DIMENSION THEORY AND NONSTABLE K-THEORY FOR NET GROUPS

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Abstract. The article applies concepts of structure and dimension in arbitrary categories to establish that nonstable net K_1 of a net of finite Bass-Serre dimension is a nilpotent by Abelian group.

Introduction

This article together with the articles Mundkur [Mu] and Hazrat [H] apply concepts of structure and dimension in arbitrary categories to prove structural results for classical-like groups defined over rings and related objects. The current paper provides applications to net groups associated to the general linear group, the paper [Mu] to the general linear group, and the paper [H] to the general quadratic group. Each article including the current one has a short, self-contained introduction to a different aspect of the general theory of structure and dimension in categories and of group valued functors on categories with structure and dimension, which is developed in Bak [Bk4] and Bak [Bk5]. The current articles are intended to illustrate this theory. The main application in the present article is to prove that nonstable K_1 of a major net of rank $\geqslant 4$ and finite Bass-Serre dimension is a nilpotent by Abelian group.

We describe the general theory as it pertains to the current paper. An arbitrary category $\mathcal{C}$ is structured by fixing a class of commutative diagrams called structure diagrams and a class of functors called infrastructure functors on directed quasi-ordered sets with values in $\mathcal{C}$, whose direct limits exist in $\mathcal{C}$. In the current article

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the diagrams are commutative squares. A function $d : \mathcal{O}bj(\mathcal{C}) \to \mathbb{Z}^{\geqslant 0} \cup \{\infty\}$ is called a dimension function, if it satisfies a certain property called reduction relating it to the structure on $\mathcal{C}$. Let $\tau : \mathcal{S} \to \mathcal{G}$ denote a natural transformation of group valued functors $\mathcal{S}$ and $\mathcal{G}$ on $\mathcal{C}$. The dimension filtration of τ on $\mathcal{G}$ is defined by

$$\mathcal{G}^i(A) = \bigcap_{\substack{A \to B \\ d(B) \leqslant i}} \operatorname{Ker}\Big(\mathcal{G}(A) \to \mathcal{G}(B)/\operatorname{Im}\big(\mathcal{S}(B) \to \mathcal{G}(B)\big)\Big)$$

for any integer $i \geqslant 0$ and object A of $\mathcal{C}$. Let $\mathcal{E} = \operatorname{Im}(\tau)$. The main result used in the current paper for group valued functors on a category with structure and dimension is the following: If τ is good with respect to the structure on $\mathcal{C}$ then $\mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots \geqslant \mathcal{E}$ is a descending central series such that $\mathcal{G}^i(A) = \mathcal{E}(A)$ for all $i \geqslant d(A)$. Moreover, if the coset space $\mathcal{G}(A)/\mathcal{E}(A)$ is an Abelian group for all 0-dimensional objects A then the quotient functor $\mathcal{G}/\mathcal{G}^0$ takes its values in Abelian groups. In this case, the quotient functor $\mathcal{G}/\mathcal{E}$ is a nilpotent by Abelian group on all finite dimensional objects of $\mathcal{C}$.

Let ((ν-nets)) denote the category of all nets corresponding to an equivalence relation ν on a subset of the natural numbers. Let ((major ν-nets)) denote the full subcategory of all major ν-nets. Let $\mathrm{G} : ((\nu - \text{nets})) \to ((\text{groups}))$, $\sigma \mapsto \mathrm{G}(\sigma)$, denote the usual group valued functor associating to each ν-net σ, its group $\mathrm{G}(\sigma)$. Let $\mathrm{E}(\nu, \sigma)$ denote the elementary subgroup of $\mathrm{G}(\sigma)$ and $\mathrm{St}(\nu, \sigma)$ the Steinberg group of σ. The main results concerning ν-nets are obtained by exhibiting ((ν-nets)) as a category with structure and dimension whose dimension function is the Bass-Serre dimension of a net. It is then shown that the natural transformation $\pi : \mathrm{St}(\nu, \_\,) \to \mathrm{G}(\_\,)$ is good on the category ((major ν-nets)) and that for any 0-dimensional ν-net σ, $\mathrm{G}(\sigma)/\mathrm{E}(\nu, \sigma)$ is an Abelian group. It follows now from the general theory above that the dimension filtration $\mathrm{G} \geqslant \mathrm{G}^0 \geqslant \mathrm{G}^1 \geqslant \cdots \geqslant \mathrm{E}(\nu, \_\,)$ on G makes the quotient functor $K_1 G := \mathrm{G}/\mathrm{E}(\nu, \_\,)$ into a nilpotent by Abelian group valued functor on finite dimensional major ν-nets. This allows one to deduce that each sandwich $\mathrm{E}(\nu, \sigma) \leqslant H \leqslant \mathrm{C}(\nu, \sigma)$ in the sandwich classification theorem for subgroups H of the general linear group normalized by a block diagonal elementary subgroup, has the property that the quotient $\mathrm{C}(\nu, \sigma)/\mathrm{E}(\nu, \sigma)$ is a $G^0([\nu] + \sigma)$-nilpotent group for any finite dimensional ν-net σ.

The rest of the paper is organized as follows. In section 1, a certain group homomorphism χ is constructed, which plays a crucial role in section 2 in showing that the filtration $\mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots$ is a descending central series. When χ is bijective, it turns out that its inverse is the connecting map in a certain nonabelian Mayer-Vietoris sequence. This result is included for completeness, but is not necessary for proving the main results. Section 2 develops notions of structure and dimension in arbitrary categories and proves the main results for dimension filtrations on group valued functors. Section 3 recalls nets and net groups and establishes stable properties of these groups. Section 4 proves the injective stability theorem for the functor K_1 of nets. The surjective stability theorem is known by work of Vavilov [Vv]. Section 5 studies nets over quasi-finite rings. In particular, it shows that elementary subgroups of net groups are normal and that the so-called standard commutator formula holds. Golubchik's sandwich classification theorem is recalled. Section 6 exhibits the categories ((ν-nets)) and ((major ν-nets)) as categories with structure and dimension and shows that the natural transformation $\pi : \mathrm{St}(\nu, \_\,) \to \mathrm{G}(\_\,)$ is good on the category ((major ν-nets)). A nonstable K_2–K_1 Mayer-Vietoris sequence for nets is also proved. Section 7 puts together the results of sections 2, 5, and 6 to prove our main results concerning net groups.

1. The map χ and a nonabelian Mayer-Vietoris sequence

The materials in this section are taken in a selfcontained way from Bak [Bk2] and [Bk3] (unpublished). We begin by sharpening the Mayer-Vietoris sequences in[Bk2], following the exposition given in [Bk3]. In the second half of the section, we investigate the equivariant properties of the connecting map in the Mayer-Vietoris sequence.

1.1. Let $\mathcal{R}$ be a category and let $\mathcal{G}, \mathcal{E}, \mathcal{S}$ be functors from $\mathcal{R}$ to ((groups)). Suppose that for any object $A \in \mathcal{R}$, these functors satisfy the following conditions.

(i) $\mathcal{E}(A)$ is a normal subgroup of $\mathcal{G}(A)$ and the inclusion map defines a natural transformation of functors.

(ii) $\mathcal{S}(A)$ is a central extension of $\mathcal{E}(A)$ and the covering $\mathcal{S}(A) \xrightarrow{\pi} \mathcal{E}(A)$ defines a natural transformation of functors.

1.2. For each object A in $\mathcal{R}$, define the groups

$$\mathrm{K}_1(A) = \mathcal{G}(A)/\mathcal{E}(A) \quad \text{and} \quad \mathrm{K}_2(A) = \mathrm{Ker}(\mathcal{S}(A) \to \mathcal{E}(A)).$$

Clearly K_1 and K_2 are functors from $\mathcal{R}$ to ((groups)).

For a commutative square

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ C & \xrightarrow{\varphi'} & D \end{array}$$

in $\mathcal{R}$ consider the corresponding squares of groups

$$(\mathcal{G}-square) \quad \begin{array}{ccc} \mathcal{G}(A) & \xrightarrow{\varphi} & \mathcal{G}(B) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \mathcal{G}(C) & \xrightarrow{\varphi'} & \mathcal{G}(D) \end{array} \qquad (\mathcal{S}-square) \quad \begin{array}{ccc} \mathcal{S}(A) & \xrightarrow{\varphi} & \mathcal{S}(B) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ \mathcal{S}(C) & \xrightarrow{\varphi'} & \mathcal{S}(D). \end{array}$$

Factoring $\mathrm{K}_2(A)$, $\mathrm{K}_2(B)$ and $\mathrm{K}_2(C)$ out of the right hand square, one obtains the commutative square

$$(factor\text{-}square) \qquad \begin{array}{ccc} \mathcal{E}(A) & \xrightarrow{\varphi} & \mathcal{E}(B) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathcal{E}(C) & \xrightarrow{\omega} & \mathcal{S}(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C)) \end{array} .$$

Note that we denote functorial group homomorphisms in the diagrams above with the same letters used for the corresponding morphisms in the category $\mathcal{R}$, e.g. we write φ instead of $\mathcal{G}(\varphi)$, $\mathcal{S}(\varphi)$ and $\mathcal{E}(\varphi)$, but we introduce into the factor-square the new letters ω and ρ to denote the group homomorphisms induced from the functorial ones on the $\mathcal{S}$-level. We shall use such notation when it cannot cause confusion.

1.3. A commutative diagram

$$\begin{tikzcd}
\mathrm{K}_2(B) \arrow[dr] & & & \mathrm{K}_1(B) \arrow[dr] & \\
& \mathrm{K}_2(D) \arrow[r, "\partial"] & \mathrm{K}_1(A) \arrow[ur] \arrow[dr] & & \mathrm{K}_1(D) \\
\mathrm{K}_2(C) \arrow[ur] & & & \mathrm{K}_1(C) \arrow[ur] &
\end{tikzcd}$$

will be called a **Mayer–Vietoris sequence** if the following conditions are satisfied.

(i) The square is exact, i.e. the canonical homomorphism from $\mathrm{K}_1(A)$ to the pullback of the diagram $\mathrm{K}_1(C) \to \mathrm{K}_1(D) \leftarrow \mathrm{K}_1(B)$ is surjective.

(ii) The intersection of the kernels of the maps $\mathrm{K}_1(A) \to \mathrm{K}_1(B)$ and $\mathrm{K}_1(A) \to \mathrm{K}_1(C)$ coincides with $\mathrm{Im}(\partial)$.

(iii) The set $\{bc \mid b \in \mathrm{Im}(\mathrm{K}_2(B) \to \mathrm{K}_2(D)), c \in \mathrm{Im}(\mathrm{K}_2(C) \to \mathrm{K}_2(D))\}$ formed by the product of the images of $\mathrm{K}_2(B)$ and $\mathrm{K}_2(C)$ in $\mathrm{K}_2(D)$ coincides with $\mathrm{Ker}(\partial)$.

The purpose of this section is to construct for certain commutative squares $(*)$ in $\mathcal{R}$, a Mayer–Vietoris sequence which is functorial over these squares.

1.4. Following [Bk2] we define the kinds of squares that will be needed. The square $(*)$ will be called:

(i) **weak $\mathcal{E}$-fibred**, if *factor-square* is a fibre square.

(ii) **$\mathcal{E}$-surjective**, if given $b \in \mathcal{G}(B)$ and $c \in \mathcal{G}(C)$ such that $\psi'(b)\varphi'(c) \in \mathcal{E}(D)$, there are elements $b' \in \mathcal{E}(B)$ and $c' \in \mathcal{E}(C)$ such that

$$\psi'(b')\varphi'(c') = \psi'(b)\varphi'(c).$$

(iii) **$\mathcal{S}$-surjective**, if given $d \in \mathrm{K}_2(D)$, there are elements $b \in \mathcal{S}(B)$ and $c \in \mathcal{S}(C)$ such that $d = \psi'(b)\varphi'(c)$.

(iv) **$\mathcal{G}$-fibred**, if $\mathcal{G}$-square is a fibre square.

The next lemma provides conditions which are easier to check in practice than those in (1.4)(i)-(iii). It will be used in the Goodness Lemma 6.9.

1.5. Excision Lemma. *Let $(*)$ be a $\mathcal{G}$-fibred square and $\theta : \mathcal{S}(B)/\operatorname{Im}\mathcal{S}(\varphi) \to \mathcal{S}(D)/\operatorname{Im}\mathcal{S}(\varphi')$ the map of coset spaces induced by $\mathcal{S}$-square.*

If θ is injective and

$$\operatorname{Ker}\mathcal{E}(\varphi') \leqslant \mathcal{E}(\psi)\big(\operatorname{Ker}\mathcal{E}(\varphi)\big)$$

then the square $(*)$ is weak $\mathcal{E}$-fibred.

If θ is surjective then the square $(*)$ is $\mathcal{E}$-surjective and $\mathcal{S}$-surjective.

Proof. Let $b \in \mathcal{E}(B)$ and $c \in \mathcal{E}(C)$ be such that $\omega(c) = \rho(b)$ and let b' and c' be preimages in $\mathcal{S}(B)$ and $\mathcal{S}(C)$, respectively. Obviously we can choose b' and c' such that $\mathcal{S}(\varphi')(c') = \mathcal{S}(\psi')(b')$. We have $\theta(b' \operatorname{Im}\mathcal{S}(\varphi)) = 1 \cdot \operatorname{Im}\mathcal{S}(\varphi')$ and since θ is injective, there is an element $a' \in \mathcal{S}(A)$ such that $\mathcal{S}(\varphi)(a') = b'$. Let $\bar{a}$ denote the image of a' in $\mathcal{E}(A)$. Since the element $\mathcal{E}(\psi)(\bar{a})^{-1}c$ belongs to $\operatorname{Ker}\mathcal{E}(\varphi')$, it follows from the hypotheses in the lemma that it has a preimage $\tilde{a} \in \operatorname{Ker}\mathcal{E}(\varphi)$. If $a = \bar{a}\tilde{a}$ then it is easy to check that $\mathcal{E}(\varphi)(a) = b$ and $\mathcal{E}(\psi)(a) = c$. The uniqueness of such an element a follows immediately from condition (1.4)(iv). Thus we have shown that *factor-square* is fibred.

The surjectivity of θ immediately implies that each element from $\mathcal{S}(D)$ can be written as product $\mathcal{S}(\varphi)(b')\mathcal{S}(\psi)(c')$ for some $b' \in \mathcal{S}(B)$, $c' \in \mathcal{S}(C)$. The rest of the proof is very easy and will be left to the reader.

1.6. Let

$$\widetilde{\mathcal{E}} = \mathcal{G}(\varphi)^{-1}\big(\mathcal{E}(B)\big) \cap \mathcal{G}(\psi)^{-1}\big(\mathcal{E}(C)\big).$$

Obviously $\mathcal{E}(A) \leqslant \widetilde{\mathcal{E}}$. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathcal{E}(B) \arrow[r, "\rho"] & \mathcal{S}(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C)) \arrow[dr] & \\
\widetilde{\mathcal{E}} \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & & & \mathcal{E}(D) \quad . \\
 & \mathcal{E}(C) \arrow[r, "\omega"] & \mathcal{S}(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C)) \arrow[ur] &
\end{tikzcd}$$

Define a function $\chi : \widetilde{\mathcal{E}} \to \mathcal{S}(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C))$ by the formula

$$\chi(a) = \rho\varphi(a^{-1})\omega\psi(a).$$

1.7. Lemma. *χ is a homomorphism from $\widetilde{\mathcal{E}}$ to $\mathrm{K}_2(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C))$. Suppose that $(*)$ is a $\mathcal{G}$-fibred square.*

If $(*)$ is $\mathcal{S}$-surjective then χ is an epimorphism.

If $(*)$ is weak $\mathcal{E}$-fibred then $\operatorname{Ker}\chi = \mathcal{E}(A)$.

Proof. Since $\mathcal{G}$-*square* is commutative, the image of χ lies in the subgroup $\mathrm{K}_2(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C))$ which is central in $\mathcal{S}(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C))$ by (1.1)(ii). Hence

$$\chi(a')\chi(a) = \rho\varphi(a^{-1})\chi(a')\omega\psi(a) = \rho\varphi(a^{-1})\rho\varphi({a'}^{-1})\omega\psi(a')\omega\psi(a) = \chi(a'a)$$

for any $a, a' \in \widetilde{\mathcal{E}}$. Thus χ is a homomorphism.

Let $\bar{d} \in \mathrm{K}_2(D)/(\mathrm{K}_2(B)\,\mathrm{K}_2(C))$ and d be a preimage of $\bar{d}$ in $\mathrm{K}_2(D)$. By (1.4)(iii) there are elements $b \in \mathcal{S}(B)$ and $c \in \mathcal{S}(C)$ such that $d = \psi'(b)\varphi'(c)$. Hence, $\bar{d} = \rho\pi(b)\omega\pi(c)$ (where π denotes as in (1.1) the natural transformation $\mathcal{S} \to \mathcal{E}$). Since the image of $\bar{d}$ in $\mathcal{E}(D)$ is trivial, it follows that $\mathcal{E}(\psi')\pi(b)^{-1} = \mathcal{E}(\varphi')\pi(c)$. By (1.4)(iv), there is an element $a \in \mathcal{G}(A)$ such that $\varphi(a) = \pi(b)^{-1}$ and $\psi(a) = \pi(c)$. Obviously $a \in \widetilde{\mathcal{E}}$, since $\pi(b)^{-1} \in \mathcal{E}(B)$ and $\pi(c) \in \mathcal{E}(C)$. But $\chi(a) = \rho\varphi(a^{-1})\omega\psi(a) = \rho\pi(b)\omega\pi(c) = \bar{d}$. Thus χ is surjective.

Clearly $\mathcal{E}(A) \leqslant \operatorname{Ker}(\chi)$ because *factor-square* is commutative. Suppose $\omega\psi(a) = \rho\varphi(a)$ for some $a \in \widetilde{\mathcal{E}}$. Since $(*)$ is weak $\mathcal{E}$-fibred, there is by definition (1.4)(i) an element $a' \in \mathcal{E}(A)$ such that $\varphi(a') = \varphi(a)$ and $\psi(a') = \psi(a)$. But, since $\mathcal{G}$-*square* is a fibre square, $a = a'$ which proves the last assertion of the lemma.

Suppose that $(*)$ is $\mathcal{G}$-fibred, weak $\mathcal{E}$-fibred, and $\mathcal{S}$-surjective. Define the homomorphism $\partial : \mathrm{K}_2(D) \to \mathrm{K}_1(A)$ as the composite map

$$\mathrm{K}_2(D) \twoheadrightarrow \mathrm{K}_2(D)/\big(\mathrm{K}_2(B)\,\mathrm{K}_2(C)\big) \xrightarrow{\chi^{-1}} \widetilde{\mathcal{E}}/\mathcal{E}(A) \rightarrowtail \mathrm{K}_1(A).$$

1.8. Theorem. *There is a functorial Mayer–Vietoris sequence (1.3) for $\mathcal{G}$-fibred, weak $\mathcal{E}$-fibred, $\mathcal{S}$-surjective, $\mathcal{E}$-surjective squares $(*)$.*

Proof. The proof is absolutely the same as that of Theorem 5.31 of [Bk2] (using (1.7) instead of [Bk2, Lemma 5.29]).

In fact, exactness of the K_1-square follows from (1.4)(iv) and $\mathcal{E}$-surjectivity; exactness at the term $\mathrm{K}_2(D)$ follows from $\mathcal{S}$-surjectivity; and exactness at $\mathrm{K}_1(A)$ follows from the condition that $(*)$ is weak $\mathcal{E}$-fibred. Of course, in the last two cases we must change the definition of χ' in the obvious way if χ is not an isomorphism.

In subsequent sections, the Mayer-Vietoris sequence in (1.8) will not be fully used, but the homomorphism χ which is used to construct it, will play a crucial role. Furthermore it will be important that χ is equivariant in a certain sense. For the sake of precision and clarity, we define and establish the equivariance of χ in a purely group theoretic context. The arguments refine those given already above and are taken from [Bk3].

1.9. For the rest of this section, let

$$\begin{tikzcd}
 & S_A \arrow[rr] \arrow[dl] \arrow[dd] & & S_B \arrow[dl] \arrow[dd] \\
S_C \arrow[rr] \arrow[dd] & & S_D \arrow[dd] & \\
 & G_A \arrow[rr] \arrow[dl] & & G_B \arrow[dl] \\
G_C \arrow[rr] & & G_D &
\end{tikzcd}$$

be a commutative cube of groups. For $X = A, B, C$ or D, let $E_X = \mathrm{Im}(S_X \to G_X)$ and $K_X = \mathrm{Ker}(S_X \to G_X)$. Assume that $K_D \leqslant \mathrm{center}(S_D)$. Define the group $\widetilde{E}_A = \{a \in G_A \mid \text{image of } a \text{ in } G_X\ (X = B, C) \text{ lies in } E_X\}$. Define the homomorphisms φ, ψ, ρ, and ω as in the commutative diagram

$$\begin{tikzcd}
 & E_B \arrow[r, "\rho"] & S_D/(K_BK_C) \arrow[dr] & \\
\widetilde{E}_A \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & & & E_D \\
 & E_C \arrow[r, "\omega"] & S_D/(K_BK_C) \arrow[ur] &
\end{tikzcd}$$

and set

$$\begin{aligned}
\chi_A : \widetilde{E}_A &\to K_D/K_BK_C \\
a &\mapsto (\rho\varphi(a))^{-1}(\omega\psi(a)).
\end{aligned}$$

Since $K_D \leqslant \mathrm{center}(S_D)$, it follows that χ_A is a group homomorphism.

1.10. Key Lemma. *Suppose that the cube in (1.9) satisfies the following conditions.*

(i) *The group K_D is central in S_D.*
(ii) *The groups E_X are perfect for $X = B, C$.*
(iii) *The maps $S_B/S_A \to S_D/S_C$ and $E_C \to E_D$ are injective.*
(iv) *The G-square is fibred.*

Let $J_A \leqslant G_A$ be a subgroup satisfying the following conditions.

(v) $\mathrm{Im}(J_A \to G_X)$ $(X = B, C)$ *normalizes* E_X.

(vi) $\mathrm{Im}(J_A \to G_D) \leqslant E_D$. *Lift in the obvious way the action of J_A on E_D by conjugation to an action of J_A on S_D.*

Then J_A normalizes $\widetilde{E}_A$ and leaves each element of K_D fixed, the homomorphism $\chi_A : \widetilde{E}_A \to K_D/K_BK_C$ is J_A-equivariant, and $\operatorname{Ker}\chi_A = E_A$. *In particular the action of J_A on $\widetilde{E}_A/E_A$ is trivial.*

Proof. Since J_A normalizes E_B and E_C, it follows from the definition of $\widetilde{E}_A$ that J_A normalizes $\widetilde{E}_A$. Since $K_D \leqslant \mathrm{center}(S_D)$, the obvious lifting to S_D of the conjugation action of J_A on E_D leaves each element of K_D fixed. Let J_A act on E_B and E_C by conjugation. To show that χ_A is J_A-equivariant, it suffices to show that all the homomorphisms in the second diagram in (1.9) are J_A-equivariant. The homomorphisms $E_B \to E_D$ and $E_C \to E_D$ are clearly J_A-equivariant. Since E_B and E_C are perfect and $K_D/K_BK_C \leqslant \mathrm{center}(S_D/K_BK_C)$, it follows that the homomorphisms $\rho : E_B \to S_D/K_BK_C$ and $\omega : E_C \to S_D/K_BK_C$ are J_A-equivariant (cf. [M, lemma 5.4]). The remaining homomorphisms in the second diagram in (1.9) are obviously J_A-equivariant and thus χ_A is J_A-equivariant. From the J_A-equivariance of χ_A and the triviality of the J_A-action on K_D/K_BK_C, it follows that the J_A-action on $\widetilde{E}_A/\operatorname{Ker}\chi_A$ is trivial. But the third and fourth conditions in the lemma show as in the proof of (1.7) that $\operatorname{Ker}\chi_A = E_A$.

There are frequently situations in which all of the hypotheses of (1.10) hold except the one that the map $E_C \to E_D$ is injective. In order to get around this problem and still have the final conclusion of (1.10) hold, one develops a relative version of the above.

Let G and $\bar{G}$ be groups with an action of G on $\bar{G}$ by automorphisms of $\bar{G}$. The **semidirect** or **smash product** of this action is denoted by $\bar{G} \rtimes G$. By definition, $\bar{G} \rtimes G$ is a group whose underlying set is the Cartesian product $\bar{G} \times G$ and whose multiplication is given by $(\bar{\sigma}, \sigma)(\bar{\rho}, \rho) = (\bar{\sigma}(\sigma \circ \bar{\rho}), \sigma\rho)$. A **precrossed module** of the action is a group homomorphism $f : \bar{G} \to G$ which is G-equivariant under the action of G on itself by conjugation. This implies of course that $Im(f)$ is normal in G. The group $\bar{G} \rtimes G$ is called the smash product group **associated** to f. A precrossed module $f : \bar{G} \to G$ is called a **crossed module** if the action of $\bar{G}$ on itself induced from that of G on $\bar{G}$ is the conjugation action. This implies of course that $\mathrm{Ker}(f) \leqslant$ center $(\bar{G})$. A **homomorphism** of precrossed modules is a commutative diagram

$$\begin{array}{ccc} \bar{G} & \longrightarrow & G \\ {\scriptstyle \bar{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \bar{H} & \longrightarrow & H \end{array}$$

of groups such that the group homomorphism $\bar{f}$ is G-equivariant. A homomorphism of crossed modules defines a homomorphism $\bar{G} \rtimes G \to \bar{H} \rtimes H, (\bar{\sigma}, \sigma) \mapsto (\bar{f}(\bar{\sigma}), f(\sigma))$, of groups. Let $G \xrightarrow{1} G$ denote the crossed module defined by the action of G on itself by conjugation and the identity map $1 : G \to G$.

1.11. Corollary. [Bk3] *Let*

$$
\begin{tikzcd}
 & (S_{\bar A} \to S_A) \arrow[rr] \arrow[dl] \arrow[dd] & & (S_B \xrightarrow{1} S_B) \arrow[dl] \arrow[dd] \\
(S_{\bar C} \to S_C) \arrow[rr] \arrow[dd] & & (S_D \xrightarrow{1} S_D) \arrow[dd] & \\
 & (G_{\bar A} \to G_A) \arrow[rr] \arrow[dl] & & (G_B \xrightarrow{1} G_B) \arrow[dl] \\
(G_{\bar C} \to G_C) \arrow[rr] & & (G_D \xrightarrow{1} G_D) &
\end{tikzcd}
$$

be a commutative cube of precrossed modules such that the associated cube of groups

$$
\begin{tikzcd}
 & S_{\bar A} \rtimes S_A \arrow[rr] \arrow[dl] \arrow[dd] & & S_B \rtimes S_B \arrow[dl] \arrow[dd] \\
S_{\bar C} \rtimes S_C \arrow[rr] \arrow[dd] & & S_D \rtimes S_D \arrow[dd] & \\
 & G_{\bar A} \rtimes G_A \arrow[rr] \arrow[dl] & & G_B \rtimes G_B \arrow[dl] \\
G_{\bar C} \rtimes G_C \arrow[rr] & & G_D \rtimes G_D &
\end{tikzcd}
$$

and the subgroup $J_A \leqslant G_A \leqslant G_{\bar A} \rtimes G_A$ of $G_{\bar A} \rtimes G_A$ satisfy the assumptions of (1.10), except possibly the third and fourth assumptions. Let $E_{\bar X} = \operatorname{Im}(S_{\bar X} \to G_{\bar X})$ $(X = A, C)$ and let $\widetilde E_{\bar A} = \{\bar a \in G_{\bar A} \mid$ image of $\bar a$ in G_B (resp. $G_{\bar C}$) lies in E_B (resp. $E_{\bar C}$)$\}$. Assume the following.

(i) *The maps $S_B/S_{\bar A} \to S_D/S_{\bar C}$ and $E_{\bar C} \to E_D$ are injective.*

(ii) $\begin{tikzcd} G_{\bar A} \arrow[r] \arrow[d] & G_B \arrow[d] \\ G_{\bar C} \arrow[r] & G_D \end{tikzcd}$ *is a fibred subsquare of* $\begin{tikzcd} G_A \arrow[r] \arrow[d] & G_B \arrow[d] \\ G_C \arrow[r] & G_D \end{tikzcd}$ *of normal subgroups and the action of G_X on $G_{\bar X}$ $(X = A, C)$ is by conjugation.*

(iii) *The map $\widetilde E_{\bar A}/E_{\bar A} \to \widetilde E_A/E_A$ is surjective.*

(iv) *E_A is J_A-invariant.*

Then the action of J_A on the coset space $\widetilde E_A/E_A$ is trivial. (No assertion is being made concerning the normality of E_A in $\widetilde E_A$.)

Proof. Obviously $\widetilde E_{\bar A}$ is J_A-invariant. We show next that $E_{\bar A}$ is J_A-invariant and normal in $\widetilde E_{\bar A}$ and that the action of J_A on $\widetilde E_{\bar A}/E_{\bar A}$ is trivial. Let $K_{\bar X} = \operatorname{Ker}(S_{\bar X} \to G_{\bar X})$ $(X = A, C)$. Define $\chi_{\bar A} : \widetilde E_{\bar A} \to K_D/K_B K_{\bar C}$ and $\chi : \widetilde E_{\bar A} \rtimes \widetilde E_A \to K_D \rtimes K_D/(K_B \rtimes K_B)(K_{\bar C} \rtimes K_C)$ as in (1.9). Then $\chi = \chi_{\bar A} \rtimes \chi_A$. The proof of (1.10) shows that χ is J_A-equivariant. Since the action of of J_A on $K_D \rtimes K_D$ is trivial, it follows that the action of J_A on $\widetilde E_{\bar A} \rtimes \widetilde E_A/\operatorname{Ker}\chi_{\bar A} \rtimes \operatorname{Ker}\chi_A$ is trivial. But the first and second assumptions in the corollary imply as in the proof of (1.7) that $\operatorname{Ker}\chi_{\bar A} = E_{\bar A}$. Thus $E_{\bar A}$ is J_A-invariant and normal in $\widetilde E_{\bar A}$ and the action of J_A on $\widetilde E_{\bar A}/E_{\bar A}$ is trivial. The map $\widetilde E_{\bar A}/E_{\bar A} \to \widetilde E_A/E_A$ of coset spaces is obviously J_A-equivariant and by the third assumption in the corollary, it is surjective. Thus the action of J_A on $\widetilde E_A/E_A$ is trivial.

2. Dimension Theory and group valued functors

The materials in this section are taken from Bak [Bk3] (unpublished). We define the notion of a category with structure and introduce one kind of dimension function on a category with structure. A category with structure equipped with a dimension function is called a category with dimension. Let $\tau : \mathcal{S} \to \mathcal{G}$ be a natural transformation of group valued functors $\mathcal{S}, \mathcal{G} : \mathcal{C} \to ((\text{groups}))$ on a category $\mathcal{C}$ with dimension. We define the dimension filtration $\mathcal{G} = \mathcal{G}^{-1} \geqslant \mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots \geqslant \mathcal{G}^i \geqslant \cdots \geqslant \operatorname{Im}(\mathcal{S} \to \mathcal{G})$ of τ on $\mathcal{G}$ and show under certain conditions that $\mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots \geqslant \mathcal{G}^i \geqslant \cdots$ is a descending central series such that $\mathcal{G}^i(A) = \operatorname{Im}(\mathcal{S}(A) \to \mathcal{G}(A))$ whenever $i \geqslant \text{dimension}(A)$.

Recall that a **quasi-ordered set** is a set I together with a reflexive, transitive relation $\leqslant$ on I. A quasi-ordered set $(I, \leqslant)$ is called **directed** if given $i, j \in I$, there is a $k \in I$ such that $i \leqslant k$ and $j \leqslant k$. An equivalent definition of a quasi-ordered set is a category whose objects form a set and for any pair i, j of objects, $Mor(i, j)$ has at most one element.

2.1. Definition. A **category with structure** is a category $\mathcal{C}$ together with a class $\mathcal{S}(\mathcal{C})$ of commutative squares in $\mathcal{C}$ called **structure squares** and a class $\mathcal{I}(\mathcal{C})$ of functors called **infrastructure functors** whose domain categories are directed quasi-ordered sets and whose target category is $\mathcal{C}$, satisfying the following conditions.

(i) $\mathcal{S}(\mathcal{C})$ is closed under isomorphism of commutative squares. For each object A of $\mathcal{C}$, the **constant** or **trivial** square

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow[1]{} & A \end{array}$$

is in $\mathcal{S}(\mathcal{C})$.

(ii) $\mathcal{I}(\mathcal{C})$ is closed under isomorphism of functors. For each object A of $\mathcal{C}$, the **constant** or **trivial** functor $F_A : \{*\} \to \mathcal{C}$, $* \mapsto A$, is in $\mathcal{I}(\mathcal{C})$, where $\{*\}$ denotes the directed quasi-ordered set with precisely one element $*$. For each $(F : I \to \mathcal{C})$ in $\mathcal{I}(\mathcal{C})$, the direct limit $\varinjlim_I F$ exists in $\mathcal{C}$.

2.2. Definition. Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure. Let $d : \mathcal{O}bj(\mathcal{C}) \to \mathbb{Z}^{\geqslant 0} \cup \{\infty\}$ be a function which is constant on isomorphism classes of objects. Let $A \in \mathcal{O}bj(\mathcal{C})$ such that $0 < d(A) < \infty$. A d-**reduction** of A is a set

$$\begin{array}{ccc} A & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & D_i \end{array} \qquad (i \in I)$$

of structure squares where I is a directed quasi-ordered set and $B : I \to \mathcal{C}, i \mapsto B_i$, is an infrastructure functor such that the following holds.

(i) If $i \leqslant j \in I$ then the triangle

$$\begin{tikzcd} & B_j \\ A \arrow[ur] \arrow[dr] & \\ & B_i \arrow[uu] \end{tikzcd}$$

commutes.

(ii) $d(\varinjlim_I B) = 0$.

(iii) $d(C_i) < d(A)$ for all $i \in I$.

The function d is called a **dimension function** on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ if each object A of $\mathcal{C}$ such that $0 < d(A) < \infty$ has a d-reduction. In this case, the quadruple $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ is called a **category with dimension**.

In [Bk3], a more general concept of dimension function is developed, which allows conditions other than those above to be placed on $d(\varinjlim_I B_i)$ and the $d(X_i)$'s $(X = B, C, D)$, e.g. $d(\varinjlim_I B_i) < d(A)$, $d(C_i) < d(A)$, and $d(D_i) < d(A)$.

A dimension function d on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ is called **tame** or **universal**, if the existence of a d-reduction

$$\begin{array}{ccc} A & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & D_i \end{array} \qquad (i \in I)$$

for A such that $d(C_i) \leqslant n$ for all $i \in I$ implies that $d(A) \leqslant n+1$.

The next result is fundamental for appreciating the concept of a dimension function, but will not be applied in the sequel.

2.3. Theorem. [Bk4] *Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ be a category with structure and $\mathcal{C}^0$ a nonempty class of objects of $\mathcal{C}$, closed under isomorphism. Then there is a universal dimension function δ on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ such that $\mathcal{C}^0$ is the class of 0-dimensional objects of δ and such that if d is any other dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ whose 0-dimensional objects are contained in $\mathcal{C}^0$ then $\delta \leqslant d$, i.e. $\delta(A) \leqslant d(A)$ for all $A \in \mathcal{O}bj(\mathcal{C})$.*

The example below of a category with dimension will play a role in the current paper.

2.4. Example. Let $\mathcal{C}$ denote the category of all algebras A_R over Noetherian commutative rings R such that A_R is module finite over R. A morphism $A_R \to A'_{R'}$ is a pair of ring homomorphisms $A_R \to A'_{R'}$ and $R \to R'$ such that the diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ A_R & \longrightarrow & A'_{R'} \end{array}$$

commutes. If $s \in R$, let $\langle s \rangle$ denote the multiplicative set generated by s. For any Noetherian R-module M, let $\langle s \rangle^{-1} M$ denote the module of $\langle s \rangle$-fractions of M and let $\hat{M}_{(s)}$ denote the completion $\varprojlim_{n \in \mathbb{N}} M/s^n M$ of M. Let $\langle s \rangle^{-1} A_R$ denote the

$\langle s\rangle^{-1}R$ algebra $((\langle s\rangle^{-1}A_R)_{\langle s\rangle^{-1}R}$ and $(\hat{A}_R)_{(s)}$ the $\hat{R}_{(s)}$-algebra $((\hat{A}_R)_{(s)})_{\hat{R}_{(s)}}$. Let $\mathcal{S}(\mathcal{C})$ denote the class of commutative squares in $\mathcal{C}$ isomorphic to a square of the kind

$$\begin{tikzcd} A_R \arrow[r] \arrow[d] & \langle s\rangle^{-1}A_R \arrow[d] \\ (\hat{A}_R)_{(s)} \arrow[r] & \langle s\rangle^{-1}(\hat{A}_R)_{(s)} \end{tikzcd} .$$

Such squares are called **localization-completion** squares. The Noetherian and finiteness conditions guarantee that they are pullback diagrams in $\mathcal{C}$. If $S \leqslant R$ is a multiplicative set, let S have the directed quasi-ordering defined by $s \leqslant t \Leftrightarrow$ there is a $u \in S$ such that $su = t$. Let $F_{S,A_R} : S \to \mathcal{C}$ denote the functor $s \mapsto \langle s\rangle^{-1}A_R$. Clearly $\varinjlim_S F_{S,A_R}$ exists and is the $S^{-1}R$ algebra $(S^{-1}A_R)_{S^{-1}R}$. Let $\mathcal{I}(\mathcal{C})$ be the class of all functors isomorphic to some functor F_{S,A_R}. Then Bass-Serre dimension BS and Jacobson-Krull dimension JK are dimension functions on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$. A definition of Bass-Serre dimension is found in [Bk1] where it is shown that BS is a dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$. Jacobson-Krull dimension is defined as follows. An ideal of R is called Jacobson if it is the intersection of the maximal ideals containing it. $JK(A_R)$ is by definition the largest nonnegative integer n such that there is a chain $\mathfrak{P}_0 \subsetneqq \mathfrak{P}_1 \subsetneqq \cdots \subsetneqq \mathfrak{P}_n$ of prime Jacobson ideals $\mathfrak{P}_i$ in R. It is not difficult to show that $JK(A_R) =$ dimension (maxspec (R)) where maxspec (R) is the space of all maximal ideals of R under the Zariski topology. Using this fact, one can show easily that $JK(A_R) \geqslant BS(A_R)$ and can adapt to JK the proof in [Bk1] that BS is a dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$. It can also be shown, but is beyond the scope of the current article, that neither JK nor BS is the universal dimension function on $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}))$ for their class of 0-dimensional algebras, namely all A_R such that R is semilocal.

2.5. Definition-Lemma. *Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ be a category with dimension. Let $\tau : \mathcal{S} \to \mathcal{G}$ be a natural transformation of functors $\mathcal{S}, \mathcal{G} : \mathcal{C} \to$ ((groups)). Let $\mathcal{E} = \operatorname{Im}(\tau)$. Define the* **dimension filtration** *or* d-**filtration** *$\mathcal{G} = \mathcal{G}^{-1} \geqslant \mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots \geqslant \mathcal{G}^i \geqslant \cdots \geqslant \mathcal{E}$ of τ on $\mathcal{G}$ by*

$$\mathcal{G}^i(A) = \bigcap_{\substack{f:A\to B\\ d(B)\leqslant i}} \operatorname{Ker}(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B))$$

for any $i \geqslant 0$. In general $\mathcal{G}(B)/\mathcal{E}(B)$ is just a coset space (not a group) and by definition, $\operatorname{Ker}(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B)) = \{a \in \mathcal{G}(A) \mid$ image of a in $\mathcal{G}(B)$ lies in $\mathcal{E}(B)\}$. It is clearly a subgroup of $\mathcal{G}(A)$. If there are no morphisms $f : A \to B$ such that $d(B) \leqslant i$ then by definition $\mathcal{G}^i(A) = \mathcal{G}(A)$. A trivial, but important consequence of the definition of $\mathcal{G}^i$ is that if $i \geqslant \dim(A)$ then $\mathcal{G}^i(A) = \mathcal{E}(A)$. Furthermore, if A and i are fixed and $\mathcal{M}$ is a nonempty set of morphisms $A \to B$ such that for each B, $\dim(B) \leqslant i$ and $\mathcal{G}^i(A) = \bigcap_{(A\to B)\in\mathcal{M}} \operatorname{Ker}\big(\mathcal{G}(A) \to \mathcal{G}(B)/\mathcal{E}(B)\big)$ then the induced map $\mathcal{G}(A)/\mathcal{G}^i(A) \to \prod_{(A\to B)\in\mathcal{M}} \mathcal{G}(B)/\mathcal{E}(B)$ is injective.

2.6. Definition. Let $\tau : \mathcal{S} \to \mathcal{G}$ be as in (2.5). An arbitrary commutative square

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
$$

is called τ**-good**, if the cube of groups

$$
\begin{tikzcd}
& \mathcal{S}(A) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{S}(B) \arrow[dl] \arrow[dd] \\
\mathcal{S}(C) \arrow[rr, crossing over] \arrow[dd] & & \mathcal{S}(D) & \\
& \mathcal{G}(A) \arrow[rr] \arrow[dl] & & \mathcal{G}(B) \arrow[dl] \\
\mathcal{G}(C) \arrow[rr] & & \mathcal{G}(D) \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

extends not necessarily functorially to a commutative cube of precrossed modules

$$
\begin{tikzcd}
& (\mathcal{S}_{\bar{A}} \to \mathcal{S}(A)) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{S}(B) \xrightarrow{1} \mathcal{S}(B)) \arrow[dl] \arrow[dd] \\
(\mathcal{S}_{\bar{C}} \to \mathcal{S}(C)) \arrow[rr, crossing over] \arrow[dd] & & (\mathcal{S}(D) \xrightarrow{1} \mathcal{S}(D)) & \\
& (\mathcal{G}_{\bar{A}} \to \mathcal{G}(A)) \arrow[rr] \arrow[dl] & & (\mathcal{G}(B) \xrightarrow{1} \mathcal{G}(B)) \arrow[dl] \\
(\mathcal{G}_{\bar{C}} \to \mathcal{G}(C)) \arrow[rr] & & (\mathcal{G}(D) \xrightarrow{1} \mathcal{G}(D)) \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

which satisfies all the conditions in (1.11) with $J_A = 1$ and $\operatorname{Im}\big(\mathcal{S}_{\bar{C}} \rtimes \mathcal{S}(C) \to \mathcal{G}_{\bar{C}} \rtimes \mathcal{G}(C)\big)$ is a normal subgroup of $\mathcal{G}_{\bar{C}} \rtimes \mathcal{G}(C)$.
The natural transformation τ is called **good**, if $\mathcal{S}$ and $\mathcal{G}$ commute with arbitrary direct limits in $\mathcal{C}$ and if each object A of $\mathcal{C}$ such that $0 < d(A) < \infty$ has a d-reduction

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B_i \arrow[d] \\
C_i \arrow[r] & D_i
\end{tikzcd}
\qquad (i \in I)
$$

consisting of structure squares which are direct limits of τ-good squares.

2.7. Theorem. [Bk4] *Let $\tau : \mathcal{S} \to \mathcal{G}$ be a good natural transformation of functors $\mathcal{S}, \mathcal{G} : \mathcal{C} \to$((groups)) on a category with dimension $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ such that for each finite dimensional object A, $\mathcal{E}(A)$ is normal in $\mathcal{G}(A)$. Then the dimension filtration $\mathcal{G} \geqslant \mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots \geqslant \mathcal{G}^n \geqslant \cdots \geqslant \mathcal{E}$ of τ on $\mathcal{G}$ has the property that each $\mathcal{G}^n$ is normal in $\mathcal{G}$, $\mathcal{G}^n(A) = \mathcal{E}(A)$ whenever $n \geqslant d(A)$, and $\mathcal{G}^0 \geqslant \mathcal{G}^1 \geqslant \cdots$ is a descending central series i.e. for all objects A of $\mathcal{C}$, the mixed commutator group $[\mathcal{G}^0(A), \mathcal{G}^n(A)] \leqslant \mathcal{G}^{n+1}(A)$. Moreover, if $\mathcal{G}(A)/\mathcal{E}(A)$ is Abelian for all 0-dimensional objects A then $\mathcal{G}(A)/\mathcal{G}^0(A)$ is Abelian for all objects A.*

Proof. The normality of each $\mathcal{G}^n(A)$ in $\mathcal{G}(A)$ follows from the definition of $\mathcal{G}^n(A)$ and from the normality of $\mathcal{E}(B)$ in $\mathcal{G}(B)$ for each finite dimensional object B. Clearly $\mathcal{E}(A) \leqslant \mathcal{G}^n(A)$ and if $d(A) \leqslant n$ then $\mathcal{G}^n(A) \leqslant \operatorname{Ker}(\mathcal{G}(A) \to \mathcal{G}(A)/\mathcal{E}(A)) = \mathcal{E}(A)$. Thus $d(A) \leqslant n$ implies $\mathcal{G}^n(A) = \mathcal{E}(A)$. To show that $[\mathcal{G}^0(A), \mathcal{G}^n(A)] \leqslant \mathcal{G}^{n+1}(A)$

$(n \geqslant 0)$, it suffices, by the last assertion in (2.5), to consider the case $d(A) \leqslant n+1$. If $d(A) \leqslant n$ then $\mathcal{G}^n(A) = \mathcal{E}(A) = \mathcal{G}^{n+1}(A)$ and the inclusion $[\mathcal{G}^0(A), \mathcal{G}^n(A)] \leqslant \mathcal{G}^{n+1}(A)$ follows from the normality of $\mathcal{E}(A)$. Suppose $d(A) = n+1$. Let $x \in \mathcal{G}^0(A)$ and $y \in \mathcal{G}^n(A)$. We must show that $[x, y] \in \mathcal{G}^{n+1}(A)$. Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B_i \arrow[d] \\ C_i \arrow[r] & D_i \end{tikzcd} \qquad (i \in I)$$

be a d-reduction of A consisting of structure squares which are direct limits of τ-good squares. There is canonical morphism $A \to \varinjlim_I B$, thanks to (2.2) (i). Since $d(\varinjlim_I B) = 0$ by (2.2) (ii), both x and y vanish in $\mathcal{G}(\varinjlim_I B)/\mathcal{E}(\varinjlim_I B)$. Since $\mathcal{S}$ and $\mathcal{G}$ commute with limits of infrastructure functors, so do $\mathcal{E}$ and $\mathcal{G}/\mathcal{E}$. Thus there is a $j \in I$ such that x and y vanish in $\mathcal{G}(B_j)/\mathcal{E}(B_j)$. Let $\widetilde{\mathcal{E}}(A) = \{z \in \mathcal{G}(A) \mid z \text{ vanishes in } \mathcal{G}(B_j)/\mathcal{E}(B_j) \text{ and } \mathcal{G}(C_j)/\mathcal{E}(C_j)\}$. Since $d(C_j) \leqslant n$ by (2.2) (iii), y vanishes in $\mathcal{G}(C_j)/\mathcal{E}(C_j)$. Thus $y \in \widetilde{\mathcal{E}}(A)$. Since

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B_j \arrow[d] \\ C_j \arrow[r] & D_j \end{tikzcd}$$

is a direct limit of τ-good squares and since $\mathcal{S}$ and $\mathcal{G}$ commute with arbitrary direct limits in $\mathcal{C}$, we can assume that the square above is τ-good instead of a structure square and that x and y vanish in $\mathcal{G}(B_j)/\mathcal{E}(B_j)$, and $y \in \widetilde{\mathcal{E}}(A)$. Let $J(A)$ denote the subgroup of $\mathcal{G}(A)$ generated by x. Since the square above is τ-good, there is a cube of precrossed modules

$$\begin{tikzcd}[column sep=small]
 & (\mathcal{S}_{\bar{A}} \to \mathcal{S}(A)) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{S}(B_j) \xrightarrow{1} \mathcal{S}(B_j)) \arrow[dl] \arrow[dd] \\
(\mathcal{S}_{\bar{C}_j} \to \mathcal{S}(C_j)) \arrow[rr] \arrow[dd] & & (\mathcal{S}(D_j) \xrightarrow{1} \mathcal{S}(D_j)) \arrow[dd] & \\
 & (\mathcal{G}_{\bar{A}} \to \mathcal{G}(A)) \arrow[rr] \arrow[dl] & & (\mathcal{G}(B_j) \xrightarrow{1} \mathcal{G}(B_j)) \arrow[dl] \\
(\mathcal{G}_{\bar{C}_j} \to \mathcal{G}(C_j)) \arrow[rr] & & (\mathcal{G}(D_j) \xrightarrow{1} \mathcal{G}(D_j)) &
\end{tikzcd}$$

such that $J(A)$ and this cube satisfy the hypotheses of (1.11). By the conclusion of (1.11), the action of $J(A)$ on $\widetilde{\mathcal{E}}(A)/\mathcal{E}(A)$ by conjugation is trivial. Thus $[x, y] \in \mathcal{E}(A) = \mathcal{G}^{n+1}(A)$.

The last assertion of the theorem follows immediately from the last assertion of (2.5).

The theorem above says that the nilpotent class of $\mathcal{G}^0(A)/\mathcal{E}(A)$ is $\leqslant d(A)$. Let $N \in \mathbb{Z}^{\geqslant 0}$. If $\mathcal{G}^0(A) = \mathcal{E}(A)$ whenever $d(A) \leqslant N$ then it turns out that the nilpotent class of $\mathcal{G}^0(A)/\mathcal{E}(A)$ is $\leqslant |d(A) - N|$. One proves this in a trivial way, using dimension shifting which is defined as follows.

2.8. Definition. Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{S}(\mathcal{C}), d)$ be a category with dimension. Let $N \in \mathbb{Z}^{\geqslant 0}$. For an object A of $\mathcal{C}$, define

$$d[-N](A) = \begin{cases} 0 & \text{if } d(A) \leqslant N \\ d(A) - N & \text{if } d(A) \geqslant N. \end{cases}$$

Obviously $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{S}(\mathcal{C}), d[-N])$ is again a category with dimension.

2.9. Corollary. *Let $(\mathcal{C}, \mathcal{S}(\mathcal{C}), \mathcal{I}(\mathcal{C}), d)$ be a category with dimension and $\tau : \mathcal{S} \to \mathcal{G}$ a good natural transformation of functors $\mathcal{S}, \mathcal{G} : \mathcal{C} \to$ ((groups)). Let $N \in \mathbb{Z}^{\geqslant 0}$. Let $\mathcal{G} = \mathcal{G}^{[-1]} \geqslant \mathcal{G}^{[0]} \geqslant \cdots \geqslant \mathcal{G}^{[n]} \geqslant \cdots \geqslant \mathcal{E}$ be the dimension filtration of τ on $\mathcal{G}$ with respect to the dimension function $d[-N]$. Then the conclusions of (2.7) are valid for the filtration above. In particular $\mathcal{G}^{[n]}(A) = \mathcal{E}(A)$ whenever $n \geqslant d[-N](A)$.*

Proof. The corollary is an immediate consequence of (2.7).

3. Nets and net subgroups

3.1. Let J be a subset of the set $\mathbb{N}$ of natural numbers, of order $|J| = n$ (where n can be infinite) and let ν be an equivalence relation on J. Following [BV2], we denote by $\mathrm{h}(\nu)$ the minimal order (possibly infinite) of the equivalence classes of ν. $\mathrm{h}(\nu)$ is called **the rank of** ν. Recall that a square table $\sigma = (\sigma_{ij})_{i,j \in J}$ is called a **net** over a ring R, with index set J if each σ_{ij} is an additive subgroup of R and $\sigma_{ij}\sigma_{jk} \subseteq \sigma_{ik}$ for all $i, j, k \in J$. Let $\mathrm{M}(n, R)$ denote the ring, without unit element if $n = \infty$, of all $n \times n$ matrices with coefficients in R such that each row and each column contains only a finite number of nonzero elements. Enumerate the entries of a matrix in $\mathrm{M}(n, R)$ by elements in $J \times J$. We shall usually identify a net σ with its **net ring** $\mathrm{M}(\sigma)$ which by definition is the subring of $\mathrm{M}(n, R)$ consisting of all matrices a such that $a_{ij} \in \sigma_{ij}$ for all $i, j \in J$. Even if n is finite, $\mathrm{M}(\sigma)$ does not necessarily have a unit. A net σ is called a ***ν*-net**,if all σ_{ij} are ideals in R and $\sigma_{ij} = \sigma_{kl}$ whenever $i \overset{\nu}{\sim} k$ and $j \overset{\nu}{\sim} l$.

3.2. The **principal net subgroup** $\mathrm{G}(\sigma)$ is the largest subgroup of $\mathrm{GL}(n, R)$ such that each matrix $a \in \mathrm{G}(\sigma)$ has the property that $a_{ij} \equiv \delta_{ij} \mod \sigma_{ij}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Let e denote the identity matrix and e^{ij} the matrix with 1 in position (i, j) and 0 elsewhere. Let $\mathrm{E}(\sigma)$ denote the group generated by all **elementary transvections** $t_{ij}(\xi) = e + \xi e^{ij}$ where $i \neq j$ and $\xi \in \sigma_{ij}$. Denote by $[\nu, R]$ or simply $[\nu]$ the following net:

$$[\nu]_{ij} = \begin{cases} R, & \text{if } i \overset{\nu}{\sim} j, \\ 0, & \text{otherwise.} \end{cases}$$

The net subgroups $\mathrm{G}(\nu) = \mathrm{G}(\nu, R) := \mathrm{G}([\nu, R])$ and $\mathrm{E}(\nu) = \mathrm{E}(\nu, R) := \mathrm{E}([\nu, R])$ corresponding to this net are called the **principal** and **elementary block-diagonal groups**, respectively. Thus the elementary block diagonal group $\mathrm{E}(\nu)$ over a ring R is the subgroup of $\mathrm{GL}(n, R)$ generated by all elementary transvections $t_{ij}(\xi)$ such that $i \neq j$, $i \overset{\nu}{\sim} j$, $\xi \in R$. The **elementary net subgroup** $\mathrm{E}(\nu, \sigma)$ for a net σ corresponding to an equivalence ν is the normal closure $\mathrm{E}(\sigma)^{\mathrm{E}(\nu)}$ of the subgroup $\mathrm{E}(\sigma)$ by $\mathrm{E}(\nu)$.

3.3. Clearly $\mathrm{E}(\nu, \sigma) \leqslant \mathrm{G}(\sigma)$ if and only if σ is a ν-net. Clearly $\mathrm{G}(\sigma)$ is normalized by $\mathrm{E}(\nu)$ if and only if σ is a ν-net. For a ν-net σ, denote by $\sigma + [\nu]$ the net with ideals $(\sigma+[\nu])_{ij}=\sigma_{ij}$ if $i \overset{\nu}{\not\sim} j$ and $(\sigma+[\nu])_{ij}=R$ otherwise. If $\sigma=\sigma+[\nu]$ then σ is called a **major ν-net**. Clearly if σ is a major ν-net then $\mathrm{E}(\sigma)=\mathrm{E}(\nu,\sigma)$.

For the rest of this article, all nets are ν-nets.

Define $\mathrm{C}(\nu,\sigma)$ to be the largest subgroup C of $\mathrm{GL}(n,R)$ such that

$$[C,\mathrm{E}([\nu]+\sigma)]\leqslant \mathrm{G}(\sigma).$$

If σ is major then $\mathrm{C}(\nu,\sigma)$ obviously coincides with the normalizer $\mathrm{N}(\sigma)$ of the principal net subgroup $\mathrm{G}(\sigma)$ in $\mathrm{GL}(n,R)$. On the other hand, if ν is the trivial equivalence (i.e. $i \overset{\nu}{\sim} j$ for all $i,j\in J$) then σ is a constant net (i.e. $\sigma_{ij}=I$ for some ideal I and all $i,j\in J$) and $\mathrm{E}(\nu,\sigma)=\mathrm{E}(n,R,I)$, $\mathrm{G}(\sigma)=\mathrm{GL}(n,R,I)$ and $\mathrm{C}(\nu,\sigma)=\mathrm{C}(n,R,I)$ are the corresponding relative subgroups of level I. In the above situation, σ is major means $I=R$.

The sandwich classification theorem shows that a subgroup of $\mathrm{GL}(n,R)$ is normalized by $\mathrm{E}(\nu)$ if and only if it fits into a sandwich $\mathrm{E}(\nu,\sigma)\leqslant\ldots\leqslant\mathrm{C}(\nu,\sigma)$ for some ν-net σ. The subgroup $\mathrm{G}(\sigma)$ divides the sandwich $\mathrm{E}(\nu,\sigma)\leqslant\ldots\leqslant\mathrm{C}(\nu,\sigma)$ into two parts. We shall consider only the group $\mathrm{G}(\sigma)/\mathrm{E}(\nu,\sigma)$ in situations when $\mathrm{E}(\nu,\sigma)$ is normal in $\mathrm{G}(\sigma)$. Denote this group by $\mathrm{K}_1(\nu,\sigma)$. The kind of behavior we expect for net K_1- and K_2-groups is described in the next two propositions in the special case $\mathrm{h}(\nu)=\infty$.

3.4. Proposition. *Let ν be an equivalence on $\mathbb{N}$ with a finite number of equivalence classes each of infinite order.*

(i) *For a subgroup $H\leqslant\mathrm{GL}(R)$ normalized by $\mathrm{E}(\nu)$, there exists a unique ν-net σ such that $\mathrm{E}(\nu,\sigma)\leqslant H\leqslant\mathrm{G}(\sigma)$.*
(ii) $[\mathrm{G}(\sigma),\mathrm{G}(\nu)]=[\mathrm{G}(\sigma),\mathrm{G}([\nu]+\sigma)]=\mathrm{E}(\nu,\sigma)$.

Proof. The second assertion of the proposition follows in the usual way from the block version of Whitehead lemma. We leave to the reader the formulation and proof of the block version. The first assertion follows easily from lemma 2 of [BV1] or lemma 3 of [S2].

The proposition shows that if $\mathrm{h}(\nu)=\infty$ then:

- $\mathrm{C}(\nu,\sigma)$ coincides with $\mathrm{G}(\sigma)$.
- In terms of [S1], $\mathrm{E}(\nu)$ is strongly polynormal in $\mathrm{GL}(R)$ and the subgroups $\mathrm{E}(\nu,\sigma)$ as σ ranges over all ν-nets account for all $\mathrm{E}(\nu)$-perfect subgroups of $\mathrm{GL}(R)$.
- $\mathrm{K}_1(\nu,\sigma)=\mathrm{G}(\sigma)/\mathrm{E}(\nu,\sigma)$ lies in the center of $\mathrm{G}([\nu]+\sigma)/\mathrm{E}(\nu,\sigma)$. In particular, the group $\mathrm{K}_1(\nu,\sigma)$ is Abelian.

3.5. Let $\sigma\geqslant[\nu]$ be a major ν-net over a ring R, with index set J. Define the Steinberg group $\mathrm{St}(\nu,\sigma)$ to be the group with generators $x_{ij}(\xi)$, $i\neq j\in J$, $\xi\in\sigma_{ij}$ and the ordinary Steinberg relations: For any $i,j,k,l\in J$, $i\neq j$, $k\neq l$, $i\neq l$, $j\neq k$, one has:

(1) $x_{ij}(\xi)x_{ij}(\eta)=x_{ij}(\xi+\eta)$.
(2) $[x_{ij}(\xi),x_{kl}(\eta)]=1$.
(3) $[x_{ij}(\xi),x_{jk}(\eta)]=x_{ik}(\xi\eta)$.

Let $\mathrm{St}(\nu)=\mathrm{St}(\nu,R):=\mathrm{St}([\nu,R])$. For an arbitrary ν-net σ, define $\mathrm{St}(\sigma)$ to be the subgroup of $\mathrm{St}(\nu,\sigma+[\nu])$ generated by all generators $x_{ij}(\xi)$ such that $i\neq j\in J$ and

$\xi \in \sigma_{ij}$, and define $\mathrm{St}(\nu, \sigma)$ to be the normal closure of $\mathrm{St}(\sigma)$ in $\mathrm{St}(\nu, \sigma + [\nu])$, or equivalently the normal closure $\mathrm{St}(\sigma)^{\mathrm{St}(\nu,R)}$ of $\mathrm{St}(\sigma)$ by the subgroup $\mathrm{Im}(\mathrm{St}(\nu, R) \to \mathrm{St}(\nu, \sigma + [\nu]))$ of $\mathrm{St}(\nu, \sigma + [\nu])$.

Let π denote the homomorphism $\pi : \mathrm{St}(\nu, \sigma) \to \mathrm{E}(\nu, \sigma)$, $x_{ij}(\xi) \mapsto t_{ij}(\xi)$, and define $\mathrm{K}_2(\nu, \sigma)$ to be the kernel of π.

3.6. Proposition. *Let R be a ring, ν be an equivalence on J and σ a major ν-net over R. If $\mathrm{h}(\nu) \geqslant 5$ then the group $\mathrm{St}(\nu, \sigma)$ is centrally closed (i.e. any central extension of this group splits). If $\mathrm{h}(\nu) = \infty$ then $\mathrm{St}(\nu, \sigma)$ is the universal perfect central extension of $\mathrm{E}(\nu, \sigma)$.*

Proof. The proof is absolutely the same as in [M, § 5]. (Note that to establish perfectness, we use $\sigma \geqslant [\nu]$.) The rest of the paper is devoted to investigating K_1 and K_2 of nets with finite index sets.

4. In the stable range.

Throughout this section, R denotes a ring and $\mathrm{sr}\, R$ its stable rank.

The standard description as in (3.4)(i) of $\mathrm{E}(\nu)$-normal subgroups of $\mathrm{GL}(n, R)$ is not known under a stable rank condition on rings (it was proved in [VvS] only under a weaker condition). Nevertheless the structure of the standard sandwiches is similar to that described in the previous section.

The normality of $\mathrm{E}(\nu, \sigma)$ in $\mathrm{N}(\sigma)$ was proved in [VvS] for any net $\sigma \geqslant [\nu]$ whenever $\mathrm{h}(\nu) \geqslant \sup(\mathrm{sr}\, R + 1,\ 3)$. Together with Corollary 5.2 this implies the commutator formula $[\mathrm{C}(\nu, \sigma), \mathrm{E}([\nu] + \sigma)] = \mathrm{E}(\nu, \sigma)$. In this section, we shall prove a stronger commutator formula (4.3)(iii). The standard commutator formula will be obtained in the next section under a weaker condition.

4.1. In the rest of this section, ν will be an equivalence on $\mathbb{N}$ and σ a ν-net over a ring R. Denote by $\nu^{(kh)}$ the equivalence on the set $J = \{k, \dots, h\}$ got by restricting ν to J and by $\sigma^{(kh)}$, the $\nu^{(kh)}$-net with index set J such that $\sigma^{(kh)}_{ij} = \sigma_{ij}$ for all $i, j \in J$. If $k = 1$, we write $\nu^{(h)}$ and $\sigma^{(h)}$ instead of $\nu^{(1h)}$ and $\sigma^{(1h)}$, respectively. For an equivalence χ, denote by $|\chi(m)|$ the order of the equivalence class of m.

4.2. Theorem. *If $|\nu^{(n+1)}(n + 1)| \geqslant \mathrm{sr}\, R + 1$ then*

$$\mathrm{G}(\sigma^{(n+1)}) = \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)})\, \mathrm{G}(\sigma^{(n)}).$$

If $\mathrm{h}(\nu^{(n)}) \geqslant \mathrm{sr}\, R$ and $\mathrm{E}(\nu^{(n)}, \sigma^{(n)})$ is normal in $\mathrm{G}(\sigma^{(n)})$ then the canonical homomorphisms

$$\mathrm{K}_1(\nu, \sigma^{(n)}) \longrightarrow \mathrm{K}_1(\nu, \sigma^{(n+1)}) \longrightarrow \cdots \longrightarrow \mathrm{K}_1(\nu, \sigma)$$

are surjective.

Proof. The proof is easy (see [Vv1]).

4.3. Theorem. (Injective stability for K_1)

(i) *If $|\nu^{(n+1)}(n + 1)| \geqslant \mathrm{sr}\, R + 2$ then*

$$\mathrm{E}(\nu^{(n)}, \sigma^{(n)}) = \mathrm{G}(\sigma^{(n)}) \cap \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)}).$$

(ii) *If $\mathrm{h}(\nu^{(n)}) \geqslant \mathrm{sr}\, R + 1$ and the set $\{1, \dots, n\}$ has nonempty intersection with each equivalence class of ν then $\mathrm{K}_1(\nu, \sigma^{(n)}) \cong \mathrm{K}_1(\nu, \sigma)$.*

(iii) *If χ is an equivalence with* $\mathrm{h}(\chi) \geqslant \operatorname{sr} R + 1$ *and τ is a χ-net over R then*

$$[\mathrm{G}(\tau), \mathrm{G}([\chi] + \tau)] = \mathrm{E}(\chi, \tau).$$

Items (ii) and (iii) follow easily from (i). The proof of (i) will be given below and is based on ideas in [SuTu]. In the special case of the general linear group, it will appear in [Vv2]. The key step in [Vv2] and in our proof is the so-called Dennis–Vaserstein decomposition.

4.4. We introduce the following notation. Let a be a matrix and τ a net. Then

- R^n (${}^nR$) is the set of all rows (columns) of length n over R.
- a_{i*} (a_{*i}) is the i-th row (column) of a matrix a.
- a'_{i*} (a'_{*i}) is the i-th row (column) of a matrix a^{-1}.
- $\tau_{i*} = \{v \in R^n |\ v_j \in \tau_{ij}\}$.
- $\tau_{*i} = \{u \in {}^nR |\ u_j \in \tau_{ji}\}$.
- $t_{i*}(v) = \prod_{j \neq i} t_{ij}(v_j)$ where $v \in R^n$.
- $t_{*i}(u) = \prod_{j \neq i} t_{ji}(u_j)$ where $u \in {}^nR$.

4.5. Without lost of generality we may suppose that $1 \overset{\nu}{\sim} n+1$. Consider the subgroups $P = P(\nu^{(n+1)}, \sigma^{(n+1)})$ and $Q = Q(\nu^{(n+1)}, \sigma^{(n+1)})$ of the group $H = \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)})$ defined as follows:

$$P = \left\{ \begin{pmatrix} 1 & 0 \\ s & a \end{pmatrix} \in H \,\middle|\, s \in {}^nR,\ a \in \mathrm{E}(\nu^{(2,n+1)}, \sigma^{(2,n+1)}) \right\}$$

$$Q = \left\{ \begin{pmatrix} b & 0 \\ v & 1 \end{pmatrix} \in H \,\middle|\, v \in R^n,\ b \in \mathrm{E}(\nu^{(n)}, \sigma^{(n)}) \right\}.$$

4.6. Theorem. *(Dennis–Vaserstein decomposition) Let* $|\nu^{(n+1)}(n+1)| \geqslant \operatorname{sr} R + 2$. *Then every element* $g \in \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)})$ *can be written in the form* $g = yt_{1,n+1}(\lambda)z$, *where* $y \in P$, $\lambda \in \sigma_{11}$, $z \in Q$.

We start with some lemmas.

4.7. Lemma. *Suppose that* $|\nu^{(n+1)}(n+1)| \geqslant \operatorname{sr} R + 2$. *Let* $u \in \sigma_{*1}^{(n+1)}$, $w \in \sigma_{1*}^{(n+1)}$, $w_{n+1} = 0$, $wu = 1$ *and* $\lambda \in \sigma_{11}$. *Then there exists* $x \in Q$ *such that* $(xu)_{n+1} = 0$ *and* $t_{1,n+1}(-\lambda) x t_{1,n+1}(\lambda) \in P$.

Proof. Since $|\nu^{(n)}(1)| \geqslant \operatorname{sr} R + 1$ and the column

$$(w_1u_1, u_k, w_ju_j)^t_{k \overset{\nu}{\sim} 1,\ j \overset{\nu}{\not\sim} 1,\ 2 \leqslant j,k \leqslant n}$$

is unimodular, there exists an $\alpha_k \in R$ such that the column $(u_k + \alpha_k w_1 u_1, u_j)^t$ is also unimodular. Hence for some $\beta_i \in R$ $(2 \leqslant i \leqslant n)$

$$\sum_{k \overset{\nu}{\sim} 1,\ k \neq 1} u_{n+1}\beta_k(\alpha_k w_1 u_1 + u_k) + \sum_{j \overset{\nu}{\not\sim} 1} u_{n+1}\beta_j w_j u_j = u_{n+1}.$$

Clearly the elements $\beta'_k = -u_{n+1}\beta_k$ are in $\sigma_{n+1,n+1} = \sigma_{n+1,k}$ for $k \overset{\nu}{\sim} 1$ and $\beta'_j = -u_{n+1}\beta_j w_j$ are in $\sigma_{n+1,1}\sigma_{1j} \subseteq \sigma_{n+1,j}$ for $j \overset{\nu}{\not\sim} 1$. Thus the matrices

$$x_1 = \prod_{k \overset{\nu}{\sim} 1,\ k \neq 1} t_{k1}(\alpha_k w_1) \quad \text{and} \quad x_2 = \prod_{i=2}^{n} t_{n+1,i}(\beta'_i) t_{1i}(\lambda\beta'_i)$$

are in Q. It is easy to see that $(x_1 x_2 u)_{n+1} = 0$ and $t_{1,n+1}(-\lambda) x_h t_{1,n+1}(\lambda) \in P$ for $h = 1, 2$. Hence $x = x_1 x_2$ fits the assertion of the lemma.

Let X denote the subset in $\mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)})$ consisting of those elements g which can be expressed in the form $g = y t_{1,n+1}(\lambda) z$, where $y \in P$, $\lambda \in \sigma_{11}$, $z \in Q$.

4.8. Lemma. *Assume that* $|\nu^{(n+1)}(n+1)| \geqslant \operatorname{sr} R + 2$. *Fix* $j \overset{\nu}{\sim} 1$. *Then every element* $g = y t_{1,n+1}(\lambda) z \in X$ *can be expressed in the form* $g = \bar{y} t_{1,n+1}(\bar{\lambda}) \bar{z}$ *as above where in addition* $\bar{z}_{n+1,j} = 0$.

Proof. Let $u = z_{*j} \in \sigma_{*j}$ and $w = z'_{j*} \in \sigma_{j*}$. By the previous lemma, there exists a matrix $x \in Q$ such that $(xu)_{n+1} = 0$ and $t_{1,n+1}(-\lambda) x t_{1,n+1}(\lambda) \in P$. Then

$$g = \big(y t_{1,n+1}(\lambda) x^{-1} t_{1,n+1}(-\lambda)\big) t_{1,n+1}(\lambda) \big(xz\big)$$

is the desired factorization.

We are now ready to prove Dennis–Vaserstein decomposition.

Proof. We shall prove that X is normalized by $\mathrm{E}(\nu^{(n+1)})$. Since

$$Q^{\mathrm{E}(\nu^{(n+1)})} = \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)}),$$

this will imply $X = \mathrm{E}(\nu^{(n+1)}, \sigma^{(n+1)})$.

Fix $j \overset{\nu}{\sim} 1$, $j \neq 1$, $j \neq n+1$. Clearly $\mathrm{E}(\nu^{(n+1)})$ is generated by all transvections $t_{ik}(\xi)$ with $\xi \in R$ and either $i \overset{\nu}{\sim} k \overset{\nu}{\not\sim} 1$ or $i = j \neq k$, $k \overset{\nu}{\sim} 1$ or $k = j \neq i$, $i \overset{\nu}{\sim} 1$. Obviously if $i \neq 1$ and $k \neq n+1$ then $t_{ik}(\xi)$ normalizes X. Thus we only have to show that

$$c = (y t_{1,n+1}(\lambda) z)^{t_{j,n+1}(\xi)} \in X$$

for all $y \in Q, z \in P, \lambda \in \sigma_{11}$ and $\xi \in R$ (the case $(i,k) = (1,j)$ can be proved in the same way).

By the last lemma, one may suppose that $z_{n+1,j} = 0$. Straightforward calculation shows that

$$c = \left[y \prod_{i=2}^{n} t_{i,n+1}((z_{ij} - \delta_{ij})\xi) \right] t_{1,n+1}(\lambda + z_{1j}\xi) \left[\left(\prod_{i=1}^{n} t_{i,n+1}(-z_{ij}\xi) \right) z t_{j,n+1}(\xi) \right].$$

Since P is normalized by $t_{j,n+1}(\xi)$ and $z_{ij} - \delta_{ij} \in \sigma_{ij} = \sigma_{i,n+1}$, the factor in the first square brackets belongs to P. It is easy to verify that the factor in the last square brackets is of the form $\begin{pmatrix} a & 0 \\ v & 1 \end{pmatrix}$, where $z = \begin{pmatrix} b & 0 \\ v & 1 \end{pmatrix}$ and $a = b \prod_{i=1}^{n} t_{ji}(\xi v_i)$. Therefore it belongs to Q. Hence $c \in X$.

An analogue of this result holds also for the Steinberg group and allows one to prove *surjective* stability for net K_2 (surjective stability for K_2 is practically equivalent to injective stability for K_1, see [SuTu] for the case of $\mathrm{GL}(n,R)$).

Proof of injective stability of K_1. It is obvious that $\mathrm{E}(\nu^{(n)},\sigma^{(n)}) \subseteq \mathrm{GL}(\sigma^{(n)}) \cap \mathrm{E}(\nu^{(n+1)},\sigma^{(n+1)})$. We want to prove the opposite inclusion. Indeed, take a matrix $g\oplus 1 \in \mathrm{GL}(\sigma^{(n)}) \cap \mathrm{E}(\nu^{(n+1)},\sigma^{(n+1)})$. Then by the decomposition theorem (4.6), it can be expressed as a product

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ y^{21} & y^{22} & y^{23} \\ y^{31} & y^{32} & y^{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & \lambda \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^{11} & z^{12} & 0 \\ z^{21} & z^{22} & 0 \\ z^{31} & z^{32} & 1 \end{pmatrix}$$

where the matrices y, $t_{1,n+1}(\lambda)$ and z are written in block form with respect to the partition $(1, n-1, 1)$. Multiplying the matrices on the right, we get $\lambda = 0$, $y^{23}=0$ and $y^{33}=1$. Hence, $1\oplus y^{22} \in \mathrm{E}(\nu^{(n)},\sigma^{(n)})$. Thus by definition of P and Q,

$$\begin{pmatrix} y^{22} & y^{23} \\ 0 & 1 \end{pmatrix} \in \mathrm{E}(\nu^{(2,n+1)},\sigma^{(2,n+1)}) \quad \text{and} \quad b = \begin{pmatrix} z^{11} & z^{12} \\ z^{21} & z^{22} \end{pmatrix} \in \mathrm{E}(\nu^{(n)},\sigma^{(n)}).$$

Consequently $g = \begin{pmatrix} 1 & 0 \\ y^{21} & y^{22} \end{pmatrix} b = \begin{pmatrix} 1 & 0 \\ y^{21} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & y^{22} \end{pmatrix} b \in \mathrm{E}(\nu^{(n)},\sigma^{(n)})$ Q.E.D.

5. Over a quasi-finite ring

Following [Bk1], we call a ring **quasi-finite**, if it is a direct limit of module finite rings. In this section, we shall prove the standard classification of $\mathrm{E}(\nu)$-normal subgroups $\mathrm{GL}(n,A)$ over quasi-finite rings A, where ν is an equivalence on the set $J = \{1,\cdots,n\}$ and $\mathrm{h}(\nu) \geqslant 3$. It will also be shown that $\mathrm{K}_2(\nu,\sigma)$ is central in $\mathrm{St}(\nu,\sigma)$ for any ν-net σ over a quasi-finite ring.

The standard classification theorem breaks into two parts, the sandwich theorem and the standard net commutator formula. The former follows by a direct limit argument from the sandwich theorem of I. Z. Golubchik [G] over module finite rings. So there is nothing to do here. The proof of the commutator formula over quasi-finite rings is reduced also by a direct limit argument to the case of a module finite ring. Module finite rings A over a commutative ring R have the property that if M is a maximal ideal in R and $S = R\setminus M$ then $S^{-1}A$ is semilocal (by definition, $S^{-1}A/\big(\text{Jacobson radical}(S^{-1}A)\big)$ is semisimple) and thus its stable rank $\mathrm{sr}(S^{-1}A) = 1$. We shall prove the commutator formula over R-algebras A such that given any maximal ideal M of R, there is a multiplicative set $S \subseteq R\setminus M$ such that $\sup(\mathrm{sr}(S^{-1}A)+1, 3) \leqslant \mathrm{h}(\nu)$. The proof will use a localization method of Bak [Bk1]. Before beginning the proof, we want to reduce to the case where $1\in\sigma_{ii}$ for any $i \in J = \{1,\cdots,n\}$.

5.1. Lemma. *Suppose that subgroups D and $H \leqslant C$ of a group* G *satisfy the following conditions:*

(i) *D is perfect, i.e.* $[D,D]=D$.
(ii) *H is D-perfect, i.e.* $[H,D]=H$.
(iii) *C is normalized by D.*
(iv) $D^C \leqslant HD$.
(v) $[C\cap D, D] = D\cap H$.

Then $[C,D]=[C,DH]=H$.

Proof. First, note that $[CD,HD]\leqslant HD$ by (ii) and (iv) and that the converse inclusion holds by (i) and (ii). In particular, HD is normal in CD. We are interested in the subgroup $L=[C,D]$. It is contained in C by (iii) and in $[CD,HD]=HD$. It follows that $L\leqslant HD\cap C=H(C\cap D)$, and hence $[L,D]\leqslant[H,D][C\cap D,D]\leqslant H$ by (ii) and (v). On the the other hand, $[L,D]\geqslant H$ by (ii). Lemma 1 from [S1] says that if D is perfect and $\big[[C,D],D\big]$ is normal in C then $\big[[C,D],D\big]=[C,D]$. But

$$\big[[C,D],D\big]=[L,D]=[L,D^{DL}]=[L,D^{DH}]=[L,DH]$$

and the last group is normal in CD because L and DH are. Thus $[C,D]=H$. It follows that H is normalized by C and hence $[C,DH]=H$.

Recall that if σ is a net then by definition, $\mathrm{N}(\sigma)$ is the normalizer in $\mathrm{GL}(n,A)$ of $\mathrm{G}(\sigma)$.

5.2. Corollary. *Let σ be a ν-net such that* $\mathrm{h}(\nu)\geqslant 3$. *Suppose that* $\mathrm{E}(\nu)^{\mathrm{N}([\nu]+\sigma)}\leqslant\mathrm{E}([\nu]+\sigma)$ *and the standard commutator formula* $[\mathrm{C}(|\nu(i)|,R,\sigma_{ii}),\mathrm{E}(|\nu(i)|,R)]=\mathrm{E}(|\nu(i)|,R,\sigma_{ii})$ *holds for any $i\in J$. Then*

$$[\mathrm{C}(\nu,\sigma),\mathrm{E}(\nu)]=[\mathrm{C}(\nu,\sigma),\mathrm{E}([\nu]+\sigma)]=\mathrm{E}(\nu,\sigma)$$

and $\mathrm{E}(\nu,\sigma)$ *is normal in* $\mathrm{G}([\nu]+\sigma)$.

Proof. Let $D=\mathrm{E}(\nu)$, $C=\mathrm{C}(\nu,\sigma)$ and $H=\mathrm{E}(\nu,\sigma)$. Clearly $CD\leqslant\mathrm{N}([\nu]+\sigma)$ and $C\cap D$ is contained in the block diagonal group with diagonal blocks $\mathrm{C}(|\nu(i)|,R,\sigma_{ii})$. The first assertion follows now from the previous lemma. To prove the second one, note that $\mathrm{G}(\sigma)$ is normal in $G=\mathrm{G}([\nu]+\sigma)$. Also $\mathrm{E}([\nu]+\sigma)$ is normal in G, because it equals $[G,D]$. It follows that

$$[\mathrm{C}(\nu,\sigma)^G,\mathrm{E}([\nu]+\sigma)]\leqslant\big[\mathrm{C}(\nu,\sigma),\mathrm{E}([\nu]+\sigma)^G\big]^G\leqslant\mathrm{G}(\sigma)^G\leqslant\mathrm{G}(\sigma)\,.$$

By definition of $\mathrm{C}(\nu,\sigma)$, the above implies that $\mathrm{C}(\nu,\sigma)^G\leqslant\mathrm{C}(\nu,\sigma)$. Hence $\mathrm{E}(\nu,\sigma)$ is normal in G.

5.3. Let R be a commutative ring, A an R-algebra, and σ a net over A. For a multiplicative subset S in R, we denote by $S^{-1}R$ the localization of R in S, by $S^{-1}A$ the $S^{-1}R$-algebra $A\otimes_R S^{-1}R$, and by $S^{-1}\sigma$ the net over $S^{-1}A$ of ideals $(S^{-1}\sigma)_{ij}=\sigma_{ij}\otimes_R S^{-1}R$. In the sequel, quotation marks will denote the image of an ideal or group under a homomorphism induced by the canonical homomorphism $A\to S^{-1}A$, where S will be specified by context.

5.4. Lemma. *Let A be an R-algebra, $\sigma\geqslant[\nu]$ a major net over A such that* $\mathrm{h}(\nu)\geqslant 3$, *and $s\in R$. Then for any $u\in\mathrm{E}(\nu,\langle s\rangle^{-1}\sigma)$ there exists a positive integer m such that* “$\mathrm{E}(s^m\sigma)$”$^u\leqslant$ “$\mathrm{E}(s\sigma)$”.

Proof. The proof is the same as for Lemma 4.6 of [Bk1], because σ is major.

5.5. Lemma. *Let A be an R-algebra such that the nilpotent radical of A is trivial and let $s\in R$. Then the ideal sA injects into $\langle s\rangle^{-1}A$ and hence,* $\mathrm{GL}(n,A,sA)$ *injects into* $\mathrm{GL}(n,\langle s\rangle^{-1}A)$.

Proof. Suppose that $s\xi\in sA$ goes to 0 in $\langle s\rangle^{-1}A$. Then there is $m\in\mathbb{N}$ such that $s^ms\xi=0$. It follows that $(s\xi A)^{m+1}=0$. Hence by the condition of the lemma, $s\xi A=0$. Thus $s\xi=0$.

5.6. Lemma. (Vavilov, Stepanov [VvS]). *Let A be a ring such that $\mathrm{h}(\nu) \geqslant \sup(\operatorname{sr} A+1,3)$, $\sigma \geqslant [\nu]$ a major net over A, K a set of representatives of the equivalence classes of ν, and $i \in J$. Then each matrix $a \in \mathrm{N}(\sigma)$ can be decomposed as a product $a = bu$ such that $u \in \mathrm{E}(\nu,\sigma)$, $b \in \mathrm{N}(\sigma)$, and $b_{ik} = 0$ for $k \in K$.*

5.7. Theorem. *Let A be an R-algebra and ν be an equivalence on $J = \{1,\dots,n\}$ (where n can be infinite). Suppose that for any maximal ideal M of R there is a multiplicative set $S \subseteq R \setminus M$ such that $\mathrm{h}(\nu) \geqslant \sup(\operatorname{sr} S^{-1}A+1,3)$. Then for any ν-net σ*

$$[\mathrm{C}(\nu,\sigma),\mathrm{E}(\nu)] = [\mathrm{C}(\nu,\sigma),\mathrm{E}([\nu]+\sigma)] = \mathrm{E}(\nu,\sigma)$$

and $\mathrm{E}(\nu,\sigma)$ is normal in $\mathrm{G}([\nu]+\sigma)$. In particular if A is quasi-finite then the conclusions above hold for $\mathrm{h}(\nu) \geqslant 3$.

Proof. The last assertion of the theorem follows from the other assertions, because a quasi-finite ring is a direct limit $\varinjlim A_i$ of module finite rings A_i, $\mathrm{C}(\nu,\_\,)$, $\mathrm{E}(\nu,\_\,)$, and $\mathrm{G}(\_)$ commute with direct limits, and if A_i is module finite over R_i and $S = R_i \backslash M$ then $\operatorname{sr}(S^{-1}A_i) = 1$. Thus $\sup(\operatorname{sr}(S^{-1}A_i)+1,3) = 3$.

We prove now the rest of the theorem. If ν is the trivial equivalence (cf. 3.3), the result has been proved by L.N.Vaserstein in [Vs1]. By Corollary 5.2, we have to prove only that $\mathrm{E}(\nu)^{\mathrm{N}([\nu]+\sigma)} \leqslant \mathrm{E}([\nu]+\sigma)$. Thus we can assume that $\sigma \geqslant [\nu]$ is major.

Suppose first that the Jacobson radical $\operatorname{Rad} A$ is trivial. Let M be a maximal ideal in R, $h \overset{\nu}{\sim} i$ be distinct indexes from J, $\xi \in A$, and $a \in \mathrm{N}(\sigma)$. We want to show that there exists an $\alpha \in R \setminus M$ such that $d = t_{hi}(\xi\alpha\eta)^a \in \mathrm{E}(\sigma)$ for any $\eta \in R$. Choose a set $K \subset J$ of representatives of the equivalence classes of ν and a multiplicative set S satisfying the conditions of the theorem. By Lemma 5.6, we can decompose the image of a in $\mathrm{GL}(n,S^{-1}A)$ into a product bu where $u \in \mathrm{E}(\nu,S^{-1}\sigma)$, $b \in \mathrm{N}(S^{-1}\sigma)$ and $b_{ik} = 0$ for all $k \in K$. Since $S^{-1}R = \varinjlim \langle t\rangle^{-1}R$ over all $t \in S$, there is an $s \in S$ such that $u \in \mathrm{E}(s^{-1}\sigma)$ and b'_{pq} "σ_{qr}" $b_{rj} \in s^{-1}$ "σ_{pj}". By Lemma 5.5, $\mathrm{GL}(n,A,sA)$ injects into $\mathrm{GL}(n,\langle s\rangle^{-1}A)$. Thus by Lemma 5.4, there is an $m \in \mathbb{N}$ such that "$\mathrm{E}(s^m\sigma)$"$^{u} \leqslant$ "$\mathrm{E}(s\sigma)$".

For an index $l \in J$ consider the matrix

$$c^{(l)} = t_{hi}(\xi s^{2m+2}\eta b_{hl}b'_{lh})^b = e + b'_{*h}\xi s^{2m+2}\eta b_{hl}b'_{lh}b_{i*} =$$

$$= \left[t_{*k}(b'_{*h}\xi s^{m+1}\eta b_{hl})\,,\, t_{k*}(b'_{lh}s^{m+1}b_{i*})\right] t_{k*}(b'_{kh}\xi s^{2m+2}\eta b_{hl}b'_{lh}b_{i*}) \in \text{“}\mathrm{E}(s^m\sigma)\text{”}$$

where $k \in K$, $k \overset{\nu}{\sim} l$ (the formula above is just a variation of the Whitehead lemma). Set $\alpha = s^{2m+2}$. Then $d \in \mathrm{GL}(n,A,sA)$ and the image of d in $\mathrm{GL}(n,\langle s\rangle^{-1}A)$ is equal to

$$t_{hi}\big(\xi s^{2m+2}\eta\big)^a = \left(\prod_{l\in J} c(l)\right)^u \in \text{“}\mathrm{E}(s^m\sigma)\text{”}^{\,u} \leqslant \text{“}\mathrm{E}(s^{m'}\sigma)\text{”}\,.$$

Thus $d \in \mathrm{E}(s\sigma) \leqslant \mathrm{E}(\sigma)$.

Let U be the set of all $t \in R$ such that $t_{hi}(\xi t\eta)^a \in \mathrm{E}(\sigma)$ for any $\eta \in R$. Obviously, U is an ideal in R. If $U \neq R$ then there exists a maximal ideal $M \leqslant R$ containing U. But this is impossible because we have just proved that there exists an element

$\alpha \in U$ which does not belong to M. The contradiction shows that $U = R$ and hence $t_{hi}(\xi)^a \in \mathrm{E}(\sigma)$ for any $h \neq j \in J$, $\xi \in A$ and $a \in \mathrm{N}(\sigma)$.

Let now A be an arbitrary R-algebra, satisfying the conditions of the theorem. Clearly $A/\operatorname{Rad} A$ fulfills the conditions of the theorem as well. Hence $H = [\mathrm{E}(\nu), \mathrm{N}(\sigma)]$ is contained in $H' = \mathrm{E}(\sigma)\,\mathrm{GL}(n, A, \operatorname{Rad} R) \cap \mathrm{G}(\sigma)$ and is normal in $\mathrm{N}(\sigma)$. It is easy to see that $H' \leqslant \mathrm{E}(\sigma)\,\mathrm{D}(n, R)$ where $\mathrm{D}(n, R)$ denotes the subgroup of diagonal matrices and therefore $[H, H] \leqslant [H', H'] \leqslant \mathrm{E}(\sigma)$. On the other hand $[H, H] \geqslant \mathrm{E}(\sigma)$, because $[\mathrm{E}(\sigma), \mathrm{E}(\nu)] = \mathrm{E}(\sigma)$. Thus $\mathrm{E}(\sigma)$ is normal in $\mathrm{N}(\sigma)$, since it is the commutator subgroup [H, H] of the normal subgroup H of $\mathrm{N}(\sigma)$.

5.8. Standard Classification Theorem. *Let A be a quasi-finite ring, ν an equivalence on $J = \{1, \cdots, n\}$ (where n can be infinite) and $\mathrm{h}(\nu) \geqslant 3$. Then the $\mathrm{E}(\nu)$-normal subgroups of $\mathrm{GL}(n, A)$ are in one to one correspondence with the subgroups H of the sandwiches $\mathrm{E}(\nu, \sigma) \leqslant H \leqslant \mathrm{C}(\nu, \sigma)$ where σ ranges over all ν-nets. Moreover each $\mathrm{E}(\nu)$-normal subgroup of $\mathrm{GL}(n, A)$ belongs to precisely one sandwich and the mixed commutator group $[\mathrm{E}(\nu), \mathrm{C}(\nu, \sigma)] = [\mathrm{E}([\nu] + \sigma), \mathrm{C}(\nu, \sigma)] = \mathrm{E}(\nu, \sigma)$.*

Proof. Since the functors $\mathrm{E}(\nu, \_\,), \mathrm{G}(\_)$, and $\mathrm{GL}(n, \_\,)$ commute with direct limits, we can reduce routinely the proof of the theorem to the case A is module finite over a commutative ring R. The commutator formula follows now from Theorem 5.7 and the fact that if M is a maximal ideal in R and $S = R \setminus M$ then $\mathrm{sr}(S^{-1}A) = 1$ because $S^{-1}A$ is semilocal. From the commutator formula, it follows that if a subgroup H is contained in a sandwich $\mathrm{E}(\nu, \sigma) \leqslant H \leqslant \mathrm{C}(\nu, \sigma)$ then $[\mathrm{E}(\nu), H] \leqslant H$. Thus H is $\mathrm{E}(\nu)$-normal. Furthermore a sandwich containing H must be unique, since $\mathrm{E}(\nu, \sigma) = [\mathrm{E}(\nu), \mathrm{E}(\nu, \sigma)] \leqslant [\mathrm{E}(\nu), H] \leqslant [\mathrm{E}(\nu), \mathrm{C}(\nu, \sigma)] = \mathrm{E}(\nu, \sigma)$, i.e. $\mathrm{E}(\nu, \sigma) = [\mathrm{E}(\nu), H]$, and obviously $\mathrm{E}(\nu, \sigma) = \mathrm{E}(\nu, \rho) \Leftrightarrow \sigma = \rho$. The fact that each $\mathrm{E}(\nu)$-normal subgroup of $\mathrm{GL}(n, A)$ is contained in a sandwich is a special case of Golubchik's theorem [G].

Let K be an overgroup of $\mathrm{E}(\nu)$ in $\mathrm{G}(\nu)$. The K-normal subgroups of $\mathrm{GL}(n, A)$ are among the $\mathrm{E}(\nu)$-normal subgroups and the $\mathrm{E}(\nu)$-normal subgroups are classified in Theorem 5.8. If the conjugation action of K on $\mathrm{C}(\nu, \sigma)/\mathrm{E}(\nu, \sigma)$ is trivial for each ν-net σ over A then the set of K-normal subgroups of $\mathrm{GL}(n, A)$ is the same as the set of $\mathrm{E}(\nu)$-normal subgroups. Thus the nilpotent class of the action of K on $\mathrm{C}(\nu, \sigma)/\mathrm{E}(\nu, \sigma)$ is an obstruction to the sets above being equal and provides a rough measure of how much smaller the set of K-normal subgroups is. We shall investigate this nilpotent class in the next section.

5.9. Theorem. *Let A be a quasi-finite ring, ν be an equivalence on $J = \{1, \ldots, n\}$ (where n can be infinite) and $\mathrm{h}(\nu) \geqslant 4$. Then for any major net $\sigma \geqslant [\nu]$, the group $\mathrm{K}_2(\nu, \sigma)$ is contained in the center of $\mathrm{St}(\nu, \sigma)$. Moreover there is a natural action of $\mathrm{G}(\sigma)$ on $\mathrm{St}(\nu, \sigma)$ extending the action of $\mathrm{G}(\sigma)$ on $\mathrm{E}(\nu, \sigma)$ via conjugation.*

Proof. For module finite rings one can take the set

$$V = \{(\alpha, \beta)|\ \alpha = a_{*i} \text{ for some } a \in \mathrm{E}(\nu, \sigma), i \in J,\ \beta \in \sigma_{i*},\ \beta\alpha = 0\}$$

and run it through the proof of the main theorem of [Tu], because σ is major. The quasi-finite case can be obtained by the standard direct limit procedure.

6. ν-NETS AS A CATEGORY WITH DIMENSION

In this section, we define the category of ν-nets over algebras and make it and its full subcategory of major ν-nets into categories with dimension. Then we show that the natural transformation $\pi : \mathrm{St}(\nu, \_\,) \to \mathrm{G}(\_)$ is good on the category of major ν-nets. In the next section, we shall combine the goodness of π with Theorem 2.7 to conclude that for a major ν-net σ, the group $\mathrm{G}^0(\sigma)/\mathrm{E}(\nu, \sigma)$ has nilpotent class $\leqslant \dim(\nu, \sigma)$ whenever $\dim(\nu, \sigma) < \infty$ and $\mathrm{h}(\nu) \geqslant 4$, and that for any ν-net σ, the group $\mathrm{G}^0(\sigma)/\mathrm{E}(\nu, \sigma)$ has nilpotent class $\leqslant \dim(\nu, \sigma)+1$, under the same conditions.

6.1. It will be important in this section to know exactly what is meant by an algebra and a morphism of algebras, because ground rings will frequently change. An **algebra** will mean a pair (R, A) where A is an associative ring with identity and R is a commutative ring with identity, together with a fixed ring homomorphism $R \to \mathrm{center}(A)$ which preserves the identity. We let **((alg))** denote the category whose objects are all pairs (R, A) as above and whose morphisms $f : (R, A) \to (R', A')$ are all pairs of identity preserving ring homomorphisms $R \to R'$ and $A \to A'$ such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

commutes. An algebra (R, A) is called **module finite** (over R), if A is finitely generated as an R-module. Following [Bk1], we shall call an algebra (R, A) **quasi-finite** (over R), if (R, A) is a direct limit $\varinjlim(R_i, A_i)$ in ((alg)) of module finite algebras (R_i, A_i). It is easy to check that this definition is equivalent to the one obtained by postulating that each $R_i = R$. The concept of quasi-finiteness for algebras is stronger than that for rings in §3, because it is not required in §3 that the ring homomorphisms $A_i \to A_j$, where $i \leqslant j$, take the center(A_i) to the center(A_j). Let ν be a fixed equivalence on a subset J of $\mathbb{N}$. Define the category **((ν-nets))** as follows. An object is a triple (R, A, σ) where (R, A) is an algebra and σ is a ν-net in the sense of §3 over A. We shall frequently denote the triple (R, A, σ) by σ, if this does not lead to confusion. A morphism $(R, A, \sigma) \to (R', A', \sigma')$ is an algebra homomorphism $f : (R, A) \to (R', A')$ such that $f(\sigma_{ij}) \leqslant \sigma'_{ij}$ for all $i, j \in J$. We let **((major ν-nets))** denote the full subcategory of ((ν-nets)) consisting of all ν-nets (R, A, σ) such that σ is major in the sense of §3.

We want our structure squares in ((ν-nets)) to be pullback squares, since the functor $\mathrm{G}(\_)$ on ((ν-nets)) preserves pullback squares, and we want localization-completion squares to be included among our structure squares, since such squares behave well under Bass-Serre dimension. However localization-completion squares are not in general pullback squares, but only when R is Noetherian and A is module finite over R. To get around this problem, Bak [Bk4] has introduced the notion of finite completion and constructed localization-finite-completion squares which are always pullback squares and are well behaved under Bass-Serre dimension. We recall what is necessary for the current paper.

6.2. Let ((mod)) denote the category of modules M over commutative rings R. By definition, an object of ((mod)) is a pair (R, M). A morphism $(R, M) \to (R', M')$ is a pair (f, g) consisting of a ring homomorphism $f : R \to R'$ and a homomorphism

$g : M \to M'$ of Abelian groups such that $g(rm) = f(r)g(m)$ for all $r \in R$ and $m \in M$. For an element $s \in R$, let $\hat{M}_{(s)} = \varprojlim_{i \geqslant 0} M/s^iM$ denote the **completion** of M at s and let $\langle s\rangle^{-1}M$ denote the module of $\langle s\rangle$-fractions of M at the multiplicative set $\langle s\rangle = \{s^i \mid i \geqslant 0\}$. Let $I(s,R,M)$ denote $\{(R_i, M_i) \mid R_i \subseteq R$ a finitely generated $\mathbb{Z}$-subalgebra such that $s \in R_i$, $M_i \subseteq M$ a finitely generated R_i-submodule$\}$. Obviously $I(s,R,M)$ is a directed, partially ordered set under inclusion. Define the **finite completion** $\widetilde{M}_{(s)}$ of M at s by $\widetilde{M}_{(s)} = \varinjlim_{i \in I(s,R,M)} (\hat{M}_i)_{(s)}$. Clearly $\widetilde{M}_{(s)}$ is a module over $\widetilde{R}_{(s)}$. Furthermore if R is finitely generated over $\mathbb{Z}$ and M is finitely generated over R then $\widetilde{M}_{(s)} = \hat{M}_{(s)}$. The basic facts concerning finite completion are contained in the next four lemmas.

For a fixed commutative ring R, let $((R\text{-mod}))$ denote the subcategory of $((\text{mod}))$ of all R-modules (R,M) and all morphisms $(f,g) : (R,M) \to (R,M')$ such that $f =$ identity.

6.3. Lemma. [Bk 4] *Finite completion is an exact functor on ((R-mod)).*

Proof. Any 3-term exact sequence in $((R\text{-mod}))$ is a direct limit of 3-term exact sequences of finitely generated modules over finitely generated $\mathbb{Z}$- subalgebras of R. Ordinary completion preserves such exact sequences and a direct limit of exact sequences is exact. The conclusion of the lemma follows.

6.4. If $(R,M) \in ((\text{mod}))$ and $s \in R$ then

$$LFC(s,R,M) := \begin{array}{ccc} (R,M) & \longrightarrow & (\langle s\rangle^{-1}R, \langle s\rangle^{-1}M) \\ \big\downarrow & & \big\downarrow \\ (\widetilde{R}_{(s)}, \widetilde{M}_{(s)}) & \longrightarrow & (\langle s\rangle^{-1}\widetilde{R}_{(s)}, \langle s\rangle^{-1}\widetilde{M}_{(s)}) \end{array}$$

is called the **localization-finite-completion square** of (R,M) at s.

6.5. Lemma. [Bk 4] *Localization-finite-completion squares are fibre squares.*

Proof. $LFC(s,R,M) = \varinjlim_{i \in I(s,R,M)} LFC(s,R_i,M_i)$ and it is well known classically that each $LFC(s,R_i,M_i)$ is a fibre square.

6.6. Definition-Lemma. [Bk 4] *Let (R,A) be an R-algebra and $s \in R$. Define a multiplication on $\widetilde{A}_{(s)}$ as follows. Let $x, y \in \widetilde{A}_{(s)}$. Choose $\alpha, \beta \in I(s,R,A)$ and elements $x' \in (\hat{A}_\alpha)_{(s)}$ and $y' \in (\hat{A}_\beta)_{(s)}$ such that x' and y' represent x and y, respectively. Neither A_α nor A_β is necessarily closed under multiplication in A. Let $\prod_{i\geqslant 0} x_i \in \prod_{i\geqslant 0} A_\alpha$ represent x' and $\prod_{i\geqslant 0} y_i \in \prod_{i\geqslant 0} A_\beta$ represent y'. Choose $\gamma \in I(s,R,A)$ such that $\alpha \leqslant \gamma, \beta \leqslant \gamma$, and $A_\alpha A_\beta \subseteq A_\gamma$. Define $x \circ y$ to be the class in $\widetilde{A}_{(s)}$ of the element of $(\hat{A}_\gamma)_{(s)}$ defined by $\prod_{i\geqslant 0} x_iy_i \in \prod_{i\geqslant 0} A_\gamma$. Then the product $x \circ y$ is independent of all choices made and makes $\widetilde{A}_{(s)}$ into an $\widetilde{R}_{(s)}$-algebra.*

Proof. Straightforward.

6.7. Corollary. [Bk4] *If (R, A) is an R-algebra, $s \in R$, and $\mathfrak{A} \subseteq A$ is an ideal then $\widetilde{\mathfrak{A}}_{(s)} \subseteq \widetilde{A}_{(s)}$ is an ideal.*

Proof. This follows routinely from (6.3), (6.6), and the proof of (6.6).

6.8. Definition-Lemma. *If $(R, A, \sigma) \in ((\nu - nets))$ and $s \in R$ then the square*

$$LFC(s, R, A, \sigma) := \begin{array}{ccc} (R, A, \sigma) & \longrightarrow & (\langle s\rangle^{-1}R, \langle s\rangle^{-1}A, \langle s\rangle^{-1}\sigma) \\ \Big\downarrow & & \Big\downarrow \\ (\widetilde{R}_{(s)}, \widetilde{A}_{(s)}, \widetilde{\sigma}_{(s)}) & \longrightarrow & (\langle s\rangle^{-1}\widetilde{R}_{(s)}, \langle s\rangle^{-1}\widetilde{A}_{(s)}, \langle s\rangle^{-1}\widetilde{\sigma}_{(s)}) \end{array}$$

is called the **localization-finite-completion square** *of (R, A, σ) at s. Let LFC $((\nu - nets))$ (resp. $LFC(($ major $\nu - nets))$) denote the class of all commutative squares in ((ν-nets)) (resp. ((major ν-nets))) which are isomorphic to a localization-finite-completion square. If $S \subseteq R$ is a multiplicative set, give S the directed quasi-ordering (cf. §2) defined by $s \leqslant r \Leftrightarrow \exists u \in S$ such that $su = t$. Let Frac((ν-nets)) (resp. Frac((major ν-nets))) denote the class of all functors $F_{(S,R,A,\sigma)} : S \to$ ((ν-nets)) (resp. ((major ν-nets))), $s \mapsto (\langle s\rangle^{-1}R, \langle s\rangle^{-1}A, \langle s\rangle^{-1}\sigma)$, where (R, A, σ) ranges over all objects of ((ν-nets)) (resp. ((major ν-nets))) and S over all multiplicative subsets of R. Then (((ν-nets)), LFC((ν-nets)), Frac((ν-nets))) and (((major ν-nets)), LFC((major ν-nets)), Frac((major ν-nets))) are categories with structure in the sense of (2.1) whose structure squares are pullback squares. Let $BS(R)$ denote the Bass-Serre dimension of R, cf. [Bk1]. Define*

$$\dim(R, A, \sigma) = \begin{cases} BS(R) & \textit{if } A \textit{ is quasi-finite over } R \\ \infty & \textit{otherwise.} \end{cases}$$

Then dim *is a dimension function in the sense of (2.2) on the categories above. Moreover if* $\dim(R, A, \sigma) = 0$ *then the stable rank* $\mathrm{sr}(A) = 1$.

Remark An important part of the definition of dim is that $\dim(R, A, \sigma) < \infty$ implies (R, A) in quasi-finite.

Proof. Obviously we have categories with structure. Lemma 6.5 shows that all structure squares are pullback squares. It is clear that if (R, A) is quasi-finite then for any element $s \in R$ and any multiplicative set $S \subseteq R$, every algebra in the localization-finite-completion square $LFC(s, R, A)$ and the algebra $(S^{-1}R, S^{-1}A)$ are quasi-finite. It follows therefore from the Induction Lemma 4.17 of [Bk1] that dim is a dimension function on both categories with structure. Suppose $\dim(R, A, \sigma) = 0$. Since A is quasi-finite over R, A is a direct limit of module finite R-subalgebras A_i. Since $BS(R) = 0$, it follows that R is semilocal. Thus A_i is semilocal. Thus $\mathrm{sr}(A_i) = 1$. But a direct limit of stable rank 1 rings has stable rank 1. Thus $\mathrm{sr}(A) = 1$.

For the proof of the goodness lemma below we shall use smash products of rings and ν-nets. By definition, the **smash product** $A \rtimes A$ of a ring A with itself has as elements those of the Cartesian product $A \times A$. Addition is defined componentwise and multiplication by the rule $(b, a)(b', a') = (bb' + ab' + ba', aa')$. If σ and ρ are ν-nets over A such that $\sigma\rho \subseteq \rho$ and $\rho\sigma \subseteq \rho$ then one defines the **smash product** $\rho \rtimes \sigma$ similarly and $\rho \rtimes \sigma$ becomes a ν-net over $A \rtimes A$. If σ and ρ are major

ν-nets, then so is $\rho \rtimes \sigma$. Let $f : A \rtimes A \to A$, $(b, a) \mapsto a$, and $g : A \rtimes A \to A$, $(b, a) \mapsto b + a$. Both f and g are ring homomorphisms and are commonly split by the homomorphism $A \to A \rtimes A$, $a \mapsto (0, a)$. They induce morphisms $f : \rho \rtimes \sigma \to \sigma$ and $g : \rho \rtimes \sigma \to \rho + \sigma$ of ν-nets, and f is split by the ν-net map $\sigma \to \rho \rtimes \sigma$, $a \mapsto (0, a)$. Moreover if $\rho \subseteq \sigma$ then g is split by the same ν-net map.

Let $\rho \subseteq \sigma$ be ν-nets over A such that $\sigma\rho \subseteq \rho$ and $\rho\sigma \subseteq \rho$. If $L : ((\nu\text{-nets})) \longrightarrow ((\text{groups}))$ is a functor, set $L(\sigma, \rho) = \operatorname{Ker}\bigl(L(f) : L(\rho \rtimes \sigma) \to L(\sigma)\bigr)$. Give $L(\sigma, \rho)$ the $L(\sigma)$-action defined by the conjugation action of $L(\rho \rtimes \sigma)$ on $L(\sigma, \rho)$ and the split exact sequence $L(\sigma, \rho) \hookrightarrow L(\rho \rtimes \sigma) \xrightarrow{L(f)} L(\sigma)$. Let $h_{\sigma,\rho}$ denote the composition of the homomorphisms $L(\sigma, \rho) \hookrightarrow L(\rho \rtimes \sigma) \xrightarrow{L(g)} L(\rho + \sigma) = L(\sigma)$. Obviously $h_{\sigma,\rho} : L(\sigma, \rho) \to L(\sigma)$ is a precrossed module whose associated smash product group $L(\sigma, \rho) \rtimes L(\sigma)$ (defined just prior to (1.11)) has a canonical identification $L(\sigma, \rho) \rtimes L(\sigma) \cong L(\rho \rtimes \sigma)$ with $L(\rho \rtimes \sigma)$.

Let $\mathrm{G}(\_)$, $\mathrm{E}(\nu, \_\,)$, and $\mathrm{St}(\nu, \_\,) : ((\nu\text{-nets})) \to ((\text{groups}))$ be defined as in §3 and let $\pi : \mathrm{St}(\nu, \_\,) \to \mathrm{G}(\_)$ denote the canonical natural transformation. Let $\rho \subseteq \sigma$ be ν-nets on A such that $\sigma\rho \subseteq \rho$ and $\rho\sigma \supseteq \rho$. The proof of the goodness lemma below will use tacitly the following facts.

Since the square

$$\begin{array}{ccc}
\rho \rtimes \sigma & \xrightarrow{f} & \sigma \\
{\scriptstyle g}\big\downarrow & & \big\downarrow \\
\sigma & \longrightarrow & \sigma/\rho
\end{array}$$

is a fibre square of ν-nets and the functor $\mathrm{G}(\_)$ preserves fibre squares, it follows that the map $h_{\sigma,\sigma} : \mathrm{G}(\sigma, \sigma) \to \mathrm{G}(\sigma)$ above is an isomorphism. Although the functor $\mathrm{St}(\nu, \_\,)$ does not preserve fibre squares, using the relations (3.5) defining $\mathrm{St}(\nu, \_\,)$, one can show (see for example the proof of [M] (6.1)) that $h_{\sigma,\sigma} : \mathrm{St}(\nu, \sigma, \sigma) \to \mathrm{St}(\nu, \sigma)$ is an isomorphism providing $\mathrm{h}(\nu) \geqslant 4$ and σ is major. It follows that $h_{\sigma,\sigma} : \mathrm{E}(\nu, \sigma, \sigma) \to \mathrm{E}(\nu, \sigma)$ is an isomorphism under the same conditions.

6.9. Goodness Lemma. *The natural transformation π above is good in the sense of (2.6) on the category with structure $\bigl($((major ν-nets)), LFC((major ν-nets)), Frac((major ν-nets))$\bigr)$, providing* $\mathrm{h}(\nu) \geqslant 4$.

Proof. It is straightforward to check that St and G commute with arbitrary direct limits. Any localization-finite-completion square $LFC(s, R, A, \sigma)$ such that $\dim(R, A, \sigma) < \infty$ is a direct limit of subsquares $LFC(s, R_i, A_i, \sigma_i)$ where R_i is Noetherian and A_i is module finite over R_i. It suffices to show that each square $LFC(s, R_i, A_i, \sigma_i)$ is π-good. Given i, one can show as in the proof of Lemma 4.10 in [Bk1] that there is an $n \geqslant 0$ such that $s^n A_i$ is s-torsion free. It suffices therefore to prove the following:

(**6.9.1**) If (R, A) is quasi-finite, $n \geqslant 0$, and σ is a major ν-net such that $s^n\sigma$ is s-torsion free then $LFC(s, R, A, \sigma)$ is a π-good square.

By definition, it suffices to show that the commutative cube of precrossed modules

$$\begin{tikzcd}[column sep=small]
 & \big(\mathrm{St}(\nu,\sigma,s^n\sigma) \xrightarrow{h} \mathrm{St}(\nu,\sigma)\big) \arrow[rr] \arrow[dl] \arrow[dd] & & \big(\mathrm{St}(\nu,\langle s\rangle^{-1}\sigma) \xrightarrow{1} \mathrm{St}(\nu,\langle s\rangle^{-1}\sigma)\big) \arrow[dl] \arrow[dd] \\
\big(\mathrm{St}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)}) \xrightarrow{\tilde h} \mathrm{St}(\nu,\tilde\sigma_{(s)})\big) \arrow[rr] \arrow[dd] & & \big(\mathrm{St}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)}) \xrightarrow{1} \mathrm{St}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)})\big) \arrow[dd] & \\
 & \big(\mathrm{G}(s^n\sigma) \xrightarrow{h} \mathrm{G}(\sigma)\big) \arrow[rr] \arrow[dl] & & \big(\mathrm{G}(\langle s\rangle^{-1}\sigma) \xrightarrow{1} \mathrm{G}(\langle s\rangle^{-1}\sigma)\big) \arrow[dl] \\
\big(\mathrm{G}(s^n\tilde\sigma_{(s)}) \xrightarrow{\tilde h} \mathrm{G}(\tilde\sigma_{(s)})\big) \arrow[rr] & & \big(\mathrm{G}(\langle s\rangle^{-1}\tilde\sigma_{(s)}) \xrightarrow{1} \mathrm{G}(\langle s\rangle^{-1}\tilde\sigma_{(s)})\big) &
\end{tikzcd}$$

satisfies (1.11)(i), (ii) and (iii), and that the associated cube of smash product groups

$$\begin{tikzcd}[column sep=small]
 & \mathrm{St}(\nu,s^n\sigma\rtimes\sigma) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{St}(\nu,\langle s\rangle^{-1}\sigma\rtimes\langle s\rangle^{-1}\sigma) \arrow[dl] \arrow[dd] \\
\mathrm{St}(\nu,s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)}) \arrow[rr] \arrow[dd] & & \mathrm{St}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)}\rtimes\langle s\rangle^{-1}\tilde\sigma_{(s)}) \arrow[dd] & \\
 & \mathrm{G}(s^n\sigma\rtimes\sigma) \arrow[rr] \arrow[dl] & & \mathrm{G}(\langle s\rangle^{-1}\sigma\rtimes\langle s\rangle^{-1}\sigma) \arrow[dl] \\
\mathrm{G}(s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)}) \arrow[rr] & & \mathrm{G}(\langle s\rangle^{-1}\tilde\sigma_{(s)}\rtimes\langle s\rangle^{-1}\tilde\sigma_{(s)}) &
\end{tikzcd}$$

satisfies (1.10)(i) and (ii) and $\mathrm{Im}\big(\mathrm{St}(\nu,s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})\to\mathrm{G}(s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})\big)$ is a normal subgroup of $\mathrm{G}(s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})$. Condition (1.10)(i) follows from Theorem 5.9. Condition (1.10) (ii) follows from the standard equations for elementary transvections, which are spelled out under (3.5)(1)–(3), and the fact that $\mathrm{h}(\nu)\geqslant 3$ and σ is major. A little extra care has to be taken in the case of $\mathrm{E}(\nu,s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})$, because $s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)}$ is not major, but only $\tilde\sigma_{(s)}$. The normality of $\mathrm{E}(\nu,s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})$ in $\mathrm{G}(s^n\tilde\sigma_{(s)}\rtimes\tilde\sigma_{(s)})$ follows from Theorem 5.7. Condition (1.11)(ii) follows from the fact that localization-finite-completion squares are fibre squares by Lemma 6.5 and G preserves fibre squares. Condition (1.11)(i) has 2 parts. We demonstrate first the injectivity of $\mathrm{E}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)})\to\mathrm{E}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)})$. Since $s^n\tilde\sigma_{(s)}$ is s-torsion free and finite-completion is exact by Lemma 6.3, it follows that $s^n\tilde\sigma_{(s)}=\big(\widetilde{s^n\sigma}\big)_{(s)}$ is s-torsion free. Thus the canonical map $s^n\tilde\sigma_{(s)}\to\langle s\rangle^{-1}\tilde\sigma_{(s)}$ is injective. Thus the induced map $\mathrm{G}(s^n\tilde\sigma_{(s)})\to\mathrm{G}(\langle s\rangle^{-1}\tilde\sigma_{(s)})$ is injective. Thus $\mathrm{E}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)})\to\mathrm{E}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)})$ is injective. To demonstrate that the rest of condition (1.11)(i) and all of condition (1.11)(iii) are satisfied, we can reduce to the case that A is module finite over a Noetherian ring R, because any quasi-finite algebra is a direct limit of such algebras and the functors involved commute with direct limits. Under the assumption that A is module finite over a Noetherian ring R, conditions (1.11)(i) and (iii) are verified respectively in (6.12) and (6.13) below. This completes the proof Lemma 6.9.

6.10. Suppose that A is module finite over a Noetherian ring R. Let σ be a major ν-net over A. Let $s\in R$ and let

$$\begin{tikzcd}
\sigma \arrow[r, "\varphi"] \arrow[d, "\Psi"'] & \langle s\rangle^{-1}\sigma \arrow[d, "\Psi'"] \\
\hat\sigma_{(s)} \arrow[r, "\varphi'"'] & \langle s\rangle^{-1}\hat\sigma_{(s)}
\end{tikzcd}$$

denote the localization-completion square of (R, A, σ) at s. Our next goal is to show that for any nonnegative integer n, the canonical map

$$\theta : \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma) \to \mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)})/\,\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)})$$

of coset spaces is bijective.

For a major ν-net τ on A, let $Y^{\pm 1}(\tau)$ denote the set of all formal products $\prod_{k=1}^{l} y_{i_k j_k}(\xi_k)^{\eta_k}$ where $\eta_k = \pm 1$ and $\xi_k \in \tau_{i_k j_k}$. Consider the following formal products

(i) $y_{ij}(\xi)y_{ij}(\xi)^{-1}$, $y_{ij}(\xi)^{-1}y_{ij}(\xi)$
(ii) $y_{ij}(\xi)y_{ij}(\zeta)y_{ij}(\xi+\zeta)^{-1}$
$y_{ij}(\xi)y_{pq}(\zeta)y_{ij}(\xi)^{-1}y_{pq}(\zeta)^{-1}$, $i \neq q$ and $j \neq p$
$y_{ij}(\xi)y_{jq}(\zeta)y_{ij}(\xi)^{-1}y_{jq}(\zeta)^{-1}y_{iq}(\xi\zeta)^{-1}$, $i \neq q$.

Let $\mathbf{Y}^{\pm 1}(\tau)$ denote the union of $Y^{\pm 1}(\tau)$ and the empty product. $\mathbf{Y}^{\pm 1}(\tau)$ has an obvious rule of composition $\mathbf{Y}^{\pm 1}(\tau) \times \mathbf{Y}^{\pm 1}(\tau) \to \mathbf{Y}^{\pm 1}(\tau)$, $\left(\prod_{k=1}^{\ell} y_{i_k j_k}(\xi_k)^{\eta_k}, \prod_{k=1}^{\ell'} y'_{i_k j_k}(\xi'_k)^{\eta'_k}\right) \mapsto \prod_{k=1}^{\ell} y_{i_k j_k}(\xi_k)^{\eta_k} \prod_{k=1}^{\ell'} y'_{i_k j_k}(\xi'_k)^{\eta'_k}$, such that if $\emptyset \in \mathbf{Y}^{\pm 1}(\tau)$ denotes the empty product and $y \in \mathbf{Y}^{\pm 1}(\tau)$ then $y\emptyset = \emptyset y = y$. If we form equivalence classes on $\mathbf{Y}^{\pm 1}(\tau)$ by deleting formal products under (i) from elements of $\mathbf{Y}^{\pm}(\tau)$ or inserting formal products under (i) into elements of $\mathbf{Y}^{\pm 1}(\tau)$ then the rule of composition above is well defined on equivalence classes and the result is the free group on the symbols $y_{ij}(\xi)$ such that $\xi \in \tau_{ij}$. If we form equivalence classes by deleting formal products under (i) and (ii) from elements of $\mathbf{Y}^{\pm 1}(\tau)$ or inserting formal products under (i) and (ii) into elements of $\mathbf{Y}^{\pm 1}(\tau)$ then the composition on $\mathbf{Y}^{\pm 1}(\tau)$ is well defined on these equivalence classes and the result is the group $\mathrm{St}(\nu, \tau)$. If X is any set then it is evident that a map $\theta' : \mathbf{Y}^{\pm 1}(\tau) \to X$ induces a map $\mathrm{St}(\nu, \tau) \to X \Leftrightarrow$ it is constant on the equivalence classes above. (Note that if θ' is constant on equivalence classes above then all of the formal products under (i) and (ii) and the empty product $\emptyset$ go to the same element of X. It is often helpful to think of this element as the base point of X.)

Let $Y^{\pm 1} = Y^{\pm 1}(\langle s\rangle^{-1}\hat{\sigma}_{(s)})$ and $\mathbf{Y}^{\pm 1} = \mathbf{Y}^{\pm 1}(\langle s\rangle^{-1}\hat{\sigma}_{(s)})$. Following Bak [Bk2] (7.13), we call an element $x = \prod_{k=1}^{\ell} x_{i_k j_k}(\xi_k) \in \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$ a **good approximation** for $y = \prod_{k=1}^{\ell} y_{i_k j_k}(\hat{\xi}_k)^{\eta_k} \in Y^{\pm 1}$ if $\eta_k\hat{\xi}_k - \Psi'(\xi_k) \in \varphi'((s^{2^k+n}\hat{\sigma}_{(s)})_{i_k j_k})$. It is routine to check that given $k \geqslant 0$ and $\hat{\xi} \in (\langle s\rangle^{-1}\hat{\sigma}_{(s)})_{ij}$, there is an element $\xi \in (\langle s\rangle^{-1}\sigma)_{ij}$ such that $\hat{\xi} - \Psi'(\xi) \in \varphi'((s^{2^k+n}\hat{\sigma}_{(s)})_{ij})$. Thus any element of $Y^{\pm 1}$ has a good approximation. The **good approximation** of the empty formal product $\emptyset \in \mathbf{Y}^{\pm 1}$ is by definition the element $1 \in \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$. The next lemma shows that the map

$$\begin{aligned} \theta' : \mathbf{Y}^{\pm 1} &\longrightarrow \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma) \\ y &\longmapsto \varphi(\mathrm{St}(\nu, s^n\sigma))\big(\text{good approximation}(y)\big) \end{aligned}$$

is well defined, providing $\mathrm{h}(\nu) \geqslant 3$, and that it induces a retract for θ.

6.11. Lemma. *Suppose in the setting of (6.10) that* $\mathrm{h}(\nu) \geqslant 3$. *If* $x, x' \in \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$ *are good approximations for the same element of* $\mathbf{Y}^{\pm 1}$ *then* $x'x^{-1} \in \varphi\big(\mathrm{St}(\nu, \sigma, s^n\sigma)\big)$. *It follows that the map* θ' *is well defined. Furthermore* θ' *is constant on equivalence classes of* $\mathbf{Y}^{\pm 1}(\tau)$ *and sends formal products representing naturally elements of* $\varphi'(\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)}))$ *to the trivial coset* $\varphi(\mathrm{St}(\nu, \sigma, s^n\sigma))$ *of* $\mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma)$. *It follows that* θ' *induces a map*

$$\mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)})/\,\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)}) \to \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma)$$

which is a retract of the canonical map θ *in the other direction.*

Proof. Suppose $x = \prod\limits_{k=1}^{\ell} x_{i_k j_k}(\xi_k)$ and $x' = \prod\limits_{k=1}^{\ell} x_{i_k j_k}(\xi'_k)$. For $1 \leqslant h \leqslant \ell$, set

$$x_h = \prod_{k=h}^{l} x_{i_k j_k}(\xi_k) \qquad \textit{and} \qquad x'_h = \prod_{k=h}^{l} x_{i_k j_k}(\xi'_k),$$

so that $x = x_1$ and $x' = x'_1$. Since x and x' are good approximations of the same element of $Y^{\pm 1}$, we have the inclusion $\psi'\left((\xi'_k - \xi_k)s^{-2^k - n}\right) \in \varphi'\left((\hat{\sigma}_s)_{i_k j_k}\right)$. Since the localization-completion square of ν-nets in (6.10) is fibred, it follows that $(\xi'_k - \xi_k)s^{-2^k - n} \in \varphi(\sigma_{i_k j_k})$. Thus $\xi'_k - \xi_k \in \varphi(s^{2^k+n}\sigma_{i_k j_n})$. We shall prove by induction on $h = l, l-1, \dots, 1$ that $x'_h x_h^{-1} \in \varphi(\mathrm{St}(\nu, \sigma, s^{2^h+n}\sigma))$. It has just been shown that the assertion is true if $h = l$. Consider the element

$$x'_h x_h^{-1} = x_{i_h j_h}(\xi'_h - \xi_h) x_{i_h j_h}(\xi_h) x'_{h+1} x_{h+1}^{-1} x_{i_h j_h}(-\xi_h).$$

Obviously the first factor belongs to $\varphi(\mathrm{St}(\nu, \sigma, s^{2^h+n}\sigma))$. By the induction assumption, $x'_{h+1} x_{h+1}^{-1} \in \varphi(\mathrm{St}(\nu, \sigma, s^{2^{h+1}+n}\sigma))$. Now, the same arguments as in Lemma 4.6 of [Bk1] or in Lemma 7.9 of [Bk2] show that $x'_h x_h^{-1} \in \varphi(\mathrm{St}(\nu, \sigma, s^{2^h+n}\sigma))$. This is where the assumption that $\mathrm{h}(\nu) \geqslant 3$ is required.

It follows now that θ' induces a well defined map $\mathbf{Y}^{\pm 1} \to \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma)$. If the formal product y' is obtained from the formal product y by either deleting from y a formal product under (i) or (ii) or inserting into y a formal product under (i) or (ii) then it is clear that we can pick a good approximation x for y and x' for y' such that $x = x'$ in $\mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$. For example, suppose $y' = \emptyset$ and $y = y_{ij}(\hat{\xi}) y_{ij}(\hat{\zeta}) y_{ij}(\hat{\xi} + \hat{\zeta})^{-1}$. Then we could pick $x = x_{ij}(\xi) x_{ij}(\zeta) x_{ij}(-\xi - \zeta)$ where $\hat{\xi} - \Psi'(\xi), \hat{\zeta} - \Psi'(\zeta) \in \varphi'(s^{8+n}\hat{\sigma}_{(s)})$ and $x' = 1$. Clearly $x = 1 = x'$. Thus θ' induces a well defined map $\mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)}) \to \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma)$. Suppose that $y = \prod\limits_{k=1}^{\ell} y_{i_k j_k}(\hat{\xi}_k) \in Y^{\pm 1}$ such that $\hat{\xi}_k \in \varphi'(s^n\hat{\sigma}_{(s)})$ $(1 \leqslant k \leqslant \ell)$. Then any good approximation $x = \prod\limits_{k=1}^{\ell} x_{i_k j_k}(\xi_k)$ of y has the property that each $\xi_k \in \varphi(s^n\sigma)$. The proof has been given already in the first paragraph above. Thus $x \in \varphi(\mathrm{St}(\nu, \sigma, s^n\sigma))$. Finally suppose $y \in Y^{\pm 1}$ represents naturally an element of $\varphi'\big(\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)})\big)$ and x is a good approximation for y. Then one checks similarly that $x \in \varphi\big(\mathrm{St}(\nu, \sigma, s^n\sigma)\big)$. Thus θ' induces a well defined map

$$\mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)})/\,\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)}) \to \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)/\,\mathrm{St}(\nu, \sigma, s^n\sigma).$$

Moreover if $x = \prod_{k=1}^{\ell} x_{i_k j_n}(\xi_k) \in \mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$ then x is a good approximation of the formal product $\prod_{k=1}^{\ell} y_{i_k j_k}(\Psi'(\xi_k))$ which represents $\Psi'(x)$. Thus θ' induces a retract of θ.

The next result is a generalization of the special case in [Bk2] (7.16) where ν has only one equivalence class.

6.12. Theorem. *Suppose in the setting of (6.10) that* $\mathrm{h}(\nu) \geqslant 3$. *Then* θ *is a bijection, with inverse the map induced by* θ' *in (6.11).*

Proof. In view of (6.11), it suffices to show that θ is surjective. Let $1 \neq \hat{y} \in \mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)})$ and let $y = \prod_{k=1}^{\ell} y_{i_k j_k}(\hat{\xi}_k) \in Y^{\pm 1}$ be a representative of $\hat{y}$. Let x be a good approximation of y. The proof will be complete, if we can show that $\hat{y}\Psi'(x)^{-1} \in \varphi'(\mathrm{St}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)}))$. But this is proved routinely using the methodology in the first paragraph of the proof of (6.11).

6.13. Lemma. *Suppose in the setting of (6.10) that* $\mathrm{h}(\nu) \geqslant 3$. *Then the canonical map* $\tilde{\mathrm{E}}(\nu,\sigma,s^n\sigma)/\mathrm{E}(\nu,\sigma,s^n\sigma) \to \tilde{\mathrm{E}}(\nu,\sigma)/\mathrm{E}(\nu,\sigma)$ *is surjective.*

Proof. Let $z \neq 1 \in \tilde{\mathrm{E}}(\nu,\sigma)$. By definition, $\Psi(z) \in \mathrm{E}(\nu, \tilde{\sigma}_{(s)})$. Reformulate the concept good approximation above by replacing $\mathrm{St}(\nu, \langle s\rangle^{-1}\hat{\sigma}_{(s)})$ by $\mathrm{St}(\nu, \hat{\sigma}_{(s)})$, $\mathrm{St}(\nu, \langle s\rangle^{-1}\sigma)$ by $\mathrm{St}(\nu,\sigma)$, and $Y^{\pm 1}(\langle s\rangle^{-1}\hat{\sigma}_{(s)})$ by $Y^{\pm 1}(\hat{\sigma}_{(s)})$. Let $y \in Y^{\pm 1}(\hat{\sigma}_s)$ represent $\Psi(z)$. Let $x' \in \mathrm{St}(\nu,\sigma)$ be a good approximation for y. Let x denote the image of x' in $\mathrm{E}(\nu,\sigma)$. The methodology in the first paragraph of the proof of (6.11) shows that $\Psi(z)\Psi(x)^{-1} \in \mathrm{E}(\nu, \hat{\sigma}_{(s)}, s^n\hat{\sigma}_{(s)})$. Thus $zx^{-1} \in \tilde{\mathrm{E}}(\nu,\sigma,s^n\sigma)$, by definition. Clearly zx^{-1} is a representative of the class of z in $\tilde{\mathrm{E}}(\nu,\sigma)/\mathrm{E}(\nu,\sigma)$.

Let σ and ρ be ν-nets over A such that $\sigma\rho \subseteq \rho$ and $\rho\sigma \subseteq \rho$. Define

$$\begin{aligned}
\mathrm{K}_1(\nu,\sigma,\rho) &= \mathrm{Coker}\,(\mathrm{St}(\nu,\sigma,\rho) \to \mathrm{G}(\sigma,\rho)) \\
\mathrm{K}_2(\nu,\sigma,\rho) &= \mathrm{Ker}\,(\mathrm{St}(\nu,\sigma,\rho) \to \mathrm{G}(\sigma,\rho))\,.
\end{aligned}$$

Note that $h_{\sigma,\rho} : \mathrm{G}(\sigma,\rho) \to \mathrm{G}(\sigma+\rho)$ maps $\mathrm{G}(\sigma,\rho)$ isomorphically onto $\mathrm{G}(\rho) \leqslant \mathrm{G}(\sigma+\rho)$.

6.14. Theorem. *Let* A *be a quasi-finite* R*-algebra and* σ *a major* ν*-net over* A *such that* $\mathrm{h}(\nu) \geqslant 4$. *Let* $n \geqslant 0$ *and* $s \in R$ *be such that* $s^n\sigma$ *is* s*-torsion free. Then corresponding to a localization-finite-completion square* $LFC(s,R,A,\sigma)$, *there is a Mayer-Vietoris sequence of nonstable net K-groups*

$$\begin{tikzcd}
\mathrm{K}_2(\nu,\langle s\rangle^{-1}\sigma) \arrow[dr] & & & \mathrm{K}_1(\nu,\langle s\rangle^{-1}\sigma) \arrow[dr] & \\
& \mathrm{K}_2(\nu,\langle s\rangle^{-1}\tilde{\sigma}_{(s)}) \arrow[r] & \mathrm{K}_1(\nu,\sigma,s^n\sigma) \arrow[ur] \arrow[dr] & & \mathrm{K}_1(\nu,\langle s\rangle^{-1}\tilde{\sigma}_{(s)}). \\
\mathrm{K}_2(\nu,\tilde{\sigma}_{(s)},s^n\tilde{\sigma}_{(s)}) \arrow[ur] & & & \mathrm{K}_1(\nu,\tilde{\sigma}_{(s)},s^n\tilde{\sigma}_{(s)}) \arrow[ur] &
\end{tikzcd}$$

Moreover the Mayer-Vietoris sequence is natural in LFC's.

Proof. The conclusion of Theorem 1.8 provides the natural Mayer-Vietoris sequence we want. The hypotheses of Theorem 1.8 demand precisely that the commutative cube

$$\begin{tikzcd}
& \mathrm{St}(\nu,\sigma,s^n\sigma) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{St}(\nu,\langle s\rangle^{-1}\sigma) \arrow[dl] \arrow[dd] \\
\mathrm{St}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)}) \arrow[rr] \arrow[dd] & & \mathrm{St}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)}) \arrow[dd] & \\
& \mathrm{G}(s^n\sigma) \arrow[rr] \arrow[dl] & & \mathrm{G}(\langle s\rangle^{-1}\sigma) \arrow[dl] \\
\mathrm{G}(s^n\tilde\sigma_{(s)}) \arrow[rr] & & \mathrm{G}(\langle s\rangle^{-1}\tilde\sigma_{(s)}) &
\end{tikzcd}$$

satisfy conditions (1.4) (i) - (iv). By Lemma 1.5, it suffices to show that the canonical map $\mathrm{St}(\nu,\langle s\rangle^{-1}\sigma)/\mathrm{St}(\nu,\sigma,s^n\sigma)\to\mathrm{St}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)})/\mathrm{St}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)})$ is bijective, the map $\mathrm{E}(\nu,\tilde\sigma_{(s)},s^n\tilde\sigma_{(s)})\to\mathrm{E}(\nu,\langle s\rangle^{-1}\tilde\sigma_{(s)})$ injective, and condition (1.4)(iv) satisfied. But this has been shown already in the proof of (6.9.1).

7. Nilpotent structure of net K_1 and sandwich classification

Let $\mathrm{St}(\nu,\_\,)$ and $\mathrm{G}(\_)$: ((ν-nets)) →((groups)) denote the Steinberg net group functor and net group functor, respectively, defined in §3. Let $\pi:\mathrm{St}(\nu,\_\,)\to\mathrm{G}(\_)$ denote the canonical natural transformation. Let $\mathrm{G}=\mathrm{G}^{-1}\geqslant\mathrm{G}^0\geqslant\cdots\geqslant\mathrm{G}^i\geqslant\cdots\geqslant\mathrm{E}(\nu,\_\,)$ denote the dim-filtration of π on G defined in (2.5) for the dimension function dim defined in (6.8).

7.1. Lemma. *If* $\mathrm{G}=\mathrm{G}^{\langle -1\rangle}\geqslant\mathrm{G}^{\langle 0\rangle}\geqslant\mathrm{G}^{\langle 1\rangle}\geqslant\cdots$ *denotes the dim-filtration of* π *restricted to ((major* ν*-nets)) on G restricted to ((major* ν*-nets)) then* $\mathrm{G}^{\langle i\rangle}$ *and* G^i *agree on ((major* ν*-nets)).*

Proof. If $\sigma\to\rho$ is a morphism of ν-nets and σ is major then so is ρ.

Let J be a group which is operating automorphically on a group H. A chain $H=H^0\geqslant H^1\geqslant\cdots$ of subgroups of H is called a **descending J-central series**, if each H^i is invariant under the action of J and the subgroup $D_J(H^i):=\langle({}^{x}h)h^{-1}\,|\,x\in J,\,h\in H\rangle\leqslant H^{i+1}$.

The main result of the section is the following theorem which generalizes results in [Bk1] §5, particularly Theorem 6.5, concerning the special case that ν has only one equivalence class.

7.2. Theorem. *Let* σ *denote any* ν*-net. If* $\mathrm{h}(\nu)\geqslant 2$ *then* $\mathrm{G}^0(\sigma)$ *is a normal subgroup of* $\mathrm{G}([\nu]+\sigma)$ *and the action*

$$\mathrm{G}([\nu]+\sigma)\curvearrowright\mathrm{G}(\sigma)/\mathrm{G}^0(\sigma)$$

of $\mathrm{G}([\nu]+\sigma)$ *on* $\mathrm{G}(\sigma)/\mathrm{G}^0(\sigma)$ *by conjugation is trivial. If* $\mathrm{h}(\nu)\geqslant 4$ *then the filtration*

$$\begin{aligned}\mathrm{G}(\sigma)\geqslant\mathrm{G}(\sigma)\cap\mathrm{G}^0([\nu]+\sigma)\geqslant\mathrm{G}(\sigma)\cap\mathrm{G}^1([\nu]+\sigma)\geqslant\cdots\\ \geqslant\mathrm{G}(\sigma)\cap\mathrm{G}^i([\nu]+\sigma)\geqslant\cdots\geqslant\mathrm{E}(\nu,\sigma)\end{aligned}$$

is a descending $\mathrm{G}^0([\nu]+\sigma)$*-central series of normal subgroups of* $\mathrm{G}([\nu]+\sigma)$ *such that if* $i\geqslant\dim(\sigma)$ *then*

$$\mathrm{G}(\sigma)\cap\mathrm{G}^i([\nu]+\sigma)=\mathrm{G}(\sigma)\cap\mathrm{E}(\nu,[\nu]+\sigma),\quad\textit{and}$$
$$[\mathrm{G}(\sigma)\cap\mathrm{G}^i([\nu]+\sigma),\mathrm{C}(\nu,\sigma)]=\mathrm{E}(\nu,\sigma).$$

Furthermore under the notation of §4, the stabilization maps

$$\mathrm{G}(\sigma^{(n)})/\,\mathrm{G}^i(\sigma^{(n)}) \to \mathrm{G}(\sigma^{(n+1)})/\,\mathrm{G}^i(\sigma^{(n+1)}), \text{ and}$$
$$\mathrm{G}(\sigma^{(n)})/\,\mathrm{G}(\sigma^{(n)}) \cap \mathrm{G}^i(([\nu]+\sigma)^{(n)}) \to \mathrm{G}(\sigma^{(n+1)})/\,\mathrm{G}(\sigma^{(n+1)}) \cap \mathrm{G}^i(([\nu]+\sigma)^{(n+1)})$$

are injective whenever $|\nu(n)| \geqslant \sup(i+2,2)$. *(Notice that this an injective stability result without any stability conditions on* σ*.) Finally if the underlying ring* A *of* σ *is commutative,* σ *is major, and* $\mathrm{SL}_n(A)$ *denotes the usual special linear group of rank* n *then*

$$\mathrm{G}(\sigma^{(n)}) \cap \mathrm{G}^0([\nu]+\sigma) = \mathrm{G}(\sigma^{(n)}) \cap \mathrm{SL}_n(A).$$

Proof. The proof of the last assertion is the same as that in [Bk1] (3.7) for the special case that ν has just one equivalence class.

The stabilization assertions follow directly from Theorem 4.3 and the definition of G^i.

To prove the first assertion, it suffices, by the last assertion in (2.5), to establish the special case that $\dim(\sigma)=0$. Let A denote the underlying ring of σ. By (6.8), $\mathrm{sr}(A)=1$. Thus the special case follows from (4.3)(iii).

It remains to prove the second assertion. To establish the descending $\mathrm{G}^0([\nu]+\sigma)$-central series, we can assume that $\sigma=[\nu]+\sigma$ is major. We would be finished, if we could apply Theorem 2.7. The hypotheses of (2.7) are precisely that the natural transformation $\pi : \mathrm{St}(\nu,\_\,) \to \mathrm{G}(\_)$ is good on ((major ν-nets)) and that $\mathrm{E}(\nu,\sigma)$ is normal in $\mathrm{G}(\sigma)$ whenever $\dim(\sigma)<\infty$. But the former hypothesis is proved in Lemma 6.9 and the latter in Theorem 5.7. It follows also from Theorem 2.7 that $\mathrm{G}^i(\sigma)=\mathrm{E}(\nu,\sigma)$ whenever $i \geqslant \dim(\sigma)$. Thus for an arbitrary ν-net σ, $\mathrm{G}(\sigma)\cap \mathrm{G}^i([\nu]+\sigma)=\mathrm{G}(\sigma)\cap\mathrm{E}(\nu,[\nu]+\sigma)$ whenever $i\geqslant\dim(\sigma)$, because $\dim(\sigma)=\dim([\nu]+\sigma)$. The inclusion $[\mathrm{G}(\sigma)\cap\mathrm{E}(\nu,[\nu]+\sigma),\mathrm{C}(\nu,\sigma)] \leqslant \mathrm{E}([\nu]+\sigma)$ follows immediately from Theorem 5.7.

Let $N\geqslant 0$. Define $\dim[-N]$: $((\nu\text{-nets})) \to \mathbb{Z}^{\geqslant 0}\cup\{\infty\}$ by

$$\dim[-N](\sigma)=\begin{cases} 0, & \text{if } \dim(\sigma)\leqslant N \\ \dim(\sigma)-N, & \text{if } \dim(\sigma)\geqslant N \end{cases}.$$

By (2.8) and (6.8), $\dim[-N]$ is a dimension function on the category with structure $\big(((\nu\text{-nets})),\ \mathrm{LFC}((\nu\text{-nets})),\ \mathrm{Frac}((\nu\text{-nets}))\big)$. Let $\mathrm{G}=\mathrm{G}^{[-1]}\geqslant\mathrm{G}^{[0]}\geqslant\mathrm{G}^{[1]}\geqslant\cdots$ denote the $\dim[-N]$-filtration of the natural transformation π on G.

7.3. Corollary. *Let* $N\geqslant 2$. *Let* (R,A,σ) *be a* ν*-net such that* $\mathrm{h}(\nu)\geqslant N\geqslant \mathrm{sr}\,A+1$. *Then all of the conclusions of Theorem 7.2 hold after* G^i *is replaced by* $\mathrm{G}^{[i]}$.

Proof. The proof is identical to that of Theorem 7.2.

7.4 Theorem. *Let* σ *denote any* ν*-net. If* $\mathrm{h}(\nu)\geqslant 3$ *then the action*

$$\mathrm{G}^0([\nu]+\sigma) \curvearrowright \mathrm{C}(\nu,\sigma)/\,\mathrm{G}^0(\sigma)$$

of $\mathrm{G}^0([\nu]+\sigma)$ *on* $\mathrm{C}(\nu,\sigma)/\,\mathrm{G}^0(\sigma)$ *by conjugation is trivial. If* $\mathrm{h}(\nu)\geqslant 4$ *then the filtration* $\mathrm{C}(\nu,\sigma)\geqslant\mathrm{C}(\nu,\sigma)\cap\mathrm{G}^0([\nu]+\sigma)\geqslant\mathrm{C}(\nu,\sigma)\cap\mathrm{G}^1([\nu]+\sigma)\geqslant\cdots\geqslant\mathrm{E}(\nu,\sigma)$ *is a descending* $\mathrm{G}^0([\nu]+\sigma)$*-central series of normal subgroups of* $\mathrm{C}(\nu,\sigma)$ *such that if* $i\geqslant\dim(\sigma)$ *then*

$$\mathrm{C}(\nu,\sigma)\cap\mathrm{G}^i([\nu]+\sigma)=\mathrm{C}(\nu,\sigma)\cap\mathrm{E}(\nu,[\nu]+\sigma),\ \textit{and}$$
$$[\mathrm{C}(\nu,\sigma)\cap\mathrm{G}^i([\nu]+\sigma),\mathrm{C}(\nu,\sigma)]=\mathrm{E}(\nu,\sigma).$$

Proof. To prove the first assertion, it suffices, by the last assertion of (2.5), to establish the special case that $\dim(\sigma)=0$. Under this assumption, $\mathrm{G}^0([\nu]+\sigma)=\mathrm{E}(\nu,[\nu]+\sigma)$ and by Theorem 5.7, $[\mathrm{E}(\nu,[\nu]+\sigma),\mathrm{C}(\nu,\sigma)]=\mathrm{E}(\nu,\sigma)$, i.e. the action of $\mathrm{E}(\nu,[\nu]+\sigma)$ on $\mathrm{C}(\nu,\sigma)/\mathrm{G}^0(\sigma)$ by conjugation is trivial. The rest of the proof is the same as that of Theorem 7.2.

7.5 Corollary. *Let $N\geqslant 0$. Let σ denote any ν-net. Then all of the conclusions of Theorem 7.4 hold after G^i is replaced by $\mathrm{G}^{[i]}$.*

Proof. The proof is identical to that of Theorem 7.4.

The next result generalizes [Bk1] Theorem 6.25 concerning the special case ν has just one equivalence class.

7.6. Nilpotent Sandwich Classification Theorem. *Let A be a quasi-finite ring and ν an equivalence on $J=\{1,\cdots,n\}$ (where n can be infinite). Suppose $\mathrm{h}(\nu)\geqslant 4$. Then the $\mathrm{E}(\nu)$-normal subgroups of $\mathrm{GL}(n,A)$ are in one to one correspondence with the subgroups H of the disjoint sandwiches $\mathrm{E}(\nu,\sigma)\leqslant H\leqslant\mathrm{C}(\nu,\sigma)$ where σ ranges over all ν-nets. Furthermore if $F\leqslant\mathrm{G}^0([\nu]+\sigma)$ and F normalizes H then*

$$H\geqslant H\cap\mathrm{G}^0([\nu]+\sigma)\geqslant H\cap\mathrm{G}^1([\nu]+\sigma)\geqslant\cdots\geqslant\mathrm{E}(\nu,\sigma)$$

is a descending F-central series such that if $i\geqslant\dim(\sigma)$ then

$$H\cap\mathrm{G}^i([\nu]+\sigma)=H\cap\mathrm{E}(\nu,[\nu]+\sigma),\ \textit{and}$$
$$\big[H\cap\mathrm{G}^i([\nu]+\sigma),\ F\cap\mathrm{C}(\nu,\sigma)\big]\leqslant\mathrm{E}(\nu,\sigma).$$

In particular, if $F\leqslant\mathrm{C}(\nu,\sigma)\cap\mathrm{G}^0([\nu]+\sigma)$ normalizes H and $\dim(\sigma)<\infty$ then

$$H\geqslant H\cap\mathrm{G}^0([\nu]+\sigma)\geqslant H\cap\mathrm{G}^1([\nu]+\sigma)\geqslant\cdots\geqslant H\cap\mathrm{G}^{\dim(\sigma)}([\nu]+\sigma)\geqslant\mathrm{E}(\nu,\sigma)$$

is a descending F-central series.

Proof. The theorem is an immediate consequence of Theorems 5.8 and 7.4.

7.7. Corollary. *Let $N\geqslant 0$. If $\mathrm{h}(\nu)\geqslant 4$ then the conclusions of Theorem 7.6 remain valid after replacing* dim *by* $\dim[-N]$ *and* G^i *by* $\mathrm{G}^{[i]}$.

Proof. The proof is exactly the same as that of Theorem 7.6.

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