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RELATIONSHIP OF NON-ABELIAN TENSOR PRODUCTS AND NON-ABELIAN HOMOLOGY OF GROUPS WITH WHITEHEAD'S GAMMA FUNCTOR

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ABSTRACT. Brown and Loday's exact sequence about the relationship of Whitehead's gamma functor with the non-abelian tensor product and the exterior product of groups is generalized for precrossed modules and is extended using their non-abelian left derived functors and non-abelian homology of groups.

Sufficient conditions of finite generation of the non-abelian tensor product of groups and sufficient conditions of finiteness of the non-abelian homology of groups with coefficients in groups are established.

INTRODUCTION

Let $\mu : M \to P$ and $\nu : N \to P$ be crossed modules over a group P and let M and N act on each other via P. Then R. Brown and J.-L. Loday get the following result [5]: there exists an exact sequence of groups

$$\Gamma(M \times_P N/ < M, N >) \xrightarrow{\Psi} M \otimes N \to M \wedge^P N \to 1,$$

where Γ is Whitehead's universal quadratic gamma functor [20] (see definition below), $M \times_P N = \{(m,n) \in M \times N \mid \mu(m) = \nu(n)\}$, $< M, N >= Im((\lambda, \lambda') : M \otimes N \to M \times_P N, (\lambda, \lambda')(m \otimes n) = (m \cdot {}^n m^{-1}, {}^m n n^{-1}))$, $M \otimes N$ and $M \wedge^P N$ are non-abelian tensor and non-abelian exterior products of M and N respectively, $\Psi(\gamma(m,n)) = m \otimes n$. Also Ψ has central image.

The non-abelian tensor product of groups was introduced by R. Brown and J.-L. Loday [3-5] following works of A. Lue [15] and R.K. Dennis [6]. It arises in applications to homotopy theory of a generalized Van Kampen theorem. It was defined for any

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groups A and B which act on themselves by conjugation (${}^{x}y = xyx^{-1}$) and each of which acts on the other, but the theory got development when actions satisfy the following compatibility conditions:

$$ {}^{({}^{a}b)}a' = {}^{aba^{-1}}a', \qquad {}^{({}^{b}a)}b' = {}^{bab^{-1}}b' \tag{1}$$

for all $a, a' \in A$ and $b, b' \in B$.

Guin generalized the tensor product of Brown and Loday for $\mu : M \to P$, $\nu : N \to P$ precrossed modules [8] in the sense that the groups A and B acting on each other compatibly form in a natural way a pair of crossed modules for which Guin's tensor product coincides with Brown and Loday's tensor product. We always denote Guin's non-abelian tensor product of precrossed modules by $M \overset{G}{\otimes} N$ (see definition below).

Using the non-abelian tensor product Guin [8] defined non-abelian homology of a group G with coefficients in G-crossed module A in low dimensions $H_0(G, A)$, $H_1(G, A)$ and which have important applications to K-theory of non-commutative local rings.

The definition of non-abelian homology of a group G with coefficients in any group A in any dimensions $H_n(G, A)$, $n \geq 0$ necessitated another generalization of the non-abelian tensor product of Brown and Loday (see [11]).

Let A and B be arbitrary groups which act on themselves by conjugation and each of which acts on the other. The non-abelian tensor product $A \otimes B$ of the groups A and B [11] is the group generated by the symbols $a \otimes b$, ($a \in A$, $b \in B$) subject to the relations

$$\begin{aligned} aa' \otimes b &= ({}^{a}a' \otimes {}^{a}b)(a \otimes b) \\ a \otimes bb' &= (a \otimes b)({}^{b}a \otimes {}^{b}b') \\ (a \otimes b)(a' \otimes b') &= ({}^{[a,b]}a' \otimes {}^{[a,b]}b')(a \otimes b) \\ (a' \otimes b')(a \otimes b) &= (a \otimes b)({}^{[b,a]}a' \otimes {}^{[b,a]}b') \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$,where $[a, b] = aba^{-1}b^{-1} \in A * B$. Note that when the actions are compatible then this tensor product coincides the Brown-loday tensor product.

As the non-abelian left derived functors of the non-abelian tensor product of groups a non-abelian homology of groups is constructed with coefficients in any group [11] which generalizes the classical Eilenberg- MacLane homology theory of groups [1] and extends Guin's non-abelian homology [8]. For non-commutative local rings the relationship of Milnor's algebraic K-functor K_2 with the symbol group Sym is given in terms of a long exact sequence of non-abelian homology of groups [11] extending Guin's six terms exact sequence [8]. In fact, let R be a non-commutative local ring such that $R/\operatorname{Rad} R \neq \mathbb{F}_2$, then one has the following exact sequence of groups

$$\begin{aligned} \cdots \longrightarrow H_3(R^*, [R^*, R^*]) \longrightarrow H_2(R^*, U(R), [R^*, R^*]) \longrightarrow H_2(R, U(R)) \longrightarrow \\ \longrightarrow H_2(R^*, [R^*, R^*]) \longrightarrow H_1(R^*, U(R), [R^*, R^*]) \longrightarrow H_1(R, U(R)) \longrightarrow \\ \longrightarrow H_1(R^*, [R^*, R^*]) \longrightarrow K_2(R) \longrightarrow \mathrm{Sym}(R) \longrightarrow [R^*, R^*]/[R^*, [R^*, R^*]] \longrightarrow 1. \end{aligned}$$

In Section 1 we continue to study the abovementioned generalized non-abelian tensor product of groups. It is shown the natural relation of the non-abelian tensor product of groups with any actions with the non-abelian tensor product of groups with compatible actions.

Sufficient conditions are given for the non-abelian tensor product of groups to be finitely generated. As the consequence of this fact we get that $GL_n(\mathbb{Z}) \otimes GL_n(\mathbb{Z})$, $E_n(\mathbb{Z}) \otimes E_n(\mathbb{Z})$ and $St_n(\mathbb{Z}) \otimes St_n(\mathbb{Z})$, $n \geq 5$ are finitely generated groups, where $GL_n(\mathbb{Z})$, $E_n(\mathbb{Z})$ and $St_n(\mathbb{Z})$ are the linear group of $n \times n$-matrices over $\mathbb{Z}$, the elementary group of $n \times n$-matrices over $\mathbb{Z}$ and the non-stable Steinberg group of $\mathbb{Z}$ respectively.

It is proved that the non-abelian tensor product of groups $M \otimes N$, of precrossed modules $\mu : M \to P$, $\nu : N \to P$ over the same group P is also a precrossed module over P.

We find nontrivial generalizations (Theorem 10 and Theorem 14) of the abovementioned theorem Brown and Loday about relationship of Whitehead's gamma functor with the non-abelian tensor product and the exterior product of groups for precrossed modules. A long exact sequence of their left derived functors and non-abelian homology of groups extends the natural epimorphism $M \otimes N \to M \wedge^P N$ and this is another application of non-abelian homology of groups.

Guin's homology with coefficients in precrossed modules ${}^G H_n(M, N)$, $n \geq 0$ is constructed using the non-abelian left derived functors of Guin's tensor product. It is easy to see that when (M, μ) and (N, ν) are crossed modules then ${}^G H_0(M, N) = H_0(M, N)$ and ${}^G H_1(M, N) = H_1(M, N)$.

Using the left derived functors of Γ, $-\otimes N$ and $\wedge^P N$ we obtain long exact sequences extending Brown and Loday's abovementioned three terms right exact sequence about relationship of Whitehead's gamma functor with non-abelian tensor product and non-abelian exterior product of crossed modules and making this relationship more precise.

Section 2 is concerned with finiteness of the non-abelian homology of groups. Using the loop functor $\mathbb{G}$ from the category of reduced complex to the category of simplicial groups [13] we construct a new simplicial projective resolution of a group G which give the possibility to prove that if G and A are finite groups, A acts trivially on G and G acts on A such that $G \otimes A$ is a finite group (see sufficient conditions in [12]) then $H_n(G, A)$, $n \geq 0$ is also a finite group.

1. Non-abelian Tensor Products of Precrossed Modules

First we recall some definitions from [12]

Definition 1. Let A and B be groups, which act on each other. The Comp-subgroup of A with B, denoted by $CompA(B)$, is the normal subgroup of A generated by the elements

$$^{(^a b)}a' (^{aba^{-1}} a')^{-1}$$

where $a, a' \in A$, $b, b' \in B$.

Definition 2. It will be said that A acts on B perfectly if the action of A on B induces an action of A on $\operatorname{Comp} B(A)$, and that A, B are groups with perfect actions if under these actions A acts on B perfectly and B acts on A perfectly.

Proposition 3. *Let A and B be groups. If they act on each other perfectly and the following conditions hold:*

$$^{a'}bb^{-1} \in \operatorname{Comp} B(A), \quad ^{b'}aa^{-1} \in \operatorname{Comp} A(B)$$

for each $a \in A$, $b \in B$, $a' \in \operatorname{Comp} A(B)$, $b' \in \operatorname{Comp} B(A)$, then there exists an isomorphism

$$\overline{A} \otimes \overline{B} \approx (A \otimes B)/H,$$

where H is a normal subgroup of $A \otimes B$ generated by $a' \otimes b'$, $a' \in \operatorname{Comp} A(B)$ or $b' \in \operatorname{Comp} B(A)$, and the groups $\overline{A} = A/\operatorname{Comp} A(B)$, $\overline{B} = B/\operatorname{Comp} B(A)$ act on each other compatibly by the induced actions.

Proof. It is easy to see that the actions of the groups A and B induce compatible actions of the groups $\overline{A} = A/\operatorname{Comp} A(B)$ and $\overline{B} = B/\operatorname{Comp} B(A)$.

We have a natural homomorphism $f' : A \otimes B \longrightarrow \overline{A} \otimes \overline{B}$. It is clear that $f'(H) = 1$. This induces a homomorphism

$$f : (A \otimes B)/H \longrightarrow \overline{A} \otimes \overline{B}.$$

Define a map $g' : \overline{A} \times \overline{B} \longrightarrow (A \otimes B)/H$ by the formula $g'([a],[b]) = [a \otimes b]$. Since

$$\begin{aligned} aa' \otimes bb' = ({}^a a' \otimes {}^a b \cdot {}^a b')(a \otimes bb') = ({}^a a' \otimes {}^a b)({}^{({}^a b)}({}^a a') \otimes {}^{({}^a b)}({}^a b')) \cdot \\ \cdot (a \otimes b)({}^b a \otimes {}^b b'), \end{aligned}$$

for every $a \in A$, $b \in B$, $a' \in \operatorname{Comp} A(B)$, $b' \in \operatorname{Comp} B(A)$, the map g' is defined correctly. It is easy to see that g' satisfies the conditions of crossed pairing. Therefore g' induces a homomorphism

$$\begin{gathered} g : \overline{A} \otimes \overline{B} \longrightarrow (A \otimes B)/H, \\ g([a] \otimes [b]) = [a \otimes b]. \end{gathered}$$

It is clear that fg and gf are identity homomorphisms. The proof is complete. ■

Theorem 4. *Let M and N be normal subgroups of some group G acting on each other by conjugation in G. If the groups $[M,N]$, $M \cap N$, $H_2(MN)$, $H_3(MN/M)$, $H_3(MN/N)$ are finitely generated, then $M \otimes N$ is a finitely generated group.*

Proof. From [5, Theorem 4.5] we have an exact sequence

$$\begin{gathered} \longrightarrow H_3(MN/M) \oplus H_3(MN/N) \longrightarrow \operatorname{Ker}([,] : M \wedge N \to [M,N]) \longrightarrow \\ H_2(MN) \longrightarrow \cdots, \end{gathered} \tag{2}$$

where $M \wedge N$ is the quotient of $M \otimes N$ by the subgroup generated by the elements $x \otimes x$ with $x \in M \cap N$ [5] and the homomorphism $[,] : M \wedge N \to [M,N]$ is defined by $m \otimes n \mapsto [m,n]$.

Under the conditions of the theorem the exact sequence (2) implies that $\mathrm{Ker}([,]: M\wedge N \to [M,N])$ is finitely generated and hence $M\wedge N$ is a finitely generated group.

From [5, Theorem 2.12] we have the exact sequence

$$\Gamma((M\cap N)/[M,N]) \longrightarrow M\otimes N \longrightarrow M\wedge N \longrightarrow 1. \tag{3}$$

Now we have to show that if A is a finitely generated abelian group, then the group $\Gamma(A)$ is finitely generated. Let consider the natural epimorphism $F\to A$, where F is the free abelian group generated by the generators of A. From [20] the induced homomorphism $\gamma(F)\to\gamma(A)$ is surjective and $\gamma(F)$ is a finitely generated free abelian group, therefore $\gamma(A)$ is a finitely generated abelian group.

Since $(M\cap N)/[M,N]$ is a finitely generated abelian group, this implies that $\Gamma((M\cap N)/[M,N])$ is also finitely generated. From (3) one deduces that $M\otimes N$ is a finitely generated group. The proof is complete. ■

Corollary 5. *Let G be a finitely generated group such that its abelization is finite and $H_2(G)$ is finitely generated then the non-abelian tensor square $G\otimes G$ is finitely generated.*

Note that the linear group $GL_n(\mathbb{Z})$, the elementary group $E_n(\mathbb{Z})$ and the Steinberg group $St(\mathbb{Z})$, $n\geq 5$, where $\mathbb{Z}$ is the ring of integers, satisfy conditions of Corollary 5. For instance it will be shown for the elementary group.

It is known that the group $E_n(\mathbb{Z})$ is perfect and finitely generated for $n\geq 3$ [2,10]. The non-stable Steinberg group $St_n(\mathbb{Z})$ of the ring $\mathbb{Z}$ is the universal central extension of $E_n(\mathbb{Z})$ for $n\geq 5$ [16] and we have the exact sequence

$$0\longrightarrow K_2(n,\mathbb{Z})\longrightarrow St_n(\mathbb{Z})\longrightarrow E_n(\mathbb{Z})\longrightarrow 1,$$

where $K_2(n,\mathbb{Z})$ is the non-stable Milnor functor of the ring $\mathbb{Z}$. Therefore $H_2(E_n(\mathbb{Z}))=K_2(n,\mathbb{Z})$, $n\geq 5$. By the stability theorem we have the isomorphism $K_2(n,\mathbb{Z})\xrightarrow{\approx} K_2(\mathbb{Z})$ for $n\geq 3$ [19]. By Milnor's theorem [16] the group $K_2(\mathbb{Z})$ is cyclic of order 2.

A precrossed module (M,μ) over P is a group homomorphism $\mu: M\to P$ together with an action of P on M, which satisfies the following condition

$$\mu({}^p m) = p\mu(m)p^{-1}, \ \ p\in P, \ \ m\in M.$$

If in addition

$${}^{\mu(m)}m' = mm'm^{-1}, \ \ m,m'\in M,$$

then (M,μ) is a crossed module over P.

Proposition 6. *Let $\mu\colon M\to P$, $\nu\colon N\to P$ be precrossed modules over the group P. Let M and N act on each other via P and themselves by conjugation. Then*

(i) *The group P acts on the group $M\otimes N$ as follows ${}^p(m\otimes n) = {}^pm\otimes {}^pn$,*
(ii) *$\lambda: M\otimes N\to P$ defined by the formula $\lambda(m\otimes n)=[\mu(m),\nu(n)]$ with the actions of P on $M\otimes N$ as in (i) is a precrossed module over P.*

Proof. (i) We only have to show that every element $p \in P$ induces a homomorphism

$$\widetilde{p} : M \otimes N \longrightarrow M \otimes N,$$
$$\widetilde{p}(m \otimes n) = {}^p(m \otimes n) = {}^pm \otimes {}^pn.$$

It follows from the equalities

$$\begin{gathered}
{}^pm \cdot {}^pm' \otimes {}^pn = ({}^{({}^pm)}({}^pm') \otimes {}^{\mu({}^pm)}({}^pn))({}^pm \otimes {}^pn) = ({}^{pmp^{-1}p}m' \otimes {}^{pmp^{-1}p}n)\cdot \\
({}^pm \otimes {}^pn) = ({}^{pm}m' \otimes {}^{pm}n)({}^pm \otimes {}^pn), \\
{}^pm \otimes {}^pn \cdot {}^pn' = ({}^pm \otimes {}^pn)({}^{\nu({}^pn)}({}^pm) \otimes {}^{({}^pn)}({}^pn')) = ({}^pm \otimes {}^pn)({}^{pnp^{-1}p}m \otimes {}^{pnp^{-1}p}n') \\
= ({}^pm \otimes {}^pn)({}^{pn}m \otimes {}^{pn}n'), \\
{}^p({}^{[m,n]}m') \otimes {}^p({}^{[m,n]}n') = {}^{[{}^pm,{}^pn]}({}^pm') \otimes {}^{[{}^pm,{}^pn]}({}^pn') = ({}^pm \otimes {}^pn)\cdot \\
({}^pm' \otimes {}^pn')({}^pm \otimes {}^pn)^{-1}, \\
{}^p({}^{[n,m]}m') \otimes {}^p({}^{[n,m]}n') = {}^{[{}^pn,{}^pm]}({}^pm') \otimes {}^{[{}^pn,{}^pm]}({}^pn') = ({}^pm \otimes {}^pn)^{-1}\cdot \\
({}^pm' \otimes {}^pn')({}^pm \otimes {}^pn).
\end{gathered}$$

(ii) Let define a map $\widetilde{\lambda} : M \times N \to P$, $\widetilde{\lambda}(m,n) = [\mu(m), \nu(n)]$.

We have

$$\begin{gathered}
\widetilde{\lambda}({}^mm', {}^mn)\widetilde{\lambda}(m,n) = [\mu({}^mm'), \nu({}^mn)] \cdot [\mu(m), \nu(n)] = \mu(m)\mu(m')\mu(m)^{-1}\mu(m)\cdot \\
\nu(n)\mu(m)^{-1} \cdot \mu(m)\mu(m')^{-1}\mu(m)^{-1}\mu(m)\nu(n)^{-1}\mu(m)^{-1} \cdot \mu(m)\nu(n)\mu(m)^{-1}\nu(n)^{-1} = \\
\mu(m)\mu(m')\nu(n)\mu(m')^{-1}\mu(m)^{-1}\nu(n)^{-1} = \widetilde{\lambda}(mm', n).
\end{gathered}$$

The prove of the rest three conditions of the crossed pairing (for the definition see [11]) is similar. Therefore $\widetilde{\lambda}$ is a crossed pairing and induces a homomorphism $\lambda : M \otimes N \to P$ defined by the formula $\lambda(m \otimes n) = [\mu(m), \nu(n)]$.

Show that λ is precrossed module over P with the action of P on $M \otimes N$ defined in (i). In effect

$$\begin{gathered}
\lambda({}^p(m \otimes n)) = [\mu({}^pm), \nu({}^pn)] = [p\mu(m)p^{-1}, p\nu(n)p^{-1}] = p[\mu(m), \nu(n)]p^{-1} = \\
p\lambda(m \otimes n)p^{-1}.
\end{gathered}$$

The proof is complete. ■

Now we recall Guin's definition of the non-abelian tensor product of precrossed modules. Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over the group P. Let M and N act on each other and themselves via the group P i.e.

$$\begin{aligned}
{}^mm' &= {}^{\mu(m)}m', & {}^mn &= {}^{\mu(m)}n, \\
{}^nm &= {}^{\nu(n)}m, & {}^nn' &= {}^{\nu(n)}n',
\end{aligned}$$

for any $m, m' \in M$, $n, n' \in N$.

Definition 7 ([8]). Guin's non-abelian tensor product of precrossed modules (M, μ) and (N, ν) over P is the group $M \overset{G}{\otimes} N$ generated by the symbols $m \otimes n$ and defined by the relations

$$mm' \otimes n = ({}^mm' \otimes {}^mn)(m \otimes n),$$
$$m \otimes nn' = (m \otimes n)({}^nm \otimes {}^nn'),$$

for every $m, m' \in M$ and $n, n' \in N$.

Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over P. The associated crossed modules are

$$\mu' : M/M' \longrightarrow P, \qquad \nu' : N/N' \longrightarrow P,$$

where M' is a normal subgroup of M generated by the elements ${}^{\mu(m)}m'mm'^{-1}m^{-1}$, for $m, m' \in M$, N' is a normal subgroup of N generated by the elements ${}^{\nu(n)}n'nn'^{-1}n^{-1}$, for $n, n' \in N$, μ' and ν' are induced by μ and ν respectively.

One has the canonical epimorphism

$$M \overset{G}{\otimes} N \xrightarrow{f} M/M' \otimes N/N'.$$

It is easy to verify that $\operatorname{Ker} f$ is a normal subgroup generated by the elements $m \otimes n$, where $m \in M'$ or $n \in N'$.

Let consider the fibre product $M \times_p N = \{(m,n) \in M \times N : \mu(m) = \nu(n)\}$.

Definition 8. The (non-abelian) exterior product $M \wedge^P N$ is obtained from the tensor product $M \overset{G}{\otimes} N$ by imposing the additional relations $m \otimes n = 1$ for all $(m,n) \in M \times_p N$.

Now we recall the definition of Whitehead's Γ-functor [20].

Definition 9. Let A be an abelian group. Then $\Gamma(A)$ is the group with generators $\gamma(a)$, $a \in A$ and the following relations:

$$\gamma(a) = \gamma(a^{-1}),$$
$$\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ac),$$

for every $a, b, c \in A$.

Theorem 10. *Let (M,μ) and (N,ν) be precrossed modules over the group P such that $\mu M \cap \nu N$ acts trivially on X, where X is the subgroup of $M \overset{G}{\otimes} N$ generated by the elements $m \otimes n$ such that $\mu(m) = \nu(n)$. Then there is an exact sequence of groups*

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \overset{G}{\otimes} N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m,n)) = m \otimes n$. Also Ψ has central image.

Proof. Let $\widetilde{\Psi} : M \times_p N \to M \overset{G}{\otimes} N$ be the map $(m,n) \mapsto m \otimes n$; Therefore $X = \operatorname{Ker}(M \overset{G}{\otimes} N \to M \wedge^p N)$ is the subgroup of $M \overset{G}{\otimes} N$ generated by $\operatorname{Im} \widetilde{\Psi}$.

Let consider a homomorphism $\lambda : M \overset{G}{\otimes} N \to P$, $\lambda(m \otimes n) = [\mu(m), \nu(n)]$. From [8, Proposition 2.4] it is known that $(M \overset{G}{\otimes} N, \lambda)$ is crossed module over P.

We have to show that X is central. Since $\lambda(m \otimes n) = [\mu(m), \nu(n)] = 1$ for $(m,n) \in M \times_p N$, one has $X \subseteq \operatorname{Ker} \lambda$. But $\operatorname{Ker} \lambda$ is central in $M \overset{G}{\otimes} N$ and therefore X is also central.

Let $(m,n),(u,v) \in M \times_p N$. Since $\mu(m) = \nu(n)$, $m \otimes n$, $u \otimes v \in Z(M \overset{G}{\otimes} N)$ and $\mu M \cap \nu N$ acts trivially on X, one has

$$mu \otimes nv = ({}^m u \otimes {}^m n {}^m v)(m \otimes nv) = ({}^m u \otimes {}^m n)({}^{({}^m n)}({}^m u) \otimes {}^{({}^m n)}({}^m v))\cdot$$
$$(m \otimes n)({}^n m \otimes {}^v n) = {}^m\left\{(u \otimes n)(m \otimes v)\right\}(m \otimes n)(u \otimes v).$$

Hence $(u \otimes n)(m \otimes v) \in X$ and

$$mu \otimes nv = (u \otimes n)(m \otimes v)(m \otimes n)(u \otimes v), \tag{4}$$

Let $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then by (4) one has
$m[m_1,m_2] \otimes n[n_1,n_2] = (m_1 m_2 m_1^{-1} m_2^{-1} \otimes n)(m \otimes n_1 n_2 n_1^{-1} n_2^{-1})(m \otimes n)\cdot$
$(m_1 m_2 m_1^{-1} m_2^{-1} \otimes n_1 n_2 n_1^{-1} n_2^{-1}) = {}^{m_1 m_2 m_1^{-1}}(m_2^{-1} \otimes n) \cdot {}^{m_1 m_2}(m_1^{-1} \otimes n)\cdot$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1}(m \otimes n_2) \cdot {}^{n_1 n_2}(m \otimes n_1^{-1}) \cdot {}^{n_1 n_2 n_1^{-1}}(m \otimes n_2^{-1})$
$(m \otimes n)(m_2 m_1^{-1} m_2^{-1} \otimes n_1)(m_1 \otimes n_2 n_1^{-1} n_2^{-1})(m_1 \otimes n_1)$
$(m_2 m_1^{-1} m_2^{-1} \otimes n_2 n_1^{-1} n_2^{-1}) = \left\{(m_1 \otimes n)(m \otimes n_1)\right\} \cdot {}^{m_1}\left\{(m_2 \otimes n)(m \otimes n_2)\right\}\cdot$
${}^{m_1 m_2}\left\{(m_1^{-1} \otimes n)(m \otimes n_1^{-1})\right\} \cdot {}^{m_1 m_2 m_1^{-1}}\left\{(m_2^{-1} \otimes n)(m \otimes n_2^{-1})\right\}(m \otimes n)$
$\left\{(m_2 \otimes n_1)(m_1 \otimes n_2)\right\} \cdot {}^{m_2}\left\{(m_1^{-1} \otimes n_1)(m_1 \otimes n_1^{-1})\right\}\cdot$
${}^{m_2 m_1^{-1}}\left\{(m_2^{-1} \otimes n_1) \cdot (m_1 \otimes n_2^{-1})\right\}(m_1 \otimes n_1)(m_1^{-1} m_2^{-1} \otimes n_2)(m_2 \otimes n_1^{-1} n_2^{-1})\cdot$
$(m_2 \otimes n_2)(m_1^{-1} m_2^{-1} \otimes n_1^{-1} n_2^{-1}) = \left\{(m_1 \otimes n)(m \otimes n_1)\right\}\left\{(m_1^{-1} \otimes n)(m \otimes n_1^{-1})\right\}\cdot$
$\left\{(m_2 \otimes n)(m \otimes n_2)\right\}\left\{(m_2^{-1} \otimes n)(m \otimes n_2^{-1})\right\}(m \otimes n)\left\{(m_2 \otimes n_1)(m_1 \otimes n_2)\right\}\cdot$
$\left\{(m_2^{-1} \otimes n_1) \cdot (m_1 \otimes n_2^{-1})\right\}\left\{(m_1^{-1} \otimes n_1)(m_1 \otimes n_1^{-1})\right\}(m_1 \otimes n_1)\cdot$
$\left\{(m_1^{-1} \otimes n_2)(m_2 \otimes n_1^{-1})\right\} \cdot {}^{m_1^{-1}}\left\{(m_2^{-1} \otimes n_2)(m_2 \otimes n_2^{-1})\right\}(m_2 \otimes n_2)\cdot$
$\left\{(m_2^{-1} \otimes n_1^{-1})(m_1^{-1} \otimes n_2^{-1})\right\}(m_1^{-1} \otimes n_1^{-1})(m_2^{-1} \otimes n_2^{-1}) = (m \otimes n)(m_1 \otimes n_1)^{-1}\cdot$
$(m_1 \otimes n_2)^{-1} \cdot (m_2 \otimes n_1)^{-1}(m_2 \otimes n_2)^{-1}(m_2 \otimes n_2)^{-1} \cdot (m_2 \otimes n_2)(m_2 \otimes n_1)\cdot$
$(m_1 \otimes n_2)(m_1 \otimes n_1)(m_2 \otimes n_2) = (m \otimes n)$.

This proves that $\widetilde{\Psi}$ induces a map on $(M \times_p N)^{ab}$.

We now start the proof that $\widetilde{\Psi}$ induces a homomorphism on $\Gamma((M \times_p N)^{ab})$. Let $(m,n) \in M \times_p N$, then

$$m^{-1} \otimes n^{-1} = ({}^{m^{-1}}m \otimes {}^{m^{-1}}(n^{-1}))^{-1} = {}^{m^{-1}}({}^{n^{-1}}m \otimes {}^{n^{-1}}n) = {}^{m^{-1}n^{-1}}(m \otimes n) = m \otimes n.$$

Therefore $\widetilde{\Psi}[(m,n)] = \widetilde{\Psi}[(m^{-1},n^{-1})]$ for $(m,n) \in M \times_p N$.

Let $(m,n),(m_1,n_1),(m_2,n_2) \in M \times_p N$. Then by (4)
$mm_1m_2 \otimes nn_1n_2 = (m_1m_2 \otimes n)(m \otimes n_1n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1}(m \otimes n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
$\{(m_1 \otimes n)(m \otimes n_1)\}\{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_1m_2 \otimes n_1n_2)$,
$mm_1 \otimes nn_1 = \{(m_1 \otimes n)(m \otimes n_1)\}(m \otimes n)(m_1 \otimes n_1)$,
$mm_2 \otimes nn_2 = \{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_2 \otimes n_2)$.

Hence

$$(mm_1m_2 \otimes nn_1n_2)(m \otimes n)(m_1 \otimes n_1)(m_2 \otimes n_2) = (mm_1 \otimes nn_1)\cdot$$
$$(m_1m_2 \otimes n_1n_2)(mm_2 \otimes nn_2),$$

i.e.

$$\widetilde{\Psi}[(mm_1m_2, nn_1n_2)]\widetilde{\Psi}[(m,n)]\widetilde{\Psi}[(m_1,n_1)]\widetilde{\Psi}[(m_2,n_2)] = \widetilde{\Psi}[(mm_1,nn_1)]\cdot \widetilde{\Psi}[(m_1m_2,n_1n_2)]\widetilde{\Psi}[(mm_2,nn_2)].$$

Thus $\widetilde{\Psi}$ induces a homomorphism $\Psi : \Gamma((M \times_p N)^{ab}) \to M \overset{G}{\otimes} N$ given by $\Psi(\gamma(m,n)) = m \otimes n$. The proof is complete. ■

Corollary 11. *Let (M,μ) and (N,ν) be precrossed P-modules and let $[\mu M, \nu N] = \mu M \cap \nu N$. Then there is an exact sequence of groups*

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \overset{G}{\otimes} N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m,n)) = m \otimes n$. Also Ψ has central image.

Proof. We have only to show that $\mu M \cap \nu N$ acts trivially on $X = Ker(M \overset{G}{\otimes} N \to M \wedge^p N)$.

Let $p \in \mu M \cap \nu N$ and $l \in \operatorname{Ker}\lambda$, where $\lambda : M \overset{G}{\otimes} N \to P$, $\lambda(m \otimes n) = [\mu(m), \nu(n)]$. Then there exists $l' \in M \overset{G}{\otimes} N$ such that $\lambda(l') = p$. Since [8] $(M \overset{G}{\otimes} N, \lambda)$ is a crossed module over P and therefore l is in the center of $M \otimes N$, one gets

$${}^{p}l = {}^{\lambda(l')}l = l'll'^{-1} = l.$$

Hence $\mu M \cap \nu N$ acts trivially on $Ker\lambda$. Since $X \subseteq \operatorname{Ker}\lambda$, the group $\mu M \cap \nu N$ acts trivially on X. Then from Theorem 10 follows the assertion. The proof is complete. ■

We have a non-trivial

Example 12. Let P be a perfect group i.e. $P = [P,P]$, A be any group and P act on A non trivially. Let $(M,\mu) = (N,\nu) = (A \rtimes P, f)$, where $A \rtimes P$ is a semi-direct product of A and P , f is the natural surjective homomorphism of $A \rtimes P$ on P. The action of P on $A \rtimes P$ is given by the formula: ${}^{p'}(a,p) = ({}^{p'}a, {}^{p'}p)$. It is easy to verify that $(M,\mu) = (N,\nu)$ is a precrossed module, not crossed module and $[\mu M, \nu N] = \mu M \cap \nu N$.

A similar result holds for our non-abelian tensor product of precrossed modules extending Brown and Loday's Theorem 2.12 [5] for crossed modules.

Lemma 13. *Let A and B be groups which act on each other and on themselves by conjugation.*

(i) *If $a \in A$ such that ${}^{({}^{a}\bar{b})}\bar{a} = {}^{a\bar{b}a^{-1}}\bar{a}$, for every $\bar{a} \in A$, $\bar{b} \in B$, then a induces the homomorphism*

$$\widetilde{a} : A \otimes B \longrightarrow A \otimes B,$$
$$\widetilde{a}(a' \otimes b') = {}^{a}(a' \otimes b') = {}^{a}a' \otimes {}^{a}b'.$$

(ii) *If $b \in B$ such that ${}^{({}^{b}\bar a)}\bar b = {}^{b\bar a b^{-1}}\bar b$, for every $\bar a \in A$, $\bar b \in B$, then b induces the homomorphism*

$$\begin{aligned} &\widetilde{b} : A \otimes B \longrightarrow A \otimes B, \\ &\widetilde{b}(a' \otimes b') = {}^{b}(a' \otimes b') = {}^{b}a' \otimes {}^{b}b'. \end{aligned}$$

(iii) *If $a, a' \in A$ such that ${}^{({}^{a}\bar b)}\bar a = {}^{a\bar b a^{-1}}\bar a$, ${}^{({}^{a'}\bar b)}\bar a = {}^{a'\bar b a'^{-1}}\bar a$, for every $\bar a \in A$, $\bar b \in B$ and if $b, b' \in B$ such that ${}^{({}^{b}\bar a)}\bar b = {}^{b\bar a b^{-1}}\bar b$, ${}^{({}^{b'}\bar a)}\bar b = {}^{b'\bar a b'^{-1}}\bar b$, for every $\bar a \in A$, $\bar b \in B$, then*

$$\begin{aligned} &g \cdot {}^{b}a^{-1} \otimes b' = (a \otimes b) \cdot {}^{b'}(a \otimes b)^{-1}, \\ &a' \otimes {}^{a}bb^{-1} = {}^{a'}(a \otimes b)(a \otimes b)^{-1}. \end{aligned}$$

Proof. **(i)** Follows from the equalities

$$\begin{aligned} &{}^{a}(a'a'' \otimes b') = {}^{a}a' \cdot {}^{a}a'' \otimes {}^{a}b' = ({}^{({}^{a}a')a}a'' \otimes {}^{({}^{a}a')a}b')({}^{a}a' \otimes {}^{a}b') = \\ &= ({}^{aa'}a'' \otimes {}^{aa'}b')({}^{a}a' \otimes {}^{a}b') = {}^{a}({}^{a'}a'' \otimes {}^{a'}b') \cdot {}^{a}(a' \otimes b'), \\ &{}^{a}(a' \otimes b'b'') = {}^{a}a' \otimes {}^{a}b' \cdot {}^{a}b'' = ({}^{a}a' \otimes {}^{a}b')({}^{({}^{a}b')a}a' \otimes {}^{({}^{a}b')a}b'') = \\ &= ({}^{a}a' \otimes {}^{a}b')({}^{ab'}a' \otimes {}^{ab'}b'') = {}^{a}(a' \otimes b') \cdot {}^{a}({}^{b'}a' \otimes {}^{b'}b''), \\ &{}^{a}(a' \otimes b') \cdot {}^{a}(a'' \otimes b'') \cdot {}^{a}(a' \otimes b')^{-1} = ({}^{a}a' \otimes {}^{a}b')({}^{a}a'' \otimes {}^{a}b'')({}^{a}a' \otimes {}^{a}b')^{-1} = \\ &= {}^{[{}^{a}a', {}^{a}b']a}a'' \otimes {}^{[{}^{a}a', {}^{a}b']a}b'' = {}^{a[a',b']}a'' \otimes {}^{a[a',b']}b'' = {}^{a}({}^{[a',b']}a'' \otimes {}^{[a',b']}b''), \\ &{}^{a}(a' \otimes b')^{-1} \cdot {}^{a}(a'' \otimes b'') \cdot {}^{a}(a' \otimes b') = ({}^{a}a' \otimes {}^{a}b')^{-1}({}^{a}a'' \otimes {}^{a}b'')({}^{a}a' \otimes {}^{a}b') = \\ &= {}^{[{}^{a}b', {}^{a}a']a}a'' \otimes {}^{[{}^{a}b', {}^{a}a']a}b'' = {}^{a[b',a']}a'' \otimes {}^{a[b',a']}b'' = {}^{a}({}^{[b',a']}a'' \otimes {}^{[b',a']}b''). \end{aligned}$$

(ii) Can be proved analogously.

(iii)

$$\begin{aligned} &a \cdot {}^{b}a^{-1} \otimes b' = ({}^{ab}a^{-1} \otimes {}^{a}b')(a \otimes b') = ({}^{ab}a^{-1} \otimes {}^{a}b^{-1})({}^{({}^{a}b^{-1})ab}a^{-1} \otimes {}^{({}^{a}b^{-1})ab}(b'b)) \\ &(a \otimes n') = (a \otimes b)(a^{-1} \otimes {}^{a}b' \cdot {}^{a}b)(a \otimes b') = (a \otimes b)(a^{-1} \otimes {}^{a}b')({}^{({}^{a}b')}a^{-1} \otimes {}^{({}^{a}b')a}b) \\ &(a \otimes b') = (a \otimes b)(a \otimes b')^{-1}({}^{ab'}a^{-1} \otimes {}^{ab'}b)(a \otimes b') = (a \otimes b) \cdot {}^{b'}(a \otimes b)^{-1}. \end{aligned}$$

The second part of **(iii)** is proved similarly. The proof is complete. ■

Theorem 14. *Let $\mu : M \to P$ and $\nu : N \to P$ be precrossed modules over the group P. Let M and N act on each other via P and on themselves by conjugation and the following conditions hold:*
${}^{\mu(m)}m' = mm'm^{-1}$ *if* $\exists\ n \in N$ *such that* $\mu(m) = \nu(n)$,
${}^{\nu(n)}n' = nn'n^{-1}$ *if* $\exists\ m \in M$ *such that* $\nu(n) = \mu(m)$.
Then there is an exact sequence of groups

$$\Gamma((M \times_p N)^{ab}) \xrightarrow{\Psi} M \otimes N \to M \wedge^p N \to 1,$$

where $\Psi(\gamma(m,n)) = m \otimes n$. Also Ψ has central image.

Proof. Let $\widetilde{\Psi} : M \times_p N \to M \otimes N$ be the map given by $(m,n) \mapsto m \otimes n$. Then $X = \mathrm{Ker}(M \otimes N \to M \wedge^p N)$ is the subgroup of $M \otimes N$ generated by $\mathrm{Im}\, \widetilde{\Psi}$.

Now we have to show that X is central in $M \otimes N$. Let $m \otimes n$, $m' \otimes n' \in M \otimes N$ and $\mu(m) = \nu(n)$. Then $m \cdot {}^n m^{-1} = 1$, ${}^m n n^{-1} = 1$ and

$$(m \otimes n)(m' \otimes n')(m \otimes n)^{-1} = {}^{mnm^{-1}n^{-1}}m' \otimes {}^{mnm^{-1}n^{-1}}n' = {}^{m \cdot {}^n m^{-1}}m' \otimes {}^{{}^m n n^{-1}}n' = \\ = m' \otimes n'$$

i.e. we have $(m \otimes n)(m' \otimes n') = (m' \otimes n')(m \otimes n)$ and X is central.

If $(m,n) \in M \times_p N$ then m and n verify conditions of Lemma 13, therefore they act on $M \otimes N$ and trivially on X.

Let (m,n), $(u,v) \in M \times_p N$. Since $\mu(m) = \nu(n)$ and $m \otimes n$, $u \otimes v \in Z(M \otimes N)$,

$$mu \otimes nv = ({}^m u \otimes {}^m n^m v)(m \otimes nv) = {}^m \{(u \otimes n)(m \otimes v)\} (m \otimes n)(u \otimes v).$$

It follows that $(u \otimes n)(m \otimes v) \in X$ and

$$mu \otimes nv = (u \otimes n)(m \otimes v)(m \otimes n)(u \otimes v). \tag{5}$$

Suppose $(m,n), (m_1,n_1), (m_2,n_2) \in M \times_p N$. Then by using (5) one has

$$m[m_1,m_2] \otimes n[n_1,n_2] = m \otimes n.$$

This proves that $\widetilde{\Psi}$ induces a map defined on $(M \times_p N)^{ab}$.

We now start the proof that $\widetilde{\Psi}$ induces a homomorphism on $\Gamma((M \times_p N)^{ab})$. Let $(m,n) \in M \times_p N$, then

$$m^{-1} \otimes n^{-1} = ({}^{m^{-1}}m \otimes {}^{m^{-1}}(n^{-1}))^{-1} = {}^{m^{-1}}({}^{n^{-1}}m \otimes {}^{n^{-1}}n) = {}^{m^{-1}n^{-1}}(m \otimes n) = m \otimes n.$$

Hence $\widetilde{\Psi}[(m,n)] = \widetilde{\Psi}[(m^{-1},n^{-1})]$.

Let $(m,n), (m_1,n_1), (m_2,n_2) \in M \times_p N$. Then from (5)
$mm_1m_2 \otimes nn_1n_2 = (m_1m_2 \otimes n)(m \otimes n_1n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
${}^{m_1}(m_2 \otimes n)(m_1 \otimes n)(m \otimes n_1) \cdot {}^{n_1} (m \otimes n_2)(m \otimes n)(m_1m_2 \otimes n_1n_2) =$
$\{(m_1 \otimes n)(m \otimes n_1)\}\{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_1m_2 \otimes n_1n_2)$,
$mm_1 \otimes nn_1 = \{(m_1 \otimes n)(m \otimes n_1)\}(m \otimes n)(m_1 \otimes n_1)$,
$mm_2 \otimes nn_2 = \{(m_2 \otimes n)(m \otimes n_2)\}(m \otimes n)(m_2 \otimes n_2)$.
Hence we have

$$(mm_1m_2 \otimes nn_1n_2)(m \otimes n)(m_1 \otimes n_1)(m_2 \otimes n_2) = (mm_1 \otimes nn_1) \cdot \\ (m_1m_2 \otimes n_1n_2)(mm_2 \otimes nn_2),$$

i.e.

$$\widetilde{\Psi}[(mm_1m_2, nn_1n_2)]\widetilde{\Psi}[(m,n)]\widetilde{\Psi}[(m_1,n_1)]\widetilde{\Psi}[(m_2,n_2)] = \widetilde{\Psi}[(mm_1,nn_1)] \cdot \\ \widetilde{\Psi}[(m_1m_2,n_1n_2)]\widetilde{\Psi}[(mm_2,nn_2)].$$

Thus $\widetilde{\Psi}$ induces a homomorphism Ψ:$\Gamma((M \times_p N)^{ab}) \to M \otimes N$, $\Psi(\gamma(m,n)) = m \otimes n$.
The proof is complete. ■

Example 15. Let A be metabelian (not abelian) group i.e. $[A, A] \subset Z(A)$ (center of A). Let $M = A$, $N = A$, $P = (A/Z(A)) \times A$, the actions of P on M and N be given: $\forall\ p = ([a], a')$, $m = a''$, $n = a'''$

$$
{}^{([a],a')}a'' = {}^{a'}a'',
$$
$$
{}^{([a],a')}a''' = {}^{a'}a'''
$$

and $\mu : M \to P$, $\mu(a) = ([a], 1)$, $\nu : N \to P$, $\nu(a) = (1, a)$. Then it is easy to verify that (M, μ) is a precrossed module and not a crossed module, (N, ν) is a crossed module and conditions (1) hold.

Consider PCM(P) the category of precrossed modules over the group P. The objects of this category are (M, μ) precrossed modules over P and morphisms $(M, \mu) \xrightarrow{f} (M', \mu')$ of the category are homomorphisms $M \xrightarrow{f} M'$ such that 1. $f({}^p m) = {}^p f(m)$, 2. $\mu' f = \mu$.

Now we construct a cotriple $\mathcal{F} = (F, \tau, \delta)$ in the category PCM(P) in the following manner. Let $F : \mathrm{PCM}(P) \to \mathrm{PCM}(P)$ be an endofunctor defined as follows: for $(M, \mu) \in \operatorname{ob} \mathrm{PCM}(P)$ the object $F(M, \mu) \equiv (F(M), \widetilde{\mu})$ is a free precrossed module $\widetilde{\mu} : F(M) \to P$ generated by the precrossed module (M, μ), where $F(M)$ is the free group generated by M, $\widetilde{\mu}$ is the homomorphism induced by μ and the action of P on $F(M)$ is given by the formula: ${}^p(|m_1|^{\epsilon_1}|m_2|^{\epsilon_2} \cdots |m_s|^{\epsilon_s}) = |{}^p m_1|^{\epsilon_1}|{}^p m_2|^{\epsilon_2} \cdots |{}^p m_s|^{\epsilon_s}$, for $p \in P$, $|m_1|^{\epsilon_1}|m_2|^{\epsilon_2} \cdots |m_s|^{\epsilon_s} \in F(M)$ and $\epsilon_i = \pm 1$. If $f : (M, \mu) \to (M', \mu')$ is a morphism of PCM(P), then $F(f) : (F(M), \widetilde{\mu}) \to (F(M'), \widetilde{\mu'})$ is the natural homomorphism induced by f.

Let $\tau : F \to 1_{\mathrm{PCM}(P)}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for $(M, \mu) \in \operatorname{ob} \mathrm{PCM}(P)$ by the injection $M \to F(M)$. It is easy to verify that the homomorphisms $\tau_{(M,\mu)}$ and $\delta_{(M,\mu)}$ are morphisms of the category PCM(P) and that $\mathcal{F} = (F, \tau, \delta)$ is a cotriple which we call the free cotriple in the category PCM(P) of precrossed modules over P.

It is easy to see that there exist simplicial kernels in the category PCM(P).

Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(- \otimes N)$, $L_n^{\mathcal{P}}(- \wedge^P N)$, $n \geq 0$, of the functors $- \otimes N$, $- \wedge^P N : \mathrm{PCM}(P) \to \mathfrak{Gr}$ respectively relative to the projective class $\mathcal{P}$ [9,14] induced by the free cotriple $\mathcal{F}$.

Theorem 16. *Let $M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 1$ be an exact sequence of precrossed modules. Then we have the following exact sequence of groups*

$$
M_1 \wedge^P N \xrightarrow{f'} M \wedge^P N \xrightarrow{g'} M_2 \wedge^P N \to 1,
$$

where $f' = f \wedge^P 1_N$, $g' = g \wedge^P 1_N$.

Proof. According to the right exactness of the non-abelian tensor product of groups [11, Theorem 1] we have an exact sequence of groups

$$
M_1 \otimes N \xrightarrow{\overline{f'}} M \otimes N \xrightarrow{\overline{g'}} M_2 \otimes N \to 1.
$$

Let X be the subgroup of $M \otimes N$ generated by $m \otimes n$ such that $\mu(m) = \nu(n)$ and X_2 be the subgroup of $M_2 \otimes N$ generated by $m_2 \otimes n$ such that $\mu_2(m_2) = \nu(n)$. Then the restriction of $\overline{g'}$ on X is surjective on X_2.

We have a commutative diagram of groups

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & X & \longrightarrow & X_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
M_1 \otimes N & \xrightarrow{\overline{f'}} & M \otimes N & \xrightarrow{\overline{g'}} & M_2 \otimes N & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
M_1 \wedge^P N & \xrightarrow{f'} & M \wedge^P N & \xrightarrow{g'} & M_2 \wedge^P N & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array} .$$

Since columns and two top rows are exact, the bottom row is also exact. The proof is complete. ■

Theorem 17. *There exists a natural isomorphism*

$$L_0^{\mathcal{P}}(- \wedge^P N) \approx - \wedge^P N$$

Proof. Consider a $\mathcal{P}$-projective pseudosimplicial resolution of an object M of the category PCM(P)

$$\cdots \begin{smallmatrix}\to\\ \vdots \\ \to\end{smallmatrix} X_n \ \begin{smallmatrix}\xrightarrow{\partial_0^n}\\ \vdots \\ \xrightarrow[\partial_n^n]{}\end{smallmatrix} \cdots \rightrightarrows X_1 \ \begin{smallmatrix}\xrightarrow{\partial_0^1}\\ \xrightarrow[\partial_1^1]{}\end{smallmatrix} X_0 \xrightarrow{\partial_0^0} M.$$

Then the sequence of precrossed modules

$$\operatorname{Ker} \partial_0^1 \xrightarrow{\partial_1^1 \sigma} X_0 \xrightarrow{\partial_0^0} M \to 1$$

is exact, where $\sigma : \operatorname{Ker} \partial_0^1 \to X_1$ is the inclusion and from Theorem 16 it follows that the sequence

$$\operatorname{Ker} \partial_0^1 \wedge^P N \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0 \wedge^P 1_N} M \wedge^P N \to 1$$

is also exact.

We have the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Ker} \partial_0^1 \wedge^P N & \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} & X_0 \wedge^P N & & & & \\
\big\downarrow f & & \big\| & & & & \\
\operatorname{Ker}(\partial_0^1 \wedge^P 1_N) & \xrightarrow{\partial_1^1 \wedge^P 1_N} & X_0 \wedge^P N & \xrightarrow{\partial_0^0 \wedge^P 1_N} & M \wedge^P N & \longrightarrow & 1,
\end{array}$$

where the homomorphism f induced by the inclusion σ is surjective by

Theorem 16. Thus the bottom sequence of this diagram is exact. The proof is complete. ■

Theorem 18. *Let $\mu : M \to P$, $\nu : N \to P$ be precrossed modules over the group P. Then we have the following exact sequence of groups*

$$\begin{gathered}\cdots \longrightarrow H_3(M,N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f,N) \longrightarrow \\ H_2(M,N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f,N) \longrightarrow M \otimes N \xrightarrow{f} \\ M \wedge^P N \longrightarrow 1\end{gathered},$$

where

$$\begin{gathered}L_n^{\mathcal{P}}(f,N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0, \\ \operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F^n(M) \otimes N \to F^n(M) \wedge^P N,\ n \geq 1\}.\end{gathered}$$

Proof. Let consider the cotriple $\mathcal{F}$- resolution of an object M of the category $\mathrm{PCM}(P)$

$$\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} F^n(M) \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} F^2(M) \rightrightarrows F^1(M) \to M,$$

where $F^n(B) = F(F^{n-1}(B))$, $\partial_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$.

It is known [17] that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple, $L_n^{\mathcal{P}}(-) \approx L_n^{\mathcal{F}}(-)$.

We have the commutative diagram of groups

$$\begin{array}{ccccccc}\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(M) \otimes N & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \rightrightarrows & F^1(M) \otimes N & \to & M \otimes N \\ & \downarrow f_n & & \downarrow f_1 & & \downarrow f \\ \cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(M) \wedge^P N & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \rightrightarrows & F^1(M) \wedge^P N & \to & M \wedge^P N\end{array}.$$

and hence an exact sequence of groups

$$\begin{gathered}\cdots \longrightarrow L_2^{\mathcal{P}}(M \otimes N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f,N) \longrightarrow \\ L_1^{\mathcal{P}}(M \otimes N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f,N) \longrightarrow L_0^{\mathcal{P}}(M \otimes N) \xrightarrow{f} \\ L_0^{\mathcal{P}}(M \wedge^P N) \longrightarrow 1.\end{gathered}$$

It is clearly that the derived functors $L_n^{\mathcal{P}}(- \otimes N)$ coincide with the derived functors of the non-abelian tensor product as a functor from the category $\mathcal{A}_N$ to the category $\mathfrak{Gr}$ relative to the projective class induced by the free cotriple. Hence $L_0^{\mathcal{P}}(M \otimes N) \approx M \otimes N$ and $L_n^{\mathcal{P}}(M \otimes N) = H_{n+1}(M,N)$, $n \geq 1$ [11] (see definition in Section 2). ∎

Now we consider Guin's non-abelian tensor product $- \overset{G}{\otimes} N$ and its exterior product $- \wedge^P N$ as functors from the category $\mathrm{PCM}(P)$ to the category $\mathfrak{Gr}$ of groups and their non-abelian left derived functors relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

Theorem 19. *Let $M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 1$ be an exact sequence of precrossed modules. Then we have the following exact sequence of groups*

$$M_1 \wedge^P N \xrightarrow{f'} M \wedge^P N \xrightarrow{g'} M_2 \wedge^P N \to 1,$$

(the exterior product $M \wedge^P N$ is taken for the Guin's tensor product) where $f' = f \wedge^P 1_N$, $g' = g \wedge^P 1_N$.

Proof. Similar to the proof of Theorem 16. ■

Theorem 20. *There exist natural isomorphisms*

$$L_0^{\mathcal{P}}(- \overset{G}{\otimes} N) \approx - \overset{G}{\otimes} N,$$
$$L_0^{\mathcal{P}}(- \wedge^P N) \approx - \wedge^P N,$$

where the exterior product $- \wedge^P N$ is taken for the Guin's tensor product.

Proof. Similar to the proof of Theorems 4[11] and Theorem 17. ■

Definition 21. Let (M, μ) be an object of the category PCM(P). Then we define Guin's homology ${}^G H_*(M, N)$ of a precrossed module (M, μ) with coefficients in any precrossed module (N, ν) over P as follows

$$\begin{aligned} {}^G H_n(M,N) &= L_{n-1}^{\mathcal{P}}(M \overset{G}{\otimes} N), \qquad n \geq 2, \\ {}^G H_1(M,N) &= \operatorname{Ker} \lambda', \qquad {}^G H_0(M,N) = \operatorname{Coker} \lambda', \end{aligned}$$

where $\lambda' : M \overset{G}{\otimes} N \to N/N'$, $\lambda'(m \otimes n) = [{}^{\mu(m)}n n^{-1}]$ and N' is the normal subgroup generated by the elements ${}^{\nu(n)}n' n n'^{-1} n^{-1}$ for $n, n' \in N$.

Remark 22. It is clear that the groups ${}^G H_n(M, N)$ are abelian for $n \geq 2$. When (M, μ) and (N, ν) are crossed modules then ${}^G H_1(M, N) = H_1(M, N)$ is also abelian and ${}^G H_0(M, N) = H_0(M, N)$. We will show that $\operatorname{Im} \lambda'$ is a normal subgroup of N/N'.
In fact, let $[n'] \in \operatorname{Im} \lambda'$ and $[n] \in N/N'$, then we have
$[n][n'][n] =$
$= [n]\lambda'((m_1 \otimes n_1) \cdots (m_k \otimes n_k))[n]^{-1} = [n][{}^{\mu(m_1)}n_1 n_1^{-1} \cdots {}^{\mu(m_k)} n_k n_k^{-1}][n]^{-1} =$
$= [{}^{\nu(n)\mu(m_1)\nu(n)^{-1}}({}^{\nu(n)}n_1)({}^{\nu(n)}n_1)^{-1}] \cdots [{}^{\nu(n)\mu(m_k)\nu(n)^{-1}}({}^{\nu(n)}n_k)({}^{\nu(n)}n_k)^{-1}] =$
$= [{}^{\mu({}^{\nu(n)}m_1)}({}^{\nu(n)}n_1)({}^{\nu(n)}n_1)^{-1}] \cdots [{}^{\mu({}^{\nu(n)}m_k)}({}^{\nu(n)}n_k)({}^{\nu(n)}n_k)^{-1}] =$
$= \lambda'(({}^{\nu(n)}m_1 \otimes {}^{\nu(n)} n_1) \cdots ({}^{\nu(n)}m_k \otimes {}^{\nu(n)} n_k))$

Theorem 23. *Let (M, μ), (N, ν) be precrossed modules over the same group P. Then we have the following exact sequence of groups*

$$\begin{gathered} \cdots \longrightarrow {}^G H_3(M,N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f, N) \longrightarrow \\ {}^G H_2(M,N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f, N) \longrightarrow M \overset{G}{\otimes} N \xrightarrow{f} , \\ M \wedge^P N \longrightarrow 1 \end{gathered}$$

where

$$\begin{gathered} L_n^{\mathcal{P}}(f, N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0 \\ \operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F^n(M) \overset{G}{\otimes} N \to F^n(M) \wedge^P N, \ n \geq 1\}. \end{gathered}$$

Proof. Similar to the proof of the Theorem 18 (i). ■

Consider the category $\mathrm{CM}(P)$ of crossed modules over the group P. The objects of this category are (M,μ) crossed modules over P, and morphisms $(M,\mu)\xrightarrow{f}(M',\mu')$ of the category are the homomorphisms $M\xrightarrow{f}M'$ such that 1. $f({}^pm)={}^pf(m)$, 2. $\mu' f=\mu$.

Now we construct a cotriple $\mathcal{F}'=(F',\tau',\delta')$ in the category $\mathrm{CM}(P)$. Let $F':\mathrm{CM}(P)\to\mathrm{CM}(P)$ be an endofunctor defined as follows: for $(M,\mu)\in\mathrm{ob}\,\mathrm{CM}(P)$ the object $F'(M,\mu)$ is the crossed module $(F(M)/H,\widetilde{\mu})$, where $F(M)$ is the free group generated by M and H is the normal subgroup generated by the elements ${}^{\widetilde{\mu}(x)}x'xx'^{-1}x^{-1}$ for each $x,x'\in F(M)$. If $f:(M,\mu)\to(M',\mu')$ is a morphism of $\mathrm{CM}(P)$, then $F'(f):(F(M)/H,\widetilde{\mu})\to(F(M')/H',\widetilde{\mu'})$ is induced by the natural homomorphism $F(f)$.

Let (M,μ) be a crossed module. Then the natural epimorphism $F(M)\to M$ induces a morphism $(F(M)/H,\widetilde{\mu})\xrightarrow{\tau'(M)}(M,\mu)$ of the category $\mathrm{CM}(P)$ and $\tau':F'\to 1_{\mathrm{CM}(P)}$ satisfies the condition of natural transformation.

Let $\delta':F'\to F'^2$ be the natural transformation induced for $(M,\mu)\in\mathrm{ob}\,\mathrm{CM}(P)$ by the composition of the injection $M\to F(M)$ and the natural epimorphism $F(M)\to F(M)/H$ (because any map of sets $f:M\to M'$ which satisfies conditions $f({}^pm)={}^p f(m)$, $\mu' f=\mu$ induces a morphism $F(M)/H\to F(M')/H'$ of crossed modules).

It is easy to see that there exist simplicial kernels in the category $\mathrm{CM}(P)$.

From [8] it is known that $-\overset{G}{\otimes}N$ and $-\otimes N$ are the same functors on the category $\mathrm{CM}(P)$. Consider the non-abelian left derived functors $\mathbb{L}_n^{\mathcal{P}'}(-\otimes N)$ and $\mathbb{L}_n^{\mathcal{P}'}(-\wedge^P N)$, $n\geq 0$ of the functors $-\otimes N$ and $-\wedge^P N$ respectively relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$. Then we have

Theorem 24. *There exist natural isomorphisms*

$$\mathbb{L}_0^{\mathcal{P}'}(-\otimes N)\approx -\otimes N,$$
$$\mathbb{L}_0^{\mathcal{P}'}(-\wedge^P N)\approx -\wedge^P N.$$

Proof. Consider a projective pseudosimplicial resolution of an object M of the category $\mathrm{CM}(P)$ relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$

$$\cdots\ \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} X_n\ \begin{smallmatrix}\xrightarrow{\partial_0^n}\\ \vdots\\ \xrightarrow[\partial_n^n]{}\end{smallmatrix}\cdots\ \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix} X_1\ \begin{smallmatrix}\xrightarrow{\partial_0^1}\\ \xrightarrow[\partial_1^1]{}\end{smallmatrix} X_0\xrightarrow{\partial_0^0} M.$$

Since $\mathcal{P}'$-epimorphism is surjective, then

$$\operatorname{Ker}\partial_0^1\xrightarrow{\partial_1^1\sigma}X_0\xrightarrow{\partial_0^0}M\to 1$$

is an exact sequence of crossed modules, where $\sigma:\operatorname{Ker}\partial_0^1\to X_1$ is the inclusion. From Theorem 1[11] and Theorem 16 it follows that the sequences of groups

$$\operatorname{Ker}\partial_0^1\otimes N\xrightarrow{\partial_1^1\sigma\otimes 1_N}X_0\otimes N\xrightarrow{\partial_0^0\otimes 1_N}M\otimes N\to 1,$$

$$\operatorname{Ker} \partial_0^1 \wedge^P N \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0 \wedge^P 1_N} M \wedge^P N \to 1$$

are also exact.

We have the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Ker} \partial_0^1 \otimes N & \xrightarrow{\partial_1^1 \sigma \otimes 1_N} & X_0 \otimes N & & & & \\
\big\downarrow f & & \big\| & & & & \\
\operatorname{Ker}(\partial_0^1 \otimes 1_N) & \xrightarrow{\partial_1^1 \otimes 1_N} & X_0 \otimes N & \xrightarrow{\partial_0^0 \otimes 1_N} & M \otimes N & \longrightarrow & 1,
\end{array}$$

where the homomorphism f induced by the inclusion σ is surjective by Theorem 1[11]. Thus the bottom sequence of this diagram is exact. The exactness of the sequence $\operatorname{Ker} \partial_0^1 \wedge^P N \xrightarrow{\partial_1^1 \sigma \wedge^P 1_N} X_0 \wedge^P N \xrightarrow{\partial_0^0 \wedge^P 1_N} M \wedge^P N \to 1$ is proved analogously. The proof is complete. ■

It is easy to see that $(- \times_P N)/ < -, N >$ is a covariant functor from the category $\mathrm{CM}(P)$ to the category $\mathfrak{Ab}$ of abelian groups, then $\Gamma(- \times_P N/ < -, N >)$ will be a functor from the category $\mathrm{CM}(P)$ to the category $\mathfrak{Ab}$. Consider the non-abelian left derived functors $\mathbb{L}_n^{\mathcal{P}'} \Gamma(-\times_P N/ < -, N >), n \geq 0$ of the functor $\Gamma(-\times_P N)/ < -, N >$ relative to the projective class $\mathcal{P}'$ induced by the cotriple $\mathcal{F}'$.

Theorem 25. *Let $(M, \mu), (N, \nu) \in \operatorname{ob} \mathrm{CM}(P)$, then we have the following exact sequences of groups*

$$\begin{array}{c}
\cdots \longrightarrow \mathbb{L}_2^{\mathcal{P}'}(M \wedge^P N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(M \otimes N) \longrightarrow \\
\mathbb{L}_1^{\mathcal{P}'}(M \wedge^P N) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(f, N) \longrightarrow M \otimes N \xrightarrow{f} M \wedge^P N \longrightarrow 1
\end{array} ,$$

where

$$\begin{array}{c}
\mathbb{L}_n^{\mathcal{P}'}(f, N) = \pi_n(\operatorname{Ker} f_*), \quad n \geq 0, \\
\operatorname{Ker} f_* = \{\operatorname{Ker} f^n \mid f^n : F'^n(M) \otimes N \longrightarrow F'^n(M) \wedge^P N, \quad n \geq 1\}
\end{array}$$

and

$$\begin{array}{c}
\cdots \longrightarrow \mathbb{L}_2^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(\psi, N) \longrightarrow \\
\mathbb{L}_1^{\mathcal{P}'} \Gamma(M \times_P N/ < M, N >) \longrightarrow \mathbb{L}_1^{\mathcal{P}'}(f, N) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(\psi, N) \\
\longrightarrow \mathbb{L}_0^{\mathcal{P}'} \Gamma(M \times_P N/ < M, N >) \longrightarrow \mathbb{L}_0^{\mathcal{P}'}(f, N) \longrightarrow 1
\end{array} ,$$

where

$$\begin{array}{c}
\mathbb{L}_n^{\mathcal{P}'}(\psi, N) = \pi_n(\operatorname{Ker} \psi_*), \quad n \geq 0, \\
\operatorname{Ker}\psi_* = \{\operatorname{Ker}\psi^n \mid \psi^n : \Gamma(F'^n(M) \times_P N/ < F'^n(M), N >) \to F'^n(M) \otimes N, \ n \geq 1\};
\end{array}$$

Proof. Consider $\mathcal{F}'$ cotriple resolution of an object M of the category $\mathrm{CM}(P)$

$$\cdots \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} F'^n(M) \begin{smallmatrix}\to \\ \vdots \\ \to\end{smallmatrix} \cdots \begin{smallmatrix}\to \\ \to \\ \to\end{smallmatrix} F'^2(M) \rightrightarrows F'^1(M) \to M.$$

Put $\Omega^n \equiv \Gamma(F'^n(M) \times_P N/ < F'^n(M), N >)$. Then from [5, Theorem 2.12] we have the comutative diagram

$$
\begin{array}{ccccccc}
\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & \Omega^n & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \rightrightarrows & \Omega^1 & \to & \Gamma(M \times_P N/ < M, N >) & \\
 & \downarrow \psi_n & & \downarrow \psi_1 & & \downarrow \psi & \\
\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F'^n(M) \otimes N & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \rightrightarrows & F'^1(M) \otimes N & \to & M \otimes N & . \\
 & \downarrow f_n & & \downarrow f_1 & & \downarrow f & \\
\cdots \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F'^n(M) \wedge^P N & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} \cdots \rightrightarrows & F'^1(M) \wedge^P N & \to & M \wedge^P N &
\end{array}
$$

which has exact vertical rows. The proof is complete. ■

Remark 26. There exists the natural epimorphism

$$\mathbb{L}_0^{\mathcal{P}'} \Gamma(M \times_P N/ < M, N >) \longrightarrow \Gamma(M \times_P N/ < M, N >).$$

2. Finiteness of a Non-abelian Homology of Groups

The non-abelian homology of groups with coefficients in any group was introduced in [11]. The construction was made possible by using the left derived functors of the non-abelian tensor product of groups. Now we recall the definition of the non-abelian homology of groups from [11].

Let consider the category $\mathcal{A}_A$ of groups acting on a fixed group A and the group A acts on these groups. Morphisms of the category $\mathcal{A}_A$ are homomorphisms $\alpha : G \to H$, G, $H \in \operatorname{ob} \mathcal{A}_A$ of groups which preserve the actions in sense that: $\alpha({}^a g) = {}^a\alpha(g)$ and ${}^g a = {}^{\alpha(g)} a$, for all $a \in A$, $g \in G \in \operatorname{ob} \mathcal{A}_A$.

Let $F : \mathcal{A}_A \to \mathcal{A}_A$ be an endofunctor defined as follows: for $G \in \operatorname{ob} \mathcal{A}_A$ the object $F(G)$ is the free group generated by G with actions: ${}^{|g_1|^{\epsilon_1} \cdots |g_s|^{\epsilon_s}} a = {}^{g_1^{\epsilon_1}}(\ldots({}^{g_s^{\epsilon_s}} a)\ldots)$, and ${}^a(|g_1|^{\epsilon_1} \cdots |g_s|^{\epsilon_s}) = |{}^a g_1|^{\epsilon_1} \cdots |{}^a g_s|^{\epsilon_s}$ for $a \in A$, $|g_1|^{\epsilon_1} \cdots |g_s|^{\epsilon_s} \in F(G)$ and $\epsilon_i = \pm 1$; for $\alpha : G \to G'$ the morphism $F(\alpha)$ is the canonical homomorphism from $F(G)$ to $F(G')$ induced by α.

Let $\tau : F \to 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $G \in \operatorname{ob} \mathcal{A}_A$ by the injection $G \to F(G)$. We obtain a cotriple $\mathcal{F} = (F, \tau, \delta)$, which we call the free cotriple in the category $\mathcal{A}_A$.

The non-abelian tensor product of groups defines a covariant functor $- \otimes A$ from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups. Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(- \otimes A)$, $n \geq 0$, of the functor $- \otimes A$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

Definition 27. Let G be an object of the category $\mathcal{A}_A$. Then we define

$$
\begin{aligned}
&H_n(G, A) = L_{n-1}^{\mathcal{P}}(G \otimes A), \qquad n \geq 2, \\
&H_1(G, A) = \operatorname{Ker} \lambda', \qquad H_0(G, A) = \operatorname{Coker} \lambda',
\end{aligned}
$$

where $\lambda' : G \otimes A \to A/H$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H is the normal subgroup generated by the elements ${}^{({}^a g)}a' \cdot {}^{aga^{-1}}a'^{-1}$, for each $a, a' \in A$, $g \in G$.

The present section is concerned solely with finiteness of homology of groups.

Let G be any group and A a finite group acting on each other. It is easy to see that $H_0(G, A)$ is a finite group.

Proposition 28. *Let G and A be finite groups which act on each other compatibly i.e. conditions (1) hold. Then $H_1(G, A)$ is a finite group.*

Proof. If the finite groups G and A act on each other compatibly then from [7] $G \otimes A$ is also finite. From the definition $H_1(G, A)$ is a normal subgroup of $G \otimes A$. Thus $H_1(G, A)$ is a finite group. The proof is complete. ■

Remark 29. If finite groups G and A act on each other such that one of the following sufficient conditions for the finiteness of the non-abelian tensor product of groups [12] holds:

(i) G and A act on each other half compatibly (see [12]);
(ii) A is soluble;
(iii) $[G, A]^n$ is abelian for some $n \geq 1$, where $[G, A]^n = [G, [G, A]^{n-1}]$ and $[G, A]$ is a normal subgroup of A generated by the elements ${}^g a a^{-1}$ for all $g \in G$, $a \in A$.

Then $H_1(G, A)$ is a finite group.

It is interesting to establish the finiteness of non-abelian homology in higher dimensions. Below we only concern the non-abelian homology of a group G with coefficients in a G-group A. From Definition 3[11] the homology of G with coefficient in G-group A is $H_*(G, A)$ with trivial action of A on G.

Now we construct a new simplicial $\mathcal{P}$-projective resolution of G in this case. For this we recall the construction of the loop functor $\mathbb{G}$ from the category of reduced complexes to the category of simplicial groups [13].

Definition 30. Let K be a reduced complex (i.e. K is simplicial set which has only one 0-simplex ϕ). Then we define a simplicial group $\mathbb{G}K$ as follows: the group of the n-simplices is a group which has

(i) one generator $\overline{\sigma}$ for every $n+1$-simplex $\sigma \in K_{n+1}$,
(ii) one relation $\overline{s_n^n \tau} = e_n$ for every n-simplex $\tau \in K_n$.

The face and degeneracy homomorphisms $\partial_i^n : \mathbb{G}_n K \to \mathbb{G}_{n-1} K$ and $s_i^n : \mathbb{G}_n K \to \mathbb{G}_{n+1} K$ are given by the formulas

$$\begin{aligned}\partial_i^n \overline{\sigma} &= \overline{\partial_i^n \sigma}, 0 \leq i < n, \\ \partial_n^n \overline{\sigma} &= \overline{\partial_n^n \sigma} \cdot \overline{\partial_{n+1}^n \sigma}^{-1}, \\ s_i^n \overline{\sigma} &= \overline{s_i^n \sigma}, 0 \leq i \leq n.\end{aligned}$$

Clearly the groups $\mathbb{G}_n K$ are free and from [13] it is known that $\pi_n(K) \approx \pi_{n-1}(\mathbb{G}K)$ for $n > 0$.

Let $\underline{G}$ be a category which has only one object O and $Hom(O, O) = G$, and consider the nerve $M_*(\underline{G})$ of the category $\underline{G}$. It is easy to see that $M_*(\underline{G})$ is a reduced complex. It is known that $\pi_1(M_*(\underline{G})) \approx G$ and $\pi_n(M_*(\underline{G})) = 0$, $n \neq 1$. Let consider the simplicial group $\mathbb{G}M_*(\underline{G})$. Then $\pi_0(\mathbb{G}M_*(\underline{G})) \approx \pi_1(M_*(\underline{G})) \approx G$ and

$\pi_{n-1}(\mathbb{G}M_*(\underline{G})) \approx \pi_n(M_*(\underline{G})) = 0$, $n > 1$. Hence $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is a $\mathcal{P}$-projective simplicial resolution in the category of groups, where ∂_0^0 is the natural epimorphism, $\mathcal{P}$ is the projective class in the category of groups induced by the free cotriple.

Remark 31. An epimorphism of groups $\alpha : G \to H$, $G, H \in A_A$, which preserves the actions, is a $\mathcal{P}$-epimorphism in the category $\mathcal{A}_A$ if A acts trivially on G and H, and any object of the projective class $\mathcal{P}$ in the category $\mathcal{A}_A$ induced by free cotriple (F, τ, δ) is a free group $F(X)$, where X is a set on which A acts such that ${}^a x \in X$ for every $x \in X$.

Thus, if we define the action of A on $\mathbb{G}_n M_*(\underline{G})$ trivially and the action of $\mathbb{G}_n M_*(\underline{G})$ on A as follows: ${}^m a = {}^{\partial_0^0 \partial_0^1 \cdots \partial_0^n m} a$, $m \in \mathbb{G}_n M_*(\underline{G})$, then from Remark $\mathbb{G}_n M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is the $\mathcal{P}$- projective simplicial resolution in the category $\mathcal{A}_A$.

Theorem 32. *Let G and A be finite groups. A acts on G trivially and G acts on A such that the non-abelian tensor product $G \otimes A$ is finite* (*see Remark* 29, *conditions* (i-iii)).

If F is a finitely generated free group which acts on A and A acts on F trivially and α is an epimorphism $\alpha : F \to G$ preserving the actions, then $F \otimes A$ is a finite group.

Proof. Consider the exact sequence of groups

$$1 \to \operatorname{Ker} \alpha \to F \xrightarrow{\alpha} G \to 1$$

From the right exactness of the non-abelian tensor product [11, Theorem 1], we have the exact sequence of groups

$$\operatorname{Ker} \alpha \otimes A \to F \otimes A \xrightarrow{\alpha \otimes 1_A} G \otimes A \to 1. \tag{6}$$

The group $\operatorname{Ker} \alpha$ is the subgroup with finite index of the finitelly generated free group F and from [18] it follows that $\operatorname{Ker} \alpha$ is a finitely generated free group. Since $\operatorname{Ker} \alpha$ and A act on each other trivially, from [3] we have

$$\operatorname{Ker} \alpha \otimes A \approx \operatorname{Ker} \alpha^{ab} \otimes_{\mathbb{Z}} A^{ab} \approx \sum_m \mathbb{Z} \otimes_{\mathbb{Z}} A^{ab} \approx \sum_m A^{ab}.$$

Hence $\operatorname{Ker} \alpha \otimes A$ is a finite group. From (6) one gets that $F \otimes A$ is a finite group. The proof is complete. ∎

Theorem 33. *Let G and A be finite groups. If A acts on G trivially and G acts on A such that the non-abelian tensor product $G \otimes A$ is finite, then $H_n(G, A)$, $n \geq 0$, is a finite group.*

Proof. Consider the $\mathcal{P}$-projective pseudosimplicial resolution $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ in the category $\mathcal{A}_A$ for G constructed above. Every $\mathbb{G}_n M_*(\underline{G})$ is a finitely generated free group and since the face operators are epimorphisms, $\alpha : \mathbb{G}_n M_*(\underline{G}) \to G$, $\alpha = \partial_0^0 \partial_0^1 \cdots \partial_0^n$ is an epimorphism which preserve the actions. Thus from Theorem 32 $\mathbb{G}_n M_*(\underline{G}) \otimes A$ is a finite group. Clearly $H_n(G, A)$, $n > 1$, is a finite group. The finiteness of $H_1(G, A)$ and $H_0(G, A)$ is shown above. The proof is complete. ∎

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