GENERALIZED HIGSON'S HOMOTOPY THEOREM

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ABSTRACT. It is proved Higson's homotopy invariant theorem for complex and real C^*-algebras with compact group action. It is given various consequences.

INTRODUCTION

The purpose of this article is investigation possibilities to generalize Higson homotopy invariant theorem and give some elementary applications.

In [8] it has been proved that if E is a functor from an admissible subcategory of the category of *complex* C^*-algebras to the category of abelian groups, and E is stable and split exact, then E is homotopy invariant. The proof of this theorem is successful mixture of ideas of Kasparov's proof of homotopy invariance of Kasparov KK-theory and Cuntz's machinary of quasi-homomorphisms to characterization of KK-theory of C^*-algebras [9], [4], [5], [8], [1].

Our main aim is to generalize this theorem for both cases *real* and *complex* in more general category then category of C^*-algebras with value of an additive category. Our approach is based on the Cuntz-Skandalis view on Kasparov's homotopy invariant theorem (HIT) for KK-groups [7].

In first section, using functional calculus, which exists in both case real or complex, we modify Cuntz-skandalis HIT in the form which will be applicable in the next sections, and in the second section we proof HIT for functor E on the admissible category of C^*-algebras, with value in an additive category, which has M-pairing with the set of unitary Fredholm modules. M-pairing satisfies different conditions then conditions of pairings from [8]. After same simflications we get HIT for E which has pairing with the set Fredholm pairs (in both real and complex case).

In the next section, after some preparatory material we proof that if E is stable and split additive functor from an admissible category

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of C^*-algebras (real or complex) to an additive category, then E is homotopy invariant functor.

In next step, we define an abstract category $\mathbf{A}$ with *split exact sequences*, and *split action* of admissible category of C^*- algebras on the $\mathbf{A}$. Using this action we define what functors from $\mathbf{A}$ to an additive category are homotopy invariant, stable or split exact. From this data and above result we deduce more abstract form of HIT.

Finally, using abstract form of homotopy invariant theorem we deduce HIT for admissible subcategories of category real or complex C^*-algebras with compact group action, particularly for Z_2-graded , for admissible subcategories of category of pro-C^*-algebras, and HIT for some class of m-algebras. It is given an elementary applications in KK-theory, bivariant periodic cyclic cohomology, algebraic K-theory.

1. On The Cuntz-Skandalis HIT

In this section we shortly recall one of the main results of [7], homotopy invariant theorem (HIT) 3.7, which we modify slightly for our purposes in next section, to deduce generalized Higson's homotopy invariant theorem (HIT).

Kasparov KK-groups

In this subsection it will be assumed that A is separable and B is σ-unital Z_2-graded real (or complex) C^*-algebras.

We recall definition of Kasparov KK-group for C^*algebras [9], [13]. For such algebras, let $\mathcal{E}(A,B)$ be the set of *Kasparov A, B-modules* (E,φ,F), where E is a countable generated graded Hilbert B-module; $\varphi : A \to \mathcal{L}(E)$ is a graded $*$-homomorphism, F is in $\mathcal{L}(E)$ of degree 1, such that

$$[\varphi(a),F] \in \mathcal{K}(E),\ (F-F^*)\varphi(a) \in \mathcal{K}(E),\ (F^2-1)\varphi(a) \in \mathcal{K}(E), \quad \forall a \in A. \tag{1}$$

A Kasparov module (E,φ,F) will be said degenerate iff

$$[\varphi(a),F] = 0,\ \ (F-F^*)\varphi(a) = 0,\ \ (F^2-1)\varphi(a) = 0, \quad \forall a \in A. \tag{2}$$

An element of $\mathcal{E}(A,B(I))$ given by a family (E_t,φ_t,F_t) will be called a homotopy between (E_0,φ_0,F_0) and (E_1,φ_1,F_1), where $B(I)$ is C^*algebra of continuous function from $I=[0,1]$ to B. The addition of two Kasparov modules is defined by

$(E_0,\varphi_0,F_0) \oplus (E_1,\varphi_1,F_1) = (E_0 \oplus E_1, \varphi_0 \oplus \varphi_1, F_0 \oplus F_1)$

Let $KK(A,B)$ be the quotient of $\mathcal{E}(A,B)$ by homotopy equivalence. Then it is abelian group relative the sum defined by the sum of Kasparov modules.

Cuntz-Skandalis homotopy invariance theorem

Let $\overline{\mathcal{E}}(A,B)$ be the abelian monoid of classes of Kasparov (A,B)-bimodules, where (E,φ,F) and

$$(E',\varphi',F')$$

are identified if there exists a invariant unitary $u\in\mathcal{L}(E,E')$ of degree zero, interwining the action of A i.e.

$$u\varphi(a)u^*=\varphi'(a),$$

and such that

$$uFu^*-F'\in\mathcal{K}(E,E')$$

Such type equivalence will be called *cs-isomorphism.* Let $\overline{KK}(A,B)$ be the *cancellation* monoid associated with $\overline{\mathcal{E}}(A,B)$. Theorem 3.7 of [7] states that the natural homomorphism

$$\tau:\overline{KK}(A,B)\overset{\simeq}{\to}KK(A,B) \tag{3}$$

is isomorphism.

In the next subsections we are need following

LEMMA 1. *Let $\bar{\mathcal{E}}_d(A,B)$ be abelian monoid of unitary isomorphic degenerate Kasparov A,B-modules, and $K\bar{K}_d(A,B)$ be its cancellation monoid. Then $K\bar{K}_d(A,B)=0$.*

PROOF. For each degenerate A,B-module (E,φ,F) one can form A,B-module $(E^\infty,\varphi^\infty,F^\infty)$ such that $(E,\varphi,F)\oplus(E^\infty,\varphi^\infty,F^\infty)$ is unitary isomorphic to $(E^\infty,\varphi^\infty,F^\infty)$ where $E^\infty=E\oplus E\oplus\cdots$, $\varphi^\infty=\varphi\oplus\varphi\cdots$, $F^\infty=F\oplus F\oplus\cdots$. This means that $(E,\varphi,F)=0$ in $K\bar{K}_d(A,B)$.

Note to functional calculus in real algebra case

As it is known each a real C^*-algebras can be realized as involutive closed subalgebra of of a complex C^*-algebra, considered later algebra as the real involutive algebra. More exactly, Let A be a real C^*-algebra, consider complex involutive algebra $A\otimes_R C$ with involution $(a\otimes c)^*=a^*\otimes\bar{c}$. Then there exist C^*-norm, relative which $A\otimes_R C$ is complex C^*-algebra and canonical $*$-imbedding $i:A\to A\otimes_R C$ defined by $a\mapsto a\otimes 1$ is isometry (cf. Theorem 2 and Corollary 2 of [12], also [10]). From this easy follows that for real C^*-algebras there exists functional calculus for self adjoint elements. Let us to be more precisely. if $r\in A$ is self-adjoint element then let $A(r)$ be the real closed subalgebra in A generated by r. It is real part of the sub-C^*-algebra $C^*(r\otimes 1)$ of $A\otimes_R C$ generated by element $r\otimes 1$. By functional calculus of complex C^*algebras if r is normal i.e. $r=s^*s$ for some $s\in A$ then there is canonical isomorphism

$C^*(r\otimes 1) \simeq C(sp(r\otimes 1))$, and thus if $R(sp_R(r))$ is algebra of continuous real functions on the spectra of r which is $sp(r \otimes 1)$ by definition (cf. [12]), then there exist canonical isomorphism $\kappa : R(sp_R(r)) \simeq R(r)$ such that $\kappa(id_{sp(r)}) = r$. In particular, we can use functional calculus on self-adjoint elements both in real and complex cases.

An interpretation of KK-groups

Thanks to remarks of subsection above, we able to proof, in the same way, analogies of lemma 17.4.2.-17.4.3 from [1] *in both cases real or complex*. We shorty recall contents those lemmas.

Lemma 2. *Kasparov's KK-groups will be unchanged if in definition of it will be taken:*
1. only Kasparov modules (E, φ, F) with property $F = F^*$ and one take only homotopy equivalence or only cs-isomorphisms
2. The same is true if one take Kasparov modules (E, φ, F) with properties $F = F^*$ and $||F|| \leq 1$.

According on discussion from 17.6 of [1], in the definition of KK-group one can to take Kasparov A, B-modules (E, φ, F) with properties:

$$[\varphi(a), F] \in \mathcal{K}(E),\ F^* = F,\ F^2 = 1,\ a \in A, \tag{4}$$

in both case, real or complex C^*-algebras.

Let $\widehat{\mathcal{E}}(A, B)$ be monoid of Kasparov A, B-bimodules (E, φ, F), with properties 4, identified by cs-isomorphism. Let $\widehat{KK}(A, B)$ be cancellation monoid associated with $\widehat{\mathcal{E}}(A, B)$.

Proposition 3. *The natural homomorphism*

$$\widehat{\tau} : \widehat{KK}(A, B) \to \overline{KK}(A, B)$$

Proof. (cf. [1]) The homomorphism $\widehat{\tau}$ is epimorphism, because Kasparov modules (E, φ, F) and $(E \oplus E, \varphi \oplus 0, F \oplus F^*)$ are equal in $\overline{KK}(A, B)$ because of lemma 1, and last module is compact perturbation of

$$\left(E \oplus E, \varphi \oplus 0, \begin{pmatrix} F & (1 - FF^*)^{\frac{1}{2}} \\ (1 - FF^*)^{\frac{1}{2}} & F^* \end{pmatrix}\right),$$

which naturally defines an element in $\widehat{KK}(A, B)$. The homomorphism $\widehat{\tau}$ is a monomorphism: Let (E, φ, F) be Kasparov A, B-bimodule such that $F^* = F$ and $F^2 = 1$. Let $(E, \varphi, F) \oplus (E', \varphi', F')$ is compact perturbation (this is partial case of cs-isomorphism) of (E', φ', F') where $F'^* = F'$ and $||F'|| \leq 1$ by lemma 2. Then $(E, \varphi, F) \oplus (E' \oplus E', \varphi' \oplus 0, G)$

is compact perturbation of $(E' \oplus E', \varphi' \oplus 0, G)$, where

$$G = \begin{pmatrix} F' & (1 - F'F'^*)^{\frac{1}{2}} \\ (1 - F'F'^*)^{\frac{1}{2}} & F'^* \end{pmatrix}.$$

But A, B-module

$$(E' \oplus E', \varphi' \oplus 0, G)$$

has property 4. Thus $\{(E, \varphi, F)\} = 0$ in $\widehat{KK}(A, B)$, because the last one is abelian cancelation monoid.

From this proposition follows that $\widehat{KK}(A, B)$ is abelian group.

COROLLARY 4. . *The group $\widehat{KK}(A, B)$ is homotopy invariant relative to both arguments.*

PROOF. Trivially follows from proposition 3 and homotopy property of $\overline{KK}(A, B)$, because of homotopy property of $KK(A, B)$ and isomorphism 3.

THE CASE OF TRIVIALLY GRADED C^*-ALGEBRAS

Now we give an interpretation of $\widehat{KK}(A, B)$ for trivially graded C^*-algebras. We are need case when $B = \mathcal{C}$ ($\mathcal{C}$ is ground field of real or complex numbers). Hilbert spaces are trivially graded in this subsection.

DEFINITION 5. . A triple (φ, ψ, F) is said to be *unitary Fredholm A-module* if $\varphi : A \to \mathcal{L}(\mathcal{H}_\varphi)$ and $\varphi : A \to \mathcal{L}(\mathcal{H}_\psi)$ are representations, and $F : \mathcal{H}_\varphi \to \mathcal{H}_\psi$ is unitary isomorphism such that

$$F \cdot \varphi(a) - \psi(a) \cdot F \in \mathcal{K}(\mathcal{H}_\varphi, \mathcal{H}_\psi).$$

1) An unitary Fredholm A-module (φ, ψ, F) is *unitary isomorphic* to (φ', ψ', F') if there exists unitary isomorphisms $u : \mathcal{H}_\varphi \to \mathcal{H}_{\varphi'}$ and $v : \mathcal{H}_\psi \to \mathcal{H}_{\psi'}$ such that $u\varphi(a)u^* = \varphi'(a)$, $v\psi(a)v^* = \psi'(a)$, for all $a \in A$, and $vFu^* = F'$.

2) An unitary Fredholm A-module (φ, ψ, F) is *compact perturbation* of (φ, ψ, F') if $F - F' \in \mathcal{K}(\mathcal{H}_\varphi, \mathcal{H}_\psi)$.

3) The *sum* of unitary Fredholm A-modules is defined by formula

$$(\varphi, \psi, F) \oplus (\varphi', \psi', F') = (\varphi \oplus \varphi', \psi \oplus \psi', F \oplus F')$$

4) Let $\tilde{\mathcal{E}}(A, \mathcal{C})$ be the abelian monoid of classes of unitary Fredholm $(A, \mathcal{C})$-modules divided by the equivalence generated by unitary isomorphism and compact perturbation. Let then $\widetilde{KK}(A, \mathcal{C})$ be the cancellation monoid associated with $\widetilde{\mathcal{E}}(A, \mathcal{C})$.

Let $\mathcal{H}$ be trivially graded Hilbert space. Define by $\widehat{\mathcal{H}}$ graded Hilbert space opposite to $\mathcal{H}$. Then identity map $i_d : \mathcal{H} \to \hat{\mathcal{H}}$ has degree one.

If (φ, ψ, F) is unitary Fredholm module then define $\widehat{F} : \mathcal{H}_\varphi \to \widehat{\mathcal{H}_\psi}$ as the composition $i_d \cdot F$ and $\hat{\psi}(a) = i_d \cdot \psi(a) \cdot i_d^*$. Then correspondence

$$(\varphi, F, \psi) \mapsto \left(\mathcal{H}_\varphi \oplus \widehat{\mathcal{H}}_\psi,\ \begin{pmatrix} 0 & \hat{F}^* \\ \widehat{F} & 0 \end{pmatrix},\ \begin{pmatrix} \varphi & 0 \\ 0 & \hat{\psi} \end{pmatrix}\right)$$

defines homomorphism $\nu : \widetilde{KK}(A, \mathcal{C}) \to \widehat{KK}(A, \mathcal{C})$.

Proposition 6. . *Homomorphism ν is isomorphism.*

Proof. Inverse to ν is defined by following way. Consider Kasparov module $(\mathcal{H}, \varphi, F)$ which has property 4. Then $\mathcal{H} = \mathcal{H}^{(0)} \oplus \widehat{\mathcal{H}^{(1)}}$ where $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(1)}$ are trivially graded Hilbert spaces. In this decomposition

$$\phi = \begin{pmatrix} \varphi & \\ & \hat{\psi} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \hat{F}^* \\ \hat{F} & 0 \end{pmatrix}.$$

Let $\psi(a) = i_d^* \cdot \hat{\psi}(a) \cdot i_d$ and $F = i_d^* \cdot \hat{F}$. Then correspondence

$$(\mathcal{H}, \phi, G) \mapsto (\varphi, \psi,\ F)$$

defines inverse homomorphism.

Corollary 7. . *The cancelation abelian monoid (group) $\widetilde{KK}(A, \mathcal{C})$ is homotopy invariant relative the first argument.*

Proof. Trivially follows from corollary 4 and proposition above.

2. Pairings

Admissable category of C^*-algebras

All algebras are considered here over $\mathcal{C}$ real or complex numbers. Let $\mathcal{H}$ is a countable generated Hilbert space over $\mathcal{C}$, $\mathcal{K}$ is algebra of compact operators, and $\mathcal{L}$ is algebra of bounded operators in $\mathcal{H}$ [9],[1]. Let $\mathcal{B}$ be category of C^*-algebras over $\mathcal{C}$ real or complex numbers and their $*$-homomorphisms which contains trivial C^*-algebra, ground field of real (or complex) numbers $\mathcal{C}$, algebra of compact operators $\mathcal{K}$ and algebra $C[0, 1]$ of continuous $\mathcal{C}$ valued functions on unit interval and closed relative to all usually split exact sequences of a form

$$0 \to \mathcal{K} \to B \overset{\gamma,\beta}{\to} C([0; 1]) \to 0 \qquad (5)$$

M-PAIRING WITH UNITARY FREDHOLM MODULES

Let $\mathcal{M}$ be a category with property that $\hom(m,n)$ is abelian cancelation monoid for any $m,n \in ob\mathcal{M}$. Category $\mathcal{M}$ with such structure on morphisms is said to be *CM-category.* Let $e_0, e_1 : C([0;1]) \to C$ be evolution homomorphisms at 0 and 1 respectively, then a functor $E : \mathcal{B} \to \mathcal{M}$ is said to be *w-homotopy functor* if $E(e_0) = E(e_1)$ (or it will be written as $(e_0)_* = (e_1)_*$). Let $E : \mathcal{B} \to \mathcal{M}$ be a functor, and $E_C : \mathcal{B} \to \mathcal{M}$ be the associated constant functor defined by rules

$$E_C(B) = E(C),$$
$$E_C(f) = 1_{E(C)}.$$

for every object B and morphism f from $\mathcal{B}$.

DEFINITION 8. . (cf. [8])Let $E : \mathcal{B} \to \mathcal{M}$ a functor, where $\mathcal{M}$ is CM category. A *M-pairing* of E with the set of unitary Fredholm B-modules is rule which to each unitary Fredholm modules (φ, ψ, F), for each $B \in \mathcal{B}$, associates a morphism

$$\times(\varphi, \psi, F) : E(B) \to E_C(B) \tag{6}$$

in the category $\mathcal{M}$ with following conditions:

1. Functoriality. If (φ, ψ, F) is a unitary Fredholm B'-module, and if $f : B \to B'$ is a $*$-homomorphism from $\mathcal{B}$, then the diagram

$$\begin{array}{ccc} E(B) & \xrightarrow{\times(\varphi f, \psi f, F)} & E(C) \\ {\scriptstyle (f)_*} \downarrow & & \downarrow = \\ E(B') & \xrightarrow{\times(\varphi, \psi, F)} & E(C) \end{array}$$

commutes.

1. cs-isomorphism. If (φ, χ, F) and (ϕ, ψ, G) are cs-isomorphic then
$$\times(\varphi, \chi, F) = \times(\phi, \psi, G).$$
2. Additivity. If (φ, χ, F) and (ϕ, ψ, G) are unitary Fredholm B-modules, then
$$\times(\varphi, \chi, F) + \times(\phi, \psi, G) = \times(\varphi \oplus \phi, \chi \oplus \psi, F \oplus G).$$
3. Non-degeneracy. If $(e, 0, 1)$ is a unitary Fredholm $\mathcal{C}$-module for which $e : \mathcal{C} \to \mathcal{K}$ maps $1 \in \mathcal{C}$ to rank one projection then $\times(e, 0, 1) : E(\mathcal{C}) \to E(\mathcal{C})$ is the identity morphism.

THEOREM 9. . *Let E be a functor from category $\mathcal{B}$ to CM-category $\mathcal{M}$. If the functor E admits a M-pairing with the set of unitary Fredholm modules, then E is w-homotopy functor.*

PROOF. First step note that pairing gives homomorphism

$$\pi : K\bar{K}(C(I), \mathcal{C}) \to \hom(E(C(I)), E(\mathcal{C})),$$

defined by rule $(\varphi, \psi, F) \mapsto \times(\varphi, \psi, F)$, because of properties 2), 3) of 8 and 4) of definition 5. From homotopy invariant theorem, or more precisely from corollary 7, follows that $(e_0 e, 0, 1) = (e_1 e, 0, 1)$ as the elements of $K\bar{K}(\mathcal{C}(I), \mathcal{C})$. Using functoriality and non-degeneracy properties of M-pairing, we get

$$(e_0)_* = \times(e_0 e, 0, 1) = \times(e_1 e, 0, 1) = (e_1)_*$$

in the E-level. This means that E is w-homotopy functor.□

PAIRING WITH UNITARY FREDHOLM MODULES

DEFINITION 10. . Let $E : \mathcal{B} \to \mathcal{M}$ a functor, where $\mathcal{M}$ is CM category. A pairing of E with the set of unitary Fredholm modules is rule which to each unitary Fredholm modules (φ, ψ, F) (for $B \in \mathcal{B}$) associates a morphism

$$\times(\varphi, \psi, F) : E(B) \to E_C(B) \tag{7}$$

in the category $\mathcal{M}$, with following conditions:

1. Functoriality. If (φ, ψ, F) is a unitary Fredholm B'-module, and if $f : B \to B'$ is a $*$-homomorphism from $\mathcal{B}$, then the diagram

$$\begin{array}{ccc} E(B) & \overset{\times(\varphi f, \psi f F)}{\longrightarrow} & E(C) \\ {\scriptstyle (f)_*} \downarrow & & \downarrow {\scriptstyle =} \\ E(B') & \overset{\times(\varphi, \psi F)}{\longrightarrow} & E(C) \end{array}$$

commutes.

1. Additivity. If (φ, F) and (φ, G) are Fredholm B-modules, then
$$\times(\varphi, \chi, F) + \times(\chi, \psi, G) = \times(\varphi, \psi, F \cdot G).$$
2. Compact perturbation. If F and G are equal, module the compacts, then
$$\times(\varphi, \chi, F) = \times(\varphi, \chi, G)$$
3. Stability. If (φ, χ, F) is a unitary Fredholm B-module and $\chi : B \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism then
$$\times(\varphi, \psi, F) = \times\left(\begin{pmatrix} \varphi & 0 \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}\right).$$

4. Non-degeneracy. If $(e, 0, 1)$ is a Fredholm $\mathcal{C}$-module for which $e : \mathcal{C} \to \mathcal{K}(\mathcal{H})$ maps $1 \in \mathcal{C}$ to rank one projection then $\times(e, 0, 1) : E(\mathcal{C}) \to E(\mathcal{C})$ is the identity morphism.

THEOREM 11. . *Let E be a functor from category $\mathcal{B}$ to CM-category $\mathcal{M}$. If the functor E admits a pairing with the set of unitary Fredholm modules which satisfies conditions of 10,then E is w-homotopy functor.*

PROOF. We have to proof that functor E admits M-pairing. First we proof that unitary isomorphic unitary Fredholm modules has same pairing: Let (u, v) is isomorphism of (φ, χ, F) and (ϕ, ψ, G), i.e. $u\varphi u^* = \phi$, $v\chi v^* = \psi$ and $vFu^* = G$, then after using additivity property one gets

$$\times(\varphi, \chi, F) = \times(\varphi, \phi, u) + \times(\phi, \psi, G) + \times(\psi, \chi, v^*) = \times(\phi, \psi, G),$$

because of $\times(\varphi, \phi, u) = \times(\psi, \chi, v^*) = 0$. For the proof additivity condition of M-pairing, consider sequence of equalities, which is consequence of stability, unitarity and additivity properties:
$\times(\varphi, \chi, F) + \times(\phi, \psi, G) =$
$\times(\varphi \oplus \phi, \chi \oplus \phi, F \oplus 1) + \times(\phi \oplus \chi, \psi \oplus \chi, G \oplus 1) =$
$\times(\varphi \oplus \phi, \chi \oplus \phi, F \oplus 1) + \times(\chi \oplus \phi, \psi \oplus \chi, 1 \oplus G) =$
$\times(\varphi \oplus \phi, \chi \oplus \psi, F \oplus G)$.
Then theorem follows from the theorem 9.

PAIRING WITH FREDHOLM PAIRS

Recall that Fredholm A-pairs is pair (φ, ψ) such that $\varphi(a) - \psi(a) \in \mathcal{K}$ for any $a \in A$, where $\varphi, \psi : A \to \mathcal{L}$ are $*$-homomorphisms.

DEFINITION 12. . Let $E : \mathcal{B} \to \mathcal{M}$ a functor, where $\mathcal{M}$ is CM category. A pairing of E with the set of Fredholm pairs is rule which to each Fredholm pairs (φ, ψ) (for $B \in \mathcal{B}$) associates a morphism

$$\times(\varphi, \psi) : E(B) \to E_C(B) \tag{8}$$

in the category $\mathcal{M}$, with following conditions:

1. Functoriality. If (φ, ψ) is a Fredholm module for B', and if $f : B \to B'$ is a $*$-homomorphism from $\mathcal{B}$, then the diagram

$$\begin{array}{ccc} E(B) & \xrightarrow{\times(\varphi f, \psi f)} & E(C) \\ {\scriptstyle (f)_*}\downarrow & & \downarrow{\scriptstyle =} \\ E(B') & \xrightarrow{\times(\varphi, \psi)} & E(C) \end{array}$$

commutes.

1. Additivity. If (φ, χ) and (χ, ψ) are Fredholm modules for B, then
$$\times(\varphi,\chi) + \times(\chi,\psi) = \times(\varphi,\psi).$$
2. Stability. If (φ, ψ, F) is a unitary Fredholm bimodule for B and $\chi : B \to \mathcal{L}(\mathcal{H})$ is any $*$-homomorphism then
$$\times(\varphi,\psi) = \times\left(\begin{pmatrix} \varphi & 0 \\ 0 & \chi \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \chi \end{pmatrix}\right).$$
3. Non-degeneracy. If $(e, 0)$ is a Fredholm C-module for which $e : C \to \mathcal{K}(\mathcal{H})$ maps $1 \in C$ to rank one projection then $\times(e, 0) : E(C) \to E(C)$ is the identity morphism.
4. Unitary equivalence. If $U \in \mathcal{L}(\mathcal{H})$ is a unitary then
$$\times(\varphi,\psi) = \times(U\varphi U^*, U\psi U^*)$$
5. Stability under compact perturbations. If $U \in \mathcal{L}(\mathcal{H})$ is a unitary which is equal to the identity, module the compacts, then
$$\times(\varphi, U\varphi U^*) = 0$$

Theorem 13. . *Let E be a functor from category $\mathcal{B}$ to CM-category $\mathcal{M}$. If the functor E admits a pairing with the set of Fredholm pairs which satisfies conditions above, then E is w-homotopy functor.*

Proof. We have to construct pairing with unitary Fredholm modules. Let (φ, ψ, F) be a unitary Fredholm module, then by definition $\times(\varphi, \psi, F) = \times(\varphi, F^*\psi F)$. Checking of conditions of pairing is elementary. We give some of them:

a). *Unitary equivalence.* Using unitary equivalence condition of paring of Fredholm pairs we have

$\times(u\varphi u^*, v\psi v^*, vFu^*) = \times(u\varphi u^*, (uF^*v^*)v\psi v^*(vFu^*)) =$
$\times(u\varphi u^*, uF^*\psi Fu^*) = \times(\varphi, F^*\psi F) = \times(\varphi, \psi, F)$

where (u, v) is unitary isomorphism from (φ, ψ, F).

b). *Additivity.* Here we use unitary equivalence and additivity of pairing of Fredholm pairs:

$\times(\varphi, \psi, F) + \times(\psi, \chi, G) = \times(\varphi, F^*\psi F) + \times(\psi, G^*\chi G) =$
$\times(\varphi, F^*\psi F) + \times(F^*\psi F, F^*G^*\chi GF) = \times(\varphi, F^*G^*\chi GF) = \times(\varphi, \chi, GF).$

c). *Compact perturbation.* Let $G - F$ is compact then, using suitable property of pairing of Fredholm pairs, we get:

$\times(\varphi, \psi, G) = \times(\varphi, \psi, F) + \times(\psi, \psi, F^*G) = \times(\varphi, \psi, F)$

because F^*G is unitary which is equal identity module compact operators. Then theorem follows from theorem 11.

3. Generalized Higson HIT

Split exact sequences in the additive categories

A category A is said to be *pointed* category if there exists an object 0 which is both initial and terminal. From definition follows existing of 0-*morphism* $0_{a,b} : a \to b$ such that $0_{a,b} = 0_{0,b} \cdot 0_{a,0}$. The kernel of a morphism $\alpha : a \to b$ is kernel of the pair $(\alpha, 0_{a,b})$. This is a monomorphism $\beta : c \to a$ such that $\alpha \cdot \beta = 0$, and if $\beta' : d \to a$ is another morphism with $\alpha \cdot \beta' = 0$ then there exists unique morphism $\gamma : d \to c$ such that $\beta\gamma = \beta'$. In other words following sequence $0 \to \hom(d, c) \stackrel{\beta_*}{\to} \hom(d, a) \stackrel{\alpha_*}{\to} \hom(d, b)$ is exact in the pointed category of pointed sets and point-preserving set maps. Next $\beta = \ker \alpha$ denotes this fact. The $co\ker \alpha$ is defined dually: this is a epimorphism $\epsilon : b \to e$ such that $\epsilon\alpha = 0$ and following sequence of pointed sets $0 \to \hom(e, d) \stackrel{\beta^*}{\to} \hom(b, d) \stackrel{\alpha^*}{\to} \hom(a, d)$ is exact.

A sequence of morphisms

$$0 \to a \stackrel{\alpha}{\to} c \stackrel{\gamma,\beta}{\to} b \to 0 \tag{9}$$

is said to be *split exact* sequence if $\alpha = \ker \beta$ and $\beta = co\ker \alpha$ and $\gamma : b \to c$ is such that $\beta\gamma = 1_b$.

Lemma 14. *In a pre-additive category following conditions are equivalent*

1. A sequence

$$0 \to a \stackrel{\alpha}{\to} c \stackrel{\gamma,\beta}{\to} b \to 0$$

is split (by $\gamma : b \to c$) exact sequence;

2. For any object d sequence of abelian groups

$$0 \to \hom(d, a) \stackrel{\alpha_*}{\to} \hom(d, c) \stackrel{\gamma_*,\beta_*}{\to} \hom(d, b) \to 0 \tag{10}$$

is split exact;

3. The sequence of abelian groups 10 is split exact for $d = a$ and $d = c$;

4. There exist a morphism $\delta : c \to a$ such that

$$\delta\alpha = 1_a, \quad \alpha \cdot \delta + \gamma \cdot \beta = 1_c. \tag{11}$$

In another words, morphisms $\alpha, \beta, \gamma, \delta$ represent c as the sum of a and b.

5. For any object d sequences of abelian groups

$$0 \to \hom(d, a) \stackrel{\alpha_*}{\to} \hom(d, c) \stackrel{\gamma_*,\beta_*}{\to} \hom(d, b) \to 0$$

and

$$0 \to \hom(b, d) \stackrel{\gamma^*,\beta^*}{\to} \hom(c, d) \stackrel{\alpha^*}{\to} \hom(a, d) \to 0$$

are split exact.

PROOF. The case $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are obvious. $3 \Rightarrow 4$. Consider following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \hom(c,a) & \stackrel{\alpha_*}{\to} & \hom(c,c) & \stackrel{\gamma_*,\beta_*}{\longleftrightarrow} & \hom(c,b) & \to & 0 \\ & & {\scriptstyle\alpha^*}\downarrow & & {\scriptstyle\alpha^*}\downarrow & & {\scriptstyle\alpha^*}\downarrow & & \\ 0 & \to & \hom(a,a) & \stackrel{\alpha_*}{\to} & \hom(a,c) & \stackrel{\gamma_*,\beta_*}{\longleftrightarrow} & \hom(a,b) & \to & 0. \end{array}$$

By assumption, horizontal sequences are split exact sequences of abelian groups. There exists unique $\delta \in \hom(c,a)$ such that $\alpha_*(\delta) = 1_c - \gamma \cdot \beta$ i.e. $\alpha \cdot \delta + \gamma \cdot \beta = 1_c$. Now, consider $\delta \cdot \alpha = \alpha^*(\delta) \in \hom(a,a)$. Then $\alpha_*(\delta \cdot \alpha) = \alpha^*(1_c - \gamma \cdot \beta) = \alpha - \gamma \cdot \beta \cdot \alpha = \alpha$ because $\beta_* \alpha_*(1_a) = \beta\alpha = 0$ But $\alpha_*(1_a) = \alpha$. Thus $\delta \cdot \alpha = 1_a$, because α_* is monomorphism. $4 \Rightarrow 5$ is obvious, and $5 \Rightarrow 1$ is obvious too, only note that $\beta\gamma = \beta_*\gamma_*(1_b) = 1_b$.□

Let $\Psi_{\mathcal{B}}$ be the set of all split exact sequences of C^*-algebras of form 5. Then pair $(\mathcal{B}, \Psi_{\mathcal{B}})$ is category with split sequences.

DEFINITION 15. . Let $\mathbf{C}$ be an additive category. A functor $E : \mathcal{B} \to \mathbf{C}$ will be called

a) *w-additive*, if for each exact sequence of form 5 following sequence

$$0 \to E(\mathcal{K}) \to E(B) \to E(C([0;1])) \to 0 \tag{12}$$

is exact in additive category;

b) *w-stable*, if $E(e)$ is isomorphism, where $e : \mathcal{C} \to \mathcal{K}$ is equivariant $*$-homomorphism such that $e(1)$ is rank one projection;

c) *w-homotopy*, if $E(i_0) = E(i_1)$, where $i_0, i_1 : C[0,1] \to C$ are evolution homomorphisms at 0 and 1 respectively.

Our generalized Higson's homotopy invariant theorem asserts

THEOREM 16. . *Let $\mathcal{B}$ be an admissible category of C^*-algebras (real or complex), $\mathbf{C}$ an additive category. If functor $E : \mathcal{B} \to \mathbf{C}$ is w-additive and w-stable, then E is w-homotopy.*

A CONSTRUCTION

Let $\mathcal{H}$ is a countable generated Hilbert space, $\mathcal{K}$ is algebra of compact operators, $\mathcal{L}$ is algebra of bounded operators, and $p : \mathcal{L} \to \mathcal{Q}$ be essential projection, where $\mathcal{Q}$ is the Calkin algebra. If $\varphi : B \to \mathcal{L}$ is equivariant $*$-homomorphism, then from φ arises C^*algebra B_φ and following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \stackrel{j}{\to} & B_\varphi & \stackrel{p}{\to} & B & \to & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle \pi\varphi} & & \\ 0 & \to & \mathcal{K} & \stackrel{j}{\to} & \mathcal{L} & \stackrel{\pi}{\to} & \mathcal{Q} & \to & 0 \end{array}$$

such that second square is pullback. From definition follows that top sequence is short exact sequence of algebras,

$$B_\varphi = \{(b,x) \in B \times \mathcal{L} | \varphi(b) - x \in \mathcal{K}\},$$

p is obvious projection on the first summand, and $j(x) = (0,x)$.

Note that the exact sequence

$$0 \to \mathcal{K} \xrightarrow{j} B_\varphi \xrightarrow{p} B \to 0$$

is split by the $*$-homomorphism $\hat{\varphi} : B \to B_\varphi$ defined by $\hat{\varphi}(b) = (b, \varphi(b))$. Let $\psi : B \to \mathcal{L}(\mathcal{H})$ be an other $*$-homomorphism such that $\varphi(b) - \psi(b) \in \mathcal{K}$, then $B_\varphi = B_\psi$, but sections $\hat{\varphi} \neq \hat{\psi}$.

Now, let $E : \mathcal{B} \to \mathcal{M}$ be w-stable and split w-additive functor in additive category $\mathcal{M}$. Then following sequence

$$0 \to E(\mathcal{K}) \xrightarrow{j_*} E(B_\varphi) \overset{\varphi_*, p_*}{\longleftrightarrow} E(B) \to 0$$

is split exact sequence in the additive category. Apply lemma 14, then arises unique morphism $r_\varphi : E(B_\varphi) \to E(\mathcal{K})$ such that

$$r_\varphi \cdot j_* = 1_{E(\mathcal{K})}, \qquad j_* \cdot r_\varphi + \varphi_* \cdot p_* = 1_{E(B_\varphi)}. \tag{13}$$

From definition follows that if $\varphi - \psi \in \mathcal{K}$, then there exists unique morphism $r(\varphi,\psi) : E(B) \to E(\mathcal{K})$, such that $r(\varphi,\psi) = r_\varphi \cdot (\varphi_* - \psi_*)$ and $j_* r(\varphi,\psi) = (\varphi_* - \psi_*)$. Now, define

$$\times(\varphi,\psi) = e_*^{-1} \cdot r(\varphi,\psi). \tag{14}$$

Lemma 17. . *Let (φ,ψ) and (φ',ψ') Fredholm pairs for B in two universal Hilbert G- spaces $\mathcal{H}$ and $\mathcal{H}'$, $e : C \to \mathcal{K}$ and $\acute{e} : C \to \mathcal{K}'$ maps such that $e(1)$ and $\acute{e}(1)$ are rank one invariant projections, and the following diagram*

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{K} & \xrightarrow{j} & B_\varphi & \overset{\hat{\varphi}, p}{\rightleftarrows} & B & \to & 0 \\
 & & {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \\
0 & \to & \mathcal{K}' & \xrightarrow{j'} & B_{\varphi'} & \overset{\hat{\varphi}', p'}{\rightleftarrows} & B' & \to & 0
\end{array}$$

is commutative such that $g \cdot \hat{\varphi} = \hat{\varphi}' \cdot f$, $g \cdot \hat{\psi} = \hat{\psi}' \cdot f$ and $h \cdot e = \acute{e}$. Then $\times(\varphi,\psi) = \times(\varphi',\psi') \cdot f_*$

Proof. Firstly note that $r_{\varphi'} \cdot \varphi'_* = 0$ this is easy consequence of identities 13. This identity and 13 implies that

$h_* \cdot r_\varphi = r_{\varphi'} \cdot j'_* \cdot h_* \cdot r_\varphi = r_{\varphi'} \cdot g_* \cdot j_* \cdot r_\varphi =$

$r_{\varphi'} \cdot g_* - r_{\varphi'} \cdot \varphi'_* \cdot f_* \cdot p = r_{\varphi'} \cdot g_*$.

Then

$\times(\varphi', \psi') \cdot f_* = \acute{e}_*^{-1} \cdot r(\varphi', \psi') \cdot f_* = \acute{e}_*^{-1} \cdot r_{\varphi'} \cdot (\varphi'_* - \psi'_*) \cdot f_*$
$= \acute{e}_*^{-1} \cdot r_{\varphi'} \cdot g_* \cdot (\varphi_* - \psi_*) = \acute{e}_*^{-1} \cdot h_* \cdot r_\varphi \cdot (\varphi_* - \psi_*)$
$= e_*^{-1} \cdot r_\varphi \cdot (\varphi_* - \psi_*) = \times(\varphi, \psi).$

LEMMA 18. . *Let* $\varphi : B \to \mathcal{K} \subset \mathcal{L}$. *Then* $\times(\varphi, 0)) = e_*^{-1} \cdot \varphi_*$.

PROOF. Note that $B_\varphi = B \times K$; and $\hat{\varphi}(b) = (b, \varphi(b))$ and $\hat{0}(b) = (b, 0)$. Consider projection $p_\mathcal{K} : B \times \mathcal{K} \to \mathcal{K}$. Then $j_* \cdot p_{\mathcal{K}*} + \hat{0}_* \cdot p = 1_{B_\varphi}$ and $p_{\mathcal{K}*} \cdot j_* = 1_{\mathcal{K}*}$. Thus $r_0 = p_{\mathcal{K}_*}$ because of definition and uniqueness of $r_{0.}$ It follows that
$\times(0, \varphi) = e_*^{-1} \cdot r_0 \cdot (\hat{0}_* - \hat{\varphi}_*) = e_*^{-1} \cdot p_{\mathcal{K}*} \cdot (\hat{0} - \hat{\varphi}_*) =$ (as $p_\mathcal{K} \cdot \hat{0} = 0$)
$= -e_*^{-1} \cdot p_{\mathcal{K}*} \cdot \hat{\varphi}_* = -e_*^{-1} \cdot \varphi_*$.
But $\times(\varphi, 0) = - \times (0, \varphi)$ by properties 2) and 6) of 12. Therefore, we get that $\times(\varphi, 0) = e_*^{-1} \cdot \varphi_*$.

HOMOTOPY INVARIANT THEOREM

DEFINITION 19. . Let $\mathcal{B}$ as the above and contains $\mathcal{K}$ the algebra of compact operators of countable generated Hilbert space $\mathcal{H}$, and if $0 \to \mathcal{K} \to B \longleftrightarrow C[0;1] \to 0$ is a split extension then $B \in \mathcal{B}^G$.

1. A functor $E : \mathcal{B} \to \mathcal{M}$ is said to be w-stable if $E(e) : E(C) \to E(\mathcal{K})$ isomorphism, where $e : C \to \mathcal{K}$ is defined by formula $e(1) = p$ is rank one projection;
2. A functor $E : \mathcal{B} \to \mathcal{M}$ is said to be split w-additive if for split extension above form the following sequence

$$0 \to E(\mathcal{K}) \to E(B) \longleftrightarrow E(C([0,1])) \to 0 \tag{15}$$

is split exact sequence.

THEOREM 20. . *Let* $\mathcal{B}$ *be an admissible category real or complex* C^*-*algebras 19,* $\mathcal{M}$ *be an additive category and* $E : \mathcal{B} \to \mathcal{M}$ *be a* w-*stable and split* w-*additive. Then* E *is* w-*homotopy invariant.*

PROOF. It suffices to show that the pairing defined by 14 satisfies the hypotheses of 12. The first two are straightforward.

a) *stability.* Consider diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & B_{\varphi_+} & \to & C([0,1]) & \to & 0 \\ & & \downarrow^h & & \downarrow^f & & \downarrow^= & & \\ 0 & \to & M_2(\mathcal{K}) & \to & B_{\varphi_+ \oplus \varphi} & \to & C([0,1]) & \to & 0, \end{array} \tag{16}$$

where $f(c \oplus x) = c \oplus \begin{pmatrix} c & 0 \\ 0 & \varphi_+ \end{pmatrix}$ and $\varphi_+ \oplus \varphi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi \end{pmatrix}$. Note only does 16 commute, but also, the sections $\varphi'_\pm$ of the top short exact sequence correspond via f to the sections $\begin{pmatrix} \varphi_\pm & 0 \\ 0 & \varphi \end{pmatrix}$ for the bottom sequence. It follows from lemma 17 that

$$\times(\varphi_+, \varphi_-) = \times\left(\begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi \end{pmatrix}, \begin{pmatrix} \varphi_- & 0 \\ 0 & \varphi \end{pmatrix} \right).$$

b). *non-degeneracy.* Using lemma 18, we get that $\times(e,0) = e_*^{-1} \cdot e_* = 1_{C*}$.

c). *unitary equivalence.* Consider diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & B_{\varphi_+} & \to & C([0,1]) & \to & 0 \\ & & {\scriptstyle ad(u)}\downarrow & & \downarrow{\scriptstyle ad(u)} & & \downarrow{\scriptstyle =} & & \\ 0 & \to & \mathcal{K} & \to & B_{u\varphi_+u^*} & \to & C([0,1]) & \to & 0. \end{array}$$

From lemma 17 follows that $\times(\varphi_+, \varphi_-) = \times(u\varphi_+u^*, u\varphi_-u^*)$.

d).*compact perturbation.* Similarly, because if u is equal to the identity , module compacts, then u is an element of an subalgebra $\mathcal{L}$ which contains B_φ as the closed ideal then by lemma 2.6.12 of [8] the homomorphism

$$ad(u)_* : E(B_\varphi) \to E(B_\varphi)$$

is identity. We conclude that

$$\hat{\varphi}_* - ad(u)_*\hat{\varphi}_* = 0,$$

which implies that $\times(\varphi, ad(u)\varphi) = 0.\square$

4. Abstract form of homotopy theorem

Now, we give here an elementary, but useful variant of homotopy invariant theorem.

Category with split exact sequences

Let $\mathbf{A}$ be a category with zero object and Ψ_A be a set of (formal, without any conditions of exactness) split sequences of form

$$0 \to a \stackrel{\alpha}{\to} c \stackrel{\gamma,\beta}{\to} b \to 0, \tag{17}$$

where $\gamma : b \to c$ is a morphism . The pair $(\mathbf{A}, \Psi_A)$ will be said a *category with split exact sequences* and Ψ_A is called set of split exact sequences. Let $(\mathbf{C}, \Psi_C)$ be an other category with split exact sequences. A functor $E : \mathbf{A} \to \mathbf{C}$ is said a *split additive functor* if $E(0)$ zero object and from $\alpha \in \Psi_A$ follows that $E(\alpha) \in \Psi_C$.

Remark. a) In the next canonical structure of category with split exact sequences on an additive category $\mathbf{C}$ will be given by the set Γ_C of all split exact ∪sequences in the categorical sense.

b) In the admissible category of C^*-algebras $\mathcal{B}$ the set $\Lambda_{\mathcal{B}}$ of split exact sequences are split exact sequences of algebras of form $0 \to \mathcal{K} \to D \to C[0,1] \to 0$.

Definition 21. . Let $(\mathbf{A},\Sigma_A)$ category with split exact sequences and $\mathbf{C}$ be an additive category. A functor $E : \mathbf{A} \to \mathbf{C}$ is said to be *split exact* functor if for each split exact sequence $0 \to a \overset{\alpha}{\to} c \overset{\gamma,\beta}{\to} b \to 0$ from Σ_A the following sequences of abelian groups

$$0 \to \hom(d, E(a)) \overset{\alpha_*}{\to} \hom(d, E(c)) \overset{\gamma_*,\beta_*}{\to} \hom(d, E(b)) \to 0$$

and

$$0 \to \hom(E(b), d) \overset{\gamma^*,\beta^*}{\to} \hom(E(c), d) \overset{\alpha^*}{\to} \hom(E(a), d) \to 0$$

are exact for each object $d \in \mathbf{C}$.

Lemma 22. . *Let* $(\mathbf{A},\Sigma_A)$ *category with split exact sequences and* $\mathbf{C}$ *be an additive category. A functor* $E : \mathbf{A} \to \mathbf{C}$ *is split exact iff* E *is split additive.*

Split action

Here are given definitions which we are need to formulation of an abstract form of homotopy invariant theorem.

Definition 23. . Let $\mathbf{A}$ be and $\mathbf{B}$ be categories. We say that $\mathbf{A}$ is with right $\mathbf{B}$-action if it is given functor

$$\oslash : \mathbf{A} \times \mathbf{B} \to \mathbf{A}$$

Let $\mathbf{A}$ and $\mathbf{B}$ have of structure of split categories with set split exact sequences Ψ_A and Ψ_B respectively, we say that action is a *split action* if for any object $A \in \mathbf{A}$ and split exact sequence $0 \to B' \to B \overset{\gamma,\beta}{\to} B'' \to 0$ in $\Psi_{\mathbf{B}}$ following sequence

$$0 \to A \oslash B' \to A \oslash B \overset{\dot\gamma,\dot\beta}{\to} A \oslash B'' \to 0$$

is in $\Psi_{\mathbf{A}}$.

Definition 24. . Let $\mathbf{A}$ be a split category, $\mathcal{B}$ be an admissible category of C^*-algebras, $\mathbf{C}$ is split category and $\oslash : \mathbf{A} \times \mathcal{B} \to \mathbf{A}$ is a split action. We say that

1. functor $E : \mathbf{A} \to \mathbf{C}$ is a *homotopy functor* if $E(A \oslash -)$ is w-homotopy functor on $\mathcal{B}$ for every object $A \in \mathbf{A}$.

2. functor $E : \mathbf{A} \to \mathbf{C}$ is a *stable functor* if $E(A \oslash -)$ is w-stable on $\mathcal{B}$ for every object $A \in \mathbf{A}$

THEOREM 25. . *Let* $\mathbf{A}$ *be a split category,* $\mathcal{B}$ *an admissible split category of* C^**-algebras and* $\mathbf{C}$ *is an additive category,* $\oslash : \mathbf{A} \times \mathcal{B} \to \mathbf{A}$ *is split action. If functor* $E : \mathbf{A} \to \mathbf{C}$ *is stable and split additive, then* E *is homotopy functor.*

PROOF. For each $A \in \mathbf{A}$ functor $E(A \oslash -)$ from category $\mathcal{B}$ to $\mathbf{C}$ is w-stable and w-split additive, therefore it is w-homotopy invariant by theorem 16. Thus E is homotopy invariant by definition 24.□

REAL C^*-ALGEBRAS

Let $\mathbf{RC}^*$ (resp. SRC*) be category real C^*-algebras (resp. real separable C^*-algebras) and $*$-homomorphisms. Let $\mathcal{B}$ be an admissible category of real C^*-algebras. Consider action

$$\oslash : \mathbf{RC}^* \times \mathcal{B} \to \mathbf{RC}^*$$

defined by formulas $\oslash(A, B) = A \otimes_{\min} B$ and $\oslash(f, g) = f \otimes_{\min} g$ where $\otimes_{\min}$ is minimal C^*-tensor product. Category $\mathbf{RC}^*$ is category of split sequences relative usually split sequences and functor $A \otimes_{\min} - : \mathcal{B} \to \mathbf{RC}^*$ preserves split exact sequences for any real C^*-algebra A. Let $\mathcal{R}$ be algebra of real numbers and $\mathcal{R}[0, 1]$ real C^*- algebra of continuous real functions. Then definition ? in this case has following formulation

DEFINITION 26. Let $\mathbf{C}$ be an additive category. We say that

1. functor $E : \mathbf{RC}^* \to \mathbf{C}$ is a *homotopy functor* if $E(1_A \otimes i_0) = E(1_A \otimes i_1)$ for every object $A \in \mathbf{RC}^*$, where $i_0, i_1 : \mathcal{R}[0, 1] \to \mathcal{R}$ are evolution maps at 0 and 1
2. functor $E : \mathbf{RC}^* \to \mathbf{C}$ is a *stable functor* if $E(1_A \otimes e)$ is an isomorphism for any $A \in \mathbf{RC}^*$, where $e : \mathcal{R} \to \mathcal{K}_{\mathcal{R}}$ is such that $e(1)$ is rank one projection.
3. functor $E : \mathbf{RC}^* \to \mathbf{C}$ is a *split additive functor* if for any split exact sequence $0 \to I \to B \to A \to 0$ real algebras the sequence

$$0 \to E(I) \to E(B) \to E(A) \to 0$$

is split exact sequence in $\mathbf{C}$.

Theorem 24 gives following

COROLLARY 27. *Let* $\mathbf{RC}^*$ *be the category of real* C^**-algebras. If functor (covariant or contravariant)*

$$E : \mathbf{RC}^* \to \mathbf{C}$$

is stable and split additive, then E *is homotopy functor.*

REMARK. Corollary also is true for any admissible subcategory of $\mathbf{RC}^*$ closed relative minimal tensor product, short extensions and contains 0, R, $R[0,1]$, in particular for $\mathbf{SRC}^*$ of category of separable real C^*-algebras and their $*$-homomorphisms.

COROLLARY 28. *Let G be a compact group, $\mathbf{GRC}^*$ be the category of real $G-C^*$-algebras, $\mathcal{B}$ an admissible split category of real C^*-algebras (which are considered as the $G-C^*$-algebras with trivial G action), $\otimes_{\min} : \mathbf{GRC}^* \times \mathcal{B} \to \mathbf{GRC}^*$ is split action defined as above, $\mathbf{C}$ is an additive category. If functor $E : \mathbf{RC}^* \to \mathbf{C}$ is stable and split additive, then E is homotopy functor.*

4.1. THE CATEGORY OF PRO-C^*-ALGEBRAS. Let $\mathcal{A}^*$ denote the category of involutive algebras over C field of complex numbers, which are equipped with a Hausdorf topology such that all algebraic operations are continuous. Morphisms in this category are involutive algebra homomorphisms. We consider ungraded algebras as objects in this category.

DEFINITION 29. [15] *a)* An involutive complex algebra A is called a pro-C^*-algebra, if there exists an inverse system $\{A_i, \tau_{ij}\}$ of C^*-algebras such that $A = inv \lim A_i$. The category of pro-C^*-algebras and bounded $*$-homomor-
phisms will be denoted by $\operatorname{Pr}\mathbf{C}^*$.

b) A pro-C^*-algebra A is called countable if there exists a sequence

$$A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_n \leftarrow \cdots$$

of C^*-algebras such that $A = inv \lim A_i$. Countable pro-C^*-algebras are called σ-C^*-algebras. The full subcategory of $\operatorname{Pr}\mathbf{C}^*$ of σ-C^*-algebras is denoted by $\mathbf{S}\operatorname{Pr}\mathbf{C}^*$.

c) For A in $\mathcal{A}^*$ let $\mathcal{S}(A)$ be the set of continuous C^*-seminorms and suppose that it is Hausdorf and complete. For each $x \in \mathcal{S}(A)$ let $I_x = \{a \in A : |a|_x = 0\}$, and $A_x = A/I_x$ which is C^*-algebra. Let $x \leq y \Leftrightarrow |a|_x \leq |a|_y$, for all $a \in A$,is essential order on $\mathcal{S}(A)$. If $x \geq y$, then the identity map on A induces epimorphism $\pi^{xy} : A_x \to A_y$. Let $\pi^x : A \to A_x$ denote essential projection. Then (A_x, π^{xy}) is an inverse system of C^*-algebras which will be denoted again by $\mathcal{S}(A)$. Then $A_\infty = inv \lim \mathcal{S}(A)$ is a pro-C^*-algebra and projection π^x induce a morphism $\pi : A \to A_\infty$. Note also that $\mathcal{S}(A)$ is directed set.

In [[15]] is a characterization of such algebras which we give here:

THEOREM 30. *For a pro-C^*-algebra following properties are equivalent:*

a) A is a pro-C^*-algebra.

b) A is Hausdorf, complete and topology on A is induced by a system of C^*-seminorms.

c) The canonical homomorphism $\pi : A \to A_\infty$ is an isomorphism, i.e. π is an isomorphism of $*$-algebras and a homeomorphism.

Let $\operatorname{Pr}\mathbf{C}^*$ (resp. $\mathbf{S}\operatorname{Pr}\mathbf{C}^*$) be full subcategory of $\mathcal{A}^*$ of pro-C^*-algebras (resp. separable algebras).

Now, we are in need to construct action

$$\overset{\wedge}{\otimes}: \operatorname{Pr}\mathbf{C}^* \times \mathcal{B} \to \operatorname{Pr}\mathbf{C}^*$$

where $\mathcal{B}$ is admissible subcategory of nuclear C^*-algebras. Namely, we want to construct a tensor product of pro-C^*-algebra and nuclear C^*-algebra (cf. [[15]]).

We recall that for a C^*-algebras A and nuclear C^*-algebra B one has uniqual C^*-tensor product $A \otimes B$.

Definition 31. Let A be a pro-C^*-algebra and B be a nuclear C^*-algebra, then define the tensorproduct $A \overset{\wedge}{\otimes} B$ to be the inverse limit of the system

$$(A_x \otimes B, \pi^{xy} \otimes 1_B)\ x, y \in \mathcal{S}(A).$$

Proposition 32. *Let $0 \to B_0 \to B_2 \overset{\longleftarrow}{\longrightarrow} B_1 \to 0$ be a split exact sequence of algebras from $\mathcal{B}$ and A is a pro-C^*-algebra. Then*

$$0 \to A \overset{\wedge}{\otimes} B_0 \to A \overset{\wedge}{\otimes} B \overset{\longleftarrow}{\longrightarrow} A \overset{\wedge}{\otimes} B_1 \to 0 \tag{18}$$

is split exact sequence of pro-C^*-algebras.

Proof. Let $A = inv\lim A_\alpha$, where A_α is C^*-algebras. Then following sequence

$$0 \to A_\alpha \otimes B_0 \to A_\alpha \otimes B \overset{\longleftarrow}{\longrightarrow} A_\alpha \otimes B_1 \to 0$$

is split exact sequence of C^*-algebras, because B_0, B_1 are nuclear C^*-algebras. From this we get following long exact sequence of linear spaces

$$\cdot\cdot \overset{\longleftarrow}{\longrightarrow} \underset{\leftarrow}{\lim}\,.^{(1)}(A_\alpha \otimes B_1) \to \underset{\leftarrow}{\lim}(A_\alpha \otimes B_0) \to \underset{\leftarrow}{\lim}(A_\alpha \otimes B) \overset{\longleftarrow}{\longrightarrow} \underset{\leftarrow}{\lim}(A_\alpha \otimes B) \to 0$$

from which easy follows that the sequence 18 of pro-C^*-algebras is exact.

Now, remark that definition of general stability, homotopy and split exactness gives following definitions.

Definition 33. a) Two homomorphisms $f_0, f_1 : A \to A'$ in the category $\operatorname{Pr}\mathbf{C}^*$ are homotopic if there exist homomorphism $F : A \to A' \hat{\otimes} C([0;1])$ such that $(1 \hat{\otimes} i_0)F = f_0 \hat{\otimes} 1$ and $(1 \hat{\otimes} i_1)F = f_1 \hat{\otimes} 1$,where

$i_1, i_0 : C[0,1] \to C$ are evolution homomorphisms at 1and 0 respectively. A functor $\mathcal{F}$ on $\mathbf{B}$ is said homotopy functor if $\mathcal{F}(f_0) = \mathcal{F}(f_1)$.

b) Let $e : C \to \mathcal{K}$ be a homomorphism such that $e(1)$ is rank one projection, and $1_A \overset{\wedge}{\otimes} e : A \overset{\wedge}{\otimes} C \to A \overset{\wedge}{\otimes} \mathcal{K}$ be induced homomorphism. A functor $\mathcal{F}$ on $\Pr \mathbf{C}^*$ is said stable if $\mathcal{F}(1_A \overset{\wedge}{\otimes} e)$ is isomorphism for any A.

c). A functor $\mathcal{F}$ from $\mathbf{B}$ to additive category is split additive if from split exactness of sequence $0 \to B_0 \to B_1 \to B_2 \leftrightarrows 0$ follows split exactness of

$$0 \to \mathcal{F}(B_0) \to \mathcal{F}(B_1) \leftrightarrows \mathcal{F}(B_2) \to 0$$

THEOREM 34. *Let $\mathcal{F}$ be a stable and split additive functor from category $\Pr \mathbf{C}^*$ (or $\mathbf{S} \Pr \mathbf{C}^*$) to a pre-additive category. Then $\mathcal{F}$ is homotopy invariant functor.*

Proof. For any A from $\Pr \mathbf{C}^*$ consider functor $\mathcal{F}_A : \mathcal{B} \to \mathbf{C}$ which is defined by formulas $\mathcal{F}_A(B) = A \hat{\otimes} B$ and $\mathcal{F}_A(g) = 1_A \hat{\otimes} g$, where $\mathbf{C}$ is a additive category. By assumption $\mathcal{F}_A$ is w-stable and w-additive. Apply main theorem, we get that $\mathcal{F}_A$ is w-homotopy functor for any pro-C^*-algebra. Therefore $\mathcal{F}$ is homotopy functor.

REMARK. By analogy of theorem 34, we can proof similar theorems for other subcategories of $\Pr \mathbf{C}^*$ which are closed relative split extensions and $\mathcal{B}$-action, in particular for category of separable pro-C^*-algebras.

4.2. HOMOTOPY PROPERTY OF CUNTZ BIVARIANT THEORIES. Let $C[0,1]$ be algebra of continuous functions on the unit interval and $\mathcal{K}$ be algebra of compact operators on a separable Hilbert space. Let $\mathbf{M}$ be the category of m-algebras and their homomorphisms [2],[11].

DEFINITION 35. *For any functor $\diamond : \mathbf{M} \to \mathbf{A}$ let $\diamond^A_{\mathcal{K}} : \mathcal{B} \to \mathbf{A}$ be the functor defined by formulas*

$$\diamond^A_{\mathcal{K}}(B) = \diamond(A \otimes_\pi (\mathcal{K} \otimes_{\min} B)) \text{ and } \diamond^A_{\mathcal{K}}(f) = \diamond(1_A \otimes_\pi (1_{\mathcal{K}} \otimes_{\min} f)). \tag{19}$$

DEFINITION 36. A *algebraic* bivariant theory is a functor $\diamond$ from category $\mathbf{M}$ into an additive category $\mathbf{A}$ such that following conditions are satisfied:

1. (M_2-stability) the canonical inclusion $\iota : A \to M_2(C) \otimes A$ induces an isomorphism $\iota_\diamond \in \mathbf{A}(A, M_2(C) \otimes A)$;
2. (split exactness) for any split exact sequence $0 \to A \xrightarrow{i} B \overset{t,p}{\leftrightarrows} C \to 0$ there exists unique morphism $\tilde{j} \in \mathbf{A}(B, A)$ such that $\{A, C; i_\diamond, \tilde{j}, t_\diamond, p_\diamond\}$ is representation of B as the sum in the additive category $\mathbf{A}$.

PROPOSITION 37. *Let $\diamond : \mathbf{M} \to \mathbf{A}$ is algebraic bivariant theory. Then functor $\diamond^A_{\mathcal{K}} : \mathcal{B} \to \mathbf{A}$ defined by 19 is w-homotopy invariant functor for any $A \in \mathbf{A}$*

PROOF. According on the associativity of the tensor products bellow and nuclearity of $M_2(C)$, we get following commutative diagram

$$\begin{array}{ccc} A \otimes_\pi (\mathcal{K} \otimes_{\min} C) & \stackrel{\iota}{\to} & A \otimes_\pi (\mathcal{K} \otimes_{\min} \mathcal{K}) \\ \downarrow^j & & \downarrow^{\simeq} \\ M_2(A \otimes_\pi (\mathcal{K} \otimes_{\min} C)) & \simeq & A \otimes_\pi (\mathcal{K} \otimes_{\min} M_2(C)), \end{array}$$

where $i = 1_A \otimes_\pi (1_{\mathcal{K}} \otimes_{\min} e)$ and j is canonical imbedding in the left upper corner. Apply $M_2(C)$-stability of $\diamond$, one has that for the canonical homomorphism

$$\iota : A \otimes_\pi (\mathcal{K} \otimes_{\min} C) \to A \otimes_\pi (\mathcal{K} \otimes_{\min} \mathcal{K})$$

the morphism $\iota_\diamond$ is isomorphism, i.e. $\diamond^A_{\mathcal{K}}$ is w-stable. The w-additivity property follows from the fact that functor $A \otimes_\pi (\mathcal{K} \otimes_{\min}) : \mathcal{B} \to \mathbf{M}$ carries split exact sequences. Thus by homotopy invariance theorem functor $\diamond^A_{\mathcal{K}}$ is w-homotopy invariant.

DEFINITION 38. *Two homomorphisms $f_0, f_1 : A \to M$ in the category $\mathbf{M}$ are π-homotopic if there exists homomorphism $F : A \to M \otimes_\pi C[0,1]$ such that $f_0 = ev_0 \cdot F$ and $f_1 = ev_1 \cdot F$. Let $\diamond : \mathbf{M} \to \mathbf{A}$ be a functor and $M \in \mathbf{M}$. We say that $\diamond$ has π-homotopy property relative M, if $(1_M \otimes_\pi ev_0)_\diamond = (1_M \otimes_\pi ev_1)_\diamond$ where $ev_t : C[0,1] \to C$ is evolution at $t \in [0,1]$.*

THEOREM 39. *Let $\diamond : \mathbf{M} \to \mathbf{A}$ be $M_2(C)$-stable and splite exact functor. Then $\diamond$ has π-homotopy property relative to any m-algebra of the form $M \otimes_\pi \mathcal{K}$. In particular, for any π-homotopic homomorphisms $f_0, f_1 : A \to M \otimes_\pi \mathcal{K}$ holds $\diamond(f_0) = \diamond(f_1)$.*

PROOF. Consider commutative diagram

$$\begin{array}{ccc} A \otimes_\pi (\mathcal{K} \otimes_\pi C[0,1]) & \stackrel{c}{\to} & A \otimes_\pi (\mathcal{K} \otimes_{\min} C[0,1]) \\ \downarrow^{\simeq} & & \downarrow^{i_t} \\ (A \otimes_\pi \mathcal{K}) \otimes_\pi C[0,1] & \stackrel{ev_t}{\to} & (A \otimes_\pi \mathcal{K}) \otimes C. \end{array}$$

where c is essential homomorphism defined as the identity on the elements of form $a \otimes_\pi (|| \otimes f(t))$ and extended continuously because of $||\cdot||_{\min} \leq ||\cdot||_\pi$. By proposition 37 for vertical arrowos i_0, i_1 holds equality $\diamond(i_0) = \diamond(i_1)$. Thus $\diamond(1_{A \otimes_\pi \mathcal{K}} \otimes_\pi ev_0) = \diamond(1_{A \otimes_\pi \mathcal{K}} \otimes_\pi ev_1)$. Second fact is easy consequence of first.

COROLLARY 40. *Let $kk^*(A,B)$ be the Cuntz smooth bivariant kk-theory on the category of m-algebras* [2]. *Then*

$$kk^*_{\mathcal{K}}(A,B) = kk^*(A \otimes_\pi \mathcal{K}, B \otimes_\pi \mathcal{K})$$

has π-homotopy property for any $A, B \in \mathbf{M}$.

COROLLARY 41. *Let $HP^*(A,B)$ be the Cuntz-Quillen periodic cyclic cohomology on the category of m-algebras* [6]. *Then*

$$HP^*_{\mathcal{K}}(A,B) = HP^*(A \otimes_\pi \mathcal{K}, B \otimes_\pi \mathcal{K})$$

has π-homotopy property for any $A, B \in \mathbf{M}$.

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