CYCLIC HOMOLOGY OF SMOOTH ALGEBRAS CONTAINING $\mathbb{Z}$

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ABSTRACT. In this paper is studied the spectral sequence of Connes' double complex of smooth algebra. It is shown that the above-mentioned spectral sequence vanishes in second degree, when the ground ring contains ring of integer

INTRODUCTION

$\mathbb{Z}/p^k\mathbb{Z}$ Let A be a smooth algebra over a (commutative) ring R with $\mathbb{Z} \subseteq R$. This paper is an investigation of the first spectral sequence of the bicomplex $\mathcal{B}(A)$. In characteristic zero, as it is known, the aforementioned spectral sequence degenerates at stage 2 (see [1] or [2]):

$$E^{\infty}_{pq} = E^2_{pq} = \left\{ \begin{matrix} \Omega^q_A/d\Omega^{q-1}_A & , \text{when } p = 0 \\ H^{q-p}_{\mathrm{DR}}(A) & , \text{when } p > 0 \end{matrix} \right\} \Rightarrow CH_{p+q}(A).$$

In the first section it will be shown that if R is an integral domain and A is projective as an R-module then the aforementioned spectral sequence also collapses at stage 2. Whereas in §2 we shall show that the same holds when $R = \mathbb{Z}$ and A is flat over $\mathbb{Z}$.

§1. CYCLIC HOMOLOGY OF PROJECTIVE SMOOTH ALGEBRAS

From the theory of spectral sequences (see [4]) it is well known the construction of spectral sequence induced by filtration. Let $\mathcal{B}$ is double complex and F is filtration of $\operatorname{tot}\mathcal{B}$:

$$F_p(\operatorname{tot}\mathcal{B})_n = \oplus_{\substack{s+t=n \\ s\leq p}} \mathcal{B}_{s,t} \ .$$

The spectral sequence which is induced by this filtration we call first spectral sequence of $\mathcal{B}$.

Let R be an integral domain and let A be a smooth algebra over R. Everywhere in the sequel it will be supposed that R contains the ring of integers. In this section $\otimes$ will mean $\otimes_R$, the R-module A/R will be denoted by $\bar{A}$, and $a_1 \otimes a_2 \otimes ... \otimes a_n$ will be

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written as $(a_1, a_2, ..., a_n)$. It will be assumed that $\bar{A}$ is projective over R ($\bar{A} \in \mathcal{B}(R)$). This requirement is almost the same as $A \in \mathcal{B}(R)$.

Let denote by $\mathcal{B}(A)$ the following double chain complex:

$$\begin{array}{ccccc}
\downarrow b & & \downarrow b & & \downarrow b \\
A \otimes \overline{A}^{\otimes 2} & \overset{B}{\leftarrow} & A \otimes \overline{A} & \overset{B}{\leftarrow} & A \\
\downarrow b & & \downarrow b & & \\
A \otimes \overline{A} & \overset{B}{\leftarrow} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array},$$

where b and B are given by the formulas:

$$b(a_0, a_1, \cdots, a_n) = \sum_{i=0}^{n-1} (-1)^n (a_0, \cdots, a_i, a_{i+1}, \cdots, a_n) + (-1)^n (a_n a_0, a_1, \cdots, a_{n-1})$$

$$B(a_0, a_1, \cdots, a_n) = (1, a_0, \cdots, a_n) + \sum_{i=1}^{n} (-1)^{ni} (1, a_{n-i+1}, a_{n-i+2}, \cdots, a_n, a_0, a_1, \cdots, a_{n-i})$$

Let us fix p and q in $\mathbb{Z}^+$, $p \leq q$, and suppose $y_0 \in A \otimes \bar{A}^{q-p}$ is some cycle of stage m ($m \geq 2$) in the first spectral sequence $E = \{E^e, d^e\}$ of the bicomplex $\mathcal{B}(A)$. This means that there is a system $y_1, y_2, ..., y_{m-1}$ of elements ($y_j \in A \otimes \bar{A}^{q-p+2j}$) such that

$$b(y_0) = 0; \quad B(y_j) = -b(y_{j+1}) \ (0 \leq j < m-1) \tag{1.1}$$

where b (resp. B) is the Hochschild (resp. Connes) differential in the normalized complex.

THEOREM 1.1. *If A is smooth over R and $\bar{A} \in \mathcal{B}(R)$, then $B(y_{m-1}) \in B(\ker b_{n-1}) + \operatorname{im} b_{n+1}$, where $n = q - p + 2m - 1$.*

Rest of this section is entirely devoted to the proof of Theorem 1.1. This theorem signifies that $B(y_{m-1})$ is a stage 2 boundary in E. In this way one obtains that $d^m_{pq} : E^m_{pq} \to E^m_{p-m\ q+m-1}$ is trivial for $m \geq 2$, i. e. $E^2 = E^3 = ... = E^n = ...$

We now state without proof some theorems which can be seen in [1] or [2].

THEOREM 1.2 (Hochschild-Kostant-Rosenberg). *If A is a smooth algebra, then there are isomorphisms*

$$\gamma : \Omega^n_A \overset{\cong}{\to} HH_n(A)$$

where $\gamma(a_0 da_1 ... da_n)$ equals the class of $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(a_0, a_{\sigma(1)}, ..., a_{\sigma(n)})$.

THEOREM 1.3. *If A is a commutative algebra with unit, then there is a commutative diagram*

$$\begin{array}{ccc} \Omega^n_A & \xrightarrow{\gamma} & HH_n(A) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle B} \\ \Omega^{n+1}_A & \xrightarrow{\gamma} & HH_{n+1}A \end{array}$$

where $d(a_0da_1...da_n) = da_0da_1...da_n$.

Let X be some R-module. Define the homomorphism $\varkappa^n_X : X^{\wedge n} \to X^{\otimes n}$ (where $\bigwedge$ is exterior product w. r. t. R) by

$$\varkappa^n_X(x_1 \wedge ... \wedge x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(x_{\sigma(1)}, ..., x_{\sigma(n)}).$$

Let us introduce the symbol $t_R(\ )$ (not to be confused with the cyclic operator): $t_R(X)$ denotes the torsion submodule of X w. r. t. R.

LEMMA 1.4. *If X is a projective R-module ($X \in \mathcal{B}(R)$) or X is a directed limit of free R-modules ($X = \varinjlim_\alpha F_\alpha$, $F_\alpha \in \mathcal{B}(R)$), then $t_R(X^{\otimes n}/\operatorname{im}\varkappa^n_X) = 0$.*

Proof. If $F \in \mathcal{B}(R)$, then it can be shown in a standard way that $F^{\otimes n}/\operatorname{im}\varkappa^n_F \in \mathcal{B}(R)$. This fact elementarily implies our lemma.

LEMMA 1.5. *There is a commutative diagram*

$$\begin{array}{ccc} A \otimes \bar{A}^{\wedge n} & \xrightarrow{A\otimes\varkappa^n_{\bar{A}}} & A \otimes \bar{A}^n \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle B} \\ A \otimes \bar{A}^{\wedge(n+1)} & \xrightarrow{A\otimes\varkappa^{n+1}_{\bar{A}}} & A \otimes \bar{A}^{n+1} \end{array}$$

where A is an R-algebra with unit and $S(a_0 \otimes a_1 \wedge ... \wedge a_n) = 1 \otimes a_0 \wedge a_1 \wedge ... \wedge a_n$.

Proof will be skipped as it is completely similar to the argument required for Theorem 1.3 (see Proposition 2.2 in [2]).

In the sequel it will be convenient to denote $\operatorname{im}(A \otimes \varkappa^n_{\bar{A}})$ by G_n.

LEMMA 1.6. *Suppose $\bar{A} \in \mathcal{B}(R)$ or $\bar{A} = \varinjlim_\alpha F_\alpha$ ($F_\alpha \in \mathcal{B}(R)$, with $\{\alpha\}$ a directed system). Then $G_n \cap \ker B \subseteq B(\ker b_{n-1}) + \operatorname{im} b_{n+1}$ ($n \geq 2$).*

Proof. Conditions of the lemma ensure injectivity of the maps $f_n = A \otimes \varkappa^n_{\bar{A}}$ and f_{n+1}. Hence by Lemma 1.5 one has $G_n \cap \ker B = f_n(\ker S)$. The map S is a composite of the following two natural maps: $A \otimes \bar{A}^{\wedge n} \to \bar{A}^{\wedge(n+1)} \xrightarrow{1\otimes} A \otimes \bar{A}^{\wedge(n+1)}$. Since the second one is injective, $\ker S$ is generated (as an R-module) by the elements of the form $x_0 = a_0 \otimes a_1 \wedge a_2 \wedge ... \wedge a_n + a_1 \otimes a_0 \wedge a_2 \wedge ... \wedge a_n$ and $x_1 = a \otimes a \wedge a_2 \wedge ... \wedge a_n$. Let $\tau_n : A \otimes \bar{A}^{\wedge n} \to \Omega^n_A$ be the natural projection. Then $\tau_n(x_0) = a_0 da_1 da_2 ... da_n + a_1 da_0 da_2 ... da_n = d(a_0 a_1) da_2 ... da_n \in d\Omega^{n-1}_A$ and when $n \geq 2$, $\tau_n(x_1) = adada_2...da_n = dad(a_2 a)da_3...da_n - a_2 dadada_3...da_n \in d\Omega^{n-1}_A$, hence by Theorem 1.3 $\gamma\tau_n(x_0), \gamma\tau_n(x_1) \in B(\ker b_{n-1}) + \operatorname{im} b_{n+1}$, which implies $f_n(x_0), f_n(x_1) \in B(\ker b_{n-1}) + \operatorname{im} b_{n+1}$. This proves the Lemma.

At the beginning of this section we have fixed $p, q \in \mathbb{Z}^+$ $(p \leq q)$ and a system of elements $y_0, y_1, ..., y_{m-1}$ $(m \geq 2)$ satisfying (1.1). Let $n_0 = q - p + 2m - 1$. Construct a bicomplex $\mathcal{B}_R = \mathcal{B}_R(n_0, y_{m-1})$ as follows.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 & \ker Bb_{n_0+1}/C_{n_0} \arrow[l] \arrow[d, "I_2"'] & 0 \arrow[l] \arrow[d] & & \\
0 & A \otimes \bar{A}^{n_0+1}/D_{n_0} \arrow[l] \arrow[d, "-Bb"'] & A \otimes \bar{A}^{n_0+1}/D_{n_0} \arrow[l, "I_1"'] \arrow[d, "b"] & 0 \arrow[l] \arrow[d] & \\
0 & A \otimes \bar{A}^{n_0+1}/G_{n_0+1} \arrow[l] \arrow[d] & A \otimes \bar{A}^{n_0}/G_{n_0} \arrow[l, "B"'] \arrow[d] & L(y_{m-1}) \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where $L(y_{m-1})$ is the R-module generated by y_{m-1}, G_{n_0} is as defined before, $D_{n_0} = \{x \in A \otimes \bar{A}^{n_0+1} \mid b(x) \in G_{n_0}\}$, whereas $C_{n_0} = D_{n_0} \cap \ker Bb_{n_0+1}$. As for differentials: I_1 is the identity, and I_2 is induced by the identity. For the grading, it suffices to point out that $B(G_{n_0}) \subseteq G_{n_0+1}$. So $\mathcal{B}_R$ is rigorously determined.

LEMMA 1.7. *Suppose $\bar{A} \in \mathcal{B}(R)$ or $\bar{A} = \varinjlim_\alpha F_\alpha$, where $F_\alpha \in \mathcal{B}(R)$ and $\{\alpha\}$ is a directed system. Then entries of the bicomplex $\mathcal{B}_R$ are torsion free.*

Proof. By definition $G_n = A \otimes \operatorname{im} \varkappa^n_{\bar{A}}$. Hence $A \otimes \bar{A}^n/G_n \cong A \otimes (\bar{A}^n/\operatorname{im} \varkappa^n_{\bar{A}}$. In our situation one can apply Lemma 1.4, which gives, taking into account the above isomorphism, that $t_R(A \otimes \bar{A}^n/G_n) = 0$. The rest of the Lemma is clear as by construction $A \otimes \bar{A}^{n_0+1}/D_{n_0}$ and $\ker Bb_{n_0+1}/C_{n_0}$ embeds in $A \otimes \bar{A}^{n_0}/G_{n_0}$. So Lemma 1.7 is proved.

Denote the field of fractions of the ring R by Q. Denote by $\mathcal{B}_Q$ the bicomplex obtained by tensoring zeroth and first column of $\mathcal{B}_R$ by Q.

LEMMA 1.8. *If conditions of Lemma 1.7 are satisfied and A is smooth over R, then the natural map $\mathcal{B}_R \to \mathcal{B}_Q$ induces an isomorphism*

$$H_2(\operatorname{tot}\mathcal{B}_R) \overset{\cong}{\to} H_2(\operatorname{tot}\mathcal{B}_Q).$$

Proof. By Lemma 1.7 there is a short exact sequence of bicomplexes

$$0 \to \mathcal{B}_R \to \mathcal{B}_Q \to \bar{\mathcal{B}}_Q \to 0.$$

Let us write down the corresponding long exact sequence in homology

$$0 \to H_2(\operatorname{tot}\mathcal{B}_R) \to H_2(\operatorname{tot}\mathcal{B}_Q) \to H_2(\operatorname{tot}\bar{\mathcal{B}}_Q) \to \cdots . \tag{1.2}$$

Let us show that $H_2(\operatorname{tot}\bar{\mathcal{B}}_Q) = 0$. Therefore introduce the complexes

$$\begin{array}{lcccccl} & & 0 & & 1 & & 2 \\ \mathcal{B}_0 : & 0 \leftarrow & \operatorname{im} Bb_{n_0+1}/Bb_{n_0+1}(D_{n_0}) & \overset{Bb}{\longleftarrow} & A \otimes \bar{A}^{n_0+1}/D_{n_0} & \leftarrow & \ker Bb_{n_0+1}/C_{n_0} \leftarrow 0 \\ \mathcal{B}_1 : & 0 \leftarrow & \operatorname{im} b_{n_0} & \overset{b}{\longleftarrow} & A \otimes \bar{A}^{n_0}/G_{n_0} & \overset{b}{\longleftarrow} & A \otimes \bar{A}^{n_0+1}/D_{n_0} \leftarrow 0 \end{array}$$

Let $\bar{E} = \{\bar{E}^e, \bar{d}^e\}$ be the first spectral sequence of the bicomplex $\bar{\mathcal{B}}_Q$. It is easy to see that

$$\bar{E}^1_{02} = H_2(\bar{Q} \otimes \mathcal{B}_0);\ \bar{E}^1_{11} = H_2(\bar{Q} \otimes \mathcal{B}_1);\ \bar{E}^1_{ij} = 0\ \ (i+j=2, (i,j) \neq (0,2) \text{ or } (1,1))\,. \tag{1.3}$$

By construction $H_*(\mathcal{B}_0) = 0$. Moreover since A is smooth over R, by Theorem 1.2 one has $\ker b_{n_0} = G_{n_0} + \operatorname{im} b_{n_0}$ whence $H_*(\mathcal{B}_1) = 0$. Consequently, since $\mathcal{B}_0 \to Q \otimes \mathcal{B}_0$ and $\mathcal{B}_1 \to Q \otimes \mathcal{B}_1$ are injections (this by Lemma 1.7), one will have $H_*(\bar{Q} \otimes \mathcal{B}_0) = 0$ and $H_*(\bar{Q} \otimes \mathcal{B}_1) = 0$. By last two equalities and by equalities (1.3) one obtains that $\bar{E}^1_{ij} = 0$ for $i + j = 2$. This justifies that $H_2(\operatorname{tot}\bar{\mathcal{B}}_Q) = 0$, which by (1.2) implies that $H_2(\operatorname{tot}\mathcal{B}_R) \to H_2(\operatorname{tot}\mathcal{B}_Q)$ is an isomorphism. Lemma 1.8 is proved.

LEMMA 1.9. *Suppose that A satisfies conditions of Lemma 1.7 and A is smooth over R. Then there is an isomorphism*

$$H_2(\operatorname{tot}\mathcal{B}_R) \overset{\cong}{\to} L(y_{m-1}). \tag{1.4}$$

Proof. Let $E_R = \{(E_R)^e, d^e_R\}_{e\geq 1}$ (respectively E_Q) be the first spectral sequence of $\mathcal{B}_R$ (resp. $\mathcal{B}_Q$). Since $t_R(\mathcal{B}_R) = 0$, by construction of $\mathcal{B}_R$ one has $(E_R)^1_{02} = (E_Q)^1_{02} = (E_R)^1_{11} = (E_Q)^1_{11} = 0$. Hence by the previous lemma one has $(E_R)^\infty_{20} \cong H_2(\operatorname{tot}\mathcal{B}_R) \cong H_2(\operatorname{tot}\mathcal{B}_Q) \cong (E_Q)^\infty_{20}$. So it suffices to show

$$(E_Q)^\infty_{20} = L(y_{m-1}). \tag{1.5}$$

Since A is smooth over R, also $Q \otimes A$ is smooth over Q. By (1.1) one may consider $B(y_{m-1})$ as an m-stage boundary in the first spectral sequence of the bicomplex

$\mathcal{B}(Q \otimes A)$. This spectral sequence degenerates on the second stage, since $\mathbb{Q} \subseteq Q$. Since $m \geq 2$, by the above argument one has

$$B(y_{m-1}) \in Q \otimes B(\ker b_{n_0-1}) + Q \otimes \operatorname{im} b_{n_0+1} = Q \otimes B(G_{n_0-1}) + Q \otimes \operatorname{im} b_{n_0+1}. \quad (1.6)$$

The latter equality holds by Theorem 1.2. By (1.6) there exist $x_{n_0-1} \in Q \otimes G_{n_0-1}$ and $x_{n_0+1} \in Q \otimes A \otimes \bar{A}^{n_0+1}$ such that

$$B(y_{m-1}) = B(x_{n_0-1}) + b(x_{n_0+1}). \quad (1.7)$$

By (1.7) one has

$$Bb(x_{n_0+1}) = 0 \iff x_{n_0+1} \in Q \otimes \ker Bb_{n_0+1} \quad (1.8)$$

Moreover by Lemma 1.5 ome has $B(x_{n_0-1}) \in Q \otimes G_{n_0}$. This together with the equality (1.7) and (1.8) shows that $(x_{n_0+1}, -x_{n_0+1}, y_{m-1})$ is a cycle of $\operatorname{tot} \mathcal{B}_Q$. This means that $y_{m-1} \in (E_Q)^{\infty}_{20}$. This justifies (1.5) and proves Lemma 1.9.

LEMMA 1.10. *Suppose that A is a smooth R-algebra and conditions of Lemma 1.7 hold for A. Then one has*

$$B(L(y_{m-1})) \subset B(\ker b_{n_0-1}) + \operatorname{im} b_{n_0+1}.$$

Proof. By the previous lemma there exist $x_m \in A \otimes \bar{A}^{n_0+1}$ and $x_{m+1} \in \ker Bb_{n_0+1}$ such that

$$x_{m+1} + x_m \in D_{n_0} \quad (1.9)$$

and

$$b(x_m) + B(y_{m-1}) = z_{n_0} \in G_{n_0} \quad (1.10).$$

Now (1.9) $\Rightarrow Bb(x_m) \in Bb(D_{n_0}) = B(G_{n_0} \cap \operatorname{im} b_{n_0+1}) \Rightarrow Bb(x_m) = B(\tilde{z}_{n_0})$, where $\tilde{z}_{n_0} \in G_{n_0} \cap \operatorname{im} b_{n_0+1}$, whereas (1.10) $\Rightarrow Bb(x_m) = B(z_{n_0})$. Hence (1.9) and (1.10) give that

$$(z_{n_0} - \tilde{z}_{n_0}) \in \ker B \cap G_{n_0}, \text{ where } \tilde{z}_{n_0} \in G_{n_0} \cap \operatorname{im} b_{n_0+1}. \quad (1.11)$$

By Lemma 1.6 (which can be applied here), $(z_{n_0} - \tilde{z}_{n_0}) \in B(\ker b_{n_0-1}) + \operatorname{im} b_{n_0+1} \overset{(1.11)}{\Rightarrow} z_{n_0} \in B(\ker b_{n_0-1}) + \operatorname{im} b_{n_0+1} \overset{(1.10)}{\Rightarrow} B(y_{m-1}) \in B(\ker b_{n_0-1}) + \operatorname{im} b_{n_0+1}$. Hence Lemma 1.10 is proved.

Theorem 1.1 is a particular case of this lemma. So we have proved that if A is snooth over R and $\bar{A} \in \mathcal{B}(R)$, then the first spectral sequence of $\mathcal{B}(A)$ degenerates at stage 2. This concludes the first section.

§2. Smooth algebras over $\mathbb{Z}$

Suppose A is a flat algebra over $\mathbb{Z}$ (with unit). If $t_{\mathbb{Z}}(A/\mathbb{Z}) \neq 0$, then consider the following subring of A: $K = \{a \in A : \exists m \in \mathbb{Z} \text{ such that } m \neq 0 \text{ and } ma \in \mathbb{Z}\} = \mathbb{Q} \cap A$. One sees easily that $t_K(A/K) = 0$. Suppose $\mathcal{B}_{\mathbb{Z}}(A)$ (resp. $\mathcal{B}_K(A)$) is the bicomplex giving cyclic homology of A w. r. t. $\mathbb{Z}$ (resp. K). Denote the first spectral sequence of $\mathcal{B}_{\mathbb{Z}}(A)$ (resp. $\mathcal{B}_K(A)$) by $E_{\mathbb{Z}} = \{(E_{\mathbb{Z}})^r_{pq}\}_{r\geq 1}$ (resp. $E_K = \{(E_K)^r_{pq}\}_{r\geq 1}$).

LEMMA 2.1. *If A is smooth over $\mathbb{Z}$, then A will be also smooth over K. Moreover the equality*

$$(E_{\mathbb{Z}})^r_{pq} = (E_K)^r_{pq} \ (r \geq 1,\ p, q \in \mathbb{Z})$$

holds.

Proof. It is easy to check that the natural map $\Omega^1_{A/\mathbb{Z}} \to \Omega^1_{A/K}$ is an isomorphism of A-modules. Then by definition $\Omega^1_{A/\mathbb{Z}} \in \mathcal{B}(A) \Rightarrow \Omega^1_{A/K} \in \mathcal{B}(A)$, i. e. A is smooth over K. Hence by Theorem 1.2 one may write

$$(E_{\mathbb{Z}})^1_{pq} \cong \Omega^{q-p}_{A/\mathbb{Z}} \cong \Omega^{q-p}_{A/K} \cong (E_K)^1_{pq}. \tag{2.1}$$

Let $I : \mathcal{B}_{\mathbb{Z}}(A) \to \mathcal{B}_K(A)$ be the natural map of complexes. According to (2.1), I induces an isomorphism $(E_{\mathbb{Z}})^1 \xrightarrow{\cong} (E_K)^1$. Hence by the general theory of spectral sequences ([4]), $(E_{\mathbb{Z}})^r \xrightarrow{\cong} (E_K)^r$ $(r \geq 1)$. This proves Lemma 2.1.

THEOREM 2.2. *If A is smooth over $\mathbb{Z}$ and $t_{\mathbb{Z}}(A) = 0$, then*

$$(E_{\mathbb{Z}})^2 = (E_{\mathbb{Z}})^3 = \cdots = (E_{\mathbb{Z}})^r = \cdots.$$

Proof. By Lemma 2.1 it suffices to prove that the first spectral sequence of $\mathcal{B}_K(A)$ collapses at the second stage. To this end, it suffices to show that the algebra A over K satisfies conditions of lemma 1.7 (see Lemma 1.10). By construction

$$t_{\mathbb{Z}}(A/K) = 0 \Rightarrow A/K \overset{\mathbb{Z}-\mathrm{mod}}{\cong} \varinjlim{}_{\alpha} F_{\alpha} \tag{2.2}$$

where $\{\alpha\}$ is a directed system and $F_{\alpha} \in \mathcal{B}(\mathbb{Z})$. Then $K \otimes F_{\alpha} \in \mathcal{B}(K)$ since $K \subset \mathbb{Q}$. Moreover by (2.2) one has $\varinjlim{}_{\alpha} K \otimes F_{\alpha} \overset{\mathcal{B}-\mathrm{mod}}{\cong} K \otimes A/K \overset{\mathcal{B}-\mathrm{mod}}{\cong} A/K$. The latter isomorphism exists by virtue of the equality $t_K(A/K) = 0$. Hence conditions of Lemma 1.7 are satisfied and Theorem 2.2 is proved.

References

1. J.-L. Loday, *Cyclic Homology*, Springer Verlag, 1993.
2. J.-L. Loday and D.Quillen, *Ciclic homology and the Lie algebra hology of matrices*, Comment. Math. Helvetici 54 (1984) 565-591.
3. G.Hochsch, B.Cotant and A.Rosenberg, *Differentiol forms on regular affine algebras*, Trans.A.M.S. 102 (1962), 383-408.
4. H.Cartan and S.Eilenberg, *Homological Algebra*, Princeton University Press, 1956.