.2.3) $m = 1 + BS(R)$.
(1.2.4) R is right Noetherian and $m = 1 + K\dim(R/\mathfrak{J}(0))$.
(1.2.5) R is a PI ring, either R is finitely generated as an algebra over a Noetherian subring in center (R) or R is module finite over a J-Noetherian subring in center (R), and $m = 1 + \dim(\max\operatorname{spec}(R))$.
(1.2.6) R is strongly right J-Noetherian and $m = 1 + K\max(R)$.

In view of the theorems above, one might expect that existing surjective (resp. injective) stability results in the literature begin at m (resp. $m+1$). However, this is true only for surjective stability. Injective stability results in the literature begin at $m+2$. It follows that Theorem 1.1 provides not just a qualitative, but also a quantitative advance for stability results. Its corollary that the stabilization map is an isomorphism whenever $n \geq 2 + BS(R)$ has been sought since the late 1960's and is a direct analog for GQ of Bass' original stability theorem for GL. Moreover Theorem 1.1 as well as its analog for Hermitian forms in [BT] shows that Λ-stable range is the right analog for rings with antiinvolution of Bass' notion of stable range for rings.

The literature records several times dissatisfaction with the proof of injective stability, cf. [HO, p. 521] and [K, p. vi]. Moreover accepted proofs such as M. Saliani's in [K, VI (4.7.1)] are still long and complicated. In the current paper, we provide a one and a half page proof of injective stability, which is self contained modulo a basic elementary knowledge of quadratic modules over form rings, such as presented in the first 3 sections of chapter V of the standard text [HO] of Hahn and O'Meara or in sections 1-5 of chapter I and section 4 of chapter VI of the textbook [K] of Knus. (Either of these texts can be supplemented by sections 1-6 in chapter I and sections 1-5 in chapter II of [Bs 2], sections 1 and 2 in [BV], and sections 1B, 2 and 3 in [B3].) Although the presentations in these sources assume that the symmetry $\lambda \in$ center (R), it is a minor detail dropping this assumption, or the reader can simply assume in the current paper that $\lambda \in$ center (R).

The rest of the paper is organized as follows. In §2, we recall the notions of antiinvolution, form parameter, general quadratic group, and elementary quadratic group, and prove a couple elementary facts concerning the last 2 notions. In §3, we recall the notions of stable rank and dimension used above and prove Theorem 1.2. In §4, we provide an elementary proof of Theorem 1.1, using just the material in §2.

2 Preliminaries on GQ and EQ

In this section, we recall the definition of the general quadratic group GQ_{2n} and its elementary subgroup EQ_{2n} and prove a couple easy lemmas.

Let R denote an associative ring with identity. Recall that an **antihomomorphism** $h : R \to R$ is an additive map such that $h(ab) = h(b)h(a)$ for all $a, b \in R$. An **antiinvolution** $- : R \to R$ is an antiisomorphism for which there is an element $\lambda \in R$ with the property that $\lambda a \bar{\lambda} = \bar{\bar{a}}$ for all $a \in R$. Setting $a = 1$, we obtain $\lambda\bar{\lambda} = 1$. We claim that $\bar{\lambda}\lambda = 1$, from which it will follow that λ is invertible in R and $\lambda^{-1} = \bar{\lambda}$. Clearly $\bar{\bar{\bar{\lambda}}} = \lambda\lambda\bar{\lambda} = \lambda(\lambda\bar{\lambda}) = \lambda$. Thus $1 = (\lambda\bar{\lambda})(\lambda\bar{\lambda}) = \lambda(\bar{\lambda}\lambda)\bar{\lambda} = \overline{\overline{\bar{\lambda}\lambda}} = \bar{\bar{\bar{\lambda}}}\bar{\bar{\lambda}} = \bar{\lambda}\lambda$. Note that this argument doesn't work, if in the definition above the condition $\lambda a \bar{\lambda} = \bar{\bar{a}}$ is replaced by the condition $a = \bar{\lambda}\bar{\bar{a}}\lambda$. In the latter case, we would have to insist additionally that λ is invertible in R. The element λ is called a **symmetry** of the antiinvolution - and is obviously unique up to an element $c \in$ center (R) such that $c\bar{c} = 1$. An antiinvolution with symmetry λ will be called a **λ-involution**.

Let $- : R \to R$ denote a λ-involution on R. Let $\max^{\lambda}(R) = \{a \in R \mid a = -\bar{a}\lambda\}$ and $\min^{\lambda}(R) = \{a - \bar{a}\lambda \mid a \in R\}$. On checks straightforward that $\max^{\lambda}(R)$ and $\min^{\lambda}(R)$ are closed under the operation $a \mapsto \bar{x}ax$ for any $x \in R$. A **λ- form parameter** on R is an additive subgroup Λ of R such that

$$\min^{\lambda}(R) \subseteq \Lambda \subseteq \max^{\lambda}(R),$$

$$\bar{x}\Lambda x \subseteq \Lambda \text{ for all } x \in R.$$

Let M denote a right R-module. A **sesquilinear form** on M is a biadditive map $B : M \times M \to R$ such that $B(va, wb) = \bar{a}B(v, w)b$ for all

$v, w \in M$ and $a, b \in R$. To any sesquilinear form B on M, we associate a **λ-Hermitian form**

$$< \;,\; >_B: M \times M \to R$$

defined by

$$< v, w >_B = B(v, w) + \overline{B(w, v)}\lambda$$

and a **Λ- quadratic form**

$$q_B : M \to R/\Lambda$$

defined by

$$q_B(v) = [B(v, v)].$$

The triple $(M, < \;,\; >_B, q_B)$ is called a **Λ- quadratic module**. It is **nonsingular** if M is finitely generated and projective over R and the map $M \to \mathrm{Hom}_R(M, R), m \mapsto \langle m, \;\rangle_B$, is bijective. A morphism of Λ-quadratic modules is an R-linear map of right R-modules which preserves the associated λ-Hermitian and Λ-quadratic forms.

Let M denote a free right R-module with ordered basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$. Let φ denote the unique sesquilinear form on M such that the $2n \times 2n$-matrix

$$(\varphi(e_i, e_j)) = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

One checks straightforward that the Λ-quadratic module $(M, < \;,\; >_\varphi, q_\varphi)$ is nonsingular. Using the basis of M, one can identify Aut $(M, < \;,\; >_\varphi, q_\varphi)$ with a subgroup of the general linear group $GL_{2n}(R)$. This subgroup is denoted by

$$GQ_{2n}(R, \Lambda)$$

and is called the **general quadratic group** of rank n. Its role in the theory of nonsingular quadratic forms is analogous to that of the general linear group in the theory of finitely generated projective modules. The following lemma provides a matrix characterization of the elements of $GQ_{2n}(R, \Lambda)$ and can be used as a definition of $GQ_{2n}(R, \Lambda)$.

If $\alpha = (a_{ij})$ denotes an $m \times m$ matrix with coefficients $a_{ij} \in R$, let $\bar{\alpha} = (a'_{ij})$ where $a'_{ij} = \bar{a}_{ji}$. It is easy to see that the rule $\alpha \mapsto \bar{\alpha}$ defines a λ-involution on the ring $\mathbb{M}_m(R)$ of all $m \times m$ matrices with coefficients in R.

Lemma 2.1 [B3, (3.1) and (3.4)] An invertible $2n \times 2n$ matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}(R)$ is an element of $GQ_{2n}(R, \Lambda) \Leftrightarrow$ the following conditions hold.

$$(2.1.1) \qquad \overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{pmatrix} & \lambda I \\ I & \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} & \lambda I \\ I & \end{pmatrix}.$$

(2.1.2) The diagonal coefficients of $\bar{\gamma}\alpha$ and $\bar{\delta}\beta$ lie in Λ.
Moreover condition (2.1.1) is equivalent to the following condition.

$$(2.1.1)' \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} & \bar{\beta}\lambda \\ \bar{\lambda}\bar{\gamma} & \bar{\lambda}\bar{\alpha}\lambda \end{pmatrix}.$$

There is an obvious embedding

$$\begin{array}{ccc} GQ_{2n}(R, \Lambda) & \longrightarrow & GQ_{2(n+1)}(R, \Lambda) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \longmapsto & \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

and using this map, we shall frequently want to consider $GQ_{2n}(R, \Lambda)$ as a subgroup of $GQ_{2(n+1)}(R, \Lambda)$.

We recall next the **elementary quadratic matrices** $H\epsilon_{ij}(a), r_{ij}(a)$ and $l_{ij}(a)$ in $GQ_{2n}(R, \Lambda)$. Let

$$H\epsilon_{ij}(a) \qquad (a \in R \text{ and } 1 \leq i, j \leq n, i \neq j)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the (i, j)'th position, $-\bar{a}$ in the $(n+j, n+i)$'th position, and 0 elsewhere. Let

$$r_{ij}(a) \qquad (a \in R \text{ and } 1 \leq i, j \leq n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(i, n+j)$'th position, $-\bar{a}\lambda$ in the $(j, n+i)$'th position, and 0 elsewhere. If $i = j$, it is required additionally that $a \in \Lambda$. Let

$$l_{ij}(a) \qquad (a \in R \text{ and } 1 \leq i, j \leq n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(n+i, j)$'th position, $-\bar{\lambda}\bar{a}$ in the $(n+j, i)$'th position, and 0 elsewhere. If $i = j$, it is required additionally that $a \in \bar{\Lambda}$. Using Lemma 2.2, one checks straightforward that each matrix above lies in $GQ_{2n}(R, \Lambda)$. The subgroup generated by these matrices is denoted by

$$EQ_{2n}(R, \Lambda)$$

and is called the **elementary quadratic group**. The stabilization map $GQ_{2n}(R, \Lambda) \to GQ_{2(n+1)}(R, \Lambda)$ above induces a stabilization map $EQ_{2n}(R, \Lambda) \to EQ_{2(n+1)}(R, \Lambda)$ which will be used frequently to identify $EQ_{2n}(R, \Lambda)$ with its image in $EQ_{2(n+1)}(R, \Lambda)$.

The following subgroups of $EQ_{2n}(R, \Lambda)$ will be used to establish a decomposition of $EQ_{2n}(R, \Lambda)$ under stable range conditions. Let

$$\begin{aligned}
C_n &= < H\epsilon_{in}(a),\ 1 \leq i < n; l_{in}(a), 1 \leq i \leq n, a \in R > \\
R_n &= < H\epsilon_{nj}(a),\ 1 \leq j < n; r_{nj}(a), 1 \leq j \leq n, a \in R > \\
P_n &= \{\sigma\sigma_1 \mid \sigma \in EQ_{2(n-1)}(R, \Lambda) \text{ and } \sigma_1 \in C_n\} \\
Q_n &= < H\epsilon_{ij}(a),\ 1 \leq i, j \leq n, i \neq j; l_{ij}(a), 1 \leq i, j \leq n, a \in R > .
\end{aligned}$$

Lemma 2.2 Let $n \geq 2$. Suppose $\sigma \in GQ_{2n}(R, \Lambda)$ such that the n'th row and n'th column of σ are identical with the n'th row and n'th column of the $2n \times 2n$ identity matrix, respectively. Then the $2n$'th row and $2n$'th column of σ are identical with the $2n$'th row and $2n$'th column of the $2n \times 2n$ identity matrix, respectively. Consequently $\sigma \in GQ_{2(n-1)}(R, \Lambda)$.

Proof Let

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

By $(2.1.1)'$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \bar{\delta} & \bar{\beta}\lambda \\ \bar{\lambda}\bar{\gamma} & \bar{\lambda}\bar{\alpha}\lambda \end{pmatrix}.$$

Using the equation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \bar{\delta} & \bar{\beta}\lambda \\ \bar{\lambda}\bar{\gamma} & \bar{\lambda}\bar{\alpha}\lambda \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

one deduces straightforward the conclusion of the lemma. □

Lemma 2.3 $GQ_{2(n-1)}(R, \Lambda)$ normalizes $EQ_{2n}(R, \Lambda)$.

Proof By [B3, (3.16)], R_n and C_n generate $EQ_{2n}(R, \Lambda)$. But it is obvious that R_n and C_n are normalized by $GQ_{2(n-1)}(R, \Lambda)$. □

3 Λ-stable range condition

In this section, we recall the Λ-stable range condition and other stable range conditions and notions of dimension used in the article and prove Theorem 1.2.

Let R be an associative ring with identity. A vector $(a_1, \ldots, a_n)$ with coefficients $a_i \in R$ is called **right unimodular** if there are elements $b_1, \ldots, b_n \in R$ such that $a_1b_1 + \cdots + a_nb_n = 1$. The **stable range condition** SR_m of H. Bass in the formulation of L. Vaserstein says that if $(a_1, \ldots, a_{m+1})$ is a unimodular vector then there exist elements $b_1, \ldots, b_m \in R$ such that $(a_1 + a_{m+1}b_1, \ldots, a_m + a_{m+1}b_m)$ is unimodular. It follows easily that $SR_m \Rightarrow SR_n$ for any $n \geq m$. The **stable rank** $SR(R)$ of R is the smallest number m such that SR_m holds.

Let R be an associative ring with identity and λ-involution $a \mapsto \bar{a}$. Let Λ be a λ-form parameter on R. Let $\mathbb{M}_n(\bar{\Lambda})$ denote the set of all $n \times n$ matrices γ such that $\gamma = -\bar{\lambda}\bar{\gamma}$ and the diagonal coefficients of γ lie in $\bar{\Lambda}$. The **Λ-stable range condition** ΛS_m of A. Bak and G. Tang says that R satisfies SR_m and that given any unimodular vector $(a_1, \cdots, a_{m+1}, b_1, \cdots, b_{m+1}) \in R^{2m+2}$ there exists a matrix $\gamma \in \mathbb{M}_{m+1}(\bar{\Lambda})$ such that $(a_1, \cdots, a_{m+1}) + (b_1, \cdots, b_{m+1})\gamma$ is unimodular. By [BT, (3.3)], $\Lambda S_m \Rightarrow \Lambda S_n$ for all $n \geq m$. The **Λ-stable rank** $\Lambda S(R)$ of R is the smallest number m such that ΛS_m holds.

We recall next two other stable range conditions which have been used to prove stability results for quadratic forms and record a lemma stating that each is stronger than ΛS_m. The lemma has the consequence that stability results based on ΛS_m are stronger that those based on the other stable range conditions.

If S is a set of elements of R, let

$$\mathfrak{J}(S) = \left(\bigcap_{\mathfrak{M} \supseteq S} \mathfrak{M} \right) \bigcap R$$

where $\mathfrak{M}$ runs through all maximal right ideals of R. Note that a vector $(a_1, \ldots, a_n)$ is unimodular $\iff$ $\mathfrak{J}\{a_1, \ldots, a_n\} = R$. The **absolute stable range condition** AS_m of M. Stein, W. van der Kallen, B. Magurn, and L. Vaserstein says if $(a_1, \cdots, a_{m+1})$ is a vector then there are elements $x_1, \cdots, x_m \in R$ such that $a_{m+1} \in \mathfrak{J}\{a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m\}$, i.e. $\mathfrak{J}\{a_1, \cdots, a_{m+1}\} = \mathfrak{J}\{a_1 + a_{m+1}x_1, \cdots, a_m + a_{m+1}x_m\}$. It is not difficult to check that $AS_m \Rightarrow AS_n$ for all $n \geq m$. The **absolute stable rank** $AS(R)$ of R is the smallest number m such that AS_m holds.

Let Λ be a form parameter on R. The **Λ-unitary stable range condition** ΛUS_m (cf. [HO, p. 526]) says R satisfies SR_m and given any unimodular vector $(a_1, \ldots, a_m, b_1, \cdots, b_m)$ there exists a vector $(x_1, \ldots, x_m, y_1, \cdots, y_m)$ such that $x_1\bar{y}_1 + \ldots + x_m\bar{y}_m \in \Lambda$ and $a_1x_1 + \ldots + a_mx_m + b_1y_1 + \cdots + b_my_m = 1$. It is not difficult to show that $\Lambda US_m \Rightarrow \Lambda US_n$ for all $n \geq m$. The **Λ-unitary stable rank** $\Lambda US(R)$ is the smallest number m such that ΛUS_m holds.

Lemma 3.1 [BT, (3.4) - (3.5)] The following holds for any ring R with form parameter Λ.
(3.1.1) $AS_m \Rightarrow \Lambda S_m$
(3.1.2) $\Lambda US_m \Rightarrow \Lambda S_m$.

If P denotes a partially ordered set, let $\operatorname{dev}(P)$ denote its deviation, cf. [MR, 6.1.2], and let $\ell(P)$ denote the length of the longest chain in P. Let R denote an associative ring with identity. A right ideal $\mathfrak{A}$ in an associative ring R with identity is called a ***J*-ideal** (J stands for Jacobson), if $\mathfrak{A} = \mathfrak{J}(\mathfrak{A})$. Let $\mathfrak{L}(R)$ (resp. $\mathfrak{L}_J(R)$) denote the lattice of all right ideals (resp. J-ideals) of R. Define the **Krull dimension** (in the sense of Rentschler and Gabriel) $K\dim(R) = \operatorname{dev}\mathfrak{L}(R)$, cf. [MR, 2.2], and the **max Krull dimension** (in the sense of Stafford) $K\max(R) = \operatorname{dev}\mathfrak{L}_J(R)$, cf. [S, §1]. Let $\operatorname{Prime}_J(R)$ denote the partially ordered set of all prime J-ideals of R and define $\dim(\max\operatorname{spec}(R)) = \ell(\operatorname{Prime}_J(R))$, cf. [S, §0]. Let k denote a commutative ring. Let $\max\operatorname{spec}(k)$ denote the set of all maximal ideals of k under the Zariski topology. If Y is a subspace of $\max\operatorname{spec}(k)$, let $Irr(Y)$ denote the partially ordered set of all irreducible closed subsets of Y. Define $bs(k)$ to be the smallest natural number m such that $\max\operatorname{spec}(k)$ is a finite union of Noetherian subspaces Y such that $\ell(Y) \leq m$. If no such natural number exists then set $bs(k) = \infty$. For an associative ring R with identity, define the **Bass-Serre dimension** $BS(R)$ to be the smallest number m such that R is module finite over a commutative subring $k \subseteq$ center (R) and $bs(k) \leq m$.

Proof of Theorem 1.2 (1.2.1) and (1.2.2) are the assertions of Lemma 3.1.

(1.2.3) By [KMV, Theorem 3.7], R satisfies AS_m and so we are finished thanks to (1.2.1).
(1.2.4) (resp. (1.2.5), (1.2.6)) By [S, Theorem A (i)] (resp. [S, Theorem A (ii)], [S, Theorem B]), R satisfies AS_m and so we are finished again by (1.2.1). □

4 Proof of Theorem 1.1

Throughout this section, R is an associative ring with identity and λ-involution $a \mapsto \bar{a}$ and Λ is a λ-form parameter on R. It will be assumed throughout that

$$R \text{ satisfies the } \Lambda\text{-stable range condition } \Lambda S_m.$$

This implies by definition that R satisfies Bass' stable range condition SR_m.

Recall the subgroup Q_n of $EQ_{2n}(R, \Lambda)$, which was defined in §2.

Lemma 4.1 Let $n \geq m+1$. Then for any $\sigma \in GQ_{2n}(R, \Lambda)$, there is an element $\tau \in Q_n$ such that $\sigma\tau$ has 0 in the first $(n-1)$ entries of its n'th row and 1 in the n'th entry of this row.

Proof Let

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the 2×2 block matrix description of σ provided in Lemma 2.2. Clearly (α, β) is a unimodular vector in $(\mathbb{M}_n(R))^2$. Let v denote the bottom row of α and w the bottom row of β. Then (v, w) is the bottom row of (α, β) and hence is unimodular in R^{2n}.

Since R satisfies ΛS_m and $n \geq m+1$, there exists a matrix $\gamma \in \mathbb{M}_n(\overline{\Lambda})$ such that $v + w\gamma$ is unimodular in R^n. Set

$$\tau_1 = \begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix} = \prod_{1 \leq i \leq j \leq n} l_{ij}(a_{ij}) \in Q_n$$

where a_{ij} is the (i, j)'th coefficient of γ. Since R satisfies SR_m and $n \geq m+1$, there is by [Bs1, Theorem 5.3.3] a product ϵ of elementary $n \times n$ matrices such that $(v + w\gamma)\epsilon = (0, \cdots, 0, 1)$. Set

$$\tau_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \bar{\epsilon}^{-1} \end{pmatrix} \in Q_n.$$

Then $\sigma\tau_1\tau_2$ has n'th row $((0, \cdots, 0, 1), w)$. □

Recall the subgroups P_n, Q_n, and R_n of $EQ_{2n}(R, \Lambda)$, which are defined in §2.

Definition 4.2 Let $\phi \in EQ_{2n}(R, \Lambda)$. A PRQ**-decomposition** of ϕ is a product decomposition $\phi = \sigma\alpha\tau$ where $\sigma \in P_n, \alpha \in R_n$, and $\tau \in Q_n$.

Decomposition Theorem 4.3 Let $n \geq m+2$. Then every element of $EQ_{2n}(R,\Lambda)$ has a PRQ-decomposition, i.e. $EQ_{2n}(R,\Lambda) = P_nR_nQ_n$.

Proof Let $\phi \in EQ_{2n}(R,\Lambda)$. A PRQ-decomposition $\sigma\alpha\tau$ of ϕ will be called **reduced** if the $(n-1,n)$'th coefficient of σ is 0. The strategy of the proof is as follows. First we show that if ϕ has a PRQ-decomposition then it has a reduced one. Then we identify a set of generators θ of $EQ_{2n}(R,\Lambda)$ and show using reduced PRQ-decompositions that $\theta P_nR_nQ_n \subseteq P_nR_nQ_n$. It follows trivially that $EQ_{2n}(R,\Lambda) = P_nR_nQ_n$.

Let $\sigma\alpha\tau$ be a PRQ-decomposition of ϕ. Write

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 \\ 0 & 1 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & 1 \end{pmatrix}$$

and set

$$\sigma_1 = \begin{pmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{pmatrix}.$$

By definition, $\sigma_1 \in EQ_{2(n-1)}(R,\Lambda)$. Since $n \geq m+2$, it follows from (4.1) that there is a $\tau_1 \in Q_{n-1}$ such that the $(n-1,n-1)$'th coefficient of $\sigma_1\tau_1$ is 1. It is obvious that if τ_1 is identified with its image in $EQ_{2n}(R,\Lambda)$ (under the stabilization map $EQ_{2(n-1)}(R,\Lambda) \longrightarrow EQ_{2n}(R,\Lambda)$) then $\tau_1 \in Q_n$ and the $(n-1,n-1)$'th coefficient of $\sigma\tau_1$ is 1. Furthermore $\tau_1 \in P_n \cap Q_n$ and τ_1 normalizes R_n. Thus $(\sigma\tau_1)(\tau_1^{-1}\alpha\tau_1)(\tau_1^{-1}\tau)$ is a PRQ-decomposition of ϕ such that the $(n-1,n-1)$'th coefficient of $\sigma\tau_1$ is 1. Choose $x \in R$ such that the $(n-1,n)$'th coefficient of $\sigma\tau_1 H\epsilon_{n-1,n}(x)$ is 0. Choose $y \in R$ such that the $(n,n-1)$'th coefficient of $\tau_1^{-1}\alpha\tau_1 H\epsilon_{n,n-1}(y)$ is 0. Let $\tau_2 = H\epsilon_{n-1,n}(x)$ and $\tau_3 = H\epsilon_{n,n-1}(y)$. Then $\tau_2^{-1}(\tau_1^{-1}\alpha\tau_1\tau_3)\tau_2 = \sigma_2\alpha_1$ for some $\sigma_2 \in EQ_{2(n-1)}(R,\Lambda) \subseteq P_n$ and some $\alpha_1 \in R_n$. Thus $\phi = \sigma\alpha\tau = (\sigma\tau_1\tau_2)(\tau_2^{-1}(\tau_1^{-1}\alpha\tau_1\tau_3)\tau_2)(\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau) = (\sigma\tau_1\tau_2\sigma_2)\alpha_1(\tau_2^{-1}\tau_3^{-1}\tau_1^{-1}\tau)$ which is a reduced PRQ-decomposition of ϕ.

The relations

$$H\epsilon_{ni}(a) = [H\epsilon_{n,n-1}(a), H\epsilon_{n-1,i}(1)] \qquad (a \in R \text{ and } 1 \leq i \leq n-1),$$
$$r_{nj}(a) = [H\epsilon_{nj}(a), r_{ji}(1)] \qquad (a \in R \text{ and } 1 \leq i \neq j \leq n-1),$$
$$r_{nn}(a)r_{n-1,n}(-a) = [r_{n-1,n-1}(a), H\epsilon_{n,n-1}(1)], \qquad (a \in \Lambda),$$

show that P_n and the matrices $H\epsilon_{n,n-1}(a)$ $(a \in R)$ generate $EQ_{2n}(R,\Lambda)$. Obviously $P_n(P_nR_nQ_n) \subseteq P_nR_nQ_n$. Let $\sigma\alpha\tau$ be a reduced PRQ-decomposition. Since the $(n-1,n)$'th coefficient of σ is 0, σ can be expressed as a product $\sigma = \sigma_3\sigma_4$ where $\sigma_3 \in C_n$ such that the $(n-1,n)$'th coefficient of σ_3

is 0 and $\sigma_4 \in EQ_{2(n-1)}(R,\Lambda)$. A straightforward computation shows that $H\epsilon_{n,n-1}(a)\sigma_3 H\epsilon_{n,n-1}(-a) \in P_n$ and it is clear that $EQ_{2(n-1)}(R,\Lambda)$ normalizes R_n. Thus $H\epsilon_{n,n-1}(a)\sigma\alpha\tau = (H\epsilon_{n,n-1}(a)\sigma_3 H\epsilon_{n,n-1}(-a)\sigma_4)$ $(\sigma_4^{-1}H\epsilon_{n,n-1}(a)\sigma_4\alpha)\tau$ which is a PRQ-decomposition. □

Proof of Theorem (1.1) Let $\sigma \in GQ_{2n}(R,\Lambda)$. By (4.1), there is a $\tau_1 \in Q_n \subseteq EQ_{2n}(R,\Lambda)$ such that $\sigma\tau_1$ has 0 in the first $(n-1)$ entries of its n'th row and 1 in the n'th entry of this row. It follows that there is a matrix $\tau_3 = \prod_{i=1}^{n} \ell_{in}(y_i) \prod_{i=1}^{n-1} H\epsilon_{in}(y_i')$ such that $\tau_3\sigma\tau_1$ has the same n'th row as $\sigma\tau_1\tau_2$ and the same n'th column as the $2n \times 2n$ identity matrix. Now we can find a matrix $\tau_2 = \prod_{i=1}^{n} r_{in}(z_i)$ such that $\tau_3\sigma\tau_1\tau_2$ has the same n'th row and n'th column as $2n \times 2n$ identity matrix. Thus the element $\rho = \tau_3\sigma\tau_1\tau_2 \in GQ_{2(n-1)}(R,\Lambda)$, by (2.2). So ρ normalizes $EQ_{2n}(R,\Lambda)$, by (2.3). Since $\sigma = \tau_3^{-1}\tau_4^{-1}\rho\tau_2^{-1}\tau_1^{-1}$, it follows that σ normalizes $EQ_{2n}(R,\Lambda)$. Thus $KQ_{1,n}(R,\Lambda)$ is a group and the map $KQ_{1,n-1}(R,\Lambda) \longrightarrow KQ_{1,n}(R,\Lambda)$ is surjective.

Let $\phi \in GQ_{2n}(R,\Lambda) \cap EQ_{2(n+1)}(R,\Lambda)$. Let $\sigma\alpha\tau$ be a $P_{(n+1)}R_{(n+1)}Q_{(n+1)}$-decomposition of ϕ. Since the $(n+1)$'th row of σ coincides with that of the $2(n+1) \times 2(n+1)$ identity matrix, it follows that the $(n+1)$'th row of $\sigma\alpha\tau$ coincides with the $(n+1)$'th row of $\alpha\tau$. Thus the $(n+1)$'th row of $\alpha\tau$ coincides with that of the $2(n+1) \times 2(n+1)$ identity matrix. Write

$$\tau = \begin{pmatrix} \epsilon & 0 \\ \gamma & \bar{\epsilon}^{-1} \end{pmatrix}.$$

If (v,w) denotes the $(n+1)$'th row of α then the $(n+1)$'th row of $\alpha\tau$ is

$$(v,w)\begin{pmatrix} \epsilon & 0 \\ \gamma & \bar{\epsilon}^{-1} \end{pmatrix} = (v\epsilon + w\gamma, w\bar{\epsilon}^{-1}).$$

Thus $w\bar{\epsilon}^{-1} = 0$. Since $\bar{\epsilon}^{-1}$ is invertible, $w = 0$. Thus $\alpha \in Q_{n+1}$. Write $\sigma = \sigma_1\tau_1$ where $\sigma_1 \in EQ_{2n}(R,\Lambda)$ and $\tau_1 \in C_{(n+1)} \subseteq Q_{(n+1)}$. Obviously $\phi = \sigma_1(\tau_1\alpha\tau)$ and $\tau_1\alpha\tau \in Q_{(n+1)} \cap GQ_{2n}(R,\Lambda)$. It suffices to show that $\tau_1\alpha\tau \in EQ_{2n}(R,\Lambda)$. In fact, we shall show that $\tau_1\alpha\tau \in Q_n$.

Write

$$\tau_1\alpha\tau = \begin{pmatrix} \epsilon_1 & 0 \\ \gamma_1 & \bar{\epsilon}_1^{-1} \end{pmatrix}.$$

Since $\tau_1\alpha\tau \in GQ_{2n}(R,\Lambda)$, it follows that the last row and column of γ_1 are zero and $\epsilon_1 \in GL_n(R)$. From the definition of Q_{n+1}, ϵ_1 is an $(n+1)\times(n+1)$ matrix in the elementary group $E_{n+1}(R)$. Thus $\epsilon_1 \in GL_n(R) \cap E_{n+1}(R)$.
To complete the proof, it suffices to show that $\epsilon_1 \in E_n(R)$. Since R satisfies SR_m (because it satisfies ΛS_m) and $n \geq m+1$, it follows by stability for K_1

of the general linear group [Bs1, Theorem 5.4.2] that $GL_n(R) \cap E_{n+1}(R) = E_n(R)$. $\square$

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