Dimension Theory and Group Valued Functors

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Abstract

This article defines concepts of structure and dimension in arbitrary categories and illustrates them with well known examples. It shows that arbitrary group and coset valued functors define in a natural way notions of structure and dimension on their source categories and that this data predicts group theoretic properties of the functors over finite dimensional objects, such as solvability, nilpotence, or normality of subfunctors. The results are illustrated with applications.

1 Introduction

The dimension theory of categories provides valuable information concerning the properties of functors on categories. This article focuses on group and coset valued functors and examines group theoretic properties of these functors such as solvability, nilpotence, or normality of subfunctors. Further articles will treat functors with values in other categories and investigate other properties, such as splitting and cancellation for functors with values in additive categories.

The current article begins by defining a concept of structure in an arbitrary category and for a category with structure, it defines many kinds of dimension functions. It compares dimension functions and shows that for a given kind of dimension function, there is a unique smallest one which is called the universal one of its kind. Well known dimension functions are discussed in this context.

It is then shown how arbitrary group and coset valued functors lead naturally to specific structures and dimension functions on their source categories. Further it is shown that this data predicts group theoretic properties of the functors on finite dimensional objects, such as solvability, nilpotence, or normality of subfunctors. Applications to classical-like groups and K-theory are given.

A generalization of a coset valued functor is a natural transformation valued functor of group valued functors. It is shown that such functors lead also to structures and dimension functions on their source categories and that this information predicts group theoretic properties of the target of the natural transformation on finite dimensional objects. Applications are also given to classical-like groups and K-theory.

The current article has a sequel [ ] titled *Dimension Theory and Linked Sequences of Group Valued Functors*. A sequence $\cdots G_i, G_{i+1} \cdots$ of group valued functors is called linked if each consecutive pair G_i, G_{i+1} is a linked pair. Such sequences occur very naturally in homotopical settings. The current article defines the concept of a linked pair and shows how to associate to a natural transformation of group valued functors a linked pair of functors. We then use linked pairs to recover results above and as a bridge to the next paper.

The rest of the article is organized as follows. In §2, concepts of structure and dimension in arbitrary categories are defined. In §3, some well known and not so well known examples are given. In §4, it is shown how coset valued functors lead to specific structures and dimension functions on their source categories and this data is used to determine group theoretic properties of the functors on finite dimensional objects. In §5 applications of the above are made to classical-like groups and K-theory. In §6, it is shown how natural transformation valued functors of group valued functors lead also to structures and dimension functions on their source categories and how this information determines group theoretic properties of the target of the natural transformation. In §8, the concepts of linking diagram and linked pair of functors are introduced. It is shown that group theoretic properties of the second functor in a linked pair influence those of the first. In §9, it is shown how to construct a linked pair of functors from a natural transformation valued functor of group valued functors. In §10, the results of §8 and §9 are applied to classical-like groups and K-theory.

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2 Categories with Structure and Dimension

This section defines the concept of a category with structure and of a dimension function on a category with structure. On a given category with structure, there are many kinds of dimension functions, corresponding to a notion called type. A general category $\mathcal{C}$ becomes a category with structure when we fix in it four ingredients of structure. A function $d : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ is a dimension function if it satisfies certain properties with respect to the structure on $\mathcal{C}$. Different structures determine usually different classes of dimension functions. The four ingredients of structure are the following:

- A class $\mathcal{V}(\mathcal{C})$ of morphisms in $\mathcal{C}$, called virtual isomorphisms.
- A class $\mathcal{I}(\mathcal{C})$ of functors on directed preordered sets with values in $\mathcal{C}$, called infrastructure functors.
- A class of commutative squares in $\mathcal{C}$, called structure squares.

The dimension theory developed in this article is geared to show that a class of functors on a general category $\mathcal{C}$ will have a given property, if $\mathcal{C}$ has a structure which is tied to the functors themselves and the property as follows:

- The property is invariant under $\mathcal{V}(\mathcal{C})$.
- The functors preserve direct limits of functors in $\mathcal{I}(\mathcal{C})$.

- The behavior of the functors in the upper left hand object of a structure square is suitably tied to its behavior on the rest of the square.

Under the relationship above of functors to structure, the results of the article will show that the property holds over all finite dimensional objects.

We define now in detail the ingredients of a category with structure.

2.1

Throughout let $\mathcal{C}$ denote an arbitrary category.

Let **Iso** $(\boldsymbol{\mathcal{C}})$ denote the class of all isomorphisms in $\mathcal{C}$. A class $\boldsymbol{\mathcal{V}(\mathcal{C})}$ of morphisms in $\mathcal{C}$ is called a class of **virtual isomorphisms**, if $\boldsymbol{\mathcal{V}(\mathcal{C})}$ contains $Iso(\mathcal{C})$ and if $\mathcal{V}(\mathcal{C})$ is closed under composition.

Let $x = s, t$, or ϕ. The letter s stands for source, the letter t for target, and the symbol ϕ stands for the empty letter. A class $\mathcal{C}_0$ of objects of $\mathcal{C}$ is called ***x*-closed under virtual isomorphism**, if one of the following holds:

(i) $x = s$ and given $B \in \mathcal{C}_0$ and $A \to B \ \in \ \mathcal{V}(\mathcal{C})$, it follows that $A \in \mathcal{C}_0$.

(ii) $x = t$ and given $A \in \mathcal{C}_0$ and $A \to B \ \in \ \mathcal{V}(\mathcal{C})$, it follows that $B \in \mathcal{C}_0$.

(iii) $x = \phi$ and $\mathcal{C}_0$ is both s-closed and t closed under virtual isomorphisms.

A class $\mathcal{C}_0$ is called **closed under virtual isomorphism**, if it is ϕ-closed.

Let A and A' be objects of $\mathcal{C}$. A **virtual chain equivalence** from A to A' is by definition a chain of objects $A = A_0, A_1, \cdots, A_n = A'$ and morphisms $f_0, f_1, \cdots, f_{n-1} \in \mathcal{V}(\mathcal{C})$ such that for each i $(0 \leqq i \leqq n-1)$ either source $(f_i) = A_i$ and target $(f_i) = A_{i+1}$ or source $(f_i) = A_{i+1}$ and target $(f_i) = A_i$. Clearly virtual chain equivalence is an equivalence relation on $\mathrm{Ob}(\mathcal{C})$.

Obviously a class $\mathcal{C}_0$ of objects of $\mathcal{C}$ is closed under virtual isomorphisms $\Leftrightarrow$ it is closed under virtual chain equivalence.

2.2

Recall that a **preordered set** I is a set together with a reflexive, transitive relation $\leqq$. Equivalently I is a category whose objects form a set and for any pair i, j of objects of I, there is at most one morphism whose source is i and target is j. A **partially ordered set** I is a preordered set whose relation is also anitsymmetric, i.e. $i \leqq j$ and $j \leqq i$ implies $i = j$. A preordered set I is called **directed**, if given elements $i, j \in I$, there is an element $k \in I$ such that $i \leqq k$ and $j \leqq k$.

Let $\{*\}$ denote the directed preordered set with precisely one element $*$. If $A \in \mathrm{Ob}(\mathcal{C})$, let F_A denote the functor

$$\begin{array}{rccc} F_A : & \{*\} & \to & \mathcal{C} \\ & * & \mapsto & A \\ & 1_* & \mapsto & 1_A \end{array}$$

where 1_* and 1_A denote the identity morphisms on $*$ and A, respectively. F_A will be called a **trivial infrastructure functor**.

A class $\mathfrak{I}(\mathcal{C})$ of functors with values in $\mathcal{C}$ is called a **class of infrastructure functors**, if it contains all trivial infrastructure functors and is closed under natural isomorphisms and if each $F \in \mathfrak{I}(\mathcal{C})$ has the property that its source category is a directed, preordered set I and the direct limit $\varinjlim_I F$ exists in $\mathcal{C}$. By definition, $\varinjlim_I F$ is the colimit $\operatorname{colim}_I F$.

2.3

If $A \in \mathrm{Ob}(\mathcal{C})$ then the commutative square

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"'] & A \arrow[d, "1"] \\ A \arrow[r, "1"'] & A \end{tikzcd}$$

will be called a **trivial square**.

A class $\boldsymbol{\mathcal{S}(\mathcal{C})}$ of commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in $\mathcal{C}$ is called a **class of structure squares**, if it contains all trivial squares and is closed under isomorphism of commutative squares.

A class of $\boldsymbol{\mathcal{GS}(\mathcal{C})} \subseteqq \mathcal{S}(\mathcal{C})$ of commutative squares in $\mathcal{C}$ is called a **class of generating squares for $\boldsymbol{\mathcal{S}(\mathcal{C})}$**, if it is closed under isomorphism of commutative squares, and every square in $\mathcal{S}(\mathcal{C})$ is a direct limit in a prescribed way of squares in $\mathcal{GS}(\mathcal{C})$. This means that we have a class $\mathcal{FGS}(\mathcal{C})$ of functors S whose source categories are directed, preordered sets J, whose values are in $\mathcal{GS}(\mathcal{C})$, and whose direct limits $\varinjlim_J S$ exist and are in $\mathcal{S}(\mathcal{C})$ and every square in $\mathcal{S}(\mathcal{C})$ is some direct limit $\varinjlim_J S$. The purpose for introducing $\mathcal{FGS}(\mathcal{C})$ rather than allowing all direct limits of squares in $\mathcal{GS}(\mathcal{C})$, is that the latter would permit us to treat only functors on $\mathcal{C}$ which commute with all direct limits of generating squares.

2.4

A **category with structure** is a quadruple $\boldsymbol{(\mathcal{C}, \mathcal{V}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C}))}$ consisting of a category $\mathcal{C}$, a class $\mathcal{V}(\mathcal{C})$ of virtual isomorphisms, a class $\mathcal{I}(\mathcal{C})$ of infrastructure functors, and a class $\mathcal{S}(\mathcal{C})$ of structure squares. A **functor** or **morphism** $\mathcal{F} : (\mathcal{C}, \mathcal{V}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C})) \to (\mathcal{D}, \mathcal{V}(\mathcal{D}), \mathcal{I}(\mathcal{D}), \mathcal{S}(\mathcal{D}))$ of categories with structure is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ which preserves virtual isomorphisms, infrastructure functors, and structure squares. This means that

- if f is a virtual isomorphism then so is $\mathcal{F}(f)$,
- if $F : I \to \mathcal{C}$ is an infrastructure functor then so is the composition $\mathcal{F}F : I \to \mathcal{D}$ and the canonical morphism $\varinjlim_I \mathcal{F}F \to \mathcal{F}\left(\varinjlim_I F\right)$ is an isomorphism,

- if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a structure square in $\mathcal{C}$ then

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r] \arrow[d] & \mathcal{F}(B) \arrow[d] \\ \mathcal{F}(C) \arrow[r] & \mathcal{F}(D) \end{tikzcd}$$

is one in $\mathcal{D}$.

2.5

Let $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{V}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C}))$ denote a category with structure. Let ∞ denote the first infinite ordinal. Let $x = s, t$, or ϕ be as in (2.1). An $\boldsymbol{x}$**−dimension function** on $\underline{\mathcal{C}}$ is a function

$$d : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$$

such that one of the following holds:

(i) $x = s$ and given $A \to B \in \mathcal{V}(\mathcal{C})$, it follows that $d(A) \leqq d(B)$.

(ii) $x = t$ and given $A \to B \in \mathcal{V}(\mathcal{C})$, it follows that $d(A) \geqq d(B)$.

(iii) $x = \phi$ and given $A \to B \in \mathcal{V}(\mathcal{C})$, it follows that $d(A) = d(B)$, i.e. d is both an s- and a t-dimension function.

A **dimension function** on $\underline{\mathcal{C}}$ is by definition a ϕ-dimension function on $\underline{\mathcal{C}}$. If d and e are x-dimension functions on $\mathcal{C}$ then $\boldsymbol{d} \leqq \boldsymbol{e}$ means that $d(A) \leqq e(A)$ for each $A \in \mathrm{Ob}(\mathcal{C})$. A **category with** $\boldsymbol{x}$**-dimension** is a pair $(\underline{\mathcal{C}}, d)$

consisting of a category with structure $\underline{\mathcal{C}}$ and an x-dimension function d on $\underline{\mathcal{C}}$. A **functor** or **morphism** $\mathcal{F} : (\underline{\mathcal{C}}, d) \to (\underline{\mathcal{D}}, e)$ of categories with x-dimension is a functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ of categories with structure such that $d \geqq e\mathcal{F}$.

The most important concepts for an x-dimension function are the notions of type and reduction. The problems we can solve using dimension theory depend on these notions. They are introduced next.

2.6

A **type** t is a triple $t_B, t_C, t_D : \mathbb{Z}^{\geq 0} \cup \{\infty\} \to \mathbb{N} \cup \{\infty, \infty + 1\}$ of functions. It is usually written in the form

$$t = \begin{pmatrix} & t_B \\ t_C & t_D \end{pmatrix}.$$

If t and t' are types then $\boldsymbol{t} \leqq \boldsymbol{t'}$ means that $t(x) \leqq t'(x)$ for each element $x \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, i.e. if $X = A, B,$ or C then $t_X(x) \leqq t'_X(x)$.

Let $\underline{\mathcal{C}}$ be a category with structure. Let $A \in \mathrm{Ob}(\mathcal{C})$. A **reduction** of A consists of

(i)

$$\left\{ \begin{array}{c} \text{a morphism } A \to A' \in \mathcal{V}(\mathcal{C}), \text{if } x = s. \\ \text{a morphism } A' \to A \in \mathcal{V}(\mathcal{C}),\ \text{if } x = t \\ \text{a virtual chain equivalence from } A \text{ to } A',\ \text{if } x = \phi \end{array} \right\} \textit{and}$$

(ii) an infrastructure functor $B : I \to \mathcal{C}, i \mapsto B_i$, and

(iii) a set

$$\begin{array}{ccc} A' & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & D_i \end{array} \quad (i \in I)$$

of structure squares indexed by I (C and D are not necessarily functorial in I) with the property that for each relationship $(i \leqq j) \in I$, the triangle

$$\begin{tikzcd} & B_j \\ A' \arrow[ur] \arrow[dr] & \\ & B_i \arrow[uu] \end{tikzcd}$$

commutes.

Let d be an x-dimension function on $\underline{\mathcal{C}}$. Let t be a type and $n \in \mathbb{N}$. A reduction of A is said to be a ***d*-reduction of type *t* at *n***, if in addition to the above

(iv) $d\left(\varinjlim_I B\right) < t_B(n)$

and for each $i \in I$,

(v) $d(C_i) < t_C(n)$, and

(vi) $d(D_i) < t_D(n)$.

Suppose now that $0 < d(A) < \infty$. Then a d-reduction of type t at $d(A)$ for A will be simply called a ***d*-reduction of type *t*** for A. An x-dimension function d on $\underline{\mathcal{C}}$ is said to **have type *t***, if each $A \in \mathrm{Ob}(\mathcal{C})$ such that $0 < d(A) < \infty$ has a d-reduction of type t. An x-dimension function can have many types and obviously if it has type t and $t \leqq t'$ then it also has type t'. The applicability of a given x-dimension function d to a given problem depends on the minimal types of d.

2.7

Let d be an x-dimension function of type t on the category with structure $\underline{\mathcal{C}}$. Let $A \in \mathrm{Ob}(\mathcal{C})$ and $n \in \mathbb{N}$. Even if A has a d-reduction of type t at n, it does not follow that $d(A) \leqq n$. However if the conclusion above is true for any A and any n then we shall say that d is a **tame** x-dimension function of type t.

2.8

Let $\underline{\mathcal{C}}$ be a category with structure. Define $\boldsymbol{x - \mathrm{dim}(\underline{\mathcal{C}})}$ to be the partially ordered class of all x-dimension functions on $\underline{\mathcal{C}}$, with partial ordering given by $\leqq$. If t is a type, let $\boldsymbol{x - \mathrm{dim}(\underline{\mathcal{C}}, t)}$ denote the partially ordered subclass of $x - \mathrm{dim}(\underline{\mathcal{C}})$ consisting of all x-dimension functions of type t.

2.9

THEOREM *The partially ordered class $x - \mathrm{dim}(\underline{\mathcal{C}})$ has arbitrary greatest lower bounds. If $S \subseteqq x - \mathrm{dim}(\underline{\mathcal{C}})$ is a nonempty subclass then its greatest lower bound glb_S is computed by the formula*

$$glb_S(A) = \inf\{d(A) \mid d \in S\}.$$

Moreover if $S \subseteqq x - \mathrm{dim}(\underline{\mathcal{C}}, t)$ is a nonempty subclass then $glb_S \in x - \mathrm{dim}(\underline{\mathcal{C}}, t)$.

PROOF The function $\mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$, $A \mapsto \inf\{d(A) \mid d \in S\}$, obviously satisfies the conditions of definition 2.5 of an x-dimension function. Clearly this function is the greatest lower bound for S. Suppose $S \subseteqq x - \mathrm{dim}(\underline{\mathcal{C}}, t)$. It will be shown that $glb_S \in x - \mathrm{dim}(\underline{\mathcal{C}}, t)$. Let $A \in \mathrm{Ob}(\mathcal{C})$ such that $glb_S(A) \in \mathbb{N}$. We must show that A has a glb_S-reduction of type t at $glb_S(A)$. Let $d \in S$ such that $glb_S(A) = d(A)$. By definition, A has a d-reduction of type t at $d(A)$. Since $glb_S(X) \leqq d(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it follows that any d-reduction above of A is also a glb_S-reduction of A of type t at $d(A) = glb_S(A)$. Thus by definition, glb_S has type t.$\square$

2.10

THEOREM *Let d be an x-dimension function of type t on the category with structure $\underline{\mathcal{C}}$. Then there is a tame x-dimension function $\boldsymbol{d_t}$ of type t, called the* **tame closure of $\boldsymbol{d}$ at $\boldsymbol{t}$**, *which is the unique maximum among all tame x-dimension functions d' of type t such that $d' \leqq d$. Moreover d_t is computed according to the following rule: Let $T(d) = \{d' \mid d'$ a tame x-dimension function of type $t, d' \leqq d\}$ and $S(d) = \{d'' \mid d''$ an x-dimension function of type $t, d'' \leqq d, d' \leqq d''\ \forall\ d' \in T(d)\}$. Then*

$$d_t = glb_{S(d)}.$$

PROOF Clearly $d' \leqq glb_{S(d)} \leqq d$ for all $d' \in T(d)$ and by Theorem 2.9, $\mathrm{glb}_{S(d)}$ is an x-dimension function of type t. To complete the proof of the theorem, it suffices to show that $glb_{S(d)}$ is tame of type t. Suppose $glb_{S(d)}$ is not tame. Define $f : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ by

$$f(A) = \inf(\{glb_{S(d)}(A)\} \cup \{n \in \mathbb{N} \mid A \text{ has a } glb_{S(d)}\text{-reduction of type } t \text{ at } n\}).$$

Since $glb_{S(d)}$ is not tame, $f < glb_{S(d)}$, i.e. $f \leqq glb_{S(d)}$ and there is an $B \in \mathrm{Ob}(\mathcal{C})$ such that $f(B) < glb_{S(d)}(B)$. Thus $f \notin S(d)$. It will be shown next that $f \leqq d$. Since $glb_{S(d)}$ is an x-dimension function, it follows by construction that f is one also. Let $A \in \mathrm{Ob}(\mathcal{C})$ such that $0 < f(A) < \infty$. By definition A has a $glb_{S(d)}$-reduction of type t at $f(A)$. Since $f(X) \leqq glb_{S(d)}(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it follows that each $glb_{S(d)}$-reduction of A above is also an f-reduction of A of type t at $f(A)$. Thus f has type t, by definition. Suppose $d' \in T(d)$. We must show that $d' \leqq f$. From the definition of $glb_{S(d)}$, it follows that $d' \leqq glb_{S(d)}$. It suffices to show that if $f(A) < \infty$, then $d'(A) \leqq f(A)$.If $f(A) = 0$ then from the definition of f, it follows $glb_{S(d)}(A) = 0$ and so $d'(A) = 0$. Suppose $f(A) > 0$. From the definition of f, it follows that A has a $glb_{S(d)}$-reduction of type t at $f(A)$. Since $d'(X) \leqq glb_{S(d)}(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$, it follows that any $glb_{S(d)}$-reduction above of A is also a d'-reduction of A of type t at $f(A)$. Since d' is tame, it follows that $d'(A) \leqq f(A)$.$\square$

2.11

Let $\underline{\mathcal{C}}$ be a category with structure. Let t be a type. An x-dimension function δ of type t is called **universal of type $\boldsymbol{t}$**, if $\delta \leqq d$ for any x-dimension function d of type t, whose 0-dimensional objects are also 0-dimensional under δ.

2.12

COROLLARY *Let $\underline{\mathcal{C}}$ be a category with structure. Let $\mathcal{C}^0$ be a class of objects of $\mathcal{C}$, which is x-closed under virtual isomorphism. Then there is a unique universal x-dimension function of type t, whose 0-dimensional objects are precisely those of $\mathcal{C}^0$. Furthermore it is tame. This function is often denoted by*

$$\boldsymbol{d}_{\underline{\mathcal{C}},\mathcal{C}^0,\boldsymbol{t}}.$$

PROOF Let S denote the class of all x-dimension functions of type t, whose 0-dimensional objects are in $\mathcal{C}^0$. This class is nonempty and contains an x-dimension function d whose 0-dimensional objects are precisely $\mathcal{C}^0$, namely the function $d : \ \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ defined by

$$d(A) = \begin{cases} 0 & \text{if } A \in \mathcal{C}^0 \\ \infty & \text{if } A \notin \mathcal{C}^0. \end{cases}$$

By Theorem 2.10, glb_S exists and is an x-dimension function of type t. Obviously glb_S has $\mathcal{C}^0$ as its class of 0-dimensional objects, is universal of type t and unique. Furthermore by Theorem 2.8, it must be tame. $\square$

2.13

THEOREM *Let $\underline{\mathcal{C}}$ be a category with structure. Let t be a type such that given $X \in \{A, B, C\}$ either $t_X \leqq id$ or $t_X = \infty + 1$. Let d be a tame x-dimension function of type t on $\underline{\mathcal{C}}$. Then d is universal of type t and is computed by the following rule: Define recursively disjoint, x-closed classes $\mathcal{C}^n$ as follows. Let $\mathcal{C}^0 = \{A \in \ Ob(\mathcal{C}) \mid d(A) = 0\}$. Let $n \in \mathbb{N}$ and assume* *that disjoint, x-closed classes $\mathcal{C}^m$ have been defined for all $0 \leqq m < n$. Define $\delta^{n-1} : Ob(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ by*

$$\delta^{n-1}(A) = \begin{cases} m & \text{if} \quad A \in \mathcal{C}^m \qquad \text{where } 0 \leqq m < n, \\ \infty & \text{if} \quad A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m. \end{cases}$$

Obviously δ^{n-1} is an x-dimension function. Define $\mathcal{C}^n = \{A \in Ob(\mathcal{C}) \mid A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m, A$ has a δ^{n-1}-reduction of type t at $n\}$. Define $\mathcal{C}^\infty = Ob(\mathcal{C}) \setminus \bigcup_{n \in \mathbb{Z}^{\geq 0}} \mathcal{C}^n$. Then for any $A \in Ob(\mathcal{C})$ and any $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$,

$$d(A) = n \Leftrightarrow A \in \mathcal{C}^n.$$

Furthermore if $n \in \mathbb{Z}^{\geq 0}$ and $\mathcal{C}^n = \phi$ then $\mathcal{C}^{n'} = \phi$ for all $n' \geq n$, i.e. $d = \delta^{n-1}$.

Proof We shall treat only the case $t \leqq \begin{pmatrix} & id \\ id & id \end{pmatrix}$. The other cases are done similarly.

Let $\mathcal{C}^n \quad (n \in \mathbb{Z}^{\geq 0} \cup \{\infty\})$ be defined as in the theorem. Clearly these classes are disjoint and x-closed under virtual isomorphisms. Define $\delta : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ by $\delta(A) = n \Leftrightarrow A \in \mathcal{C}^n$. By construction, δ is a tame type t x-dimension function.

We shall show that δ is universal. Let $\delta' = d_{\underline{\mathcal{C}}, \mathcal{C}^0, t}$ be the universal x-dimension function of type t whose 0-dimensional objects form the class $\mathcal{C}^0$. We must show that $\delta \leqq \delta'$. It suffices to show that if $\delta'(A) < \infty$ then $\delta(A) \leqq \delta'(A)$. We proceed by induction on $\delta'(A)$. If $\delta'(A) = 0$ then $\delta(A) = 0$ by definition of δ'. Let $n \in \mathbb{N}$ such that $\delta'(A) < n$ implies $\delta(A) \leqq \delta'(A)$. Suppose that $\delta'(A) = n$. By definition, A has a δ'-reduction of type t at n. Since $t \leqq \begin{pmatrix} & id \\ id & id \end{pmatrix}$ and since for any $X \in \mathrm{Ob}(\mathcal{C}), \delta(X) \leqq \delta'(X)$ whenever $\delta'(X) < n$, it follows that any δ'-reduction above of A is also a δ-reduction of A of type t at n. Since δ is tame, it follows that $\delta(A) \leqq n = \delta'(A)$. Thus $\delta \leqq \delta'$.

We show next that $d = \delta$. Since δ is universal, it suffices to show that if $\delta(A) < \infty$ then $d(A) \leqq \delta(A)$. We proceed by induction on $\delta(A)$. If $\delta(A) = 0$

then $d(A) = 0$, by definition of δ. Let $n \in \mathbb{N}$ such that $\delta(A) = n$ implies $d(A) \leqq \delta(A)$. Suppose $\delta(A) = n$. By definition, A has a δ-reduction of type t at n. Since $t \leqq \begin{pmatrix} & id \\ id & id \end{pmatrix}$ and since for any $X \in \mathrm{Ob}(\mathcal{C}), d(X) \leqq \delta(X)$ whenever $\delta(X) < n$, it follows that the reduction of A above is also a d-reduction of type t at n. Since d is tame, this implies $d(A) \leqq n = \delta(A)$.

Finally suppose that $\mathcal{C}^n = \phi$. We shall show that $\mathcal{C}^{n+1} = \phi$. Suppose $A \in \mathcal{C}^{n+1}$. Then A has a δ^n-reduction of type t at $n+1$. Since $\mathcal{C}^n = \phi$ and $t \leqq \begin{pmatrix} & id \\ id & id \end{pmatrix}$, this reduction must also be a δ^{n-1}-reduction of type t at n. Since $\mathcal{C}^n = \phi$, it follows that $A \in \bigcup_{m=0}^{n-1} \mathcal{C}^m$, which contradicts the fact that the $\mathcal{C}^k$'s are disjoint. Thus $\mathcal{C}^{n+1} = \phi$. By induction, we conclude that $\mathcal{C}^{n'} = \phi$ for all $n' \in \mathbb{Z}^{\geq 0}$ such that $n' \geq n$.$\square$

2.14

Let $(\underline{\mathcal{C}}, d)$ be a category with structure and d an x-dimension function on $\underline{\mathcal{C}}$. For $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, define the classes

$$\begin{aligned} \boldsymbol{\mathcal{C}_d^n} &= \{A \in \mathrm{Ob}(\mathcal{C}) \mid d(A) = n\} \\ \boldsymbol{\mathcal{C}_d^{\leqq n}} &= \{A \in \mathrm{Ob}(\mathcal{C}) \mid d(A) \leqq n\}. \end{aligned}$$

The next lemma is left as an easy exercise to the reader.

2.15

Lemma *The classes above are x-closed under virtual isomorphisms and $Ob(\mathcal{C}) = \bigcup_{n=0}^{\infty} \mathcal{C}_d^n$. Conversely any partition of $Ob(\mathcal{C})$ as a disjoint union $Ob(\mathcal{C}) = \bigcup_{n=0}^{\infty} \mathcal{C}^n$ of x-closed classes $\mathcal{C}^n$ defines an x-dimension function d on $\underline{\mathcal{C}}$ such that $\mathcal{C}_d^n = \mathcal{C}^n$. Partially order the members of $\{x -$ closed classes in $Ob(\mathcal{C})\}$ by inclusion. Then the function $\{x$-dimension functions of type t on $\underline{\mathcal{C}}\}$ $\to \{x$-closed classes in $Ob(\mathcal{C})\}$, $d \mapsto \mathcal{C}_d^0$, preserves partial orderings and* *induces a bijection between universal x-dimension functions of type t and x-closed classes in $Ob(\mathcal{C})$. If $(\underline{\mathcal{D}}, e)$ is a category with x-dimension and $\mathcal{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a functor of categories with structure then $\mathcal{F}$ defines a morphism $(\underline{\mathcal{C}}, d) \to (\underline{\mathcal{D}}, e)$ of categories with x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}_d^{\leq n}) \subseteqq \mathcal{D}_e^{\leqq n}$ for all $n \in \mathbb{Z}^{\geq 0}$. Furthermore if $d = d_{\underline{\mathcal{C}},\mathcal{C}^0,t}$ and $e = d_{\underline{\mathcal{D}},\mathcal{D}^0,t}$ for some x-closed classes $\mathcal{C}^0 \subseteq Ob(\mathcal{C})$ and $\mathcal{D}^0 \subseteq Ob(\mathcal{D})$ then $\mathcal{F}$ induces a morphism of categories with x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}^0) \subseteqq \mathcal{D}^0$.*

2.16

Let $\underline{\mathcal{C}}$ be a category with structure. If $S \subseteqq x - \dim(\underline{\mathcal{C}})$, define the partially ordered set **type(S)** $= \{\text{types } t \mid \text{ each } d \in S \text{ has type } t\}$, with partial ordering that of types.

Since the results we prove for group valued functors are formulated using the concept of type for x-dimension functions, it is useful to have some general guidelines how large the set type(S) above is, in particular when S has just one element. The following lemmas provide such information. They are given without proof.

2.17

LEMMA *The partially ordered set type(S) has the property that every element is bounded below by a minimal element. Furthermore if $t \in$ type(S) and t' is a type such that $t' \geq t$ then $t' \in$ type(S). Thus the minimal elements of type(S) define it uniquely.*

2.18

LEMMA *Let $\mathcal{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a functor of categories with structure. Let $T \subseteqq x - \dim(\underline{\mathcal{D}})$. Define $S = \{d\mathcal{F} \mid d \in T\}$.*

- *(i) Then $S \subseteqq x - \dim(\underline{\mathcal{C}})$ and type(S) $\subseteqq$ type(T).*
- *(ii) Let $x \in \{s, t, \phi\}$ as in 2.1. If $x = s$, suppose that for any $(\mathcal{F}(A) \to D) \in \mathcal{V}(\mathcal{D})$ where $A \in Ob(\mathcal{C})$ and $D \in Ob(\mathcal{D})$, there is a $(A \to B) \in$*

$\mathcal{V}(\mathcal{C})$ and an isomorphism $\mathcal{F}(B) \to D$. If $x = t$, suppose that for any $(D \to \mathcal{F}(A)) \in \mathcal{V}(\mathcal{D})$ where $A \in Ob(\mathcal{C})$ and $D \in Ob(\mathcal{D})$, there is a $(B \to A) \in \mathcal{V}(\mathcal{C})$ and an isomorphism $\mathcal{F}(B) \to D$. If $x = \phi$, suppose that for any virtual chain equivalence in $\underline{\mathcal{D}}$ from $\mathcal{F}(A)$ to D where $A \in Ob(\mathcal{C})$ and $D \in Ob(\mathcal{D})$, there is a virtual chain equivalence in $\underline{\mathcal{C}}$ from A to some $B \in Ob(\mathcal{C})$ and an isomorphism from $\mathcal{F}(B)$ to D. Suppose that any data consisting of an infrastructure functor $(B : I \to \mathcal{D}) \in \mathcal{I}(\mathcal{D})$ and a set

$$\begin{array}{ccc} A & \longrightarrow & B_i \\ \downarrow & & \downarrow \\ C_i & \longrightarrow & D_i \end{array} \quad (i \in I)$$

of structure squares in $\mathcal{S}(\mathcal{D})$, which together satisfy (2.6) (ii) and (iii) and have the property that all objects A, B_i, C_i, D_i $(i \in I)$ lie in $\mathcal{F}(Ob(\mathcal{C}))$, must be the image under $\mathcal{F}$ of analogous data in $\mathcal{C}$. Then type$(T) \subseteqq$ type(S).

3 Categories with Global Reduction and Dimension

The previous section introduced concepts of structure and dimension in categories for the purpose of studying properties of functors, in subsequent sections and papers. The current section develops further these concepts in order to expand their range of application. Once the section is complete, all of the constructions and results of the previous one will be special cases of those of the current one. Proofs for the current results will be omitted, since they are obvious extensions of their counterparts in the previous section. Examples of concepts introduced in the current section are provided in section 5.

The key construction for extending the scope of the previous section is that of a global reduction. In the previous section, reductions were used only one at a time. In the current section, we want to specify for each object in a category, certain sets of reductions to be called global reductions. The concept of dimension will be formulated such that certain properties hold across an entire set of reductions instead of just for one reduction. The constructions and results of the previous section are recovered from those of the current, by letting each global reduction consist of just one ordinary reduction and by assigning to each object in the category all of its single member global reductions.

Throughout this section, let $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{V}(\mathcal{C}), \mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C}))$ denote a category with structure.

3.1

A **category with global reduction** consists of a category with structure $\underline{\mathcal{C}}$ and for each $A \in \mathrm{Ob}(\mathcal{C})$ a class $\boldsymbol{glrd(A)}$ of sets X such that each member of X is a reduction of A in the sense of (2.6) (i) - (iii). Each member $X \in glrd(A)$ is called a **global reduction of $\boldsymbol{A}$**. It will be assumed that if $X \in glrd(A)$ and Y is a set of reductions of A such that each reduction in X is isomorphic (in the straightforward sense) to a reduction in Y and conversely then $Y \in glrd(A)$. In this case, we say that X is **isomorphic** to Y. A category with global reduction will be denoted by a pair

$$(\underline{\mathcal{C}}, glrd)$$

where $glrd$ assigns to each $A \in \mathrm{Ob}(\mathcal{C})$ a class $glrd(A)$ of global reductions of A, which is closed under isomorphism. We explicitly allow the possibility that for some $A's, glrd(A) = \phi$.

Let $\mathcal{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ be a functor of categories with structure. It follows from the definition of such functors that $\mathcal{F}$ takes any reduction of an object A of $\mathcal{C}$ to a reduction of $\mathcal{F}(A)$. If R is a reduction of A, let $\mathcal{F}(R)$ denote the corresponding reduction of $\mathcal{F}(A)$. If X is a global reduction of A in $(\underline{\mathcal{C}}, glrd)$, let $\mathcal{F}(X) = \{\mathcal{F}(R) \mid R \in X\}$. We say that $\mathcal{F}$ defines a **functor** or **morphism** $(\underline{\mathcal{C}}, glrd) \to (\underline{\mathcal{C}'}, glrd')$ of categories with global reduction, if for each $A \in \mathrm{Ob}(\mathcal{C})$ and $X \in glrd(A), \mathcal{F}(X) \in glrd'(\mathcal{F}(A))$.

3.2

Let $x = s$ (source), t (target), or ϕ (no condition), be as in (2.1). An ***x*-dimension function** d on $(\underline{\mathcal{C}}, glrd)$ is by definition an x-dimension function on $\underline{\mathcal{C}}$, as in (2.5). A **category with *x*-dimension** is a pair $((\underline{\mathcal{C}}, glrd), d)$ consisting of a category with global reduction $(\underline{\mathcal{C}}, glrd)$ and an x-dimension function d on $(\underline{\mathcal{C}}, glrd)$. A **functor** or **morphism** $\mathcal{F} : ((\underline{\mathcal{C}}, glrd), d) \to ((\underline{\mathcal{C}'}, glrd'), d')$ of categories with x-dimension is a functor $\mathcal{F} : (\underline{\mathcal{C}}, glrd) \to (\underline{\mathcal{C}'}, grld')$ of categories with global reduction such that $d \geqq d'\mathcal{F}$.

As in section 2, the most important properties of x-dimension functions are described using the notions of type and global reduction. This is done in the following paragraphs.

3.3

Let d be an ***x*-dimension function** on the category with global reduction $(\underline{\mathcal{C}}, glrd)$. Let t denote a type, as in (2.6), and let $n \in \mathbb{N}$. A global reduction $X \in glrd(A)$ is called a **global *d*-reduction of type *t* at *n***, if $X \neq \phi$ and each reduction in X is a d-reduction of type t at n, as in (2.6) (iv) - (vi). A **global *d*-reduction of type *t* at *A*** is by definition an $X \in glrd(A)$, which is a global d-reduction of type t at $d(A)$. Note that this definition makes sense only when $d(A) > 0$, because t is defined on $\mathbb{N} \cup \{\infty\}$. An x-dimension function d is said **to have type *t***, if each $A \in \mathrm{Ob}(\mathcal{C})$ such that $0 < d(A) < \infty$ has a global d-reduction of type t at A.

An x-dimension function on $(\underline{\mathcal{C}}, glrd)$ can have many types and obviously if it has type t and $t \leqq t'$ then it also has type t'. The applicability of a given x-dimension function d to a given problem depends on the minimal types of d.

For the rest of this section, let

$$\overline{\mathcal{C}} = (\underline{\mathcal{C}}, glrd)$$

denote a category with global reduction. We continue developing the properties of x-dimension functions.

3.4

Let d be an x-dimension function of type t on the category with global reduction $\overline{\mathcal{C}}$. Let $A \in \mathrm{Ob}\ (\mathcal{C})$ and $n \in \mathbb{N}$. Even if A has a global d-reduction of type t at n, it does not follow that $d(A) \leqq n$. However if the conclusion above is true for any A and any n then we shall say that d is a **tame** x-dimension function of type t.

3.5

Let $\overline{\mathcal{C}}$ be a category with global reduction. Define $\boldsymbol{x - \dim(\overline{\mathcal{C}})}$ to be the partially ordered class of all x-dimension functions on $\overline{\mathcal{C}}$, with partial ordering given by $\leqq$. If t is a type, let $\boldsymbol{x - \dim(\overline{\mathcal{C}}, t)}$ denote the partially ordered subclass of $x - \dim(\overline{\mathcal{C}})$ consisting of all x-dimension functions of type t.

3.6

The next result generalizes Theorem 2.9.

THEOREM *The partially ordered class $x - \dim(\overline{\mathcal{C}})$ has arbitrary greatest lower bounds. If $S \subseteqq x - \dim(\overline{\mathcal{C}})$ is a nonempty subclass then its greatest lower bound glb_S is computed by the formula*

$$glb_S(A) = \inf\{d(A) \mid d \in S\}.$$

Moreover if $S \subseteqq x - \dim(\overline{\mathcal{C}}, t)$ is a nonempty subclass then $glb_S \in x - \dim(\overline{\mathcal{C}}, t)$.

3.7

Let next result generalizes Theorem 2.10.

THEOREM *Let d be an x-dimension function of type t on the category with global reduction $\overline{\mathcal{C}}$. Then there is a tame x-dimension $\boldsymbol{d_t}$ of type t, called the* **tame closure of $\boldsymbol{d}$ at $\boldsymbol{t}$**, *which is the unique maximum among all tame x-dimension functions d' of type t such that $d' \leqq d$. Moreover d_t is computed according to the following rule: Let $T(d) = \{d' \mid d'$ a tame x-dimension function* *of type* $t, d' \leqq d\}$ *and* $S(d) = \{d'' \mid d''$ *an* x*-dimension function of type* $t, d'' \leqq d, d' \leqq d'' \ \forall \ d' \in T(d)\}$. *Then*

$$d_t = glb_{S(d)}.$$

3.8

Let $\overline{\mathcal{C}}$ be a category with global reduction. Let t be a type. An x-dimension function δ of type t is called **universal of type t**, if $\delta \leqq d$ for any x-dimension function d of type t, whose 0-dimensional objects are also 0-dimensional under δ.

3.9

The next result generalizes Corollary 2.12.

COROLLARY *Let* $\overline{\mathcal{C}}$ *be a category with global reduction. Let* $\mathcal{C}^0$ *be a class of objects of* $\mathcal{C}$*, which is* x*-closed under virtual isomorphism. Then there is a unique universal* x*-dimension function of type* t*, whose 0-dimensional objects are precisely those of* $\mathcal{C}^0$*. Furthermore it is tame. This function is often denoted by*

$$\boldsymbol{d}_{\overline{\mathcal{C}},\mathcal{C}^0,\boldsymbol{t}}.$$

3.10

The next result generalizes Theorem 2.13.

THEOREM *Let* $\overline{\mathcal{C}}$ *be a category with global reduction. Let* t *be a type such that given* $X \in \{A, B, C\}$ *either* $t_X \leqq id$ *or* $t_X = \infty + 1$*. Let* d *be a tame* x*-dimension function of type* t *on* $\overline{\mathcal{C}}$*. Then* d *is universal of type* t *and is computed by the following rule: Define recursively disjoint,* x*-closed classes* $\mathcal{C}^n$ *as follows. Let* $\mathcal{C}^0 = \{A \in Ob(\mathcal{C}) \mid d(A) = 0\}$*. Let* $n \in \mathbb{N}$ *and assume that disjoint,* x*-closed classes* $\mathcal{C}^m$ *have been defined for all* $0 \leqq m < n$*. Define* $\delta^{n-1} : Ob(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ *by*

$$\delta^{n-1}(A) = \begin{cases} m & \text{if} \quad A \in \mathcal{C}^m \quad \text{where } 0 \leqq m < n, \\ \infty & \text{if} \quad A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m. \end{cases}$$

Obviously δ^{n-1} is an x-dimension function. Define $\mathcal{C}^n = \{A \in Ob(\mathcal{C}) \mid A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m, A$ has a δ^{n-1}-reduction of type t at $n\}$. Define $\mathcal{C}^\infty = Ob(\mathcal{C}) \setminus \bigcup_{n \in \mathbb{Z}^{\geq 0}} \mathcal{C}^n$. Then for any $A \in Ob(\mathcal{C})$ and any $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$,

$$d(A) = n \Leftrightarrow A \in \mathcal{C}^n.$$

Furthermore if $n \in \mathbb{Z}^{\geq 0}$ and $\mathcal{C}^n = \phi$ then $\mathcal{C}^{n'} = \phi$ for all $n' \geq n$, i.e. $d = \delta^{n-1}$.

3.11

Let $(\overline{\mathcal{C}}, d)$ be a category with global reduction and d an x-dimension function on $\overline{\mathcal{C}}$. For $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, define the classes

$$\boldsymbol{\mathcal{C}_d^n} = \{A \in \text{ Ob}(\mathcal{C}) \mid d(A) = n\}$$
$$\boldsymbol{\mathcal{C}_d^{\leqq n}} = \{A \in \text{ Ob}(\mathcal{C}) \mid d(A) \leqq n\}.$$

3.12

The next result generalizes Lemma 2.15.

LEMMA *The classes above are x-closed under virtual isomorphisms and $Ob(\mathcal{C}) = \bigcup_{n=0}^{\infty} \mathcal{C}_d^n$. Conversely any partition of $Ob(\mathcal{C})$ as a disjoint union $Ob(\mathcal{C}) = \bigcup_{n=0}^{\infty} \mathcal{C}^n$ of x-closed classes $\mathcal{C}^n$ defines an x-dimension function d on $\overline{\mathcal{C}}$ such that $\mathcal{C}_d^n = \mathcal{C}^n$. Partially order the members of $\{x -$ closed classes in $Ob(\mathcal{C})\}$ by inclusion. Then the function $\{x$-dimension functions of type t on $\overline{\mathcal{C}}\} \to \{x$-closed classes in $Ob(\mathcal{C})\}$, $d \mapsto \mathcal{C}_d^0$, preserves partial orderings and induces* *a bijection between universal x-dimension functions of type t and x-closed classes in $Ob(\mathcal{C})$. If $(\overline{\mathcal{D}}, e)$ is a category with x-dimension and $\mathcal{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ is a functor of categories with global reduction then $\mathcal{F}$ defines a morphism $(\overline{\mathcal{C}}, d) \to (\overline{\mathcal{D}}, e)$ of categories with x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}_d^{\leq n}) \subseteqq \mathcal{D}_e^{\leq n}$ for all $n \in \mathbb{Z}^{\geq 0}$. Furthermore if $d = d_{\overline{\mathcal{C}}, \mathcal{C}^0, t}$ and $e = d_{\overline{\mathcal{D}}, \mathcal{D}^0, t}$ for some x-closed classes $\mathcal{C}^0 \subseteq Ob(\mathcal{C})$ and $\mathcal{D}^0 \subseteq Ob(\mathcal{D})$ then $\mathcal{F}$ induces a morphism of categories with x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}^0) \subseteqq \mathcal{D}^0$.*

3.13

Let $\overline{\mathcal{C}}$ be a category with global reduction. If $S \subseteqq x - \dim(\overline{\mathcal{C}})$, define the partially ordered set $\mathbf{type(}\boldsymbol{S}\mathbf{)} = \{\text{types } t \mid \text{ each } d \in S \text{ has type } t\ \}$, with partial ordering that of types.

Since the results we prove for group valued functors are formulated using the concept of type for x-dimension functions, it is useful to have some general guidelines how large the set type(S) above is, in particular when S has just one element. The following lemmas provide such information.

3.14

The next lemma generalizes Lemma 2.17.

Lemma *The partially ordered set type(S) has the property that every element is bounded below by a minimal element. Furthermore if $t \in$ type(S) and t' is a type such that $t' \geq t$ then $t' \in$ type(S). Thus the minimal elements of type(S) define it uniquely.*

3.15

The next lemma generalizes Lemma 2.18.

Lemma *Let $\mathcal{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ be a functor of categories with global reduction. Let $T \subseteqq x - \dim(\overline{\mathcal{D}})$. Define $S = \{d\mathcal{F} \mid d \in T\}$.*

(i) Then $S \subseteqq x - \dim(\underline{\mathcal{C}})$ and type$(S) \subseteqq$ type(T).

Suppose that for each $A \in Ob(\mathcal{C})$, the function $glrd_{\overline{\mathcal{C}}}(A) \to glrd_{\overline{\mathcal{D}}}(A)$ induced by $\mathcal{F}$ is surjective up to isomorphism of global reductions in $glrd_{\overline{\mathcal{D}}}(A)$. Then $type(T) \subseteqq type(S)$.

4 Categories with Costructure, Global Coreduction, and Dimension

This section dualizes the concepts and results of the previous two sections. Proofs are the duals of their predecessors and will be omitted. Examples of the concepts are provided in section 5.

The principle of dualization reverses arrows in categories and replaces each functor by its dual. I.e., if one starts with a functor $F : \mathcal{C} \to \mathcal{D}$ of categories then dualizing replaces each arrow $f : A \to B$ in either $\mathcal{C}$ or $\mathcal{D}$ by an arrow $f^\circ : B^\circ \to A^\circ$ and the functor F by the functor $F^\circ : \mathcal{C}^\circ \to \mathcal{D}^\circ$ where F° sends each arrow $f^\circ : B^\circ \to A^\circ$ in $\mathcal{C}^\circ$ to the arrow $(F(f))^\circ : F(B)^\circ \to F(A)^\circ$ in $\mathcal{D}^\circ$. Thus if F is covariant (resp. contravariant) then so is F°. Note that whereas dualization reverses the orientation of arrows in categories, it does not reverse the orientation of functors between categories being dualized.

Applying dualization to a category with structure $\underline{\mathcal{C}}$, we get a category with costructure $\underline{\mathcal{C}}^\circ$ such that each coreduction in $\underline{\mathcal{C}}^\circ$ is the dual of a reduction in $\underline{\mathcal{C}}$. Applying dualization to a category with global reduction $\overline{\mathcal{C}}$, we get a category $\overline{\mathcal{C}}^\circ$ with global coreduction such that each global coreduction in $\overline{\mathcal{C}}^\circ$ is the dual of a global reduction in $\overline{\mathcal{C}}$. Dimension in a category with costructure or category with global coreduction is defined by reversing arrows and applying the existing definition of dimension in the corresponding category with structure or global reduction.

In order to fix terminology and avoid confusion later one, we carry out the details of the program above, but leave as mentioned already the proofs of the results as an exercise in dualization.

4.1

Throughout let $\mathcal{C}$ denote an arbitrary category.

Let **Iso ($\mathfrak{C}$)** denote the class of all isomorphisms in $\mathfrak{C}$. A class $\boldsymbol{\mathcal{V}(\mathfrak{C})}$ of morphisms in $\mathfrak{C}$ is called a class of **virtual isomorphism**, if $\boldsymbol{\mathcal{V}(\mathfrak{C})}$ contains $Iso(\mathfrak{C})$ and if $\mathcal{V}(\mathfrak{C})$ is closed under composition.

Let $x = s, t$, or ϕ, as in (2.1). Recall that the letter s stands for source, the letter t for target, and the symbol ϕ stands for the empty letter. In dualizing, s and t get interchanged. Define the concepts ***x*-closed under virtual isomorphism**, **closed under virtual isomorphism**, and **virtual chain equivalence**, as in (2.1).

4.2

A preordered set I is called **inverse directed**, if given elements $i, j \in I$, there is an element $k \in I$ such that $i \geqq k$ and $j \geqq k$. In dualizing, directed preordered sets are replaced by inverse directed ones, and conversely. This implies that direct limits $\varinjlim$ are replaced by inverse limits $\varprojlim$, and conversely.

Let $\{*\}$ denote the inverse directed preordered set with precisely one element $*$. If $A \in \mathrm{Ob}(\mathfrak{C})$, let F_A denote the functor

$$\begin{array}{rccc} F_A : & \{*\} & \to & \mathfrak{C} \\ & * & \mapsto & A \\ & 1_* & \mapsto & 1_A \end{array}$$

where 1_* and 1_A denote the identity morphisms on $*$ and A, respectively. F_A will be called a **trivial coinfrastructure functor**.

A class $\boldsymbol{co\mathfrak{I}(\mathfrak{C})}$ of functors with values in $\mathfrak{C}$ is called a **class of coinfrastructure functors**, if it contains all trivial coinfrastructure functors, is closed under natural isomorphism, and each $F \in co\mathfrak{I}(\mathfrak{C})$ has the property that its source category is an inverse directed, preordered set I and the inverse limit $\varprojlim_I F$ exists in $\mathfrak{C}$. By definition, $\varprojlim_I F$ is the limit $\lim_I F$.

4.3

If $A \in \mathrm{Ob}(\mathfrak{C})$ then the commutative square

$$\begin{array}{ccc} A & \xleftarrow{1} & A \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle 1} \\ A & \xleftarrow[1]{} & A \end{array}$$

will be called a **trivial square**.

A class $\mathbf{\mathcal{S}(\mathcal{C})}$ of commutative squares

$$\begin{array}{ccc} A & \longleftarrow & B \\ \uparrow & & \uparrow \\ C & \longleftarrow & D \end{array}$$

in $\mathcal{C}$ is called a **class of structure squares**, if it contains all trivial squares and is closed under isomorphism of commutative squares.

A class of $\mathbf{\mathcal{GS}(\mathcal{C})} \subseteqq \mathcal{S}(\mathcal{C})$ of commutative squares in $\mathcal{C}$ is called a **class of generating squares for $\mathcal{S}(\mathcal{C})$**, if it is closed under isomorphism of commutative squares, and every square in $\mathcal{S}(\mathcal{C})$ is an inverse limit in a prescribed way of squares in $\mathcal{GS}(\mathcal{C})$. This means that we have a class $\mathcal{FGS}(\mathcal{C})$ of functors S whose source categories are inverse directed, preordered sets J, whose values are in $\mathcal{GS}(\mathcal{C})$, and whose inverse limits $\varprojlim_J S$ exist and are in $\mathcal{S}(\mathcal{C})$ and every square in $\mathcal{S}(\mathcal{C})$ is some inverse limit $\varprojlim_J S$.

4.4

A **category with costructure** is a quadruple $\mathbf{(\mathcal{C}, \mathcal{V}(\mathcal{C}), co\mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C}))}$ consisting of a category $\mathcal{C}$, a class $\mathcal{V}(\mathcal{C})$ of virtual isomorphisms, a class $co\mathcal{I}(\mathcal{C})$ of coinfrastructure functors, and a class $\mathcal{S}(\mathcal{C})$ of structure squares. A **functor** or **morphism** $\mathcal{F} : (\mathcal{C}, \mathcal{V}(\mathcal{C}), co\mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C})) \to (\mathcal{D}, \mathcal{V}(\mathcal{D}), co\mathcal{I}(\mathcal{D}), \mathcal{S}(\mathcal{D}))$ of categories with costructure is a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ which preserves virtual isomorphisms, coinfrastructure functors, and structure squares. This means that

- if f is a virtual isomorphism then so is $\mathcal{F}(f)$,
- if $F : I \to \mathcal{C}$ is a coinfrastructure functor then so is the composition $\mathcal{F}F : I \to \mathcal{D}$ and the canonical morphism $\underset{\overleftarrow{I}}{\lim}\,\mathcal{F}F \to \mathcal{F}\left(\underset{\overleftarrow{I}}{\lim}\,F\right)$ is an isomorphism,
- if

$$\begin{tikzcd} A & B \arrow[l] \\ C \arrow[u] & D \arrow[l] \arrow[u] \end{tikzcd}$$

is a structure square in $\mathcal{C}$ then

$$\begin{tikzcd} \mathcal{F}(A) & \mathcal{F}(B) \arrow[l] \\ \mathcal{F}(C) \arrow[u] & \mathcal{F}(D) \arrow[l] \arrow[u] \end{tikzcd}$$

is one in $\mathcal{D}$.

4.5

Let $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{V}(\mathcal{C}), co\mathcal{I}(\mathcal{C}), \mathcal{S}(\mathcal{C}))$ denote a category with costructure. Let ∞ denote the first infinite ordinal. Let $x = s, t$, or ϕ be as in (4.1). The concepts of an ***x*−dimension function** on $\underline{\mathcal{C}}$ and of a **dimension function** on $\underline{\mathcal{C}}$ are defined exactly as in (2.5) for a category with structure. This means that dualizing interchanges s-dimension and t-dimension functions, whereas dimension functions remain dimension functions. A **category with costructure and *x*-dimension** is a pair $(\underline{\mathcal{C}}, d)$ consisting of a category with costructure $\underline{\mathcal{C}}$ and an x-dimension function d on $\underline{\mathcal{C}}$. A **functor** or **morphism** $\mathcal{F} : (\underline{\mathcal{C}}, d) \to (\underline{\mathcal{D}}, e)$ of categories with costructure and x-dimension is a functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ of categories with costructure such that $d \geqq e\mathcal{F}$.

The most important concepts for an x-dimension function are the notions of type and coreduction. The problems we can solve using dimension theory depend on these notions. They are discussed next.

4.6

As in (2.6), a **type** t is a triple $t_B, t_C, t_D : \mathbb{Z}^{\geq 0} \cup \{\infty\} \to \mathbb{N} \cup \{\infty, \infty+1\}$ of functions and is usually written in the form

$$t = \begin{pmatrix} & t_B \\ t_C & t_D \end{pmatrix}.$$

Let $\underline{\mathcal{C}}$ be a category with costructure. Let $A \in \operatorname{Ob}(\mathcal{C})$. A **coreduction** of A consists of

(i)

$$\left\{ \begin{array}{c} \text{a morphism } A \to A' \in \mathcal{V}(\mathcal{C}), \text{if } x = s. \\ \text{a morphism } A' \to A \in \mathcal{V}(\mathcal{C}),\ \text{if } x = t \\ \text{a virtual chain equivalence from } A \text{ to } A',\ \text{if } x = \phi \end{array} \right\} \mathit{and}$$

(ii) an infrastructure functor $B : I \to \mathcal{C}, i \mapsto B_i$, and

(iii) a set

$$\begin{array}{ccc} A' & \longleftarrow & B_i \\ \uparrow & & \uparrow \\ C_i & \longleftarrow & D_i \end{array} \quad (i \in I)$$

of costructure squares indexed by I (C and D are not necessarily functorial in I) with the property that for each relationship $(i \leqq j) \in I$, the triangle

$$\begin{tikzcd}
 & B_j \\
A' \arrow[ur] \arrow[dr] & \\
 & B_i \arrow[uu]
\end{tikzcd}$$

commutes.

Let d be an x-dimension function on $\underline{\mathcal{C}}$. Let t be a type and $n \in \mathbb{N}$. A coreduction of A is said to be a ***d*-coreduction of type *t* at *n***, if in addition to the above

(iv) $d\left(\lim\limits_{\overleftarrow{I}} B\right) < t_B(n)$

and for each $i \in I$,

(v) $d(C_i) < t_C(n)$, and

(vi) $d(D_i) < t_D(n)$.

Suppose now that $0 < d(A) < \infty$. Then a d-coreduction of type t at $d(A)$ for A will be simply called a ***d*-coreduction of type *t*** for A. An x-dimension function d on $\underline{\mathcal{C}}$ is said to **have type *t***, if each $A \in \mathrm{Ob}(\mathcal{C})$ such that $0 < d(A) < \infty$ has a d-coreduction of type t. An x-dimension function can have many types and obviously if it has type t and $t \leqq t'$ then it also has type t'. The applicability of a given x-dimension function d to a given problem depends on the minimal types of d.

4.7

Next we generalize the notion of a category with costructure to that of a category with global coreduction and define the notion of a dimension function on such categories, generalizing the existing notion on categories with costructure.

A **category with global coreduction** consists of a category with costructure $\underline{\mathcal{C}}$ and for each $A \in \mathrm{Ob}(\mathcal{C})$ a class $\boldsymbol{gcrd(A)}$ of sets X such that each member of X is a coreduction of A. Each member $X \in gcrd(A)$ is called a **global coreduction of $\boldsymbol{A}$**. It will be assumed that if $X \in gcrd(A)$ and Y is a set of coreductions of A such that each coreduction in X is isomorphic (in the straightforward sense) to a coreduction in Y and conversely then $Y \in gcrd(A)$. In this case, we say that X is **isomorphic** to Y. A category with global coreduction will be denoted by a pair

$$(\underline{\mathcal{C}}, gcrd)$$

where $gcrd$ assigns to each $A \in \mathrm{Ob}(\mathcal{C})$ a class $gcrd(A)$ of global coreductions of A, which is closed under isomorphism. We explicitly allow the possibility that for some $A's, gcrd(A) = \phi$.

Let $\mathcal{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}'$ be a functor of categories with costructure. It follows from the definition of such functors that $\mathcal{F}$ takes any coreduction of an object A of $\mathcal{C}$ to a coreduction of $\mathcal{F}(A)$. If R is a coreduction of A, let $\mathcal{F}(R)$ denote the corresponding coreduction of $\mathcal{F}(A)$. If X is a global coreduction of A in $(\underline{\mathcal{C}}, gcrd)$, let $\mathcal{F}(X) = \{\mathcal{F}(R) \mid R \in X\}$. We say that $\mathcal{F}$ defines a **functor** or **morphism** $(\underline{\mathcal{C}}, gcrd) \to (\underline{\mathcal{C}'}, gcrd')$ of categories with global coreduction, if for each $A \in \mathrm{Ob}(\mathcal{C})$ and $X \in gcrd(A), \mathcal{F}(X) \in gcrd'(\mathcal{F}(A))$.

4.8

Let $x = s$ (source), t (target), or ϕ (no condition), be as in (2.1). An $\boldsymbol{x}$ **-dimension function** d on $(\underline{\mathcal{C}}, gcrd)$ is by definition an x-dimension function on $\underline{\mathcal{C}}$, as in (2.5). A **category with global coreduction and $\boldsymbol{x}$-dimension** is a pair $((\underline{\mathcal{C}}, gcrd), d)$ consisting of a category with global coreduction $(\underline{\mathcal{C}}, gcrd)$ and an x-dimension function d on $(\underline{\mathcal{C}}, gcrd)$. A **functor** or **morphism** $\mathcal{F} : ((\underline{\mathcal{C}}, gcrd), d) \to ((\underline{\mathcal{C}'}, gcrd'), d')$ of categories with global coreduction and x-dimension is a functor $\mathcal{F} : (\underline{\mathcal{C}}, gcrd) \to (\underline{\mathcal{C}'}, gcld')$ of categories with global coreduction such that $d \geqq d'\mathcal{F}$.

As in sections 2 and 3, the most important properties of x-dimension functions are described using the notions of type and global coreduction. This is done in the following paragraphs.

4.9

Let d be an **x-dimension function** on the category with global coreduction $(\underline{\mathcal{C}}, gcrd)$. Let t denote a type, as in (4.6), and let $n \in \mathbb{N}$. A global coreduction $X \in gcrd(A)$ is called a **global d-coreduction of type t at n**, if $X \neq \phi$ and each coreduction in X is a d-coreduction of type t at n, as in (4.6) (iv) - (vi). A **global d-coreduction of type t at A** is by definition an $X \in gcrd(A)$, which is a global d-coreduction of type t at $d(A)$. Note that this definition makes sense only when $d(A) > 0$, because t is defined on $\mathbb{N} \cup \{\infty\}$. An x-dimension function d is said **to have type t**, if each $A \in \mathrm{Ob}(\mathcal{C})$ such that $0 < d(A) < \infty$ has a global d-coreduction of type t at A.

An x-dimension function on $(\underline{\mathcal{C}}, gcrd)$ can have many types and obviously if it has type t and $t \leqq t'$ then it also has type t'. The applicability of a given x-dimension function d to a given problem depends on the minimal types of d.

For the rest of this section, let

$$\overline{\mathcal{C}} = (\underline{\mathcal{C}}, gcrd)$$

denote a category with global coreduction. We continue developing the properties of x-dimension functions.

4.10

Let d be an x-dimension function of type t on the category with global coreduction $\overline{\mathcal{C}}$. Let $A \in \mathrm{Ob}\ (\mathcal{C})$ and $n \in \mathbb{N}$. Even if A has a global d-coreduction of type t at n, it does not follow that $d(A) \leqq n$. However if the conclusion above is true for any A and any n then we shall say that d is a **tame** x-dimension function of type t.

4.11

Let $\overline{\mathcal{C}}$ be a category with global coreduction. Define $\boldsymbol{x - \mathrm{dim}(\overline{\mathcal{C}})}$ to be the partially ordered class of all x-dimension functions on $\overline{\mathcal{C}}$, with partial ordering given by $\leqq$. If t is a type, let $\boldsymbol{x - \mathrm{dim}(\overline{\mathcal{C}}, t)}$ denote the partially ordered subclass of $x - \mathrm{dim}(\overline{\mathcal{C}})$ consisting of all x-dimension functions of type t.

4.12

The next result is the dual of Theorem 3.6.

THEOREM *The partially ordered class $x - \dim(\overline{\mathfrak{C}})$ has arbitrary greatest lower bounds. If $S \subsetneqq x - \dim(\overline{\mathfrak{C}})$ is a nonempty subclass then its greatest lower bound glb_S is computed by the formula*

$$glb_S(A) = \inf\{d(A) \mid d \in S\}.$$

Moreover if $S \subsetneqq x - \dim(\overline{\mathfrak{C}}, t)$ is a nonempty subclass then $glb_S \in x - \dim(\overline{\mathfrak{C}}, t)$.

4.13

Let next result is the dual of Theorem 3.7.

THEOREM *Let d be an x-dimension function of type t on the category with global coreduction $\overline{\mathfrak{C}}$. Then there is a tame x-dimension $\boldsymbol{d_t}$ of type t, called the* **tame closure of $\boldsymbol{d}$ at $\boldsymbol{t}$**, *which is the unique maximum among all tame x-dimension functions d' of type t such that $d' \leqq d$. Moreover d_t is computed according to the following rule: Let $T(d) = \{d' \mid d'$ a tame x-dimension function of type $t, d' \leqq d\}$ and $S(d) = \{d'' \mid d''$ an x-dimension function of type $t, d'' \leqq d, d' \leqq d'' \ \forall \ d' \in T(d)\}$. Then*

$$d_t = glb_{S(d)}.$$

4.14

Let $\overline{\mathfrak{C}}$ be a category with global coreduction. Let t be a type. An x-dimension function δ of type t is called **universal of type $\boldsymbol{t}$**, if $\delta \leqq d$ for any x-dimension function d of type t, whose 0-dimensional objects are also 0-dimensional under δ.

4.15

The next result is the dual of Corollary 3.9.

COROLLARY *Let $\overline{\mathcal{C}}$ be a category with global coreduction. Let $\mathcal{C}^0$ be a class of objects of $\mathcal{C}$, which is x-closed under virtual isomorphism. Then there is a unique universal x-dimension function of type t, whose 0-dimensional objects are precisely those of $\mathcal{C}^0$. Furthermore it is tame. This function is often denoted by*

$$\boldsymbol{d}_{\overline{\mathcal{C}},\mathcal{C}^0,\boldsymbol{t}}.$$

4.16

The next result is the dual of Theorem 3.10.

THEOREM *Let $\overline{\mathcal{C}}$ be a category with global coreduction. Let t be a type such that given $X \in \{A, B, C\}$ either $t_X \leqq id$ or $t_X = \infty + 1$. Let d be a tame x-dimension function of type t on $\overline{\mathcal{C}}$. Then d is universal of type t and is computed by the following rule: Define recursively disjoint, x-closed classes $\mathcal{C}^n$ as follows. Let $\mathcal{C}^0 = \{A \in Ob(\mathcal{C}) \mid d(A) = 0\}$. Let $n \in \mathbb{N}$ and assume that disjoint, x-closed classes $\mathcal{C}^m$ have been defined for all $0 \leqq m < n$. Define $\delta^{n-1} : Ob(\mathcal{C}) \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ by*

$$\delta^{n-1}(A) = \begin{cases} m & \text{if} \quad A \in \mathcal{C}^m \quad \text{where } 0 \leqq m < n, \\ \infty & \text{if} \quad A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m. \end{cases}$$

Obviously δ^{n-1} is an x-dimension function. Define $\mathcal{C}^n = \{A \in Ob(\mathcal{C}) \mid A \notin \bigcup_{m=0}^{n-1} \mathcal{C}^m, A$ has a δ^{n-1}-reduction of type t at $n\}$. Define $\mathcal{C}^\infty = Ob(\mathcal{C}) \setminus \bigcup_{n \in \mathbb{Z}^{\geq 0}} \mathcal{C}^n$. Then for any $A \in Ob(\mathcal{C})$ and any $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$,

$$d(A) = n \Leftrightarrow A \in \mathcal{C}^n.$$

Furthermore if $n \in \mathbb{Z}^{\geq 0}$ and $\mathcal{C}^n = \phi$ then $\mathcal{C}^{n'} = \phi$ for all $n' \geq n$, i.e. $d = \delta^{n-1}$.

4.17

Let $(\overline{\mathcal{C}}, d)$ be a category with global coreduction and d an x-dimension function on $\overline{\mathcal{C}}$. For $n \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, define the classes

$$\boldsymbol{\mathcal{C}^n_d} = \{A \in \ \mathrm{Ob}(\mathcal{C}) \mid d(A) = n\}$$
$$\boldsymbol{\mathcal{C}^{\leqq n}_d} = \{A \in \ \mathrm{Ob}(\mathcal{C}) \mid d(A) \leqq n\}.$$

4.18

The next result is the dual of Lemma 3.12.

Lemma *The classes above are x-closed under virtual isomorphisms and $Ob(\mathcal{C}) = \overset{\infty}{\underset{n=0}{\cup}} \mathcal{C}^n_d$. Conversely any partition of $Ob(\mathcal{C})$ as a disjoint union $Ob(\mathcal{C}) = \dot{\bigcup}_{n=0}^{\infty} \mathcal{C}^n$ of x-closed classes $\mathcal{C}^n$ defines an x-dimension function d on $\overline{\mathcal{C}}$ such that $\mathcal{C}^n_d = \mathcal{C}^n$. Partially order the members of $\{x -$ closed classes in $Ob(\mathcal{C})\}$ by inclusion. Then the function $\{x$-dimension functions of type t on $\overline{\mathcal{C}}\} \to \{x$-closed classes in $Ob(\mathcal{C})\}$, $d \mapsto \mathcal{C}^0_d$, preserves partial orderings and induces a bijection between universal x-dimension functions of type t and x-closed classes in $Ob(\mathcal{C})$. If $(\overline{\mathcal{D}}, e)$ is a category with global coreduction and x-dimension and $\mathcal{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ is a functor of categories with global coreduction then $\mathcal{F}$ defines a morphism $(\overline{\mathcal{C}}, d) \to (\overline{\mathcal{D}}, e)$ of categories with global coreduction and x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}^{\leq n}_d) \subseteqq \mathcal{D}^{\leqq n}_e$ for all $n \in \mathbb{Z}^{\geq 0}$. Furthermore if $d = d_{\overline{\mathcal{C}},\mathcal{C}^0,t}$ and $e = d_{\overline{\mathcal{D}},\mathcal{D}^0,t}$ for some x-closed classes $\mathcal{C}^0 \subseteq \ Ob(\mathcal{C})$ and $\mathcal{D}^0 \subseteq \ Ob(\mathcal{D})$ then $\mathcal{F}$ induces a morphism of categories with global coreduction and x-dimension $\Leftrightarrow \mathcal{F}(\mathcal{C}^0) \subseteqq \mathcal{D}^0$.*

4.19

Let $\overline{\mathcal{C}}$ be a category with global coreduction. If $S \subseteqq x - \dim(\overline{\mathcal{C}})$, define the partially ordered set $\mathbf{type}(\boldsymbol{S}) = \{$types $t \mid$ each $d \in S$ has type $t\ \}$, with partial ordering that of types.

Since the results we prove for group valued functors are formulated using the concept of type for x-dimension functions, it is useful to have some general guidelines how large the set type(S) above is, in particular when S has just one element. The following lemmas provide such information.

4.20

The next lemma is the dual of Lemma 3.14.

LEMMA *The partially ordered set type(S) has the property that every element is bounded below by a minimal element. Furthermore if $t \in$ type(S) and t' is a type such that $t' \geq t$ then $t' \in$ type(S). Thus the minimal elements of type(S) define it uniquely.*

4.21

The next lemma is the dual of Lemma 3.15.

LEMMA *Let $\mathcal{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ be a functor of categories with global coreduction. Let $T \subseteqq x - \dim(\overline{\mathcal{D}})$. Define $S = \{d\mathcal{F} \mid d \in T\}$.*

(i) Then $S \subseteqq x - \dim(\underline{\mathcal{C}})$ and type(S) $\subseteqq$ type(T).

Suppose that for each $A \in Ob(\mathcal{C})$, the function $gcrd_{\overline{\mathcal{C}}}(A) \to gcrd_{\overline{\mathcal{D}}}(A)$ induced by $\mathcal{F}$ is surjective up to isomorphism of global coreductions in $glrd_{\overline{\mathcal{D}}}(A)$. Then type($T$) $\subseteqq$ type(S).