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Cancellation over Rings of Dimension $\leqq 1$

ANTHONY BAK
Department of Mathematics
University of Bielefeld
W-4800 Bielefeld, Germany

MASAHARU MORIMOTO
Department of Mathematics
Okayama University
Okayama 700, Japan

Abstract Let A be a module finite R-algebra such that the Bass-Serre dimension $(R) \leqq 1$. Let M, M' and P be A-modules. Then $M \oplus A \oplus P \cong M' \oplus A \oplus P$ implies $M \oplus A \cong M' \oplus A$, providing the following holds: (1) P is finitely generated and projective. (2) M is finitely presented. (3) There is a 2-sided ideal I in A such that the general linear group $GL_2(A)$ acts transitively on the (A/I)-unimodular vectors in $A/I \oplus A/I$ and for almost all maximal ideals $\mathfrak{m}$ of R there is locally an $A_\mathfrak{m}$-homomorphism $f^\mathfrak{m} : M_\mathfrak{m} \longrightarrow A_\mathfrak{m}$ such that modulo the Jacobson radical$(A_\mathfrak{m})$, image $(f^\mathfrak{m}) \supseteq I_\mathfrak{m}$.

1. Introduction

The purpose of this note is to extend a recent cancellation result of Hambleton and Kreck [H-K, Theorem A] for modules over a separable order to modules over a module finite R-algebra where dimension $(R) \leq 1$. We define the **dimension** $\dim(R)$ of a commutative ring R to be 0 (resp. 1) if it is semilocal (resp. there is a finite set $\mathfrak{M}$ of maximal ideals of R such that for each element $s \in R \backslash \underset{\mathfrak{m} \in \mathfrak{M}}{\cup} \mathfrak{m}$, the quotient ring R/Rs is semilocal.) This notion of dimension is weaker than that of Bass-Serre dimension which was used by H. Bass in his fundamental work on cancellation, cf. [B, IV].

Our main result is the following.

□ **THEOREM** 1.1 Let A be a module finite R-algebra such that $\dim(R) \leqq 1$. Let M, M', and P be A-modules. Suppose the following conditions hold.

(1.1.1) There is a 2-sided ideal I in A such that the general linear group $GL_2(A)$ acts transitively on the (A/I)-unimodular vectors in $A/I \oplus A/I$.
(1.1.2) M is finitely presented (this is automatic if A is Noetherian and M is finitely generated) and for all but a finite number of maximal ideals $\mathfrak{m}$ of R, there is locally an $A_{\mathfrak{m}}$-homomorphism $f^{\mathfrak{m}} : M_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ such that modulo the Jacobson radical$(A_{\mathfrak{m}})$ the image $(f^{\mathfrak{m}}) \supseteq I_{\mathfrak{m}}$.
If P is finitely generated and projective then $P \oplus A \oplus M \cong P \oplus A \oplus M'$ implies $A \oplus M \cong A \oplus M'$. □

It is very likely that there are appropriate generalizations of (1.1) to module finite R-algebras A where the only condition imposed on A is that R is finite dimensional. As Hambleton and Kreck [H-K] have shown, one can expect such results to find applications in the classification of 2-dimensional C.W. complexes.

□ **COROLLARY** 1.2 Let A be a module finite R-algebra such that $\dim(R) \leqq 1$. Let M, M' and P be A-modules. Suppose the following conditions hold.

(1.2.1) There is a 2-sided ideal I in A such that A/I is commutative and each element of the special linear group $SL_2(A/I)$ lifts to $GL_2(A)$; e.g., $SL_2(A/I)$ is equal to the elementary group $E_2(A/I)$.

(1.2.2) Condition (1.1.2) above.

If P is finitely generated and projective then $P \oplus A \oplus M \cong P \oplus A \oplus M'$ implies $A \oplus M \cong A \oplus M'$. □

PROOF The conclusion above will follow from (1.1), once we show that condition (1.1.1) is satisfied. It suffices to show that $SL_2(A/I)$ acts transitively on unimodular vectors of $A/I \oplus A/I$. Let $a, c \in A/I$ such that $(a, c) \in A/I \oplus A/I$ is unimodular. Choose elements $b, d \in A/I$ such that $ad + bc = 1$. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant 1, i.e. $\in SL_2(A/I)$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$. It follows that $SL_2(A/I)$ acts transitively on the unimodular vectors in $A/I \oplus A/I$. Q.E.D.

□ **COROLLARY** 1.3 Let R be a Dedekind ring with field of fractions F. Let A be an R-order on a finite separable semisimple F-algebra. Let M, M', and N be finitely generated A-modules. Let I be a 2-sided ideal of A such that conditions (1.1.1) and (1.1.2) are satisfied. Suppose that N is R-torsion free and that there is a natural number r such that for each maximal ideal $\mathfrak{m}$ of $R, N_{\mathfrak{m}}$ is a direct summand of $(A_{\mathfrak{m}} \oplus M_{\mathfrak{m}})^r$. Then $N \oplus A \oplus M \cong N \oplus A \oplus M'$ implies $A \oplus M \cong A \oplus M'$. □

PROOF Clearly, $N \oplus A \oplus A \oplus M \cong N \oplus A \oplus A \oplus M'$. By Swan's cancellation theorem [S, (9.4) and (9.7)], $A \oplus A \oplus M \cong A \oplus A \oplus M'$. Since $\dim(R) \leqq 1$, it follows now from Theorem (1.1) that $A \oplus M \cong A \oplus M'$. Q.E.D.

□ **COROLLARY** 1.4 (Hambleton - Kreck [H-K, Theorem A]) Let A be a separable R-order as in (1.3). Let M, M' and N be finitely generated A-modules where N is as in (1.3). Suppose there is a 2-sided ideal I in A such that the ring A/I is also a separable R-order and the following conditions hold.

(1.4.1) $GL_2(A)$ acts transitively on the (A/I)-unimodular vectors of $A/I \oplus A/I$.

(1.4.2) There is a natural number k such that for all but a finite number of maximal ideals $\mathfrak{m}$ of R, $((A/I)^k \oplus M)_{\mathfrak{m}}$ has a direct summand isomorphic to $A_{\mathfrak{m}}$. Then $N \oplus A \oplus M \cong N \oplus A \oplus M'$ implies $A \oplus M \cong A \oplus M'$. □

PROOF The conclusion of Hambleton - Kreck will follow from (1.3), once we show that condition (1.1.2) is satified.

Let $B = A/I$. For almost all maximal ideals $\mathfrak{m}$ of R, there are by hypothesis $A_{\mathfrak{m}}$-homomorphisms $f : A_{\mathfrak{m}} \longrightarrow B^k_{\mathfrak{m}} \oplus M_{\mathfrak{m}}$ and $g : B^k_{\mathfrak{m}} \oplus M_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ such that $gf = 1_{A_{\mathfrak{m}}}$. Write $f = (f_1, f_2)$ where $f_1 : A_{\mathfrak{m}} \longrightarrow B^k_{\mathfrak{m}}$ and $f_2 : A_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}}$ and write $g = (g_1, g_2)$ where $g_1 : B^k_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ and $g_2 : M_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$. For $a \in I_{\mathfrak{m}}, f_1(a) = f_1(1)a = 0$; thus, $a = gf(a) = g_2 f_2(a)$. Thus, $g_2 f_2|_{I_{\mathfrak{m}}} = 1_{I_{\mathfrak{m}}}$. Thus, $M_{\mathfrak{m}}$ contains a direct summand isomorphic to $I_{\mathfrak{m}}$. Q.E.D.

In the next section, we recall a few definitions and then prove Theorem (1.1). Our methods are elementary and require little beyond a familiarity with semilocal rings, Nakayama's lemma, and localization.

2. Proof of Theorem 1.1

We begin by recalling a few definitions.

Let A be an associative ring with identity and $\mathfrak{q}$ a 2-sided ideal in A. Let $M = M_1 \oplus \cdots \oplus M_n$ be a direct sum of right A-modules. If $f : M_i \longrightarrow M_j (i \neq j)$ is an A-homomorphism and $\overline{f}$ its unique extension to an A-endomorphism of M such that $\overline{f}(M_k) = 0$ for all $k \neq i$, we set $\epsilon(f) = 1_M + \overline{f}$. Clearly $\epsilon(f)$ is an A-automorphism of M with inverse $\epsilon(-f)$. $\epsilon(f)$is called the **elementary transformation** defined by f. If image $(f) \subseteq M_j\mathfrak{q}$ then $\epsilon(f)$ is called a **$\mathfrak{q}$-elementary transformation**. Let $E(M_1 \ldots, M_n)$ denote the subgroup of $\mathrm{Aut}_A(M)$ generated by all elementary transformations $\epsilon(f)$ where f ranges over all A-homomorphisms $f : M_i \longrightarrow M_j$ such that $i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$. Let $E(M_1, \ldots, M_n; \mathfrak{q})$ denote the normal subgroup of $E(M_1, \ldots, M_n)$

generated by the $\mathfrak{q}$-elementary transformations. If $M_1 = \cdots = M_n = A$ then by definition $E_n(A) = E_n(M_1, \ldots, M_n)$ and $E_n(A, \mathfrak{q}) = E(M_1, \ldots, M_n; \mathfrak{q})$.

Let M be a right A-module. An element $m \in M$ is called **unimodular** if there is an A-homomorphism $f : M \longrightarrow A$ such that $f(m) = 1$. It follows that $m = (m_1, \ldots, m_n) \in M_1 \oplus \cdots \oplus M_n$ is unimodular $\Leftrightarrow$ there are A-homomorphisms $f_i : M_i \longrightarrow A (i = 1, \ldots, n)$ such that $\sum_{i=1}^{n} f_i(m) = 1$ $\Leftrightarrow$ there are A-homomorphisms $f_i : M_i \longrightarrow A (i = 1, \ldots, n)$ such that $(f_1(m_1), \ldots, f_n(m_n)) \in A^n = A \oplus \cdots \oplus A$ (n times) is unimodular. A vector $(a_1, \ldots, a_n) \in A^n$ is unimodular $\Leftrightarrow$ there are elements $b_1, \ldots, b_n \in A$ such that $\sum_{i=1}^{n} b_i a_i = 1$.

If M is a right A-module and $m \in M$, one defines $o_M(m) = \{f(m) | f \in \operatorname{Hom}_A(M, A)\}$. Clearly, $o_M(m)$ is a left ideal in A and m is unimodular $\Leftrightarrow o_M(m) = A$.

□ **LEMMA** 2.1 Let A be an associative ring with identity. Let $(a_1, \ldots, a_n) \in A^n$ be unimodular. Then there is an element $b \in A$ such that $(a_1, \ldots, a_{n-1}, (ba_n)^2)$ is unimodular. □

PROOF By definition, there are elements $c_1, \ldots, c_n \in A$ such that $1 = c_1 a_1 + \cdots + c_n a_n$. Thus, $a_n = a_n(c_1 a_1 + \cdots + c_n a_n)$. Thus, $1 = c_1 a_1 \cdots + c_n a_n = c_1 a_1 + \cdots + c_{n-1} a_{n-1} + c_n [a_n(c_1 a_1 + \cdots + c_n a_n)] = (c_n a_n c_1 + c_1) a_1 + \cdots + (c_n a_n c_{n-1} + c_{n-1}) a_{n-1} + (c_n a_n)^2$. Thus, $(a_1, \ldots, a_{n-1}, (c_n a_n)^2)$ is unimodular. Q.E.D.

□ **LEMMA** 2.2 Let A be a semilocal ring and let $\mathfrak{q}$ be a 2-sided ideal in A. Let $\mathfrak{a}$ be a left ideal in A. Let M be a right A-module. Let $(a, m) \in A \oplus M$ such that $m \in M\mathfrak{q}$ and $\mathfrak{a} + o_{A \oplus M}(a, m) = A$. Then there is an A-homomorphism $f : M \longrightarrow A\mathfrak{q}$ such that $\mathfrak{a} + o_A(a + f(m)) = A$. □

PROOF By definition, there is an A-homomorphism $g : M \longrightarrow A$ such that $\mathfrak{a} + o_{A \oplus A}(a, g(m)) = A$. Since $m \in M\mathfrak{q}, g(m) \in \mathfrak{q}$. It follows from (2.1) that $\mathfrak{a} + o_{A \oplus A}(a, (bg(m))^2) = A$ for some $b \in A$. By [B, III (2.8)], there is an element $c \in A$ such that $\mathfrak{a} + o_A(a + c(bg(m))^2) = A$. Define f to be the compositon of the A-homomorphisms $M \xrightarrow{g} A \xrightarrow{cbg(m)b} A$ where $cbg(m)b$ denotes left multiplication by $cbg(m)b$. Clearly, f has the desired properties. Q.E.D.

□ **LEMMA** 2.3 Let A be a module finite R-algebra. Let $\mathfrak{q}$ be a 2-sided ideal in A. Let $\mathfrak{M}$ be a finite set of maximal ideals in R. Let M be a right A-module. If $(a, m) \in A \oplus M$ is a unimodular element such that $m \in M\mathfrak{q}$ then there is an A-homomorphism $f: M \longrightarrow A\mathfrak{q}$ such that $A(a+f(m)) \supseteqq As$ for some $s \in R \setminus \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$. □

PROOF Let $\mathfrak{p} = \bigcap_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$. $R/\mathfrak{p}$ is a semilocal ring (with maximal ideals $\{\mathfrak{m}/\mathfrak{p} | \mathfrak{m} \in \mathfrak{M}\}$). Since A is module finite over R, $A/A\mathfrak{p}$ is module finite over $R/R\mathfrak{p}$ and hence semilocal. By hypothesis, there is an A-homomorphism $g : M \longrightarrow A$ such that $(a, g(m))$ is unimodular. Since $m \in M\mathfrak{q}, g(m) \in \mathfrak{q}$. By (2.1), $(a, (bg(m))^2)$ is unimodular for some $b \in A$. By [B, III (2.8)], there is an element $c \in A$ such that $a+c(bg(m))^2$ is a unit mod $A\mathfrak{p}$. Let f denote the composition of $M \xrightarrow{g} A \xrightarrow{cbg(m)b} A$ where $cbg(m)b$ denotes left multiplication by $cbg(m)b$. Then $a+f(m)$ is a unit in $A/A\mathfrak{p}$. Let S denote the multiplicative set $R \setminus \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$. Since the ideal $S^{-1}A\mathfrak{p}$ in $S^{-1}A$ is contained in the Jacobson radical($S^{-1}A$) and $S^{-1}A/S^{-1}A\mathfrak{p} = A/\mathfrak{p}$, it follows from Nakayama's lemma [B, III (2.2)] that $a+f(m)$ is a unit in $S^{-1}A$. Thus, there are elements $d \in A$ and $s \in S$ such that $s^{-1}d(a+f(m)) = 1$ in $S^{-1}A$. Thus, there is an element $t \in S$ such that the equality $td(a+f(m)) = ts$ holds in A. Q.E.D.

□ **PROPOSITION** 2.4 Let A be a module finite R-algebra such that $\dim(R) \leqq 1$. Let M be a finitely presented right A-module. Let I be a 2-sided ideal in A with the following properties.

(2.4.1) There is a subgroup G of the general linear group $GL_2(A)$, which acts transitively on the (A/I)-unimodular vectors in $A/I \oplus A/I$.

(2.4.2) there is a finite set $\mathfrak{M}$ of maximal ideals of R such that for each maximal ideal $\mathfrak{m} \notin \mathfrak{M}$, there is locally an $A_\mathfrak{m}$-homomorphism $f^\mathfrak{m} : M_\mathfrak{m} \longrightarrow A_\mathfrak{m}$ such that modulo the Jacobson radical($A_\mathfrak{m}$), the image $(f^\mathfrak{m}) \supseteq I_\mathfrak{m}$.

Let $\mathfrak{q}$ be a 2-sided ideal in A and let $G(\mathfrak{q})$ be a subgroup of the $\mathfrak{q}$-relative general linear group $GL_2(A, \mathfrak{q})$, which acts transitively on the (A/I)-unimodular vectors u of $A/I \oplus A/I$ such that $u \equiv (1,0) \mod (\mathfrak{q}+I)/I$. (If $\mathfrak{q} = A$, one can take $G(\mathfrak{q}) = G$.) If $v, w \in A \oplus A \oplus M$ are unimodular elements such that $v \equiv w \mod \mathfrak{q}$, i.e $v - w \in (A \oplus A \oplus M)\mathfrak{q}$, then there is an automorphism σ in the normal closure of $< E(A, A, M; \mathfrak{q}), G(\mathfrak{q}) >$ by $< E(A, A, M), G >$ such that $\sigma v = w$. □

PROOF The proof will be divided into two steps.

Step 1: There is an element $\rho \in < E(A, A, M), G >$ such that $\rho w = (1, 0, 0) \in A \oplus A \oplus M$.

Step 2: If $w = (1, 0, 0)$ then there is an element $\tau \in < E(A, A, M; \mathfrak{q}), G(\mathfrak{q}) >$ such that $\tau v = (1, 0, 0)$.

Assume Steps 1 and 2 have been established. The proof is then completed as follows. By Step 1, there is a ρ such that $\rho w = (1, 0, 0)$. Clearly, $\rho v \equiv \rho w$

mod $\mathfrak{q}$. Thus, according to Step 2, there is a $\tau \in < E(A, A, M; \mathfrak{q}), G(\mathfrak{q}) >$ such that $\tau\rho v = \rho w$. Clearly, $(\rho^{-1}\tau\rho)v = w$ and $\rho^{-1}\tau\rho$ is in the normal closure of $< E(A, A, M; \mathfrak{q}), G(\mathfrak{q}) >$ under $< E(A, A, M), G >$.

Step 1 is the special case of Step 2 where $\mathfrak{q} = A$. Thus, it suffices to prove Step 2.

Let $v = (1 + a, b, m)$ be a unimodular element in $A \oplus A \oplus M$ such that $a, b \in \mathfrak{q}$ and $m \in M\mathfrak{q}$. Enlarge $\mathfrak{M}$ to a finite set, denoted again by $\mathfrak{M}$, such that if m is a maximal ideal $\notin \mathfrak{M}$ and $s \in R \setminus \mathfrak{m}$ then A/As is semilocal. This can be done, since $\dim(R) \leqq 1$. By Lemma (2.3), there is an A-homomorphism $f : A \oplus M \longrightarrow A\mathfrak{q}$ such that $A(a + f(b, m)) \supseteqq As$ for some $s \in R \setminus \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$. Clearly, $\epsilon(f)v = (1 + a + f(b, m), bm)$. Thus, we can assume right from the start that $A(1 + a) \supseteqq As$ for some $s \in R \setminus \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$.

Since $(1 + a, b, m)$ is unimodular, there is an A-homomorphism $f : M \longrightarrow A$ such that $(1 + a, b, f(m)) \in A \oplus A \oplus A$ is unimodular. Applying Lemma (2.2) to the vector $(1 + a, b, f(m))$ over the semilocal ring A/As, we can find an A-homomorphism $g : A \longrightarrow A\mathfrak{q}$ such that $(1 + a, b + gf(m))$ is unimodular over A/As. But, since $A(1 + a) \supseteqq As$, it follows that $(1 + a, b + gf(m))$ is unimodular over A. Clearly, $\epsilon(gf)v = (1 + a, b + gf(m), m)$. Thus, we can assume right from the start that $v = (1 + a, b, m)$ where $(1 + a, b)$ is unimodular. By hypothesis, there is an element $\tau \in G(\mathfrak{q})$ such that $\tau \oplus 1_M(v) = (1 + a', b', m)$ where $a', b' \in I \cap \mathfrak{q}$. Thus, we can assume $v = (1 + a, b, m)$ where $a, b \in I \cap \mathfrak{q}$ and $(1 + a, b)$ is unimodular. Moreover, by applying if necessary an elementary transformation $\epsilon(f)$ to v, where $f : A \longrightarrow A(I \cap \mathfrak{q})$ has the property that $A(1 + a + f(b)) \supseteq As$ for some $s \in R \setminus \bigcup_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$, we can assume that $A(1 + a) \supseteq As$.

Let $V(Rs) = \{\mathfrak{m} | \mathfrak{m} \text{ a maximal ideal of } R, \mathfrak{m} \supseteq Rs\}$. Evidently, $V(Rs) \cap \mathfrak{M} = \varnothing$. Thus, R/Rs is semilocal and $V(Rs)$ is finite. Let $\mathfrak{m} \in V(Rs)$. Let $f^{\mathfrak{m}} : M_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ be as in the hypothesis of the proposition. Since M is finitely presented, we can apply [B, III (4.5)] to find an A-homomorphism $f^{[\mathfrak{m}]} : M \longrightarrow A$ and an element $s^{[\mathfrak{m}]} \in R \setminus \mathfrak{m}$ such that $(s^{[\mathfrak{m}]})^{-1} f^{[\mathfrak{m}]} = f^{\mathfrak{m}}$. Let $t^{[\mathfrak{m}]} \in R \setminus \mathfrak{m}$ such that $t^{[\mathfrak{m}]} \equiv (s^{[\mathfrak{m}]})^{-1} \mod (R_{\mathfrak{m}}\mathfrak{m})$. Let $g^{[\mathfrak{m}]} = t^{[\mathfrak{m}]} f^{[\mathfrak{m}]}$. Let $J^{[\mathfrak{m}]}$ denote the inverse image in A of the Jacobson radical $(A_{\mathfrak{m}}/A_{\mathfrak{m}}s)$. Each $g^{[\mathfrak{m}]}$ has the property that $\mod J^{[\mathfrak{m}]}$, image $(g^{[\mathfrak{m}]}) \supseteq I$. Let $x^{[\mathfrak{m}]} \in M$ such that $\mod J^{[\mathfrak{m}]}, g^{[\mathfrak{m}]} x^{[\mathfrak{m}]} = b - g^{[\mathfrak{m}]}(m)$. Let $r^{[\mathfrak{m}]} \in R$ such that $r^{[\mathfrak{m}]} \equiv 1 \mod \mathfrak{m}$ and $r^{[\mathfrak{m}]} \equiv 0 \mod \mathfrak{m}'$ for each $\mathfrak{m}' \neq \mathfrak{m} \in V(Rs)$. Let $x = \sum_{\mathfrak{m}\in V(Rs)} x^{[\mathfrak{m}]} r^{[\mathfrak{m}]}$. Since $(1 + a, b)$ is unimodular, we can find an A-homomorphism $h : A \oplus A \longrightarrow M$ such that $h(1 + a, b) = x$. Clearly, $\epsilon(h)(1 + a, b, m) = (1 + a, b, m + h(1 + a, b)) = (1 + a, b, m + x)$. Since $(1 + a, b)$ is unimodular and

$g^{[\mathfrak{m}]}(m+x) \equiv g^{[\mathfrak{m}]}(m)+g^{[\mathfrak{m}]}x^{[\mathfrak{m}]} \equiv g^{[\mathfrak{m}]}(m)+(b-g^{[\mathfrak{m}]}(m) = b \mod J^{[\mathfrak{m}]}$, we see that $(1+a, m+x) \in A \oplus M$ is unimodular $\mod J^{[\mathfrak{m}]}$. Thus, $(1+a, m+x)$ is unimodular over $A_{\mathfrak{m}}/A_{\mathfrak{m}}s = (A/As)_{\mathfrak{m}}$ for each $\mathfrak{m} \in V(Rs)$. Thus, by Nakayama, $(1+a, m+x)$ is unimodular over A/As. (More specifically, one can argue as follows. Choose $c, d \in A$ such that $c(1+a)+db = 1$ and let $g = \sum_{\mathfrak{m} \in V(Rs)} x^{[\mathfrak{m}]}g^{[\mathfrak{m}]}$. It suffices to show that $A(ca + dg(m+x)) \equiv A \mod As$. By the local-global principle, it suffices to show that for all maximal ideals of R/Rs, equivalently for all $\mathfrak{m} \in V(Rs), A_{\mathfrak{m}}(ca + dg(m+x)) \equiv A_{\mathfrak{m}} \mod (A_{\mathfrak{m}}s)$. By Nakayama's lemma [B, III (2.2)], it suffices to show that $A_{\mathfrak{m}}(ca+dg(m+x)) \equiv A_{\mathfrak{m}} \mod A_{\mathfrak{m}}J^{[\mathfrak{m}]}$. But $ca+dg(m+x)) \equiv ca+db = 1 \mod A_{\mathfrak{m}} \mod J^{[\mathfrak{m}]}$.) Since $A(1+a) \supseteq As$, it follows that $(1+a, m+x)$ is unimodular over A. Choose $h' : A \oplus M \longrightarrow A$ such that $h'(1+a, m+x) = 1-b$. Clearly, $\epsilon(h')\epsilon(h)(1+a, b, m) = (1+a, 1, m+x)$. Choose $h'' : A \longrightarrow A$ such that $h''(1) = -a$. If $\tau = \epsilon(h)^{-1}\epsilon(h')^{-1}\epsilon(h'')\epsilon(h')\epsilon(h)$ then $\tau(1+a, b, m) = (1, b', m')$ for suitable b' and m'. Furthermore, since image $(h'') \subseteq I \cap \mathfrak{q}, \tau \in E(A, A, , M; \mathfrak{q})$. Thus, $b' \equiv 0 \mod \mathfrak{q}$ and $m' \equiv 0 \mod M\mathfrak{q}$. Letting $h_1 : A \longrightarrow A$ such that $h_1(1) = -b'$ and $h_2 : A \longrightarrow M$ such that $h_2(1) = -m'$, we obtain that $\epsilon(h_2)\epsilon(h_1)\tau(1+a, b, m) = (1, 0, 0)$. Q.E.D.

□ **THEOREM** 2.5 Let A be a module finite R-algebra such that $\dim(R) \leqq 1$. Let M be a finitely presented right A-module. Let I be a 2-sided ideal in A satisfying (2.4.1) and (2.4.2). If M' and P are right A-modules and P is finitely generated and projective then $P \oplus A \oplus M \cong P \oplus A \oplus M'$ implies $A \oplus M \cong A \oplus M'$. □

PROOF The proof follows the pattern of that in Bass [B, IV (3.5)]. Choose Q such that $P \oplus Q \cong A^n$ for some n. If $n = 0$ then $P = 0$ and we are done. Thus, we can assume $n > 0$. It suffices now to show that $A^{n+1} \oplus M \cong A^{n+1} \oplus M'$ implies $A^n \oplus M \cong A^n \oplus M'$ for any $n > 0$. Let $v = (1, 0, \ldots, 0) \in A^{n+1} \oplus M, w = (1, 0, \ldots, 0) \in A^{n+1} \oplus M'$, and identify $A^{n+1} \oplus M$ with $A^{n+1} \oplus M'$. By Proposition (2.4), there is a transformation $\sigma \in < E(A, \ldots, A, M), G >$ such that $\sigma v = w$. σ induces an isomorphism $A \oplus A^n \oplus M/vA \xrightarrow{\cong} A \oplus A^n \oplus M'/wA$. But $A^n \oplus M \cong A \oplus A^n \oplus M/vA$ and $A^n \oplus M' \cong A \oplus A^n \oplus M'/wA$. Q.E.D.

□ **THEOREM** 2.6 Let A be a module finite R-algebra such that $\dim(R) \leq 1$ and R is Noetherian. Let M, M', and N be finitely generated right A-modules (and therefore finitely presented, because A is Noetherian). Let B denote the A-endomorphism ring $\mathrm{End}_A(N)$ of N and suppose that the canonical A-homomorphisms $\mathrm{Hom}_A(N, M) \otimes_B N \longrightarrow M, f \otimes n \mapsto f(n)$, and $\mathrm{Hom}_A(N, M') \otimes_B N \longrightarrow M', f \otimes n \mapsto f(n)$ are isomorphisms; e.g., M

and M' are direct summands of a direct sum of N's. Let $\mathfrak{J}$ be a 2-sided ideal in B such that $\mathfrak{J}$ and the right B-module $\mathrm{Hom}_A(N, M)$ satisfy conditions (2.4.1) and (2.4.2). Let Q be a right A-module which is a direct summand of a direct sum of finitely many copies of N. Then $Q \oplus N \oplus M \cong Q \oplus N \oplus M'$ implies $N \oplus M \cong N \oplus M'$. $\square$

PROOF Since N finitely generated over A and A is module finite over R with R Noetherian, it follows that B is module finite over R. Consider the functor ((right A-modules)) $\longrightarrow$ ((right B-modules)), $X \mapsto \mathrm{Hom}_A(N, X)$. Applying the functor to the isomorphism $Q \oplus N \oplus M \cong Q \oplus N \oplus M'$, we obtain an isomorphism $\mathrm{Hom}_A(N, Q) \oplus B \oplus \mathrm{Hom}_A(N, M) \cong \mathrm{Hom}_A(N, Q) \oplus B \oplus \mathrm{Hom}_A(N, M')$. Since Q is a direct summand of a direct sum of finitely many copies of N, it follows that $\mathrm{Hom}_A(N, Q)$ is finitely generated and projective over B. $\mathrm{Hom}_A(N, M)$ is finitely presented over B, since it is finitely generated already over R and B is Noetherian. Thus, we can apply Theorem (2.5). By the conclusion of that theorem, $B \oplus \mathrm{Hom}_A(N, M) \cong B \oplus \mathrm{Hom}_A(N, M')$. Applying the functor $- \otimes_B N$ to the isomorphism above, we obtain an isomorphism $N \oplus \mathrm{Hom}_A(N, M) \otimes_B N \cong N \oplus \mathrm{Hom}_A(N, M') \otimes_B N$. But by hypothesis, $\mathrm{Hom}_A(N, M) \otimes_B N \cong M$ and $\mathrm{Hom}_A(N, M') \otimes_B N \cong M'$. Q.E.D.

REMARK 2.7 One can replace in (2.4) and (2.5) (resp. (2.6)) the hypothesis that M is finitely presented by the weaker hypothesis that M contains a direct summand M_0 such that M_0 is finitely presented and the ideal I (resp. $\mathfrak{J}$) in A (resp. B) satisfies (2.4.2) with respect to the submodule M_0 (rersp. $\mathrm{Hom}_A(N, M_0)$). The details are a little tedious, but not difficult. We shall skip them.

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