EQUIVARIANT INTERSECTION THEORY AND SURGERY THEORY FOR MANIFOLDS WITH MIDDLE DIMENSIONAL SINGULAR SETS

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In memory of Yu Solovyov

Abstract. Let G denote a finite group and $n = 2k \geqq 6$ an even integer. Let X denote a simply connected, compact, oriented, smooth G-manifold of dimension n. Let L denote a union of connected, compact, neat submanifolds in X of dimension $\leqq k$. We invoke the hypothesis that L is a G-subcomplex of a G-equivariant smooth triangulation of X and contains the singular set of the action of G on X. If the dimension of the G-singular set is also k then the ordinary equivariant self-intersection form is not well defined, although the equivariant intersection form is well defined. The first goal of the paper is to eliminate the deficiency above by constructing a new, well defined, equivariant, self-intersection form, called the generalized (or doubly parametrized) equivariant self-intersection form. Its value at a given element agrees with that of the ordinary equivariant self-intersection form when the latter value is well defined. Let $\mathcal{F}$ denote a finite family of immersions with trivial normal bundle of k-dimensional, connected, closed, orientable, smooth manifolds into X. Assume that the integral (and mod 2) intersection forms applied to members of $\mathcal{F}$ and to orientable (and nonorientable) k-dimensional members of L are trivial. Then the vanishing of the equivariant intersection form on $\mathcal{F} \times \mathcal{F}$ and the generalized equivariant self-intersection form on $\mathcal{F}$ is a necessary and sufficient condition that $\mathcal{F}$ is regularly homotopic to a family of disjoint embeddings, each of which is disjoint from L. This property, when $\mathcal{F}$ is a finite family of immersions of the k-dimensional sphere S^k into X, is just what is needed for constructing an equivariant surgery theory for G-manifolds X as above whose G-singular set has dimension less than or equal to k. What is new for surgery theory is that the equivariant surgery obstruction is defined for an almost arbitrary singular set of dimension $\leqq k$ and in particular, the k-dimensional components of the singular set can be nonorientable.

1. Introduction

Equivariant surgery solves the following problem. Let G denote a finite group and $n = 2k \geqq 6$. Let $f : X \to Y$ denote a k-connected, degree-one, G-framed map of one-connected, compact, n-dimensional, smooth G-manifolds such that the fixed point set of X satisfies the following assumption.

(A1) Any connected component of a fixed point set X^g, where $g \in G \smallsetminus \{1\}$, has dimension $\leqq k$ and no connected component of dimension $(k-1)$ of a fixed point

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set X^H, $H \subseteqq G$, is totally contained in a connected component of dimension k of some fixed point set X^K, $K \subseteqq G$.

Assume further

(A2) $\partial f\,(:= f|_{\partial X}) : \partial X \to \partial Y$ is a homology equivalence, and

(A3) $f^H\,(:= f|_{X^H}) : X^H \to Y^H$ is a homology equivalence for each nontrivial hyperelementary subgroup $H \subseteqq G$.

Since f is G-framed, it comes equipped with a G-vector bundle isomorphism $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$, where $T(X)$ denotes the tangent bundle over X with its natural G-action and η and ξ are G-vector bundles over Y. Let $\varepsilon_Y(\mathbb{R}^n) = (Y \times \mathbb{R}^n \to Y)$ denote the product G-vector bundle on Y where G acts trivially on $\mathbb{R}^n$. After replacing η by $\eta \oplus \varepsilon_Y(\mathbb{R}^n)$, it can be assumed and will be assumed throughout the paper that

(A4) η contains the product bundle $\varepsilon_Y(\mathbb{R}^n)$ as a G-vector subbundle.

The problem of equivariant surgery is finding necessary and sufficient conditions when (f, b) is G-framed cobordant rel $\partial X \cup X_{\mathrm{sing}}$ to a G-framed map (f', b') such that $f' : X' \to Y$ is a homotopy equivalence, where X_{sing} denotes the singular set $\bigcup_{g \in G \smallsetminus \{1\}} X^g$ of X. This problem is solved along the usual lines by constructing a group $W(G, Y, X_{\mathrm{sing}})_{\mathrm{free}}$ called the *surgery obstruction group* and an element $\sigma(f, b)$ in $W(G, Y, X_{\mathrm{sing}})_{\mathrm{free}}$ called the *surgery obstruction* of (f, b), having the property that $\sigma(f, b) = 0$ if and only if (f, b) is G-framed cobordant rel $\partial X \cup X_{\mathrm{sing}}$ to a G-framed map (f', b') such that $f' : X' \to Y$ is a homotopy equivalence. However the construction of the group $W(G, Y, X_{\mathrm{sing}})_{\mathrm{free}}$ and the surgery obstruction $\sigma(f, b)$ is much more intricate than those of previous theories, because the singular set is now much more general.

We summarize next the literature on equivariant surgery and then state our main surgery results.

The best results to date in the literature are in [6] and [30]. They are obtained when Assumption (A1) above is strengthened (considerably) by adding the conditions that each k-dimensional connected component of X^H, $H \subseteqq G$, is orientable, that for each $g \in G$ the translation $X^H \to X^{gGg^{-1}}$, $x \mapsto gx$, is orientation preserving, and that 3 further technical assumptions [6, (2.1.2)–(2.1.4)] are satisfied. The surgery theory in [25] and [26] is obtained when the assumptions above are further strengthened by adding the condition $\dim X^g \leqq k - 1$ for all $g \in G \smallsetminus \{1\}$. Finally the surgery theories of T. Petrie [39], [40], [12], [41] and W. Lück–I. Madsen [19], [20] are obtained when the assumptions above are strengthened again by adding either the condition $\dim X^g \leqq k - 2$ for all $g \in G \smallsetminus \{1\}$ or the condition that G has odd order. Applications of the surgery results above to transformation groups are found in [4], [7], [16], [17], [22], [24], [27], [28], [32], [33], [40], [41] and have motivated all of the work above on equivariant surgery theory. Applications of results of the current paper will be forthcoming.

Our main surgery results are as follows.

Theorem 1.1. *Let X and Y be compact, connected, oriented, smooth G-manifolds of dimension $n = 2k \geqq 6$. Suppose X satisfies (A1) and Y is simply connected. Let $\boldsymbol{f} = (f, b)$ be a degree-one G-framed map satisfying (A2)–(A4), where $f : (X, \partial X) \to (Y, \partial Y)$ and $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$. Then one can perform G-surgery rel $\partial X \cup X_{\text{sing}}$ on $\boldsymbol{f}$ to obtain a G-framed map $\boldsymbol{f}' = (f', b')$ with $f' : (X', \partial X') \to (Y, \partial Y)$ and $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta)$ such that $f' : X' \to Y$ is a homotopy equivalence if and only if the element $\sigma(\boldsymbol{f})$ in $W(G, Y, X_{\text{sing}})_{\text{free}}$ is trivial.*

This follows from Theorem 7.8.

The subscript free on $W(G, Y, X_{\text{sing}})_{\text{free}}$ signifies that the underlying modules used in constructing the Witt group $W(G, Y, X_{\text{sing}})_{\text{free}}$ are free (or stably free) over the integral group ring $\mathbb{Z}[G]$. Under a weaker assumption than (A3), namely

(A3′) $f^P : X^P \to Y^P$ is a $\mathbb{Z}_{(p)}$-homology equivalence for each prime p and any nontrivial p-subgroup P of G,

we get also an equivariant surgery theory, except the modules used in constructing the surgery obstruction group here are only projective over $\mathbb{Z}[G]$. Accordingly, the surgery obstruction group here is denoted by $W(G, Y, X_{\text{sing}})_{\text{proj}}$.

Theorem 1.2. *Let X and Y be as in the previous theorem. Let $\boldsymbol{f} = (f, b)$ denote a degree-one G-framed map satisfying (A2), (A3′) and (A4), where $f : (X, \partial X) \to (Y, \partial Y)$ and $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$. If the element $\sigma(\boldsymbol{f})$ in $W(G, Y, X_{\text{sing}})_{\text{proj}}$ is trivial, then one can perform G-surgery rel $\partial X \cup X_{\text{sing}}$ on $\boldsymbol{f}$ to obtain a G-framed map $\boldsymbol{f}' = (f', b')$ where $f' : (X', \partial X') \to (Y, \partial Y)$ and $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta)$ such that $f' : X' \to Y$ is a homotopy equivalence.*

This follows from Theorem 6.3.

It is worth noting that there is a canonical homomorphism

$$W(G, Y, X_{\text{sing}})_{\text{free}} \to W(G, Y, X_{\text{sing}})_{\text{proj}}$$

so that a part of Theorem 1.1 follows directly from Theorem 1.2. Moreover the homomorphism is injective, according to Lemma 3.34.

We describe how the rest of the paper is organized, while at the same time providing insight how the surgery obstruction groups $W(G, Y, X_{\text{sing}})_\varepsilon$, $\varepsilon =$ free or proj, and the surgery obstruction $\sigma(\boldsymbol{f})$ are constructed.

The key to constructing an equivariant, smooth, surgery theory on $2k$-dimensional ($k \geqq 3$) G-manifolds X is having good equivariant, geometric, intersection tools. This means the following: Given a finite set $\mathcal{F}$ of smooth immersions with trivial normal bundle of 1-connected, k-dimensional, closed, oriented manifolds into X, the tools should provide necessary and sufficient conditions that $\mathcal{F}$ is regularly homotopic to a set of equivariantly disjoint embeddings, each of which is disjoint from the G-singular set X_{sing} of the action of G on X. The usual tools for doing this are the equivariant, geometric, intersection form $\#_G$ and the equivariant, geometric, self-intersection form μ_G. However since X_{sing} can have k-dimensional components, μ_G is not always defined. To get around this problem, we construct in Section 4 a generalized, equivariant, self-intersection form $\natural_G$, which is always defined and agrees with μ_G whenever it is defined. The construction of $\natural_G$ uses 2 parameters instead of just one form parameter as in the case of μ_G. The forms $\#_G$ and $\natural_G$ are related by certain equalities which generalize the relationship between $\#_G$ and μ_G and the pair $(\#_G, \natural_G)$ defines a so-called doubly parametrized form on the surgery kernel $K_k(X) := \mathrm{Ker}[f_* : H_k(X) \to H_k(Y)]$ of a G-framed map $\boldsymbol{f} = (f, b)$ such that $f : X \to Y$ is k-connected. Section 2 develops for reference purposes, several algebraic concepts including that of doubly parametrized forms on modules. Section 5 constructs the surgery module $M_{\boldsymbol{f}}$ of $\boldsymbol{f}$, the surgery obstruction $\sigma(\boldsymbol{f})$ of $\boldsymbol{f}$, and the surgery obstruction group $W(G, Y, X_{\mathrm{sing}})_\varepsilon$, $\varepsilon =$ proj or free. By definition, $\sigma(\boldsymbol{f})$ is the class of $M_{\boldsymbol{f}}$ in $W(G, Y, X_{\mathrm{sing}})_\varepsilon$. To define $M_{\boldsymbol{f}}$, we need to take into account additional structure on $K_k(X)$, arising from X_{sing}. This is done as follows. Let $\widetilde{\Theta}$ (resp. Θ_2) denote the G-set of all connected, k-dimensional, oriented (resp. nonoriented) components of X_{sing}. There are canonically defined a $G \times \{\pm 1\}$-map $\theta : \widetilde{\Theta} \to K_k(X)$ and a G-map $\theta_2 : \Theta_2 \to K_k(X)/2K_k(X)$ called positioning maps, making $K_k(X)$ into a so-called positioning module. Section 2 also develops the algebraic concepts of positioning module and doubly parametrized positioning module. The 5-tuple $M_{\boldsymbol{f}} = (K_k(X), \#_G, \natural_G, \theta, \theta_2)$ is an example of a doubly parametrized positioning module and is by definition the surgery module of $\boldsymbol{f}$. One might at this point that the surgery obstruction group is the Witt group of all nonsingular doubly parametrized positioning modules, but this is not the case. It turns out that a subtle invariant ∇ is playing a role here. This invariant vanishes on all surgery modules $M_{\boldsymbol{f}}$ and the correct definition of the surgery obstruction group is the Witt group of all nonsingular, doubly parametrized positioning modules with trivial ∇-invariant. (This vanishing of ∇ affects the concept of Lagrangian and gives a distinctly different Witt group.) The algebraic concepts underlying ∇ are developed in Section 3, as well as the algebraic constructions of the kinds of Witt groups needed in surgery theory.

Our main surgery result Theorem 1.2 is proved in Section 6. The cobordism invariance of the surgery obstruction and Theorem 1.1 are proved in Section 7.

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2. Doubly parametrized forms on positioning modules

This section introduces the concepts of doubly parametrized form and positioning module and develops their elementary K-theory. Both concepts are motivated by and needed in the surgery theory developed in Sections 5–7.

The concept of a doubly parametrized form is a nonobvious synthesis of the concepts of Λ-quadratic form and Λ-Hermitian form in [2]. Both older concepts are special cases of the new one and the K-theory developed in this section for the new concept generalizes that in Section 2 and Sections 5–6 of [2] for the older ones. The geometry in Sections 5–7 requires that the underlying modules of our doubly parametrized forms contain positioning information. So we shall develop, right from the beginning, our K-theory of doubly parametrized forms, over positioning modules rather than just ordinary modules. We shall see that ordinary modules are the special case when the positioning information is empty. This has the consequence that our results for doubly parametrized forms on positioning modules are also valid for doubly parametrized forms on ordinary modules.

For functorial reasons, it is better to define our forms over associative algebras rather than associative rings, because in practice a ground ring in the center of our ring will play a special role and it will be necessary to restrict ring homomorphisms to those which preserve the ground rings. By definition, an *associative algebra* A will mean an associative ring A with identity together with a fixed commutative ring $R \subseteqq \mathrm{Center}(A)$, called the *ground ring* of A. An *R-algebra* is an associative algebra whose ground ring is R. An *algebra homomorphism* $A \to A'$ is any ring homomorphism $f : A \to A'$ which preserves the identity and has the property that $f(R) \subseteqq R'$ where R and R' are the ground rings of A and A', respectively. Throughout this article, we shall use the word *algebra* to mean associative algebra in the sense above.

An involution on an R-algebra A is a map $A \to A$, $a \mapsto \overline{a}$, such that $\overline{r} \in R$ for all $r \in R$ and such that $\overline{\overline{a}} = a$ and $\overline{ab} = \overline{b}\overline{a}$ for all a, $b \in A$. An *antistructure* in the sense of Wall [44] on an R-algebra A is a pair $(-, \lambda)$ consisting of a map $A \to A, a \mapsto \overline{a}$, and a unit $\lambda \in A$, which we shall call the *symmetry* of the antistructure, such that $\overline{r} \in R$ for all $r \in R$ and such that $\overline{\overline{a}} = \lambda^{-1} a \lambda$ and $\overline{ab} = \overline{b}\overline{a}$ for all $a, b \in A$. A *morphism* $(A, (-, \lambda)) \to (A', (-, \lambda'))$ of algebras with antistructure is an algebra homomorphism $f : A \to A'$ such that $f(\overline{a}) = \overline{f(a)}$ for all $a \in A$ and $f(\lambda) = \lambda'$.

Let A be an R-algebra with antistructure $(-, \lambda)$. The additive subgroups

$$\min_\lambda(A) = \{a - \lambda\overline{a} \mid a \in A\},$$
$$\max_\lambda(A) = \{a \in A \mid a = -\lambda\overline{a}\}$$

are introduced in Section 1 of [2].

Definition 2.1. A $(-,\lambda)$-*form parameter* in the sense of [2, §13, p.255] on the R-algebra A is an additive subgroup $\Lambda \subseteqq A$ such that

(2.1.1) $\min_\lambda(A) \subseteqq \Lambda \subseteqq \max_\lambda(A)$,
(2.1.2) $ax\overline{a} \in \Lambda$ for all $x \in \Lambda$ and $a \in A$, and
(2.1.3) $R_0\Lambda \subseteqq \Lambda$, where $R_0 = \{r \in R \mid r = \overline{r}\}$.

One checks easily that $\min_\lambda(A)$ and $\max_\lambda(A)$ satisfy the closure conditions (2.1.2–3) so that both $\min_\lambda(A)$ and $\max_\lambda(A)$ are form parameters.

Note that if $(-,\lambda)$ is an antistructure on the R-algebra A then $(-,-\lambda)$ is also an antistructure on A.

Definition 2.2. We define a $(-,\lambda)$-*symmetric parameter* on the R algebra A to be a $(-,-\lambda)$-form parameter on A.

The next definition is motivated by geometric considerations in Sections 4–5.

Definition 2.3. A *prepared algebra* is a quadruple $(A,(-,\lambda),G,A_s)$ where

(2.3.1) A is an R-algebra with antistructure $(-,\lambda)$,
(2.3.2) G is a multiplicative subset of A such that A is generated over R by G and for each $g \in G$, there are elements $r \in R$ and $g' \in G$ with the property that $\overline{g} = rg'$, and
(2.3.3) A_s is an R-submodule of A which is closed under the operations $a \mapsto \lambda\overline{a}$ and $a \mapsto ga\overline{g}$ for all $a \in A_s$ and $g \in G$.

A *morphism* $(A,(-,\lambda),G,A_s) \to (A',(-,\lambda'),G',A'_s)$ of prepared algebras is a homomorphism $f : (A,(-,\lambda)) \to (A',(-,\lambda'))$ of algebras with antistructure such that $f(G) \subseteqq G'$ and $f(A_s) \subseteqq A'_s$.

In the geometric situation, each element $a \in A_s$ has the property $a = \lambda\overline{a}$, i.e. is λ-symmetric. This is our justification for introducing the notation A_s.

Let $(A,(-,\lambda),G,A_s)$ be a prepared algebra. Define

$$\max_{\lambda,A_s}(A) = \{a \in A |\ a \equiv -\lambda\overline{a} \mod A_s\}.$$

Since the operations $a \mapsto \lambda\overline{a}$ and $a \mapsto ga\overline{g} (g \in G)$ on A preserve A_s, they induce operations on the quotient module A/A_s. This fact is used in making the following definition.

Definition 2.4. Let $(A,(-,\lambda),G,A_s)$ be a prepared algebra. A *generalized form parameter* or $((-,\lambda),G,A_s)$-*form parameter* on A is an additive subgroup Λ of A such that

(2.4.1) $A_s + \min_\lambda(A) \subseteqq \Lambda \subseteqq \max_{\lambda,A_s}(A)$,
(2.4.2) $gx\overline{g} \in \Lambda$ for all $x \in \Lambda$ and $g \in G$, and
(2.4.3) $R_0\Lambda \subseteqq \Lambda$, where $R_0 = \{r \in R |\ r = \overline{r}\}$.

One checks easily that $A_s + \min_\lambda(A)$ and $\max_{\lambda,A_s}(A)$ satisfy the closure conditions ((2.4.2–3) so that both $A_s + \min_\lambda(A)$ and $\max_{\lambda,A_s}(A)$ are generalized form parameters.

Recall that a *form algebra* [2, §1, B] is a triple $(A,(-,\lambda),\Lambda)$ where $(A,(-,\lambda))$ is an R-algebra with antistructure and Λ is a $(-,\lambda)$-form parameter in the sense of Definition 2.1 on A. The next concept generalizes that of a form algebra.

Definition 2.5. A *double parameter algebra* or simply *parameter algebra* is a 6-tuple $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$ where Γ a $(-,\lambda)$-symmetric parameter, $(A,(-,\lambda),G,A_s)$ is a prepared algebra, and Λ a $((-,\lambda),G,A_s)$-form parameter. A *morphism*

$$(A,(-,\lambda),\Gamma,G,A_s,\Lambda) \longrightarrow (A',(-,\lambda'),\Gamma',G',A'_s,\Lambda')$$

of parameter algebras is a morphism $f : (A,(-,\lambda),G,A_s) \to (A',(-,\lambda'),G',A'_s)$ of prepared algebras such that $f(\Gamma) \subseteqq \Gamma'$ and $f(\Lambda) \subseteqq \Lambda'$.

Let $(A,(-,\lambda))$ be an algebra with antistructure. Let M be a left A-module. Recall that a *sesquilinear form* on M is a biadditive map $B : M \times M \to A$ such that $B(am,bn) = bB(m,n)\overline{a}$ for all $a,\ b \in A$ and $m,\ n \in M$. Let B be a sesquilinear form on M. Define $\overline{B} : M \times M \to A$; $(m,n) \mapsto \overline{B(n,m)}$. Although $\overline{B}$ is not necessarily a sesquilinear form on M, one checks easily

$$\lambda\overline{B} : M \times M \to A; \quad (m,n) \mapsto \lambda(\overline{B}(m,n)),$$

is a sesquilinear form on M. B is called $(-,\lambda)$-*Hermitian* or simply λ-*Hermitian* if $B = \lambda\overline{B}$. Let Γ be a $(-,\lambda)$-symmetric parameter on A. A $(-,\lambda)$-Hermitian form is called Γ-*Hermitian* in the sense of [2, §1, C] or Γ-*symmetric* if $B(m,m) \in \Gamma$ for all $m \in M$. A Γ-Hermitian form B on M is called *nonsingular* if M is finitely generated and projective over A and the map $M \to M^*$; $m \mapsto B(m,-)$ is bijective, where

$$M^* = \mathrm{Hom}_A(M,A).$$

The next definition generalizes [2, p.251, 13.1].

Definition 2.6. Let $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$ be a parameter algebra. Let M be a left A-module. An $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$-*form* or simply (Γ,Λ)-*form* on M is a pair (B,q) where B is a Γ-Hermitian form on M and $q : M \to A/\Lambda$ is a map called a $\mathfrak{q}$-*form*, satisfying the following:

(2.6.1) $q(am) = aq(m)\overline{a}$ for all $m \in M$ and $a \in R \cup G$.
(2.6.2) $q(m+n) - q(m) - q(n) = B(m,n)$ in A/Λ for all $m,\ n \in M$.
(2.6.3) If $\alpha \in A/\Lambda$, let $\widetilde{\alpha}$ denote a lifting of α to A. Then for all $m \in M$,

$$\widetilde{q(m)} + \lambda\overline{\widetilde{q(m)}} \equiv B(m,m) \mod A_s.$$

The triple (M,B,q) is called an $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$-*module* or a (Γ,Λ)-*module* over $(A,(-,\lambda),G,A_s)$. A *doubly parametrized form* is a (Γ,Λ)-form for some pair (Λ,Γ). A *doubly parametrized module* is a (Γ,Λ)-module for some pair (Γ,Λ).

A (Γ,Λ)-module (M,B,q) is called *nonsingular* if the Hermitian module (M,B) is nonsingular. This includes the condition that M is finitely generated and projective over A. The *orthogonal sum* of two (Γ,Λ)-modules is defined by

$$(M,B,q)\oplus(M',B',q')=(M\oplus M',B\oplus B',q\oplus q')$$

where $M\oplus M'$ is the direct sum of M and M', $(q\oplus q')(m,m')=q(m)+q'(m')$, and $B\oplus B'((m,m'),(n,n'))=B(m,n)+B'(m',n')$ for all $m,n\in M$ and $m',n',\in M'$. A *morphism* $(M,B,q)\to(M',M',q')$ is an A-linear map $f:M\to M'$ which preserves the Hermitian forms and $\mathfrak{q}$-forms, i.e. $B'(f(m),f(n))=B(m,n)$ and $q'(f(m))=q(m)$ for all $m,\ n\in M$.

The symmetric monoidal category of all nonsingular, doubly parametrized modules over $(A,(-,\lambda),\Gamma,A_s,\Lambda)$, with morphisms restricted to all isomorphisms will be denoted by

$$\boldsymbol{\mathcal{Q}}(A,(-,\lambda),\Gamma,G,A_s,\Lambda).$$

The next result generalizes Theorem 1.1 of [2].

Lemma 2.7. *Let $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$ be a parameter algebra such that $\Lambda=\max_{\lambda,A_s}(A)$. Then an A-linear map $f:M\to M'$ of A-modules defines a morphism $(M,(B,q))\to(M',(B',q'))$ of (Γ,Λ)-modules if and only if f preserves the Hermitian forms.*

Proof. The only if part is trivial. We prove now the if part. Let $m\in M$. Let $\widetilde{q(m)}$ and $\widetilde{q'(f(m))}$ be liftings of $q(m)$ and $q'(f(m))$, respectively, to A. It suffices to show that $\widetilde{q(m)}-\widetilde{q'(f(m))}\in\Lambda$. By (2.6.3),

$$\widetilde{q(m)}+\lambda\overline{\widetilde{q(m)}}\equiv B(m,m)\mod A_s$$

and

$$\widetilde{q'(f(m))}+\lambda\overline{\widetilde{q'(f(m))}}\equiv B'(f(m),f(m))\mod A_s.$$

By assumption, $B(m,m)\equiv B'(f(m),f(m))$. Thus,

$$\widetilde{q(m)}-\overline{\widetilde{q'(f(m))}}\equiv-\lambda(\overline{\widetilde{q(m)}}-\overline{\widetilde{q'(f(m))}})\mod A_s.$$

Hence, $\widetilde{q(m)}-\widetilde{q'(f(m))}\in\Lambda$. $\square$

Lemma 2.8. *Let $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$ be a parameter algebra. Let (M_1,B_1,q_1) and (M_2,B_2,q_2) be (Γ,Λ)-modules. Let B be a Γ-Hermitian form on $M_1\oplus M_2$ such that $B|_{M_1}=B_1$ and $B|_{M_2}=B_2$. Then*

$$q:M_1\oplus M_2\to A/\Lambda;\quad(m_1,m_2)\mapsto q_1(m_1)+q_2(m_2)+[B(m_1,m_2)]$$

is the unique $\mathfrak{q}$-form such that $q|_{M_1}=q_1$, $q|_{M_2}=q_2$ and (M,B,q) is a (Γ,Λ)-module.

Proof. It is easy to check that (M,B,q) is a (Γ,Λ)-module. The uniqueness follows from property (2.6.2) of (Γ,Λ)-modules. $\square$

Lemma 2.9. *Let $\boldsymbol{A} = (A, (-,\lambda), \Gamma, G, A_s, \Lambda)$ be a parameter R-algebra. If either $\Lambda = \max_{\lambda, A_s}(A)$ or each element of A is a unique R-linear combination $\sum_{g\in G} r_g g$ then the following property is satisfied:*

(2.9.1) *If $x \in A_s \cap \Gamma$ and $\sum\limits_{i=1}^{n} r_i g_i = 0$, $r_i \in R$, $g_i \in G$, then $\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j \in \Lambda$.*

Furthermore, if $\boldsymbol{A}$ satisfies (2.9.1) then every quotient of $\boldsymbol{A}$ and every localization

$$S^{-1}\boldsymbol{A} := (S^{-1}A, (-,\lambda), S^{-1}\Gamma, G, S^{-1}A_s, S^{-1}\Lambda)$$

of $\boldsymbol{A}$ where S is a multiplicative set in $R_0 = \{r \in R \mid r = \overline{r}\}$ also satisfies (2.9.1).

Proof. If each element of A is a unique R-linear combination of elements of G then $\sum_{i=1}^{n} r_i g_i = 0$ implies each $r_i = 0$. Thus, $\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j = 0 \in \Lambda$. Suppose now that $\Lambda = \max_{\lambda, A_s}(A)$. By $\overline{\overline{g_j}} = \lambda^{-1} g_j \lambda$,

$$\begin{aligned} 0 &= \left(\sum_{i=1}^{n} r_i g_i\right) x \overline{\left(\sum_{i=1}^{n} r_i g_i\right)} \\ &= \sum_{i=1}^{n} r_i \overline{r}_i g_i x \overline{g}_i + \sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j + \lambda \overline{\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j}. \end{aligned}$$

Since each $r_i \overline{r}_i g_i x \overline{g}_i \in A_s$ by (2.3.3), it follows that

$$\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j + \lambda \overline{\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j} \equiv 0 \mod A_s.$$

Thus, $\sum_{i<j} r_i \overline{r}_j g_i x \overline{g}_j \in \Lambda$. The second assertion in the lemma is clear. □

The next result is a generalization of [2, Lemma 13.6].

Lemma 2.10. *Let $\boldsymbol{A} = (A, (-,\lambda), \Gamma, G, A_s, \Lambda)$ be a parameter algebra. Suppose that $\boldsymbol{A}$ satisfies (2.9.1) and $\Gamma + A_s = \min_{-\lambda}(A) + A_s$. If (M, B) is a Γ-Hermitian module and M is finitely generated and projective over A then there is a map $q : M \to A/\Lambda$ such that (M, B, q) is a (Γ, Λ)-module.*

Proof. Let M' be an A-module such that $M \oplus M'$ is finitely generated and free over A and let B' be the Γ-Hermitian form on M' with constant value 0. Let $(N, C) = (M \oplus M', B \oplus B')$. If there is a map $q : N \to A/\Lambda$ such that (N, C, q) is a (Γ, Λ)-module then $(M, B, q|_M)$ is a (Γ, Λ)-module. Thus, we can reduce to the case M is finitely generated and free over A, with A-basis $\{v_i, \dots, v_n\}$. Let B_0 and B_k $(1 \leqq k \leqq r)$ denote the unique A-sesquilinear forms on M such that

$$B_0(v_i, v_j) = \begin{cases} B(v_i, v_j) & (i < j) \\ 0 & (\text{otherwise}) \end{cases}$$

$$B_k(v_i, v_j) = \begin{cases} B(v_i, v_j) & (k = i = j) \\ 0 & (\text{otherwise}). \end{cases}$$

One checks easily that $B = B_0 + \lambda\overline{B}_0 + \sum\limits_{i=1}^{r} B_i$ where $\lambda\overline{B}_0(m, m') = \lambda(\overline{B_0(m'.m)})$.

We define now $\mathfrak{q}$-forms for $B_0 + \lambda\overline{B}_0$, B_1, $\dots$, B_r. Define $q_0 : M \to A/\Lambda$ by $q(m) = [B_0(m, m)]$. For $1 \leqq i \leqq r$, it follows from the assumption on Γ and the fact that B is Γ-Hermitian that $B_i(v_i, v_i) = c_i + \lambda\bar{c}_i + d_i$ for some $c_i \in A$ and $d_i \in A_s$. Since

$$c_i + \lambda\bar{c}_i + d_i \in \Gamma \quad \text{and} \quad c_i + \lambda\bar{c}_i \in \min\nolimits_{-\lambda}(A) \subseteqq \Gamma,$$

we have that $d_i \in \Gamma$. Let $a \in A$ and suppose $a = \sum r_k g_k$ $(r_k \in R, g_k \in G)$. By (2.9.1), the class in A/Λ of the element $\sum_{k<\ell} r_\ell \bar{r}_k g_\ell d_i \bar{g}_k$ does not depend on the decomposition of the element a. For $1 \leqq i \leqq r$, define

$$q_i : M \to A/\Lambda; \quad \sum_{j=1}^{r} a_j v_j \longmapsto [a_i c_i \bar{a}_i + \sum_{k<\ell} r_\ell \bar{r}_k g_\ell d_i \bar{g}_k]$$

where $a_i = \sum_k r_k g_k$. One shows straightforward that $(M, q_0, B_0 + \lambda\overline{B}_0)$ is a (Γ, Λ)-module and using the manipulations in the proof of Lemma 2.9 that (M, B_i, q_i) $(1 \leqq i \leqq r)$ is a (Γ, Λ)-module. Thus,

$$(M, B, \sum_{i=0}^{r} q_i) = (M, B_0 + \lambda\overline{B}_0 + \sum_{i=1}^{r} B_i, \sum_{i=0}^{r} q_i)$$

is a (Γ, Λ)-module. $\square$

Let $(A, (-, \lambda))$ be an R-algebra with antistructure and let Λ (resp. Γ) be a $(-, \lambda)$-form parameter (resp. $(-, \lambda)$-symmetric parameter) on A. Let

$$\boldsymbol{\mathcal{Q}}(A, (-, \lambda), \Lambda)$$

denote the category of nonsingular, Λ-quadratic modules (cf. Sections 1, B and 13 of [2]) and let

$$\boldsymbol{\mathcal{H}}(A, (-, \lambda), \Gamma)$$

denote the category of nonsingular Γ-Hermitian modules. Both categories above are symmetric monoidal categories under orthogonal sum.

Lemma 2.11. *Let* $\boldsymbol{A} = (A, (-, \lambda), \Gamma, G, A_s, \Lambda)$ *be a parameter algebra.*

(2.11.1) *If* $A_s = 0$*, there is a canonical equivalence*

$$\begin{aligned} \boldsymbol{\mathcal{Q}}(A, (-, \lambda), \Lambda) &\longrightarrow \boldsymbol{\mathcal{Q}}(\boldsymbol{A}) \\ (M, B, q) &\longmapsto (M, B, q) \\ [f : (M, B, q) \to (M_1, B_1, q_1)] &\longmapsto [f : (M, B, q) \to (M_1, B_1, q_1)] \end{aligned}$$

of symmetric monoidal categories.

(2.11.2) *Let* $\boldsymbol{A}' = (A, (-,\lambda), G, \Gamma', A'_s, \Lambda')$ *be a parameter algebra such that* $\Gamma \subseteqq \Gamma'$*,* $A_s \subseteqq A'_s$*, and* $\Lambda \subseteqq \Lambda'$*. Let* $\pi : A/\Lambda \to A/\Lambda'$ *denote the canonical map. Then there is a canonical functor*

$$\begin{aligned} \boldsymbol{\mathcal{Q}}(\boldsymbol{A}) &\longrightarrow \boldsymbol{\mathcal{Q}}(\boldsymbol{A}') \\ (M, B, q) &\longmapsto (M, B, \pi q) \\ [f : (M, B, q) \to (M_1, B_1, q_1)] &\longmapsto [f : (M, B, \pi q) \to (M_1, B_1, \pi q_1)] \end{aligned}$$

of symmetric monoidal categories.

(2.11.3) *If* $A_s = A$*, there is a canonical equivalence*

$$\begin{aligned} \boldsymbol{\mathcal{Q}}(\boldsymbol{A}) &\longrightarrow \boldsymbol{\mathcal{H}}(A, (-,\lambda), \Gamma) \\ (M, B, q) &\longmapsto (M, B) \\ [f : (M, B, q) \longrightarrow (M_1, B_1, q_1)] &\longmapsto [f : (M, B) \longrightarrow (M_1, B_1)] \end{aligned}$$

of symmetric monoidal categories.

The proof is straightforward.

Lemma 2.12. *Let* $\boldsymbol{A} = (A, (-,\lambda), \Gamma, G, A_s, \Lambda)$ *be a parameter algebra. Let*

$$\mathcal{F} : \boldsymbol{\mathcal{Q}}(\boldsymbol{A}) \longrightarrow \boldsymbol{\mathcal{H}}(A, (-,\lambda), \Gamma); \quad (M, q, B) \longmapsto (M, B)$$

denote the forgetful functor. If $\boldsymbol{A}$ *satisfies (2.9.1) (resp.* $\Lambda = \max_{\lambda, A_s}(A)$*) then* $\mathcal{F}$ *is surjective on isomorphism classes of objects (resp. an isomorphism of symmetric monoidal categories).*

Proof. The assertions are an immediate consequence of Lemmas 2.7, 2.9, and 2.10. ☐

Corollary 2.13. *Let* $\boldsymbol{A} = (A, (-,\lambda), \Gamma, G, A_s, \Lambda)$ *be a parameter algebra. If there is an element* $x \in \mathrm{Center}(A)$ *such that* $x + \overline{x} = 1$ *then* $\Gamma = \max_{-\lambda}(A)$*,* $\Lambda = \max_{\lambda, A_s}(A)$*, and the forgetful functor*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}) \longrightarrow \boldsymbol{\mathcal{H}}(A, (-,\lambda), \Gamma); \quad (M, q, B) \longmapsto (M, B)$$

is an isomorphism of symmetric monoidal categories.

Proof. By the proof of Lemma 13.8 of [2], $\Gamma = \max_{-\lambda}(A)$ and $\Lambda = \max_{\lambda, A_s}(A)$. The last assertion of the corollary follows now from Lemmas 2.9 and 2.12. ☐

Our next goal is to introduce the notion of positioning module and to study doubly parametrized forms on positioning modules. A positioning module with empty positioning data is essentially just an ordinary module, so that our study of doubly parametrized forms on positioning modules will include as a special case the doubly parametrized modules in Definition 2.6.

Suppose G is a monoid, i.e. a nonempty set with an associative, binary operation having an identity element 1_G. Recall that a *G-set* is a set X together with a map $G \times X \to X$; $(g, x) \mapsto gx$, such that $(gf)x = g(fx)$ and $1_G x = x$ for all g, $f \in G$ and $x \in X$.

Definition 2.14. A *positioning algebra* is a triple

$$(A, G, \Theta)$$

where A is an algebra, G a monoidal subset of A, and Θ a G-set. A *morphism* $(A, G, \Theta) \to (A', G', \Theta')$ of positioning algebras is a pair (f_A, τ) where $f_A : A \to A'$ is a homomorphism of algebras such that $f_A(G) \subsetneqq G'$ and $\tau : \Theta \to \Theta'$ an equivariant map, i.e. $\tau(gx) = f_A(g)\tau(x)$ for all $g \in G$ and $x \in \Theta$.

Definition 2.15. Let (A, G, Θ) be a positioning algebra. A *positioning module* over (A, G, Θ) is a pair

$$(M,\ \theta : \Theta \to M)$$

where M is a left A-module and $\theta : \Theta \to M$ a G-equivariant map, i.e. $\theta(gx) = g\theta(x)$ for all $g \in G$ and $x \in X$. This θ will be called a *positioning map* of the module. Define

$$\boldsymbol{\mathcal{P}}(A, G, \Theta)$$

to be the category whose objects are all positioning modules $(P, \theta : \Theta \to P)$ over (A, G, Θ) where P is a finitely generated, projective A-module. A *morphism* $(P, \theta : \Theta \to P) \to (P', \theta' : \Theta \to P')$ of positioning modules is an A-linear isomorphism $f : P \to P'$ such that $f\theta = \theta'$. (We use only isomorphisms so that we can define later the hyperbolic functor.) The category $\boldsymbol{\mathcal{P}}(A, G, \Theta)$ has a sum operation

$$(P, \theta) \oplus (P', \theta') = (P \oplus P', \theta \oplus \theta')$$

where $P \oplus P'$ denotes the direct sum of P and P' and $\theta \oplus \theta'$ is defined by

$$(\theta \oplus \theta')(x) = (\theta(x), \theta'(x)).$$

Let

$$\boldsymbol{\mathcal{P}}(A)$$

denote as usual the category whose objects are all finitely generated, projective A-modules and whose morphisms are all A-linear isomorphisms between such modules. $\boldsymbol{\mathcal{P}}(A)$ has a sum operation defined by direct sum.

Lemma 2.16. *Let (A, G, Θ) be a positioning algebra such that Θ is the empty G-set. Then the canonical functor*

$$\begin{aligned} \boldsymbol{\mathcal{P}}(A) &\longrightarrow \boldsymbol{\mathcal{P}}(A, G, \Theta) \\ P &\longmapsto (P, \phi) \\ [f : P \to P'] &\longmapsto [f : (P, \phi) \to (P', \phi)] \end{aligned}$$

where $\phi : \Theta \to P$ denotes the empty map is an equivalence of symmetric monoidal categories.

This is evident.

If $\boldsymbol{\mathcal{M}}$ is a symmetric monoidal category, let

$$K_0(\boldsymbol{\mathcal{M}})$$

denote its Grothendieck group.

Lemma 2.17. *Let (A, G, Θ) be a positioning algebra. Then the functor*

$$\begin{aligned} \boldsymbol{\mathcal{P}}(A) &\longrightarrow \boldsymbol{\mathcal{P}}(A, G, \Theta) \\ P &\longmapsto (P, 0) \\ [f : P \to P'] &\longmapsto [f : (P, 0) \to (P', 0)] \end{aligned}$$

where $0 : \Theta \to P$ denotes the constant map with value 0 preserves sums and the forgetful functor

$$\begin{aligned} \boldsymbol{\mathcal{P}}(A, G, \Theta) &\longrightarrow \boldsymbol{\mathcal{P}}(A) \\ (P, \theta) &\longmapsto P \\ [f : (P, \theta) \to (P', \theta')] &\longmapsto [f : P \to P'] \end{aligned}$$

is a sum preserving retract. Moreover, the induced homomorphism

$$K_0(\boldsymbol{\mathcal{P}}(A)) \longrightarrow K_0(\boldsymbol{\mathcal{P}}(A, G, \Theta)); \quad [P] \longmapsto [P, 0]$$

of Grothendieck groups is an isomorphism.

Proof. The assertions for the functor $\boldsymbol{\mathcal{P}}(A) \to \boldsymbol{\mathcal{P}}(A, G, \Theta)$; $P \mapsto (P, 0)$, and the forgetful functor $\boldsymbol{\mathcal{P}}(A, G, \Theta) \to \boldsymbol{\mathcal{P}}(A)$; $(P, \theta) \mapsto P$, are clear. Thus, the induced homomorphism $K_0(\boldsymbol{\mathcal{P}}(A)) \to K_0(\boldsymbol{\mathcal{P}}(A, G, \Theta))$; $[P] \mapsto [P, 0]$, is injective, since it has a retract defined by the forgetful functor. Let $(P, \theta) \in \boldsymbol{\mathcal{P}}(A, G, \Theta)$. The A-linear isomorphism $P \oplus P \to P \oplus P$; $(p, p') \mapsto (p, p' + p)$, defines an isomorphism

$$(P, \theta) \oplus (P, 0) \xrightarrow{\cong} (P, \theta) \oplus (P, \theta) \tag{2.17.1}$$

of positioning modules. Thus, $[P, 0] = [P, \theta]$. Thus, the homomorphism $K_0(\boldsymbol{\mathcal{P}}(A)) \to K_0(\boldsymbol{\mathcal{P}}(A, G, \Theta))$ is surjective. □

Definition 2.18. A *parameter algebra with positioning data* is a 7-tuple

$$(A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$$

where $(A, (-, \lambda), \Gamma, G, A_s, \Lambda)$ is a parameter algebra and (A, G, Θ) a positioning algebra. A *morphism* of parameter algebras with positioning date is defined in the obvious way, i.e. it is an algebra homomorphism which induces morphisms of parameter and positioning algebras.

Definition 2.19. Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. A *doubly parametrized positioning module* over $(\boldsymbol{A}, \Theta)$ is a quadruple (M, B, q, θ) where (M, θ) is a positioning module and (B, q) a doubly parametrized form on M over $\boldsymbol{A}$. *Morphisms* (resp. *orthogonal sums*) of doubly parametrized positioning modules over $(\boldsymbol{A}, \Theta)$ are defined in the obvious way, i.e. so that they induce morphism (resp. sums) on the underlying doubly parametrized modules and positioning modules.

The symmetric monoidal category of all nonsingular, doubly parametrized positioning modules over $(\boldsymbol{A}, \Theta)$, with morphisms restricted to all isomorphisms will be denoted by

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta).$$

Lemma 2.20. *If (M, B, q, θ) be a doubly parametrized positioning module over $(\boldsymbol{A}, \Theta)$ then*

$$(M, B, q, \theta) \cong (M, B, q, -\theta).$$

Proof. The map $M \to M$; $m \mapsto -m$, is an isomorphism $(M, B, q, \theta) \to (M, B, q, -\theta)$. □

Lemma 2.21. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. If Θ is the empty set then the canonical functor*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}) \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta); \quad (M, B, q) \mapsto (M, B, q, \phi)$$

where $\phi : \Theta \to M$ is the empty map is an equivalence of symmetric monoidal categories.

This is clear.

Lemma 2.22. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Then the functor*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}) \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta); \quad (M, B, q) \mapsto (M, B, q, 0)$$

where $0 : \Theta \to M$ denotes the constant map with value 0 preserves sums and the forgetful functor

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}); \quad (M, B, q, \theta) \mapsto (M, B, q)$$

is a sum preserving retract. In particular, the induced homomorphism

$$K_0(\boldsymbol{\mathcal{Q}}(\boldsymbol{A})) \to K_0(\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)); \quad [M, B, q] \mapsto [M, B, q, 0]$$

of Grothendieck groups has a retract (but is not in general an isomorphism, as in Lemma 2.17.

The proof is straightforward.

In order to obtain the analogs of Lemmas 2.11, 2.12, and Corollary 2.13 for positioning modules, we introduce notions of quadratic and Hermitian forms on positioning modules.

Definition 2.23. A *form algebra with positioning data* is a quintuple

$$(A, (-, \lambda), \Lambda, G, \Theta)$$

where $(A, (-, \lambda), \Lambda)$ is a form algebra and (A, G, Θ) a positioning algebra. A *morphism* of form algebras with positioning data is a homomorphism of algebras which induces morphisms of form algebras and positioning algebras.

Definition 2.24. A *symmetric parameter algebra with positioning data* is a quintuple

$$(A, (-, \lambda), \Gamma, G, \Theta)$$

such that $(A, (-, -\lambda), \Gamma, G, \Theta)$ is a form algebra with positioning data.

Definition 2.25. Let $(A, (-, \lambda), \Lambda, G, \Theta)$ be a form algebra with positioning data. A *quadratic positioning module* over $(A, (-, \lambda), \Lambda, G, \Theta)$ is a quintuple (M, B, q, θ) where (M, θ) is a positioning module and (B, q) a Λ-quadratic form on M. *Morphisms* (resp. *orthogonal sums*) of quadratic positioning modules over $(A, (-, \lambda), \Lambda, G, \Theta)$ are defined in the usual way, i.e. so that they induce morphisms (resp. sums) on the underlying quadratic modules and positioning modules.

The symmetric monoidal category of all nonsingular, quadratic forms on positioning modules over $(A, (-, \lambda), \Lambda, G, \Theta)$, with morphisms restricted to all isomorphisms will be denoted by

$$\boldsymbol{\mathcal{Q}}(A, (-, \lambda), \Lambda, G, \Theta).$$

Definition 2.26. Let $(A, (-, \lambda), \Gamma, G, \Theta)$ be a symmetric parameter algebra with positioning data. A *Hermitian positioning module* over $(A, (-, \lambda), \Gamma, G, \Theta)$ is a triple (M, B, θ) where (M, θ) is a positioning module and B a Γ-Hermitian form on M. *Morphsims* (resp. *orthogonal sums*) of Hermitian positioning modules over $(A, (-, \lambda), \Gamma, G, \Theta)$ are defined in the usual way, i.e. so that they induce morphisms (resp. sums) on the underlying Hermitian modules and positioning modules.

The symmetric monoidal category of all nonsingular, Hermitian forms on positioning modules over $(A, (-, \lambda), \Gamma, G, \Theta)$, with morphisms restricted to all isomorphisms will be denoted by

$$\boldsymbol{\mathcal{H}}(A, (-, \lambda), \Gamma, G, \Theta).$$

The next result generalizes Lemma 2.11.

Lemma 2.27. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data.*

(2.27.1) *If $A_s = 0$, there is a canonical equivalence*

$$\boldsymbol{\mathcal{Q}}(A, (-, \lambda), \Lambda, G, \Theta) \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta);\ (M, B, q, \theta) \mapsto (M, B, q, \theta)$$

of symmetric monoidal categories.

(2.27.2) *Let $(\boldsymbol{A}', \Theta) = (A, (-, \lambda), \Gamma', G, A'_s, \Lambda', \Theta)$ be a parameter algebra with positioning data such that $\Gamma \subseteqq \Gamma'$, $A_s \subseteqq A'_s$, and $\Lambda \subseteqq \Lambda'$. Let $\pi : A/\Lambda \to A/\Lambda'$ denote the canonical map. Then there is a canonical functor*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}', \Theta);\ (M, B, q, \theta) \mapsto (M, B, \pi \circ q, \theta)$$

of symmetric monoidal categories.

(2.27.3) *If $A_s = A$, there is a canonical equivalence*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) \to \boldsymbol{\mathcal{H}}(A, (-, \lambda), \Gamma, G, \Theta);\ (M, B, q, \theta) \mapsto (M, B, \theta)$$

of symmetric monoidal categories.

The proof is straightforward.

The next result generalizes Lemma 2.12.

Lemma 2.28. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let*

$$\mathcal{F} : \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) \to \boldsymbol{\mathcal{H}}(A, (-, \lambda), \Gamma, G, \Theta);\ (M, B, q, \theta) \mapsto (M, B, \theta)$$

denote the forgetful functor. If the underlying parameter algebra $\boldsymbol{A}$ satisfies (2.9.1) *(resp. $\Lambda = \max_{\lambda, A_s}(A)$) then $\mathcal{F}$ is surjective on isomorphism classes of objects (resp. an isomorphism of symmetric monoidal categories).*

Proof. The proof is the same as that of Lemma 2.12. The positioning maps are only extra baggage. □

The next result generalizes Corollary 2.13.

Corollary 2.29. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. If there is an element $x \in \mathrm{Center}(A)$ such that $x + \overline{x} = 1$ then $\Gamma = \max_{-\lambda}(A)$, $\Lambda = \max_{\lambda, A_s}(A)$, and the forgetful functor*

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) \to \boldsymbol{\mathcal{H}}(A, (-, \lambda), \Gamma, G, \Theta);\ (M, (B, q), \theta) \mapsto (M, B, \theta)$$

is an isomorphism of symmetric monoidal categories.

Proof. The proof is the same as that of Corollary 2.13, except the reference to Lemma 2.12 is replaced by one to Lemma 2.28. □

Let $\boldsymbol{\mathcal{C}}$ be a subcategory of $\boldsymbol{\mathcal{P}}(A)$ satisfying the next assumption.

Assumption 2.30. $\boldsymbol{\mathcal{C}}$ contains A and is closed under isomorphisms, direct sums, and dualization $P \mapsto P^*$.

Definition 2.31. Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let $(P, \theta) \in \boldsymbol{\mathcal{P}}(A, G, \Theta)$. Let B'_P denote the sesquilinear form on $P \oplus P^*$ defined by $B'_P((p, f), (p', f')) = f(p')$. Let

$$B_P = B'_P + \lambda \overline{B}'_P$$

where $\lambda \overline{B}'_P(m, m') = \lambda \overline{(B'_P(m', m))}$. Let

$$q_P : P \oplus P^* \to A/\Lambda;\quad m \mapsto [B'_P(m, m)].$$

One checks easily that the construction $(P, \theta) \mapsto \mathbb{H}(P, \theta)$, where

$$\mathbb{H}(P, \theta) = (P \oplus P^*, B_P, q_P, \theta),$$

defines a sum preserving functor

$$\begin{aligned} \mathbb{H} : \boldsymbol{\mathcal{P}}(A,G,\Theta) &\longrightarrow \boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta) \\ (P,\theta) &\longmapsto \mathbb{H}(P,\theta) \\ [f:(P,\theta)\to(P',\theta')] &\longmapsto [f\oplus f^{*-1} : \mathbb{H}(P,\theta)\to\mathbb{H}(P',\theta')] \end{aligned}$$

called the *hyperbolic functor*. The object $\mathbb{H}(P,\theta)$ is called the *hyperbolic module* on (P,θ) and $\mathbb{H}(A,\theta)$ is a called a *hyperbolic plane*. $\mathbb{H}(P,\theta)$ is called a $\boldsymbol{\mathcal{C}}$*-hyperbolic module* whenever $P \in \boldsymbol{\mathcal{C}}$.

Definition 2.32. Let $(\boldsymbol{A},\Theta) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ be a parameter algebra with positioning data. Let

$$\boldsymbol{\mathcal{P}}(\boldsymbol{A},\Theta)$$

denote the symmetric monoidal category whose objects are all quadruples (P,b_0,q_0,θ) where $(P,\theta)\in\boldsymbol{\mathcal{P}}(A,G,\Theta)$ and (b_0,q_0) is a doubly parametrized form on P^* over $\boldsymbol{A}$. A *morphism* $(P,b_0,q_0,\theta)\to(P',b_0',q_0',\theta')$ is an A-linear isomorphism $f:P\to P'$ such that f defines a morphism $(P,\theta)\to(P',\theta')$ of positioning modules and f^{*-1} defines a morphism $(P^*,b_0,q_0)\to(P'^*,b_0',q_0')$ of doubly parametrized modules over $\boldsymbol{A}$. Sums are defined as in Definitions 2.6 and 2.15.

Let $(P,b_0,q_0,\theta)\in\boldsymbol{\mathcal{P}}(\boldsymbol{A},\Theta)$. Let B_{P,b_0} denote the Γ-Hermitian form on $P\oplus P^*$ defined by

$$B_{P,b_0}((p,f),(p',f')) = B_P((p,f),(p',f')) + b_0(f,f')$$

where B_P is defined as in the paragraph above. Define

$$q_{P,q_0} : P\oplus P^* \to A/\Lambda;\ (p,f)\mapsto [f(p)] + q_0(f).$$

By Lemma 2.8, (B_{P,b_0},q_{P,q_0}) is a doubly parametrized form on $P\oplus P^*$ over $\boldsymbol{A}$. One checks easily that the construction $(P,b_0,q_0,\theta)\mapsto\mathbb{M}(P,b_0,q_0,\theta)$, where

$$\mathbb{M}(P,b_0,q_0,\theta) = (P\oplus P^*, B_{P,b_0}, q_{P,q_0},\theta),$$

defines a sum preserving functor

$$\begin{aligned} \mathbb{M} : \boldsymbol{\mathcal{P}}(\boldsymbol{A},\Theta) &\longrightarrow \boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta) \\ (P,b_0,q_0,\theta) &\longmapsto \mathbb{M}(P,b_0,q_0,\theta) \\ [f:(P,b_0,q_0,\theta)\to(P',b_0',q_0',\theta')] &\longmapsto [f\oplus f^{*-1} : \mathbb{M}(P,b_0,q_0,\theta)\to\mathbb{M}(P',b_0',q_0',\theta')] \end{aligned}$$

called the *metabolic functor*. The object $\mathbb{M}(P,b_0,q_0,\theta)$ is called the *metabolic module* on (P,b_0,q_0,θ) and $\mathbb{M}(A,b_0,q_0,\theta)$ is called a *metabolic plane*. $\mathbb{M}(P,b_0,q_0,\theta)$ is called a $\boldsymbol{\mathcal{C}}$*-metabolic module* whenever $P\in\boldsymbol{\mathcal{C}}$.

Lemma 2.33. *Let* $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ *be a parameter algebra with positioning data. Then the diagram*

$$\begin{tikzcd} \boldsymbol{\mathcal{P}}(A, G, \Theta) \arrow[rr, "\mathcal{I}"] \arrow[dr, "\mathbb{H}"'] & & \boldsymbol{\mathcal{P}}(\boldsymbol{A}, \Theta) \arrow[dl, "\mathbb{M}"] \\ & \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta) & \end{tikzcd}$$

of symmetric monoidal categories commutes, where $\mathcal{I}((P, \theta)) = (P, 0, 0, \theta)$.

This lemma is obvious.

Definition 2.34. Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. An A-submodule $L \subsetneqq M$ is called *totally isotropic* if B and q are trivial on L, i.e. $B(m, m') = 0$ and $q(m) = 0$ for all $m, m' \in L$. The positioning function θ is called *totally isotropic* if $B(\theta(x), \theta(x'))$ and $q(\theta(x)) = 0$ for all $x, x' \in \Theta$. A totally isotropic, A-direct summand $L \subsetneqq M$ which contains Image(θ) is called a *sublagrangian*. If, in addition, $L \in \boldsymbol{\mathcal{C}}$ then L is called a $\boldsymbol{\mathcal{C}}$-*sublagrangian*. A sublagrangian (resp. $\boldsymbol{\mathcal{C}}$-sublagrangian) L such that $L = L^\perp$, where

$$L^\perp = \{m \in M \mid B(m, \ell) = 0 \ \forall \ell \in L\},$$

is called a *Lagrangian* (resp. $\boldsymbol{\mathcal{C}}$-*Lagrangian*). (M, B, q, θ) is called a *null module* (resp. $\boldsymbol{\mathcal{C}}$-*null module*), if it contains a Lagrangian (resp. $\boldsymbol{\mathcal{C}}$-Lagrangian).

The next result generalizes Lemma 2.8 of [2].

Lemma 2.35. *Let* $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ *be a parameter algebra with positioning data. Let* $P \subsetneqq M$ *be a Lagrangian in* (M, B, q, θ) *and let* $i : P \hookrightarrow M$ *denote the natural embedding. Then the* A*-linear map* $M \mapsto P^*$*;* $m \mapsto B(m, -)$*, has a splitting* $i' : P^* \to M$ *and for any splitting* i'*, if* $b_0 = B|_{\mathrm{Image}(i')}$ *and* $q_0 = q|_{\mathrm{Image}(i')}$ *then the* A*-linear map* $i \oplus i' : P \oplus P^* \to M$ *defines an isomorphism* $\mathbb{M}(P, b_0, q_0, \theta) \to (M, B, q, \theta)$ *of doubly parametrized, positioning modules.*

Proof. Since B is nonsingular and P is a direct summand of M, it follows that the A-linear map $M \to P^*$; $m \longmapsto B(m, -)$, is surjective. Since M is projective, it follows that P is projective and hence, P^* is projective. Thus, the map $M \to P$ above has a splitting, say i'. Let $K = \mathrm{Ker}[M \to P^*]$. By definition,

$$K = \{m \in M \mid B(m, p) = 0 \ \ \forall p \in P\} = P^\perp.$$

Since P is a Lagrangian, it follows that $P = P^\perp = K$. Thus, the A-linear map $i \oplus i'$: $P \oplus P^* \to M$ is an isomorphism of A-modules. Defining (b_0, q_0) as in the lemma, one checks that $i \oplus i'$ defines an isomorphism $\mathbb{M}(P, b_0, q_0, \theta) \to (M, B, q, \theta)$ of doubly parametrized, positioning modules. $\square$

The next result generalizes Lemmas 2.6 and 2.7 of [2].

Corollary 2.36. *Let $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and let $P \subsetneqq M$ be a sublagrangian. Let $i : P \hookrightarrow M$ denote the natural embedding and let $i' : P^* \to M$ be a splitting of the A-linear map $M \to P^*$; $m \mapsto B(m, -)$. Let $b_0 = B|_{\mathrm{Image}(i')}$ and $q_0 = q|_{\mathrm{Image}(i')}$. Then the A-linear map $i \oplus i' : P \oplus P^* \to M$ defines an embedding $\mathbb{M}(P, b_0, q_0, \theta) \to (M, B, q, \theta)$ and if $N = \mathrm{Image}(i \oplus i')$ then*

$$(M, B, q, \theta) = (N, B|_N, q|_N, \theta) \oplus (N^\perp, B|_{N^\perp}, q|_{N^\perp}, 0).$$

Furthermore, the Hermitian form B and the map q are well defined on $P^\perp/P$ and if $j : P^\perp \hookrightarrow M$ denotes the natural embedding then j induces an isomorphism

$$(P^\perp/P, B|_{P^\perp}, q|_{P^\perp}, 0) \to (N^\perp, B|_{N^\perp}, q|_{N^\perp}, 0).$$

Proof. As in the proof of Lemma 2.35, one shows that $i \oplus i'$ defines an embedding

$$\mathbb{M}(P, b_0, q_0, \theta) \to (M, B, q, \theta).$$

By Corollary 2.3 of [2] and Lemma 2.8,

$$(M, B, q, \theta) = (N, B|_N, q|_N, \theta) \oplus (N^\perp, B|_{N^\perp}, q_{N^\perp}, 0).$$

It follows that $P^\perp = P \oplus N^\perp$. One checks easily that the isomorphism $j : P^\perp \to P \oplus N^\perp$ followed by the canonical projection $P \oplus N^\perp \to N^\perp$ defines an isomorphism $(P^\perp/P, B|_{P^\perp}, q|_{P^\perp}, 0) \to (N^\perp, B|_{N^\perp}, q|_{N^\perp}, 0)$. □

The next result generalizes Lemma 2.9 of [2].

Lemma 2.37. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. Then the diagonal submodule $\{(m, m) |\ m \in M\}$ of $M \oplus M$ is a Lagrangian in $(M, B, q, \theta) \oplus (M, -B, -q, \theta)$.*

The proof is straightforward.

Corollary 2.38. *The family of metabolic planes in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ is cofinal in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$, i.e. given $\boldsymbol{M} \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$, there is an $\boldsymbol{M}' \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and metabolic planes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_r \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ such that*

$$\boldsymbol{M} \oplus \boldsymbol{M}' \cong \boldsymbol{M}_1 \oplus \cdots \oplus \boldsymbol{M}_r.$$

Proof. From Lemmas 2.37 and 2.35, it follows that metabolic modules $\mathbb{M}(P', b_0', q_0', \theta)$ are cofinal in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. Choosing $P \in \boldsymbol{\mathcal{P}}(A)$ such that $P' \oplus P$ is free and forming the orthogonal sum $\mathbb{M}(P', b_0', q_0', \theta) \oplus \mathbb{M}(P, 0, 0, 0)$, one obtains that metabolic modules $\mathbb{M}(F, b_0, q_0, \theta)$ on finitely generated, free modules F are cofinal in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. We shall show that $\mathbb{M}(F, b_0, q_0, \theta)$ is isomorphic to an orthogonal sum of metabolic planes. Let (B, q) denote the doubly parametrized form (B_{F,b_0}, q_{F,q_0}) on $\mathbb{M}(F, b_0, q_0, \theta)$.

Let $\{e_1, \ldots, e_r\}$ be a basis for F and $\{f_1, \ldots, f_r\} \in F^*$ its dual basis. Let c denote the (unique) sesquilinear form on F^* such that

$$c(f_i, f_j) = \begin{cases} -b_0(f_i, f_j) & (i < j) \\ 0 & (\text{otherwise}). \end{cases}$$

Let $\alpha : F^* \to F$ denote the (unique) A-linear map such that

$$B(\alpha(f_i), f_j) = c(f_i, f_j) \quad (1 \leqq i,\ j \leqq r).$$

Let $id : F \to F^*$ denote the identity map. Define $f'_i = (id + \alpha)f_i$, $M_i = Ae_i + Af'_i$, $B_i = B|_{M_i}$ and $q_i = q|_{M_i}$ $(1 \leqq i \leqq r)$. Further define $\theta_i : \Theta \to Ae_i$ such that $\theta(x) = \sum_{i=1}^r \theta_i(x)$ for all $x \in \Theta$. Then $\mathbb{M}(F, b_0, q_0, \theta) = (F \oplus F^*, B, q, \theta) = \bigoplus_{i=1}^r (M_i, B_i, q_i, \theta_i)$. Moreover, one sees easily (see the proof of Lemma 2.35) that each $(M_i, B_i, q_i, \theta_i)$ is isomorphic to a metabolic plane. □

Definition 2.39. Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let (M, B, q, θ) be a doubly parametrized, positioning module over $(\boldsymbol{A}, \Theta)$. An ordered pair (e, f) of elements in M is called a *metabolic pair* if

$$B(e, e) = 0,\ B(f, e) = 1,\ q(e, e) = q(f, f) = 0,$$

and

$$B(\theta(x), e) = 0 \quad (\forall x \in \Theta).$$

If a metabolic pair (e, f) generates M as an A-module then it is called a *metabolic basis* for (M, B, q, θ). In this case, (M, B, q, θ) is a metabolic plane in the sense of the paragraph subsequent to Definition 2.32.

Lemma 2.40. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data.*

(2.40.1) *A doubly parametrized, positioning module over $(\boldsymbol{A}, \Theta)$ has a metabolic basis if and only if it is isomorphic to a metabolic plane.*

(2.40.2) *If (e, f) is a metabolic basis for (M, B, q, θ) and $a \in A_s$ then $(e, f + ae)$ is a metabolic basis for (M, B, q, θ) and $B(f + ae, f + ae) = B(f, f) + a + \lambda\bar{a}$.*

Proof. We prove the only if part of (2.40.1). Let (e, f) be a metabolic basis for the doubly parametrized, positioning module (M, B, q, θ). Let $x = 1 \in A$ and let $y \in A^*$ be its dual. Let $\alpha : A \oplus A^* \to M$; $cx + dy \mapsto ce + df$. Let $b_0 : A^* \times A^* \to A$; $(dy, d'y) \mapsto B(df, d'f)$ and let $q_0 : A^* \to A/\Lambda$; $dy \mapsto q(df)$. Let $\tau : \Theta \to A$ be the unique map such that the diagram

$$\begin{tikzcd} \Theta \arrow[r, "\tau"] \arrow[dr, "\theta"'] & A \oplus A^* \arrow[d, "\alpha", "\cong"'] \\ & M \end{tikzcd}$$

commutes. Then (b_0, q_0) is a doubly parametrized form on A^* and α defines an isomorphism $\mathbb{M}(A, b_0, q_0, \tau) \to (M, B, q, \theta)$.

Next we prove the if part of (2.40.1). Let (B, q) denote the doubly parametrized form on the metabolic plane $\mathbb{M}(A, b_0, q_0, \theta)$. Let $e = 1 \in A$ and $y \in A$ its dual. By definition, $B(e, e) = 0$, $q(e) = 0$, and $B(\theta(t), e) = 0$ for all $t \in \Theta$. Let $a \in A_q$ be a representative of

$q(y)$ and let $f = y - ae$. Then

$$\begin{aligned} B(f,e) &= B(y,e) - B(e,e)\overline{a} = B(y,e) = 1 \quad \text{and} \\ q(f) &= q(y) + q(-ae) + [B(y,-ae)] \quad \text{(by (2.6.2))} \\ &= q(y) + [-aB(y,e)] = q(y) - [a] = 0. \end{aligned}$$

Thus, (e,f) is a metabolic basis for $\mathbb{M}(A, b_0, q_0, \theta)$.

The assertion (2.40.2) can be proved by a straightforward computation. □

Lemma 2.41. *Let $(\boldsymbol{A}, \Theta) = (A, (-,\lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let $(P, b_0, q_0, \theta) \in \boldsymbol{\mathcal{P}}(\boldsymbol{A}, \Theta)$. Let $\beta : P^* \to P$ denote the (unique) A-linear map such that $f'(\beta(f)) = b_0(f', f)$ for all $f' \in P^*$. Then the A-linear isomorphism*

$$\begin{pmatrix} I_P & 0 & 0 & 0 \\ 0 & I_{P^*} & 0 & I_{P^*} \\ 0 & 0 & I_P & 0 \\ 0 & 0 & 0 & I_{P^*} \end{pmatrix} \begin{pmatrix} I_P & 0 & 0 & 0 \\ 0 & I_{P^*} & 0 & 0 \\ -I_P & 0 & I_P & 0 \\ 0 & 0 & 0 & I_{P^*} \end{pmatrix} \begin{pmatrix} I_P & 0 & 0 & 0 \\ 0 & I_{P^*} & 0 & 0 \\ 0 & -\beta & I_P & 0 \\ 0 & 0 & 0 & I_{P^*} \end{pmatrix} :$$

$$P \oplus P^* \oplus P \oplus P^* \longrightarrow P \oplus P^* \oplus P \oplus P^*;$$

$$\begin{pmatrix} p \\ f \\ p' \\ f' \end{pmatrix} \longmapsto \begin{pmatrix} p \\ f + f' \\ p' - p - \beta(f) \\ f' \end{pmatrix},$$

defines an isomorphism

$$\mathbb{M}(P, b_0, q_0, \theta) \oplus \mathbb{H}(P, 0) \xrightarrow{\cong} \mathbb{M}(P, b_0, q_0, \theta) \oplus \mathbb{M}(P, -b_0, -q_0, -\theta)$$

of doubly parametrized, positioning modules over $(\boldsymbol{A}, \Theta)$.

Proof. One checks first that the A-linear map in the lemma preserves the positioning maps and Hermitian forms. The one can use Lemma 2.8 to simplify checking that it preserves the $\mathfrak{q}$-forms. □

Let A be an algebra with antistructure $(-,\lambda)$.

Definition 2.42. Let $(\boldsymbol{A}, \Theta) = (A, (-,\lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Define

$$\boldsymbol{\mathcal{P}}(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}$$
$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}$$

to be the full, symmetric monoidal subcategories of $\boldsymbol{\mathcal{P}}(\boldsymbol{A}, \Theta)$ and $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$, respectively, such that the underlying A-module of each object lies in $\boldsymbol{\mathcal{C}}$.

Let

$$\boldsymbol{\mathcal{D}} \subseteqq \boldsymbol{\mathcal{C}}$$

be a subcategory which contains A and has the same closure properties as $\boldsymbol{\mathcal{C}}$ (see Assumption 2.30).

Definition 2.43. Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Define

$$\begin{aligned}
&KQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}} = K_0(\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}) \\
&WQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}} = KQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}/\langle \boldsymbol{\mathcal{D}}\text{-null modules}\rangle \\
&WQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}} = WQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{C}}} \\
&KQ_0(\boldsymbol{A}, \Theta) = KQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{P}}(\boldsymbol{A})} \quad \text{and} \\
&WQ_0(\boldsymbol{A}, \Theta) = WQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{P}}(\boldsymbol{A})}.
\end{aligned}$$

Proposition 2.44. *Let $(\boldsymbol{A}, \Theta) = (A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ be a parameter algebra with positioning data. Let $\mathbb{H} : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}$; $P \mapsto \mathbb{H}(P, 0) = \mathbb{M}(P, 0, 0)$. Then the canonical homomorphisms below are isomorphisms:*

$$\begin{aligned}
\operatorname{Coker}[\mathbb{H} : K_0(\boldsymbol{\mathcal{D}}) \to KQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}] &\xrightarrow{\cong} \operatorname{Coker}[\mathbb{M} : K_0(\boldsymbol{\mathcal{P}}(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{D}}}) \to KQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{C}}}] \\
&\xrightarrow{\cong} WQ_0(\boldsymbol{A}, \Theta)_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}}.
\end{aligned}$$

Proof. The first isomorphism follows from Lemma 2.41 and the second from Lemma 2.35. □

3. Invariants of forms and G-surgery obstruction groups

In geometry, not all nonsingular, doubly parametrized, positioning modules can arise. Precisely those modules which are trivial under a certain invariant will occur. This means that from a geometric point of view, we must study K-groups of nonsingular, doubly parametrized, positioning modules vanishing under a certain kind of invariant. This section describes first in a general context the notion of invariant that will be used and then sets up the K-theoretic framework needed to deal with forms of invariant zero. This is done abstractly in order to focus attention on essentials. Then we define the specific invariant that is required in the geometric context and classify the metabolic planes of invariant zero. The classification is crucial for proving our main surgery results. At the end of the section, we use the geometric invariant to define the G-equivariant surgery obstruction groups.

Definition 3.1. Let $\boldsymbol{\mathcal{Q}}$ be a symmetric monoidal category. (We have in mind the symmetric monoidal category $\boldsymbol{\mathcal{Q}}(A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$ of all nonsingular, doubly parametrized, positioning modules.) Let ((abelian groups)) denote the symmetric monoidal category of all abelian groups, under direct sum. An *invariant* ∇ on $\boldsymbol{\mathcal{Q}}$ with values in a symmetric monoidal subcategory $\boldsymbol{\mathcal{A}}$ of ((abelian groups)) consists of a functor

$$\alpha : \boldsymbol{\mathcal{Q}} \longrightarrow \boldsymbol{\mathcal{A}}$$

(covariant or contravariant) of symmetric monoidal categories and for each object $M \in \boldsymbol{\mathcal{Q}}$, an element

$$\nabla(M) \in \alpha(M)$$

such that the following holds:

(3.1.1) If $f : M \to M'$ is a morphisms in $\boldsymbol{Q}$ then $\alpha(f)(\nabla(M)) = \nabla(M')$ if α is covariant and $\nabla(M) = \alpha(f)(\nabla(M'))$ if α is contravariant.

(3.1.2) The natural identification $\alpha_{M \oplus M'} : \alpha(M \oplus M') \to \alpha(M) \oplus \alpha(M')$ included in the definition of α has the property that $\alpha_{M\oplus M'}(\nabla(M \oplus M')) = (\nabla(M), \nabla(M'))$.

If ∇ is an invariant on $\boldsymbol{Q}$, let

$$\nabla\boldsymbol{Q}$$

denote the full subcategory of $\boldsymbol{Q}$ consisting of all objects $M \in \boldsymbol{Q}$ such that $\nabla(M) = 0$. Call an invariant ∇ on $\boldsymbol{Q}$ a *trivial* invariant, if $\nabla(M) = 0 \in \alpha(M)$ for all objects $M \in \boldsymbol{Q}$, in which case $\nabla\boldsymbol{Q} = \boldsymbol{Q}$.

Below, we shall be confronted with a family ∇ of invariants ∇_i where i ranges through an index set I. In this situation, one constructs in the obvious way an invariant

$$\nabla' = \bigoplus\nolimits_{i\in I} \nabla_i \ \text{ by } \ \nabla'(M) = (\nabla_i(M))_{i\in I} \in \bigoplus_{i\in I} \alpha_i(M)$$

and defines

$$\nabla\boldsymbol{Q} = \nabla'\boldsymbol{Q}.$$

Lemma 3.2. *Let $\boldsymbol{Q}$ be a symmetric monoidal category and ∇ invariant on $\boldsymbol{Q}$. Then $\nabla\boldsymbol{Q}$ is closed under isomorphism classes, sums and summands. In particular, $\nabla\boldsymbol{Q}$ is a symmetric monoidal subcategory of $\boldsymbol{Q}$.*

Proof. The assertions follow immediately from the definition of an invariant. □

Lemma 3.3. *Let $\boldsymbol{Q}$ denote the symmetric monoidal category $\boldsymbol{Q}(A, (-, \lambda), \Gamma, G, A_s, \Lambda, \Theta)$. Let $\boldsymbol{\mathcal{M}}$ denote the full subcategory of $\boldsymbol{Q}$ generated under orthogonal sum by all metabolic planes. Let ∇ be an invariant on $\boldsymbol{Q}$. Suppose that for each object $(M, B, q, \theta) \in \boldsymbol{Q}$, $\nabla(M, -B, -q, \theta) = -\nabla(M, B, q, \theta)$ and $\nabla(\mathbb{H}(A, 0)) = 0$. Then $\nabla\boldsymbol{\mathcal{M}}$ is cofinal in $\nabla\boldsymbol{Q}$. (In practice, this means that $\nabla\boldsymbol{Q}$ has a good class of cofinal objects.)*

Proof. Given an object $(M, B, q, \theta) \in \nabla\boldsymbol{Q}$, we must show there is an object (M', B', q', θ') $\in \nabla\boldsymbol{Q}$ such that

$$(M, B, q, \theta) \oplus (M', B', q', \theta') \in \nabla\boldsymbol{\mathcal{M}}.$$

Suppose $(M, B, q, \theta) \in \nabla\boldsymbol{Q}$. By assumption, $\nabla(M, -B, -q, \theta) = -\nabla(M, B, q, \theta) = 0$. Thus, by Lemma 3.2,

$$\nabla((M, B, q, \theta) \oplus (M, -B, -q, \theta)) = 0.$$

By Lemma 2.37, $(M, B, q, \theta) \oplus (M, -B, -q, \theta)$ has a Lagrangian and hence, by Lemma 2.35, is isomorphic to a metabolic module. The proof of Corollary 2.38 shows there is a hyperbolic module $\mathbb{H}(P, 0)$ and metabolic planes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_r$ such that

$$(M, B, q, \theta) \oplus (M, -B, -q, \theta) \oplus \mathbb{H}(P, 0) \cong \boldsymbol{M}_1 \oplus \cdots \oplus \boldsymbol{M}_r.$$

By assumption and Lemma 3.2, $\nabla(\mathbb{H}(P,0)) = 0$ and

$$\nabla((M,B,q,\theta) \oplus (M,-B,-q,\theta) \oplus \mathbb{H}(P,0)) = 0.$$

Thus, again by Lemma 3.2, $\nabla(\boldsymbol{M}_i) = 0$ for all $1 \leqq i \leqq r$. □

Lemma 3.4. *Let $\boldsymbol{\mathcal{Q}}$ denote the symmetric monoidal category $\boldsymbol{\mathcal{Q}}(A,(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ of nonsingular, doubly parametrized, positioning modules over the parameter algebra $(A,$ $(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ with positioning data. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}$. If ∇ vanishes on the metabolic module $\mathbb{M}(P,b_0,q_0,\theta)$ and on the hyperbolic module $\mathbb{H}(P,0)$ then ∇ vanishes on $\mathbb{M}(P,-b_0,-q_0,-\theta)$.*

Proof. By Lemma 2.41, there is an isomorphism

$$\mathbb{M}(P,b_0,q_0,\theta) \oplus \mathbb{H}(P,0) \xrightarrow{\cong} \mathbb{M}(P,b_0,q_0,\theta) \oplus \mathbb{M}(P,-b_0,-q_0,-\theta).$$

Since ∇ vanishes on $\mathbb{M}(P,b_0,q_0,\theta)$ and $\mathbb{H}(P,0)$, it follows from Lemma 3.2 that ∇ vanishes on $\mathbb{M}(P,-b_0,-q_0,-\theta)$. □

Let A be an algebra with antistructure $(-,\lambda)$ and let $\boldsymbol{\mathcal{D}} \subseteqq \boldsymbol{\mathcal{C}}$ be as in Definition 2.43.

Let $(\boldsymbol{A},\Theta) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ be a parameter algebra with positioning data. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$. Define

$$\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}}$$

to be the full, symmetric monoidal subcategory of $\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$ such that the underlying A-module of each object lies in $\boldsymbol{\mathcal{C}}$, i.e. $\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}} = \nabla(\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}})$.

Definition 3.5. Let $(\boldsymbol{A},\Theta) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ be a parameter algebra with positioning data. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$ such that $\nabla(\mathbb{H}(A,0)) = 0$. Define

$$\nabla KQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}} = K_0(\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}})$$

$$\nabla WQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}} = \nabla KQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}}/\langle\boldsymbol{\mathcal{D}}\text{-null modules in } \nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)\rangle.$$

Define

$$\nabla WQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}} = \nabla WQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{C}}}$$

and

$$\nabla KQ_0(\boldsymbol{A},\Theta) = \nabla KQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{P}(A)}$$

$$\nabla WQ_0(\boldsymbol{A},\Theta) = \nabla WQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{P}(A)}.$$

Proposition 3.6. *Let $(\boldsymbol{A},\Theta) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\Theta)$ be a parameter algebra with positioning data. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$ such that $\nabla(\mathbb{H}(A,0)) = 0$. Then ∇ vanishes on all $\boldsymbol{\mathcal{D}}$-null modules in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$ and the canonical homomorphism*

$$\mathrm{Coker}[\mathbb{H} : K_0(\boldsymbol{\mathcal{D}}) \to \nabla KQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{C}}}] \xrightarrow{\cong} \nabla WQ_0(\boldsymbol{A},\Theta)_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}}$$

is an isomorphism.

Proof. The assertions follow directly from Lemmas 2.41, 3.2 and 3.4. □

In order to define surgery obstruction groups, we shall need a refinement of the situation above. For this, we adopt the following notation and assumption.

Assumption 3.7. We shall assume that $(\boldsymbol{A},\widetilde{\Theta}) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\widetilde{\Theta})$ is a parameter algebra with positioning data such that $\widetilde{\Theta}$ has a free action of the multiplicative group $\{\pm 1\}$ of two elements, which commutes with the action of G. (Hence $\widetilde{\Theta}$ is regarded as a $G\times\{\pm 1\}$-set.) Let

$$(\boldsymbol{A}_2,\Theta_2) = (A_2,(-\lambda),\Gamma_2,G_2,A_{2,s},\Lambda_2,\Theta_2)$$

be a parameter algebra with positioning data where $A_2 = A/2A, (-,\lambda)$ denotes the antistructure on A_2 induced by that on A, $\Gamma_2 = \mathrm{Image}[\Gamma\to\Lambda_2]$, $G_2 = \mathrm{Image}[\mathrm{G}\to\mathrm{A}_2]$, $A_{2,s} = \mathrm{Image}[\mathrm{A_s}\to\mathrm{A}_2]$, $\Lambda_2 = \mathrm{Image}[\Lambda\to\mathrm{A}_2]$, and Θ_2 is a G_2-set. We shall assume that the pair $(\widetilde{\Theta},\Theta_2)$ is equipped with an equivariant map

$$p:\widetilde{\Theta}\to\Theta_2,$$

namely $p((g,\epsilon)x) = [g]p(x)$ for $g\in G$, $\epsilon\in\{\pm 1\}$ and $x\in\widetilde{\Theta}$, where $[g]\in A_2$ is the image of g. (The action of $\{\pm 1\}$ on $\widetilde{\Theta}$ and Θ_2 is introduced in order to handle orientation considerations in the next section.) If M is an A-module, let $M_2 = M/2M$. If (M,B,q) is a doubly parametrized module over $(A,(-,\lambda),\Gamma,G,A_s,\Lambda)$, let (M_2,B_2,q_2) be the induced doubly parametrized module over $(A_2,(-,\lambda),\Gamma_2,A_{2,s},\, G_2,\Lambda_2)$.

Let $(\boldsymbol{A},\widetilde{\Theta}) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,\widetilde{\Theta})$ be a parameter algebra with positioning data. Let

$$\widetilde{\boldsymbol{Q}}(\boldsymbol{A},\widetilde{\Theta})$$

denote the full subcategory of $\boldsymbol{Q}(\boldsymbol{A},\widetilde{\Theta})$ of all objects (M,B,q,θ) such that θ is $\{\pm\}$-equivariant, i.e. $\theta(\epsilon x) = \epsilon\theta(x)$ for all $\epsilon\in\{\pm 1\}$ and $x\in\Theta$. Then $\widetilde{\boldsymbol{Q}}(\boldsymbol{A},\widetilde{\Theta})$ is closed under orthogonal sums and orthogonal summands and in particular, is a symmetric monoidal subcategory of $\boldsymbol{Q}(\boldsymbol{A},\widetilde{\Theta})$. Finally, if ∇ is an invariant on $\boldsymbol{Q}(\boldsymbol{A},\widetilde{\Theta})$ then ∇ is defined on $\widetilde{\boldsymbol{Q}}(\boldsymbol{A},\widetilde{\Theta})$ and so the symmetric monoidal subcategory

$$\nabla\widetilde{\boldsymbol{Q}}(\boldsymbol{A},\widetilde{\Theta})$$

of $\widetilde{\boldsymbol{Q}}(\boldsymbol{A},\widetilde{\Theta})$ is defined by Definition 3.1 and closed under orthogonal summands by Lemma 3.2. If Θ is a G-set and $\widetilde{\Theta} = \Theta\times\{\pm 1\}$ then

$$\widetilde{\boldsymbol{Q}}(\widetilde{\boldsymbol{A}},\widetilde{\Theta}) = \boldsymbol{Q}(\boldsymbol{A},\Theta).$$

Let $\boldsymbol{A} = (A,(-,\lambda),\Gamma,G,A_s,\Lambda)$ be a parameter algebra and

$$\boldsymbol{\Theta} = (\widetilde{\Theta},p,\Theta_2).$$

Let

$$\boldsymbol{Q}(\boldsymbol{A},\boldsymbol{\Theta})$$

denote the symmetric monoidal category whose objects are all quintuples $(M, B, q, \theta, \theta_2)$ such that $(M, B, q, \theta) \in \widetilde{\boldsymbol{\mathcal{Q}}}(\boldsymbol{A}, \widetilde{\Theta})$, $(M_2, B_2, q_2, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}_2, \Theta_2)$, and the diagram

$$\begin{tikzcd} \widetilde{\Theta} \arrow[r, "\theta"] \arrow[d, "p"] & M \arrow[d] \\ \Theta_2 \arrow[r, "\theta_2"'] & M_2 \end{tikzcd}$$

commutes. A *morphism* $\alpha : (M, B, q, \theta, \theta_2) \to (M', B', q', \theta', \theta_2')$ is an A-linear map $\alpha : M \to M'$ which defines a morphism $(M, B, q, \theta) \to (M', B', q', \theta')$ in $\widetilde{\boldsymbol{\mathcal{Q}}}(\boldsymbol{A}, \widetilde{\Theta})$ and makes the diagram

$$\begin{tikzcd} \Theta_2 \arrow[r, "\theta_2"] \arrow[dr, "\theta_2'"'] & M_2 \arrow[d, "\alpha_2"] \\ & M_2' \end{tikzcd}$$

commute where α_2 is the A_2-linear map $M_2 \to M_2'$ induced by α. The *orthogonal sum* of objects is defined in the obvious way, namely

$$(M, B, q, \theta, \theta_2) \oplus (M', B', q', \theta', \theta_2') = (M \oplus M', B \oplus B', q \oplus q', \theta \oplus \theta', \theta_2 \oplus \theta_2').$$

Furthermore, if ∇ is an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ then the symmetric monoidal subcategory

$$\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$$

of $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ is defined in Definition 3.1 and is closed under orthogonal summands by Lemma 3.2.

If the equivariant map $p : \widetilde{\Theta} \to \Theta_2$ is surjective, the forgetful functor

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}) \to \widetilde{\boldsymbol{\mathcal{Q}}}(\boldsymbol{A}, \widetilde{\Theta})$$

is an isomorphism of symmetric monoidal categories.

Our next goal is to prove the analogs of Lemmas 3.3, 3.4 and Proposition 3.6, after replacing $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ by $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. This requires extending the language we have for the category $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ to the category $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$.

Let $\boldsymbol{\mathcal{D}} \subsetneqq \boldsymbol{\mathcal{C}}$ be the symmetric monoidal subcategories of $\boldsymbol{P}(A)$ appearing in Definition 2.43.

Definition 3.8. Let $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. An A-submodule $L \subsetneqq M$ is called *totally isotropic* if q and B are trivial on L (and so B_2 and q_2 are trivial on $\mathrm{Image[L \to M_2]}$). The positioning function θ (resp. θ_2) is called *totally isotropic* if $B(\theta(t), \theta(t')) = 0$ and $q(\theta(t)) = 0$ for all $t, t' \in \widetilde{\Theta}$ (resp. $B_2(\theta_2(u), \theta_2(u')) = 0$ and $q(\theta_2(u)) = 0$ for all $u, u' \in \Theta_2$). A totally isotropic, A-direct summand $L \subsetneqq M$ such that $\mathrm{Image}(\theta) \subsetneqq \mathrm{L}$ and $\mathrm{Image}(\theta_2) \subsetneqq \mathrm{Image[L \to M_2]}$ is called a *sublagrangian*. If, in addition, $L \in \boldsymbol{\mathcal{C}}$ then L is called a *$\boldsymbol{\mathcal{C}}$-sublagrangian*. A sublagrangian (resp. $\boldsymbol{\mathcal{C}}$-sublagrangian) L such that $L = L^\perp := \{m \in M \mid B(m, \ell) = 0 \ \forall\ \ell \in L\}$ is called a *Lagrangian* (resp. *$\boldsymbol{\mathcal{C}}$-Lagrangian*). $(M, B, q, \theta, \theta_2)$

is called a *null module* (resp. $\boldsymbol{\mathcal{C}}$-*null module*), if it contains a Lagrangian (resp. $\boldsymbol{\mathcal{C}}$-Lagrangian). $(M, B, q, \theta, \theta_2)$ is called *hyperbolic*, if $(M, B, q, \theta) = \mathbb{H}(P, \theta)$ and $\mathrm{Image}(\theta_2) \subseteqq \mathrm{P}_2$; in this case, we write $(M, B, q, \theta, \theta_2) = \mathbb{H}(P, \theta, \theta_2)$. $(M, B, q, \theta, \theta_2)$ is called *metabolic*, if $(M, B, q, \theta) = \mathbb{M}(P, b_0, q_0, \theta)$ and $\mathrm{Image}(\theta_2) \subseteqq \mathrm{P}_2$; in this case, we write $(M, B, q, \theta, \theta_2) = \mathbb{M}(P, b_0, q_0, \theta, \theta_2)$. Objects of the kind $\mathbb{H}(A, \theta, \theta_2)$ (resp. $\mathbb{M}(A, b_0, q_0, \theta, \theta_2)$) are called *hyperbolic* (resp. *metabolic*) *planes*.

Definition 3.9. Let $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. An ordered pair (e, f) of elements of M is called a *metabolic pair* if it is a metabolic pair in the sense of Definition 2.39 for (M, B, q, θ) and if $B_2(\theta_2(y), e) = 0$ for all $y \in \Theta_2$. If a metabolic pair (e, f) generates M as an A-module then it is called a *metabolic basis* for $(M, B, q, \theta, \theta_2)$.

Proposition 3.10. *The analog of anyone of Lemmas 2.35, 2.37, 2.40, 2.41 and Corollaries 2.36, 2.38 is valid when $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ is replaced by $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$.*

The proofs are analogous to those of the original results. There are no pitfalls. Details are left to the reader.

Lemma 3.11. *Let $\boldsymbol{\mathcal{M}}$ denote the full subcategory of $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ generated under orthogonal sum by all metabolic planes. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. Suppose that for each object $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$, $\nabla(M, -B, -q, \theta, \theta_2) = -\nabla(M, B, q, \theta, \theta_2)$ and $\nabla(\mathbb{H}(A, 0, 0)) = 0$. Then $\nabla\boldsymbol{\mathcal{M}}$ is cofinal in $\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$.*

Proof. Thanks to Proposition 3.10, the proof is the same as that of Lemma 3.3. □

Lemma 3.12. *Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. If ∇ vanishes on the metabolic module $\mathbb{M}(P, b_0, q_0, \theta, \theta_2)$ and on the hyperbolic module $\mathbb{H}(P, 0, 0)$ then ∇ vanishes also on $\mathbb{M}(P, -b_0, -q_0, -\theta, -\theta_2)$ and*

$$\mathbb{M}(P, b_0, q_0, \theta, \theta_2) \oplus \mathbb{H}(P, 0, 0) \cong \mathbb{M}(P, b_0, q_0, \theta, \theta_2) \oplus \mathbb{M}(P, -b_0, -q_0, -\theta, -\theta_2)$$

under the isomorphism in Lemma 2.41.

Proof. Thanks to Proposition 3.10, the proof is the same as that of Lemma 3.4. □

Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. Let

$$\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}}$$

denote the full, symmetric monoidal subcategory of $\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ such that the underlying A-module of each object lies in $\boldsymbol{\mathcal{C}}$. Let

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}} = T\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}}$$

where T denotes a trivial invariant.

Definition 3.13. Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ such that $\nabla(\mathbb{H}(A, 0, 0)) = 0$. Define

$$\nabla KQ_0(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}} = K_0(\nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}})$$

$$\nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}} = \nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}}/\langle \boldsymbol{\mathcal{D}}\text{-null modules in } \nabla\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta})\rangle.$$

Define

$$\nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}} = \nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{C}}}$$

and

$$\nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta}) = \nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{P}(A)}$$

$$\nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta}) = \nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{P}(A)}.$$

The next proposition is a generalization of Proposition 3.6.

Proposition 3.14. *Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta})$ such that $\nabla(\mathbb{H}(A,0,0)) = 0$ where $P \in \boldsymbol{\mathcal{D}}, \theta = 0$, and $\theta_2 = 0$. Then ∇ vanishes on all $\boldsymbol{\mathcal{D}}$-null modules in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta})$ and the canonical homomorphism*

$$\mathrm{Coker}[\mathbb{H} : K_0(\boldsymbol{\mathcal{D}}) \to \nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}}] \longrightarrow \nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}}$$

is an isomorphism.

Proof. The assertions follow directly from Lemmas 3.2 and 3.12. □

Corollary 3.15. *Let ∇ be an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta})$ such that for any $(M,B,q,\theta,\theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta})$, $\nabla(M,B,q,\theta,\theta_2) = -\nabla(M,-B,-q,\theta,\theta_2)$ and $\nabla(\mathbb{H}(A,0,0)) = 0$. Let $\boldsymbol{\mathcal{B}}$ denote a subcategory of $\boldsymbol{P}(A)$, which contains A, is closed under isomorphisms, direct sums, and dualization, and $\boldsymbol{\mathcal{C}} \subsetneqq \boldsymbol{\mathcal{B}}$. Then the canonical homomorphisms*

$$\nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}} \longrightarrow \nabla KQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{B}}} \quad \textit{and}$$
$$\nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{C}}} \longrightarrow \nabla WQ_0(\boldsymbol{A},\boldsymbol{\Theta})_{\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{B}}}$$

are injective.

Proof. The injectivity of the first homomorphism follows from Lemma 3.11 and of the second from Lemma 3.11 and Proposition 3.14. □

The rest of this section will be concerned with certain invariants which arise geometrically. A prototype for these invariants is given in the following lemma.

Lemma 3.16. *Let $(\boldsymbol{A},\Theta) = (A,(-,\lambda),\Gamma,G,A_s,\Lambda,,\Theta)$ be a parameter algebra with positioning data. Let $\Gamma' \subsetneqq \Gamma$ be a $(-,\lambda)$-symmetric parameter on A. Give Γ/Γ' the A-module structure defined by $a[\gamma] = [a\gamma\overline{a}]$ for all $a \in A$ and $\gamma \in \Gamma$. For each object $(M,B,q,\theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$, let*

$$\alpha_{\Gamma'}(M,B,q,\theta) = \mathrm{Hom}_A(M,\Gamma/\Gamma') \quad \textit{and}$$
$$\nabla_{\Gamma'}(M,B,q,\theta) : M \longrightarrow \Gamma/\Gamma'; \quad m \longmapsto [B(m,m)].$$

Then the pair $(\nabla_{\Gamma'},\alpha_{\Gamma'})$ defines an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\Theta)$.

The proof is straightforward.

Geometric invariants are obtained by modifying the basic idea above and taking into account the positioning functions. To simplify details, we shall restrict to the case A is a group ring. We make for the rest of this section, the following assumption concerning our setup.

Assumption 3.17. Let G denote a group, $w : G \to \{\pm 1\}$ a group homomorphism (which will be referred to as an *orientation homomorphism*), R a commutative ring with 1 (unity) and with involution $r \mapsto \overline{r}$, and A the group ring $R[G]$ with compatible involution $a \mapsto \overline{a}$ such that $\overline{g} = w(g)g^{-1}$ for all $g \in G$. Let

$$\varepsilon : A \to R; \ \sum_{g \in G} r_g g \mapsto r_1$$

denote the projection to the coefficient of $1 \in G$. Let

$$G(2) = \{g \in G |\ g^2 = 1, g \neq 1\}.$$

Let S denote a subset of $G(2) \cup \{1\}$ such that S is closed under conjugation by elements of G. Let

$$A_s = R[S] \quad \left(:= \left\{ \sum_{g \in S} r_g g \ \middle|\ r_g \in R \ (r_g = 0 \text{ except for finitely many } g\text{'s}) \right\} \right).$$

Let $\lambda \in R$ such that $\lambda\overline{\lambda} = 1$, Λ a $(-,\lambda)$-form parameter on A, Γ a $(-,\lambda)$-symmetric parameter on A, Θ a finite G-set, and $(\boldsymbol{A}, \Theta) = (A, (-,\lambda), \Gamma, G, A_s, A_s + \Lambda, \Theta)$. Let

$$\Gamma' \subseteqq \Gamma$$

denote a $(-,\lambda)$-symmetric parameter on A and

$$\Gamma'_g = \varepsilon(\Gamma' g^{-1}) \quad \text{for } g \in G.$$

Clearly, if $g \in G(2) \cup \{1\}$ then Γ'_g is a $(-, \lambda w(g))$-symmetric parameter on R and if $g \notin G(2) \cup \{1\}$ then $\Gamma'_g = R$. Let $\mathcal{S}(G)$ denote the G-set of all subgroups of G with G-action given by conjugation, i.e.

$$G \times \mathcal{S}(G) \to \mathcal{S}(G); \ (g, H) \mapsto gHg^{-1}.$$

Let

$$\rho : \Theta \to \mathcal{S}(G)$$

denote a G-equivariant map such that for each $s \in S$, the set

$$\Theta|_s = \{t \in \Theta \mid \rho(t) \ni s\}$$

is finite.

Definition 3.18. If $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and $s \in S$, let

$$\begin{aligned}
&\Sigma^{\theta}_{\rho,s} = \sum_{x \in \Theta|_s} \theta(x) \\
&\alpha_{\Gamma',s}(M, B, q, \theta) = \mathrm{Hom}_{\mathbb{Z}}(M, R/\Gamma'_s) \\
&\nabla_{\Gamma',s}(M, B, q, \theta) : M \to R/\Gamma'_s; \quad m \longmapsto w(s)[\varepsilon\left(B(sm, \Sigma^{\theta}_{\rho,s} - m)\right)] \\
&\nabla_{\Gamma'} = \bigoplus_{s \in S} \nabla_{\Gamma',s} \quad \text{and} \quad \alpha_{\Gamma'} = \bigoplus_{s \in S} \alpha_{\Gamma',s}.
\end{aligned}$$

Lemma 3.19. *Each pair $(\nabla_{\Gamma',s}, \alpha_{\Gamma',s})$ $(s \in S)$ defines an invariant in the sense of Definition 3.1 on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. Thus, the pair $(\nabla'_{\Gamma}, \alpha_{\Gamma'})$ defines an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. Moreover, if $g \in G$ then $\Gamma'_s = \Gamma'_{gsg^{-1}}$, $\alpha_{\Gamma',s} = \alpha_{\Gamma',gsg^{-1}}$, and*

$$\nabla_{\Gamma',s}(M, B, q, \theta)(m) = w(g)\ \nabla_{\Gamma',gsg^{-1}}(M, B, q, \theta)(gm)$$

for any $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and $m \in M$.

Proof. Fix $(M, B, q, \theta) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and set $f_s = \nabla_{\Gamma',s}(M, B, q, \theta)$. First, we show that f_s is an additive map, which immediately implies $f_s \in \alpha_{\Gamma',s}(M, B, q, \theta)$. It then follows routinely that $\nabla_{\Gamma',s}$ is an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ and hence, $\nabla_{\Gamma'}$ is an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$.

Let $b = \varepsilon \circ B$. Let $m, m' \in M$. Then

$$\begin{aligned}
&f_s(m + m') - f_s(m) - f_s(m') \\
&= w(s)[b(s(m + m'), \Sigma^{\theta}_{\rho,s} - (m + m')) - b(sm, \Sigma^{\theta}_{\rho,s} - m) - b(sm', \Sigma^{\theta}_{\rho,s} - m')] \\
&= w(s)[b(s(m + m'), -(m + m')) - b(sm, -m) - b(sm', -m')] \quad \text{(biadditivity of } b\text{)} \\
&= w(s)[b(sm', -m) + b(sm, -m')] \quad \text{(biadditivity of } b\text{)} \\
&= w(s)[b(sm', -m) + \lambda\overline{b(-m', sm)}] \quad (\lambda\text{-Hermitian property of } b) \\
&= w(s)[b(sm', -m) + \lambda w(s)\overline{b(-sm', m)}] \\
&= w(s)[b(sm', -m) + \lambda w(s)\overline{b(sm', -m)}] = 0.
\end{aligned}$$

It is clear $\Gamma'_g = \Gamma'_{gsg^{-1}}$. Thus, $\alpha_{\Gamma',s} = \alpha_{\Gamma',gsg^{-1}}$. To complete the proof of the lemma, it suffices to show that for each $m \in M$, $f_{gsg^{-1}}(gm) = w(g)f_s(m)$. But,

$$\begin{aligned}
f_{gsg^{-1}}(gm) &= w(gsg^{-1})[b((gsg^{-1})gm, \Sigma^{\theta}_{\rho,gsg^{-1}} - gm)] \\
&= w(s)[b(gsm, g\Sigma^{\theta}_{\rho,s} - gm)] \quad (\text{because } \Sigma^{\theta}_{\rho,gsg^{-1}} = g\Sigma^{\theta}_{\rho,s}) \\
&= w(g)w(s)[b(sm, \Sigma^{\theta}_{\rho,s} - m)] \\
&= w(g)f_s(m).
\end{aligned}$$

□

Lemma 3.20. *The invariant $\nabla_{\Gamma'}$ on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ is trivial on modules (M, B, q, θ) such that B is Γ'-Hermitian (i.e. $B(m, m) \in \Gamma'$ for all $m \in M$) and $\theta = 0$. In particular, $\nabla_{\Gamma'}$ vanishes on all hyperbolic modules $\mathbb{H}(P, \theta)$ such that $\theta = 0$.*

If suffices to prove the assertions for the invariants $\nabla_{\Gamma',s}$ $(s \in S)$. This follows from a straightforward computation which is left to the reader.

Lemma 3.21. *Let $M(\Theta, A)^G$ denote the set of all G-equivariant maps $\Theta \to A$. Let $M(\Theta, \Theta)_G$ denote the set of all maps $f : \Theta \to R$ such that for each $x \in \Theta$, there exists a finite subset $L \subseteqq G$ satisfying $f(gx) = 0$ for all $g \in G \smallsetminus L$. Then the maps*

$$\mathrm{tr} : M(\Theta, A)^G \to M(\Theta, R)_G;\ \mathrm{tr}(\theta) = \varepsilon \circ \theta$$
$$\mathrm{sr} : M(\Theta, R)_G \to M(\Theta, A)^G;\ \mathrm{sr}(c)(x) = \sum_{g \in G} c(g^{-1}x)g \quad (x \in \Theta)$$

are mutually inverse isomorphisms.

Let $M(\widetilde{\Theta}, A)^{G\times\{\pm1\}}$ denote the set of all $G \times \{\pm1\}$-equivariant maps $\widetilde{\Theta} \to A$. Let $M(\Theta, R)_G^{\{\pm1\}}$ denote the set of all $\{\pm1\}$-equivariant maps $f : \widetilde{\Theta} \to R$ such that for each $x \in \widetilde{\Theta}$, there exists a finite subset $L \subseteqq G$ satisfying $f(gx) = 0$ for all $g \in G \smallsetminus L$. Then the maps

$$\mathrm{tr} : M(\Theta, A)^{G\times\{\pm1\}} \to M(\Theta, R)_G^{\{\pm1\}};\ \mathrm{tr}(\theta) = \varepsilon \circ \theta$$
$$\mathrm{sr} : M(\Theta, R)_G^{\{\pm1\}} \to M(\Theta, A)^{G\times\{\pm1\}};\ \mathrm{sr}(c)(x) = \sum_{g \in G} c(g^{-1}x)g \quad (x \in \Theta)$$

are mutually inverse isomorphisms.

This is proven by a straightforward computation.

Lemma 3.22. *Let $\mathbb{M}(A, b_0, q_0, \theta)$ be a metabolic plane in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$. Let $B = B_{A,b_0}$ denote the Hermitian form on $\mathbb{M}(A, b_0, q_0, \theta)$. Then $\mathbb{M}(A, b_0, q_0, \theta)$ has a metabolic basis (e, f) such that if $B(f, f) = \sum_{g\in G} r_g g$ $(r_g \in R)$ then $r_g = 0$ for all $g \notin S$.*

Proof. Let (e, f) be a metabolic basis. By Assumption 3.17 and (2.6.3), we know that $\sum_{g\notin S} r_g g = a + \lambda\overline{a}$ for some $a \in A$. Thus, by (2.40.2), $(e, f - ae)$ is a metabolic basis with the desired property. □

Lemma 3.23. *Suppose that the involution on R is trivial and that for each $r \in R, r^2 = r$, e.g. $R = \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{M}(A, b_0, q_0, \theta)$ be a metabolic plane in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \Theta)$ with metabolic basis (e, f). Let $B = B_{A,b_0}$ denote the Hermitian form on $\mathbb{M}(A, b_0, q_0, \theta)$. Let*

$$B(f, f) = \sum_{g\in G} r_g g$$

$(r_g \in R)$. Then $\nabla_{\Gamma'}(\mathbb{M}(A, b_0, q_0, \theta)) = 0$ if and only if for each $s \in S$,

$$\sum_{x\in\Theta|_s} \mathrm{tr}(\theta)(x) \equiv r_s \mod \Gamma'_s.$$

If additionally $\Gamma' = \min_{-\lambda}(A)$ then there is a metabolic basis (e, f) such that $r_g = 0$ for all $g \notin S$ and

$$r_s = \sum_{x\in\Theta|_s} \mathrm{tr}(\theta)(x)$$

for all $s \in S$*, where* $\min_{-\lambda}(A) = \{a + \lambda\overline{a} \mid a \in A\}$.

Proof. Let $\nabla = \nabla_{\Gamma'}(\mathbb{M}(A, b_0, q_0, \theta))$ and $\nabla_s = \nabla_{\Gamma',s}(\mathbb{M}(A, b_0, q_0, \theta))$ for each $s \in S$. By definition, $\nabla = 0 \iff$ for each $s \in S$, $\nabla_s = 0$. Since $\nabla_s : Ae \oplus Af \longrightarrow R/\Gamma'_s$ is an additive function by Lemma 3.19 and ∇_s vanishes on Ae, it follows that $\nabla_s = 0 \iff$ for each $\gamma \in R$ and $g \in G, \nabla_s(\gamma g f) = 0$. But, by Lemma 3.19, $\nabla_s(\gamma g f) = w(g)\nabla_{g^{-1}sg}(\gamma f)$. Thus, $\nabla = 0 \iff$ for each $\gamma \in R$ and $s \in S, \nabla_s(\gamma f) = 0$. But

$$\begin{aligned}
\nabla_s(\gamma f) &= w(s)[\varepsilon\left(B(s\gamma f, \Sigma^\theta_{\rho,s} - \gamma f)\right)] \\
&= w(s)[\varepsilon\left(B(s\gamma f, \Sigma^\theta_{\rho,s}) - B(s\gamma f, \gamma f)\right)] \quad \text{(using bilinearity of } B\text{)} \\
&= w(s)\left[\varepsilon\left(B(s\gamma f, \sum_{x\in\Theta|_s}\theta(x)) - B(s\gamma f, \gamma f)\right)\right] \\
&= w(s)\left[\varepsilon\left(B(s\gamma f, \sum_{x\in\Theta|_s}\sum_{g\in G}\mathrm{tr}(\theta)(g^{-1}x)ge) - \gamma B(f,f)\overline{\gamma s}\right)\right] \\
&= w(s)\left[\varepsilon\left(\sum_{x\in\Theta|_s}\mathrm{tr}(\theta)(s^{-1}x)s\overline{s\gamma}\right) - \gamma r_s\overline{\gamma}s\overline{s}\right] \\
&= w(s)\left[\varepsilon\left(\sum_{x\in\Theta|_s}\mathrm{tr}(\theta)(s^{-1}x)\overline{\gamma}w(s)\right) - \gamma\overline{\gamma}r_s w(s)\right] \\
&= \left[\gamma\left(\sum_{x\in\Theta|_s}\mathrm{tr}(\theta)(s^{-1}x) - r_s\right)\right].
\end{aligned}$$

The first assertion in the lemma follows.

Let (e, f) be a metabolic basis for $\mathbb{M}(A, b_0, q_0, \theta)$. By Lemma 3.21, we can assume $r_g = 0$ for all $g \notin S$. Since $\Gamma' = \min_{-\lambda}(A)$, we can find an element $a = \sum_{s\in S} r'_s s \in A$ such that for each $s \in S$,

$$\sum_{x\in\Theta|_s}\mathrm{tr}(\theta)(x) - r_s = a + \lambda\overline{a}.$$

But, then by (2.40.2) the metabolic basis $(e, f + ae)$ has the desired properties. □

We extend our list of assumptions under Assumptions 3.7 and 3.17 concerning the setup to include the following

Assumption 3.24. Let $R_2 = R/2R$ and $A_2 = A/2A$. Let $\varepsilon_2 : A_2 \to R_2$; $\sum_{g\in G} r_g g \mapsto r_1$ denote the projection to the coefficient of $1 \in G$. Let $\boldsymbol{\Theta} = (\widetilde{\Theta}, p, \Theta_2)$. Let $\rho : \widetilde{\Theta} \to \mathcal{S}(G)$ and $\rho_2 : \Theta_2 \to \mathcal{S}(G)$ denote G-equivariant functions such that $\rho((-1)x) = \rho(x)$ for all

$x \in \widetilde{\Theta}$ and the diagram

$$\begin{tikzcd} \widetilde{\Theta} \arrow[r, "\rho"] \arrow[d, "p"'] & \mathcal{S}(G) \\ \Theta_2 \arrow[ur, "\rho_2"'] & \end{tikzcd}$$

commutes. We shall assume that $\Theta_2|_s = \{x \in \Theta_2 \mid \rho(x) \ni s\}$ is a finite set for every $s \in S$. Let Γ'_2 denote a $(-,\lambda)$-symmetric parameter on A_2 such that $\Gamma'_2 \subsetneqq \mathrm{Image}[\Gamma \to \mathrm{A}_2]$ and $\Gamma'_{2,g} = \varepsilon_2(\Gamma'_2 g^{-1})$, for $g \in G$. If $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ and $g \in G$, let

$$\Sigma^{\theta_2}_{\rho_2,g} = \sum_{x \in \Theta_2|_g} \theta_2(x).$$

If $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ and $s \in S$, let $\alpha_{\Gamma'_{2,s}}(M, B, q, \theta, \theta_2) = \mathrm{Hom}_{\mathbb{Z}}\,(M_2, R_2/\Gamma'_{2,s})$ and

$$\nabla_{\Gamma'_{2,s}}(M, B, q, \theta, \theta_2) : M_2 \longrightarrow R_2/\Gamma'_{2,s}; \quad m \longmapsto [\varepsilon_2\left(B_2(sm, \Sigma^{\theta_2}_{\rho_2,s} - m)\right]\right).$$

(We remark that $w(s) = 1$ in R_2.) Set $\nabla_{\Gamma'_2} = \bigoplus_{s \in S} \nabla_{\Gamma'_{2,s}}$ and $\alpha_{\Gamma'_2} = \bigoplus_{s \in S} \alpha_{\Gamma'_{2,s}}$.

Lemma 3.25. *Each pair $(\nabla_{\Gamma_{2,s}}, \alpha_{\Gamma_{2,s}})$ $(s \in S)$ defines an invariant in the sense of Definition 3.1 on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}_2, \Theta_2)$. Thus, the pair $(\nabla_{\Gamma'_2}, \alpha_{\Gamma'_2})$ defines an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$. Moreover, if $g \in G$ then $\Gamma'_{2,s} = \Gamma'_{2,gsg^{-1}}$, $\alpha_{\Gamma'_{2,s}} = \alpha_{\Gamma'_{2,gsg^{-1}}}$, and for each $m \in M_2$ (where $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$), $\nabla_{\Gamma'_{2,s}}(M, B, q, \theta, \theta_2)(m) = w(g)\nabla_{\Gamma'_2,gsg^{-1}}\,(M, B, q, \theta, \theta_2)(gm)$.*

Proof. By Lemma 3.19, the constructions $\nabla_{\Gamma'_{2,s}}$ and $\alpha_{\Gamma'_{2,s}}$ define an invariant on

$$\boldsymbol{\mathcal{Q}}(A_2, (-,\lambda), \Gamma_2, G, A_{2,s}, \Lambda_2, \Theta_2).$$

These constructions composed with the forgetful functor

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}) \longrightarrow \boldsymbol{\mathcal{Q}}(A_2, (-,\lambda), \Gamma_2, G, A_{2,s}, \Lambda_2, \Theta_2)$$

yield the constructions in the lemma. The assertions in the lemma follow now from Lemma 3.19. □

Lemma 3.26. *The invariant $\nabla_{\Gamma'_2}$ on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ is trivial on modules $(M, B, q, \theta, \theta_2)$ such that B_2 is Γ'_2-Hermitian (i.e. $B_2(m, m) \in \Gamma'_2$ for all $m \in M_2$) and $\theta_2 = 0$. In particular, $\nabla_{\Gamma'_2}$ vanishes on all hyperbolic modules $\mathbb{H}(P, \theta, \theta_2)$ such that $\theta_2 = 0$. Furthermore, for any module $(M, B, q, \theta, \theta_2)$, $\nabla_{\Gamma'_2}(M, B, q, \theta, \theta_2) = -\nabla_{\Gamma'_2}(M, -B, -q, \theta, \theta_2)$.*

Proof. The first assertion follows from Lemma 3.20. The second and third ones are obvious. □

Lemma 3.27. *Suppose the involution on R_2 is trivial and for each $r \in R_2, r^2 = r$, e.g. $R_2 = \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{M}(A, b_0, q_0, \theta, \theta_2)$ be a metabolic plane in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ with metabolic basis (e, f). Let $B = B_{A,b_0}$ denote the Hermitian form on $\mathbb{M}(A, b_0, q_0, \theta)$. Let*

$$B(f, f) = \sum_{g \in G} r_g g \quad (r_g \in R).$$

Then the following hold.

(3.27.1) $\nabla_{\Gamma'_2}(\mathbb{M}(A, b_0, q_0, \theta, \theta_2)) = 0$ *if and only if for each* $s \in S$,

$$r_s = \sum_{x \in \Theta_2|_s} \mathrm{tr}(\theta_2)(x) \quad \text{in } R_2/\Gamma'_{2,s}.$$

(3.27.2) *If additionally* $\Gamma'_2 = \min_{-\lambda}(A_2) := \{a + \lambda\overline{a} | \ a \in A_2\}$ *then there is a metabolic basis* (e, f) *such that for each* $g \notin S$, $r_g = 0$, *and for every* $s \in S$,

$$r_s = \sum_{x \in \Theta_2|_s} \mathrm{tr}(\theta_2)(x) \quad \text{in } R_2.$$

(3.27.3) *If furthermore* $\mathcal{R}$ *is a complete set of representatives in* R *for the set* R_2 *and the subset* $\sum_{s \in S} 2Rs$ *of* A *is included in* $\min_{-\lambda}(A)$ *then there is a metabolic basis* (e, f) *such that for each* $g \notin S$, $r_g = 0$, *and for every* $s \in S$,

$$r_s = \sum_{x \in \Theta_2|_s} \widetilde{\mathrm{tr}(\theta_2)(x)} \quad \text{in } R$$

where $\widetilde{r} \in \mathcal{R}$ *denotes the lifting of* $r \in R_2$.

The proof is the same as that of Lemma 3.23, except for a few small changes in detail. We leave the checking to the reader.

Definition 3.28. Let $\boldsymbol{\Theta} = (\widetilde{\Theta}, p, \Theta_2)$, $\rho : \widetilde{\Theta} \to \mathcal{S}(G)$, and $\rho_2 : \Theta_2 \to \mathcal{S}(G)$ be as in Assumption 3.24. Let $\boldsymbol{c} = (c, c_2)$ be a pair consisting of $\{\pm 1\}$-equivariant map $c : \widetilde{\Theta} \to R$ and a map $c_2 : \Theta_2 \to R_2$ such that

$$\begin{tikzcd} \widetilde{\Theta} \arrow[r, "c"] \arrow[d, "p"'] & R \arrow[d] \\ \Theta_2 \arrow[r, "c_2"'] & R_2 \end{tikzcd}$$

commutes. Let $\widetilde{c} : \Theta_2 \longrightarrow R$ denote any map such that the diagram

$$\begin{tikzcd} \Theta_2 \arrow[r, "\widetilde{c}"] \arrow[dr, "c_2"'] & R \arrow[d] \\ & R_2 \end{tikzcd}$$

commutes. Recall the definition of metabolic plane in the paragraph subsequent to Definition 2.32 and the definition of metabolic basis in Definition 2.39. Let $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ denote the metabolic plane $\mathbb{M}(R[G], b_{0,\widetilde{c}}, q_0, \mathrm{sr}(c), \mathrm{sr}(c_2))$ defined (up to isomorphism) by a metabolic basis (e, f) such that

$$\begin{aligned} & q_0 = 0, \\ & b_{0,\widetilde{c}}(f, f) = \sum_{s \in S} \sum_{x \in \Theta_2|_s} \widetilde{c}(x)s \quad \in R[G], \\ & \mathrm{sr}(c)(x) = \sum_{g \in G} c(g^{-1}x)ge \quad \in R[G]e \quad (x \in \widetilde{\Theta}), \quad \text{and} \\ & \mathrm{sr}(c_2)(x) = \sum_{g \in G} c_2(g^{-1}x)ge \quad \in R_2[G]e \quad (x \in \Theta_2). \end{aligned}$$

If the involution on R_2 is trivial and if $r^2 = r$ for any $r \in R_2$, then $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c}) \in \nabla_{\Gamma'_2}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ by Lemma 3.27.

Lemma 3.29. *Suppose the involution on R_2 is trivial, $r^2 = r$ for any $r \in R_2$, $\sum_{s \in S} 2Rs \subseteqq \min_{-\lambda}(R[G])$, and $\Gamma'_2 = \min_{-\lambda}(R_2[G])$. Let $(\boldsymbol{c}, \widetilde{c})$ and $(\boldsymbol{c}', \widetilde{c}')$ be as in Definition 3.28. If $\boldsymbol{c} = \boldsymbol{c}'$ then $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c}) \cong \mathbb{M}(R[G], \boldsymbol{c}', \widetilde{c}')$.*

Proof. Let (e, f) and (e', f') be the metabolic bases defining $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ and $\mathbb{M}(R[G], \boldsymbol{c}', \widetilde{c}')$, respectively, as in Definition 3.28. Let $\mathcal{R}$ denote a complete set of representatives in R for the set R_2. By (3.27.3) and the proof of Lemma 3.23, there exists a metabolic basis $(e, f + ae)$ (resp. $(e', f' + ae')$), where a (resp. a') $\in R[G]$, of $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ (resp. $\mathbb{M}(R[G], \boldsymbol{c}', \widetilde{c}')$) such that

$$r_s = \widetilde{\sum_{x \in \Theta_2|_s} c_2(x)} \qquad (\text{resp. } r'_s = \widetilde{\sum_{x \in \Theta_2|_s} c'_2(x)}) \quad \in \mathcal{R},$$

where $B(f + ae, f + ae) = \sum_{s \in S} r_s s$ and $B(f' + a'e', f' + a'e') = \sum_{s \in S} r'_s s$. Since $c_2 = c'_2$, it follows that $r_s = r'_s$. Thus $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c}) \cong \mathbb{M}(R[G], \boldsymbol{c}', \widetilde{c}')$. $\square$

Definition 3.30. Let R and Γ'_2 be as in Lemma 3.29. Set $\mathbb{M}(R[G], \boldsymbol{c}) = \mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ for some $\widetilde{c}$. By Lemma 3.29, the isomorphism class of $\mathbb{M}(R[G], \boldsymbol{c})$ is uniquely defined.

Lemma 3.31. *Let R and Γ'_2 be as in Lemma 3.29. For any invertible element $r \in R$, $\mathbb{M}(R[G], \boldsymbol{c}) \cong \mathbb{M}(R[G], r\boldsymbol{c})$, where $r\boldsymbol{c} = (rc, rc_2)$.*

Proof. The maps $\mathrm{sr}(c_2)$ and $\mathrm{sr}(rc_2)$ are defined in Lemma 3.21. Let (e, f) be a metabolic basis defining $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ as in Definition 3.28. Since $r^2 \equiv r \mod 2R$ and r is invertible, $r \equiv 1 \mod 2R$. Set $e' = r^{-1}e$ and $f' = rf$. Then $\mathrm{sr}(c)(x)e = \mathrm{sr}(rc)(x)e'$, $\mathrm{sr}(c_2)(x)e = \mathrm{sr}(rc_2)(x)e'$ and $B(f, f) \equiv B(f', f') \mod \min_{-\lambda}(R[G])$. This proves Lemma 3.31. $\square$

The next lemma classifies metabolic planes vanishing under $\nabla_{\Gamma'_2}$ and plays a key role in proving our G-surgery results.

Lemma 3.32. *Let R and Γ'_2 be as in Lemma 3.29. If $\mathbb{M}(R[G], b_0, q_0, \theta, \theta_2) \in \nabla_{\Gamma'_2}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$ is a metabolic plane then*

$$\mathbb{M}(R[G], b_0, q_0, \theta, \theta_2) \cong \mathbb{M}(R[G], \boldsymbol{c})$$

for some $\boldsymbol{c} = (c, c_2)$.

Proof. This follows immediately from Lemma 3.27. $\square$

To close this section, we use the notation developed above to define G-surgery obstruction groups. They will play an important role in the rest of the paper.

Definition 3.33. We make even more precise now, the setup described under Assumptions 3.7, 3.17 and 3.24. Let

$$\begin{aligned}
&\lambda = (-1)^k \\
&G(2)_\lambda = \{g \in G(2) \mid w(g) = -\lambda\} \\
&Q = \text{ a subset of } G(2)_\lambda \cup \{1\} \smallsetminus S, \text{ which is closed under conjugation} \\
&\qquad \text{by elements of } G \\
&\Lambda(Q) = \text{the } (-, (-1)^k)\text{-form parameter on } A \text{ generated by } Q \\
&\Gamma(S) = \text{the } (-, (-1)^k)\text{-symmetric parameter on } A \text{ generated by } S \smallsetminus G(2)_\lambda \\
&\Gamma_2' = \text{the minimum symmetric parameter } \min_{(-1)^k}(A_2) \text{ on } A_2 \\
&\boldsymbol{\mathcal{F}}(A) = \text{the full, symmetric monoidal subcategory of } \boldsymbol{\mathcal{P}}(A) \text{ consisting of all} \\
&\qquad \text{finitely generated, free } A\text{-modules.}
\end{aligned}$$

Define the *G-equivariant surgery obstruction groups*

$$\begin{aligned}
W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}} &= \nabla_{\Gamma_2'} WQ_0(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{F}}(A), \boldsymbol{\mathcal{C}}} \\
W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\mathrm{proj}} &= W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{P}}(A)} \\
W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\mathrm{free}} &= W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{F}}(A)}.
\end{aligned}$$

Lemma 3.34. *Let $\boldsymbol{\mathcal{B}}$ be a subcategory of $\boldsymbol{\mathcal{P}}(A)$, which contains A, is closed under isomorphisms, direct sums, and dualization, and $\boldsymbol{\mathcal{C}} \subseteqq \boldsymbol{\mathcal{B}}$. Then the canonical homomorphism*

$$W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{C}}} \longrightarrow W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{B}}}$$

is injective.

Proof. The assertion follows immediately from Corollary 3.15 and Lemma 3.26. $\square$

Remark 3.35. Let $\boldsymbol{\mathcal{S}}(A)$ denote the full, symmetric monoidal subcategory of $\boldsymbol{\mathcal{P}}(A)$ consisting of all finitely generated, stably free A-modules. Then the canonical maps

$$\begin{aligned}
&\nabla_{\Gamma_2'} WQ_0(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{F}}(A), \boldsymbol{\mathcal{C}}} \longrightarrow \nabla_{\Gamma_2'} WQ_0(\boldsymbol{A}, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{S}}(A), \boldsymbol{\mathcal{C}}}, \quad \text{and} \\
&W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{F}}(A)} \longrightarrow W_{2k}(R, G, Q, S, \boldsymbol{\Theta})_{\boldsymbol{\mathcal{S}}(A)}
\end{aligned}$$

are isomorphisms. Henceforth we shall identify groups related by a canonical isomorphism above.

4. Geometric intersection theory

For the rest of the paper, let G denote a finite group, k an integer $\geqq 3$, and $n = 2k$.

Unless specifically mentioned otherwise, the term *submanifold* means smooth neat submanifold (cf. [15]). Similarly the term *embedding* (resp. *immersion*) means smooth neat embedding (resp. immersion). Let Z be a compact, connected, simply connected, n-dimensional, oriented, smooth G-manifold such that

$$\dim Z^g \leqq k \quad \text{for all } g \in G \smallsetminus \{1\} .$$

On the G-manifold Z, G-surgery of dimension k is the process of equivariantly replacing embedded G-handles $(G \times S^k \times D^k)_\alpha$, $\alpha \in \mathcal{A}$, by handles $(G \times D^{k+1} \times S^{k-1})_\alpha$ where the latter are obtained from the former by filling in $\{g\} \times S^k \times D^k$, $g \in G$, to get $\{g\} \times D^{k+1} \times D^k$ and emptying out $\{g\} \times D^{k+1} \times \mathrm{Interior}(D^k)$ to get $\{g\} \times D^{k+1} \times S^{k-1}$. However, one can make this replacement simultaneously for all embedded G-handles only when all handles are mutually disjoint. To get the resulting space as a manifold, the G-handles $(G \times S^k \times D^k)_\alpha$ must be disjoint from the *singular set* of Z:

$$\begin{aligned}\mathrm{Sing}(G, Z) &= \bigcup_{g \in G \smallsetminus \{1\}} Z^g \\ &= \bigcup_{H \in \mathcal{S}(G) \smallsetminus \{\{1\}\}} \left(\bigcup_{\gamma \in \pi_0(Z^H)} Z_\gamma \right),\end{aligned}$$

where Z_γ is the underlying space of the connected component γ. Let us suppose this is the case. Let $h_\alpha : S^k = \{1\} \times S^k \times \{0\} \to Z$, $\alpha \in \mathcal{A}$, and $h_\gamma : Z_\gamma \to Z$, $\gamma \in \pi_0(Z^H)$ with $H \neq \{1\}$, denote the canonical inclusions.

By performing G-surgery on the G-handles above, we kill the elements $h_{\alpha *}[S^k]$ of $H_k(Z; \mathbb{Z})$. However the problem we shall be facing starts not with the h_α's above, but with a set of elements a_α in $\mathrm{Image}[\pi_k(Z) \to H_k(Z; \mathbb{Z})]$. We are asked to realize the a_α's by embeddings $h_\alpha : S^k \to Z$ which extend to G-handles $(G \times S^k \times D^k)_\alpha$ that are mutually disjoint from one another and from the singular set. On the other hand, we know only that each a_α can be realized by an immersion h_α (cf. [15]). Getting the conditions above satisfied will rest on establishing a G-equivariant geometric self-intersection form which is defined on all immersions. Establishing such a form and showing that it does the job are the main goals of this section.

In applications outside the current paper, we shall need that the G-handles above are disjoint not only from the singular set, but also from possibly larger G-subsets L. We describe these G-subsets next.

Assumption 4.1. Let L be a G-subcomplex of a G-equivariant smooth triangulation of Z satisfying the following.

(4.1.1) $L \supseteqq \mathrm{Sing}(G, Z)$.

(4.1.2) $L = L^{(k-1)} \cup \bigcup_{\beta\in\mathcal{B}} B_\beta$ where $L^{(k-1)}$ denotes the $(k-1)$-skeleton of L and (forgetting the G-action on L) each B_β is a subcomplex of L.
(4.1.3) For each $\beta \in \mathcal{B}$, B_β is a compact connected k-dimensional submanifold of Z.
(4.1.4) For each $\beta \in \mathcal{B}$, $L^{(k-1)} \cap B_\beta \subseteqq L^{(k-2)}$.
(4.1.5) For every β, $\beta' \in \mathcal{B}$, $B_\beta \cap B_{\beta'}$ is a submanifold of Z.
(4.1.6) For every β, $\beta' \in \mathcal{B}$, if $B_\beta \neq B_{\beta'}$ as subsets of Z then $B_\beta \cap B_{\beta'} \subseteqq L^{(k-2)}$.
(4.1.7) If B_β, $\beta \in \mathcal{B}$, is orientable then B_β is oriented.

The assumption above obviously implies the next assumption on Z.

Assumption 4.2. If $Z_\gamma \neq Z_{\gamma'}$ for connected components $\gamma \in \pi_0(Z^H)$ and $\gamma' \in \pi_0(Z^{H'})$, where H, H' are nontrivial subgroups of G, then $\dim(Z_\gamma \cap Z_{\gamma'}) \leqq k-2$.

It is remarkable that if Z satisfies Assumption 4.2 then Assumption 4.1 is fulfilled by $L = \mathrm{Sing}(G, Z)$ where (forgetting orientations) the B_β's range over all underlying spaces Z_γ such that $\gamma \in \pi_0(Z^H)$ and $\dim Z^H = k$. Let $h_\beta : B_\beta \to Z$, $\beta \in \mathcal{B}$, denote the canonical inclusion maps. Since $L^{(k-1)}$ is not crucial for the general position argument for h_α, the problem we actually face is that of separating the h_α's, $\alpha \in \mathcal{A}$, from one another and from the h_β's, $\beta \in \mathcal{B}$. In case each B_β, $\beta \in \mathcal{B}$, is orientable, it is to be expected one can do this if and only if the geometric intersection numbers $\#(h_\alpha, h_{\alpha'})$ and $\#(h_\alpha, h_\beta)$ and the equivariant self-intersection numbers $\#_g(h_\alpha)$ are zero for all α, $\alpha' \in \mathcal{A}$, $\beta \in \mathcal{B}$, and $g \in G$. However, this requires that the equivariant self-intersection numbers $\#_g(h_\alpha)$ are defined. The main goal of this section is to define for certain g and arbitrary α, a replacement $\natural_g(h_\alpha)$ for $\#_g(h_\alpha)$, which agrees with $\#_g(h_\alpha)$ whenever the latter is defined, and to show (Theorems 4.19 and 4.21) that the vanishing of the geometric intersection numbers and the replacements $\natural_g$ above provides a necessary and sufficient criteria for solving the problem above. Of course, this problem is not only interesting for immersions of k-dimensional spheres, but for any finite family of immersions of closed, connected, oriented, k-dimensional manifolds and so, we shall treat the problem above in this context.

The rest of the section is organized as follows. Looking ahead, we shall want in the next section to pack all of the information above into one module called the surgery obstruction module. In order to deal with this module, it is convenient to have an algebraic description of the geometric intersection form. We begin below by recalling the algebraic intersection form on $H_k(Z; R)$ and establish some basic facts concerning it. Then we recall the geometric intersection form and compare it with the algebraic one. Next, self-intersection forms $\#_g$ are recalled and the self-intersection forms $\natural_g$ are defined. Results relating the forms $\#$ and $\natural_g$ are established. Then results concerning singular sets are proved. Finally, the main theorems are proved.

Let R be a commutative ring with unit element. For $0 \leqq \ell \leqq n$, let

$$P_{Z,\partial Z} : H^{n-\ell}(Z, \partial Z; R) \to H_\ell(Z; R)$$

and

$$P_Z : H^{n-\ell}(Z;R) \to H_\ell(Z,\partial Z;R)$$

denote the Poincaré-Lefschetz duality isomorphisms (cf. Theorems 18 and 20 of [43, VI §2]). Let

$$\cup : H^{n-\ell}(Z,\partial Z;R) \times H^\ell(Z;R) \to H^n(Z,\partial Z,;R) \text{ and}$$
$$\cup : H^{n-\ell}(Z,\partial Z;R) \times H^\ell(Z,\partial Z;R) \to H^n(Z,\partial Z;R)$$

denote the cup products [43, V §6]. If $A \in H^n(Z,\partial Z;R)$ and $a \in H_n(Z,\partial Z;R)$, let $A(a)$ denote the evaluation of A on a. Let $[Z,\partial Z]$ denote the orientation class of Z in $H_n(Z,\partial Z;\mathbb{Z})$ and $[Z,\partial Z]_R$ the image of $[Z,\partial Z]$ in $H_n(Z,\partial Z;R)$, under the canonical homomorphism. Define the *mixed R-intersection pairing*

$$\begin{aligned}\mathrm{Int}_{Z,\partial Z;R} : H_k(Z;R)\times H_k(Z,\partial Z;R) &\to R;\\ (a,b) &\mapsto (-1)^k(P^{-1}_{Z,\partial Z;R}(a) \cup P^{-1}_{Z;R}(b))([Z,\partial Z]_R)\end{aligned}$$

and the *R-intersection pairing*

$$\begin{aligned}\mathrm{Int}_{Z;R} : H_k(Z;R)\times H_k(Z;R) &\to R;\\ (a,b) &\mapsto (-1)^k(P^{-1}_{Z,\partial Z;R}(a) \cup P^{-1}_{Z,\partial Z;R}(b))([Z,\partial Z]_R).\end{aligned}$$

Of course, if $\partial Z = \varnothing$ then by definition $\mathrm{Int}_{Z,\partial Z;R} = \mathrm{Int}_{Z,R}$.

The action of G on Z induces a G-module structure on the homology groups $H_\ell(Z;R)$ and $H_\ell(Z,\partial Z;R)$. Namely if $g \in G$ and $x \in H_\ell(Z;R)$ or $H_\ell(Z,\partial Z;R)$ then $gx = \ell_{g*}(x)$, where ℓ_g denotes the left translation map $Z \to Z$; $z \mapsto gz$. Let $R[G]$ denote the group ring of G with coefficients in R. Define the *mixed G-equivariant R-intersection pairing*

$$\mathrm{Int}_{G,Z,\partial Z;R} : H_k(Z;R) \times H_k(Z,\partial Z;R) \to R[G]$$

by

$$\mathrm{Int}_{G,Z,\partial Z;R}(a,b) = \sum_{g\in G} \mathrm{Int}_{Z,\partial Z;R}(a, g^{-1}b)g,$$

and the *G-equivariant R-intersection pairing*

$$\mathrm{Int}_{G,Z;R} : H_k(Z;R) \times H_k(Z;R) \to R[G]$$

by

$$\mathrm{Int}_{G,Z;R}(a,b) = \sum_{g\in G} \mathrm{Int}_{Z;R}(a, g^{-1}b)g.$$

Lemma 4.3. *The diagram*

$$\begin{tikzcd} H_k(Z;R) \times H_k(Z;R) \arrow[r, "\mathrm{Int}_{G,Z;R}"] \arrow[d] & R[G] \\ H_k(Z,R) \times H_k(Z,\partial Z;R) \arrow[ur, "\mathrm{Int}_{G,Z,\partial Z;R}"'] & \end{tikzcd}$$

commutes.

Proof. The case of an arbitrary finite group G follows from the case $G = \{1\}$. The case $G = \{1\}$ follows from the fact that the cup product diagram corresponding to the diagram in the lemma commutes. □

Define the *orientation homomorphism* $w_Z : G \longrightarrow \{\pm 1\}$ by

$$w_Z(g) = \begin{cases} 1, & \text{if } g \text{ preserves the orientation of } Z \\ -1, & \text{if } g \text{ reverses the orientation of } Z. \end{cases}$$

Thus,

$$g[Z, \partial Z] = w_Z(g)[Z, \partial Z].$$

When working in the context of Z, we shall give the group ring $R[G]$ the involution $a \mapsto \overline{a}$ such that for all $r \in R$ and $g \in G$,

$$\overline{rg} = r w_Z(g) g^{-1}$$

and let $\lambda = (-1)^k$. The G-actions on the cohomology groups $H^\ell(Z; R)$ and $H^\ell(Z, \partial Z; R)$ are afforded by $gx = w_Z(g) {\ell_{g^{-1}}}^*(x)$ for all $g \in G$ and $x \in H^\ell(Z; R)$ or $H^\ell(Z, \partial Z; R)$.

If $f : (A, B) \to (A', B')$ is a continuous map of pairs of topological spaces ($B \subsetneqq A$, $B' \subsetneqq A'$ and $f(B) \subsetneqq B'$) then for each integer ℓ, define

$$\begin{aligned} K_\ell(A, B; R) &= \mathrm{Ker}[f_* : H_\ell(A, B; R) \longrightarrow H_\ell(A', B'; R)] \\ K_\ell(A; R) &= \mathrm{Ker}[f_* : H_\ell(A; R) \longrightarrow H_\ell(A'; R)] \\ K^\ell(A, B; R) &= \mathrm{Coker}[f^* : H^\ell(A', B'; R) \longrightarrow H^\ell(A, B; R)] \\ K^\ell(A; R) &= \mathrm{Coker}[f^* : H^\ell(A'; R) \longrightarrow H^\ell(A; R)]. \end{aligned}$$

Let $f : (X, \partial X) \to (Y, \partial Y)$ be a continuous map of compact, connected, n-dimensional manifolds. Define

$$\widehat{f}_* : H_\ell(Y; R) \longrightarrow H_\ell(X; R)$$

to be the composite of the solid arrows in the diagram

$$\begin{tikzcd} H_\ell(Y; R) \arrow[r, dotted, "\widehat{f}_*"] \arrow[d, "P^{-1}_{Y,\partial Y}"'] & H_\ell(X; R) \\ H^{n-\ell}(Y, \partial Y; R) \arrow[r, "f^*"'] & H^{n-\ell}(X, \partial X; R) \arrow[u, "P_{X,\partial X}"'] \end{tikzcd}$$

and define

$$\widehat{f}_* : H_\ell(Y, \partial Y; R) \longrightarrow H_\ell(X, \partial X; R)$$

to be the composite of the solid arrows in the diagram

$$\begin{tikzcd} H_\ell(Y, \partial Y; R) \arrow[r, dotted, "\widehat{f}_*"] \arrow[d, "P^{-1}_{Y}"'] & H_\ell(X, \partial X; R) \\ H^{n-\ell}(Y; R) \arrow[r, "f^*"'] & H^{n-\ell}(X; R). \arrow[u, "P_{X}"'] \end{tikzcd}$$

Recall that f is called a *degree-one map*, if X and Y are oriented and $f_*[X,\partial X] = [Y,\partial Y]$ in $H_n(Y,\partial Y;\mathbb{Z})$. In the remainder of this section, let $f:(X,\partial X)\to(Y,\partial Y)$ be a degree-one G-map of compact, connected, n-dimensional, oriented, smooth G-manifolds X and Y. Since f^* and the Poincaré-Lefschetz duality maps in the diagrams above are $R[G]$-homomorphisms, so is $\widehat{f}_*$.

Lemma 4.4. *The following holds:*

(4.4.1) *The homomorphism $\widehat{f}_*$ is a G-equivariant splitting for f_*.*

(4.4.2) *The sequence of $R[G]$-modules*

$$\cdots \xrightarrow{\partial} K_\ell(\partial X;R) \longrightarrow K_\ell(X;R) \longrightarrow K_\ell(X,\partial X;R) \xrightarrow{\partial} K_{\ell-1}(\partial X;R) \longrightarrow \cdots$$

is exact.

(4.4.3) *If $n=2k$ then*

$$\mathrm{Int}_{G,X;R}(K_k(X;R),\widehat{f}_*H_k(Y;R)) = 0, \quad \mathrm{Int}_{G,X;R}(\widehat{f}_*H_k(Y;R),K_k(X;R)) = 0,$$
$$\mathrm{Int}_{G,X,\partial X;R}(K_k(X;R),\widehat{f}_*H_k(Y,\partial Y;R)) = 0, \quad \mathrm{Int}_{G,X,\partial X;R}(\widehat{f}_*H_k(Y;R),K_k(X,\partial X;R)) = 0.$$

Proof. (4.4.1): Let $a\in H_\ell(Y;R)$. Then there exists an element $b\in H^{n-\ell}(Y,\partial Y;R)$ such that $P_{Y,\partial Y}(b)=a$. Thus

$$f_*\widehat{f}_*(a) = f_*(P_{X,\partial X}(f^*b)) = f_*([X,\partial X]_R\cap f^*b) = [Y,\partial Y]_R\cap b = P_{Y,\partial Y}(b) = a.$$

Similarly $f_*\widehat{f}_*(a')=a'$ for all $a'\in H_\ell(X,\partial X;R)$.

(4.4.2): The assertion follows from the definition of $K_\ell(-;R)$ and (4.4.1).

(4.4.3): This is proved by straightforward computation, cf. [8, I.2.9]. □

Lemma 4.5. *Suppose $K_k(X;R)$ and $K_k(X,\partial X;R)$ are free R-modules. Then the pairing $\mathrm{Int}_{X,\partial X;R}: K_k(X;R)\times K_k(X,\partial X;R)\to R$ is nonsingular.*

Let A denote a connected, closed, oriented, smooth manifold of dimension k, and B a connected, compact, smooth manifold of dimension k. We assume that B is oriented if B is orientable. In the following, immersions and embeddings will always mean smooth, neat immersions and embeddings (cf. [15]). Let $h_\alpha: A\to Z$ and $h_\beta: B\to Z$ be immersions such that $h_\beta|_{\partial B}$ is an embedding. (Since A is closed, it follows from the definition of neat immersion that $\mathrm{Image}(h_\alpha)\subsetneqq \mathrm{Interior}(Z)$.)

Let $\mathbb{Z}_\beta=\mathbb{Z}$ or $\mathbb{Z}_2$ $(=\mathbb{Z}/2\mathbb{Z})$ depending on whether B is oriented or nonorientable. Let

$$\#(h_\alpha,h_\beta)\in\mathbb{Z}_\beta$$

denote the geometric intersection number of h_α and h_β. Since A is oriented, the normal bundle $\nu(h_\alpha)$ of h_α is oriented such that a local frame of $T(A)$ with positive orientation followed by a local frame of $\nu(h_\alpha)$ with positive orientation becomes a local frame of $h_\alpha^*T(Z)$ with positive orientation. If both h_α and h_β are embeddings and B is oriented then our number $\#(h_\alpha,h_\beta)$ above is the same as the number $\mathrm{Image}(h_\alpha)\cdot\mathrm{Image}(h_\beta)$ in Definition 6.1 of Milnor [21]. In this connection, we stress that our $\#(h_\alpha,h_\beta)$ is the intersection number not “of Image(h_α) and Image(h_β)” but “of Image(h_β) and Image(h_α)” in the language of [21].

The geometric intersection form $\#(\ ,\ )$ has the property that $\#(h_\alpha, h_\beta) = 0 \Longleftrightarrow$ there is a regular homotopy $h_\alpha \sim h'_\alpha$ such that $\mathrm{Image}(h'_\alpha) \cap \mathrm{Image}(h_\beta) = \varnothing$. (A regular homotopy, by definition, is one such that at each level the maps are immersions.) The result above is an easy corollary of Theorem 6.6 of [21].

Define the *G-equivariant geometric intersection number* $\#_G(h_\alpha, h_\beta)$ by

$$\#_G(h_\alpha, h_\beta) = \sum_{g \in G} \#(h_\alpha, g^{-1}h_\beta)g \in \mathbb{Z}_\beta[G].$$

Lemma 4.6 (cf. [1, p.15, Theorem 1])**.** *Let $[A]$ denote the orientation class of A and $[B, \partial B]$ the orientation class of B. Then*

$$\#_G(h_\alpha, h_\beta) = \mathrm{Int}_{G,Z,\partial Z;\mathbb{Z}_\beta}(h_{\alpha *}[A], h_{\beta_*}[B, \partial B])$$

(if B is closed then this reads

$$\#_G(h_\alpha, h_\beta) = \mathrm{Int}_{G,Z;\mathbb{Z}_\beta}((h_\alpha)_*[A], (h_\beta)_*[B]).)$$

Proof. For $G = \{1\}$, the first statement is deduced from the definitions of the cup product, the Poincaré-Lefschetz duality isomorphism as derived from the dual cell decomposition, and the ordinary geometric intersection number. The result for arbitrary G follows directly from the above and the definitions of $\#_G(\ ,\ )$ and $\mathrm{Int}_{\mathrm{G,Z,\partial Z};\mathbb{Z}_\beta}(\ ,\ )$. $\square$

We want to define next the generalized (or doubly parametrized) G-equivariant geometric self-intersection form. This is a new construction, since we are going to allow fixed point manifolds of dimension k. Our definition will reduce to the one of [25, §3], when the fixed point manifolds of nontrivial subgroups have dimension $\leqq k - 1$ and to [44, Part I, §5], when the fixed point manifolds of nontrivial subgroups are empty.

For an integer ℓ, let

$$G(Z, \ell) = \{g \in G \mid Z^g \text{ has an } \ell\text{-dimensional, connected component}\}$$
$$G(2) = \{g \in G \mid g \neq 1, g^2 = 1\}$$
$$G(> 2) = \{g \in G \mid g^2 \neq 1\}.$$

Lemma 4.7. *If $g \in G(Z, \ell)$ then $w_Z(g) = (-1)^{n-\ell}$.*

Proof. Let L be an ℓ-dimensional, connected component of Z^g. Let $p \in L$ be an interior point of L. Let $T_p(L)$ (resp. $T_p(Z)$) denote the tangent space at p in L (resp. Z). Since G is finite and Z is compact, there is a G-invariant Riemannian metric on Z. Fix such a metric. Denote by $\langle g \rangle$ the subgroup generated by g. The exponential map Exp from a certain neighborhood of 0 in $T_p(Z)$ to Z (with respect to the Riemannian metric) is automatically G-equivariant and a local diffeomorphism as long as its image lies in the interior of Z. Let $B_p(Z) \subsetneqq T_p(Z)$ denote a tiny $\langle g \rangle$-invariant ball centered at $0 \in T_p(Z)$ such that $\mathrm{Exp}|_{B_p} : B_p(Z) \to Z$ is a diffeomorphism of $B_p(Z)$ onto a neighborhood U of p. Since Exp is G-equivariant, $B_p(Z)^g$ is diffeomorphic to U^g and $U^g \subseteqq L$. Clearly, $\ell = \dim L = \dim T_p(L) \leqq \dim T_p(Z)^g = \dim B_p(Z)^g = \dim U^g \leqq \dim L = \ell$. Thus, $T_p(L) = T_p(Z)^g$. Let $\nu_p(L)$ denote the normal space of $T_p(L)$, i.e. the orthogonal complement to $T_p(L)$ in $T_p(Z)$ with respect to the Riemannian metric. Clearly $\nu_p(L)$ is $\langle g\rangle$-invariant. Since $T_p(L) = T_p(Z)^g$, the group $\langle g\rangle$ acts freely on $\nu_p(L) \setminus 0$. Thus, the $\langle g\rangle$-module $\nu_p(L)$ decomposes as a product of nontrivial, 1-dimensional real representations of $\langle g\rangle$ and realifications of nontrivial, 1-dimensional complex representations of $\langle g\rangle$. In the first case, the determinant over $\mathbb{R}$ of the $\mathbb{R}$-linear automorphism defined by g is (-1) and in the second case $(+1)$. The determinant d of the $\mathbb{R}$-linear automorphism of $\nu_p(L)$ defined by g satisfies the relation $d = (-1)^{n-\ell}$. But $d = w_Z(g)$. $\square$

Corollary 4.8. *(Recall $\lambda = (-1)^k$.) If $g \in G(Z,\ell) \cap G(2)$ then*

$$g = \begin{cases} \lambda w_Z(g) g^{-1} \quad (= \lambda \overline{g}), & \text{if } \ell \equiv k \mod 2 \\ -\lambda w_Z(g) g^{-1} \quad (= -\lambda \overline{g}), & \text{if } \ell \not\equiv k \mod 2. \end{cases}$$

Proof. By Lemma 4.7, $\lambda = w_Z(g) \iff \ell \equiv k \mod 2$, and $\lambda = -w_Z(g) \iff \ell \not\equiv k \mod 2$. $\square$

Corollary 4.9. *If $\ell \not\equiv \ell' \mod 2$ then $G(Z,\ell) \cap G(Z,\ell') = \varnothing$.*

Proof. Suppose $g \in G(Z,\ell) \cap G(Z,\ell')$. By Lemma 4.7 $(-1)^{n-\ell} = w_Z(g) = (-1)^{n-\ell'}$, and hence $n - \ell \equiv n - \ell' \mod 2$. This contradicts the assumption $\ell \not\equiv \ell' \mod 2$. $\square$

Recall the notion of self-intersection form introduced in [44, pp.45–46]. It is naturally generalized as follows. Let A be a connected, closed, k-dimensional, orientable, smooth manifold and $h_\alpha : A \to Z$ an immersion. If H is a subgroup of G, let

$$\mathrm{Free}(H,Z) = \{x \in Z \mid H_x = \{1\}\},$$

where H_x is the isotropy subgroup at x of the H-action on Z. Note that $\mathrm{Free}(H,Z)$ is 1-connected. Suppose $\mathrm{Image}(h_\alpha) \subset \mathrm{Free}(H,Z)$. Since H acts freely on $\mathrm{Free}(H,Z)$, $\mathrm{Free}(H,Z)/H$ is a manifold with the fundamental group $\cong H$, and the canonical map $\pi : \mathrm{Free}(H,Z) \to \mathrm{Free}(H,Z)/H$ is a universal covering map. Using the composite mapping $\pi \circ h_\alpha : A \to \mathrm{Free}(H,Z)/H$, one defines *the H-equivariant geometric self-intersection number* $\mu_H(h_\alpha) \in \mathbb{Z}[H]/min_\lambda(\mathbb{Z}[H])$, exactly as in [44, pp.45–46], where $\min_\lambda(\mathbb{Z}[H])$ is the minimum λ-form parameter on $\mathbb{Z}[H]$. Thus, setting

$$\mathbb{Z}_g = \begin{cases} \mathbb{Z}/(1-\lambda w_Z(g))\mathbb{Z}, & \text{if } g \in G \text{ and } g^2 = 1 \\ \mathbb{Z}, & \text{if } g \in G \text{ and } g^2 \neq 1, \end{cases}$$

one can regard $\mu_H(h_\alpha)$ as an $|H|$-tuple $(\mu_H(h_\alpha)_g)_{g\in H}$ such that $\mu_H(h_\alpha)_g \in \mathbb{Z}_g$ and $\mu_H(h_\alpha)_g = \lambda w_Z(g) \mu_H(h_\alpha)_{g^{-1}}$. If H' is another subgroup of G such that $\mathrm{Free}(H',Z)$ is 1-connected and $\mathrm{Image}(h_\alpha) \subset \mathrm{Free}(H',Z)$, and if $g \in H \cap H'$, then one can check that $\mu_H(h_\alpha)_g = \mu_{H'}(h_\alpha)_g$. For any $g \in H$, we define the *g-th geometric self-intersection number*

$$\#_g(h_\alpha) = \mu_H(h_\alpha)_g \in \mathbb{Z}_g.$$

Thus, $\#_g(h_\alpha)$ is defined only when $\mathrm{Free}(\langle g\rangle, Z)$ is 1-connected and $\mathrm{Image}(h_\alpha) \subset \mathrm{Free}(\langle g\rangle, Z)$. We shall partially overcome this restriction in Definition 4.11 below. The geometric self-intersection numbers $\#_g(\quad)$ $(g \in G)$ have the property that $\#_1(h_\alpha) = 0 \iff h_\alpha$ is regularly homotopic to an embedding h'_α, and the property that $\#_1(h_\alpha) = 0$ and $\#_g(h_\alpha) = 0 \iff h_\alpha$ is regularly homotopic to an embedding h'_α such that $\mathrm{Image}(h'_\alpha) \cap \mathrm{Image}(gh'_\alpha) = \varnothing$ (cf. [1, p.17, Corollary 1]).

Let $\nu(h_\alpha)$ denote the normal bundle of the immersion $h_\alpha : A \to Z$ and $\chi(\nu(h_\alpha))$ the Euler number of $\nu(h_\alpha)$. If the normal bundle $\nu(h_\alpha)$ is trivial then $\chi(\nu(h_\alpha)) = 0$. There is a relation (cf. Theorem 4.11 (iii) of [44, p.5]) between the equivariant geometric intersection number and the equivariant geometric self-intersection number given by the equation

$$\#_H(h_\alpha, h_\alpha) = \widetilde{\mu_H(h_\alpha)} + \lambda\overline{\widetilde{\mu_H(h_\alpha)}} + \chi(\nu(h_\alpha)),$$

where $\widetilde{\mu_H(h_\alpha)}$ is a lifting of $\mu_H(h_\alpha)$ to $\mathbb{Z}[H]$. From this, one deduces easily the next lemma.

Lemma 4.10. *Let $h_\alpha : A \to Z$ be an immersion such that $\#_g(h_\alpha)$ is defined. If $g \in \{1\} \cup G(2)$, let $\widetilde{\#_g(h_\alpha)}$ denote a lifting of $\#_g(h_\alpha)$ to $\mathbb{Z}$. Then*

$$\#(h_\alpha, g^{-1}h_\alpha) = \begin{cases} \#_g(h_\alpha), & \text{if } \mathrm{order}(g) \geqq 3 \\ \widetilde{\#_g(h_\alpha)} + \lambda w_Z(g)\widetilde{\#_g(h_\alpha)}, & \text{if } g \in G(2) \\ \widetilde{\#_g(h_\alpha)} + \lambda\widetilde{\#_g(h_\alpha)} + \chi(\nu(h_\alpha)), & \text{if } g = 1. \end{cases}$$

In particular, $\chi(\nu(h_\alpha)) = 0$ whenever $\#(h_\alpha, h_\alpha) = 0$ and $\#_1(h_\alpha) = 0$.

In the following, let

$$\begin{aligned} Q(G, Z) &= G(Z, k-1) \cap G(2) \\ S(G, Z) &= G(Z, k) \cap G(2). \end{aligned}$$

Let $\Lambda(Q(G, Z))$ denote the λ-form parameter on $\mathbb{Z}[G]$ generated by $Q(G, Z)$, namely

$$\Lambda(Q(G, Z)) = \min{}_\lambda(\mathbb{Z}[G]) + \mathbb{Z}[Q(G, Z)].$$

Let

$$Z(k-1, G(2)) = \bigcup_{g \in G(2)} \left(\bigcup_{\gamma \in \pi_0(Z^g, k-1)} Z_\gamma \right)$$

where $\pi_0(Z^g, k-1)$ is the subset of $\pi_0(Z^g)$ consisting of all $(k-1)$-dimensional connected components, and Z_γ stands for the underlying space of γ. Set

$$\widehat{Z} = Z \setminus Z(k-1, G(2)).$$

Lemma 4.10 motivates making the following definition.

Definition 4.11. Let A be a connected, closed, k-dimensional, oriented, smooth manifold and $h_\alpha : A \to Z$ (resp. $h_\alpha : A \to \widehat{Z}$) an immersion. If $g \in G \setminus (Q(G,Z) \cup S(G,Z))$ (resp. $G \setminus S(G,Z)$), define

$$\natural_g(h_\alpha) = \begin{cases} \#(h_\alpha, g^{-1}h_\alpha), & \text{if } g \in G(>2) \\ \#_g(h'_\alpha), & \text{if } g \in G(2) \text{ with } \dim Z^g \leqq k-2 \\ & (\text{resp. } g \in G(2) \text{ with } \dim Z^g \leqq k-1) \\ \#_1(h_\alpha), & \text{if } g = 1 \end{cases}$$

where $h'_\alpha : A \to \mathrm{Free}(\langle g \rangle, Z)$ (resp. $h'_\alpha : A \to \mathrm{Free}(\langle g \rangle, \widehat{Z})$) is regularly homotopic in Z (resp. $\widehat{Z}$) to h_α. If $h''_\alpha : A \to \mathrm{Free}(\langle g \rangle, Z)$, $g \in G(2)$ with $\dim Z^g \leqq k-2$, (resp. $h''_\alpha : A \to \mathrm{Free}(\langle g \rangle, \widehat{Z})$, $g \in G(2)$ with $\dim Z^g \leqq k-1$,) is regularly homotopic in Z (resp. $\widehat{Z}$) to h_α then h''_α is regularly homotopic in $\mathrm{Free}(\langle g \rangle, Z)$ to h'_α, so that $\natural_g(h_\alpha)$ is well defined.

Theorem 4.12. *Let $h_\alpha : A \to Z$ (resp. $h_\alpha : A \to \widehat{Z}$) be as in Definition 4.11 and $g \in G \setminus (Q(G,Z) \cup S(G,Z))$ (resp. $G \setminus S(G,Z)$). Then $\natural_g(h_\alpha)$ is defined and $\natural_g(h_\alpha) = \#_g(h_\alpha)$ whenever $\#_g(h_\alpha)$ is defined, i.e.* $\mathrm{Image}(h_\alpha) \subsetneqq \mathrm{Free}(\langle g \rangle, Z)$. *Furthermore, if $\widetilde{\natural_g(h_\alpha)}$ denotes a lifting of $\natural_g(h_\alpha)$ to $\mathbb{Z}$ in the case $g \in \{1\} \cup G(2)$ then*

$$\#(h_\alpha, g^{-1}h_\alpha) = \begin{cases} \natural_g(h_\alpha), & \textit{if } g \in G(>2) \\ \widetilde{\natural_g(h_\alpha)} + \lambda w_Z(g)\widetilde{\natural_g(h_\alpha)}, & \textit{if } g \in G(2) \setminus (Q(G,Z,) \cup S(G,Z)) \\ & (\textit{resp. } g \in G(2) \setminus S(G,Z)) \\ \widetilde{\natural_g(h_\alpha)} + \lambda\widetilde{\natural_g(h_\alpha)} + \chi(\nu(h_\alpha)), & \textit{if } g = 1. \end{cases}$$

Proof. By definition, $\natural_g(h_\alpha)$ is always defined. The first equation for $\#(h_\alpha, g^{-1}h_\alpha)$ holds by definition and the others follow from the corresponding equations in Lemma 4.10. The assertion that $\natural_g(h_\alpha) = \#_g(h_\alpha)$ whenever the latter is defined follows now from the equations just proved and those in Lemma 4.10. ☐

Whereas the G-equivariant, self-intersection form $\#_G(\ )$ is defined only on immersions $h_\alpha : A \to \mathrm{Free}(G,Z)$, the generalized G-equivariant self-intersection form $\natural_G(\ )$ (resp. $\widehat{\natural}_G(\ )$) to be constructed below will be defined on all immersions $h_\alpha : A \to Z$ (resp. $h_\alpha : A \to \widehat{Z}$). To make this construction, we note first that each element $g \in Q(G,Z)$ (resp. $S(G,Z)$) satisfies by Corollary 4.8 the equation $g = -\lambda\overline{g}$ (resp. $g = \lambda\overline{g}$).

Definition 4.13. Let $h_\alpha : A \to Z$ (resp. $h_\alpha : A \to \widehat{Z}$) be as in Definition 4.11. Let $G(>2) = C \cup C^{-1}$ be a disjoint decomposition. For each $g \in \{1\} \cup (G(2) \setminus (Q(G,Z) \cup S(G,Z))) \cup C$ (resp. $\{1\} \cup (G(2) \setminus \cup S(G,Z)) \cup C$), let $\widetilde{\natural_g(h_\alpha)}$ be a lifting of $\natural_g(h_\alpha)$ to $\mathbb{Z}$. Define the *generalized G-equivariant self-intersection number*

$$\natural_G(h_\alpha) = \sum_g \widetilde{\natural_g(h_\alpha)}g \quad \in \mathbb{Z}[G]/(\Lambda(Q(G,Z)) + \mathbb{Z}[S(G,Z)]))$$

where g runs over $\{1\} \cup (G(2) \smallsetminus (Q(G,Z) \cup S(G,Z))) \cup C$ (resp.

$$\widehat{\natural}_G(h_\alpha) = \sum_g \widetilde{\natural_g(h_\alpha)}g \quad \in \mathbb{Z}[G]/(\min_\lambda(\mathbb{Z}[G]) + \mathbb{Z}[S(G,Z)])$$

where g runs over $\{1\} \cup (G(2) \smallsetminus S(G,Z)) \cup C$).

The number $\natural_G(h_\alpha)$ (resp. $\widehat{\natural}_G(h_\alpha)$) does not depend on the liftings $\widetilde{\natural(h_\alpha)}$, and not on the choice of C, because for $g \in G(>2)$,

$$\natural_{g^{-1}}(h_\alpha) = \lambda w_Z(g)\natural_g(h_\alpha)$$

and therefore, $\natural_g(h_\alpha)g \equiv \natural_{g^{-1}}(h_\alpha)g^{-1} \mod (\min_\lambda(\mathbb{Z}[G]))$. The number $\natural_G(h_\alpha)$ (resp. $\widehat{\natural}_G(h_\alpha)$) is invariant under regular homotopy of h_α in Z (resp. $\widehat{Z}$).

Theorem 4.14. *Let $h_\alpha : A \to Z$ (resp. $h_\alpha : A \to \widehat{Z}$) be as in Definition 4.11 and let $\widetilde{\natural_G(h_\alpha)}$ denote a lifting of $\natural_G(h_\alpha)$ (resp. $\widehat{\natural}_G(h_\alpha)$) to $\mathbb{Z}[G]/\mathbb{Z}[S(G,Z)]$. Then*

$$\#_G(h_\alpha, h_\alpha) = \widetilde{\natural_G(h_\alpha)} + \lambda\overline{\widetilde{\natural_G(h_\alpha)}} + \chi(\nu(h_\alpha)) \quad \text{in } \mathbb{Z}[G]/\mathbb{Z}[S(G,Z)].$$

Proof. It follows from the definitions of $\#_G$ and $\natural_G$ (resp. $\widehat{\natural}_G$) that the assertion of the theorem is equivalent to the assertion of Theorem 4.12. □

Definition 4.15. Let $h_i : A_i \to Z$ (resp. $h_i : A_i \to \widehat{Z}$), $i = 0, 1$, be immersions as in Definition 4.11. Then the connected sum $h_0\#h_1$ of h_0 and h_1 is defined up to regular homotopy in Z (resp. $\widehat{Z}$) as follows. Let $D_i \subset A_i$, $i = 0, 1$, be k-dimensional disks such that $h_i|_{D_i}$ are injective. Let I denote the closed unit interval $[0,1]$, D^k a k-dimensional disk, and identify the k-dimensional disks $\{i\} \times D^k \subset I \times D^k$, $i = 0, 1$, with the k-dimensional disks D_i, respectively, by orientation preserving diffeomorphisms. Let $\varphi : I \times D^k \to Z$ (resp. $\varphi : I \times D^k \to \widehat{Z}$) be an immersion such that $\varphi|_{D_i} = h_i$. Let $A_0\#A_1$ denote the usual connected sum of A_0 and A_1 formed by cutting out the interior of the disks D_i from A_i, and let T denote the tube in $A_0\#A_1$ connecting A_0 and A_1. Thus one has a diffeomorphism $\psi : I \times S^{k-1} \to T$ such that the maps $\psi|_{\{i\}\times S^{k-1}}$, $i = 0, 1$, agree with the identifications made above. Let $h = h_0\#h_1 : A_0\#A_1 \to Z$ denote the immersion such that $h|_{A_i \setminus D_i} = h_i|_{A_i \setminus D_i}$ and

$$h|_T(\psi(t,x)) = \varphi(t,x) \qquad ((t,x) \in I \times S^{k-1}).$$

If $h'_i : A_i \to Z$ (resp. $h'_i : A_i \to \widehat{Z}$), $i = 0, 1$, are immersions which are regularly homotopic in Z (resp. $\widehat{Z}$) to h_i, $i = 0, 1$, respectively, and if $h' : A_0\#A_1 \to Z$ (resp. $h' : A_0\#A_1 \to \widehat{Z}$) is an immersion enjoying the same properties relative to h'_0 and h'_2 as h relative to h_0 and h_1, then h' is regularly homotopic in Z (resp. $\widehat{Z}$) to h, because A_0 and A_1 are connected and Z (resp. $\widehat{Z}$) is 1-connected. We define $\natural_G(h_0\#h_1) = \natural_G(h)$ (resp. $\widehat{\natural}_G(h_0\#h_1) = \widehat{\natural}_G(h)$) and conclude that $\natural_G(h_0\#h_1)$ (resp. $\widehat{\natural}_G(h_0\#h_1)$) is well defined and invariant under regular homotopy of h_0 and h_1 in Z (resp. $\widehat{Z}$).

Theorem 4.16. *Let $h_i : A_i \to Z$ (resp. $h_i : A_i \to \widehat{Z}$), $i = 0, 1$, be immersions as in Definition 4.11. Then*

$$\#_G(h_0, h_1) = \natural_G(h_0 \# h_1) - \natural_G(h_0) - \natural_G(h_1)$$
$$\text{in } \mathbb{Z}[G]/(\Lambda(Q(G,Z)) + \mathbb{Z}[S(G,Z)])$$

$$(\text{resp. } \#_G(h_0, h_1) = \widehat{\natural}_G(h_0 \# h_1) - \widehat{\natural}_G(h_0) - \widehat{\natural}_G(h_1)$$
$$\text{in } \mathbb{Z}[G]/(\min_\lambda(\mathbb{Z}[G]) + \mathbb{Z}[S(G,Z)])).$$

Proof. The proof is similar to Theorem 5.2 (iv) of [44]. □

In applications, it will be sometimes necessary to perform G-surgery relative to a G-subset which is strictly larger than the singular set $\mathrm{Sing}(G, Z)$. This is of interest also in the case $G = 1$. In order to handle such situations, we introduced already in Assumption 4.1 a generalization L of the singular set. For the rest of this section, L denotes such a set. If U is a subset of Z, define

$$\rho_G(U) = \bigcap_{x \in U} G_x.$$

If $\beta \in \mathcal{B}$, set $\rho_G(\beta) = \rho_G(B_\beta)$. For a subgroup H of G, define

$$U^{=H} = \{x \in U \mid G_x = H\}.$$

Proposition 4.17. *For any $\beta \in \mathcal{B}$,*

$$B_\beta^{=\rho_G(\beta)}$$

is connected and open dense in B_β.

Proof. This follows from the property that any element $x \in B_\beta$ such that $G_x \neq \rho_G(\beta)$ lies in $L^{(k-2)}$. □

Proposition 4.18. *If* $\dim Z_\gamma = k - 1$ *or* k*, where $\gamma \in \pi_0(Z^H)$ and $H \in \mathcal{S}(G)$, then $|\rho_G(Z_\gamma) \cap G(2)| \leqq 1$.*

Proof. Let $p \in \mathrm{Interior}(Z_\gamma)$. Let $T_p(Z_\gamma)$ (resp. $T_p(Z)$) denote the tangent space at p in γ (resp. Z). Then $T_p(Z_\gamma)$ has an orthogonal complement $\nu(Z_\gamma)$ in $T_p(Z)$ with respect to some $\rho_G(Z_\gamma)$-invariant inner product. Note $\dim \nu(Z_\gamma) = n - \dim(Z_\gamma) \geqq 1$. We claim $\rho_G(Z_\gamma)$ acts freely on $\nu_p(Z_\gamma) \setminus \{0\}$. Suppose this has been shown. Then the group $\rho_G(Z_\gamma)$ injects into $\mathrm{Aut}_{\mathbb{R}[\rho_G(Z_\gamma)]}(\nu_p(Z_\gamma))$. Let $f \in \mathrm{Aut}_{\mathbb{R}[\rho_G(Z_\gamma)]}(\nu_p(Z_\gamma))$ denote an element such that $f^2 = id$. Each eigenvalue of f must be 1 or -1. Moreover if f acts freely on $\nu_p(Z_\gamma) \setminus \{0\}$ then all eigenvalues of f are -1, hence $f = -id$. In other words, $\mathrm{Aut}_{\mathbb{R}[\rho_G(Z_\gamma)]}(\nu_p(Z_\gamma))$ has exactly one element which is of order 2 and acts freely on $\nu_p(Z_\gamma) \setminus \{0\}$, namely $-id$. Thus, $|\rho_G(Z_\gamma) \cap G(2)| \leqq 1$. If $\dim Z_\gamma = k$ then it follows from $\mathrm{Sing}(G, Z) \leqq k$ that $\rho_G(Z_\gamma)$ acts freely on elements $\nu_p(Z_\gamma) \setminus \{0\}$. If $\dim \gamma = k - 1$ then there are no $\beta \in \mathcal{B}$ such that $B_\beta \supsetneqq Z_\gamma$. Thus, $\rho_G(Z_\gamma)$ acts freely on $\nu_p(Z_\gamma) \setminus \{0\}$. □

Let $h_\beta : B_\beta \to Z$, $\beta \in \mathcal{B}$, denote the canonical inclusion map afforded by L in Assumption 4.1.

Theorem 4.19. *Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a finite set of connected, closed, k-dimensional, oriented, smooth manifolds and for each $\alpha \in \mathcal{A}$, let $h_\alpha : A_\alpha \to \widehat{Z}$ be an immersion. Then (4.19.1) and (4.19.2) below are equivalent.*

(4.19.1) (i) $\#_G(h_\alpha, h_{\alpha'}) = 0$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$.
(ii) $\#(h_\alpha, h_\beta) = 0$ *for all* $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$.
(iii) $\#_G(h_\alpha, h_\alpha) = 0$ *and* $\widehat{\natural}_G(h_\alpha) = 0$ *for all* $\alpha \in \mathcal{A}$.

(4.19.2) *There is a family of regular homotopies $h_\alpha \sim h'_\alpha$ in $\widehat{Z}$ where α ranges through $\mathcal{A}$ such that the following holds:*
(a) $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_{\alpha'}) = \varnothing$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$ *and all* $g \in G$.
(b) $\mathrm{Image}(h'_\alpha) \cap B_\beta = \varnothing$ *for all* $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$.
(c) h'_α *is an embedding such that* $\chi(\nu(h'_\alpha)) = 0$ *and* $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_\alpha) = \varnothing$ *for all* $\alpha \in \mathcal{A}$ *and all* $g \in G \smallsetminus \{1\}$.

Proof. Since each h_α, $\alpha \in \mathcal{A}$, is regularly homotopic in $\widehat{Z}$ to an immersion $h''_\alpha : A_\alpha \to \widehat{Z} \setminus L^{(k-1)}$, it suffices to prove the equivalence (4.19.1) $\Longleftrightarrow$ (4.19.2) when h_α is replaced by h''_α.

It is routine to establish (i) $\Longleftrightarrow$ (a), (ii) $\Longleftrightarrow$ (b), (iii) $\Leftarrow$ (c), and (i) $\cup$ (ii) $\cup$ (iii) $\Leftarrow$ (a) $\cup$ (b) $\cup$ (c). To prove (i) $\cup$ (ii) $\cup$ (iii) $\Rightarrow$ (a) $\cup$ (b) $\cup$ (c), it suffices to show (i) $\cup$ (ii) $\cup$ (iii) $\Rightarrow$ (c). By (i) $\cup$ (ii), there is a family of regular homotopies $h''_\alpha \sim h^{(3)}_\alpha$, $\alpha \in \mathcal{A}$, such that the $h^{(3)}_\alpha$'s satisfy (a) $\cup$ (b) when h'_α is replaced by $h^{(3)}_\alpha$. In particular, each $\mathrm{Image}(h^{(3)}_\alpha) \subseteqq \widehat{Z} \smallsetminus L \subseteqq \mathrm{Free}(G, Z)$. Moreover, $\widehat{Z} \setminus L$ is 1-connected, because Z is 1-connected and $\dim Z - \dim L \geqq 3$. Hence, $\#_g(h''_\alpha)$ is defined for any $(g, \alpha) \in G \times \mathcal{A}$. Furthermore, we can treat each index α by itself. For a fixed α, (c) is equivalent to the assertion that $\#_G(h^{(3)}_\alpha, h^{(3)}_\alpha) = 0$ and $\#_G(h^{(3)}_\alpha) = 0$. It is clear that $\#_G(h^{(3)}_\alpha, h^{(3)}_\alpha) = \#_G(h_\alpha, h_\alpha) = 0$ by (iii), since $\#_G(\ ,\ )$ is invariant under homotopy. Thus it suffices to show $\#_G(h^{(3)}_\alpha) = 0$. By definition, $\#_G(h^{(3)}_\alpha) = 0 \Longleftrightarrow \#_g(h^{(3)}_\alpha) = 0$ for all $g \in G$. If $g \in G \setminus S(G, Z)$ then $\#_g(h^{(3)}_\alpha) = \natural_g(h^{(3)}_\alpha) = 0$ by (iii). Suppose $g \in S(G, Z)$. Then by Lemma 4.7, $\lambda w_Z(g) = 1$. Thus $\#_g(h^{(3)}_\alpha) \in \mathbb{Z}_g = \mathbb{Z}$. Lemma 4.10 gives $2\#_g(h^{(3)}_\alpha) = \#(h^{(3)}_\alpha, g^{-1}h^{(3)}_\alpha)$, and this is equal to 0 by (iii). Thus, $\#_g(h^{(3)}_\alpha) = 0$. $\square$

Lemma 4.20. *Let Z be as in Theorem 4.19 and let A be a closed, connected, k-dimensional, oriented, smooth manifold. If $h : A \to \widehat{Z}$ is an immersion and*

$$\tau \in \bigoplus_{g \in Q(G,Z)} \mathbb{Z}_g g$$

then there exists a regular homotopy $h \sim h'$ in Z such that $h'(A) \subseteqq \widehat{Z}$ and $\widehat{\natural}_G(h') = \widehat{\natural}_G(h) + \tau$.

We remark that $\mathbb{Z}_g = \mathbb{Z}_2$ by Lemma 4.7.

Proof. For simplicity, we treat first the case when $\tau = g$ for some element $g \in Q(G, Z)$. Let $\gamma \in \pi_0(Z^H, k-1)$ such that $\rho_G(Z_\gamma) \ni g$. (It follows from Assumption 4.1 that Z_γ is a $(k-1)$-dimensional connected component of Z^g.) Consider how $\widehat{\natural}_G(h)$ is altered by a regular homotopy $h \sim h''$ crossing γ once (i.e. the intersection number of Z_γ and the regular homotopy is equal to 1). The resulting immersion h'' can be realized by a connected sum construction as follows. Fix a point $p \in Z_\gamma^{=\rho_G(Z_\gamma^H)}$ (hence $G_p = \rho_G(Z_\gamma)$). Decompose the tangent space $T_p(Z)$ in Z into a direct sum of the tangent space $T_p(Z_\gamma)$ in Z_γ and the normal space $\nu_p(Z_\gamma)$ of Z_γ in Z. Clearly, $\dim \nu_p(Z_\gamma) = k+1$. Take a tiny ball D^{k+1} in $\nu_p(Z_\gamma)$ centered at 0 so that the exponential map Exp maps D^{k+1} diffeomorphically into Z, $\mathrm{Exp}(D^{k+1}) \cap \mathrm{Sing}(G, Z) = \{p\}$, and $\mathrm{Image}(h) \cap G(\mathrm{Exp}(D^{k+1})) = \varnothing$. Set $S^k = \partial D^{k+1}$ and $h_g = \mathrm{Exp}|_{S^k} : S^k \to \mathrm{Free}(G, Z)$. Then h'' above may be realized as $h\#h_g$ (connected sum in $\widehat{Z}$). Since $\mathrm{Image}(h_g)$ bounds the embedded disk $\mathrm{Exp}(D^{k+1})$, h_g is null homotopic in Z. Moreover, $\mathrm{Exp}(D^{k+1}) \subset Z \setminus Z^a$ if $a \notin \rho_G(Z_\gamma)$. Recall that $\rho_G(Z_\gamma) \cap G(2) = \{g\}$, by Proposition 4.18. Thus, for all $a \in G(2) \setminus \{g\}$, $\natural_a(h_g) = \#_a(h_g) = 0$. It follows that $\widehat{\natural}_G(h_g) = \#_g(h_g)g$. It is elementary to compute that $\#_g(h_g) = 1 \in \mathbb{Z}_2$. Since $\#_G(h, h_g) = 0$, $\widehat{\natural}_G(h'') = \widehat{\natural}_G(h) + \widehat{\natural}_G(h_g) = \widehat{\natural}_G(h) + g$ by Theorem 4.16. We consider now the case of a general τ. Let g_i, $i = 1, 2, \cdots, \ell$, denote all the elements such that $g_i \in Q(G, Z)$ and the g_i-th coefficient of τ is nonzero. By induction on ℓ, one shows that there is a regular homotopy $h \sim h'$ in Z such that $\widehat{\natural}_G(h') = \widehat{\natural}_G(h) + \tau$. □

Theorem 4.21. *Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a finite set of connected, closed, k-dimensional, oriented, smooth manifolds and for each $\alpha \in \mathcal{A}$, let $h_\alpha : A_\alpha \to Z$ be an immersion. Then (4.21.1) and (4.21.2) below are equivalent.*

(4.21.1) (i) $\#_G(h_\alpha, h_{\alpha'}) = 0$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$.
(ii) $\#(h_\alpha, h_\beta) = 0$ *for all* $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$.
(iii) $\#_G(h_\alpha, h_\alpha) = 0$ *and* $\natural_G(h_\alpha) = 0$ *for all* $\alpha \in \mathcal{A}$.

(4.21.2) *There is a family of regular homotopies $h_\alpha \sim h'_\alpha$ in Z where α ranges through $\mathcal{A}$ such that the following holds:*
(a) $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_{\alpha'}) = \varnothing$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$ *and for all* $g \in G$.
(b) $\mathrm{Image}(h'_\alpha) \cap B_\beta = \varnothing$ *for all* $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$.
(c) h'_α *is an embedding such that* $\chi(\nu(h'_\alpha)) = 0$ *and* $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_\alpha) = \varnothing$ *for all* $\alpha \in \mathcal{A}$ *and all* $g \neq 1 \in G$.

Proof. Since $\#_G(\ ,\ )$, $\#(\ ,\ )$ and $\natural_G(\ )$ are invariant under regular homotopy in Z, the assertion (4.21.2) $\Longrightarrow$ (4.21.1) is an obvious corollary of the assertion (4.19.2) $\Longrightarrow$ (4.19.1). We show that (4.21.1) $\Longrightarrow$ (4.21.2). It suffices to find regular homotopies $h_\alpha \sim h''_\alpha$ such that $h''_\alpha : A_\alpha \to \widehat{Z}$ and (4.21.1) (i)–(iii) are satisfied when h_α is replaced by h''_α. There are regular homotopies $h_\alpha \sim h_\alpha^{(3)}$, $A_\alpha \in \mathcal{A}$, in Z such that $h_\alpha^{(3)} : A_\alpha \to \widehat{Z}$. Clearly (4.21.1) (i) and (ii) are valid after h_α is replaced by $h_\alpha^{(3)}$. Furthermore $\#_G(h_\alpha^{(3)}, h_\alpha^{(3)}) = \#_G(h_\alpha, h_\alpha) = 0$ and $\natural_G(h_\alpha^{(3)}) = \natural_G(h_\alpha) = 0$ by assumption. It remains to find a regular homotopy $h_\alpha^{(3)} \sim h''_\alpha$

in Z such that $\widehat{\natural}_G(h''_\alpha) = 0$, for each $\alpha \in \mathcal{A}$. But, one can find such a regular homotopy by applying Lemma 4.20 with $h = h^{(3)}_\alpha$ and $\tau = -\widehat{\natural}_G(h^{(3)}_\alpha) = \widehat{\natural}_G(h^{(3)}_\alpha)$. $\square$

5. G-surgery obstruction modules

Let X and Y denote compact, connected, oriented smooth G-manifolds of dimension $n = 2k$ ($n \geqq 6$) such that $\dim \mathrm{Sing}(G, X) \leqq k$ and Y is simply connected.

This section defines the doubly parametrized positioning module $\boldsymbol{M_f}$ of a k-connected G-surgery map $\boldsymbol{f} = (f, b)$. $\boldsymbol{M_f}$ is called the *G-surgery obstruction module* of $\boldsymbol{f}$. It will be shown that a geometrically defined invariant $\nabla_{\mathbb{B}}$ vanishes on $\boldsymbol{M_f}$. The G-surgery obstruction $\sigma(\boldsymbol{f})$ is by Definition 5.11 the class of $\boldsymbol{M_f}$ in the G-surgery obstruction group. Theorem 6.3 of the next section will show that the vanishing of $\sigma(\boldsymbol{f})$ is a sufficient condition for performing G-surgery. Theorem 7.8 will show that the vanishing of $\sigma(\boldsymbol{f})$ is a necessary condition for performing G-surgery.

The module $\boldsymbol{M_f}$ will actually be defined with respect to any given G-set

$$L = L^{(k-1)} \cup \bigcup_{\beta \in \mathcal{B}} B_\beta \ (\subset X)$$

satisfying Assumption 4.1 in Section 4. By definition, $L \supseteqq \mathrm{Sing}(G, X)$ and is properly thought of as a generalization of $\mathrm{Sing}(G, X)$. See Assumption 4.2. In this setting $\boldsymbol{M_f}$ will be denoted by $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ and $\sigma(\boldsymbol{f})$ by $\sigma(\boldsymbol{f}, \mathbb{B})$, where $\mathbb{B} = \{h_\beta : B_\beta \to X \mid \beta \in \mathcal{B}\}$ denotes the set of canonical inclusion maps. Define

$$\mathcal{B}_{+,L} = \{\beta \in \mathcal{B} \mid B_\beta \text{ is orientable}\}.$$

Recall that for each $\beta \in \mathcal{B}_{+,L}$, B_β is assumed to be oriented. Its orientation class in $H_k(B_\beta, \partial B_\beta; \mathbb{Z})$ is denoted by $[\beta]$. We introduce the notion of a G-singularity structure equipped with a free $\{\pm 1\}$-action on $\mathcal{B}_{+,L}$ so that the G-action on the structure is compatible with that on the family of orientations $\{[\beta] \mid \beta \in \mathcal{B}\}$.

Definition 5.1. The set $\mathbb{B}$, or more precisely the pair $(L, \mathbb{B})$, is called a *G-singularity structure* for X if the following conditions are satisfied.

(5.1.1) $\mathcal{B}$ is a $G \times \{\pm 1\}$-set such that $\mathcal{B}^{\{\pm 1\}} = \mathcal{B} \smallsetminus \mathcal{B}_{+,L}$.
(5.1.2) For all $g \in G$ and $\beta \in \mathcal{B}$, $B_{g\beta} = gB_\beta$. (Hence the map $g : X \to X$, $x \mapsto gx$, has the well-defined restriction $g : B_\beta \to B_{g\beta}$.)
(5.1.3) For all $\beta \in \mathcal{B}_+$, $B_{(-1)\beta} = B_\beta$ as subsets of X and $[(-1)\beta] = -[\beta]$. (Hence $h_\beta = h_{(-1)\beta}$.)
(5.1.4) For all $g \in G$ and $\beta \in \mathcal{B}_{+,L}$, $[g\beta] = g[\beta]$ (namely, $g_*[\beta]$).
(5.1.5) Let β and β' be arbitrary elements of $\mathcal{B}$ such that $B_\beta = B_{\beta'}$ as subsets of X. If B_β is orientable then $\beta = \beta'$ or $\beta = (-1)\beta'$; if B_β is not orientable then $\beta = \beta'$.

For the rest of this paper, we let $(L, \mathbb{B})$ be a G-singular structure for X. Define

$$\widetilde{\Theta}_{\mathbb{B}} = \mathcal{B}_{+,L}, \quad \Theta_{2,\mathbb{B}} = \mathcal{B}/\{1, -1\}$$

and let

$$p_{\mathbb{B}} : \widetilde{\Theta}_{\mathbb{B}} \to \Theta_{2,\mathbb{B}}$$

denote the canonical map. For each element t of $\Theta_{2,\mathbb{B}}$, let $[t]$ denote the nontrivial element of $H_k(B_{\beta(t)}, \partial B_{\beta(t)}; \mathbb{Z}_2)$. It follows that for $\beta \in \widetilde{\Theta}_{\mathbb{B}}$, the $\mathbb{Z}_2$-reduction of $[\beta]$ is equal to $[p_{\mathbb{B}}(\beta)]$.

In the remainder of the paper, we often denote a lifting in $\mathcal{B}$ of $t \in \Theta_{2,\mathbb{B}}$ by $\beta(t)$ which is uniquely determined up to $\{\pm 1\}$-action. In this occasion, β is a section $\Theta_{2,\mathbb{B}} \to \mathcal{B}$ of the quotient map $\mathcal{B} \to \Theta_{2,\mathbb{B}}$.

Let R denote a commutative ring with unit element.

Definition 5.2. Let

$$\begin{aligned} &H_k(X; R) = K(X; R) \oplus K(X; R)^c \quad \text{and} \\ &H_k(X, \partial X; R) = K(X, \partial X; R) \oplus K(X, \partial X; R)^c \end{aligned}$$

be $R[G]$-direct sum decompositions such that

$$\begin{aligned} &\mathrm{Int}_{G,X,\partial X;R}(K(X; R), K(X, \partial X; R)^c) = 0 \quad \text{and} \\ &\mathrm{Int}_{G,X,\partial X;R}(K(X; R)^c, K(X, \partial X; R)) = 0. \end{aligned}$$

Let

$$\begin{aligned} &H_k(X; R_2) = K(X; R_2) \oplus K(X; R_2)^c \quad \text{and} \\ &H_k(X, \partial X; R_2) = K(X, \partial X; R_2) \oplus K(X, \partial X; R_2)^c \end{aligned}$$

be $R_2[G]$-direct decompositions compatible with the decompositions of $H_k(X; R)$ and $H_k(X, \partial X; R)$ such that

$$\begin{aligned} &\mathrm{Int}_{G,X,\partial X;R_2}(K(X; R_2), K(X, \partial X; R_2)^c) = 0 \quad \text{and} \\ &\mathrm{Int}_{G,X,\partial X;R_2}(K(X; R_2)^c, K(X, \partial X; R_2)) = 0, \end{aligned}$$

where $R_2 = R/2R$. The term compatible above means that the canonical maps

$$\begin{aligned} &\pi_X : H_k(X; R) \to H_k(X; R_2) \quad \text{and} \\ &\pi_{X,\partial X} : H_k(X, \partial X; R) \to H_k(X, \partial X; R_2) \end{aligned}$$

preserve the decompositions, i.e.

$$\begin{aligned} &\pi_X(K(X; R)) \subseteqq K(X; R_2), \\ &\pi_X(K(X; R)^c) \subseteqq K(X; R_2)^c, \\ &\pi_{X,\partial X}(K(X, \partial X; R)) \subseteqq K(X, \partial X; R_2), \quad \text{and} \\ &\pi_{X,\partial X}(K(X, \partial X; R)^c) \subseteqq K(X, \partial X; R_2)^c. \end{aligned}$$

If $x \in H_k(X, \partial X; R)$ (resp. $H_k(X, \partial X; R_2)$) then x has a decomposition

$$x = x_K + x_{K^c}$$

where $x_K \in K(X, \partial X; R)$ (resp. $K(X, \partial X; R_2)$) and $x_{K^c} \in K(X, \partial X; R)^c$ (resp. $K(X, \partial X; R_2)^c$). Define

$$\theta_K : \widetilde{\Theta}_{\mathbb{B}} \to K(X, \partial X; R);\ \beta \mapsto (h_{\beta_*}[\beta]_R)_K$$
$$\theta_{2,K} : \Theta_{2,\mathbb{B}} \to K(X, \partial X; R_2);\ t \mapsto (h_{\beta(t)_*}[t]_{R_2})_K.$$

Clearly, the diagram

$$\begin{tikzcd} \widetilde{\Theta}_{\mathbb{B}} \arrow[r, "\theta_K"] \arrow[d, "p_{\mathbb{B}}"'] & K(X, \partial X; R) \arrow[d, "\pi_{X,\partial X}"] \\ \Theta_{2,\mathbb{B}} \arrow[r, "\theta_{2,K}"'] & K(X, \partial X; R_2) \end{tikzcd}$$

commutes. The 8-tuple

$$(K(X, R), K(X, \partial X; R), \mathrm{Int}_{G,X,\partial X;R},\ \theta_K,$$
$$K(X, \partial X; R_2),\ K(X; R_2),\ \mathrm{Int}_{G,X,\partial X;R_2},\ \theta_{2,K})$$

is called a *geometric positioning module*.

The 8-tuple is usually abbreviated by $(K(X, R), K(X, \partial X; R), \mathrm{Int}_{G,X,\partial X;R}, \theta_K, \theta_{2,K})$ if the canonical maps

$$\pi_X : K(X; R) \to K(X; R_2) \text{ and}$$
$$\pi_{X,\partial X} : K(X, \partial X; R) \to K(X, \partial X; R_2)$$

are surjective.

Definition 5.3. We call the geometric positioning module above *nonsingular*, if

(5.3.1) $K(X; R)$ and $K(X, \partial X; R)$ are finitely generated, projective $R[G]$-modules,
(5.3.2) $\mathrm{Int}_{G,X,\partial X;R}$ on $K(X; R) \times K(X, \partial X; R)$ is nonsingular, and
(5.3.3) $\pi_X : K(X; R) \to K(X; R_2)$ and $\pi_{X,\partial} : K(X, \partial X;\ R) \to K(X, \partial X; R_2)$ are surjective.

If the geometric positioning module is nonsingular, it follows of course that $K(X; R_2)$ and $K(X, \partial X; R_2)$ are finitely generated $R_2[G]$-modules and $\mathrm{Int}_{G,X,\partial X;R_2}$ is nonsingular.

Definition 5.4. We shall call the geometric positioning module *Hermitian* if the canonical maps $K(X; R) \to K(X, \partial X; R)$ and $K(X; R_2) \to K(X, \partial X; R_2)$ are isomorphisms.

This make sense, since by Lemma 4.3 the diagram

$$\begin{tikzcd} K(X; R) \times K(X; R) \arrow[r, "\mathrm{Int}_{G,X;R}"] \arrow[d] & R[G] \\ K(X; R) \times K(X, \partial X; R) \arrow[ur, "\mathrm{Int}_{G,X,\partial X;R}"'] & \end{tikzcd}$$

commutes and $\mathrm{Int}_{G,X;R}$ is λ-Hermitian, where $\lambda = (-1)^k$. Similar remarks hold when R is replaced by R_2. We shall abbreviate a Hermitian geometric positioning module by

$$(K(X; R), \mathrm{Int}_{G,X;R}, \theta_K, \theta_{2,K}).$$

Theorem 5.5. *Suppose that X is simply connected. Let $\{h_\alpha : A_\alpha \to X \mid \alpha \in \mathcal{A}\}$ be a finite set of immersions h_α of connected, closed, k-dimensional, oriented, smooth manifolds A_α. Let $(K(X,\mathbb{Z}), K(X,\partial X;\mathbb{Z})$, $\mathrm{Int}_{G,X,\partial X;\mathbb{Z}}$, θ_K, $\theta_{2,K})$ be a geometric positioning module in Definition 5.2 such that $h_{\alpha *}[\alpha] \in K(X;\mathbb{Z})$ for each $\alpha \in \mathcal{A}$, where $[\alpha]$ is the orientation class of A_α. Let $\natural_G$ be the G-self-intersection form in Definition 4.13. Then (5.5.1) and (5.5.2) below are equivalent.*

(5.5.1) (i) $\mathrm{Int}_{G,X,\partial X;\mathbb{Z}}(h_{\alpha *}[\alpha], h'_{\alpha' *}[\alpha']) = 0$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$.
(ii) $\mathrm{Int}_{G,X,\partial X;\mathbb{Z}}(h_{\alpha *}[\alpha], \theta_K([\beta])) = 0$ *for all* $(\alpha,\beta) \in \mathcal{A} \times \widetilde{\Theta}_{\mathbb{B}}$ *and* $\mathrm{Int}_{G,X,\partial X;\mathbb{Z}_2}(h_{\alpha *}[\alpha], \theta_{2,K}(t)) = 0$ *for all* $(\alpha,t) \in \mathcal{A} \times \Theta_{2,\mathbb{B}}$.
(iii) $\mathrm{Int}_{G,X,\partial X;\mathbb{Z}}(h_{\alpha *}[\alpha], h_{\alpha *}[\alpha]) = 0$ *and* $\natural_G(h_\alpha) = 0$ *for all* $\alpha \in \mathcal{A}$.

(5.5.2) *There is a family of regular homotopies $h_\alpha \sim h'_\alpha$ in X where α ranges through $\mathcal{A}$ such that the following holds:*
(a) $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_{\alpha'}) = \varnothing$ *for all* $\alpha \neq \alpha' \in \mathcal{A}$ *and* $g \in G$.
(b) $\mathrm{Image}(h'_\alpha) \cap B_{\beta(t)} = \varnothing$ *for all* $(\alpha,t) \in \mathcal{A} \times \Theta_{2,\mathbb{B}}$.
(c) h'_α *is an embedding such that* $\chi(\nu(h'_\alpha)) = 0$ *and* $\mathrm{Image}(h'_\alpha) \cap g\mathrm{Image}(h'_\alpha) = \varnothing$ *for all* $\alpha \in \mathcal{A}$ *and* $g \neq 1 \in G$.

Proof. It follows from Lemma 4.6 and the observations

$$\mathrm{Int}_{G,X,\partial X;\mathbb{Z}}((h_\alpha)_*[\alpha], \theta_K(\beta)) = \mathrm{Int}_{G,X,\partial X;\mathbb{Z}}(h_{\alpha *}[\alpha], h_{\beta *}([\beta]))$$

and

$$\mathrm{Int}_{G,X,\partial X;\mathbb{Z}_2}(h_{\alpha *}[\alpha], \theta_{2,K}(t)) = \mathrm{Int}_{G,X,\partial X;\mathbb{Z}_2}(h_{\alpha *}[\alpha], h_{\beta(t)_*}([t]))$$

that (5.5.1) $\Longleftrightarrow$ (4.21.1). Thus the assertion of the theorem follows from the assertion in Theorem 4.21. □

Let $f : (X,\partial X) \to (Y,\partial Y)$ be a G-map. The map f is called an *R-boundary quasiequivalence* if the induced map $H_*(\partial X;R) \to H_*(\partial Y;R)$ is an isomorphism. If f is an R-boundary quasiequivalence then f is necessarily an R_2-boundary quasiequivalence. The map f is called a *(G,R)-singularity quasiequivalence* if the induced map

$$H_*(\mathrm{Sing}(G,X);R) \to H_*(\mathrm{Sing}(G,Y);R)$$

is an isomorphism. If f is a (G,R)-singularity quasiequivalence then f is necessarily a (G,R_2)-singularity quasiequivalence.

Lemma 5.6. *Let $f : (X,\partial X) \to (Y,\partial Y)$ be a degree-one G-map. Then*

$$(K_k(X;R), K_k(X,\partial X;R), \mathrm{Int}_{G,X,\partial X;R}, \theta_{K_k}, \theta_{2,K_k})$$

is a geometric positioning module. If f is a k-connected, R-boundary quasiequivalence and if $K_k(X;R)$ is $R[G]$-projective then $(K_k(X;R), \mathrm{Int}_{G,X;R}, \theta_{K_k}, \theta_{2,K_k})$ is a nonsingular Hermitian geometric positioning module.

Proof. By (4.4.1)–(4.4.3), the decompositions $H_k(X;R') = K_k(X;R') \oplus \widehat{f}_*(H_k(Y;R'))$ and $H_k(X,\partial X;R') = K_k(X,\partial X;R') \oplus \widehat{f}_*(H_k(Y,\partial Y;R'))$ (for $R' = R$ and R_2) satisfy the conditions required in Definition 5.2. Thus,

$$(K_k(X;R), K_k(X,\partial X;R), \mathrm{Int}_{G,X,\partial X;R}, \theta_{K_k},\\ K_k(X;R_2), K_k(X,\partial X;R_2), \mathrm{Int}_{G,X,\partial X;R}, \theta_{2,K_k})$$

is a geometric positioning module.

By k-connectedness, $K_{k-1}(X;\mathbb{Z}) = 0$. By the universal coefficient theorem, the canonical map

$$R' \otimes_{\mathbb{Z}} K_k(X;\mathbb{Z}) \to K_k(X;R')$$

is an isomorphism for an arbitrary $\mathbb{Z}$-module R'. Thus the canonical map $R_2 \otimes_R K_k(X;R) \to K_k(X;R_2)$ is an isomorphism.

Next suppose f is an R-boundary quasiequivalence. Then the maps $K_k(X;R) \to K_k(X,\partial X;R)$ and $K_k(X;R_2) \to K_k(X,\partial X;R_2)$ are isomorphisms. It follows that the map $R_2 \otimes_R K_k(X,\partial X;R) \to K_k(X,\partial X;R_2)$ is also an isomorphism. The facts above establish that $(K_k(X;R), \mathrm{Int}_{G,X;R}, \theta_{K_k}, \theta_{2,K_k})$ is a Hermitian geometric positioning module.

If $K_k(X;R)$ is an $R[G]$-projective module then by Lemma 4.5, $(K_k(X;R), \mathrm{Int}_{G,X;R})$ is a nonsingular, λ-Hermitian module. Thus $(K_k(X;R),\ \mathrm{Int}_{G,X;R}, \theta_{K_k}, \theta_{2,K_k})$ is a nonsingular Hermitian geometric positioning module. $\square$

Let $f : (X,\partial X) \to (Y,\partial Y)$ be a G-map of smooth G-manifolds X and Y. Let $T(X)$ denote the tangent bundle of X. The smooth action of G on X gives $T(X)$ a real G-vector bundle structure. Let ξ_+ and ξ_- be real G-vector bundles on Y and let $f^*(\xi_+)$ and $f^*(\xi_-)$ denote the pullback bundles of ξ_+ and ξ_-, respectively, to X. Let $\xi = \xi_+ - \xi_-$ denote the formal difference. A *G-framing* of f with respect to ξ is an isomorphism $b : T(X) \oplus f^*(\xi_- \oplus \eta) \to f^*(\xi_+ \oplus \eta)$ of real G-vector bundles where η is a real G-vector bundle over Y. Usually $\eta = \varepsilon_Y(V)$ for a finite dimensional, real G-module (i.e. real G-representation space) V. Here $\varepsilon_Y(V)$ is the product bundle on Y with fiber V (i.e. trivial G-vector bundle) and $f^*\varepsilon_Y(V) = \varepsilon_X(V)$. A *$G$-framed map* is a pair $\boldsymbol{f} = (f,b) : (X,\partial X,T(X)) \to (Y,\partial Y,\xi)$.

Definition 5.7. Let $\boldsymbol{f} = (f,b) : (X,\partial X,T(X)) \to (Y,\partial Y,\xi)$ be a k-connected, degree-one G-framed map. Suppose that f is an R-boundary quasiequivalence and $K_k(X;R)$ is $R[G]$-projective. Such a map $\boldsymbol{f}$ is called a *k-connected (R,G)-surgery map.* Define

$$B_{f,R} = \mathrm{Int}_{G,X;R|_{K_k(X;R)\times K_k(X;R)}}, \quad \theta_{f,\mathbb{B}} = \theta_{K_k}, \quad \theta_{2,f,\mathbb{B}} = \theta_{2,K_k}.$$

By Lemma 5.6, $(K_k(X;R), B_{f,R}, \theta_{f,\mathbb{B}}, \theta_{2,f,\mathbb{B}})$ is a nonsingular Hermitian geometric positioning module on the projective $R[G]$-module $K_k(X;R)$. Let

$$\begin{aligned}
Q(G,X) &= G(X,k-1) \cap G(2) \\
S(G,X) &= G(X,k) \cap G(2) \\
\Lambda(Q(G,X)) &= \min\nolimits_{\lambda}(\mathbb{Z}[G]) + \mathbb{Z}[Q(G,X)] \\
\Gamma(S(G,X)) &= \min\nolimits_{-\lambda}(\mathbb{Z}[G]) + \mathbb{Z}[S(G,X)]
\end{aligned}$$

(cf. Section 3). Since f is k-connected and of degree one, we deduce that the relative Hurewicz map $\pi_k(f) \to K_k(X;\mathbb{Z})$ is an isomorphism. Thus, each element of $K_k(X;\mathbb{Z})$ can be represented by an immersion $h : S^k \to X$ of the oriented k-sphere S^k. Since $\boldsymbol{f}$ is G-framed, we can in fact choose h such that its normal bundle $\nu(h)$ in X is trivial, cf. [44, Part 0, p.10, Theorem 1.1 and Proposition]. Such an h is unique up to regular homotopy in X, cf. [44, Part 0, Theorem 1.1] or the proposition subsequent to it. For each $x \in K_k(X;\mathbb{Z})$, let $h_x : S^k \to X$ be an immersion of the k-sphere S^k with trivial normal bundle such that $h_{x*}[S^k] = x$. By Lemma 4.6,

$$B_{f,\mathbb{Z}}(x,y) = \#_G(h_x, h_y).$$

Let

$$\natural_G : \mathrm{Immersion}(S^k, \mathrm{Free}(G,X)) \to \mathbb{Z}[G]/(\Lambda(Q(G,X)) + \mathbb{Z}[S(G,X)])$$

denote the generalized, G-equivariant, self-intersection form in Definition 4.13 and define

$$q_{\boldsymbol{f}} : K_k(X;\mathbb{Z}) \to \mathbb{Z}[G]/(\Lambda(Q(G,X)) + \mathbb{Z}[S(G,X)]);\ x \mapsto \natural_G(h_x).$$

By Theorem 4.14, we have the equality

$$B_{f,\mathbb{Z}}(x,y) = \widetilde{q}_{\boldsymbol{f},\mathbb{Z}}(x) + \widetilde{q}_{\boldsymbol{f},\mathbb{Z}}(y) \ \text{ in } \mathbb{Z}[G]/\mathbb{Z}[S(G,\mathbb{Z})]$$

where $\widetilde{q}_{\boldsymbol{f},\mathbb{Z}}(x)$ and $\widetilde{q}_{\boldsymbol{f},\mathbb{Z}}(y)$ denote liftings of $q_{\boldsymbol{f},\mathbb{Z}}(y)$ and $q_{\boldsymbol{f},\mathbb{Z}}(y)$, respectively, to $\mathbb{Z}[G]/\mathbb{Z}[S(G,\mathbb{Z})]$. By Theorem 4.16, we have the equality

$$q_{\boldsymbol{f}}(x+y) = q_{\boldsymbol{f}}(x) + q_{\boldsymbol{f}}(y) + B_{f,\mathbb{Z}}(x,y) \ \text{ in } \mathbb{Z}[G]/(\Lambda(Q(G,X)) + \mathbb{Z}[S(G,X)]).$$

Furthermore, one shows routinely that for any element $a \in \mathbb{Z}[G]$, $q_{\boldsymbol{f}}(ax) = aq_{\boldsymbol{f}}(x)\overline{a}$. It follows that the pair $(B_{f,\mathbb{Z}}, q_{\boldsymbol{f}})$ is a doubly parametrized form, in the sense of Definition 2.6, on $K_k(X;\mathbb{Z})$ over the parameter algebra

$$(\mathbb{Z}[G], (-,\lambda), \Gamma(S(G,X)), G, \mathbb{Z}[S(G,X)], \Lambda(Q(G,X)) + \mathbb{Z}[S(G,X)]).$$

By applying the exact functor $R \otimes -$, we extend $B_{f,\mathbb{Z}}$ and $q_{\boldsymbol{f},\mathbb{Z}}$, respectively, to maps

$$B_{f,R} : K_k(X,R) \times K_k(X,R) \to R[G]$$

and

$$q_{\boldsymbol{f},R} : K_k(X;R) \to R[G]/(R\Lambda(Q(G,X) + R[S(G,X)]))$$

and obtain straightforward that $(B_{f,R}, q_{\boldsymbol{f},R})$ is a doubly parametrized form on $K_k(X;R)$ over the parameter algebra

$$(R[G], (-,\lambda), R\Gamma(S(G,X)), G, R[S(G,X)], R\Lambda(Q(G,X)) + R[S(G,X)])).$$

The 5-tuple

$$\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} = (K_k(X;R), B_{f,R}, q_{\boldsymbol{f},R}, \theta_{f,\mathbb{B}}, \theta_{2,f,\mathbb{B}})$$

is called the $R[G]$-*surgery obstruction module* of $(\boldsymbol{f}, \mathbb{B})$.

Proposition 5.8. *Let* $\boldsymbol{f} : (X, \partial X; T(X)) \to (Y, \partial Y; \xi)$ *be a* k*-connected* (R,G)*-surgery map. Then its surgery obstruction module* $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ *lies in the category* $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$*, where*

$$\boldsymbol{A} = (R[G], (-,\lambda), R\Gamma(S(G,X)), G, R[S(G,X)], R\Lambda(Q(G,X)) + R[S(G,X)])$$

is as in Section 3 and $\boldsymbol{\Theta}_{\mathbb{B}} = (\widetilde{\Theta}_{\mathbb{B}}, p_{\mathbb{B}}, \Theta_{2,\mathbb{B}})$*. Moreover on* $\mathrm{Image}[K_k(X;\mathbb{Z}) \to K_k(X;R)]$*,* $B_{f,R} = \#_G$ *and* $q_{\boldsymbol{f},R} = \natural_G$*.*

Proof. The nonsingularity of $B_{f,R}$ was proved in Lemma 5.6. Everything else was shown directly above. ☐

Our next goal is to show that the surgery obstruction module $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ above lies in a certain subcategory

$$\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$$

of $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$ where $\nabla_{\mathbb{B}}$ is an invariant on the latter category, in the sense of Definition 3.1. To describe $\nabla_{\mathbb{B}}$, it is convenient to use the notation established in Section 3. By definition $A = R[G]$, $\Gamma = R\Gamma(S(G,X))$, $A_s = R[S(G,X)]$, $\Lambda = R\Lambda(Q(G,X)) + R[S(G,X)]$, $A_2 = R_2[G]$. Let $\Gamma'_2 = \min_{-\lambda}(A_2)$ $(= \min_{\lambda}(A_2)$, because $2 = 0$ in $A_2)$, $\varepsilon_2 : A_2 \to R_2$; $\sum_{g\in G} r_g g \mapsto r_1$, $\Gamma'_{2,g} = \varepsilon_2(\Gamma'_2 g^{-1})$, where $g \in G$, and

$$\rho_2 = \rho_G : \Theta_{2,\mathbb{B}} \to \mathcal{S}(G);\ t \mapsto \bigcap_{x \in B_{\beta(t)}} G_x.$$

If $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$ and $g \in S(G,X)$, let

$$\Sigma^{\theta_{2,\mathbb{B}}}_{\rho_2,g} = \sum\nolimits_{t\in\Theta_{2,\mathbb{B}}|_g} \theta_2(t) \qquad \in M_2,$$

where $\Theta_{2,\mathbb{B}}|_g = \{t \in \Theta_{2,\mathbb{B}} \mid \rho_2(t) \ni g\}$ and $M_2 = M/2M$. Define $\Theta_2(G,X)$ to be the set of all submanifolds of X obtained as k-dimensional connected components of X^H for subgroups H of G. By Assumption 4.1, $\Theta_2(G,X)$ is a subset of $\Theta_{2,\mathbb{B}}$.

Lemma 5.9. *Let* $g \in S(G,X)$*. Then the following hold.*

(5.9.1) $\Theta_{2,\mathbb{B}}|_g = \Theta_2(G,X)|_g$*, thus*

$$\Sigma^{\theta_2}_{\rho_2,g} = \sum_{t\in\Theta_2(G,X)|_g} \theta_2(t).$$

(5.9.2) *If A is a connected closed manifold and $h : A \to X$ a continuous map then*

$$\#_2(h, gh) = \sum_{t\in\Theta_2(G,X)|_g} \#_2(h, h_{\beta(t)}) \quad \text{in } \mathbb{Z}_2,$$

where $\#_2(h, gh)$ is the mod 2 geometric intersection number of h and gh.

Proof. (5.9.1): Let $t \in \Theta_{2,\mathbb{B}}$ such that $\rho_2(t) \ni g$. Then $B_{\beta(t)} \subsetneqq X^g$. Since $\dim X^g \leqq k$, $B_{\beta(t)}$ is a k-dimensional connected component of X^g. Thus t belongs to $\Theta_2(G, X)$.

(5.9.2): For the computation of the intersection number, we may assume that $h : A \to X$ is an immersion and that h, gh and X^g intersect transversely with one another. If h and gh meet each other at a point x in $X \smallsetminus X^g$ then h and gh meet at $gx \in X \smallsetminus X^g$, too. Thus the number of points in $X \setminus X^g$ at which h and gh meet each other is an even integer. Thus $\#_2(h, gh)$ is equal (mod 2) to the number of points in X^g at which h and gh meet each other. This number is equal (mod 2) to $\#_2(h, X^g)$. On the other hand

$$\#_2(h, X^g) = \sum_{t\in\Theta_2(G,X)|_g} \#_2(h, h_{\beta(t)}).$$

□

If $(M, B, q, \theta, \theta_2) \in \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$ and $s \in S(G, X)$, let

$$\alpha_{\Gamma'_{\alpha,s}}(M, B, q, \theta, \theta_2) = \mathrm{Hom}_{\mathbb{Z}}(M_2, R_2\Gamma'_{2,s}) = \mathrm{Hom}_{\mathbb{Z}}(M_2, R_2)$$

and

$$\nabla_{\Gamma'_{2,s}}(M, B, q, \theta, \theta_2) : M_2 \to R_2;\ m \mapsto \varepsilon_2(B_{f,R_2}(sm, \Sigma^{\theta_2}_{\rho_2,s} - m)).$$

Let

$$\nabla_{\Gamma'_2} = \bigoplus_{s\in S(G,X)} \nabla_{\Gamma'_{2,s}}$$
$$\alpha_{\Gamma'_2} = \bigoplus_{s\in S(G,X)} \alpha_{\Gamma'_{2,s}}$$
$$\nabla_{\mathbb{B}} = (\nabla_{\Gamma'_2}, \alpha_{\Gamma'_2}).$$

By Lemma 3.25, $\nabla_{\mathbb{B}}$ defines an invariant on $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$. It is called the *geometric invariant* on $\mathcal{Q}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$.

Lemma 5.10. *Suppose $R_2 = \mathbb{Z}/2\mathbb{Z}$ or 0. If $\boldsymbol{f} : (X, \partial X, T(X)) \to (Y, \partial Y, \xi)$ is a k-connected (R, G)-surgery map then the surgery obstruction module $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ belongs to $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$, namely*

$$\varepsilon_2(B_{f,R_2}(sm, \Sigma^{\theta_{2,f,\mathbb{B}}}_{\rho_2,s} - m))) = 0$$

for all $m \in K_k(X; R_2) = R_2 \otimes_R K_k(X; R)$.

Proof. Unwinding definitions, one obtains that

$$\begin{aligned}\varepsilon_2(B_{f,R_2}(sm, \Sigma^{\theta_{2,f,\mathbb{B}}}_{\rho_2,s} - m)) &= \mathrm{Int}_{X;R_2}(sm, \Sigma^{\theta_{2,f,\mathbb{B}}}_{\rho_2,s} - m)\\ &= \mathrm{Int}_{X,\partial X;R_2}(m, \Sigma^{\theta_{2,f,\mathbb{B}}}_{\rho_2,s} - sm).\end{aligned}$$

Choose an immersion $h : S^k \to X$ such that the normal bundle $\nu(h)$ of h in X is trivial and $h_*[S^k] = m$. By definition and (5.9.1),

$$\Sigma_{\rho_2,s}^{\theta_{2,f,\mathbb{B}}} = \sum_{t \in \Theta_2(G,X)|_s} \theta_{2,f,\mathbb{B}}(t).$$

Since $h_{\beta(t)} : B_{\beta(t)} \to X$ is the canonical embedding, it follows over R_2 that

$$\begin{aligned} \mathrm{Int}_{X,\partial X;R_2}(m, \Sigma_{\rho_2,s}^{\theta_{2,f,\mathbb{B}}} - sm) &= \sum_{t \in \Theta_2(G,X)|_s} \mathrm{Int}_{X,\partial X;R_2}(h_*[S^k], s(h_{\beta(t)})_*[t]) \\ &\qquad - \mathrm{Int}_{X,\partial X;R_2}(h_*[S^k], sh_*[S^k]) \\ &= \sum_{t \in \Theta_2(G,X)|_s} \#_2(h, sh_{\beta(t)}) - \#_2(h, sh) \quad \text{(by Lemma 4.6)} \\ &= \sum_{t \in \Theta_2(G,X)|_s} \#_2(h, h_{\beta(t)}) - \#_2(h, sh) \\ &= 0 \quad \text{(by (5.9.2))}. \end{aligned}$$

□

Definition 5.11. Suppose $R_2 = \mathbb{Z}/2\mathbb{Z}$ or 0. If $\boldsymbol{f} : (X, \partial X, T(X)) \to (Y, \partial Y, \xi)$ is a k-connected (R, G)-surgery map, define

$$\sigma(\boldsymbol{f}, \mathbb{B}) = [\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}] \in W_{2k}(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta}_{\mathbb{B}})_{\mathrm{proj}}.$$

If $K_k(X; R)$ is stably free over $R[G]$ (cf. Remark 3.35) then we can regard

$$\sigma(\boldsymbol{f}, \mathbb{B}) \in W_{2k}(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta}_{\mathbb{B}})_{\mathrm{free}}.$$

In the remainder of this paper we suppose $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$.

Lemma 5.12. *A finitely generated R-free, $R[G]$-module M is $R[G]$-projective if and only if $M_{(p)} = M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is $\mathbb{Z}_{(p)}[G]$-projective for all (rational) primes p that are not invertible in R.*

Proof. The proof is similar to that of Lemma 2 of Nakayama [34]. □

Let $\mathcal{P}(G)$ denote the set of all subgroups of G of prime power order. A G-map $f : X \to Y$ (or $(X, \partial X) \to (Y, \partial Y)$) is called a *$\mathcal{P}(G)_R$-singularity quasiequivalence* if the induced map $H_*(X^P; \mathbb{Z}_{(p)}) \to H_*(Y^P; \mathbb{Z}_{(p)})$ is an isomorphism for any prime p not invertible in R and any nontrivial p-subgroup P of G.

Lemma 5.13. *Let $f : (X, \partial X) \to (Y, \partial Y)$ be a degree-one R-boundary and $\mathcal{P}(G)_R$-singularity quasiequivalence such that $K_\ell(X; R) = 0$ for all $\ell \leqq k-1$. Then $K_\ell(X; R) = 0$ if $\ell \neq k$ and $K_k(X; R)$ is a finitely generated, projective $R[G]$-module. If moreover f is a (G, R)-singularity quasiequivalence then $K_k(X; R)$ is a stably free $R[G]$-module.*

Proof. Let C_f denote the mapping cone of f. Then $K_*(X;M) \cong \overline{H}_{*+1}(C_f;M)$ for any $\mathbb{Z}$-module M. Thus the universal coefficient theorem holds for the functors $K_*(-;-)$ and $K^*(-;-)$. Since $K_\ell(X;R) = 0$ for $\ell \leqq k-1$, we obtain $K^\ell(X;R) = 0$ for $\ell \leqq k-1$ by the universal coefficient theorem. By Poincaré-Lefschetz duality (cf. [8, I.2.8]), $K_\ell(X,\partial X;R) = 0$ for $\ell \geqq k+1$. From the exact sequence of the pair $(X,\partial X)$ with coefficient ring R, it follows that $K_\ell(X;R) = 0$ for $\ell \geqq k+1$. Thus $K_\ell(X;R) = 0$ for $\ell \neq k$. Since

$$K_k(X;R) = K_k(X,\partial X;R) = K^k(X;R) = \mathrm{Hom}(K_k(X;R),R)$$

and $K_k(X;R) = K_k(X;\mathbb{Z}) \otimes R$, it follows that $K_k(X;R)$ is a free R-module.

Next we prove that $K_k(X;R)$ is $R[G]$-projective. By Lemma 5.12, it suffices to show that $K_k(X;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}[G]$-projective for each prime p not invertible in R. It is enough to show that $K_k(X;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}[P]$-projective for any prime p as above and any Sylow p-subgroup P of G. If P is the trivial group then it is obviously the case. Suppose $P \neq \{1\}$. By hypothesis, $f|_{\mathrm{Sing}(P,X)} : \mathrm{Sing}(P,X) \to \mathrm{Sing}(P,Y)$ is a $\mathbb{Z}_{(p)}$-homology equivalence. This amounts to $K_*(X;\mathbb{Z}_{(p)}) = K_*(X,\mathrm{Sing}(P,X);\mathbb{Z}_{(p)})$. Set $K = K_k(X,\mathrm{Sing}(G,X);\mathbb{Z}_{(p)})$. By observation of the cellular chain complex with coefficient ring $\mathbb{Z}_{(p)}$ of the mapping cone of $f : (X,\mathrm{Sing}(P,X)) \to (Y,\mathrm{Sing}(P,Y))$ using [42][Proposition 3.5 (i) and (v)], we conclude that K/pK is $\mathbb{Z}_p[P]$-free. Let $\psi : \mathbb{Z}_p[P]^m \to K/pK$ be a $\mathbb{Z}_p[P]$-isomorphism and $\widetilde{\psi} : \mathbb{Z}_{(p)}[P] \to K$ be a $\mathbb{Z}_{(p)}[P]$-homomorphism covering ψ. Let $\mathfrak{A} = \{ga_i \mid g \in P,\ i = 1, \ldots, m\}$ be the canonical $\mathbb{Z}_{(p)}$-basis of $\mathbb{Z}_{(p)}[P]$. Let

$$\mathfrak{B} = \{b_{g,i} \mid g \in P,\ i = 1, \ldots, m\}$$

be a $\mathbb{Z}_{(p)}$-basis of K. Take the matrix representation τ of $\widetilde{\psi}$ with respect to the bases $\mathfrak{A}$ and $\mathfrak{B}$. Since ψ is an isomorphism, $[\det \tau]$ is an invertible element in $\mathbb{Z}_p$, which implies that $\det \tau$ is invertible in $\mathbb{Z}_{(p)}$. Thus $\widetilde{\psi}$ is an isomorphism. This implies $K_k(X;\mathbb{Z}_{(p)})$ is $\mathbb{Z}_{(p)}[P]$-free. Consequently, $K_k(X;R)$ is $R[G]$-projective.

If f is a (G,R)-singularity quasiequivalence then, by inspecting the cellular chain complex with coefficient ring R of the mapping cone of $f : (X,\mathrm{Sing}(G,X)) \to (Y,\mathrm{Sing}(G,Y))$, one can deduce that $K_k(X;R) = K_k(X,\mathrm{Sing}(G,X);R)$ is a stably free $R[G]$-module. □

Lemma 5.14. *If a degree-one G-map $f : (X,\partial X) \to (Y,\partial Y)$ is a k-connected, R-boundary, $\mathcal{P}(G)_R$-singularity quasiequivalence then*

$$K_k(X;R) = \mathrm{Ker}[f_* : H_k(X;R) \to H_k(Y;R)]$$

is a finitely generated, projective $R[G]$-module. If moreover, f is a (G,R)-singularity quasiequivalence then $K_k(X;R)$ is a stable free $R[G]$-module.

Proof. This follows immediately from Lemma 5.13. □

6. G-surgery obstructions and theorems

The present section has two main results Theorems 6.2 and 6.3. The surgery obstruction $\sigma(\boldsymbol{f}, \mathbb{B})$ is defined in Theorem 6.3 as the class of surgery module $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ in the surgery obstruction group $W_n(R, G, Q(G,X), S(G,X), \boldsymbol{\Theta}_{\mathbb{B}})_{\mathrm{proj}}$. The results are for any coefficient ring R such that $\mathbb{Z} \subseteqq R \subseteqq \mathbb{Q}$. Let $\boldsymbol{M}'$ denote the orthogonal sum $\boldsymbol{M}' = \boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \oplus \mathbb{M}$ of a G-surgery obstruction module $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ and a metabolic plane $\mathbb{M}$ which vanishes under the geometric invariant $\nabla_{\mathbb{B}}$. Theorem 6.2 says that one can convert by G-surgery the G-framed map $\boldsymbol{f}$ to a G-framed map $\boldsymbol{f}'$ such that $\boldsymbol{M}' = \boldsymbol{M}_{\boldsymbol{f}',\mathbb{B}}$. Since $\mathbb{M}$ vanishes by definition in the G-surgery obstruction group, it will follow that $\sigma(\boldsymbol{f}, \mathbb{B}) = \sigma(\boldsymbol{f}', \mathbb{B})$. Theorem 6.3 says that the vanishing of $\sigma(\boldsymbol{f}, \mathbb{B})$ is a sufficient condition for converting $\boldsymbol{f}$ by G-surgery into a G-framed homotopy equivalence $\boldsymbol{f}'$.

Let I denote the closed unit interval $[0,1]$ and $p_X : I \times X \to X$ the canonical projection on the second factor X. Recall that a *G-cobordism* W between X and X' is a compact oriented smooth G-manifold of dimension $n+1$ such that the boundary ∂W of W is a union of two G-submanifolds $\partial_+ W$ and $\partial_- W$ satisfying the properties

$$\partial_+ W = -X \amalg X' \quad (-X \text{ denotes } X \text{ with the opposite orientation})$$
$$\partial_- W \cong_G I \times \partial X' \quad (\cong_G \text{ reads } G\text{-diffeomorphic to})$$
$$\partial_+ W \cap \partial_- W = \partial(\partial_+ W) = \partial(\partial_- W) = \partial(-X) \amalg \partial(X').$$

$TW|_{\partial_+ W}$ is identified with $\varepsilon_{\partial W}(\mathbb{R}) \oplus T(\partial_+ W)$

in the standard way, namely the inward normal bundle $\nu(-X, W)$ of $-X$ in W is identified with $\varepsilon_{-X}(\mathbb{R})$ and the outward normal bundle $\nu(X', W)$ of X' in W is identified with $\varepsilon_{X'}(\mathbb{R})$. If η_i $(i = 1,\ 2)$ are bundles over X, if $b : \eta_1 \to \eta_2$ is a bundle map covering the identity map on X, and if C is a subspace of X, then $b|_C$ denotes the bundle map $b|_{\eta_1|_C} : \eta_1|_C \to \eta_2|_C$. Let $\boldsymbol{f} = (f, b) : (X, \partial X, TX) \to (Y, \partial Y, f^*\xi)$ and $\boldsymbol{f}' = (f', b') : (X', \partial X', TX') \to (Y, \partial Y, f'^*\xi)$ denote G-framed maps. Then a *G-framed cobordism* between $\boldsymbol{f}$ and $\boldsymbol{f}'$ consists of a G-cobordism W from X to X' and a G-framed map

$$(F, B) : (W, \partial W, TW) \to (I \times Y, \partial(I \times Y), (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi)),$$

where

$$B : TW \oplus (p_Y \circ F)^*(\xi_- \oplus \eta) \to \varepsilon_W(\mathbb{R}) \oplus (p_Y \circ F)^*(\xi_+ \oplus \eta),$$

such that $F(X) \subseteqq \{0\} \times Y$, $F(X') \subseteqq \{1\} \times Y$, $F|_X = f$, $F|_{X'} = f'$, $B|_X = id_{\varepsilon_X(\mathbb{R})} \oplus b$ and $B|_{X'} = id_{\varepsilon_{X'}(\mathbb{R})} \oplus b'$. Unless specifically mentioned otherwise, we assume that the stable term η has the form $\eta = \varepsilon_Y(U)$ for a real G-module U such that $\dim U^G \geqq n$.

Let C denote a G-simplicial subcomplex of X with respect to some smooth equivariant triangulation of X. The G-framed cobordism (F, B) above is said to be *relative to C*, if there exists a manifold G-neighborhood $N = N(C, X)$ of C in X such that the canonical inclusion $N \to X$ extends to a (neat) G-embedding $I \times N \to W$ with the property that $N = X \cap (I \times N) = \{0\} \times N$ (the left equality holds in X, and the right equality holds in

$I\times N)$, $X'\cap(I\times N)=\{1\}\times N$, and for each $t\in I$, $F|_{\{t\}\times C}=f|_C$ (i.e. $F(\{t\}\times C)\subseteqq\{t\}\times Y$ and the composite map

$$C\longrightarrow\{t\}\times C\xrightarrow{F}I\times Y\xrightarrow{p_Y}Y$$

is the map $f|_C$), $B|_{\{t\}\times C}=(id_{\varepsilon_X(\mathbb{R})}\oplus b)|_C$ (i.e. the composite map

$$\begin{aligned}(\varepsilon_X(\mathbb{R})\oplus TX\oplus f^*(\xi_-\oplus\eta))|_C &\longrightarrow(\varepsilon_N(\mathbb{R})\oplus TN\oplus(f|_N)^*(\xi_-\oplus\eta))|_C\\ &\longrightarrow(T(I\times N)\oplus((p_Y\circ F)|_{I\times N})^*(\xi_-\oplus\eta))|_{\{t\}\times C}\\ &\longrightarrow(TW\oplus(p_Y\circ F)^*(\xi_-\oplus\eta))|_{\{t\}\times C}\\ &\xrightarrow{B}((p_Y\circ F)^*(\varepsilon_Y(\mathbb{R})\oplus\xi_+\oplus\eta))|_{\{t\}\times C}\\ &=(\varepsilon_X(\mathbb{R})\oplus f^*(\xi_+\oplus\eta))|_C\end{aligned}$$

is the map $(id_{\varepsilon_X(\mathbb{R})}\oplus b)|_C$). If (F,B) is relative to C then one says that $\boldsymbol{f}$ is *G-framed cobordant rel C* to $\boldsymbol{f}'$.

Performing G-surgery on a G-framed map $\boldsymbol{f}=(f,b)$, $f:X\to Y$, defines a pair (F,B) of maps called the *trace* of G-surgery, which satisfies the conditions above for a G-framed cobordism, except possibly the condition that $B|_{X'}=id_{\varepsilon_{X'}(\mathbb{R})}\oplus b'$. The next lemma guarantees that we can perform the G-surgery in a way that its trace (F,B) is a G-framed cobordism in the sense above.

Lemma 6.1. *Let (M,M_0) denote an n-dimensional, finite G-CW-pair, i.e. $M_0\subseteqq M$ and $n=\dim(M\smallsetminus M_0)$. Let ω and ω' denote real G-vector bundles supplied with a G-invariant Riemannian metric over M. Let $\oplus$ denote the operation of orthogonal sum on G-vector bundles with a G-invariant Riemannian metric. If $\omega\supseteqq\varepsilon_M(\mathbb{R}^{n'})$ where $n'=\max(n,1)$ then any G-vector bundle isomorphism $b:\varepsilon_M(\mathbb{R})\oplus\omega\to\varepsilon_M(\mathbb{R})\oplus\omega'$ such that $b|_{M_0}=id_{\varepsilon_{M_0}(\mathbb{R})}\oplus b_0'$, for some G-vector bundle isomorphism $b_0':\omega_{M_0}\to\omega'_{M_0}$, is regularly G-homotopic rel M_0 to an orthogonal sum $id_{\varepsilon_M(\mathbb{R})}\oplus b'$ of G-vector bundle isomorphisms $id_{\varepsilon_M(\mathbb{R})}$ and $b':\omega\to\omega'$ such that $b'|_{M_0}=b_0'$.*

Proof. It is well known that b is regularly G-homotopic to a metric-preserving isomorphism. (This follows from the equivariant covering homotopy property and from the fact that if $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ are G-invariant Riemannian metrics on the same underlying G-vector bundle ξ, then $(1-t)\langle\ ,\ \rangle+t\langle\ ,\ \rangle'$ $(t\in I)$ is a G-invariant Riemannian metric on ξ.) Thus we may assume that b is metric preserving.

We prove Lemma 6.1 by double induction on n and the number of isotropy types of n-dimensional cells in $M\smallsetminus M_0$. Suppose $M_0\subseteqq M'$ and $M=M'\cup\bigcup_j(G/H\times D_j^n)$, where $D_j^n=D^n$. Invoke the inductive hypothesis that $b|_{M'}$ has the form $id_{\varepsilon_{M'}(\mathbb{R})}\oplus b''$ where $b'':\omega|_{M'}\to\omega'|_{M'}$. Under this hypothesis, we will find b' as in Lemma 6.1. For fixed j, set $E=H/H\times D_j^n$. Then $b(\varepsilon_M(\mathbb{R})|_{\overline{E}\smallsetminus\overset{\circ}{E}})=\varepsilon_M(\mathbb{R})|_{\overline{E}\smallsetminus\overset{\circ}{E}}$ where $\overline{E}=\mathrm{Closure}(E)$ and $\overset{\circ}{E}=\mathrm{Interior}(E)$, but it is not necessary that

$$b(\varepsilon_M(\mathbb{R})|_E)=\varepsilon_M(\mathbb{R})|_E. \tag{6.1.1}$$

Let $b^H : \varepsilon_{M^H}(\mathbb{R}) \oplus \omega^H \to \varepsilon_{M^H}(\mathbb{R}) \oplus \omega'^H$ be the restriction of b to the H-fixed point set. Then $b|_{M^H}$ decomposes as a sum $b|_{M^H} = b^H \oplus b_H$ ($N_G(H)$-orthogonal sum). We deform b keeping $b|_{M'}$ and b_H fixed. The obstruction σ to modifying b to satisfy (6.1.1) lies in $\pi_{n-1}(S^{m-1})$, where $m = \text{fiber-dim}(\omega^H) + 1$. Since $\text{fiber-dim}(\omega^H) \geqq \text{fiber-dim}(\omega^G) \geqq n$, the obstruction group $\pi_{n-1}(S^{m-1})$ is trivial. Hence the obstruction σ vanishes. If (6.1.1) is satisfied for all j then $b(\varepsilon_M(\mathbb{R})) = \varepsilon_M(\mathbb{R})$. Since b is metric preserving, we get $b(\omega) \subseteqq \omega'$. Moreover, we can arrange b so that $b|_{\varepsilon_M(\mathbb{R})} = id_{\varepsilon_M(\mathbb{R})}$, because $\text{fiber-dim}(\varepsilon_M(\mathbb{R}) \oplus \varepsilon_M(\mathbb{R}^{n'})) \geqq 2$. □

Recall that $\mathbb{Z} \subseteqq R \subseteqq \mathbb{Q}$, $\lambda = (-1)^k$,

$$\boldsymbol{A} = (R[G], (-, \lambda), R\Gamma(S(G,X)), G, R[S(G,X)], R\Lambda(Q(G,X) + R[S(G,X)]),$$

and $\Gamma'_2 = \min_{-\lambda}(R_2[G])$.

The formulation of the next theorem makes use of the classification in Lemma 3.32 of metabolic planes in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$.

Theorem 6.2. *Let $\boldsymbol{f} = (f, b) : (X, \partial X, TX) \to (Y, \partial Y, f^*\xi)$ denote a G-framed map such that f is a k-connected, $\mathcal{P}(G)_R$-singularity, R-boundary quasiequivalence. Let $(L, \mathbb{B})$ denote a G-singularity structure of X (cf. Definition 5.1). Let*

$$\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} = (K_k(X; R), B_{f,R}, q_{\boldsymbol{f},R}, \theta_{f,\mathbb{B}}, \theta_{2,f,\mathbb{B}})$$

denote the $R[G]$-surgery obstruction module (Definition 5.7) in $\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$ of $(\boldsymbol{f}, \mathbb{B})$. By Lemma 5.10, $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ lies in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$. Let $\mathbb{M}(R[G], \boldsymbol{c})$ ($\boldsymbol{c} : \boldsymbol{\Theta}_{\mathbb{B}} \to (R, R_2)$) denote a metabolic plane in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$. Then one can perform G-surgery on $\boldsymbol{f}$ along $(k-1)$-dimensional spheres in Interior$(X \smallsetminus L)$ *to obtain a k-connected G-framed map $\boldsymbol{f}' = (f', b') : (X', \partial X', TX') \to (Y, \partial Y, f'^*\xi)$ such that $\boldsymbol{M}_{\boldsymbol{f}',\mathbb{B}} \cong \boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \oplus \mathbb{M}(R[G], \boldsymbol{c})$ in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$.*

Note that in the theorem above, the resulting map f' is a $\mathcal{P}(G)_R$-singularity and R-boundary quasiequivalence, because the surgery is performed on Interior(X).

The proof of Theorem 6.2 will be given after the following application.

Theorem 6.3. *Let $\boldsymbol{f} = (f, b) : (X, \partial X, TX) \to (Y, \partial Y, f^*\xi)$ and $(L, \mathbb{B})$ denote a G-framed map and a G-singularity structure of X, respectively, in the previous theorem. Let*

$$\sigma(\boldsymbol{f}, \mathbb{B}) \in W_n(R, G, Q(G,X), S(G,X), \boldsymbol{\Theta}_{\mathbb{B}})_{\text{proj}}$$

denote the element afforded by $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$. If $\sigma(\boldsymbol{f}, \mathbb{B}) = 0$ then one can perform G-surgery on $\boldsymbol{f}$ along $(k-1)$- and k-dimensional spheres in Interior$(X \smallsetminus L)$ *to obtain a k-connected G-framed map $\boldsymbol{f}' = (f', b') : (X', \partial X', TX') \to (Y, \partial Y, f'^*\xi)$ such that $f' : X' \to Y$ is an R-homology equivalence.*

Proof. Suppose $\sigma(\boldsymbol{f}, \mathbb{B}) = 0$. Then there exists modules $\boldsymbol{M}$ and $\boldsymbol{N} \in \nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$ such that $\boldsymbol{N}$ is null in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})_{\boldsymbol{\mathcal{F}}(A)}$, and $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \oplus \boldsymbol{M} \cong \boldsymbol{N} \oplus \boldsymbol{M}$. By Lemma 3.11, the family of metabolic planes in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})_{\boldsymbol{\mathcal{F}}(A)}$ is cofinal in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta}_{\mathbb{B}})$. Thus we can assume without loss of generality that $\boldsymbol{M}$ is isomorphic to a direct sum of metabolic planes in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta}_{\mathbb{B}})_{\boldsymbol{\mathcal{F}}(A)}$. By Lemma 3.32, we may assume that

$$\boldsymbol{M} \cong \mathbb{M}(R[G], \boldsymbol{c}_1) \oplus \cdots \oplus \mathbb{M}(R[G], \boldsymbol{c}_\ell)$$

for some $\boldsymbol{c}_i = (c_i, c_{2,i})$, where $c_i : \widetilde{\Theta}_{\mathbb{B}} \to R$ and $c_{2,i} : \Theta_{2,\mathbb{B}} \to R_2$, $i = 1, \cdots, \ell$. By Theorem 6.2, one can perform G-surgery on $\boldsymbol{f}$ along $(k-1)$-dimensional spheres in $\mathrm{Interior}(X \smallsetminus L)$ to get a G-surgery map $\boldsymbol{f}''$ such that

$$\boldsymbol{M}_{\boldsymbol{f}'',\mathbb{B}} \cong \boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \oplus \mathbb{M}(R[G], \boldsymbol{c}_1) \oplus \cdots \oplus \mathbb{M}(R[G], \boldsymbol{c}_\ell).$$

Since this module is isomorphic to $\boldsymbol{N} \oplus \boldsymbol{M}$, it follows that $\boldsymbol{M}_{\boldsymbol{f}'',\mathbb{B}}$ is a null module in $\nabla_{\mathbb{B}}\boldsymbol{\mathcal{Q}}(\boldsymbol{A},\boldsymbol{\Theta}_{\mathbb{B}})_{\boldsymbol{\mathcal{F}}(A)}$. Therefore we may assume that $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ is null in the free category. In this case there is a free Lagrangian U of $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$ where $U \subset K_k(X;R)$. Let $\{x_1, \cdots, x_\ell\}$ be a basis of U. We can assume without loss of generality that each x_i is represented by an immersion $h_i : S^k \to \mathrm{Interior}(X \smallsetminus L)$ with trivial normal bundle. Note that $\#_G(h_i, h_j) = B_{f,R}(x_i, x_j)$, $\#_G(h_i, h_\beta) = B_{f,R}(x_i, \theta(\beta)) \in \mathbb{Z}$ for $\beta \in \widetilde{\Theta}_{\mathbb{B}}$, $\#_G(h_i, h_{\beta(t)}) = B_{2,f,R}(x_i, \theta_2(t)) \in \mathbb{Z}_2$ for $t \in \Theta_{2,\mathbb{B}}$ where $\beta(t) \in$ is a lifting of t, and $\natural_G(h_i) = q(x_i)$. Thus all these elements are trivial. By Theorem 5.5, we may assume that gh_i and $g'h_j$ are disjoint unless $g = g' \in G$ and $1 \leqq i = j \leqq \ell$, that $\mathrm{Image}(h_i) \cap B_\beta = \varnothing$ for all $1 \leqq i \leqq \ell$ and all $\beta \in \mathcal{B}$, and that h_i is an embedding with trivial normal bundle such that $\mathrm{Image}(h_i) \cap g\mathrm{Image}(h_i) = \varnothing$ for all $1 \leqq i \leqq \ell$ and all $g \neq 1 \in G$. Now we can perform G-surgery on $\boldsymbol{f}$ along $h_1, \ldots, h_\ell$. The resulting G-framed map $\boldsymbol{f}' = (f'; b') : (X', \partial X'; T(X')) \to (Y, \partial Y; f'^*\xi)$ is a G-surgery map in the sense of Definition 5.7 (in particular X' is 1-connected). Moreover $K_k(X'; R) = 0$ by the argument in [44, p.51, lines 12–(−7)] (cf. Proof of Theorem 7.3 Step 4 of [6, p.292]). Thus, $f' : X' \to Y$ is an R-homology equivalence. □

Proof of Theorem 6.2. The proof has 3 steps. Step 1 constructs a smooth G-embedding $\varphi : G \times S^{k-1} \times D^{k+1} \to \mathrm{Int}(X \smallsetminus L)$ satisfying certain properties. After identifying X with $\{1\} \times X$, we obtain a smooth G-embedding $\varphi : G \times S^{k-1} \times D^{k+1} \to I \times X$. Step 2 begins by attaching the G-handle $G \times D^k \times D^{k+1}$ to $I \times X$, using φ as the attaching map. Let

$$W = (I \times X) \bigcup\nolimits_{\varphi} (G \times D^k \times D^{k+1})$$

denote the resulting attaching space and let

$$X' = ((\{1\} \times X) \smallsetminus \varphi(G \times S^{k-1} \times \mathrm{Int}(D^{k+1}))) \bigcup\nolimits_{\varphi|_{G \times S^{k-1} \times S^k}} (G \times D^k \times S^k).$$

Clearly $\partial W = X' \cup (\{0\} \times X) \cup (I \times \partial X)$. Using the properties of φ and Lemma 6.1, we show that the G-framed map (f, b), $f : X \to Y$, extends to a G-framed cobordism (F, B), $F : W \to I \times Y$, rel $(\partial X \cup L)$, which satisfies the usual conditions plus an extra condition set out at the beginning of the section. Let (f', b'), $f' : X' \to Y$, denote the G-framed map obtained by restricting (F, B) to X' and by identifying canonically Y with $\{1\} \times Y$. Step 3 shows that f' is k-connected and that $\mathbb{M}_{\boldsymbol{f}',\mathbb{B}} = \mathbb{M}_{\boldsymbol{f},\mathbb{B}} \oplus \mathbb{M}(R[G], \boldsymbol{c})$.

Step 1. In order to construct an embedding φ which can be used to complete Step 3, we need to fix a model $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ in the isomorphism class of $\mathbb{M}(R[G], \boldsymbol{c})$. This is done as follows. The module $\mathbb{M}(R[G], \boldsymbol{c})$ is defined in Definition 3.30 and by definition, $\boldsymbol{c}$ is a pair (c, c_2) of maps such that the diagram

$$\begin{tikzcd} \widetilde{\Theta}_{\mathbb{B}} \arrow[r, "c"] \arrow[d, "p_{\mathbb{B}}"'] & R \arrow[d] \\ \Theta_{2,\mathbb{B}} \arrow[r, "c_2"'] & R_2 \end{tikzcd}$$

commutes. By Lemma 3.31, the isomorphism class of $\mathbb{M}(R[G], \boldsymbol{c})$ is not changed, if $\boldsymbol{c}$ is replaced by $r\boldsymbol{c} = (rc, rc_2)$ for any invertible element $r \in R$. Let $\mathbb{Z}_2' = \text{Image}[\mathbb{Z} \to R_2]$. Since $\mathbb{Z} \subseteqq R \subseteqq \mathbb{Q}$, there is an invertible element $r \in R$ such that $\text{Image}(rc) \subseteqq \mathbb{Z}$ and $\text{Image}(rc_2) \subseteqq \mathbb{Z}_2'$. After replacing $\boldsymbol{c}$ by $r\boldsymbol{c}$, we may assume $\text{Image}(c) \subseteqq \mathbb{Z}$ and $\text{Image}(c_2) \subseteqq \mathbb{Z}_2'$. Let $\beta : \Theta_{2,\mathbb{B}} \to \mathcal{B}$ denote a section of the quotient map $\mathcal{B} \to \Theta_{2,\mathbb{B}}$ (cf. the paragraph subsequent to Definition 5.1). Now let $\widetilde{c} : \Theta_{2,\mathbb{B}} \to \mathbb{Z}$ be any map such that the diagram

$$\begin{tikzcd} \Theta_{2,\mathbb{B}} \arrow[r, "\widetilde{c}"] \arrow[dr, "c_2"'] & \mathbb{Z} \arrow[d] \\ & \mathbb{Z}_2' \end{tikzcd}$$

commutes and $\widetilde{c}(t) = c(\beta(t))$ for all $t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$. This gives us the model $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$ we want.

Using $\widetilde{c}$ above, we shall first construct for each $t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$ and each $1 \leqq i \leqq |\widetilde{c}(t)|$, an embedded oriented k-dimensional disk $D_{t,i}$ in X such that the ordinary intersection number

$$B_{\beta(t)} \cdot D_{t,i} = \begin{cases} 1, & \text{if } t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}}),\ \widetilde{c}(t) > 0 \\ -1, & \text{if } t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}}),\ \widetilde{c}(t) < 0. \end{cases}$$

Then we shall construct for each $t \in \Theta_{2,\mathbb{B}} \smallsetminus p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$ and each $1 \leqq i \leqq |\widetilde{c}(t)|$, an embedded k-dimensional disk $D_{t,i}$ with arbitrary orientation such that

$$B_{\beta(t)} \cdot D_{t,i} = \pm 1.$$

Finally we shall choose a certain embedded k-dimensional disk D_0 and connect each $\partial(D_{t,i})$ to $\partial(D_0)$ by a solid k-dimensional tube $T_{t,i} \cong I \times D^{k-1}$. This leads quickly to a G-embedding $\varphi : G \times S^{k-1} \times D^{k+1} \to X$.

We fill in the details of the outline in the paragraph above.

Let $t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$ such that $\widetilde{c}(t) \neq 0$. Let $N(t)$ denote a $\rho_G(t)$-tubular neighborhood of $B_{\beta(t)}$. Take a closed, k-dimensional disk $D(t)$ in $B_{\beta(t)} \smallsetminus L^{(k-2)}$ such that $D(t) \cap gD(t) = \varnothing$ for all $g \in G \smallsetminus \rho_G(t)$. Since $D(t)$ is $\rho_G(t)$-contractible, $N(t)|_{D(t)} \cong D(t) \times M(t)$ for some $\rho_G(t)$-module $M(t)$ of dimension k. Let $D'_{t,1}, \cdots, D'_{t,|\widetilde{c}(t)|}$ denote disjoint, embedded, closed, k-dimensional disks in the interior of $D(t)$. Let $x_{t,i}$, $1 \leqq i \leqq |\widetilde{c}(t)|$, denote the center of $D'_{t,i}$. Clearly $x_{t,i} \notin B_{\beta(t')}$ for any $t' \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$ such that $t' \neq t$. Since $N(t)|_{D'_{t,i}} \cong D'_{t,i} \times M(t)$ for each i such that $1 \leqq i \leqq |\widetilde{c}(t)|$, there is a smooth section $s'_{t,i} : D'_{t,i} \to N(t)|_{D'_{t,i}}$ such that $s'_{t,i}$ intersects transversally $D'_{t,i}$ and $\mathrm{Image}(s'_{t,i}) \cap D'_{t,i} = \{x_{t,i}\}$. Note that $\rho_G(t)$ acts freely on $M(t) \smallsetminus \{0\}$. Thus we can take a δ-approximation $s_{t,i}$ of $s'_{t,i}$ in $N(t)|_{D'_{t,i}}$ such that if $y \in D'_{t,i}$ is close to $x_{t,i}$, i.e. for example, the distance between y and $x_{t,i}$ is less than the half of the radius of $D'_{t,i}$, then $s_{t,i}(y) = s'_{t,i}(y)$, and such that if $gs_{t,i}(y) = g's_{t,i}(y')$, where $g,\ g' \in G$ and $y,\ y' \in \partial D'_{t,i}$, then $g = g'$ and $y = y'$. Note that $s_{t,i}$ is not necessarily a section. If we choose δ sufficiently small (in the sense of C^1-topology) then $s_{t,i}$ is an embedding. Let $D_{t,i} = \mathrm{Image}(s_{t,i})$. Clearly, $GD_{t,i} \cap GD_{t,j} = \varnothing$ whenever $i \neq j$. Furthermore, $\partial D_{t,i} \cap g\partial D_{t',j} = \varnothing$ whenever $t \neq t'$, $i \neq j$, or $g \neq 1$. Orient each $D_{t,i}$ such that when $\widetilde{c}(t) > 0$ (resp. $\widetilde{c}(t) < 0$), the orientation at $x_{t,i}$ given by the orientation on X agrees (resp. disagrees) with the orientation at $x_{t,i}$ defined by the orientations of $B_{\beta(t)}$ and $D_{t,i}$. The constructions above guarantee that the intersection number

$$B_{\beta(t)} \cdot D_{t,i} = \begin{cases} 1, & \text{if } t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}}),\ \widetilde{c}(t) > 0 \\ -1, & \text{if } t \in p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}}),\ \widetilde{c}(t) < 0. \end{cases}$$

Now let $t \in \Theta_{2,\mathbb{B}} \smallsetminus p_{\mathbb{B}}(\widetilde{\Theta}_{\mathbb{B}})$ such that $\widetilde{c}(t) \neq 0$. Let $N(t)$ denote a $\rho_G(t)$-tubular neighborhood of $B_{\beta(t)}$. As above, we take a k-dimensional closed disk $D(t)$, k-dimensional closed disks $D'_{t,i}$, where $1 \leqq i \leqq |\widetilde{c}(t)|$, with centers $x_{t,i}$, and k-dimensional closed disks $D_{t,i}$ ($\subset N(t)|_{D'_{t,i}}$), except now we orient each $D_{t,i}$ arbitrarily. Thus

$$B_{\beta(t)} \cdot D_{t,i} = \pm 1.$$

Set

$$\boldsymbol{B} = \{(t,i) \mid t \in \Theta_{2,\mathbb{B}},\ \widetilde{c}(t) \neq 0, \text{ and } 1 \leqq i \leqq |\widetilde{c}(t)|\}.$$

We put all $D_{t,i}$'s, $(t,i) \in \boldsymbol{B}$, together as follows. Let $N(L)$ denote a G-regular neighborhood of L. Let

$$D_0 \subset \mathrm{Interior}\,(X \smallsetminus N(L))$$

denote an embedded, closed, k-dimensional disk such that $D_0 \cap gD_0 = \varnothing$ for all $g \neq 1 \in G$. Let $\{T_{t,i} \mid (t,i) \in \boldsymbol{B}\}$ be a set of disjoint, closed, k-dimensional, solid tubes ($\cong I \times D^{k-1}$) in

$$(\mathrm{Interior}(X) \smallsetminus L) \smallsetminus G\left(\mathrm{Interior}(D_0) \cup \bigcup_{(t,i)\in\boldsymbol{B}} \mathrm{Interior}(D_{t,i})\right)$$

such that $T_{t,i}$ connects ∂D_0 to $\partial D_{t,i}$ and such that $T_{t,i} \cap gT_{t',j} = \varnothing$ whenever one of the inequalities $g \neq 1$, $t \neq t'$, or $i \neq j$ holds. Let

$$D = D_0 \cup \bigcup_{(t,i)\in\boldsymbol{B}} (T_{t,i} \cup D_{t,i}).$$

Then D is an embedded, closed, k-dimensional disk such that for each $t \in \Theta_{2,\mathbb{B}}$,

$$D \cap B_{\beta(t)} = \{x_{t,1}, \cdots, x_{t,|\widetilde{c}(t)|}\}.$$

By choosing the $T_{t,i}$'s appropriately, we can assume that D is the union of D_0, $T_{t,i}$ and $D_{t,i}$ as oriented manifolds. Thus, ∂D is a $(k-1)$-dimensional sphere embedded in Interior$(X \smallsetminus L)$ and $\partial D \cap g\partial D = \varnothing$ for $g \neq 1 \in G$. Let

$$h_{\boldsymbol{c}} : S^{k-1} = \partial D \to \mathrm{Interior}(X \smallsetminus L) \tag{6.2.1}$$

denote the canonical inclusion. Since ∂D is the boundary of the embedded disk D in X and D is contractible, ∂D has a closed, tubular neighborhood $\overline{N}(\partial D)$ in X diffeomorphic to $S^{k-1} \times D^{k+1}$. Moreover, we may assume that $\overline{N}(\partial D)$ lies in Interior$(X \smallsetminus L)$, and $\overline{N}(\partial D) \cap g\overline{N}(\partial D) = \varnothing$ whenever $g \neq 1$. Let $H : S^{k-1} \times D^{k+1} \to \overline{N}(\partial D)$ denote a coordinate diffeomorphism. Define $\varphi = \mathrm{Ind}_{\{1\}}^G H : G \times S^{k-1} \times D^{k+1} \to I \times X$ where $\mathrm{Ind}_H^G(g,u) = (1, gH(u))$ for $g \in G$ and $u \in S^{k-1} \times D^{k+1}$.

Step 2. Form the attaching space

$$W = (I \times X) \bigcup\nolimits_{\varphi} (G \times D^k \times D^{k+1}),$$

and define the subset $X' \subset \partial W$ by

$$X' = \{(\{1\} \times X) \smallsetminus \mathrm{Ind}_{\{1\}}^G H(G \times S^{k-1} \times \mathrm{Interior}(D^{k+1}))\} \bigcup\nolimits_{\varphi|_{G \times S^{k-1} \times S^k}} (G \times D^k \times S^k).$$

Clearly $\partial W = X' \cup (\{0\} \times X) \cup (I \times \partial X)$. Let $\psi : D^k \to D$ denote a diffeomorphism extending $h_{\boldsymbol{c}} : \partial D^k = S^{k-1} \to \partial D$. The G-map $id_I \times f : I \times X \to I \times Y$ extends to a G-map $F : W \to I \times Y$ such that $F(X') \subsetneqq \{1\} \times Y$ and $F(g, u, 0) = (1, g(f(\psi(u))))$ for any $g \in G$ and $u \in D^k$. The obstruction σ_H to extending the G-vector bundle isomorphism

$$id_{\varepsilon_I(\mathbb{R})} \times b : \varepsilon_I(\mathbb{R}) \times (T(X) \oplus f^*\xi_- \oplus \varepsilon_X(U)) \to \varepsilon_I(\mathbb{R}) \times (f^*\xi_+ \oplus \varepsilon_X(U))$$

over $I \times X$ to a G-vector bundle isomorphism

$$B' : T(W) \oplus (p_Y \circ F)^*\xi_- \oplus \varepsilon_W(U) \to (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi_+) \oplus \varepsilon_W(U)$$

over W lies in $\pi_{k-1}(SO(k+1+m))$, where m is a certain nonnegative integer. The flexibility we have for choosing the coordinate diffeomorphism $H : S^{k-1} \times D^{k+1} \to \overline{N}(\partial D)$ above, allows us to control the element σ_H. Let $\omega : S^{k-1} \to SO(k+1)$ be a smooth map. Let H^ω denote the diffeomorphism $S^{k-1} \times D^{k+1} \to \overline{N}(\partial D)$ defined by $H^\omega(x,y) = H(x, \omega(x)y)$, where $x \in S^{k-1}$ and $y \in D^{k+1}$. Then we have the formula $\sigma_{H^\omega} = \sigma_H + [\omega]$ for the obstructions, where $[\omega] \in \pi_{k-1}(SO(k+1+m))$ is the homotopy class of the composition of $\omega : S^{k-1} \to SO(k+1)$ and the canonical map $SO(k+1) \to SO(k+1+m)$. Since the homomorphism $\pi_{k-1}(SO(k+1)) \to \pi_{k-1}(SO(k+1+m))$ induced by the canonical map is an isomorphism, we can choose H such that $\sigma_H = 0$. In other words, we can choose H such that $id_{\varepsilon_I(\mathbb{R})} \times b$ extends to B' (cf. the proof of Corollary to Theorem 1.1 in [44, Part 0, p.10]). Define $f' : X' \to Y$ by $F(u) = (1, f'(u))$ $(u \in X')$. By Lemma 6.1, B' is regularly G-homotopic rel $(I \times (\partial X \cup L)) \cup (\{0\} \times X)$ to $B : T(W) \oplus (p_Y \circ F)^*\xi_- \oplus \varepsilon_W(U) \to (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi_+) \oplus \varepsilon_W(U)$ such that $B|_{X'} = id_{\varepsilon_{X'}(\mathbb{R})} \oplus b'$ for some $b' : T(X') \oplus f'^*\xi_- \oplus \varepsilon_{X'}(U) \to f'^*\xi_+ \oplus \varepsilon_{X'}(U)$. Therefore (F, B) is a G-framed cobordism rel

$\partial X \cup L$ from $\boldsymbol{f} = (f, b)$ to $\boldsymbol{f'} = (f', b')$. The map $\boldsymbol{f'}$ is called the *G-framed map obtained from $\boldsymbol{f}$ by G-surgery along $h_{\boldsymbol{c}}$* (or more precisely *along H*).

Step 3. The step takes place in 3 parts. Part 1 constructs a pair (e_1, f_1) of elements in $K_k(X'; R)$ such that $B_{f',R}(e_1, e_1) = 0$, $B_{f',R}(f_1, e_1) = 1$ and $q_{\boldsymbol{f'},R}(e_1) = q_{\boldsymbol{f'},R}(f_1) = 0$. Suppose this has been done. Let $M(e_1, f_1)$ denote the $R[G]$-submodule of $K_k(X'; R)$ generated by e_1 and f_1. $M(e_1, f_1)$ is an $R[G]$-free module with basis $\{e_1, f_1\}$ and the canonically induced sequence

$$0 \to M(e_1, f_1)^{\perp} \to K_k(X'; R) \overset{B_{f',R}(\bullet,-)}{\longrightarrow} \mathrm{Hom}_{R[G]}(M(e_1, f_1), R[G]) \to 0$$

of $R[G]$-modules is split exact, where

$$M(e_1, f_1)^{\perp} = \{x \in K_k(X'; R) \mid B_{f',R}(x, y) = 0 \text{ for all } y \in M(e_1, f_1)\}.$$

Clearly $K_k(X'; R) = M(e_1, f_1)^{\perp} \oplus M(e_1, f_1)$ and the restriction of $B_{f',R}$ to $M(e_1, f_1)$ is nonsingular. Let $\boldsymbol{N}(e_1, f_1)$ denote the doubly parametrized module

$$(M(e_1, f_1), B_{f',R}|_{M(e_1,f_1)}, q_{\boldsymbol{f'},R}|_{M(e_1,f_1)}).$$

$(K_k(X'; R), B_{f',R}, q_{\boldsymbol{f'},R})$ obviously splits as the orthogonal sum $(K_k(X'; R), B_{f',R}, q_{\boldsymbol{f'},R}) = \boldsymbol{N}(e_1, f_1)^{\perp} \oplus \boldsymbol{N}(e_1, f_1)$, where $\boldsymbol{N}(e_1, f_1)^{\perp} = (M(e_1, f_1)^{\perp}, B_{f',R}|_{M(e_1,f_1)^{\perp}}, q_{\boldsymbol{f'},R}|_{M(e_1,f_1)^{\perp}})$. Accordingly the maps $\theta_{f',\mathbb{B}} : \widetilde{\Theta}_{\mathbb{B}} \to K_k(X'; R)$ and $\theta_{2,f',\mathbb{B}} : \Theta_{2,\mathbb{B}} \to K_k(X'; R_2)$ split canonically as direct sums $\theta_{f',\mathbb{B}} = \theta_{M^{\perp}} \oplus \theta_M$ and $\theta_{2,f',\mathbb{B}} = \theta_{2,M^{\perp}} \oplus \theta_{2,M}$ where $\theta_{M^{\perp}} : \widetilde{\Theta}_{\mathbb{B}} \to M(e_1, f_1)^{\perp}$, $\theta_M : \widetilde{\Theta}_{\mathbb{B}} \to M(e_1, f_1)$, $\theta_{2,M^{\perp}} : \Theta_{2,\mathbb{B}} \to M(e_1, f_1)_2^{\perp}$, and $\theta_{2,M} : \Theta_{2,\mathbb{B}} \to M(e_1, f_1)_2$. Let $\boldsymbol{M}(e_1, f_1) = (\boldsymbol{N}(e_1, f_1), \theta_M, \theta_{2,M})$, and let $\boldsymbol{M}(e_1, f_1)^{\perp} = (\boldsymbol{N}(e_1, f_1)^{\perp}, \theta_M^{\perp}, \theta_{2,M}^{\perp})$. By construction, $\boldsymbol{M}_{\boldsymbol{f'},\mathbb{B}} = \boldsymbol{M}(e_1, f_1)^{\perp} \oplus \boldsymbol{M}(e_1, f_1)$. Part 2 shows that $\boldsymbol{M}(e_1, f_1) \cong \mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c})$. Part 3 shows that f' is k-connected. Part 4 shows that $\boldsymbol{M}(e_1, f_1)^{\perp} \cong \boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$.

We supply now details of the above.

Part 1. Let

$$S_1 = \{1\} \times \{x_0\} \times S^k \left(\subset \{1\} \times S^{k-1} \times S^k \subset G \times D^k \times D^{k+1} \subset X'\right).$$

Using a parallel translation of $\{1\} \times D$ ($\subset \{1\} \times X$), we can find a k-dimensional closed disk D' embedded in

$$\{1\} \times (X \smallsetminus G \cdot \mathrm{Interior}(\mathrm{Image}(H)))$$

such that

$$\partial D' = \{1\} \times S^{k-1} \times \{x'_0\} \left(\subset G \times D^k \times D^{k+1} \subset X'\right)$$

for some $x'_0 \in S^k$. Let

$$S_2 = \{1\} \times D^k \times \{x'_0\} \cup_{\{1\} \times S^{k-1} \times \{x'_0\}} D' \left(\subset X'\right).$$

Let $h_{S_i} : S_i \to X'$, $i = 1$, 2, denote the inclusion maps. We shall use the h_{S_i}, $i = 1$, 2, to construct e_1 and f_1. Give S_2 the orientation which agrees with that on D' and S_1 the orientation such that $\#(h_{S_2}, h_{S_1}) = 1$, i.e. $\#(h_{S_1}, h_{S_2}) = (-1)^k$. Obviously,

$\#_G(h_{S_1}, h_{S_1}) = 0$, $\#_G(h_{S_2}, h_{S_1}) = 1$, and $\natural_G(h_{S_1}) = 0$. Since S_2 is an embedded sphere with trivial normal bundle, $\natural_1(h_{S_2}) = 0$. If

$$g \in G \smallsetminus \bigcup_{t \in \Theta_{2,\mathbb{B}}:\ \widetilde{c}(t) \neq 0} \rho_G(t),$$

then $\natural_g(h_{S_2}) = 0$. Moreover using Lemma 9.1 of [6] (as in the proof of Theorem 8.1 of [6, §9]), we can choose D so that for $g \in S(G, X)$,

$$\#(h_{S_2}, gh_{S_2}) = \sum_{t \in \Theta_{2,\mathbb{B}}|_g} (-1)^k \widetilde{c}(t).$$

Let $[S_i] \in H_k(S_i; \mathbb{Z})$ denote the orientation class of S_i. Let $e_1 = h_{S_1*}[S_1]$ and $h_2 = h_{S_2*}[S_2] \in H_k(X'; \mathbb{Z})$. Then e_1, $h_2 \in K_k(X'; \mathbb{Z})$ ($\subseteqq K_k(X'; R)$). It follows from Lemma 4.6 and the values $\#_G(h_{S_1}, h_{S_1}) = 0$ and $\#_G(h_{S_2}, h_{S_1}) = 1$ above that $B_{f',R}(e_1, e_1) = 0$ and $B_{f',R}(h_2, e_1) = 1$. We will show that $f' : X' \to Y$ is k-connected. Once this has been shown, we obtain a quadratic map $q_{\boldsymbol{f}',R} : K_k(X'; R) \to R[G]/R\Lambda(Q(G, X) + R[S(G, X)]))$ by Definition 5.7. By definition, $q_{\boldsymbol{f}',R}(e_1) = \natural_G(h_{S_1})$ and $q_{\boldsymbol{f}',R}(h_2) = \natural_G(h_{S_2})$. Thus from the values $\natural_G(h_{S_1}) = 0$ and $\natural_g(h_{S_2}) = 0$ above, it follows that $q_{\boldsymbol{f}',R}(e_1) = 0$ and $q_{\boldsymbol{f}',R}(h_2)_g = 0$ (the g-th coefficient of $q_{\boldsymbol{f}',R}(h_2)$) if $g \in \{1\} \cup (G(2) \smallsetminus (Q(G, X) \cup S(G, X)))$ (see Definition 5.7). Choose $v \in \mathbb{Z}[G \smallsetminus (\{1\} \cup G(2))]$ such that after setting $f_1 = h_2 + ve_1$, we get $q_{\boldsymbol{f}',R}(f_1) = 0$. We maintain that $B_{f',R}(f_1, e_1) = 1$ and obtain additionally that

$$B_{f',R}(f_1, f_1) = \sum_{g \in S(G,X)} \sum_{t \in \Theta_{2,\mathbb{B}}|_g} (-1)^k \widetilde{c}(t) g.$$

Part 2. By Lemma 3.31, $\mathbb{M}(R[G], \boldsymbol{c}, \widetilde{c}) \cong \mathbb{M}(R[G], (-1)^k \boldsymbol{c}, (-1)^k \widetilde{c})$. We shall show

$$\boldsymbol{M}(e_1, f_1) = (M(e_1, f_1), B_{f',R}|_{M(e_1,f_1)}, q_{\boldsymbol{f}',R}|_{M(e_1,f_1)}, \theta_M, \theta_{2,M})$$

is isomorphic to $\mathbb{M}(R[G], (-1)^k \boldsymbol{c}, (-1)^k \widetilde{c})$. According to Definition 3.28, we must show that

$$\begin{aligned} &B_{f',R}(e_1, e_1) = 0, \quad B_{f',R}(f_1, e_1) = 1, \\ &B_{f',R}(f_1, f_1) = \sum_{g \in S(G,X)} \sum_{t \in \Theta_{2,\mathbb{B}}|_g} (-1)^k \widetilde{c}(t) g, \\ &q_{\boldsymbol{f}',R}(e_1) = q_{\boldsymbol{f}',R}(f_1) = 0, \end{aligned} \tag{6.2.2}$$

and

$$\theta_M = \mathrm{sr}((-1)^k c), \quad \theta_{2,M} = \mathrm{sr}((-1)^k c_2). \tag{6.2.3}$$

The property (6.2.2) has been demonstrated already in Part 1.

Clearly $\theta_M(\beta) = B_{\boldsymbol{f}',R}(f_1, (h_{\beta_*}[\beta])_{K_k(X';R)})$ for $\beta \in \widetilde{\Theta}_{\mathbb{B}}$ (cf. Definition 5.2). But

$$\begin{aligned} B_{f',R}(f_1, (h_{\beta_*}[\beta])_{K_k(X';R)}) &= B_{f',R}(h_2, (h_{\beta_*}[\beta])_{K_k(X';R)}) \\ &= \sum_{g\in G} \mathrm{Int}(h_{S_2}, g^{-1}h_\beta)g \\ &= \sum_{g\in G} \mathrm{Int}(h_{S_2}, h_{g^{-1}\beta})g \\ &= \sum_{g\in G} (-1)^k c(g^{-1}\beta)g \\ &= \mathrm{sr}((-1)^k c)(\beta). \end{aligned}$$

Similarly, one shows that $\theta_{M_2} = \mathrm{sr}((-1)^k c_2)$.

Part 3. We show first that $f' : X' \to Y$ is k-connected. We begin by simplifying the Mayer-Vietoris exact sequences
(6.2.4)

$$\cdots \longrightarrow K_{i+1}(W, X; \mathbb{Z}) \xrightarrow{\partial} K_i(X; \mathbb{Z}) \xrightarrow{i_{X*}} K_i(W; \mathbb{Z}) \xrightarrow{\tau} K_i(W, X; \mathbb{Z}) \longrightarrow \cdots$$

and
(6.2.5)

$$\cdots \longrightarrow K_{i+1}(W, X'; \mathbb{Z}) \xrightarrow{\partial'} K_i(X'; \mathbb{Z}) \xrightarrow{i_{X'*}} K_i(W; \mathbb{Z}) \xrightarrow{\tau'} K_i(W, X'; \mathbb{Z}) \longrightarrow \cdots$$

for the pairs (W, X) and (W, X'), respectively.

By definition, $K_i(W, X; \mathbb{Z}) = \mathrm{Ker}[F_* : H_i(W, X; \mathbb{Z}) \to H_i(I \times Y, Y; \mathbb{Z})]$. Since $H_i(I \times Y, Y; \mathbb{Z}) = 0$, it follows that $K_i(W, X; \mathbb{Z}) = H_i(W, X; \mathbb{Z})$. Since the quotient space W/X is G-homotopy equivalent to $(G \times S^k)/(G \times \{pt\})$, we obtain

$$K_i(W, X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[G], & \text{if } i = k \\ 0, & \text{if } i \neq k. \end{cases} \tag{6.2.6}$$

Let $h_2 \in K_k(X; \mathbb{Z})$ be as in Part 1. Set $h_2' = i_{X'*}(h_2)$ and $h_2'' = \tau(h_2')$. From the definition of h_2, we obtain that

$$K_k(W, X; \mathbb{Z}) = \mathbb{Z}[G]h_2'', \quad \text{a free } \mathbb{Z}[G]\text{-module generated by } h_2''. \tag{6.2.7}$$

Since f is k-connected, it follows that $K_i(X; \mathbb{Z}) = 0$ if $i < k$. Thus the exact sequence for (W, X) above simplifies to a split exact sequence

$$0 \longrightarrow K_k(X; \mathbb{Z}) \xrightarrow{i_{X*}} K_k(W; \mathbb{Z}) \xrightarrow{\tau} K_k(W, X; \mathbb{Z}) \longrightarrow 0 \tag{6.2.8}$$

and canonical identifications

$$K_j(X; \mathbb{Z}) = K_j(W; \mathbb{Z}) \text{ for } j \neq k. \tag{6.2.9}$$

As above, $K_i(W, X'; \mathbb{Z}) = H_i(W, X'; \mathbb{Z})$. Since W/X' is G-homotopy equivalent to $(G \times S^{k+1})/(G \times \{pt\})$, we obtain

$$K_i(W, X'; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[G], & \text{if } i = k+1 \\ 0, & \text{if } i \neq k+1. \end{cases} \tag{6.2.10}$$

Thus the exact sequence for (W, X') above simplifies to an exact sequence

$$\cdots \longrightarrow K_{k+1}(W, X'; \mathbb{Z}) \xrightarrow{\partial'} K_k(X'; \mathbb{Z}) \xrightarrow{i_{X'*}} K_k(W; \mathbb{Z}) \xrightarrow{\tau'} 0 \tag{6.2.11}$$

and

$$K_j(X'; \mathbb{Z}) = K_j(W; \mathbb{Z}) = 0 \quad j < k. \tag{6.2.12}$$

From this fact and the fact that X' and Y are simply connected, it follows that $f' : X' \to Y$ is k-connected. Additionally, we obtain

$$K_{k+1}(W, X'; \mathbb{Z}) = \mathbb{Z}[G]e_1', \text{ a } \mathbb{Z}[G]\text{-free module generated by } e_1', \tag{6.2.13}$$

for an unique element $e_1' \in K_{k+1}(W, X'; \mathbb{Z})$ such that $\partial'(e_1') = e_1$.

Part 4. Let $\varphi = \mathrm{Ind}_{\{1\}}^G H : G \times S^{k-1} \times D^{k+1} \to X$ be as in Part 1. Let

$$X_0 = X \smallsetminus \mathrm{Interior}(\mathrm{Image}(\varphi)).$$

Let $i_0 : X_0 \to X$ and $i_1 : X_0 \ (= \{1\} \times X_0) \to X'$ denote the canonical inclusions. Let $i_X : X \to W$ and $i_{X'} : X' \to W$ denote also the canonical inclusions. Let

$$K_k(X_0; \mathbb{Z}) = \mathrm{Ker}[(f \circ i_0)_* : H_k(X_0; \mathbb{Z}) \to H(Y; \mathbb{Z})].$$

Clearly $i_X \circ i_0$ is homotopic (in fact, G-isotopic) to $i_{X'} \circ i_1$ and hence $f \circ i_0$ is homotopic to $f' \circ i_1$. Thus $K_k(X_0; \mathbb{Z}) = \mathrm{Ker}[(f' \circ i_1)_* : H_k(X_0; \mathbb{Z}) \to H(Y; \mathbb{Z})]$ and the diagram

$$\begin{array}{ccc} K_k(X_0; \mathbb{Z}) & \xrightarrow{i_{0*}} & K_k(X; \mathbb{Z}) \\ {\scriptstyle i_{1*}}\downarrow & & \downarrow{\scriptstyle i_{X*}} \\ K_k(X'; \mathbb{Z}) & \xrightarrow[i_{X'*}]{} & K_k(W; \mathbb{Z}) \end{array} \tag{6.2.14}$$

of $\mathbb{Z}[G]$-modules commutes. Moreover the map i_{0*} is surjective, because i_0 is k-connected.

Since $\mathbb{Z} \subsetneqq R \subsetneqq \mathbb{Q}$, R is a ring of fractions of $\mathbb{Z}$. Thus the functor $M \mapsto R \otimes M$, where M is a $\mathbb{Z}$-module, is exact. Thus we can canonically identify the modules $R \otimes H_j(V; \mathbb{Z}) = H_j(V; R)$ for any topological space V.

By applying the functor $R \otimes -$ to (6.2.14), we obtain a commutative diagram

$$\begin{array}{ccc} K_k(X_0; R) & \xrightarrow{i_{0*}} & K_k(X; R) \\ {\scriptstyle i_{1*}}\downarrow & & \downarrow{\scriptstyle i_{X*}} \\ K_k(X'; R) & \xrightarrow[i_{X'*}]{} & K_k(W; R) \end{array}$$

of $R[G]$-modules such that i_{0*} is surjective. By Lemma 5.13, $K_k(X;R)$ is $R[G]$-projective. Thus there is an $R[G]$-submodule $K_R \subseteqq K_k(X_0;R)$, which maps isomorphically under i_{0*} onto $K_k(X;R)$. Let $j_0 = i_{0*}|_{K_R} : K_R \to K_k(X;R)$, $j_1 = i_{1*}|_{K_R} : K_R \to K_k(X';R)$, and

$$\sigma = j_1 \circ j_0^{-1} : K_k(X;R) \to K_k(X';R).$$

From Part 1, we know that $K_k(X';R)$ has an orthogonal decomposition

$$(K_k(X';R), B_{f',R}, q_{\boldsymbol{f'},R}) = \boldsymbol{N}(e_1,f_1)^{\perp} \oplus \boldsymbol{N}(e_1,f_1).$$

Let $p^{\perp}$ denote the projection of $K_k(X';R)$ on $M(e_1,f_1)$. We shall show that the $R[G]$-homomorphism $p^{\perp} \circ \sigma : K_k(X;R) \to M(e_1,f_1)^{\perp}$ is an isomorphism $\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \to \boldsymbol{M}(e_1,f_1)^{\perp}$ of doubly parametrized modules with positioning data.

First we show that $\sigma : K_k(X;R) \to K_k(X';R)$ is a morphism $(K_k(X;R), B_{f,R}, q_{\boldsymbol{f},R}) \to (K_k(X';R), B_{f',R}, q_{\boldsymbol{f'},R})$ of doubly parametrized modules.

Let $K'_{\mathbb{Z}} = K_R \cap \operatorname{Image}[K_k(X_0;\mathbb{Z}) \to K_k(X_0;R)]$. $K'_{\mathbb{Z}}$ is a $\mathbb{Z}[G]$-submodule of $K_k(X_0;R)$ such that $R \cdot K'_{\mathbb{Z}} = K_R$. It suffices to show that the map $\sigma|_{j_0(K'_{\mathbb{Z}})}$ preserves Hermitian and quadratic forms. Let $K_{\mathbb{Z}}$ denote the preimage in $K_k(X_0;\mathbb{Z})$ of $K'_{\mathbb{Z}}$. The groups and maps above fit into a commutative diagram

(6.2.15)
$$\begin{tikzcd}
 & K_k(X_0;\mathbb{Z}) \arrow[rr, dashed, two heads, "1\otimes-", "\text{loc. epi.}"'] & & K_k(X_0;R) \arrow[dr, "i_{0*}"] \arrow[ddl, "i_{1*}"'] & \\
K_{\mathbb{Z}} \arrow[ur, hook] \arrow[r, two heads, "1\otimes-"] & K'_{\mathbb{Z}} \arrow[r, hook, dashed, "\text{loc. epi.}"'] & K_R \arrow[ur, hook] \arrow[rr, "j_0", "\cong"'] \arrow[d, "j_1"'] & & K_k(X;R) \arrow[d, "i_{X*}"] \arrow[dll, "\sigma"] \\
 & & K_k(X';R) \arrow[rr, "i_{X'*}"'] & & K_k(W;R).
\end{tikzcd}$$

Let $x \in K_{\mathbb{Z}}$. Then x is represented by an immersion $h_x : S^k \to \operatorname{Interior}(X_0)$ with trivial normal bundle. Suppose $y \in K_{\mathbb{Z}}$. By definition, $x' = 1 \otimes x$, $y' = 1 \otimes y \in K'_{\mathbb{Z}}$. Clearly h_x (resp. h_y) represents also $j_0(x')$ and $j_1(x')$ (resp. $j_0(y')$ and $j_1(y')$). By Proposition 5.8, $B_{f,R}(j_0(x'), j_0(y')) = \#_G(h_x,h_y) = B_{f',R}(j_1(x'), j_1(y')) = B_{f',R}(\sigma(j_0(x')), \sigma(j_0(y')))$ and $q_{\boldsymbol{f},R}(j_0(x')) = [\natural_G(h_x)] = q_{\boldsymbol{f'},R}(\tau_1(j_1(x'))) = q_{\boldsymbol{f'},R}(\sigma(j_0(x')))$.

Next we show that $\sigma(K_k(X;R)) \subseteqq M(e_1,f_1) \oplus R[G]e_1$. Recall that e_1 itself is totally isotropic, i.e. $B_{f,R}(e_1,e_1) = 0$ and $q_{\boldsymbol{f},R}(e_1) = 0$, and $B_{f,R}(f_1,e_1) = 0$. Obviously

$$(R[G]e_1)^{\perp} = \{u \in K_k(X';R) \mid B_{f',R}(u,e_1) = 0\}$$

coincides with $M(e_1,f_1)^{\perp} \oplus R[G]e_1$. By Diagram (6.2.15), it suffices to show that $i_1((1\otimes -)(K_k(X_0,Z))) \subseteqq (R[G]e_1)^{\perp}$. Suppose this has been done. The map $p^{\perp}|_{M(e_1,f_1)^{\perp}\oplus R[G]e_1} : M(e_1,f_1)^{\perp} \oplus R[G]e_1 \to M(e_1,f_1)^{\perp}$ preserves Hermitian and quadratic forms, because e_1 is orthogonal to $M(e_1,f_1)^{\perp}$ by definition and e_1 itself is totally isotropic. Thus the map $p^{\perp} \circ \sigma$ preserves Hermitian and quadrqatic forms and therefore $p^{\perp} \circ \sigma$ is a morphism $(K_k(X;R), B_{f,R}, q_{\boldsymbol{f},R}) \to \boldsymbol{N}(e_1,f_1)^{\perp}$ of doubly parametrized modules. Suppose

$x \in K_k(X_0;\mathbb{Z})$. Then x is represented by an immersion $h_x : S^k \to \mathrm{Interior}(X_0)$ with trivial normal bundle. On the other hand e_1 is represented by the embedding h_{S_1} appearing in Part 1. By Proposition 5.8, $B_{f',R}(i_{1*}(1 \otimes x), e_1) = \#_G(h_x, h_{S_1})$. But $\#_G(h_x, h_{S_1}) = 0$ because $\mathrm{Image}(h_x) \cap G\mathrm{Image}(h_{S_1}) = \varnothing$. Thus $i_{1*}(1 \otimes x) \in (R[G]e_1)^{\perp}$.

Next we show that $p^{\perp} \circ \sigma : (K_k(X;R), B_{f,R}, q_{\boldsymbol{f},R}) \to N(e_1, f_1)^{\perp}$ is an isomorphism of doubly parametrized modules. Since $B_{f,R}$ is nonsingular and $p^{\perp} \circ \sigma$ preserves Hermitian forms, it follows from [2, Lemma 2.3] that $p^{\perp} \circ \sigma$ is injective and $M(e_1, f_1)^{\perp}$ splits as an orthogonal sum $\mathrm{Image}(p^{\perp} \circ \sigma) \oplus M'$. Thus it suffices to show that $\mathrm{rank}_R(K_k(X;R)) = \mathrm{rank}_R(M_(e_1, f_1)^{\perp})$. We know $K_k(X';R) = M(e_1, f_1)^{\perp} \oplus M(e_1, f_1)$. Therefore it suffices to show

$$K_k(X';R) \cong K_k(X;R) \oplus R[G] \oplus R[G]. \tag{6.2.16}$$

Consider the diagram

(6.2.17)

$$\begin{tikzcd}
 & \cdots \arrow[r] & K_k(X;R) \arrow[r, "i_{X*}"] \arrow[d, "\sigma"'] & K_k(W;R) \arrow[r, "\tau"] \arrow[d, equal] & K_k(W,X;R) \arrow[r] & \cdots \\
\cdots \arrow[r] & K_{k+1}(W,X';R) \arrow[r, "\partial'"] & K_k(X';R) \arrow[r, "i_{X'*}"] & K_k(W;R) \arrow[r] & \cdots &
\end{tikzcd}$$

afforded by (6.2.4) and (6.2.5). We compute first the top row and then the bottom row. This will establish (6.2.14) above.

The following is a continuation of the computation in Part 3. It follows from (6.2.6) and (6.2.7) that

$$K_j(W,X;R) = \begin{cases} R[G]h_2'', & \text{if } j = k \\ 0, & \text{if } j \neq k. \end{cases} \tag{6.2.18}$$

By (6.2.10)–(6.2.13) and Poincaré-Lefschetz duality, we obtain

$$K_j(W,X';R) = \begin{cases} R[G]e_1', & \text{if } j = k+1 \\ 0, & \text{if } j \neq k+1, \end{cases} \tag{6.2.19}$$

where e_1' is an element in $K_{k+1}(W,X';R)$ such that $\partial'(e_1') = e_1$. Thus Diagram (6.2.17) simplifies to a commutative diagram

$$\begin{tikzcd}
 & K_k(X;R) \arrow[r, tail, "i_{X*}"] \arrow[d, "\sigma"'] & K_k(W;R) \arrow[r, two heads, "\tau"] \arrow[d, equal] & R[G]h_2'' \\
R[G]e_1 \arrow[r, hook] & K_k(X';R) \arrow[r, two heads, "i_{X'*}"] & K_k(W;R) &
\end{tikzcd} \tag{6.2.20}$$

Since $K_k(X;R)$ is $R[G]$-projective, so is $K_k(W;R)$. Thus the bottom row is split exact as well as the top row. We obtain $K_k(X';R) \cong K_k(X;R) \oplus R[G] \oplus R[G]$, in fact

$$K_k(X';R) = \sigma(K_k(X;R)) \oplus R[G]e_1 \oplus R[G]h_2. \tag{6.2.21}$$

We prove next that $p^\perp \circ \sigma$ preserves positioning functions. This will complete the proof of the theorem.

Let $\beta \in \widetilde{\Theta}$. We must show $p^\perp(\sigma(\theta_{f,\mathbb{B}}(\beta))) = \theta_{M^\perp}(\beta)$. By definition, $\theta_{M^\perp}(\beta) = p^\perp(\theta_{f',\mathbb{B}}(\beta))$. Thus it suffices to show $p^\perp(\sigma(\theta_{f,\mathbb{B}}(\beta))) = p^\perp(\theta_{f',\mathbb{B}}(\beta))$. Let p_1 denote the projection $M(e_1, f_1)^\perp \oplus M(e_1, f_1) \to M(e_1, f_1)^\perp \oplus R[G]f_1$ and p_2 denote the projection $M(e_1, f_1)^\perp \oplus R[G]f_1 \to M(e_1, f_1)^\perp$. Clearly $p^\perp = p_2 \circ p_1$. It suffices to show $p_1(\sigma(\theta_{f,\mathbb{B}}(\beta))) = p_1(\theta_{f',\mathbb{B}}(\beta))$. By definition, $\theta_{f,\mathbb{B}}(\beta) = (h_{\beta *}[\beta]_R)_K$. Thus $i_{X*}(\theta_{f,\mathbb{B}}(\beta)) = i_{X'*}(\theta_{f',\mathbb{B}}(\beta))$. From $i_{X'*} \circ \sigma = i_{X*}$ (cf. (6.2.15)) and the exactness of the bottom row in (6.2.20), it follows that $p_1(\sigma(\theta_{f,\mathbb{B}}(\beta))) = p_1(\theta_{f',\mathbb{B}}(\beta))$.

Replacing $\widetilde{\Theta}_{f,\mathbb{B}}$ above by $\Theta_{2,f,\mathbb{B}}$ and applying $\mathbb{Z}_2 \otimes -$ to the argument above, we obtain that $p^\perp \circ \sigma$ preserves mod 2 positioning functions. (Note that $R_2 := \mathbb{Z}_2 \otimes R = \mathbb{Z}_2$ if $1/2 \notin R$ and $R_2 = 0$ if $1/2 \in R$.) Thus $p^\perp \circ \sigma$ preserves positioning data. □

7. G-framed $\boldsymbol{\Theta}$-cobordism invariance

The surgery obstruction in Wall's surgery theory is a framed-cobordism invariant. Similar results are expected in equivariant surgery theory. We prove in this section that our surgery obstruction is a G-framed $\boldsymbol{\Theta}$-cobordism invariant under hypotheses on the singular set and the boundary. Theorem 1.1 in the introduction follows immediately from the last result in the section, Theorem 7.8. The proof of this theorem provides the construction of the surgery obstruction element $\sigma(\boldsymbol{f})$ required in the statement of Theorem 1.1.

We introduce first the notion G-framed $\boldsymbol{\Theta}$-cobordism. Let $\boldsymbol{f} = (f, b) : (X, \partial X, TX) \to (Y, \partial Y, f^*\xi)$ and $\boldsymbol{f}' = (f', b') : (X', \partial X', TX') \to (Y, \partial Y, f'^*\xi)$ denote G-framed maps, and

$$\boldsymbol{F} = (F, B) : (W, \partial_+ W, \partial_- W, TW) \to (I \times Y, \partial I \times Y, I \times \partial Y, (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi))$$

a G-framed cobordism between $\boldsymbol{f}$ and $\boldsymbol{f}'$. Let $(L, \mathbb{B})$, $\mathbb{B} = \{h_\beta : B_\beta \to X \mid \beta \in \mathcal{B}\}$, and $(L', \mathbb{B}')$, $\mathbb{B}' = \{h'_\beta : B'_\beta \to X' \mid \beta \in \mathcal{B}'\}$, denote G-singularity structures for X and X', respectively. We shall assume throughout $\mathcal{B} = \mathcal{B}'$ and $\mathcal{B}_{+,L} = \mathcal{B}_{+,L'}$ as $G \times \{\pm 1\}$-sets (see Definition 5.1). To simplify notation, we shall use the abbreviations $\mathcal{B}_+ = \mathcal{B}_{+,L} (= \mathcal{B}_{+,L'})$, $\widetilde{\Theta} = \widetilde{\Theta}_{\mathbb{B}} (= \widetilde{\Theta}_{\mathbb{B}'})$, $\Theta_2 = \Theta_{2,\mathbb{B}} (= \Theta_{2,\mathbb{B}'})$, and $\boldsymbol{\Theta} = \boldsymbol{\Theta}_{\mathbb{B}} (= \boldsymbol{\Theta}_{\mathbb{B}'})$. We remind the reader that the G-framed cobordism (F, B) above is called *trivial*, if W is just a cylinder, i.e. $W = I \times X$. If (F, B) is trivial then f is G-homotopic to f' and b is regularly G-homotopic to b'.

Definition 7.1. Let W be a G-cobordism between X and X'. A $\boldsymbol{\Theta}$-*cobordism* on W between $(L, \mathbb{B})$ and $(L', \mathbb{B}')$ is a pair $(\widehat{L}, \widehat{\mathbb{B}})$ consisting of a G-simplicial subcomplex $\widehat{L}$ of W and a set

$$\begin{aligned}\widehat{\mathbb{B}} = \{&\text{inclusion maps } H_\beta : (\widehat{B}_\beta, \partial_+ \widehat{B}_\beta, \partial_- \widehat{B}_\beta) \to (W, \partial_+ W, \partial_- W) \\ &\text{of submanifolds } \widehat{B}_\beta \subset W \mid \beta \in \mathcal{B}\}\end{aligned}$$

satisfying the following conditions.

(7.1.1) $\widehat{L} \supseteqq \mathrm{Sing}(G, W)$.

(7.1.2) $\widehat{L} = \widehat{L}^{(k)} \cup \bigcup_{\beta \in \mathcal{B}} \widehat{B}_\beta$, where $\widehat{L}^{(k)}$ is the k-skeleton of $\widehat{L}$.

(7.1.3) For each $\beta \in \mathcal{B}$, $\widehat{B}_\beta$ is a $(k+1)$-dimensional connected submanifold of W.
(7.1.4) If $\beta \in \mathcal{B} \smallsetminus \mathcal{B}_+$ (resp. $\mathcal{B}_+$) then $\widehat{B}_\beta$ is a cobordism (resp. oriented cobordism) between B_β and B'_β (hence $\partial\widehat{B}_\beta = \partial_+\widehat{B}_\beta \cup \partial_-\widehat{B}_\beta$, $\partial_+\widehat{B}_\beta = B_\beta \amalg B'_\beta$, $\partial_+\widehat{B}_\beta \cap \partial_-\widehat{B}_\beta = \partial B_\beta \amalg \partial B'_\beta$).
(7.1.5) $H_\beta|_{\partial_+\widehat{B}_\beta} = h_\beta \cup h'_\beta$.
(7.1.6) For each $\beta \in \mathcal{B}$, $\widehat{L}^{(k)} \cap \widehat{B}_\beta \subseteqq \widehat{L}^{(k-1)}$.
(7.1.7) For every β, $\beta' \in \mathcal{B}$, $\widehat{B}_\beta \cap \widehat{B}_{\beta'}$ is a submanifold of W.
(7.1.8) For every β, $\beta' \in \mathcal{B}$, if $\widehat{B}_\beta \neq \widehat{B}_{\beta'}$ as subsets of W then $\widehat{B}_\beta \cap \widehat{B}_{\beta'} \subseteqq \widehat{L}^{(k-1)}$.
(7.1.9) For all $g \in G$ and $\beta \in \mathcal{B}$, $\widehat{B}_{g\beta} = g\widehat{B}_\beta$. (Hence the map $g : W \to W$ has the well-defined restriction $g : \widehat{B}_\beta \to \widehat{B}_{g\beta}$.)
(7.1.10) For all $\beta \in \mathcal{B}_+$, $\widehat{B}_{(-1)\beta} = \widehat{B}_\beta$ as subsets of W (hence $H_\beta = H_{(-1)\beta}$) and $[\widehat{B}_{(-1)\beta}, \partial\widehat{B}_{(-1)\beta}] = -[\widehat{B}_\beta, \partial\widehat{B}_\beta]$.
(7.1.11) For all $g \in G$ and $\beta \in \mathcal{B}_+$, $[\widehat{B}_{g\beta}, \partial\widehat{B}_{g\beta}] = g[\widehat{B}_\beta, \partial\widehat{B}_\beta]$.

A pair $(\boldsymbol{F}, (\widehat{L}, \widehat{\mathbb{B}}))$ is called a *G-framed $\boldsymbol{\Theta}$-cobordism* between $(\boldsymbol{f}, (L, \mathbb{B}))$ and $(\boldsymbol{f}', (L', \mathbb{B}'))$ if $\boldsymbol{F} = (F, B)$ is a G-framed cobordism between $\boldsymbol{f} = (f, b)$ and $\boldsymbol{f}' = (f', b')$, and $(\widehat{L}, \widehat{\mathbb{B}})$ is a $\boldsymbol{\Theta}$-cobordism on W between $(L, \mathbb{B})$ and $(L', \mathbb{B}')$.

For the remainder of the section, let $(\boldsymbol{F}, (\widehat{L}, \widehat{\mathbb{B}}))$ denote a G-framed $\boldsymbol{\Theta}$-cobordism between $(\boldsymbol{f}, (L, \mathbb{B}))$ and $(\boldsymbol{f}', (L', \mathbb{B}'))$. In addition, let $A = R[G]$, $\lambda = (-1)^k$,

$$\boldsymbol{A} = (A, (-, \lambda), R\Gamma(S(G, X)), G, R[S(G, X)], R\Lambda(Q(G, X)) + R[S(G, X)]),$$

and $\Gamma'_2 = \min_{-\lambda}(R_2[G])$. Γ'_2 defines $\nabla = \nabla_{\mathbb{B}}$ (cf. Definition 3.18, Definition 3.33 and the notation prior to Lemma 5.10).

Lemma 7.2. *One can perform G-surgery on $\boldsymbol{F}$ along spheres of dimension $\leqq k-1$ in* Interior$(W \smallsetminus \widehat{L})$ *to obtain a G-framed $\boldsymbol{\Theta}$-cobordism $(\boldsymbol{F}', (\widehat{L}, \widehat{\mathbb{B}}))$, $\boldsymbol{F}' = (F', B')$, between $(\boldsymbol{f}, (L, \mathbb{B}))$ and $(\boldsymbol{f}', (L', \mathbb{B}'))$ such that the resulting map $F' : W' \to I \times Y$ is k-connected.*

Proof. This is clear from the observation

$$\dim \widehat{L} + (k-1) < \dim W \quad \text{(for any } \beta \in \mathcal{B}).$$

□

Recall the assumption $\mathbb{Z} \subseteqq R \subseteqq \mathbb{Q}$. Suppose $\boldsymbol{f}$ and $\boldsymbol{f}'$ are k-connected, R-boundary quasiequivalences, F is k-connected, and $\partial_- F : \partial_- W \to I \times \partial Y$ is an R-homology equivalence. Then, the sequence

$$0 \to K_{k+1}(W) \to K_{k+1}(W, \partial W) \to K_k(\partial W) \to K_k(W) \to K_k(W, \partial W) \to 0$$

is exact, where

$$K_\ell(W, \partial W) = \mathrm{Ker}[F_* : H_\ell(W, \partial W; R) \to H_\ell(I \times Y, \partial(I \times Y); R)]$$
$$K_\ell(W) = \mathrm{Ker}[F_* : H_\ell(W; R) \to H_\ell(I \times Y; R)], \text{ etc.}$$

Lemma 7.3. *Suppose f and f' are k-connected, $\mathcal{P}(G)_R$-singularity, R-boundary quasiequivalences, $\partial_- F = F|_{\partial_- W} : \partial_- W \to I \times \partial Y$ is an R-homology equivalence, and F is k-connected. Then $\boldsymbol{F}$ is G-framed cobordant rel $X \cup \partial_- W \cup \widehat{L}$ to $\boldsymbol{F'} = (F', B')$, where $F' : (W', \partial_+ W', \partial_- W') \to (I \times Y, \partial I \times Y, I \times \partial Y)$, satisfying the following properties.*

(7.3.1) *$(\boldsymbol{F'}, (\widehat{L}, \widehat{\mathbb{B}}))$ is a G-framed $\boldsymbol{\Theta}$-cobordism between $(\boldsymbol{f}, (L, \mathbb{B}))$ and $(\boldsymbol{f''}, (L', \mathbb{B}'))$ such that $\boldsymbol{f''} = (f'', b'') : (X'', \partial X'', TX'') \to (Y, \partial Y, f''^{*}\xi)$ is a k-connected G-framed map.*

(7.3.2) *$F' : W' \to I \times Y$ is k-connected and $K_k(W', \partial W'; R) = 0 = K_{k+1}(W'; R)$.*

(7.3.3) *$\boldsymbol{M}_{\boldsymbol{f''}, \mathbb{B}'} \cong \boldsymbol{M}_{\boldsymbol{f'}, \mathbb{B}'} \oplus \mathbb{M}(R[G], \boldsymbol{c}_1) \oplus \cdots \oplus \mathbb{M}(R[G], \boldsymbol{c}_\ell)$ in $\nabla \boldsymbol{Q}(\boldsymbol{A}, \boldsymbol{\Theta})$ for some $\boldsymbol{c}_i : \boldsymbol{\Theta} \to (\mathbb{Z}, \mathbb{Z}_2)$, $i = 1, \cdots, \ell$, (cf. Definitions 3.28 and 3.30).*

Proof. Fix a small, closed, n-dimensional disk D_0 in general position in Interior$(X' \smallsetminus L')$. Thus $D_0 \cap gD_0 = \varnothing$ if $g \in G \smallsetminus \{1\}$. Let $t \in \Theta_2$. Take a point $x_t \in B'_{\beta(t)}$ such that $G_{x_t} = \rho_G(t)$. Further choose small, closed, n-dimensional, G_{x_t}-invariant disks D_t with center x_t in $X' \smallsetminus GD_0$ such that $gD_t \cap g'D_{t'} = \varnothing$ unless $t = t'$ and $gG_{x_t}g^{-1} = g'G_{x_{t'}}g'^{-1}$, where t, $t' \in \Theta_2$, and g, $g' \in G$. Thus each GD_t is a G-equivariant tubular neighborhood of Gx_t. For each t, take a connecting tube T_t from ∂D_0 to ∂D_t in general position in $(X' \smallsetminus L') \smallsetminus \bigcup_{t \in \Theta_2} G(\mathrm{Interior}(D_t))$. Then,

$$D = D_0 \cup \bigcup_{t \in \Theta_2} (D_t \cup T_t)$$

is homeomorphic to the n-dimensional disk D^n. Since $K_k(W) \to K_k(W, \partial W)$ is surjective, we can take finitely many embeddings with trivial normal bundle, $h_i : S^k \to \mathrm{Interior}(W)$, $i = 1, \cdots, \ell$ say, which together generate $K_k(W, \partial W)$. Without loss of generality, we can assume gh_i and $g'h_j$ are disjoint unless $i = j$. For each i, take a path $p_i : [0,1] \to W \smallsetminus \widehat{L}$ in general position starting in Interior(D_0) and ending in Im(h_i) (i.e., $p_i(0) \in \mathrm{Interior}(D_0)$ and $p_i(1) \in \mathrm{Im}(h_i)$). Choose a thin band $u_i : [0,1] \times D^k \hookrightarrow X$ along the path p_i (i.e., $u_i(t, 0) = p_i(t)$ for all $t \in [0,1]$) in general position such that $u_i(0, D^k) \subset \mathrm{Interior}(D_0)$ and $u_i(1, D^k) \subset \mathrm{Image}(h_i)$. Then taking the union of $u_i([0,1] \times S^{k-1})$ and $\mathrm{Image}(h_i) \smallsetminus u_i(1, \mathrm{Interior}(D^k))$, we obtain an embedding $h'_i : (D^k, S^{k-1}) \to (W, X' \smallsetminus L')$. Here the restriction $\partial h'_i : S^{k-1} \to X' \smallsetminus L'$ of h'_i is a trivial embedding close to $p_i(0)$. We may suppose gh_i and $g'h_j$ are disjoint unless $i = j$. For $\beta \in \widetilde{\Theta}$, let $c_i(\beta)$ denote the ordinary intersection number in $\mathbb{Z}$ afforded by W_β and Image(h_i). Set $\widetilde{c}_i(t) = c_i(\beta(t))$ and $c_{2,i}(t) = [c_i(\beta(t))] \in \mathbb{Z}_2$ for $t \in p_{\mathbb{B}}(\widetilde{\Theta})$, where $\beta(t) \in \widetilde{\Theta}$ denotes as usual a lifting of t. For $t \in \Theta_2 \smallsetminus p_{\mathbb{B}}(\widetilde{\Theta})$, let $\widetilde{c}_i(t)$ denote the number ($\in \mathbb{Z}$) of intersection points of $\widehat{B}_t$ with Image(h_i). Set $c_{2,i}(t) = [\widetilde{c}_i(t)] \in \mathbb{Z}_2$. Thus $\boldsymbol{c}_i = (c_i, c_{2,i}) : \boldsymbol{\Theta} \to (\mathbb{Z}, \mathbb{Z}_2)$ and $\widetilde{c}_i : \Theta_2 \to \mathbb{Z}$. Without loss of generality, we can assume that for $t \in \Theta_2$, h_i meets with $\widehat{B}_{\beta(t)}$ at $m_{i,t}$-points $A(i,t,1), \cdots, A(i,t,m_{i,t})$, each having isotropy subgroup G_{x_t}, where $m_{i,t} = |\widetilde{c}_i(t)|$. Take points $B(i,t,1), \cdots, B(i,t,m_{i,t})$ in $h'_i(S^{k-1})$ and $C(i,t,1), \cdots, C(i,t,m_{i,t})$ in $D_t \cap B'_{\beta(t)}$. Let $\triangle(i,t,j) = \triangle A(i,t,j)B(i,t,j)C(i,t,j)$ denote embedded triangles in W (each of which is homeomorphic to D^2) such that

$$\overline{A(i,t,j)B(i,t,j)} \subset \mathrm{Im}(h'_i),$$
$$\overline{A(i,t,j)C(i,t,j)} \subset \widehat{B}^{=\rho_G(t)}_{\beta(t)}, \text{ and}$$
$$\overline{B(i,t,j)C(i,t,j)} \subset \mathrm{Interior}(D).$$

We may assume $g(\mathrm{Interior}(\triangle(i,t,j))) \cap g'(\mathrm{Interior}(\triangle(i',t',j'))) = \emptyset$ unless $i = i'$, $t = t'$, $j = j'$ and $g = g'$. Moreover we may assume $\triangle(i,t,j)$ is perpendicular to $\widehat{B}_{\beta(t)}$, X' and $\mathrm{Image}(h'_i)$. For each i, delete the intersection points $A(i,t,j)$ of h'_i with $B'_{\beta(t)}$ along the triangle $\triangle(i,t,j)$ (see Lemma 1.2 of [30]), where $t \in \Theta_2$ and $j = 1, \cdots, m$, and obtain an embedding $h''_i : (D^k, S^{k-1}) \to (W \smallsetminus \widehat{L}, X' \smallsetminus L')$. It is remarkable that the restriction $\partial h''_i : S^{k-1} \to X' \smallsetminus L'$ of h''_i is the connected sum of $\partial h'_i$ and $h_{\boldsymbol{c}_i}$ (cf. (6.2.1)). Thickening h''_i, we obtain embeddings

$$H''_i : (D^k \times D^{k+1}, S^{k-1} \times D^{k+1}) \to (W \smallsetminus \widehat{L}, X' \smallsetminus L'),$$

$i = 1, \cdots, \ell$. Now we may suppose gH''_i and $g'H''_j$ are disjoint unless $g = g'$ and $i = j$. Further we can suppose $F(\mathrm{Image}(H''_i))$ is a point in $\{1\} \times Y$. Set

$$\begin{aligned}
V &= \bigcup_{i=1}^{\ell} G(\mathrm{Image}(H''_i)),\\
\partial_+ V &= \bigcup_{i=1}^{\ell} G(H''_i(S^{k-1} \times D^{k+1})),\\
W' &= \mathrm{Closure}(W \smallsetminus V),\\
X'' &= (X' \cup V) \smallsetminus \left\{ \bigcup_{i=1}^{\ell} G(H''_i(D^k \times (\mathrm{Interior}(D^{k+1})))) \right\},\\
F' &= F|_{W'} : W' \to I \times Y,\\
B' &= B|_W : T(W') \to (p_Y \circ F')^*(\varepsilon_Y(\mathbb{R}) \oplus \xi),\\
f'' &= F|_{X''} : X'' \to \{1\} \times Y = Y.
\end{aligned}$$

Deforming B by a regular G-homotopy (cf. Lemma 6.1) if necessary, we can assume without loss of generality that $B'|_{X''}$ has the form

$$id_{X''} \oplus b'' : \varepsilon_{X''}(\mathbb{R}) \oplus T(X'') \to \varepsilon_{X''}(\mathbb{R}) \oplus f''^*\xi.$$

We check that $\boldsymbol{F}' = (F', B')$ and $\boldsymbol{f}'' = (f'', b'')$ satisfy (7.3.1)–(7.3.3). Clearly, by construction, f'' and F' are k-connected. By excision, $K_k(W', \partial W'; R) = K_k(W, \partial W \cup V; R)$. There is an exact sequence

$$K_k(\partial W \cup V, \partial W; R) \longrightarrow K_k(W, \partial W; R) \longrightarrow K_k(W, \partial W \cup V; R) \longrightarrow 0.$$

Since the first arrow is surjective, $K_k(W, \partial W \cup V; R) = 0$. Thus we get $K_k(W', \partial W'; R) = 0$. By the universal coefficient theorem, $K^k(W', \partial W'; R) = 0$. The Poincaré-Lefschetz duality implies $K_{k+1}(W'; R) = 0$. Note that $\boldsymbol{f}''$ can be obtained from $\boldsymbol{f}'$ by G-surgery along the embeddings $\partial h''_1, \cdots, \partial h''_\ell : S^{k-1} \to X'$. These embeddings are isotopic in Interior$(X' \smallsetminus L')$ to $h_{\boldsymbol{c}_1}, \cdots, h_{\boldsymbol{c}_\ell}$, respectively (see (6.2.1)). Thus we obtain

$$\boldsymbol{M}_{\boldsymbol{f}'',\mathbb{B}} \cong \boldsymbol{M}_{\boldsymbol{f}',\mathbb{B}'} \oplus \mathbb{M}(R[G], \boldsymbol{c}_1) \oplus \cdots \oplus \mathbb{M}(R[G], \boldsymbol{c}_\ell).$$

□

Suppose $\boldsymbol{f}$ and $\boldsymbol{f}'$ are k-connected, $\mathcal{P}(G)_R$-singularity, R-boundary quasiequivalences. Suppose F is k-connected, $\partial_- F : \partial_- W \to I \times \partial Y$ is an R-homology equivalence, and furthermore $K_{k+1}(W; R) = 0 = K_k(W, \partial W; R)$. Note that $\partial_+ \boldsymbol{F}$ is the disjoint union of two G-framed maps $-\boldsymbol{f}$ and $\boldsymbol{f}'$, where $-\boldsymbol{f}$ is a copy of $\boldsymbol{f}$ whose underlying manifold has orientation opposite to the original one. Let $\overline{\theta}_{\partial(F,\widehat{\mathbb{B}})}$ (resp., $\overline{\theta}_{2,\partial(F,\widehat{\mathbb{B}})}$) denote the composition:

$$\widetilde{\Theta} \xrightarrow{id \cup id} \widetilde{\Theta}_{\mathbb{B}} \amalg \widetilde{\Theta}_{\mathbb{B}'} \xrightarrow{\theta_{\partial_+ F, \mathbb{B} \cup \mathbb{B}'}} K_k(\partial_+ W; R) = K_k(\partial W; R)$$

$$(\text{resp., } \Theta_2 \xrightarrow{id \cup id} \Theta_{2,\mathbb{B}} \amalg \Theta_{2,\mathbb{B}'} \xrightarrow{\theta_{2,\partial_+ F, \mathbb{B} \cup \mathbb{B}'}} K_k(\partial_+ W; R_2) = K_k(\partial W; R_2)).$$

Set

$$\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})} = (K_k(\partial W; R), B_{\partial F,R}, q_{\partial \boldsymbol{F},R}, \overline{\theta}_{\partial(F,\widehat{\mathbb{B}})}, \overline{\theta}_{2,\partial(F,\widehat{\mathbb{B}})}).$$

Clearly this module is isomorphic to $\boldsymbol{M}_{-(\boldsymbol{f},\mathbb{B})} \oplus \boldsymbol{M}_{\boldsymbol{f}',\mathbb{B}'}$.

Lemma 7.4. *Suppose $\boldsymbol{f}$ and $\boldsymbol{f}'$ are k-connected, $\mathcal{P}(G)_R$-singularity, R-boundary quasiequivalences, $\partial_- F : \partial_- W \to I \times \partial Y$ is an R-homology equivalence, $F : W \to I \times Y$ is k-connected, and $K_{k+1}(W; R) = 0 = K_k(W, \partial W; R)$. Then the following hold.*

(7.4.1) $\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})} = -\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} \oplus \boldsymbol{M}_{\boldsymbol{f}',\mathbb{B}'}$ *(hence $\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})} \in \nabla \boldsymbol{\mathcal{Q}}(\boldsymbol{A}, \boldsymbol{\Theta})$), where*

$$-\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}} = (K_k(X, R), -q_{\boldsymbol{f},R}, -B_{f,R}, -\theta_{f,\mathbb{B}}, -\theta_{2,f,\mathbb{B}}).$$

(7.4.2) *The submodule $\partial K_{k+1}(W, \partial W; R)$ of $K_k(\partial W; R)$ contains $\mathrm{Im}(\overline{\theta}_{\partial(F,\widehat{\mathbb{B}})})$; the submodule $\partial K_{k+1}(W, \partial W; R_2)$ of $K_k(\partial W; R_2)$ contains $\mathrm{Im}(\overline{\theta}_{2,\partial(F,\widehat{\mathbb{B}})})$.*

(7.4.3) *$\partial K_{k+1}(W, \partial W; R)$ is totally isotropic in $\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})}$.*

Proof. Property (7.4.1) follows from $\boldsymbol{M}_{-(\boldsymbol{f},\mathbb{B})} \cong -\boldsymbol{M}_{\boldsymbol{f},\mathbb{B}}$.

Property (7.4.2) is obtained by chasing the following commutative diagram

$$\begin{tikzcd}
H_{k+1}(\widehat{B}_\beta, \partial \widehat{B}_\beta) \arrow[r] \arrow[d, "\partial"] & H_{k+1}(W, \partial W) \arrow[r] \arrow[d, "\partial"] & K_{k+1}(W, \partial W) \arrow[d, "\partial"] \\
H_k(\partial \widehat{B}_\beta) \arrow[r] \arrow[d] & H_k(\partial W) \arrow[r] \arrow[d] & K_k(\partial W) \arrow[d, "="] \\
H_k(\partial \widehat{B}_\beta, \partial_- \widehat{B}_\beta) \arrow[r] & H_k(\partial W, \partial_- W) \arrow[r] & K_k(\partial W, \partial_- W) \\
H_k(-B_\beta, \partial(-B_\beta)) \oplus H_k(B'_\beta, \partial B'_\beta). \arrow[u, "="] & &
\end{tikzcd}$$

for the coefficient rings R and R_2.

We prove (7.4.3). Here the coefficient ring of homology groups is R. Regard $K_{k+1}(W, \partial W)$ as a submodule of $K_k(\partial W)$ via the connecting homomorphism ∂. It is well known that the ordinary intersection form $\mathrm{Int}_{\partial W;R} : K_k(\partial W) \times K_k(\partial W) \to R$ satisfies

$$\mathrm{Int}_{\partial W;R}(K_{k+1}(W, \partial W), K_{k+1}(W, \partial W)) = 0.$$

Since $K_{k+1}(W, \partial W)$ is G-invariant,

$$B_{\partial F,R}(K_{k+1}(W, \partial W), K_{k+1}(W, \partial W)) = 0.$$

We conclude the proof by showing $q_{\partial F}$ vanishes on $\partial K_{k+1}(W, \partial W)$. By Definition 5.7, $q_{\partial \boldsymbol{F}}$ is a map

$$K_k(\partial W) \longrightarrow (R/(1-\lambda))[\{1\}] \oplus R_2[G(2)_\lambda \smallsetminus Q] \oplus R[G(2)_{-\lambda} \smallsetminus S] \oplus R[G(\geq 3)_h],$$

where $G(2)_\lambda = \{a \in G(2) \mid w(a) = -\lambda\}$, $G(2)_{-\lambda} = \{a \in G(2) \mid w(a) = \lambda\}$, $Q = Q(G, X)$, $S = S(G, X)$, and $G(\geqq 3)_h$ is a subset of G such that

$$G = \{1\} \amalg G(2) \amalg G(\geqq 3)_h \amalg G(\geqq 3)_h{}^{-1}.$$

Let $x \in K_{k+1}(W, \partial W)$. For $g \in G(\geqq 3)_h$, the g-th coefficient $q_{\partial \boldsymbol{F},R}(x)_g$ of $q_{\partial \boldsymbol{F},R}(x)$ is equal to 0, because

$$q_{\partial \boldsymbol{F},R}(x)_g = \varepsilon(\mathrm{Int}_{\partial W;R}(x, g^{-1}x)).$$

For $g \in G(2)_{-\lambda} \smallsetminus S$, $q_{\partial \boldsymbol{F},R}(x)_g = 0$, because

$$\varepsilon(\mathrm{Int}_{\partial W;R}(x, g^{-1}x)) = 2q_{\partial \boldsymbol{F},R}(x)_g.$$

Thus we have reduced to the case $g \in \{1\} \cup (G(2)_\lambda \smallsetminus Q)$. Let $\langle g \rangle$ denote the subgroup generated by g in G. By definition,

$$q_{\partial \boldsymbol{F},R}(x)_g = q_{\partial(\mathrm{Res}^G_{\langle g \rangle} \boldsymbol{F}),R}(x)_g$$

for any $x \in K_k(\partial W) = K_k(\partial(\mathrm{Res}^G_{\langle g \rangle} W))$. But

$$\begin{aligned} \dim(\mathrm{Sing}(\langle g \rangle, \mathrm{Res}^{\mathrm{G}}_{\langle \mathrm{g} \rangle}\mathrm{W})) + (\mathrm{k}+1) &\leq \{(k-2)+1\} + (k+1) \\ &= 2k \\ &< \dim(\mathrm{Res}^G_{\langle g \rangle} W). \end{aligned}$$

In other words, the strong gap hypothesis holds for the group $\langle g \rangle$. In this case, an argument similar to [44, p.53, lines 7–11] proves that $q_{\partial(\mathrm{Res}^G_{\langle g \rangle} \boldsymbol{F}),R}(x) = 0$ for all $x \in K_{k+1}(\mathrm{Res}^G_{\langle g \rangle} W, \partial(\mathrm{Res}^G_{\langle g \rangle} W))$. Consequently $q_{\partial \boldsymbol{F},R}(x) = 0$ for all $x \in K_{k+1}(W, \partial W)$. □

Corollary 7.5. *Suppose $\boldsymbol{f}$, $\boldsymbol{f}'$ and $\boldsymbol{F}$ are as in Lemma 7.4. Let $\sigma(\boldsymbol{f}, \mathbb{B})$ and $\sigma(\boldsymbol{f}', \mathbb{B}')$ $\in W_n(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta})_{\mathrm{proj}}$ denote their surgery obstructions defined in Theorem 6.3. If $K_{k+1}(W, \partial W; R)$ is a stably free $R[G]$-module then*

$$\sigma(-(\boldsymbol{f}, \mathbb{B})) + \sigma(\boldsymbol{f}', \mathbb{B}') = 0 \ \in W_n(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta})_{\mathrm{proj}}.$$

Proof. First note that $K_k(\partial W; R) = K_k(X; R) \oplus K_k(X; R)$ is a projective $R[G]$-module and has the nonsinglar Hermitian form $B_{\partial F,R}$.

Since $K_{k+1}(W; R) = 0 = K_k(W, \partial W; R)$, we obtain from the Mayer-Vietoris exact sequence for the pair $(W, \partial W)$, a short exact sequence

$$\partial K_{k+1}(W, \partial W; R) \hookrightarrow K_k(\partial W; R) \xrightarrow{i_{\partial W*}} K_k(W; R) \longrightarrow 0. \tag{7.5.1}$$

of $R[G]$-modules. By Poincaré-Lefschetz duality, $K_k(W, R) \cong K^{k+1}(W, \partial W; R)$ as R-modules. By the universal coefficient theorem, $K^{k+1}(W, \partial W; R) \cong \mathrm{Hom}(K_{k+1}(W, \partial W; \mathbb{Z}), R)$ as R-modules. Thus $K_k(W; R)$ is a free R-module. In addition, we get $\mathrm{rank}_R K_k(W; R) = \mathrm{rank}_R(\partial K_{k+1}(W, \partial W; R))$. The exact sequence (7.5.1) splits over R and

$$K_k(\partial W; R) \cong \partial K_{k+1}(W, \partial W; R) \oplus K_k(W; R) \tag{7.5.2}$$

as R-modules. Let ε_1 denote the projection map $R[G] \to R$ on the coefficient of $1 \in G$. The ordinary intersection pairing, i.e. $\varepsilon_1 \circ B_{\partial f,R}$, yields an $R[G]$-homomorphism

$$\psi : K_k(\partial W; R) \to \mathrm{Hom}_R(\partial K_{k+1}(W, \partial W; R), R),\ x \mapsto \varepsilon_1(B_{\partial f,R}(x, -)). \tag{7.5.3}$$

Since $\varepsilon_1 \circ B_{\partial f,R}$ is nonsingular, ψ is epic. Thus $K_k(\partial W; R)$ possesses an $R[G]$-direct sum decomposition

$$K_k(\partial W; R) = \mathrm{Ker}(\psi) \oplus M \tag{7.5.4}$$

where $M \cong \mathrm{Hom}_R(\partial K_{k+1}(W, \partial W; R), R)$ via $\psi|_M$. It follows from (7.5.2) and (7.5.4) that

$$\mathrm{rank}_R(\mathrm{Ker}(\psi)) = \mathrm{rank}_R K_k(W; R) = \mathrm{rank}_R(\partial K_{k+1}(W, \partial W; R)).$$

By Lemma 7.4, $\partial K_{k+1}(W, \partial W; R) \subseteqq \mathrm{Ker}(\psi)$. Since $\mathrm{Ker}(\psi)$ and $\partial K_{k+1}(W, \partial W; R)$ are both R-free modules having the same R-rank, we obtain $\mathrm{Ker}(\psi) = \partial K_{k+1}(W, \partial W; R)$. Set

$$\partial K_{k+1}(W, \partial W; R)^{\perp} = \{x \in K_k(\partial W; R) \mid B_{\partial F,R}(x, y) = 0 \text{ for all } y \in \partial K_{k+1}(W, \partial W; R)\}.$$

Then the equality $\partial K_{k+1}(W, \partial W; R)^{\perp} = \mathrm{Ker}(\psi)$ follows from the formula

$$B_{\partial F,R}(x, y) = \sum_{g \in G} \varepsilon_1(B_{\partial F,R}(x, g^{-1}y))g \quad (x,\ y \in K_k(\partial W; R)).$$

Thus we obtain

$$\partial K_{k+1}(W, \partial W; R) = \mathrm{Ker}(\psi) = \partial K_{k+1}(W, \partial W; R)^{\perp}. \tag{7.5.5}$$

By (7.5.4) and (7.5.5), $\partial K_{k+1}(W, \partial W; R)$ is an $R[G]$-direct summand of $K_k(\partial W; R)$. Hence it follows from Lemma 7.4 that $K_k(W, \partial W; R)$ is an $R[G]$-stably free Lagrangian of $\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})}$. Thus $\overline{\boldsymbol{M}}_{\partial(\boldsymbol{F},\widehat{\mathbb{B}})}$ vanishes in $W_n(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta})_{\mathrm{free}}$, and consequently $\sigma(-(\boldsymbol{f}, \mathbb{B})) + \sigma(\boldsymbol{f}', \mathbb{B}') = 0$. □

Remark 7.6. Let $K_k(W; R)$, $K_k(\partial W; R)$ and $K_{k+1}(W, \partial W; R)$ be as in Corollary 7.5. Then $K_k(W; R) \cong \mathrm{Hom}_R(\partial K_{k+1}(W, \partial W; R), R)$ as $R[G]$-modules. Moreover $K_k(W; R)$ and $K_k(\partial W; R)$ are both stably $R[G]$-free modules.

Proof. Let M be as in (7.5.4). Then $i_{\partial W*}$ maps M isomorphically onto $K_k(W;R)$. Thus $K_k(W;R) \cong \mathrm{Hom}_R(\partial K_{k+1}(W,\partial W;R),R)$ as $R[G]$-modules. Since $\partial K_{k+1}(W,\partial W;R)$ is stably free over $R[G]$, so is $\mathrm{Hom}_R(\partial K_{k+1}(W,\partial W;R),R)$. Thus $K_k(W;R)$ and $K_k(\partial W;R)$ are both stably free over $R[G]$. □

Let $\mathcal{H}(G)$ denote the set of all hyperelementary subgroups of G. The map $f:(X,\partial X)\to(Y,\partial Y)$ is called an $(\mathcal{H}(G),R)$-*singularity quasiequivalence* if $f^H: X^H \to Y^H$ is an R-homology equivalence for all nontrivial $H\in\mathcal{H}(G)$, i.e. $H\neq\{1\}$.

Lemma 7.7. *Suppose f and f' are k-connected, R-boundary, $(\mathcal{H}(G),R)$-singularity equivalences such that $\partial X=\partial X'$, $\partial f=\partial f'$ and $\partial b=\partial b'$. If $(\boldsymbol{F},\widehat{\mathbb{B}})$ is a G-framed $\boldsymbol{\Theta}$-cobordism rel ∂X between $\boldsymbol{f}$ and $\boldsymbol{f}'$ and F is an $(\mathcal{H}(G),R)$-singularity quasiequivalence then the G-surgery obstructions*

$$\sigma(\boldsymbol{f},\mathbb{B}) \textit{ and } \sigma(\boldsymbol{f}',\mathbb{B}') \textit{ lie in } W_n(R,G,Q(G,X),S(G,X),\boldsymbol{\Theta})_{\mathrm{free}}$$

$(\subseteqq W_n(R,G,Q(G,X),S(G,X),\boldsymbol{\Theta})_{\mathrm{proj}})$ *and furthermore*

$$\sigma(\boldsymbol{f},\mathbb{B})=\sigma(\boldsymbol{f}',\mathbb{B}').$$

Proof. For each hyperelementary subgroup H of G, $K_k(X;R)$ is stably free over $R[H]$, by Lemma 5.14. Thus by Swan's induction theorem, $K_k(X;R)$ is stably free over $R[G]$. Hence the G-surgery obstruction $\sigma(\boldsymbol{f},\mathbb{B})$ lies in $W_n(R,G,Q(G,X),S(G,X),\boldsymbol{\Theta})_{\mathrm{free}}$. By Lemmas 2.20 and 2.37, $\sigma(-(\boldsymbol{f},\mathbb{B}))=-\sigma(\boldsymbol{f},\mathbb{B})$. Similarly $\sigma(\boldsymbol{f}',\mathbb{B}')$ lies in the same abelian group. Since F is a cobordism rel ∂X and f is an R-boundary quasiequivalence, $\partial_- F$ is also an R-homology equivalence. Thus $K_k(\partial W;R)\cong K_k(X;R)\oplus K_k(X';R)$ as $R[G]$-modules and $K_k(\partial W;R)$ is stably free over $R[G]$. Moreover, by Lemma 7.3 we can assume F satisfies the hypotheses of Lemma 7.4. Then $K_j(W;R)=0$ except for $j=k$, and $K_k(W;R)$ is free over R. For each hyperelementary subgroup H of G, $F: W\to I\times Y$ is an (H,R)-singularity quasiequivalence in the sense prior to Lemma 5.6. One can adapt now the proof of Lemma 5.14 to show that $K_k(W;R)$ is stably free over $R[H]$. Thus by Swan's induction theorem, $K_k(W;R)$ is stably free over $R[G]$. By the short exact sequence (7.5.1), $K_{k+1}(W,\partial W;R)$ is also stably free over $R[G]$. Thus by Corollary 7.5, $\sigma(\boldsymbol{f},\mathbb{B})=\sigma(\boldsymbol{f}',\mathbb{B}')$. □

We remind the reader once again of the assumption $\mathbb{Z}\subseteqq R\subseteqq\mathbb{Q}$.

Theorem 7.8. *Let X and Y denote compact, connected, oriented, smooth G-manifolds of dimension $n=2k\geqq 6$. Let $(L,\mathbb{B})$ denote a G-singularity structure for X. Let $\boldsymbol{f}=(f,b):(X,\partial X,TX)\to(Y,\partial Y,f^*\xi)$ denote a degree one G-framed map. Suppose Y is simply connected and $f:X\to Y$ is an R-boundary, $(\mathcal{H}(G),R)$-singularity quasiequivalence. Then $\sigma(\boldsymbol{f},\mathbb{B})$ possesses a complete G-surgery obstruction*

$$\sigma(\boldsymbol{f},\mathbb{B})\in W_n(R,G,Q(G,X),S(G,X),\boldsymbol{\Theta}_{\mathbb{B}})_{\mathrm{free}},$$

i.e., $\sigma(\boldsymbol{f}, \mathbb{B}) = 0$ *if and only if one can perform G-surgery on $\boldsymbol{f}$ along spheres of dimension $\leqq k$ in* Interior$(X \smallsetminus L)$ *to obtain a k-connected G-framed map $\boldsymbol{f}' = (f', b')$: $(X', \partial X, TX') \to (Y, \partial Y, f'^*\xi)$ such that $f' : X' \to Y$ is an R-homology equivalence.*

The definition of $\sigma(\boldsymbol{f}, \mathbb{B})$ above is given in the proof below. Note that $\sigma(\boldsymbol{f}, \mathbb{B})$ has been defined previously in Theorem 6.3, but only for k-connected maps $\boldsymbol{f}$.

Proof. Since $\dim B_\beta \leqq k$, we can perform G-surgery on $\boldsymbol{f}$ along spheres of dimension $\leqq k-1$ in Interior$(X \smallsetminus L)$ to obtain a k-connected (R, G)-surgery map $\boldsymbol{f}_1 = (f_1, b_1)$, $f_1 : (X_1, \partial X_1) \to (Y, \partial Y)$. When f itself is k-connected, we allow surgery along an empty set of spheres and get $\boldsymbol{f}_1 = \boldsymbol{f}$. Note that $\partial X_1 = \partial X$ and $\mathrm{Sing}(G, X_1) = \mathrm{Sing}(G, X)$. Thus, f_1 is also an R-boundary, $(\mathcal{H}(G), R)$-singularity quasiequivalence. Using Swan's induction theorem (cf. proof of Lemma 7.7), we get

$$\sigma(\boldsymbol{f}_1, \mathbb{B}) \in W_n(R, G, Q(G, X), S(G, X), \boldsymbol{\Theta}_{\mathbb{B}})_{\mathrm{free}}.$$

We define $\sigma(\boldsymbol{f}, \mathbb{B})$ to be the element $\sigma(\boldsymbol{f}_1, \mathbb{B})$. We check that the element $\sigma(\boldsymbol{f}, \mathbb{B})$ is independent of the choice of $\boldsymbol{f}_1$. Let $\boldsymbol{f}_2 = (f_2, b_2)$ be another k-connected (R, G)-surgery map obtained in the same way as $\boldsymbol{f}_1$. Then there exists a G-framed $\boldsymbol{\Theta}_{\mathbb{B}}$-cobordism $(\boldsymbol{F}, \widehat{\mathbb{B}})$ rel ∂X between $(\boldsymbol{f}_1, (L, \mathbb{B}))$ and $(\boldsymbol{f}_2, (L, \mathbb{B}))$ such that $\widehat{L} = I \times L$ and $\widehat{\mathbb{B}} = I \times \mathbb{B}$. In this situation, $F : W \to I \times Y$ is automatically an $(\mathcal{H}(G), R)$-singularity quasiequivalence and $\partial_-(F, B) = I \times \partial(f, b)$. Thus by Lemma 7.7, $\sigma(\boldsymbol{f}_1, \mathbb{B}) = \sigma(\boldsymbol{f}_2, \mathbb{B})$.

Suppose $\sigma(\boldsymbol{f}, \mathbb{B}) = 0$. By definition, this means that $\sigma(\boldsymbol{f}_1, \mathbb{B}) = 0$. By Theorem 6.3, we can perform G-surgery on $\boldsymbol{f}_1$ along spheres of dimension $\leqq k$ in Interior$(X \smallsetminus L)$ to obtain a G-framed map $\boldsymbol{f}'$ as described in the theorem.

On the other hand, if we can obtain by a G-surgery in Interior$(X \smallsetminus L)$, a G-framed map $\boldsymbol{f}'$ as described in the theorem then by independence of the choice of $\boldsymbol{f}_1$ in constructing $\sigma(\boldsymbol{f}, \mathbb{B})$, we have $\sigma(\boldsymbol{f}, \mathbb{B}) = \sigma(\boldsymbol{f}', \mathbb{B})$. But $\sigma(\boldsymbol{f}', \mathbb{B}) = 0$, because now $K_k(X'; R) = 0$. □

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