Global Actions:
The Algebraic Counterpart of a Topological Space

Dedicated to P. S. Aleksandrov on his 100'th anniversary

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1 Introduction

This article introduces a general algebraic concept of space with motion. The spaces consist of a set X together with a collection of group actions $G_\alpha \curvearrowright X_\alpha$ where G_α is a group acting on a subset $X_\alpha \subseteqq X$. It is possible that a given subset of X is acted on by several different groups. The group actions are tabulated by letting α above run through an index set Φ called a coordinate system. We structure the set $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ of group actions by equipping Φ with a transitive reflexive relation $\leqq$ and imposing the condition that G defines a functor $\Phi \to$ ((groups)), $\alpha \mapsto G_\alpha$, such that if $\sigma \in G_\alpha$ and ρ its image in G_β under the homomorphism $G_{\alpha \leqq \beta}$ then for any $\mathrm{x} \in X_\alpha \cap X_\beta, \sigma\mathrm{x} = \rho\mathrm{x}$. The resulting triple (Φ, G, X) is called a global action. Motion is provided by the concept of path. A path is a sequence $\mathrm{x}_0, \cdots, \mathrm{x}_p$ of points in X such that for each $0 \leqq i \leqq p-1$, there is a coordinate $\alpha_i \in \Phi$ with the property that $\mathrm{x}_i, \mathrm{x}_{i+1} \in X_{\alpha_i}$ and $\sigma_i \mathrm{x}_i = \mathrm{x}_{i+1}$ for some $\sigma_i \in G_{\alpha_i}$. It turns out that there is a global action L called a line such that paths in a given global action A are determined by morphisms from L to A. We shall use this natural and intuitive construction of paths to carry over to algebra all of the experience we have with paths in topological spaces. One consequence of this program will be a homotopy theory for algebraic structures which includes a natural, intuitive, as well as rigorous concept of algebraic deformation of morphisms. There are two prerequisites for realizing this goal and they are supplied in the current article. First one must show how to make the set $Mor(A, B)$ of all morphisms from a global action A to a global action B, into a global action. This allows one to deform a morphism $f : A \to B$ to a morphism $g : A \to B$ by a path from f to g. The second is to formulate a general condition for global actions, which guarentees that the exponential map $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$

is an isomorphism of global actions. This implies that the cylinder method for deforming a morphism f to a morphism g is equivalent to the path procedure above. With this equivalence, one develops fundamental constructions and principles of algebraic homotopy theory along the lines of their topological precedents. This will be done in a sequel to the current paper.

In order to apply the algebraic homotopy theory above to developing a theory of deformation for morphisms in arbitrary categories, the notion of global object will be used. This concept is introduced also in the current article. It is a generalization to arbitrary categories of the notion of global action and serves in the current article to provide depth and perspective for the notion of global action.

The remainder of the article is organized as follows. §2 introduces the notion of global action and provides numerous examples. These include the line action L mentioned above, other related, geometrically inspired examples, and global actions which we christen Volodin models. The Volodin models will be used in a future paper to provide an algebraic definition of higher Volodin K-groups and algebraic foundations of algebraic K-theory. Next the concept of global object is introduced, as well as the concept of a representation of a global object by a global action. Examples of both concepts are given and it turns out that the geometrically inspired global actions at the beginning of the section are representations of global simplicial complexes, i.e. of global objects in the category of (abstract) simplicial complexes. The section closes with two results showing how to functorially construct global objects from primitive data. These constructions will be enormously important in constructing a deformation theory for morphisms in arbitrary categories.

§3 studies the concept of morphism for global actions. There is a general notion of morphism and two important special kinds of morphisms, namely normal morphisms and regular morphisms. The regular morphisms provide the strongest notion of morphism and preserve all the structural concepts in the definition of a global action. The general notion takes individually into account, the group actions making up a global action, but does not reflect the coherence among the actions, given by the transitive reflexive relation on the coordinate system Φ and the functoriality of the global group functor G. Normal morphisms lie somewhere between regular and general. All regular morphisms are normal, but not conversely.

We define first the general notion of morphism and then that of regular morphism. The notion of chart is introduced and used to define a global structure on the set $Mor(A, B)$ of all morphisms from a global action A to a global action B. As a global action, $Mor(A, B)$ is a contravariant functor in the first variable, but is not defined over all morphisms in the second variable. The notion of normal morphism is introduced so that $Mor(A,B)$ becomes a covariant functor in the first variable over all normal morphisms. This result will be very important for algebraic homotopy theory, since it will imply that algebraic homotopy groups are functorial over a large class of normal morphisms called $\infty-$L-morphisms. Next the notions of infimum and strong infimum global action are introduced. Volodin models and the geometrically inspired global actions in §2 are examples of strong infimum actions. It is shown that any morphism whose target is an infimum or strong infimum action is $\infty-$normal and that the exponential morphism $E : Mor(A, Mor(B,C)) \to Mor(A \times B, C)$ is an $\infty-$ normal isomorphism if C is an infimum action and a regular isomorphism if C is a strong infimum action. These results will be also required in developing algebraic homotopy theory.

§4 introduces the notion of subaction of a global action and the notion of relative action. A relative action is a pair consisting of a global action and a subaction. Relative actions are required in the homotopy theory of global actions. §4 repeats the entire program of §3, with relative actions replacing global actions. The details are not routine, as in the case of topological spaces. The added complications arise from the notion of relative chart which is needed to put a relative global structure on the set of all morphisms $Mor(A,B)$ from a relative action A to a relative action B. Relative charts are subtler than their absolute counterparts and this added subtlety has to be followed up throughout the entire section. This done, one gets the same results as in §3.

The current article is written in an elementary style and is selfcontained. From a technical standpoint, an advanced undergraduate level familiarity with sets, groups, simplicial complexes, and categories is all that is required. From an appreciation standpoint, experience in algebraic topology and algebraic K-theory are helpful.

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2 Global actions

A global action is an algebraic object which is formed by fitting or gluing together various group actions. The construction resembles that of several well known mathematical objects which are formed by fitting certain building blocks together, in our case group actions, to form more complicated structures. Examples include simplicial complexes where the building blocks are simplices, CW-Complexes where the building blocks are closed disks, manifolds where the building blocks are open disks of a fixed dimension, and varieties (resp. schemes) where the building blocks are affine varieties (resp. affine schemes). Furthermore the homotopy theory of global actions resembles that of the topological examples above in so far as the building blocks turn out to be homotopically trivial, being n-connected for all n > 0.

DEFINITION 2.1 A global action is a set $\{G_\alpha \curvearrowright X_\alpha | \alpha \in \Phi\}$ of groups G_α acting on subsets X_α of some set $|X|$, subject to the following conditions.

(2.1.1) Φ is equipped with a transitive, reflexive relation $\leqq$. If the relation $\leqq$ is a partial ordering, i.e. $\alpha \leqq \beta$ and $\beta \leqq \alpha$ imply $\alpha = \beta$, then the global action is called a **partially ordered action**.

Φ equipped with the relation $\leqq$ will be frequently considered as a category. As such, there is at most one morphism $\alpha \leqq \beta$ between any two objects $\alpha, \beta \in \Phi$. If $\alpha \leqq \beta$ and $\beta \leqq \alpha$ then these morphisms are inverse to each other.

(2.1.2) The function $X : \Phi \to$ subsets $|X|$, $\alpha \mapsto X_\alpha$, has in general no special properties. If the relation $\leqq$ is a partial ordering, if the set subsets $|X|$ is partially ordered by inclusion, and if the function X above is order reversing, i.e. a contravariant functor, then the global action is called **contravariant**.

(2.1.3) G is a covariant functor $\Phi \to ((groups)), \alpha \mapsto G_\alpha$.

(2.1.4) **Compatibility condition.** If $\alpha \leqq \beta$ then G_α leaves $X_\alpha \cap X_\beta$ invariant and for all $\sigma \in G_\alpha$ and all x $\in X_\alpha \cap X_\beta$, $\sigma x = G_{\alpha \leqq \beta}(\sigma)$x. (Note that if $X_\alpha \cap X_\beta$ is empty, the compatibility condition is automatically satisfied.)

Φ is called the **coordinate system** of the action and each element of Φ is called a **coordinate**. The functor G is called the **global group functor** of the action and the function X the **global set function**. $|X|$ is called the **enveloping set**. If α is a coordinate then G_α is called the **local group** at α and X_α the **local set** at α. Let x $\in |X|$. The local group G_α or an element $\sigma \in G_\alpha$ is said to be defined at x whenever x in X_α. A **group element** of a global action is an element of some local group G_α.

REMARK Many global actions satisfy the additional property that $|X| = \cup_{\alpha\in\Phi} X_\alpha$ or even the property that $|X| = X_\alpha$ for some $\alpha \in \Phi$. If a global action doesn't have this property one can introduce it by enlarging Φ with an element $*$ such that $* < \alpha$ for all $\alpha \in \Phi$ and then setting $X_* = |X|$ and $G_* = \{1\}$. However, this will change subtly the structure of the global action, as we shall see for example in the definition of an $\infty-$ exponential action in (3.18) and in Theorem 3.23 and its proof.

It is allowed that $X_\alpha = \phi$. This will be convenient when making certain constructions, since it is not necessary to check whether or not X_α is empty.

The examples below illustrate the concept of a global action.

EXAMPLE 2.2 Let G be a group acting on a set $|X|$. Let Φ be a set which indexes a set $\{G_\alpha | \alpha \in \Phi\}$ of subgroups of G. Assume that $G_\alpha = G_\beta \Leftrightarrow \alpha = \beta$. Partially order $\{G_\alpha | \alpha \in \Phi\}$ by inclusion and give Φ the induced partial ordering. Clearly the rule $\alpha \mapsto G_\alpha$ defines a functor $\Phi \to ((\text{groups}))$. Define the function $X : \Phi \to$ subsets $|X|$, $\alpha \mapsto X_\alpha$, by $X_\alpha = |X|$ for all $\alpha \in \Phi$. Then one obtains a global action (Φ, G, X) which is contravariant in X.

DEFINITION 2.3 Suppose that in (2.2), $G = |X|$ and the action of G on $|X|$ is by multiplication. Suppose that $G_* = \{1\}$ for some $* \in \Phi$, that $\{G_\alpha | \alpha \in \Phi\}$ is closed under arbitrary intersections, and that the following condition is satisfied: If G_α and G_β are contained in some subgroup G_γ then the subgroup $\langle G_\alpha, G_\beta \rangle$ of G generated by G_α and G_β is identical with a subgroup G_γ. Then (Φ, G, X) is called a **Volodin model**. (It turns out that the Volodin K-groups of rings or of rings with extra structure such as an involution are algebraic homotopy groups of certain Volodin models, cf. § 6. The intersection property of Volodin models is needed to show that the algebraic homotopy groups of a Volodin model agree with the ordinary homotopy groups used by Volodin of a related topological space.)

If U is a set, let

$$\begin{aligned} \mathrm{Perm}(U) =& \mathrm{Aut}_{\mathrm{sets}}(U) \\ f\mathrm{Perm}(U) =& \{\sigma \in \mathrm{Perm}(U) | \sigma \text{ fixes all but a finite number of elements of } U\}. \end{aligned}$$

If U is a well ordered nonempty finite set, let

$$\begin{aligned} \mathrm{cPerm(U)} = \; & \text{cyclic subgoups of } \mathrm{Perm}(U) \text{ generated by the cyclic} \\ & \text{permutation which sends each element of } U, \\ & \text{except for the last, to its successor and} \\ & \text{sends the last element to the first.} \end{aligned}$$

EXAMPLE 2.4 This example is called the **line action** and is important for the homotopy theory of global actions. Let $\Phi = \mathbb{Z} \cup \{*\}$. Give Φ the partial ordering such that there is no relation between elements of $\mathbb{Z}$ and such that $* \leqq$ for all $n \in \mathbb{Z}$. Let $|X| = \mathbb{Z}$ and define $X : \Phi \to$ subsets $|X|, \alpha = n \mapsto \{n, n+1\}$ and $\alpha = * \mapsto |X|$. Define $G : \Phi \mapsto$ ((groups)),$\alpha = n \mapsto G_\alpha = \text{Perm}(\{n, n+1\})$ and $\alpha = * \mapsto G_\alpha = \{1\}$. Then the triple (Φ, G, X) is a contravariant action.

The next example generalizes the one above.

EXAMPLE 2.5 Let S denote an abstract simplicial complex and let $|X|$ denote the set of vertices of S. If α is a subcomplex of S, let X_α denote the set of its vertices. Call a subcomplex α simple, if X_α has a partition into subsets U such that any finite subset of U is a simplex in α and such that any simplex of α is a subset of some U. Clearly if α is simple then the partition above of (X_α) is unique; let $\text{Part}(X_\alpha)$ denote this partition. Let Φ denote the set of all simple subcomplexes of S. Partially order Φ by defining $\alpha \leqq \beta \Leftrightarrow X_\alpha \supseteqq X_\beta$ and every member of $\text{Part}(X_\beta)$ is a union of members of the $Part(X_\alpha)$. Clearly the subcomplex whose vertices are $|X|$ and whose simplices are the singleton subsets of $|X|$ is the smallest element of Φ. For $\alpha \in \Phi$, define

$$\begin{aligned} G_\alpha &= \prod_{U \in \text{Part}(X_\alpha)} \text{Perm}(U) \\ fG_\alpha &= \prod_{U \in \text{Part}(X_\alpha)} f\text{Perm}(U) \end{aligned}$$

There is a canonical action of G_α (resp. fG_α) on X_α defined by the action of each permutation group Perm(U) (resp. fPerm (U)) on U. Define

$$\begin{aligned} gl(S) &= (\Phi, G, X) \\ fgl(S) &= (\Phi, fG, X). \end{aligned}$$

Then $gl(S)$ and $fgl(S)$ are global actions called **simplicial actions**.

Well order now the vertices $|X|$ of S and let $c\Phi$ denote the subset of Φ of all simple subcomplexes α such that $Part(X_\alpha)$ contains only finite sets. The smallest element of Φ, say $*$, clearly lies in $c\Phi$. Give $c\Phi$ a new partial ordering such that $\alpha \leqq \beta \Leftrightarrow \alpha = *$. Thus if $\alpha \neq * \neq \beta$ then either $\alpha = \beta$ or there is no relation between α and β. For $\alpha \in c\Phi$, define $(cX)_\alpha = X_\alpha$ and

$$cG_\alpha = \prod_{U \in Part(X_\alpha)} cPerm(U).$$

Define

$$cgl(S) = (c\Phi, cG, cX).$$

Then $cgl(S)$ is a global action called a **cyclic simplicial action**.

To prepare for further examples, a few concepts from category theory are recalled.

Let $\mathcal{C}$ be a category. Let O be an object in $\mathcal{C}$. Let $O' \rightarrowtail O$ be a subobject of O. If $\sigma \in Aut_{\mathcal{C}}(O)$ then one says that σ **leaves** O' **invariant** or **stabilizes** O', if there is a $\rho \in Aut_{\mathcal{C}}(O')$ such that the diagram

$$\begin{tikzcd} O' \arrow[r, tail] \arrow[d, "\rho"'] & O \arrow[d, "\sigma"] \\ O' \arrow[r, tail] & O \end{tikzcd}$$

commutes. Clearly if ρ exists, it is unique. The set of all automorphisms of O which stabilize O' form a subgroup

$$Stab_O(O')$$

of $Aut_{\mathcal{C}}(O)$ called the **stabilizer** of O' in O. There is a canonical group homomorphism

$$\begin{array}{ccc} Stab_O(O') & \to & Aut_{\mathcal{C}}(O') \\ \sigma & \mapsto & \rho \end{array} .$$

Let P be an object of $\mathcal{C}$. A **P-point** of $\mathcal{C}$ is an element of $Mor_{\mathcal{C}}(P, O)$ where O is any object of $\mathcal{C}$. For a fixed P, the concept of P-point allows one to associate to an arbitrary object O of $\mathcal{C}$, an **underlying set**, namely the set $Mor_{\mathcal{C}}(P, O)$ of all **P-points in** O.

Moreover given any set $\mathcal{P}$ of objects of $\mathcal{C}$ (for the current purposes, it can be assumed that no two distinct objects in $\mathcal{P}$ are isomorphic), it makes sense to define the **underlying set of $\mathcal{P}$-points of** O as the set $\bigcup_{P\in\mathcal{P}} Mor_{\mathcal{C}}(P, O)$. If $O' \rightarrowtail O$ is a subobject then there is a canonical injection

$$\mathcal{P}\text{-points } (O') \subseteq \mathcal{P}\text{-points } (O)$$

of sets which will be frequently used to identify the former set with a subset of the latter.

Next we generalize the concept of global action to arbitrary categories by the concept of global object. Then we define the notion of representing a global object by a global action. After that a canonical method of constructing global objects from simple data is developed. All of this provides a wealth of examples of global objects and global actions and paves the way for applying global action methods to many different kinds of problems.

Definition 2.6 Let $\mathcal{C}$ be a category. A **global object** in $\mathcal{C}$ consists of a set $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ of subobjects $O_\alpha \rightarrowtail O$ of an object O of $\mathcal{C}$ and a set $\{G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha) | \alpha \in \Phi\}$ of groups G_α and group homomorphisms $G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha)$ satisfying the following conditions.

(2.6.1) Φ is equipped with a transitive, reflexive relation $\leqq$. If the relation $\leqq$ is a partial ordering then the global object is called a **partially ordered object**.

(2.6.2) The function $O : \Phi \longrightarrow \text{subobjects}(O), \alpha \mapsto (O_\alpha \rightarrowtail O)$, has in general no special properties.

If the relation $\leqq$ is a partial ordering, if the class subobjects (O) is given the canonical transitive, reflexive relation defined by inclusion among subobjects, and if the function O above is relation reversing, i.e. a contravariant functor, then the global object is called **contravariant**.

(2.6.3) G is a covariant functor $\Phi \longrightarrow ((groups)), \alpha \mapsto G_\alpha$.

(2.6.4) **Compatibility condition**. If $\alpha \leqq \beta$ then the pullback diagram

$$\begin{tikzcd}
O_\alpha \cap O_\beta \arrow[r, tail, "\varphi_\alpha"] \arrow[d, tail, "\varphi_\beta"'] & O_\alpha \arrow[d, tail] \\
O_\beta \arrow[r, tail] & O
\end{tikzcd}$$

exists in $\mathcal{C}$ and there is a (necessarily unique) group homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\alpha \cap O_\beta)$ such that φ_α and φ_β are G_α -equivariant.

Clearly the concept global action is identical with that of global set.

Definition 2.7 Let (Φ, G, O) be a global object in the category $\mathcal{C}$. Let P be a set of objects in $\mathcal{C}$. A P-**representation** of (Φ, G, O) is a set $\{X_\alpha | \alpha \in \Phi, X_\alpha \subseteqq \text{P-points}(O_\alpha), X_\alpha \text{ is } G_\alpha\text{-invariant}\}$. **The** P-representation of (Φ, G, O) is the set $\{\text{P-points}O_\alpha | \alpha \in \Phi\}$. It is easy to check that (Φ, G, X) is a global action. The P-representation of a contravariant object is a contravariant action, but the same is not true of an arbitrary P-representation.

Example 2.8 Let $\mathcal{C}$ denote the category of abstract simplicial complexes. Let S be an object of $\mathcal{C}$. Let Φ be as in (2.5) and for each $\alpha \in \Phi$, set $S_\alpha = \alpha$. Thus S_α is a simple subcomplex of S. Let G_α and fG_α be defined as in (2.5). Thus G_α and fG_α are subgroups of $Aut_{\mathcal{C}}(S_\alpha)$. One checks routinely that (Φ, G, S) and (Φ, fG, S) are global simplicial complexes. Moreover if P denotes the simplicial complex with precisely one vertex then the P-representation of (Φ, G, S) $(resp.(\Phi, fG, S))$ is the global action $\mathrm{gl}(S)$ $(resp.fgl(S))$ defined in (2.5).

Let $c\Phi$ be as in (2.5) and for each $\alpha \in c\Phi$, set $(cS)_\alpha = \alpha$ and let $(cG)_\alpha$ be as in (2.5). Then $(c\Phi, cG, cS)$ is a global simplicial complex and the P-representation of $(c\Phi, cG, cS)$ is the global action $cgl(S)$ defined in (2.5).

The following method of constructing global objects from data generalizes Example 2.2, even in the case of sets, and is very useful.

Construction-Lemma 2.9 Let $\mathcal{C}$ be a category and O an object in $\mathcal{C}$.

(2.9.1) **Global data** for O consists of a set $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ of subobjects $O_\alpha \rightarrowtail O$ of O and a set $\{G_\alpha \subseteqq Aut_{\mathcal{C}}(O_\alpha) | \alpha \in \Phi\}$ of subgroups $G_\alpha \subseteqq Aut_{\mathcal{C}}(O_\alpha)$.

(2.9.2) Given global data, define a transitive reflexive relation $\leqq_{cc}$ on Φ, called the **canonical contravariant relation** , as follows: $\alpha \leqq_{cc} \beta \Leftrightarrow$ there is a commutative diagram

$$\begin{tikzcd}
O_\alpha \arrow[r, tail] & O \\
O_\beta \arrow[u, tail] \arrow[ur, tail] &
\end{tikzcd}$$

such that $G_\alpha \subseteqq Stab_{O_\alpha}(O_\beta)$ and the canonical homomorphism $G_\alpha \longrightarrow Aut_{\mathcal{C}}(O_\beta)$ has its image in G_β. One checks straightforward that the triple (Φ, G, O) is a global object such that O is contravariant. Moreover, if $\leqq$ is any transitive reflexive relation on Φ such that

$((\Phi, \leqq), G, O)$ is a global object with O contravariant then the identity map $\Phi \to \Phi$ is a morphism $(\Phi, \leqq) \longrightarrow (\Phi, \leqq_{cc})$ of partially ordered sets.

(2.9.3) Given global data, define a reflexive relation $\leqq_{cr}$ on Φ, called the **canonical relation** , as follows: $\alpha \leqq_{cr} \beta \Longleftrightarrow$ there is a pullback diagram

$$\begin{tikzcd} O_\beta \cap O_\alpha \arrow[r, tail] \arrow[d, tail] & O_\alpha \arrow[d, tail] \\ O_\beta \arrow[r, tail] & O \end{tikzcd}$$

in $\mathcal{C}$ such that G_α and $G_\beta \subseteqq Stab_{O_\alpha}(O_\beta \cap O_\alpha)$, the canonical homomorphism $G_\beta \longrightarrow Aut_{\mathcal{C}}(O_\beta \cap O_\alpha)$ is injective, and the image $(G_\alpha \longrightarrow Aut_{\mathcal{C}}(O_\beta \cap O_\alpha))$ is contained in the image of the previous homomorphism. It follows that if $\alpha \leqq_{cr} \beta$ then there is a unique homomorphism $G_\alpha \longrightarrow G_\beta$ such that the morphism $O_\beta \cap O_\alpha \longrightarrow O_\beta$ is G_α-equivariant. Let $\leqq$ be a transitive reflexive subrelation of $\leqq_{cr}$, for example the relation $\leqq_{cc}$ defined above. Let Φ have the relation $\leqq$. One checks straightforward that the triple (Φ, G, O) is a global object. Moreover, if $\leqq$ is any transitive reflexive relation on Φ such that $((\Phi, \leqq), G, O)$ is a global object then the identity map $\Phi \to \Phi$ is a morphism $(\Phi, \leqq) \longrightarrow (\Phi, \leqq_{cr})$ of relations. In particular $(\Phi, \leqq_{cc}) \longrightarrow (\Phi, \leqq_{cr})$ is a morphism of relations.

PROOF The only assertions that were left to prove are those concerning the relation $\leqq_{cc}$ in (2.9.2) and $\leqq_{cr}$ in (2.9.3). The proofs are similar and we carry out only that for $\leqq_{cc}$. Suppose $\alpha \leqq \beta$. By contravariantness, there is a commutative diagram

$$\begin{tikzcd} O_\alpha \arrow[r, tail] & O \\ O_\beta \arrow[u, tail] \arrow[ur, tail] & \end{tikzcd} .$$

¿From the compatibility condition (2.6.4), it follows that the morphism $O_\beta \rightarrowtail O_\alpha$ is G_α-equivariant. But this says that G_α leaves O_β invariant. Furthermore it is clear that the homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\alpha)$ must have its image in G_β, because there is exactly one homomorphism $G_\alpha \to Aut_{\mathcal{C}}(O_\alpha)$ which makes $O_\beta \rightarrowtail O_\alpha$ $G_\alpha-$equivariant and the functorially given homomorphism $G_\alpha \longrightarrow G_\beta$ makes $O_\beta \rightarrowtail O_\alpha$ $G_\alpha-$equivariant. Thus $\alpha \leqq_{cc} \beta$, by definition (2.9.3). $\square$

Let Ψ be an index set. Let O be an object in a category $\mathcal{C}$ and let $O : \Psi \to$ subobjects(O), $\alpha \mapsto (O_\alpha \rightarrowtail O)$, be a function. Let

$$Sub_\Psi(O)$$

denote the category whose objects are $\{O_\alpha | \alpha \in \Psi\}$ and whose morphisms are the unique morphisms $O_\alpha \rightarrowtail O_\beta$ such that the diagram

$$\begin{tikzcd} O_\alpha \arrow[rr, tail] \arrow[dr, tail] & & O_\beta \arrow[dl, tail] \\ & O & \end{tikzcd}$$

commutes. If $S \subseteqq Sub_\Psi(O)$ is a subcategory and if $colim S$ exists in $\mathcal{C}$ then there is a canonical morphism $colim S \to O$.

The following method of constructing global objects generalizes Example 2.8.

CONSTRUCTION-LEMMA 2.10 Let $\mathcal{C}$ be a category and O an object in $\mathcal{C}$.

(2.10.1) Let $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ be a set of subobjects $O_\alpha \rightarrowtail O$ of O. For each $\alpha \in \Phi$, let $\{O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha | (\alpha, \mathrm{i}) \in \Phi_\alpha\}$ be a set of subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha$ of O_α such that there is a subcategory $S_\alpha \subseteqq Sub_{\Phi_\alpha}(O_\alpha)$ with the property that the colim (S_α) exists in $\mathcal{C}$ and the canonical morphism colim $(S_\alpha) \to O_\alpha$ is an isomorphism. Let $G_\alpha = \{\sigma \in \mathrm{Aut}_{\mathcal{C}}(O_\alpha) | \sigma \in \mathrm{Stab}_{O_\alpha}(O_{\alpha,\mathrm{i}}) \forall (\alpha, \mathrm{i}) \in \Phi_\alpha\}$. The sets $\{O_\alpha \rightarrowtail O | \alpha \in \Phi\}$ and $\{G_\alpha \subseteqq \mathrm{Aut}_{\mathcal{C}}(O_\alpha) | \alpha \in \Phi\}$ define global data in the sense of (2.9.1).

(2.10.2) Given the data above, define a transitive reflexive relation $\leqq_{cc}$ on Φ as follows: $\alpha \leqq_{cc} \beta \Leftrightarrow$ there is a commutative diagram

$$\begin{tikzcd} O_\alpha \arrow[r, tail] & O \\ O_\beta \arrow[u, tail] \arrow[ur, tail] & \end{tikzcd}$$

such that for each $(\beta, j) \in \Phi_\beta$, the object $O_{\beta,j}$ is a colimit of not necessarily all subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_{\beta,j}$ for which there is a commutative diagram

$$\begin{tikzcd} O_{\alpha,i} \arrow[r, tail] \arrow[d, tail] & O_\alpha \arrow[d, tail] \\ O_{\beta,j} \arrow[r, tail] & O_\beta \end{tikzcd} .$$

This implies $G_\alpha \subseteqq \mathrm{Stab}_{O_\alpha}(O_\beta)$ and the canonical homomorphism $G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\beta)$ takes its image in G_β. Applying (2.9.2), one obtains that the triple (Φ, G, O) is a global object such that O is contravariant.

(2.10.3) Given the data above, define a reflexive relation $\leqq_{cr}$ on Φ as follows: $\alpha \leqq_{cr} \beta \Leftrightarrow$ there is a pullback diagram

$$\begin{tikzcd} O_\alpha \cap O_\beta \arrow[r, tail] \arrow[d, tail] & O_\alpha \arrow[d, tail] \\ O_\beta \arrow[r, tail] & O \end{tikzcd}$$

in $\mathcal{C}$ such that $O_\alpha \cap O_\beta$ is a colimit of not necessarily all subobjects $O_{\alpha,\mathrm{i}} \rightarrowtail O_\alpha \cap O_\beta$(resp.$O_{\beta,j} \rightarrowtail O_\alpha \cap O_\beta$) for which there is a commutative diagram

$$\begin{tikzcd} O_{\alpha,i} \arrow[d, tail] \arrow[dr, tail] & \\ O_\alpha \cap O_\beta \arrow[r, tail] & O_\alpha \end{tikzcd}$$

(resp.

$$\begin{tikzcd} O_{\beta,j} \arrow[d, tail] \arrow[dr, tail] & \\ O_\alpha \cap O_\beta \arrow[r, tail] & O_\beta \end{tikzcd} \text{),}$$

the canonical homomorphism $G_\beta \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha \cap O_\beta)$ is injective, and the image $(G_\alpha \to \mathrm{Aut}_{\mathcal{C}}(O_\alpha \cap O_\beta))$ is contained in the image of the previous homomorphism. Let $\leqq$ be a transitive reflexive subrelation of $\leqq_{cr}$, for example the relation $\leqq_{cc}$ defined above. Let Φ have the relation $\leqq$. Applying (2.9.3), one obtains that the triple (Φ, G, O) is a global object. Moreover, the identity map $\Phi \to \Phi$ defines a morphism $(\Phi, \leqq) \to (\Phi, \leqq_{cr})$ of relations.

3 Morphisms and morphism spaces

There is a general notion of morphism for global actions and two important special kinds of morphisms, namely normal morphisms and regular morphisms. The set $Mor(A, B)$ of all morphisms from a global action A to a global action B will be given the structure of a global action such that $Mor(,)$ defines a contravariant functor with values in global actions with respect to the first variable over all morphisms and a contravariant functor with values in global actions with respect to the second variable over all normal morphisms.

Regular morphisms provide the strongest notion of morphism and preserve all the essential structural concepts in the definition of a global action. Two global actions which are regularly isomorphic are essentially the same. On the other hand, two global actions which are only isomorphic can behave very differently, since their structures are not necessarily in $1-1$ correspondence. For example, they can have different higher algebraic homotopy groups because the construction of such groups is functorial only over a certain class of morphisms containing the regular morphisms. This class is called the ∞ -L-normal morphisms and will also be defined below.

The notion of morphism depends on the concepts of path, local path, and local frame. The concepts local path and local frame are really the same, but the notion local frame suggests possible directions for movement rather than a definite direction of movement and this will be helpful in developing the notion of normal morphism.

If A is a global action, let

$$\begin{aligned}\Phi_A &= \text{coordinate system of } A\\ G_A &= \text{global group functor of } A\\ X_A &= \text{global set function of } A\\ |A| &= \text{enveloping set of } A.\end{aligned}$$

DEFINITION 3.1 Let A be a global action.

(3.1.1) A **path** in A is a sequence $\mathrm{x}_o, \cdots, \mathrm{x}_p$ of points in $|A|$ such that for each i $(0 \leqq i \leqq p-1)$, there is a group element g_i defined at x_i with the property that $g_i\mathrm{x}_i = \mathrm{x}_{i+1}$. If $0 = p$, it is assumed that x_o lies in some local set $(X_A)_\alpha$.

(3.1.2) A **local path** at $\alpha \in \Phi_A$ is a path $\mathrm{x}_o, \cdots, \mathrm{x}_p$ in A such that each $\mathrm{x}_i \in (\mathrm{X}_A)_\alpha$ and each $g_{\mathrm{i}} \in (G_A)_\alpha$. (Clearly if $\mathrm{x}_o, \cdots, \mathrm{x}_p$ is a local path then so is $\mathrm{x}_{\pi(\mathrm{x}_o)}, \cdots, \mathrm{x}_{\pi(p)}$ where π is any permutation of $(p+1)$ letters.)

(3.1.3) Let $\mathrm{x} \in (X_A)_\alpha$. A **local frame** at x in α or simply an α-**frame** at x is a sequence $\mathrm{x} = \mathrm{x}_o, \cdots, \mathrm{x}_p$ of points in $(X_A)_\alpha$ such that for each i $(1 \leqq i \leqq p)$ there is a $g_i \in (G_A)_\alpha$ such that $g_i\mathrm{x} = \mathrm{x}_i$. (Clearly $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is an α-frame at x $\Leftrightarrow \mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is a local path at α.)

DEFINITION 3.2 A **morphism** f $:A \to B$ of global actions is a function f $:|A| \to |B|$ which preserves local frames or equivalently local paths. Specifically if $\mathrm{x}_o, \cdots, \mathrm{x}_p$ is an α-frame at x_o then $f(\mathrm{x}_o), \cdots, f(\mathrm{x}_p)$ is an β-frame at $f(\mathrm{x}_o)$ for some $\beta \in \Phi_B$.

DEFINITION 3.3 A **regular** morphism $\eta : A \to B$ of global actions is a triple $(\eta_\Phi, \eta_G, \eta_X)$ satisfying the following conditions.

(3.3.1) $\eta_\Phi : \Phi_A \to \Phi_B$ is a relation preserving function, i.e. a covariant functor.

(3.3.2) $\eta_G : G_A \to (G_B)_{\eta_\Phi(\,)}$ is a natural transformation of group valued functors on Φ_A where $(G_B)_{\eta_\Phi(\,)}$ denotes the composition of η_Φ with G_B.

(3.3.3) $\eta_X : |A| \to |B|$ is a function such that $\eta_X((X_A)_\alpha) \subseteqq (X_B)_{\eta_\Phi(\alpha)}$ for all $\alpha \in \Phi_A$.

(3.3.4) For each $\alpha \in \Phi_A$, the pair $(\eta_G, \eta_X) : (G_A)_\alpha \curvearrowright (X_A)_\alpha \to (G_B)_{\eta_\Phi(\alpha)} \curvearrowright (X_B)_{\eta_\Phi(\alpha)}$ is a morphism of group actions, i.e. for $\sigma \in (G_A)_\alpha$ and $\mathrm{x} \in (X_A)_\alpha, \eta_X(\alpha)(g\mathrm{x}) = \eta_G(\alpha)(\sigma)(\eta_X(\alpha)(\mathrm{x}))$. (This implies that a regular morphism is one in the usual sense).

A **regular isomorphism** $\eta : A \to B$ is a regular morphism such that there is a regular morphism $\eta' : B \to A$ called the **regular inverse** of η with the property that η'_Φ is inverse to η_Φ, η'_X is inverse to η_X, and for each $\alpha \in \Phi_A$ $\eta'_G(\eta_\Phi(\alpha))$ is inverse to $\eta_G(\alpha)$.

It is of course not true in general that a regular morphism which is an isomorphism is a regular isomorphism.

The notion of chart, to be introduced next, will be used to put a global action structure on the set $Mor(A, B)$ of all morphisms from a global action A to be a global action B.

DEFINITION 3.4 Let A and B be global actions. An A-**chart** in B is a morphism $f : A \to B$ of global actions and a function $\beta : |A| \to \Phi_B$ such that the following conditions are satisfied.

(3.4.1) $\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$ for all $\mathrm{x} \in |A|$.

(3.4.2) If $\mathrm{x}, \mathrm{x}_1, \cdots, \mathrm{x}_p$ is an a-frame at $\mathrm{x} \in |A|$ then $\mathrm{f}(\mathrm{x}), \mathrm{f}(\mathrm{x}_1), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\mathrm{f}(\mathrm{x})$ for some $b \geq \beta(\mathrm{x}), \beta(\mathrm{x}_1), \cdots, \beta(\mathrm{x}_p)$.

DEFINITION-LEMMA 3.5 Let (f, β) be an A-chart in B.

If

$$\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}$$

define

$$\begin{aligned} \sigma\mathrm{f} : |A| &\to |B|. \\ \mathrm{x} &\mapsto \sigma_{\mathrm{x}}\mathrm{f}(\mathrm{x}) \end{aligned}$$

Then $\sigma\mathrm{f}$ is a morphism $A \to B$ of global actions and $(\sigma\mathrm{f}, \beta)$ is an A-chart in B.

PROOF Since $\sigma_{\mathrm{x}} \in (G_B)_{\beta(\mathrm{x})}$, it follows that $\sigma\mathrm{f}(\mathrm{x}) \in (X_B)_{\beta(\mathrm{x})}$. Thus the pair $(\sigma\mathrm{f}, \beta)$ satisfies (3.4.1). To show that $\sigma\mathrm{f}$ is a morphism of global actions and that $(\sigma\mathrm{f}, \beta)$ is an A-frame in B, it suffices to show that (3.4.2) is satisfied. Let $\mathrm{x}_o, \cdots, \mathrm{x}_p$ be a local frame at $\mathrm{x}_o \in |A|$. By definition $\mathrm{f}(\mathrm{x}_o), \cdots, \mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\mathrm{f}(\mathrm{x}_o)$ for some $b \geq \beta(\mathrm{x}_o), \cdots, \beta(\mathrm{x}_p)$. Let $\rho_{\mathrm{x}_o}, \cdots, \rho_{\mathrm{x}_p}$ denote respectively the images of $\sigma_{\mathrm{x}_o}, \cdots, \sigma_{\mathrm{x}_p}$ in $(G_B)_{\mathrm{b}}$ under the canonical homomorphisms $(G_B)_{\beta(\mathrm{x}_i)} \to (G_B)_{\mathrm{b}}$ $(0 \leqq \mathrm{i} \leqq \mathrm{p})$. Clearly $\rho_{\mathrm{x}_o}\mathrm{f}(\mathrm{x}_o), \cdots, \rho_{\mathrm{x}_p}\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\rho_{\mathrm{x}_o}\mathrm{f}(\mathrm{x}_o)$. But $\rho_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i) = \sigma_{\mathrm{x}_i}\mathrm{f}(\mathrm{x}_i)$ by (2.1.4). Thus $\sigma\mathrm{f}(\mathrm{x}_o), \cdots, \sigma\mathrm{f}(\mathrm{x}_p)$ is a b-frame at $\sigma\mathrm{f}(\mathrm{x}_o)$ and $b \geqq \beta(\mathrm{x}_o), \cdots, \beta(\mathrm{x}_p)$. □

DEFINITION 3.6 Let (f, β) be an A-chart in B. An A-**frame at** f **on** (f, β) is a set $\mathrm{f} = \mathrm{f}_o, \mathrm{f}_1, \cdots, \mathrm{f}_p : A \to B$ of morphisms for which there are elements $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A|}$

$(G_B)_{\beta(\mathrm{x})}$ such that $\sigma_i\mathrm{f} = \mathrm{f}_i$ $(1 \leqq i \leqq \mathrm{p})$. (In view of Lemma (3.5), $\mathrm{f} = \mathrm{f}_o, f_1, \cdots, \mathrm{f}_\mathrm{p}$ is also an A-frame at f_i on (f_i, β) for any i $(0 \leqq \mathrm{i} \leqq \mathrm{p})$.)

The next lemma will be very useful.

LOCAL-GLOBAL LEMMA 3.7 Let (f, β) be an A-chart in B. Then $\mathrm{f} = \mathrm{f}_o, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is an A-frame at f on $(\mathrm{f}, \beta) \Leftrightarrow$ for each $\mathrm{x} \in |A|, \mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local frame at $\mathrm{f}(\mathrm{x})$ in $\beta(\mathrm{x})$.

PROOF The assertions are trivial consequences of Lemma (3.5).

DEFINITION 3.8 An A-**normal** morphism $\mathrm{g} : B \to C$ of global actions is one which preserves A-frames, i.e. if $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ is an A-frame at f on (f, β) then $\mathrm{gf}, \mathrm{gf}_1, \cdots, \mathrm{gf}_\mathrm{p}$ is an A-frame at gf on (gf, γ) for some A-chart (gf, γ) in C. A **normal** morphism $\mathrm{g} : B \to C$ is one which preserves A-frames for any global action A. An A-**normal** (resp. **normal**) **isomorphism** is an A-normal (resp. normal) morphism which has an A-normal (resp. normal) inverse.

It is not true in general that an A-normal (resp. normal) morphism which is an isomorphism in the usual sense is an A-normal (resp. normal) isomorphism.

LEMMA 3.9 A regular morphism is normal.

PROOF Let $\eta : B \to C$ be a regular morphism. If (f, β) is an A-chart in B then it follows straightforward that $(\eta_X\mathrm{f}, \eta_\Phi\beta)$ is an A-chart in C. Let $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at f on (f, β) and let $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}$ such that $\sigma_\mathrm{i}\mathrm{f} = \mathrm{f}_\mathrm{i}$ $(1 \leqq \mathrm{i} \leqq \mathrm{p})$. If $\sigma = (\sigma_\mathrm{x}) \in \prod_{\mathrm{x}\in|A|} (G_B)_{\beta(\mathrm{x})}$, define $\eta_G(\sigma) = (\eta_G(\beta(\mathrm{x}))(\sigma_\mathrm{x})) \in \prod_{\mathrm{x}\in|A|} (G_C)_{\eta_\Phi(\beta(\mathrm{x}))}$. Then $\eta_G(\sigma_i)(\eta_X\mathrm{f}) = \eta_X\mathrm{f}_\mathrm{i}$ $(1 \leqq i \leqq \mathrm{p})$, by (3.3.4). Thus $\eta_X\mathrm{f}, \eta_X\mathrm{f}_1, \cdots, \eta_X\mathrm{f}_p$ is an A-frame at $\eta_X\mathrm{f}$ on $(\eta_X\mathrm{f}, \eta_\Phi\beta)$. $\square$

Next the set $Mor(A, B)$ of all morphisms from a global action A to a global action B is given the structure of a global action.

DEFINITION 3.10 Let A and B be global actions. Let $|Mor(A, B)|$ denote the set of all morphisms from A to B. Define a global action

$$(\Phi_{(A,B)}, G_{(A,B)}, X_{(A,B)})$$

whose enveloping set is $|Mor(A, B)|$ as follows. This global action will be denoted by $Mor(A, B)$. Define

$$\Phi_{(A,B)} = \{\beta : |A| \to \Phi_B\}.$$

Give $\Phi_{(A,B)}$ the transitive reflexive relation defined by $\beta \leqq \beta' \Leftrightarrow \beta(\mathrm{x}) \leqq \beta'(\mathrm{x})\ \forall \mathrm{x} \in |A|$. For $\beta \in \Phi_{(A,B)}$, define

$$(G_{(A,B)})_\beta = \prod_{\mathrm{x} \in |A|} (G_B)_{\beta(\mathrm{x})}.$$

If $\beta \leqq \beta'$, there is for each $\mathrm{x} \in |A|$ a functorially defined homomorphism $(G_B)_{\beta(\mathrm{x})} \rightarrow (G_B)_{\beta'(\mathrm{x})}$ and therefore a homomorphism $(G_{(A,B)})_\beta \longrightarrow (G_{(A,B)})_{\beta'}$ which is obviously functiorial in β. For $\beta \in \Phi_{(A,B)}$, define

$$(X_{(A,B)})_\beta = \{\mathrm{f} : |A| \rightarrow |B| | (\mathrm{f}, \beta)\ A - \text{chart in } B\}.$$

By (3.5), if $\sigma \in (G_{(A,B)})_\beta$ and $\mathrm{f} \in (X_{(A,B)})_\beta$ then $\sigma \mathrm{f} \in (X_{(A,B)})_\beta$ and so there is an action of $(G_{(A,B)})_\beta$ on $(X_{(A,B)})_\beta$. All the conditions for a global action are obviously satisfied except possibly the compatibility condition (2.1.4) which can be easily checked.

Proposition 3.11 As a functor taking values in global actions, $Mor(,)$ is contravariant and regular over all morphisms in the first variable and covariant over all normal morphisms in the second variable. More precisely the following holds.

(3.11.1) Let C be a global action and let $\mathrm{f} : A \rightarrow B$ be a morphism of global actions. Then f defines a regular morphism

$$\eta = Mor(\mathrm{f}, 1_C) : Mor(B, C) \rightarrow Mor(A, C)$$

as follows. Define the relation preserving morphism

$$\begin{aligned} \eta_\Phi : \Phi_{(B,C)} &\rightarrow \Phi_{(A,C)}. \\ \beta &\mapsto \beta \mathrm{f} \end{aligned}$$

Define the natural transformation

$$\eta_G : G_{(B,C)} \rightarrow G_{(A,C)}$$

by

$$\begin{array}{ccc}
\eta_G(\beta) : (G_{(B,C)})_\beta & \longrightarrow & (G_{(A,C)})_{\eta_\Phi(\beta)} \\
\| & & \| \\
\prod_{y\in|B|} (G_C)_{\beta(y)} & & \prod_{\mathrm{x}\in|A|} (G_C)_{\beta\mathrm{f}(\mathrm{x})}
\end{array}$$

where

$$\eta_G(\beta)|_{(G_C)_{\beta(y)}} \text{ is the diagonal homomorphism}$$
$$(G_C)_{\beta(y)} \to \prod_{\mathrm{x}\in|A|,\ f(\mathrm{x})=y} (G_C)_{\beta\mathrm{f}(\mathrm{x})},$$

under the convention that the empty product of groups, which can occur on the right hand side of the arrow above, is the trivial group. Define

$$\begin{aligned}\eta_X : |Mor(B,C)| &\to |Mor(A,C)|.\\ \mathrm{g} &\mapsto \mathrm{gf}\end{aligned}$$

Then $\eta = (\eta_\Phi, \eta_G, \eta_X)$ is a morphism of global actions.

(3.11.2) Let A be a global action and let $\mathrm{g} : B \to C$ be a morphism of global actions. Then the function

$$Mor(1_A, \mathrm{g}) : |Mor(A,B)| \to |Mor(A,C)|$$

is a morphism $Mor(A,B) \to Mor(A,C)$ of global actions $\Leftrightarrow$ g is A-normal.

PROOF (3.11.1) Straightforward and routine. Details are left to the reader.

(3.11.2) Let (f,β) be an A-chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ be an A-frame on (f,β). By definition of the term local frame, $\mathrm{f}_0, \cdots, \mathrm{f}_\mathrm{p}$ is also a local β-frame in the global action $Mor(A,B)$ and conversely, any local frame in $Mor(A,B)$ is an A-frame on some A-chart in B. Thus the function $Mor(1_A, \mathrm{g}) : |Mor(A,B)| \to |Mor(A,C)|$ is a morphism of global actions $\Leftrightarrow$ it preserves A-frames $\Leftrightarrow$ g is A-normal.□

REMARK If B is a global action then letting $\Phi_{B'}$ denote a subcategory of Φ_B whose objects exhaust those of Φ_B, one obtains a global action $B' = (\Phi_{B'}, G_B, X_B)$ which at first glance looks very much like B, in fact the identity map $|B| \to |B|$ defines a regular morphism $B' \to B$ which is an isomorphism of global actions, but not in general a regular isomorphism. Consequences of the structural difference between B' and B can be observed by comparing the global action $Mor(A, B')$ with the global action $Mor(A, B)$, via the canonical morphism $Mor(A, B') \to Mor(A, B)$. The set of A-charts in B' is in general smaller than the set of A-charts in B, which has the consequence that the domain of a local group $(G_{(A,B')})_\beta$ is in general smaller than the domain of the corresponding group $(G_{(A,B)})_\beta$, i.e. $(X_{(A,B')})_\beta \subsetneqq (X_{(A,B)})_\beta$. Of course the corresponding comparison between the domain of the local group $(G_{B'})_b$ and that of $(G_B)_b$ is equality, i.e. $(X_{B'})_b = (X_B)_b$. It is worth noting that if B satisfies the condition that for each coordinate b, the canonical homomorphism $(G_B)_b \to \mathrm{Perm}((X_B)_b)$ is injective then the construction in (2.9.3) shows how to enlarge the set of morphisms in Φ_B to an absolute maximum for the data (see (2.9.1)) provided by B.

Definition 3.12 Let $g : B \to C$ be a morphism of global actions. A sequence $A_n, \cdots, A_1$ of global actions is called a **normal chain** of length n for g if g is A_1-normal and if for each i $(1 \leqq i \leqq n-1)$, the morphism $Mor(1_{A_{i-1}}, \cdots, Mor(1_{A_1}, g)) \cdots) : Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, B)) \cdots) \to Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, C)) \cdots)$ is A_{i+1}-normal. Let $\mathcal{N}$ be a class of global actions. The morphism g is called ***n*-$\mathcal{N}$-normal** if every sequence of n objects from $\mathcal{N}$ forms a normal chain for g. The morphism g is called **$\mathcal{N}$-normal** (resp. **∞-$\mathcal{N}$-normal**) if it is 1-$\mathcal{N}$-normal (resp. n-$\mathcal{N}$-normal for all $n > 0$). If $\mathcal{N} = \{A\}$ (resp. $\mathcal{N}$ = all global actions), we shall write **∞-*A*-normal** (resp. **∞-normal**) in place of ∞-$\mathcal{N}$-normal.

If the expression t-morphism denotes anyone of the notions of normality above or the notion of regularity then a t-isomorphism is a t-morphism which has a t-morphism as its inverse.

In order to associate to a morphism g : $A \to B$ of global actions a long exact sequence of algebraic homotopy groups, we shall need that g is ∞-L-normal where L is the line action defined in Example (2.4).

Lemma 3.13 If g : $B \to C$ is a regular morphism then for any global action A, the morphism $Mor(1_A, \mathrm{g}) : Mor(A, B) \to Mor(A, C)$ is regular. Thus g is ∞-normal.

PROOF By (3.9)and (3.11.2), the morphism $Mor(1_A, \mathrm{g}) : Mor(A, B) \to Mor(A, C)$ exists. Let $(\eta_\Phi, \eta_G, \eta_X = \mathrm{g})$ be the regular structure of g. We define a regular structure $(\mu_\Phi, \mu_g, \mu_X = Mor(1_A, \mathrm{g}))$ for $Mor(1_A, \mathrm{g})$ as follows.

Define the coordinate morphism

$$\mu_\Phi : \Phi_{(A,B)} \to \Phi_{(A,C)}.$$
$$\beta \mapsto \eta_\Phi \beta$$

Define the natural transformation

$$\mu_G : G_{(A,B)} \to G_{(A,C)}$$

by the commutative diagram

$$\begin{tikzcd}
(G_{(A,B)})_\beta \arrow[r, "\mu_G(\beta)"] \arrow[d, equal] & (G_{(A,C)})_{\mu_\Phi(\beta)} \arrow[d, equal] \\
\prod\limits_{\mathrm{x}\in|A|} (G_{(A,B)})_{\beta(\mathrm{x})} \arrow[r, "\prod\limits_{\mathrm{x}\in|A|} \eta_G(\beta(\mathrm{x}))"'] & \prod\limits_{\mathrm{x}\in|A|} (G_{(A,C)})_{\eta_\Phi(\beta(\mathrm{x}))}
\end{tikzcd}$$

One checks straightforward that $(\mu_\Phi, \mu_G, Mor(1_A, \mathrm{g}))$ is a regular morphism.

That g is ∞-normal follows by a trivial induction argument from the result just proved. □

DEFINITION 3.14 Let N denote the name of a kind of morphism defined in (3.12). A global action is called an **N action** if it has the property that every morphism to it is an N morphism. For example an ∞ **-normal action** has the property that every morphism to it is ∞-normal.

For the results below on the exponential law, the notion of product is needed. We construct this next.

DEFINITION-LEMMA 3.15 Let A and B be global actions. Their **product** $A \times B$ is constructed as follows.

$$\Phi_{A\times B} = \Phi_A \times \Phi_B$$

and $(\alpha, \beta) \leqq (\alpha', \beta') \Leftrightarrow \alpha \leqq \alpha' \text{and} \beta \leqq \beta'$.

$$
\begin{aligned}
G_{A\times B} &= G_A \times G_B \\
|A \times B| &= |A| \times |B| \\
X_{A\times B} &= X_A \times X_B.
\end{aligned}
$$

For any coordinate $(\alpha, \beta) \in \Phi_{A\times B}$, there is an obvious action of $(G_{A\times B})_{(\alpha,\beta)}$ on $(X_{A\times B})_{(\alpha,\beta)}$, namely the one defined coordinatewise. One checks easily that $A\times B$ satisfies the universal property of a product.

The following notation will be used below. If S and T are sets, let

$$(S, T) = Mor_{((sets))}(S, T).$$

If U is also a set then there is a canonical isomorphism

$$
\begin{aligned}
E : (U, (S, T)) &\xrightarrow{\cong} (U \times S, T) \\
f &\longmapsto Ef
\end{aligned}
\tag{3.16}
$$

of sets such that $Ef(u, s) = f(u)(s)$. Its inverse is obviously the function

$$
\begin{aligned}
E' : (U \times S, T) &\longrightarrow (U, (S, T)) \\
f &\longmapsto E'f
\end{aligned}
$$

where $(E'f(u))(s) = f(u, s)$.

DEFINITION 3.17 Let A, B and C be global actions. We define a regular morphism

$$E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$$

as follows. Denote the structural components of the global action $Mor(A, Mor(B, C))$ by $(\Phi_{(A,(B,C))}, G_{(A,(B,C))}, X_{(A,(B,C))})$. Define

$$
\begin{array}{ccc}
E_\Phi : \Phi_{(A,(B,C))} & \longrightarrow & \Phi_{(A\times B,C)} \\
\| & & \| \\
(|A|, (|B|, \Phi_C)) & & (|A| \times |B|, \Phi_C)
\end{array}
$$

to be the set theoretic exponential isomorphism (3.16). Clearly E_Φ preserves the transitive reflexive relation. Define the natural transformation

$$E_G : G_{(A,(B,C))} \to (G_{(A\times B,C)})_{E_\Phi(\ )}$$

such that

$$\begin{array}{ccc} E_G(\alpha) : (G_{(A,(B,C))})_\alpha & \longrightarrow & (G_{(A\times B,C)})_{E_\Phi(\alpha)} \\ \| & & \| \\ \prod\limits_{\mathrm{x}\in|A|}(\prod\limits_{y\in|B|}(G_C)_{\alpha(\mathrm{x})(y)}) & & \prod\limits_{(\mathrm{x},y)\in|A|\times|B|}(G_C)_{(E_\Phi\alpha)(\mathrm{x},y)} \end{array}$$

maps the factor $(G_C)_{\alpha(\mathrm{x})(y)}$ via the identity map onto the factor $(G_C)_{(E_\Phi\alpha)(\mathrm{x},y)} = (G_C)_{\alpha(\mathrm{x})(y)}$. One verifies easily that the composite mapping $|Mor(A, Mor(B,C))| \to (|A|,(|B|,|C|))$ @ > (3.16) >> $(|A| \times |B|, |C|)$ takes its image in $|Mor(A\times B, C)|$ and we define

$$E_X : |Mor(A, Mor(B,C))| \to |Mor(A\times B, C)|$$

to be the resulting mapping. One checks straightforward that

$$E = (E_\Phi, E_G, E_X)$$

is a regular morphism. (It fails in general to be an isomorphism (resp. regular isomorphism) because E_X is not necessarily surjective (resp. $E_X((X_{(A,(B,C))})_\alpha)$ is not necessarily all of $(X_{(A\times B,C)})_{E_\Phi(\alpha)}$).

Let $A_n, \cdots, A_1$ be an arbitrary sequence of global actions. Iterating the procedure above, one defines for any $n \geq 2$ a regular morphism

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C))\cdots) \to Mor(A_n \times \cdots \times A_1, C)$$

as follows. For $n = 2$, the morphism is defined above. Suppose $n > 2$ and that the morphism has been defined for every natural number N where $2 \leqq N \leqq n-1$. Let E_{n-1} denote the morphism for the sequence $A_{n-1}, \cdots, A_1$. Define E_n for the sequence $A_n, A_{n-1}, \cdots, A_1$ as the composite of the regular morphism $Mor(1_{A_n}, E_{n-1})$ (see (3.13)) and the regular morphism $E_2 : Mor(A_n, Mor(A_{n-1} \times \cdots \times A_1, B)) \to Mor(A_n \times \cdots \times A_1, B)$.

The next definition is made to cope with the problem of finding an inverse to the morphism E_n above.

DEFINITION 3.18 Let $\mathcal{P}$ be a class of global actions closed under finite products. A global action C is called **∞-$\mathcal{P}$-exponential** if the morphism $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ is an ∞-$\mathcal{P}$-normal isomorphism for all pairs $A, B \in \mathcal{P}$. C is called **regularly ∞-$\mathcal{P}$-exponential** if E is a regular isomorphism for all pairs $A, B \in \mathcal{P}$. If $\mathcal{P}$ = all finite products of A (resp. $\mathcal{P}$ = all global actions A such that $|A| = \cup_{\alpha\in\Phi_A} X_\alpha$) then C is called **$\infty$-$A$-exponential** (resp. **$\infty$-exponential**) if it is ∞-$\mathcal{P}$-exponential.

LEMMA 3.19 Suppose the global action C is ∞-$\mathcal{P}$-exponential (resp. regularly ∞-$\mathcal{P}$-exponential). Then for any sequence $A_n, \cdots, A_1 \in \mathcal{P}$ such that $n \geq 2$, the morphism E_n in (3.17) is an ∞-$\mathcal{P}$-normal (resp. regular) isomorphism.

PROOF For $n = 2$, the conclusion holds by hypothesis. Proceeding by induction on n, we can assume that the result holds for $n - 1$. By definition $E_n = E_2 Mor(1, E_{n-1})$. By induction E_2 and $Mor(1, E_{n-1})$ are ∞-$\mathcal{P}-$isomorphisms (resp. regular isomorphisms). The conclusion of the lemma follows. □

The next condition provides a useful criterion for guaranteeing that a global action is ∞-normal and either ∞-exponential or regularly ∞-exponential.

DEFINITION 3.20 Let A be a global action. If $\Delta \subseteqq \Phi_A$, let $\Phi_A{}^{\geqq\Delta} = \{\alpha \in \Phi_A | \alpha \geqq \beta \ \forall \ \beta \in \Delta\}$. A is called an **strong infimum action** if for any finite subset $\Delta \subseteqq \Phi_A$ and any finite nonempty set $U \subseteqq |A|$ such that $(X_A)_\beta \cap U \neq \varnothing$ for each $\beta \in \Delta$, the set $\{\alpha \in \Phi_A{}^{\geqq\Delta} | U \text{ an } \alpha - \text{frame}\}$ is either empty or contains an initial element. A is called an **infimum action** if it satisfies the condition above at least for $\Delta = \varnothing$ (empty set).

The next lemma provides a condition guaranteeing that a global action is a strong infimum action and the lemma thereafter proves the important result that if the target object in a morphism space is an infimum (resp. strong infimum) action then morphism space inherits this property.

Any global action A has the property that if α and β are coordinates such that $\alpha \leqq \beta$ then for each $\mathrm{x} \in X_\alpha \cap X_\beta$, $(G_A)_\alpha(\mathrm{x}) \subseteqq (G_A)_\beta(\mathrm{x})$. The next lemma introduces the reverse implication coupled with a certain intersection property as a sufficient condition for A being a strong infimum action.

LEMMA 3.21 Let A be a global action. Consider the following conditions.

(3.21.1) Let $\alpha, \beta \in \Phi_A$. Then $\alpha \leqq \beta \Leftrightarrow X_\alpha \cap X_\beta \neq \phi$ and for all $\mathrm{x} \in X_\alpha \cap X_\beta$, $(G_A)_\alpha(\mathrm{x}) \subseteqq (G_A)_\beta(\mathrm{x})$.

(3.21.2) Let $\alpha, \beta \in \Phi_A$. Then $\alpha \leqq \beta \Leftrightarrow X_\alpha \cap X_\beta \neq \phi$ and $\exists \mathrm{x} \in X_\alpha \cap X_\beta$ such that $(G_A)_\alpha(\mathrm{x}) \subseteqq (G_A)_\beta(\mathrm{x})$.

(3.21.3) Let $\Psi \subseteqq \Phi_A$. Then for any $\mathrm{x} \in \cap_{\alpha\in\Psi} X_\alpha$, $\cap_{\alpha\in\Psi} (G_A)_\alpha(\mathrm{x}) = (G_A)_\beta(\mathrm{x})$ for some $\beta \in \Phi_A$.

The assertion of the lemma is that if A satisfies (3.21.2) and (3.21.3) then it is a strong infimum action.

PROOF Let U be a local frame. Let $\Delta \subseteqq \Phi_A$ be a finite set such that for each $\delta \in \Delta, X_\delta \cap U \neq \phi$. Let $\Psi = \{\alpha \in {\Phi_A}^{\geqq\Delta} | U$ local $\alpha-$frame$\}$. If $u \in U$ then $U \subseteqq \cap_{\alpha\in\Psi}(G_A)_\alpha(u) = (G_A)_\beta(u)$ for some $\beta \in \Phi_A$, by (3.21.3). Clearly U is a local $\beta-$frame. Since $(G_A)_\beta(u) \subseteqq (G_A)_\alpha(u)$ it follows from (3.21.2) that $\beta \leqq \alpha$. This holds of course for all $\alpha \in \Psi$. Thus we are finished if $\Delta = \phi$. If $\Delta \neq \phi$, we must show that $\delta \leqq \beta$ for any $\delta \in \Delta$. Let $u \in X_\delta \cap U$. Since $\delta \leqq \alpha$ for any $\alpha \in \Psi$, it follows from (3.21.2) that $(G_A)_\delta(u) \subseteqq \cap_{\alpha\in\Psi}(G_A)_\alpha(u) = (G_A)_\beta(u)$. Thus $\delta \leqq \beta$. □

Remark Where as the simplicial actions $gl(S)$ and $fgl(S)$ in (2.5) and any Volodin model (2.3) satisfy the strong infimum condition, only the Volodin model satisfies the conditions in the lemma above.

Lemma 3.22 If B is an infimum (resp. strong infimum) action then for any global action $A, Mor(A, B)$ is an infimum (resp. strong infimum) action.

PROOF Let $U \subseteqq |Mor(A, B)|$ be a finite nonempty subset. Let $\Delta \subseteqq \Phi_{(A,B)}$ be a finite subset such that for each $\delta \in \Delta, (X_{(A,B)})_\delta \cap U \neq \phi$. Let $\Psi = \{\beta \in \Phi_{(A,B)} | U$ a $\beta-$frame, $\delta \leqq \beta \; \forall \; \delta \in \Delta\}$ and assume $\Psi \neq \phi$. We must show that Ψ has an initial element. For each $\mathrm{x} \in |A|$, let $U(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) | \; \mathrm{f} \in U\}, \Delta(\mathrm{x}) = \{\delta(\mathrm{x}) | \; \delta \in \Delta\}$, and $\Psi(\mathrm{x}) = \{\beta(\mathrm{x}) | \; \beta \in \Psi\}$. By hypothesis, for each $\mathrm{x} \in |A|$ there is a coordinate $b_\mathrm{x} \in \Phi_B$ such that $U(\mathrm{x})$ is a $b_\mathrm{x}-$frame and $\delta(\mathrm{x}) \leqq b_\mathrm{x} \leqq \beta(\mathrm{x})$ for all $\delta \in \Delta$ and all $\beta \in \Psi$. Let $\gamma : |A| \to \Phi_B, \mathrm{x} \mapsto b_\mathrm{x}$. Clearly $\delta \leqq \gamma \leqq \beta$ for all $\delta \in \Delta$ and all $\beta \in \Psi$. Let $u \in U$. To complete the proof, it suffices to show that (u, γ) is an $A-$chart. Let $\beta \in \Psi$. Since (u, β) is an $A-$chart and $\gamma \leqq \beta$, it follows trivially that (u, γ) is an $A-$chart. □

The next theorem is a main result.

Theorem 3.23 An infimum action is $\infty-$normal and $\infty-$exponential. A strong infimum action is $\infty-$normal and regularly $\infty-$exponential.

The proof of Theorem 3.23 will use the next lemma several times.

Lemma 3.24 Let A and B be global actions. Let (f, β) be an $A-$chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, f_p$ be an $A-$frame at (f, β). If $\mathrm{x}_0, \cdots, \mathrm{x}_q \in |A|$ is a local frame in A

then $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local $b-$frame for some $b \in \Phi_B$ such that $b \geqq \beta_{\mathrm{x}_0}, \cdots, \beta_{\mathrm{x}_q}$.

PROOF Since (f_0, β) is an A-chart in B, it follows by definition that $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_0(\mathrm{x}_q)$ is a local b-frame for some $b \in \Phi_B$ such that $b \geqq \beta_{\mathrm{x}_0}, \cdots, \beta_{\mathrm{x}_q}$. Thus $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_0(\mathrm{x}_q)$. To complete the proof, it suffices to show that $(G_B)_b$ acts transitively on $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$. Since $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A-$frame at (f, β), $(G_{(A,B)})_\beta$ acts transitively on $\mathrm{f}_0, \cdots, \mathrm{f}_p$. Thus for any $\mathrm{x} \in |A|$, $(G_B)_{\beta_\mathrm{x}}$ acts transitively on $\mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$. Using the canonical homomorphism $(G_B)_{\beta_{\mathrm{x}_j}} \to (G_B)_b$ and the observation that $\mathrm{f}_0(\mathrm{x}_j) \in (X_B)_b$, one concludes that $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_j), \cdots, \mathrm{f}_p(\mathrm{x}_j)$. Since this holds for each j such that $0 \leqq j \leqq q$ and since $(G_B)_b$ acts transitively on $\mathrm{f}_0(\mathrm{x}_0), \cdots, \mathrm{f}_p(\mathrm{x}_q)$, it follows that $(G_B)_b$ acts transitively on $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$. $\square$

PROOF of (3.23) Let C be an infimum action. We shall show that C is $\infty-$normal. Lemma 3.22 reduces the proof to showing that C is $A-$normal for any global action A. Let $g : B \to C$ be a morphism of global actions. Let (f, β) be on $A-$ chart in B. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an $A-$frame at (f, β). We must show that $g\mathrm{f}_0, \cdots, g\mathrm{f}_p$ is an $A-$frame in C. We construct first a coordinate $(\gamma : |A| \to \Phi_C) \in \Phi_{(A,C)}$ such that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C.

For $\mathrm{x} \in |A|$, let $U(\mathrm{x}) = \{g\mathrm{f}_0(\mathrm{x}), \cdots, g\mathrm{f}_p(\mathrm{x})\}$. By the Local-Global Lemma 3.7, $\mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local frame in B. Since g is a morphism, it follows that $U(\mathrm{x})$ is a local frame in C. By the infimum condition for C, the set $\Psi(\mathrm{x}) = \{c \in \Phi_C | U(\mathrm{x}) \text{ a } c-\text{frame}\}$ has an initial element c_x. Define $\gamma : |A| \to \Phi_C, \mathrm{x} \mapsto c_\mathrm{x}$. We show that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C. Let $\mathrm{x}_0, \cdots, \mathrm{x}_q$ be a local frame in A. By (3.24), $\{\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local frame in B. Thus $\{g\mathrm{f}_i(\mathrm{x}_j)|\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local $c-$frame for some $c \in \Phi_C$. Clearly $\gamma(\mathrm{x}_j) = c_{\mathrm{x}_j} \leqq c$, because c_{x_j} is initial in $\Psi(\mathrm{x}_j)$. This shows that $(g\mathrm{f}, \gamma)$ is an $A-$chart in C. By the Local-Global Lemma 3.7, $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A-$frame at $(\mathrm{f}, \gamma) \Leftrightarrow \mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})$ is a local $\gamma(\mathrm{x})-$frame for all $\mathrm{x} \in |A|$. But the right hand side of the equivalence holds by definition of $\gamma(\mathrm{x})$. This completes the proof that C is $A-$normal.

Let C denote again an infimum action. We shall show that C is $\infty-$exponential. Let A and B be global actions such that $|A| = \cup_{\alpha \in \Phi_A}(X_A)_\alpha$ and $|B| = \cup_{\beta \in \Phi_B}(X_B)_\beta$. Let $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ be the morphism in (3.17). We shall prove that E has an $\infty-$normal inverse. By (3.22), $Mor(A, Mor(B, C))$ is an infimum action and thus by the first assertion of the current theorem, it must be $\infty-$normal. Thus if an inverse to E exists, it must be $\infty-$normal. So it suffices to show that E has an inverse. There is an obvious candidate for an inverse, namely the set theoretic map $E' : |Mor(A \times B, C)| \to (A, (B, C)), \mathrm{f} \mapsto E'\mathrm{f}$, where $(E'\mathrm{f}(\mathrm{x}))(y) = \mathrm{f}(\mathrm{x}, y)$. We shall show that $E'\mathrm{f} \in |Mor(A, Mor(B, C))|$ and that the resulting map $E' : |Mor(A \times B, C)| \to$

$|Mor(A, Mor(B,C))|$ is a morphism $Mor(A \times B, C) \to Mor(A, Mor(B,C))$ of global actions. From the set theoretic definition of E', it is obvious that E' will be inverse to E.

We prove that $E'\mathrm{f} : |A| \to (B,C)$ is a morphism $A \to Mor(B,C)$ of global actions. There are two properties to verify. First, if $\mathrm{x} \in |A|$ then $E'\mathrm{f}(\mathrm{x}) : |B| \to |C|,\ y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$, is a morphism $B \to C$ of global actions. Second, the resulting map $E'\mathrm{f} : |A| \to |Mor(B,C)|$, $\mathrm{x} \mapsto E'\mathrm{f}(\mathrm{x})$, is a morphism $A \to Mor(B,C)$ of global actions.

Let $\mathrm{x} \in |A|$ and let $y_0, \cdots, y_q$ be a local frame in B. Then x is a local frame in A because $|A| = \cup_{\alpha \in \Phi_A}(X_A)_\alpha$ and so $(\mathrm{x}, y_0), \cdots, (\mathrm{x}, y_q)$ is a local frame in $A \times B$. Thus $\mathrm{f}(\mathrm{x}, y_0), \cdots, \mathrm{f}(\mathrm{x}, y_q)$ is a local frame in C. But $\mathrm{f}(\mathrm{x}, y_j) = (E'\mathrm{f}(\mathrm{x}))(y_j)$ $(0 \leqq j \leqq q)$. Thus $(E'\mathrm{f}(\mathrm{x}))(y_0), \cdots, (E'\mathrm{f}(\mathrm{x}))(y_q)$ is a local frame in C. Thus $E'\mathrm{f}(\mathrm{x}) : B \to C$ is a morphism of global actions.

Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame in A. We shall verify that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)$. For each element $y \in |B|, y$ is a local frame in B because $|B| = \cup_{\beta \in \Phi_B}(X_B)_\beta$. Thus $(\mathrm{x}_0, y), \cdots, (\mathrm{x}_p, y)$ is a local frame in $A \times B$. Thus $\mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ is a local frame in C. By the infimum condition for C, we know that the set $\{c \in \Phi_C |\ \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ a $c-$frame$\}$ has an initial element c_y. Define $\gamma : |B| \to \Phi_C, y \mapsto c_y$. We shall show that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a $B-$chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 3.7 that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a $B-$frame on $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$. But then by definition, $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)$, which is what we have to verify.

We show now that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a $B-$chart in C. Let $y_0, \cdots, y_q$ be a local frame in B. We must show that $(E'\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (E'\mathrm{f}(\mathrm{x}_0))(y_q)$ is a local $c-$frame for some $c \in \Phi_C$ such that $c \geq \gamma(y_0), \cdots, \gamma(y_q)$. Since $\mathrm{x}_0, \cdots, \mathrm{x}_p$ is a local frame in A and $y_0, \cdots, y_q$ a local frame in $B, \{(\mathrm{x}_i, y_j) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local frame in $A \times B$. Thus $\{\mathrm{f}(\mathrm{x}_i, y_j) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local $c-$frame for some $c \in \Phi_C$. But by definition of $\gamma, c \geq \gamma(y_j)$ for all j such that $0 \leqq j \leqq q$.

Next we show that the function $E' : |Mor(A \times B, C)| \to |Mor(A, Mor(B,C))|$ is a morphism $Mor(A \times B, C) \to Mor(A, Mor(B,C))$ of global actions. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p \in |Mor(A \times B, C)|$ be a local frame in $Mor(A \times B, C)$. We must show that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B,C))$. For each element (x, y) in $A \times B$, $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is a local frame in C, by the Local-Global Lemma 3.7. By the infimum condition for C, the set $\{c \in \Phi_C |\ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ a $c-$ frame$\}$ has an initial element $c_{(\mathrm{x},y)}$. Define $\gamma : |A| \to (|B|, \Phi_C), \mathrm{x} \mapsto c_{(\mathrm{x},-)}$. We claim that $(E'\mathrm{f}_0, \gamma)$ is an $A-$chart in $Mor(B,C)$. It will follow then from the definition of γ and the Local-Global Lemma 3.7 that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is an $A-$ frame at $(E'\mathrm{f}, \gamma)$. But this says by definition that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B,C))$ and we are finished.

We show now that $(E'\mathrm{f}, \gamma)$ is an $A-$chart. Let $\mathrm{x}_0, \cdots, \mathrm{x}_q$ be a local frame in A. We must show that $E'\mathrm{f}(\mathrm{x}_0), \cdots E'\mathrm{f}(\mathrm{x}_q)$ is a local $\delta-$frame in $Mor(B, C)$ for some $\delta : |B| \to \Phi_C$ such that $\delta \geqq \gamma(\mathrm{x}_0), \cdots, \gamma(\mathrm{x}_q)$. Since $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is a local frame in $Mor(A \times B, C)$, there is an $A \times B-$ chart $(\mathrm{f}_0, \varepsilon)$ in C such that $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A \times B-$ frame at $(\mathrm{f}_0, \varepsilon)$. For any fixed $y \in |B|, (\mathrm{f}_0(\ , y), \varepsilon(\ , y))$ is an $A-$chart in C and $\mathrm{f}_0(\ , y), \cdots, \mathrm{f}_p(\ , y)$ is an $A-$ frame at $(\mathrm{f}_0(\ , y), \varepsilon(\ , y))$. Since $\mathrm{x}_0, \cdots, \mathrm{x}_q$ is a local frame in A, it follows from (3.24) that the set $\{\mathrm{f}_i(\mathrm{x}_j, y) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a local frame in C. Since C satisfies the infimum condition, the set $\{c \in \Phi_C | \{\mathrm{f}_i(\mathrm{x}_j, y) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq g\}$ a local frame in $C\}$ has an initial element d_y. Clearly $d_y \geqq (\gamma(\mathrm{x}_j))(y)\ (0 \leqq j \leqq q)$. Define $\delta : |B| \to \Phi_C, y \mapsto d(y)$. Thus $\delta \geqq \gamma(\mathrm{x}_j)\ (0 \leqq j \leqq q)$. Since $(F\mathrm{f}(\mathrm{x}_j))(y) = \mathrm{f}_0(\mathrm{x}_j, y)\ (0 \leqq j \leqq q)$ and $\{\mathrm{f}_i(\mathrm{x}_j, y) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q\}$ is a $\delta(y)-$frame, it is clear that $(F\mathrm{f}(\mathrm{x}_0))(y), \cdots, (F\mathrm{f}(\mathrm{x}_q))(y)$ is a $\delta(y)-$frame. By the Local-Global Lemma 3.7, $F\mathrm{f}(\mathrm{x}_0), \cdots, F\mathrm{f}(\mathrm{x}_q)$ is a $B-$ frame at $(F\mathrm{f}(\mathrm{x}_0), \delta)$ provided that $(F\mathrm{f}(\mathrm{x}_0), \delta)$ is a $B-$chart in C. We show this next.

Let $y_0, \cdots, y_r$ be a local frame in B. We must show that $(F\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (F\mathrm{f}(\mathrm{x}_0))(y_r)$ is a $c-$frame for some $c \in \Phi_C$ such that $c \geqq \delta(y_0), \cdots, \delta(y_r)$. Since the set $\{(\mathrm{x}_j, y_k) |\ 0 \leqq j \leqq q, 0 \leqq k \leqq r\}$ is a local frame in $A \times B$ and $\mathrm{f}_0, \cdots, \mathrm{f}_p$ is an $A \times B-$frame in C, it follows from (3.24) that $\{\mathrm{f}_i(\mathrm{x}_j, y_k) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q,\ 0 \leqq k \leqq r\}$ is a c-frame for some $c \in \Phi_C$. From the definition of δ, it is obvious that $c \geqq \delta(y_0), \cdots, \delta(y_r)$. Since $(F\mathrm{f}(\mathrm{x}_0))(y_k) = \mathrm{f}_0(\mathrm{x}_0, y_k)\ (0 \leqq k \leqq r)$ and $\{\mathrm{f}_i(\mathrm{x}_j, y_k) |\ 0 \leqq i \leqq p,\ 0 \leqq j \leqq q,\ 0 \leqq k \leqq r\}$ is a c-frame, it is clear that $(F\mathrm{f}(\mathrm{x}_0))(y_0), \cdots, (F\mathrm{f}(\mathrm{x}_0))(y_r)$ is a $c-$frame. This completes the proof that C is $\infty-$exponential.

Suppose finally that C is a strong infimum action. We shall show that C is regularly $\infty-$exponential. Our task is to show that the morphism $E : Mor(A, Mor(B, C)) \to Mor(A \times B, C)$ above has a regular inverse E'. There are obvious candidates for the components (E'_Φ, E'_G, E'_X) of E'. Define

(3.25)
$$\begin{aligned} E'_X : Mor(A \times B, C)| &\longrightarrow |Mor(A, Mor(B, C))| \\ \mathrm{f} &\longmapsto E'\mathrm{f} \end{aligned}$$

where $\mathrm{f} \mapsto E'\mathrm{f}$ is the map constructed above. Define

$$E'_\Phi : \Phi_{(A \times B, C)} \longrightarrow \Phi_{(A,(B,C))}$$

as the set theoretic inverse (see (3.16)) of E_Φ. Define the natural transformation

$$E'_G : G_{(A \times B, C)} \longrightarrow (G_{(A,(B,C))})_{E'_\Phi(\ )}$$

such that

$$\begin{array}{ccc}
E'_G(\alpha) : (G_{(A\times B,C)})_\alpha & \longrightarrow & (G_{(A,(B,C))})_{E'_\Phi(\alpha)} \\
\| & & \| \\
\prod_{(\mathrm{x},y)\in|A|\times|B|}(G_C)_{\alpha(\mathrm{x},y)} & & \prod_{\mathrm{x}\in|A|}(\prod_{y\in|B|}(G_C)_{E'_\Phi(\alpha)(\mathrm{x})(y)}
\end{array}$$

maps the factor $(G_C)_{\alpha(\mathrm{x},y)}$ via the identity map to the factor $(G_C)_{E'_\Phi(\alpha)(\mathrm{x})(y)} = (G_C)_{\alpha(\mathrm{x},y)}$. If E' is a regular morphism then it is obvious from its construction that it is the regular inverse to the regular morphism E.

All the properties for E' to be a regular morphism are obvious, except the one that $E'_X(X_{(A\times B,C)})_\alpha \subseteqq (X_{(A,(B,C))})_{E'_\Phi(\alpha)}$ for any $\alpha \in \Phi_{(A\times B,C)}$. To establish this, it is enough to show that if (f,α) is an $(A\times B)$-chart in C then $(E'_X(\mathrm{f}), E'_\Phi(\alpha))$ is an $A-$chart in $Mor(B,C)$. Let $\mathrm{x}_0,\cdots,\mathrm{x}_p$ be a local frame in A. We must show that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})$ (x_p) is a $\gamma-$ frame in $Mor(B,C)$ for some $\gamma : |B| \to \Phi_C$ such that $\gamma \geqq E'_\Phi(\alpha)(\mathrm{x}_i)$ $(0 \leqq i \leqq p)$. For each $y \in |B|$, the elements $(\mathrm{x}_0,y),\cdots,(\mathrm{x}_p,y)$ form a local frame in $A\times B$. Thus $\mathrm{f}(\mathrm{x}_0,y),\cdots,\mathrm{f}(\mathrm{x}_p,y)$ is a local frame in C. By the strong infimum condition for C, the set $\{c\in\Phi_C |\ \mathrm{f}(\mathrm{x}_0,y),\cdots,\mathrm{f}(\mathrm{x}_p,y)$ a $c-$frame$, c \geqq E'_\Phi(\mathrm{x}_i)(y)\ (0\leqq i\leqq p)\}$ has an initial element c_y. Define $\gamma : |B| \to \Phi_C,\ y \mapsto c_y$. Clearly $\gamma \geqq E'_\Phi(\alpha)(\mathrm{x}_i)$ $(0\leqq i\leqq p)$. We shall show that $(E'_X(\mathrm{f})(\mathrm{x}_0),\gamma)$ is a $B-$chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 3.7 and the fact that $E'_X(\mathrm{f})(\mathrm{x}_0)(y),\cdots,E'_X(\mathrm{f})(\mathrm{x}_p)(y)$ is a $\gamma_(y)-$frame for each $y\in|B|$ that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})$ (x_p) is a $B-$frame at $(E'_X(\mathrm{f})(\mathrm{x}_0),$ $\gamma)$. But this says by definition that $E'_X(\mathrm{f})(\mathrm{x}_0),\cdots,E'_X(\mathrm{f})(\mathrm{x}_p)$ is a $\gamma-$frame in $Mor(B,C)$. This would complete the proof of the theorem.

We show now that $(E'_X(\mathrm{f})(\mathrm{x}_0),\gamma)$ is a $B-$chart in C. Let $y_0,\cdots,y_q$ be a local frame in B. Then $\{(\mathrm{x}_i,y_j)|\ 0\leqq i\leqq p,\ 0\leqq j\leqq q\}$ is a local frame in $A\times B$. Thus $\{\mathrm{f}(\mathrm{x}_i,y_j)|\ 0\leqq i\leqq p,$ $0\leqq j\leqq q\}$ is a local $c-$frame for some $c\in\Phi_C$ such that $c\geqq\alpha(\mathrm{x}_i,y_j) = E'_\Phi(\alpha)(\mathrm{x}_i)(y_j)$ $(0\leqq i\leqq p,\ 0\leqq j\leqq q)$. Since $E'_X(\mathrm{f})(\mathrm{x}_0)(y_j) = \mathrm{f}(\mathrm{x}_0,\mathrm{x}_j)$ $(0\leqq j\leqq q)$, it is clear that $E'_X(\mathrm{f})(\mathrm{x}_0)(y_0),\cdots,E'_X(\mathrm{f})(\mathrm{x}_0)(y_q)$ is a c-frame and $c\geqq E'_\Phi(\alpha)(\mathrm{x}_0)(y_j)$ $(0\leqq j\leqq q)$. $\square$

4 Relative actions and their morphism spaces

The homotopy theory of global actions will require pointed actions and more generally relative actions. These concepts will be introduced next. They are subtler than their topological counterparts and more care must be taken to define and develop them. The main result of the section is a relative version of the exponential law proved in the previous section.

The organization and development of the current section will follow closely that of the previous.

DEFINITION 4.1 Let A be a global action. A **subaction** of A is a global action B such that $|B| \subseteqq |A|$ and the inclusion above defines a morphism $B \to A$ of a global actions. If B is a subaction of A then we write $B \subseteqq A$.Let $n \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{N}$ be a class of global actions. A subaction $B \subseteqq A$ is called **n-$\mathcal{N}$-normal** (resp. **regular**) if the canonical morphism $B \to A$ is n-$\mathcal{N}$-normal (resp. regular). A **proper subaction** is a regular subaction such that the canonical morphism $\Phi_B \to \Phi_A$ is injective. A **standard subaction** is a proper one such that $\Phi_B = \{\alpha \in \Phi_A | (X_A)_\alpha \cap |B| \neq \phi\}, \Phi_B$ is a full subcategory of Φ_A, and for all $\alpha \in \Phi_B, (X_B)_\alpha = (X_A)_\alpha \cap |B|$ and $(G_B)_\alpha = Stab_{G_A}((X_B)_\alpha) := \{\sigma \in (G_A)_\alpha | \sigma(X_B)_\alpha = (X_B)_\alpha\}$.

The next lemma is obvious.

LEMMA 4.2 Let A be a global action. If $Y \subseteqq |A|$ then there is a unique global action B such that $|B| = Y$ and B is a standard subaction of A.

DEFINITION 4.3 Let A be a global action. A **base point** for A is a subaction $B \subseteqq A$ such that $|B|$ has precisely one point. Any base point is an $\infty-$normal subaction. A base point is called respectively **regular, proper**, or **standard** it it is such as a subaction.

DEFINITION 4.4 A **relative global action** A is an ordered pair $(A^{(1)}, A^{(2)})$ of global actions sucht that $A^{(2)}$ is an $\infty-$normal subaction of $A^{(1)}$. It is called respectively **regular, proper**, or **standard** if the subaction $A^{(2)} \subseteqq A^{(1)}$ is such.

DEFINTION 4.5 A **pointed global action** is a relative global action $A = (A^{(1)}, A^{(2)})$ such that $A^{(2)}$ is a base point for $A^{(1)}$. A pointed action is called respectively **regular, proper**, or **standard** if its base point is such.

DEFINITION 4.6 A **morphism** $f : A \to B$ of relative actions is a morphism $f : A^{(1)} \to B^{(1)}$ of global actions such that f takes $|A^{(2)}|$ to $|B^{(2)}|$ and defines a morphism $A^{(2)} \to B^{(2)}$ of global actions. A **morphism** $f : A \to B$ of relative actions is called **regular** if the morphisms $f : A^{(i)} \to B^{(i)}$ $(i = 1, 2)$ of global actions carry a fixed regular structure.

DEFINITION 4.7 Let A be a relative action. A **pair** $U = (U^{(1)}, U^{(2)})$ **of local frames** in A is an ordered pair of local frames $U^{(i)}$ in $A^{(i)}$ $(i = 1, 2)$ such that $U^{(2)} \subseteqq U^{(1)}$. A pair $(U^{(1)}, U^{(2)})$ is called **full**, if $U^{(2)} = U^{(1)} \cap |A^{(2)}|$. A pair $(U^{(1)}, U^{(2)})$ is said to be **at** $(\boldsymbol{\alpha^{(1)}}, \boldsymbol{\alpha^{(2)}})$$\in \Phi_{A^{(1)}} \times \Phi_{A^{(2)}}$, if $U^{(i)}$ is a local frame at $\alpha^{(i)}$. A pair $(U^{(1)}, U^{(2)})$ of local frames at $(\alpha^{(1)}, \alpha^{(2)})$ is called a **relative local frame at (α_1, α_2)** or an **$(\alpha^{(1)}, \alpha^{(2)})-$ relative local frame** if $(G_{A^{(2)}})_{\alpha^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\alpha^{(1)}}(u)$ for some and therefore any $u \in U^{(2)}$.

The concept of relative A-chart in B where A and B are relative actions is needed to put a global structure on the set $Mor(A, B)$ of all morphisms from A to B. We introduce this notion next. If $|A^{(2)}| = \phi = |B^{(2)}|$, it reduces to the concept of chart for global actions.

DEFINITION 4.8 Let A and B be relative actions. A **relative A-chart**, or simply **A-chart**, in B is a morphism $f : A \to B$ of relative actions and a function $\beta : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}$ such that the following conditions are satisfied. Set

$$|A^{(1)\backslash(2)}| = |A^{(1)}| \setminus |A^{(2)}|.$$

(4.8.0) Then β takes $|A^{(1)\backslash(2)}|$ to $\Phi_{B^{(1)}}$ and $|A^{(2)}|$ to $\Phi_{B^{(2)}}$.

(4.8.1) If $\mathrm{x} \in |A^{(1)\backslash(2)}|$ then $f(\mathrm{x}) \in (X_{B^{(1)}})_{\beta(\mathrm{x})}$. If $\mathrm{x} \in |A^{(2)}|$ then $f(\mathrm{x}) \in (X_{B^{(2)}})_{\beta(\mathrm{x})}$.

(4.8.2) If $(U^{(1)}, U^{(2)})$ is a full pair of local frames in A then $(\mathrm{f}U^{(1)}, \mathrm{f}U^{(2)})$ is a relative local frame at some $(b^{(1)}, b^{(2)}) \in \Phi_{B^{(1)}} \times \Phi_{B^{(2)}}$ such that $\beta(u) \leqq b^{(1)}$ for all $u \in U^{(1)} \backslash U^{(2)}$, and $\beta(u) \leqq b^{(2)}$ for all $u \in U^{(2)}$.

DEFINITION-LEMMA 4.9 Let A and B be relative actions. Let (f, β) be an A-chart in B. If

$$\sigma = (\sigma_{\mathrm{x}}) \in \prod_{\mathrm{x} \in |A^{(1)\backslash(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$$

define

$$\begin{aligned} \sigma\mathrm{f} : |A^{(1)}| &\to |B^{(1)}|. \\ \mathrm{x} &\mapsto \sigma_{\mathrm{x}}\mathrm{f}(\mathrm{x}) \end{aligned}$$

Then $\sigma\mathrm{f}$ is a morphism $A \to B$ of relative actions and $(\sigma\mathrm{f}, \beta)$ is an A-chart in B.

PROOF It is clear that $(\sigma\mathrm{f}, \beta)$ satisfies (4.8.0) and (4.8.1). To show that $\sigma\mathrm{f}$ is a morphism of relative actions and that $(\sigma\mathrm{f}, \beta)$ is an A-frame in B, it suffices to show that (4.8.2) is satisfied. Let $(\{\mathrm{x}_0, \cdots, \mathrm{x}_p\}, \{\mathrm{x}_0, \cdots, \mathrm{x}_q\})$, $q \leqq p$, be a full pair of local frames in A. By definition $f(\mathrm{x}_0), \cdots, f(\mathrm{x}_p)$ is a $b^{(1)}$-frame for some $b^{(1)} \in \Phi_{B^{(1)}}$ such that $b^{(1)} \geqq \beta(\mathrm{x}_{q+1}), \cdots, \beta(\mathrm{x}_p)$ and $f(\mathrm{x}_0), \cdots, f(\mathrm{x}_q)$ is a $b^{(2)}$-frame for some $b^{(2)} \in \Phi_{B^{(2)}}$ such that

$b^{(2)} \geqq \beta(\mathrm{x}_0), \cdots, \beta(\mathrm{x}_q)$ and $(G_{B^{(2)}})_{b^{(2)}} f(\mathrm{x}_0) \subsetneqq (G_{B^{(1)}})_{b^{(1)}} f(\mathrm{x}_0)$. It follows that $\sigma_{\mathrm{x}_0} f(\mathrm{x}_0)$, $\cdots, \sigma_{\mathrm{x}_p} f(\mathrm{x}_p)$ is a $b^{(1)}$-frame, that $\sigma_{\mathrm{x}_0} f(\mathrm{x}_0), \cdots, \sigma_{\mathrm{x}_q} f(\mathrm{x}_q)$ is a $b^{(2)}$-frame, and $(G_{B^{(2)}})_{b^{(2)}}(\sigma_{\mathrm{x}_0} f(\mathrm{x}_0)) \subsetneqq (G_{B^{(1)}})_{b^{(1)}}(\sigma_{\mathrm{x}_0} f(\mathrm{x}_0))$. $\square$

Definition 4.10 Let A and B be relative actions. Let (f, β) be an A-chart in B. A **(relative)** A**-frame at** f **on** (f, β) is a set $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p : A \to B$ of morphisms for which there are elements $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$ such that $\sigma_i \mathrm{f} = \mathrm{f}_i$ $(1 \leqq i \leqq \mathrm{p})$. (In view of Lemma 4.9, $\mathrm{f} = \mathrm{f}_0, f_1, \cdots, \mathrm{f}_\mathrm{p}$ is also an A-frame at f_i on (f_i, β) for any i $(0 \leqq \mathrm{i} \leqq \mathrm{p})$.)

The next lemma will be very useful, just as the analogous lemma in §3.

Local-Global Lemma 4.11 Let A and B be relative actions. Let (f, β) be an A-chart in B. Then $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ is an A-frame at f on $(\mathrm{f}, \beta) \Leftrightarrow$ for each $\mathrm{x} \in |A^{(1)}|, \mathrm{f}(\mathrm{x}), \mathrm{f}_1(\mathrm{x}), \cdots,$ $\mathrm{f}_p(\mathrm{x})$ is a local (not relative local) frame at $\mathrm{f}(\mathrm{x})$ in $\beta(\mathrm{x})$.

PROOF The assertions are trivial consequences of Lemma 4.9.

Definition 4.12 Let A, B and C be relative actions. An A- **normal** morphism $\mathrm{g} : B \to C$ of relative actions is one such that $g : B^{(2)} \to C^{(2)}$ preserves $A^{(1)}-$ frames and $g : B \to C$ preserves A-frames, i.e. if $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ is an A-frame at f on (f, β) then $\mathrm{gf}, \mathrm{gf}_1, \cdots, \mathrm{gf}_\mathrm{p}$ is an A-frame at gf on (gf, γ) for some A-chart (gf, γ) in C. A **normal** morphism $\mathrm{g} : B \to C$ of relative actions is one which preserves A-frames for any relative action A. An A-**normal** (resp. **normal**) **isomorphism** is an A-normal (resp. normal) morphism which has an A-normal (resp. normal) inverse.

It is not true in general that an A-normal (resp. normal) morphism which is an isomorphism in the usual sense is an A-normal (resp. normal) isomorphism.

Lemma 4.13 A regular morphism of relative actions is normal.

PROOF Let $\eta : B \to C$ be a regular morphism of relative actions. If (f, β) is an A-chart in B then it follows straightforward that $(\eta_X \mathrm{f}, \eta_\Phi \beta)$ is an A-chart in C. Let $\mathrm{f}, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at f on (f, β) and let $\sigma_1, \cdots, \sigma_p \in \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$ such that $\sigma_\mathrm{i} \mathrm{f} = \mathrm{f}_\mathrm{i}$ $(1 \leqq \mathrm{i} \leqq \mathrm{p})$. If $\sigma = (\sigma_\mathrm{x}) \in \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}$, define $\eta_G(\sigma) = (\eta_G(\beta(\mathrm{x}))(\sigma_\mathrm{x})) \in \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\eta_{\Phi^{(1)}} \beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{C^{(2)}})_{\eta_{\Phi^{(2)}} \beta(\mathrm{x})}$. Then $\eta_G(\sigma_i)(\eta_X \mathrm{f}) = \eta_X \mathrm{f}_\mathrm{i}$ $(1 \leqq i \leqq \mathrm{p})$, by (3.3.4). Thus $\eta_X \mathrm{f}, \eta_X \mathrm{f}_1, \cdots, \eta_X \mathrm{f}_p$ is an A-frame at $\eta_X \mathrm{f}$ on $(\eta_X \mathrm{f}, \eta_\Phi \beta)$. $\square$

Next the set $Mor(A, B)$ of all morphisms from a relative action A to a relative action B is given the structure of a relative action.

Definition 4.14 Let A and B be relative actions. Define a relative action

$$Mor(A,B) = (Mor(A,B)^{(1)}, Mor(A,B)^{(2)})$$

as follows. $Mor(A,B)^{(2)} = Mor(A^{(1)}, B^{(2)})$. $Mor(A,B)^{(1)}$ is the global action whose enveloping set is $|Mor(A,B)|$ and whose global structure

$$(\Phi_{(A,B)^{(1)}}, G_{(A,B)^{(1)}}, X_{(A,B)^{(1)}})$$

is defined as follows. Define

$$\Phi_{(A,B)^{(1)}} = \{\beta : |A| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}} | (4.8.0) \text{ satisfied}\}.$$

Give $\Phi_{(A,B)^{(1)}}$ the transitive reflexive relation defined by $\beta \leqq \beta' \Leftrightarrow \beta(\mathrm{x}) \leqq \beta'(\mathrm{x}) \; \forall \mathrm{x} \in |A^{(1)}|$. For $\beta \in \Phi_{(A,B)^{(1)}}$, define

$$(G_{(A,B)^{(1)}})_\beta = . \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{B^{(1)}})_{\beta(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{B^{(2)}})_{\beta(\mathrm{x})}.$$

If $\beta \leqq \beta'$, there is for each $\mathrm{x} \in |A^{(1)\setminus(2)}|$ a functorially defined homomorphism $(G_{B^{(1)}})_{\beta(\mathrm{x})} \to (G_{B^{(1)}})_{\beta'(\mathrm{x})}$ and for each $\mathrm{x} \in |A^{(2)}|$ a functorially defined homomorphism $(G_{B^{(2)}})_{\beta(\mathrm{x})} \to (G_{B^{(2)}})_{\beta'(x)}$ and therefore a homomorphism $(G_{(A,B)})_\beta \longrightarrow (G_{(A,B)})_{\beta'}$ which is obviously functiorial in β. For $\beta \in \Phi_{(A,B)^{(1)}}$, define

$$(X_{(A,B)^{(1)}})_\beta = \{\mathrm{f} : |A^{(1)}| \to |B^{(1)}| \mid (\mathrm{f}, \beta) \; A-\text{chart in } B\}.$$

By (4.9), if $\sigma \in (G_{(A,B)^{(1)}})_\beta$ and $\mathrm{f} \in (X_{(A,B)^{(1)}})_\beta$ then $\sigma\mathrm{f} \in (X_{(A,B)^{(1)}})_\beta$ and so there is an action of $(G_{(A,B)^{(1)}})_\beta$ on $(X_{(A,B)^{(1)}})_\beta$. It is easily established that $(Mor(A,B)^{(1)}, Mor(A,B)^{(2)})$ satisfies all the conditions for a relative action.

Proposition 4.15 As a functor taking values in relative actions, $Mor(,)$ is contravariant and regular over all morphisms in the first variable, under the condition that only regular relative actions are allowed in the second variable, and covariant over all normal morphisms in the second variable. More precisely the following holds.

(4.15.1) Let C be a regular relative action and let $\mathrm{f} : A \to B$ be a morphism of relative actions. Then f defines a regular morphism

$$\eta = Mor(\mathrm{f}, 1_C) : Mor(B,C) \to Mor(A,C)$$

as follows. Let $\nu : \Phi_{C^{(1)}} \to \Phi_{C^{(2)}}$ denote the canonical morphism determined by the regularity of C. Define the relation preserving morphism

$$\begin{aligned} \eta_{\Phi^{(1)}} : \Phi_{(B,C)^{(1)}} &\to \Phi_{(A,C)^{(1)}}. \\ \beta &\mapsto \beta \circ \mathrm{f} \end{aligned}$$

where $\beta \circ \mathrm{f}$ is defined by

$$\beta \circ \mathrm{f}(\mathrm{x}) = \begin{cases} \beta(\mathrm{f}(\mathrm{x})) \text{ if } \mathrm{x} \in |A^{(1)\setminus(2)}| \text{ and } \mathrm{f}(\mathrm{x}) \notin B^{(2)} \\ \nu\beta(\mathrm{f}(\mathrm{x})) \text{ if } \mathrm{x} \in |A^{(1)\setminus(2)}| \text{ and } \mathrm{f}(\mathrm{x}) \in B^{(2)} \\ \beta(\mathrm{f}(\mathrm{x})) \text{ if } \mathrm{x} \in |A^{(2)}|. \end{cases}$$

Define the natural transformation

$$\eta_{G^{(1)}} : G_{(B,C)^{(1)}} \to G_{(A,C)^{(1)}}$$

by

$$\begin{tikzcd} \eta_{G^{(1)}}(\beta) : (G_{(B,C)^{(1)}})_\beta \arrow[r] \arrow[d, equal] & (G_{(A,C)^{(1)}})_{\eta_{\Phi^{(1)}}(\beta)} \arrow[d, equal] \\ \prod_{y \in |B^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\beta(y)} \times \prod_{y \in |B^{(2)}|} (G_{C^{(2)}})_{\beta(y)} & \prod_{\mathrm{x} \in |A^{(1)\setminus(2)}|} (G_{C^{(1)}})_{\beta \mathrm{of}(\mathrm{x})} \times \prod_{\mathrm{x} \in |A^{(2)}|} (G_{C^{(2)}})_{\beta \mathrm{of}(\mathrm{x})} \end{tikzcd}$$

where

$\eta_{G^{(1)}}(\beta)|_{(G_{C^{(i)}})_{\beta(y)}}$ ($i = 1$ if $y \notin |B^{(2)}|$ and $i = 2$ if $y \in |B^{(2)}|$) is the diagonal homomorphism

$$(G_{C^{(2)}})_{\beta(y)} \longrightarrow \prod_{\mathrm{x} \in |A^{(1)}|, \mathrm{f}(\mathrm{x}) = y} (G_{C^{(i)}})_{\beta \mathrm{of}(\mathrm{x})},$$

under the convention that the empty product of groups, which can occur on the right hand side of the arrow above, is the trivial group. One checks straightforward that

$$\eta^{(1)} = (\eta_{\Phi^{(1)}}, \eta_{G^{(1)}}, \eta_X)$$

defines a regular morphism

$$\eta^{(1)} : Mor(B,C)^{(1)} \to Mor(A,C)^{(1)}$$

of global actions and that $\eta^{(2)} = \eta_X|_{Mor(B,C)^{(2)}}$ defines a morphism

$$\eta^{(2)} : Mor(B,C)^{(2)} \to Mor(A,C)^{(2)}$$

of global actions. Use (3.11.1) to provide $\eta^{(2)}$ with a regular structure. One checks routinely then that

$$\eta = (\eta^{(1)}, \eta^{(2)})$$

is a regular morphism

$$\eta : Mor(B,C) \to Mor(A,C)$$

of relative actions. Moreover, $Mor(B,C)$ and $Mor(A,C)$ are regular relative actions whose regular structures is induced in the obvious way from that of C.

(4.15.2) Let A be a relative action and let $\mathrm{g} : B \to C$ be a morphism of relative actions. Then the function

$$Mor(1_A, \mathrm{g}) : |Mor(A,B)^{(1)}| \to |Mor(A,C)^{(1)}|$$

is a morphism $Mor(A,B) \to Mor(A,C)$ of global actions $\Leftrightarrow$ g is A-normal.

PROOF (4.15.1) Nothing has been left to prove.

(4.15.2) Let (f, β) be an A-chart in B and let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{p}$ be an A-frame on (f, β). Let (f', β') be an $A^{(1)}$-chart in $B^{(2)}$ and let $\mathrm{f}' = \mathrm{f}'_0, \mathrm{f}'_1, \cdots, \mathrm{f}'_\mathrm{p}$, be an $A^{(1)}$-frame on (f', β'). By definition of the term local frame, $\mathrm{f}_0, \cdots, \mathrm{f}_\mathrm{p}$ is also a local β-frame in the global action $Mor(A,B)^{(1)}$ and conversely, any local frame in $Mor(A,B)^{(1)}$ is an A-frame on some A-chart in B. Similarly $\mathrm{f}'_0, \cdots, f'_{\mathrm{p}'}$ is a local β'-frame in the global action $Mor(A,B)^{(2)}$ and conversely, any local frame in $Mor(A,B)^{(2)}$ is an $A^{(1)}$-frame on some $A^{(1)}$-chart in $B^{(2)}$. Thus the function $Mor(1_A, \mathrm{g}) : |Mor(A,B)^{(1)}| \to |Mor(A,C)^{(1)}|$ defines a morphism

$Mor(A,B) \to Mor(A,C)$ of relative actions $\Leftrightarrow$ $Mor(1_A, g)$ preserves A-frames and $Mor(1_A, g)^{(2)} : |Mor(A,B)^{(2)}| \to |Mor(A,C)^{(2)}|$ preserves $A^{(1)}$-frames $\Leftrightarrow$ g is A-normal. $\square$

DEFINITION 4.16 Let $g : B \to C$ be a morphism of relative actions. A sequence $A_n, \cdots, A_1$ of relative actions is called a **normal chain** of length n for g if g is A_1-normal and if for each i ($1 \leqq i \leqq n-1$), the morphism $Mor(1_{A_{i-1}}, \cdots, Mor(1_{A_1}, g)) \cdots) : Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, B)) \cdots) \to Mor(A_i, Mor(A_{i-1}, \cdots, Mor(A_1, C)) \cdots)$ is A_{i+1}-normal. Let $\mathcal{N}$ be a class of relative actions. The morphism g is called ***n*-$\boldsymbol{\mathcal{N}}$-normal** if every sequence of n objects from $\mathcal{N}$ forms a normal chain for g. The morphism g is called **$\boldsymbol{\mathcal{N}}$-normal** (resp. **$\boldsymbol{\infty}$- $\boldsymbol{\mathcal{N}}$ -normal**) if it is 1-$\mathcal{N}$-normal (resp. n-$\mathcal{N}$-normal for all $n > 0$). If $\mathcal{N} = \{A\}$ (resp. $\mathcal{N}$ = all relative actions), we shall write **$\boldsymbol{\infty}$-*A*-normal** (resp. **$\boldsymbol{\infty}$-normal**) in place of ∞-$\mathcal{N}$-normal.

If the expression t-morphism denotes anyone of the notions of normality above or the notion of regularity then a t-isomorphism is a t-morphism which has a t-morphism as its inverse.

LEMMA 4.17 If g : $B \to C$ is a regular morphism of relative actions then for any relative action A, the morphism $Mor(1_A, \mathrm{g}) : Mor(A,B) \to Mor(A,C)$ is regular. Thus g is ∞-normal.

PROOF By (4.13)and (4.15.2), the morphism $Mor(1_A, \mathrm{g}) : Mor(A,B) \to Mor(A,C)$ exists. Let $(\eta_\Phi, \eta_G, \eta_X = \mathrm{g})$ be the regular structure of g. We define a regular structure $(\mu_\Phi, \mu_g, \mu_X = Mor(1_A, \mathrm{g}))$ for $Mor(1_A, \mathrm{g})$ as follows.

Define the coordinate morphisms

$$\begin{aligned} \mu_{\Phi^{(1)}} : \Phi_{(A,B)^{(1)}} &\to \Phi_{(A,C)^{(1)}}, \\ \beta &\mapsto \eta_{\Phi^{(1)}} \beta \\ \mu_{\Phi^{(2)}} : \Phi_{(A,B)^{(2)}} &\to \Phi_{(A,C)^{(2)}}. \\ \beta &\mapsto \eta_{\Phi^{(2)}} \beta \end{aligned}$$

Define the natural transformation

$$\mu_{G^{(1)}} : G_{(A,B)^{(1)}} \to G_{(A,C)^{(1)}}$$

by the commutative diagram

$$\begin{array}{ccc}
(G_{(A,B)^{(1)}})_\beta & \xrightarrow{\mu_{G^{(1)}}(\beta)} & (G_{(A,C)^{(1)}})_{\mu_{\Phi^{(1)}}(\beta)} \\
\| & & \| \\
\prod_{x\in|A^{(1)\setminus(2)}|}(G_{B^{(1)}})_{\beta(x)} \times \prod_{x\in|A^{(2)}|}(G_{B^{(2)}})_{\beta(x)} & \xrightarrow[\tau]{} & \prod_{x\in|A^{(1)\setminus(2)}|}(G_{C^{(1)}})_{\eta_{\Phi^{(1)}}(\beta(x))} \times \prod_{x\in|A^{(2)}|}(G_{C^{(2)}})_{\eta_{\Phi^{(2)}}}(\beta(x))
\end{array}$$

where

$$\tau = \prod_{x\in|A^{(1)\setminus(2)}|} \eta_{G^{(1)}}(\beta(x)) \times \prod_{x\in|A^{(2)}|} \eta_{G^{(1)}}(\beta(x)),$$

and define the natural transformation

$$\mu_{G^{(2)}} : G_{(A,B)^{(2)}} \to G_{(A,C)^{(2)}}$$

as in the proof of (3.13).

One checks straightforward that $(\mu_\Phi, \mu_G, Mor(1_A, g))$ is a regular morphism.

That g is ∞-normal follows by a trivial induction argument from the result just proved.
□

DEFINITION 4.18 Let N denote the name of a kind of morphism defined in (4.16). A relative action is called an **N action** if it has the property that every morphism to it is an N morphism.

The following definition is needed for the exponential law.

DEFINITION 4.19 Let A and B be relative actions. Define the relative action

$$A \bowtie B$$

as follows.

$$(A \bowtie B)^{(1)} = A^{(1)} \times B^{(1)}$$

.

The enveloping set of $(A \bowtie B)^{(2)}$ is defined by

$$|(A \bowtie B)^{(2)}| = |A^{(1)}| \times |B^{(2)}| \cup |A^{(2)}| \times |B^{(1)}|$$

where the union is taken in $|A^{(1)}| \times |B^{(1)}|$. The coordinate system of $(A \bowtie B)^{(2)}$ is defined by

$$\Phi_{(A \bowtie B)^{(2)}} = \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \cup \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}$$

where the union is now disjoint. Morphisms between elements are defined coordinatewise. The global group functor of $(A \bowtie B)^{(2)}$ is defined by

$$(G_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)} = \begin{cases} (G_{A^{(1)}})_\alpha \times (G_{B^{(2)}})_\beta \text{ if } (\alpha, \beta) \in \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \\ (G_{A^{(2)}})_\alpha \times (G_{B^{(1)}})_\beta \text{ if } (\alpha, \beta) \in \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}. \end{cases}$$

The global set function for $(A \bowtie B)^{(2)}$ is defined by

$$(X_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)} = \begin{cases} (X_{A^{(1)}})_\alpha \times (X_{B^{(2)}})_\beta \text{ if } (\alpha, \beta) \in \Phi_{A^{(1)}} \times \Phi_{B^{(2)}} \\ (X_{A^{(2)}})_\alpha \times (X_{B^{(1)}})_\beta \text{ if } (\alpha, \beta) \in \Phi_{A^{(2)}} \times \Phi_{B^{(1)}}. \end{cases}$$

The action of $(G_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)}$ on $(X_{(A \bowtie B)^{(2)}})_{(\alpha,\beta)}$ is coordinatewise.

In the next definition, the notation $(S, T) = Mor_{((sets))}(S, T)$, where S and T are sets, will be used. This notation was introduced already in (3.16).

Definition 4.20 Let A, B and C be relative actions. We define a regular morphism

$$E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$$

as follows. Denote the structural components of the relative action $Mor(A, Mor(B, C))$ by $(\Phi_{(A,(B,C))}, G_{(A,(B,C))}, X_{(A,(B,C))})$. Define $E_{\Phi^{(1)}}$ such that the diagram

$$\begin{array}{ccc} \Phi_{(A,(B,C))^{(1)}} & \xrightarrow{E_{\Phi^{(1)}}} & \Phi_{(A \bowtie B, C)^{(1)}} \\ \| \cap & & \| \cap \\ (|A^{(1)}|, (|B^{(1)}|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}})) & \xrightarrow[(3.16)]{} & (|A^{(1)}| \times |B^{(1)}|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}) \end{array}$$

commutes. Clearly $E_{\Phi^{(1)}}$ preserves the transitive reflexive relation. Define $E_{\Phi^{(2)}}$ as in (3.17), i.e. such that the diagram

$$\begin{tikzcd}
\Phi_{(A,(B,C))^{(2)}} \arrow[r, "E_{\Phi^{(2)}}"] \arrow[d, equal] & \Phi_{(A\bowtie B,C)^{(2)}} \arrow[d, equal] \\
(|A^{(1)}|, (|B^{(1)}|, \Phi_{C^{(2)}})) \arrow[r, "(3.16)"'] & (|A^{(1)}| \times |B^{(1)}|, \Phi_{C^{(2)}})
\end{tikzcd}$$

commutes.

Define the natural transformation

$$E_{G^{(1)}} : G_{(A,(B,C))^{(1)}} \to (G_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\ )}$$

such that

$$E_{G^{(1)}}(\alpha) : (G_{(A,(B,C))^{(1)}})_\alpha \longrightarrow (G_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\alpha)}$$

maps the factor of $(G_{(A,(B,C))^{(1)}})_\alpha = \prod_{\mathrm{x}\in|A^{(1)}\setminus(2)|} [(\prod_{y\in|B^{(1)}\setminus(2)|} (G_{C^{(1)}})_{\alpha(\mathrm{x})(y)}) \times (\prod_{y\in|B^{(2)}|} (G_{C^{(2)}})_{\alpha(\mathrm{x})(y)})] \times \prod_{\mathrm{x}\in|A^{(2)}|} [\prod_{y\in|B^{(1)}|} (G_{C^{(2)}})_{\alpha(\mathrm{x})(y)}]$ with the subscript $\alpha(\mathrm{x})(y)$ via the identity map onto the factor of $(G_{(A\bowtie B,C)^{(1)}})_\alpha = \prod_{(\mathrm{x},y)\in|A^{(1)\setminus(2)}|\times|B^{(1)\setminus(2)}|} (G_{C^{(1)}})_{(E_{\Phi^{(1)}}\alpha)(\mathrm{x},y)} \times \prod_{(\mathrm{x},y)\in|(A\bowtie B)^{(2)}|} (G_{C^{(2)}})_{(E_{\Phi^{(1)}}\alpha)(\mathrm{x},y)}$ with the subscript $(E_{\Phi^{(1)}}\alpha)(\mathrm{x},y)$. Define the natural transformation

$$E_{G^{(2)}} : G_{(A,(B,C))^{(2)}} \to G_{(A\bowtie B,C)^{(2)}}$$

as in (3.17), i.e. such that

$$E_{G^{(2)}}(\alpha) : (G_{(A,(B,C))^{(2)}})_\alpha \to (G_{(A\bowtie B,C)^{(2)}})_{E_{\Phi^{(2)}}(\alpha)}$$

maps the factor of $(G_{(A,(B,C))^{(2)}})_\alpha = \prod_{\mathrm{x}\in|A^{(1)}|} (\prod_{y\in|B^{(1)}|} (G_{C^{(2)}})_{\alpha(\mathrm{x})(y)})$ with the subscript $\alpha(\mathrm{x})(y)$ via the identity map onto the factor of $(G_{(A\bowtie B,C)^{(2)}})_{E_{\Phi^{(2)}}(\alpha)} = \prod_{(\mathrm{x},y)\in|A^{(1)}|\times|B^{(1)}|}$

$(G_{C^{(2)}})_{(E_{\Phi^{(2)}}\alpha)(\mathrm{x},y)}$ with the subscript $(E_{\Phi^{(2)}}\alpha)(\mathrm{x},y)$. One verifies routinely that the composite mapping $|Mor(A, Mor(B,C))^{(1)}| \to (|A^{(1)}|,(|B^{(1)}|,|C^{(1)}|))$ @ > (3.16) >> $(|A^{(1)}| \times |B^{(1)}|, |C^{(1)}|)$ takes its image in $|Mor(A \bowtie B, C)^{(1)}|$ and we define

$$E_X : |Mor(A, Mor(B,C))^{(1)}| \to |Mor(A \bowtie B, C)^{(1)}|$$

to be the resulting mapping. One checks straightforward that

$$E^{(1)} = (E_{\Phi^{(1)}}, E_{G^{(1)}}, E_X)$$

is a regular morphism

$$E^{(1)} : Mor(A, Mor(B,C))^{(1)} \to Mor(A \bowtie B, C)^{(1)}$$

of global actions. (It fails in general to be an isomorphism (resp. regular isomorphism) because E_X is not necessarily surjective (resp. $E_X((X_{(A,(B,C))^{(1)}})_\alpha)$ is not necessarily all of $(X_{(A\bowtie B,C)^{(1)}})_{E_{\Phi^{(1)}}(\alpha)}$). Let

$$E_{X^{(2)}} = E_X|_{\ |Mor(A,(B,C))^{(2)}|}$$

and

$$E^{(2)} = (E_{\Phi^{(2)}}, E_{G^{(2)}}, E_{X^{(2)}}).$$

Then by (3.17),

$$\begin{tikzcd}
E^{(2)} : Mor(A, Mor(B,C))^{(2)} \arrow[r] \arrow[d, equal] & Mor(A \bowtie B, C)^{(2)} \arrow[d, equal] \\
Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}) \arrow[r] & Mor(A^{(1)} \times B^{(1)}, C^{(2)})
\end{tikzcd}$$

is a regular morphism of global actions. One checks straightforward that

$$E = (E^{(1)}, E^{(2)})$$

is a regular morphism

$$E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$$

of relative actions.

Let $A_n, \cdots, A_1$ be an arbitrary sequence of global actions. Iterating the procedure above, one defines for any $n \geq 2$ a regular morphism

$$E_n : Mor(A_n, Mor(A_{n-1}, \cdots, Mor(A_1, C)) \cdots) \to Mor(A_n \bowtie \cdots \bowtie A_1, C)$$

as follows. For $n = 2$, the morphism is defined above. Suppose $n > 2$ and that the morphism has been defined for every natural number N where $2 \leqq N \leqq n - 1$. Let E_{n-1} denote the morphism for the sequence $A_{n-1}, \cdots, A_1$. Define E_n for the sequence $A_n, A_{n-1}, \cdots, A_1$ as the composite of the regular morphism $Mor(1_{A_n}, E_{n-1})$ (see (4.17)) and the regular morphism $E_2 : Mor(A_n, Mor(A_{n-1} \bowtie \cdots \bowtie A_1, B)) \to Mor(A_n \bowtie \cdots \bowtie A_1, B)$.

The next definition is made to cope with the problem of finding an inverse to the morphism E_n above.

Definition 4.21 Let $\mathcal{P}$ be a class of relative actions closed under finite operations by $\bowtie$. A relative action C is called **∞-$\mathcal{P}$-exponential** if the morphism $E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$ is an ∞- $\mathcal{P}$-normal isomorphism for all pairs $A, B \in \mathcal{P}$. C is called **regularly ∞- $\mathcal{P}$-exponential** if E is a regular isomorphism for all pairs $A, B \in \mathcal{P}$. If $\mathcal{P}$ = all finite $\bowtie$ $-$products of A (resp. $\mathcal{P}$ = all relative actions A such that $|A^{(i)}| = \cup_{\alpha \in \Phi_{A^{(i)}}} (X_{A^{(i)}})_\alpha$ $(i = 1, 2)$) then C is called **∞-A-exponential** (resp. **∞-exponential**) if it is ∞- $\mathcal{P}$-exponential.

Lemma 4.22 Suppose the relative action C is ∞- $\mathcal{P}$-exponential (resp. regularly ∞-$\mathcal{P}$-exponential). Then for any sequence $A_n, \cdots, A_1 \in \mathcal{P}$ such that $n \geq 2$, the morphism E_n in (4.20) is an ∞-$\mathcal{P}$-normal (resp. regular) isomorphism.

PROOF The proof is exactly the same as that of (3.19).

Definiton 4.23 Let A and B be relative actions. A morphism f : $A \to B$ is called **neat** if it preserves relative local frames, i.e. if $U^{(i)} \subsetneqq |A^{(i)}|$ $(i = 1, 2)$ are finite nonempty subsets such that $U^{(2)} \subsetneqq U^{(1)}$ and $(U^{(1)}, U^{(2)})$ forms a relative local frame in A then $(\mathrm{f}U^{(1)}, \mathrm{f}U^{(2)})$ forms a relative local frame in B.

Definition 4.24 A relative action A is called a **strong neat action** if the following condition is satisfied. The assumptions of the condition are as follows.

(4.24.1) Let $\Delta^{(2)} \subseteqq \Phi_{A^{(2)}}$ be a finite subset. Let $U^{(i)} \subseteqq |A^{(i)}|$ $(i = 1, 2)$ be finite nonempty subsets such that $U^{(2)} \subseteqq U^{(1)}$. Suppose that for each $\delta^{(2)} \in \Delta^{(2)}$, $(X_{A^{(2)}})_{\delta^{(2)}} \cap U^{(2)} \neq \phi$. Suppose that $U^{(i)}$ is a local frame at $\alpha^{(i)} \in \Phi_{A^{(i)}}$ $(i = 1, 2)$ such that$(G_{A^{(2)}})_{\delta^{(2)}} (u) \subseteqq (G_{A^{(1)}})_{\alpha^{(1)}}(u)$ for all $\delta^{(2)} \in \Delta^{(2)}$ and all $u \in (X_{A^{(2)}})_{\delta^{(2)}} \cap U^{(2)}$, and such that $\delta^{(2)} \leqq \alpha^{(2)}$ for all $\delta^{(2)} \in \Delta^{(2)}$.

The conclusion of the condition is that there is a $\beta^{(2)} \in \Phi_{A^{(2)}}$ such that $\delta^{(2)} \leqq \beta^{(2)} \leqq \alpha^{(2)}$ for all $\delta^{(2)} \in \Delta^{(2)}$ and such that $(U^{(1)}, U^{(2)})$ is a relative local frame at $(\alpha^{(1)}, \beta^{(2)})$.

A relative action A is called a **neat action** if the conclusion above is satisfied for at least $\Delta^{(2)} = \phi$.

A subspace $B \subseteqq A$ of a global action A is called a **neat** (resp. **strong neat**) **subspace** if the pair (A, B) is a neat (resp. strong neat) relative action.

Clearly a pointed action is a strong neat relative action.

LEMMA 4.25 If f : $A \to B$ is a morphism of relative actions and B is neat then f is neat.

PROOF Let $(U^{(1)}, U^{(2)})$ be a relative frame in A. Since f is a morphism of relative actions, the pair $(\mathrm{f}U^{(1)}, \mathrm{f}U^{(2)})$ fulfills the assumption (4.24.1) for $\Delta^{(2)} = \phi$. Thus $(\mathrm{f}U^{(1)}, \mathrm{f}U^{(2)})$ is a relative local frame in B. $\Box$

The next condition provides a useful criterion for guaranteeing that a relative action is ∞-normal and either ∞-exponential or regularly ∞-exponential.

DEFINITION 4.26 A relative action A is called a **strong infimum** (relative) **action** if $A^{(1)}$ and $A^{(2)}$ are strong infimum global actions and the following condition is satisfied. The assumptions of the condition are as follows.

(4.26.1) Let $\Delta^{(i)} \subseteqq \Phi_{A^{(i)}}$ $(i = 1, 2)$ be finite subsets. Let $U^{(i)} \subseteqq |A^{(i)}|$ $(i = 1, 2)$ be finite nonempty subsets such that $U^{(2)} \subseteqq U^{(1)}$. Suppose that for each $\delta^{(i)} \in \Delta^{(i)}$, $(X_{A^{(i)}})_{\delta^{(i)}} \cap U^{(i)} \neq \phi$. Suppose that the sets $\Psi^{(1)} = \{\alpha^{(1)} \in \Phi^{\geqq \Delta^{(1)}}_{A^{(1)}} \mid U^{(1)}\ \alpha^{(1)}$-frame, $\delta^{(1)} \leqq \alpha^{(1)}\ \forall\ \delta^{(1)} \in \Delta^{(1)}, (G_{A^{(2)}})_{\delta^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\alpha^{(1)}}(u)\ \forall\ \delta^{(2)} \in \Delta^{(2)}$ and $u \in (X_{A^{(2)}})_{\delta^{(2)}} \cap U^{(2)}\}$ and $\Psi^{(2)} = \{\alpha^{(2)} \in \Phi_{A^{(2)}} \mid U^{(2)}\ \alpha^{(2)}$-frame, $\delta^{(2)} \leqq \alpha^{(2)}\ \forall\ \delta^{(2)} \in \Delta^{(2)}\}$ are nonempty.

The conclusion of the condition is that $\Psi^{(i)}$ $(i = 1, 2)$ contains an initial element $\beta^{(i)}$ such that $(G_{A^{(2)}})_{\beta^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\beta^{(1)}}(u)$ for all $u \in U^{(2)}$.

A relative action A is called an **infimum** (relative) **action** if $A^{(1)}$ and $A^{(2)}$ are infimum actions and the conclusion above is satisfied for at least $(\Delta^{(1)}, \Delta^{(2)}) = (\phi, \phi)$.

The following lemma is easy to verify.

LEMMA 4.27 A standard pointed action is an infimum (resp. strong infimum) relative action $\Leftrightarrow$ the global action $A^{(1)}$ is an infimum (resp. strong infimum) action.

LEMMA 4.28 Let A be a relative action. Then A is an infimum (resp. strong infimum) action $\Leftrightarrow$ A is a neat (resp. strong neat) action and $A^{(i)}$ $(i = 1, 2)$ is an infimum (resp. strong infimum) global action.

PROOF $\Rightarrow$) Suppose that assumption (4.24.1) holds. We must show that the conclusion required for a neat or strong neat action holds. Set $\Delta^{(1)} = \phi$. Let $\Psi^{(i)}$ $(i = 1, 2)$ be defined as in (4.26.1). Since the $\alpha^{(i)}$ given in (4.24.1) belong to $\Psi^{(i)}$ $(i = 1, 2)$, the $\Psi^{(i)}$ are not empty. Let $\beta^{(i)} \in \Psi^{(i)}$ be initial elements in $\Psi^{(i)}$, respectively, which are guarenteed by the conclusion for (4.26.1). The initialness of the $\beta^{(i)}$ implies that $\beta^{(i)} \leq \alpha^{(i)}$. Moreover the conclusion for (4.26.1) says that $U^{(2)}$ is a $\beta^{(2)}$-frame, that $\delta^{(2)} \leqq \beta^{(2)}$ for all $\delta^{(2)} \in \Delta^{(2)}$, and that $(G_{A^{(2)}})_{\beta^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\beta^{(1)}}(u)$ for all $u \in U^{(2)}$. Since $\beta^{(1)} \leqq \alpha^{(1)}$, it follows that $(G_{A^{(1)}})_{\beta^{(1)}}(u) \subseteqq (G_{A^{(1)}})_{\alpha^{(1)}}(u)$. Thus A is a neat or strong neat action.

$\Leftarrow$) Suppose that the assumption (4.26.1) holds. We must show that A is an infimum or strong infimum action. By the neatness hypothesis on A, there is a $\delta^{(2)} \in \Phi_{A^{(2)}}$ such that $\delta^{(2)} \leqq \gamma^{(2)} \leqq \beta^{(2)}$ for all $\delta^{(2)} \in \Delta^{(2)}$ and $(G_{A^{(2)}})_{\gamma^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\beta^{(1)}}(u)$ for all $u \in U^{(2)}$. This says that $\gamma^{(2)} \in \Psi^{(2)}$. But since $\beta^{(2)}$ is initial in $\Psi^{(2)}$, it follows that $\beta^{(2)} \leqq \gamma^{(2)}$ and thus $(G_{A^{(2)}})_{\gamma^{(2)}}(u) = (G_{A^{(2)}})_{\beta^{(2)}}(u)$. Thus $(G_{A^{(2)}})_{\beta^{(2)}}(u) \subseteqq (G_{A^{(1)}})_{\beta^{(1)}}(u)$ for all $u \in U^{(2)}$. Thus A is an infimum or strong infimum action. $\square$

We take the opportunity now to summarize a few trivial, but useful facts. Some have been used already and others will be used soon.

LEMMA 4.29 For each fact below for global actions, there is also a corresponding one given for relative actions.

(4.29.1) Let A be a global action. If $U \subseteqq V \subseteqq |A|$ and V is a local frame then so is U.

(4.29.2) Let $\mathrm{f} : A \to B$ be a morphism of global actions. Let $\beta, \gamma \in \Phi_{(A,B)}$ such that $\mathrm{f(x)} \in \beta(\mathrm{x}) \cap \gamma(\mathrm{x})$ for all $\mathrm{x} \in |A|$. If $\beta \leqq \gamma$ and (f, γ) is an $A-$chart in B then so is (f, β).

(4.29.3) Let A be a relative action. If $U^{(i)} \subseteqq V^{(i)} \subseteqq |A^{(i)}|$ $(i = 1, 2)$ such that $U^{(2)} \subseteqq U^{(1)}$ and $(V^{(1)}, V^{(2)})$ is a relative local frame then so is $(U^{(1)}, U^{(2)})$.

(4.29.4) Let $\mathrm{f} : A \to B$ be a morphism of relative actions. Let $\beta^{(1)}, \gamma^{(1)} \in \Phi_{(A,B)^{(1)}}$ such that $\mathrm{f(x)} \in \beta^{(1)}(\mathrm{x}) \cap \gamma^{(1)}(\mathrm{x})$ for all $\mathrm{x} \in |A^{(1)}|$. If $\beta \leqq \gamma$ and (f, γ) is an $A-$chart in B then so is (f, β).

LEMMA 4.30 If B is an infimum (resp. strong infimum) relative action then for any relative action $A, Mor(A, B)$ is an infimum (resp. strong infimum) action.

PROOF Since $Mor(A,B)^{(2)} = Mor(A^{(1)},B^{(2)})$, it follows from (3.22) that Mor $(A,B)^{(2)}$ is an infimum (resp. strong infimum) action.

We show next that $Mor(A,B)^{(1)}$ is an infimum (resp. strong infimum) action. Let $U^{(1)} \subseteqq |Mor(A,B)^{(1)}|$ be a finite nonempty subset. Let $\Delta^{(1)} \subseteqq \Phi_{(A,B)^{(1)}}$ be a finite subset such that for each $\delta \in \Delta^{(1)}, (X_{(A,B)^{(1)}})_\delta \cap U^{(1)} \neq \phi$. Let $\Psi^{(1)} = \{\beta \in \Phi_{(A,B)^{(1)}} | U^{(1)}$ a β-frame, $\delta \leqq \beta \ \forall \ \delta \in \Delta^{(1)}\}$ and assume $\Psi^{(1)} \neq \phi$. We must show that $\Psi^{(1)}$ has an initial element. For each $\mathrm{x} \in |A^{(1)}|$, let $U^{(1)}(\mathrm{x}) = \{\mathrm{f}(\mathrm{x}) | \ \mathrm{f} \in U\}, \Delta^{(1)}(\mathrm{x}) = \{\delta(\mathrm{x}) | \ \delta \in \Delta^{(1)}\}$, and $\Psi^{(1)}(\mathrm{x}) = \{\beta(\mathrm{x}) | \ \beta \in \Psi^{(1)}\}$. By hypothesis, for each $\mathrm{x} \in |A^{(1)}|$ there is a coordinate $b_\mathrm{x} \in \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}$ such that $U^{(1)}(\mathrm{x})$ is a b_x-frame and $\delta(\mathrm{x}) \leqq b_\mathrm{x} \leqq \beta(\mathrm{x})$ for all $\delta \in \Delta^{(1)}$ and all $\beta \in \Psi^{(1)}$. Similarly for all $\mathrm{x} \in |A^{(2)}|$, there is a coordinate $b_\mathrm{x} \in \Phi_{B^{(2)}}$ such that $U^{(1)}(\mathrm{x})$ is a b_x-frame and $\delta(\mathrm{x}) \leqq b_\mathrm{x} \leqq \beta(\mathrm{x})$ for all $\delta \in \Delta^{(1)}$ and all $\beta \in \Psi^{(1)}$. Let $\gamma : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}, \mathrm{x} \mapsto b_\mathrm{x}$. Clearly $\gamma \in \Phi_{(A,B)^{(1)}}$ and $\delta \leqq \gamma \leqq \beta$ for all $\delta \in \Delta^{(1)}$ and all $\beta \in \Psi^{(1)}$. Let $u \in U$. To complete the proof, it suffices to show that (u,γ) is an A-chart. Let $\beta \in \Psi^{(1)}$. Since (u,β) is an A-chart and $\gamma \leqq \beta$, it follows from (4.29.4) that (u,γ) is an A-chart.

We show finally that the conclusion for (4.26.1) holds. Let $\Delta^{(i)}, U^{(i)},$ and $\Psi^{(i)}$ be as in the assumption (4.26.1). For $\mathrm{x} \in |A^{(1)}|$, define $\Delta^{(i)}(\mathrm{x}), U^{(i)}(\mathrm{x})$, and $\Psi^{(i)}(\mathrm{x})$ as in the paragraph above. For a fixed x , the data $\Delta^{(i)}(\mathrm{x}), U^{(i)}(\mathrm{x}),$ and $\Psi^{(i)}(\mathrm{x})$ satisfies (4.26.1) applied to B. Let ${b^{(i)}}_\mathrm{x} \in \Psi^{(i)}(\mathrm{x})$ be initial elements which are guarenteed by the conclusion of (4.26.1) applied to B. Let $\gamma^{(1)} : |A^{(1)}| \to \Phi_{B^{(1)}} \cup \Phi_{B^{(2)}}, \mathrm{x} \mapsto {b^{(1)}}_\mathrm{x}$, and $\gamma^{(2)} : |A^{(1)}| \to \Phi_B^{(2)}, \mathrm{x} \mapsto {b^{(2)}}_\mathrm{x}$. Clearly $\gamma^{(i)} \in \Phi_{(A,B)^{(i)}}, \delta^{(i)} \leqq \gamma^{(i)} \leqq \beta^{(i)}$ for all $\delta^{(i)} \in \Delta^{(i)}$ and all $\beta^{(i)} \in \Psi^{(i)}$, and $(G_{(A,B)^{(2)}})_{\gamma^{(2)}}(u) \subseteqq (G_{(A,B)^{(1)}})_{\gamma^{(1)}}(u)$ for all $u \in U^{(2)}$. Moreover for any $u \in U^{(1)}$, $(u,\gamma^{(1)})$ is an A-chart in B because for any $\beta^{(1)}$, $(u,\beta^{(1)})$ is an A-chart in B; and for any $u \in U^{(2)}$, $(u,\gamma^{(2)})$ is an $A^{(1)}$-chart in $B^{(2)}$ because for any $\beta^{(2)} \in \Psi^{(2)}$, $(u,\beta^{(2)})$ is an $A^{(1)}$-chart in $B^{(2)}$.□

The next theorem is a main result.

THEOREM 4.31 An infimum relative action is ∞-normal and ∞-exponential. A strong infimum relative action is ∞-normal and regularly ∞-exponential.

PROOF Let C be an infimum action. We shall show that C is ∞-normal. Lemma 4.30 reduces the proof to showing that C is A-normal for any relative action A. Let $g : B \to C$ be a morphism of relative actions. Since $C^{(2)}$ is ∞-normal as a global action, the morphism $g : B^{(2)} \to C^{(2)}$ is ∞-normal and thus $A^{(1)}$-normal. Let (f,β) be on A-chart in B. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be an A-frame at (f,β). It remains to show that $g\mathrm{f}_0, \cdots, g\mathrm{f}_p$ is an A-frame in C. We construct first a relative coordinate $(\gamma : |A^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}) \in \Phi_{(A,C)^{(1)}}$ such that $(g\mathrm{f},\gamma)$ is an A-chart in C.

Let $F^{(1)} = \{\mathrm{f}_0, \cdots \mathrm{f}_p\}$. For $\mathrm{x} \in |A^{(1)}|$, let $F^{(1)}(\mathrm{x}) = \{\mathrm{f}_0(\mathrm{x}), \cdots, \mathrm{f}_p(\mathrm{x})\}$. By the Local-Global Lemma 4.11, $F^{(1)}(\mathrm{x})$ is a local frame in $B^{(1)}$ or $B^{(2)}$ depending on whether $\mathrm{x} \in |A^{(1)\backslash(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, respectively. Thus $gF^{(1)}(\mathrm{x})$ is a local frame in either $C^{(1)}$ or $C^{(2)}$ depending on whether $\mathrm{x} \in |A^{(1)\backslash(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, respectively. Let $\Psi^{(1)}(\mathrm{x}) = \{c \in \Phi_{C^{(i)}}|\ gF^{(i)}(\mathrm{x})$ local frame in $C^{(i)}$; $i = 1$ or 2 depending on whether $\mathrm{x} \in |A^{(1)\backslash(2)}|$ or $\mathrm{x} \in |A^{(2)}|$, resp.$\}$. By the infimum condition for $C^{(i)}, \Psi^{(1)}(\mathrm{x})$ has an initial element c_{x}. Define $\gamma : |A^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, \mathrm{x} \mapsto c_{\mathrm{x}}$.

We show that $(g\mathrm{f}, \gamma)$ is an A-chart in C. Let $(U^{(1)}, U^{(2)})$ be a full pair of local frames in A. By (3.24), $(F^{(1)}U^{(1)}, F^{(1)}U^{(2)})$ is a pair of local frames in B. Thus $(gF^{(1)}U^{(1)}, gF^{(1)}U^{(2)})$ is a pair of local frames in C. By the conclusion of (4.26.1) for $C, (gF^{(1)}U^{(1)},\ gF^{(1)}U^{(2)})$ is a relative local frame at some $(c^{(1)},\ c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. Thus $(g\mathrm{f}U^{(1)}, g\mathrm{f}U^{(2)})$ is a relative local frame at $(c^{(1)},\ c^{(2)})$, by (4.29.4). But from the definition of γ, we know that for each $\mathrm{x} \in U^{(1)}\backslash U^{(2)}$ (resp. $\mathrm{x} \in U^{(2)}$), $\gamma(\mathrm{x}) = c_{\mathrm{x}} \leqq c^{(1)}$ (resp. $\gamma(\mathrm{x}) = c_{\mathrm{x}} \leqq c^{(2)}$). This shows that $(g\mathrm{f}, \gamma)$ is an A-chart in C.

¿From the definition of γ, it follows immediately that $g\mathrm{f} = g\mathrm{f}_0, g\mathrm{f}_1, \cdots, g\mathrm{f}_p$ is an A-frame on $(g\mathrm{f}, \gamma)$. This completes the proof that C is A-normal.

Let C denote again an infimum relative action. We shall show that C is ∞-exponential. Let A and B be global actions such that $|A^{(i)}| = \cup_{\alpha \in \Phi_{A^{(i)}}} (X_{A^{(i)}})_\alpha$ and $|B^{(i)}| = \cup_{\beta \in \Phi_{B^{(i)}}} (X_{B^{(i)}})_\beta$. Let $E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$ be the morphism in (4.20). We shall prove that E has an ∞-normal inverse. By (4.30), $Mor(A, Mor(B, C))$ is an infimum action and thus by the first assertion of the current theorem, it must be ∞-normal. Thus if an inverse to E exists, it must be ∞-normal. So it suffices to show that E has an inverse. There is an obvious candidate for an inverse, namely the set theoretic map $E' : |Mor(A \bowtie B, C)^{(1)}| \to (A, (B, C)), \mathrm{f} \mapsto E'\mathrm{f}$, where $(E'\mathrm{f}(\mathrm{x}))(y) = \mathrm{f}(\mathrm{x}, y)$. We shall show that $E'\mathrm{f} \in |Mor(A, Mor(B, C))^{(1)}|$ and that the resulting map $E' : |Mor(A \bowtie B, C)^{(1)}| \to |Mor(A, Mor(B, C))^{(1)}|$ is a morphism $Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B, C))^{(1)}$ of global actions. From the set theoretic definition of E', it is obvious that E' will be inverse to E.

We prove that $E'\mathrm{f} : |A^{(1)}| \to (B, C)$ is a morphism $A^{(1)} \to Mor(B, C)^{(1)}$ of global actions. There are two properties to verify. First, if $\mathrm{x} \in |A^{(1)}|$ then $E'\mathrm{f}(\mathrm{x}) : |B^{(1)}| \to |C|^{(1)},\ y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$, is a morphism $B \to C$ of relative actions. Second, the resulting map $E'\mathrm{f} : |A^{(1)}| \to |Mor(B, C)^{(1)}|, \mathrm{x} \mapsto E'\mathrm{f}(\mathrm{x})$, is a morphism $A^{(1)} \to Mor(B, C)^{(1)}$ of global actions.

The demonstration that $y \mapsto (E'\mathrm{f}(\mathrm{x}))(y)$ is a morphism $B^{(1)} \to C^{(1)}$ of global actions is the same as the analogous demonstration in the proof of Theorem 3.23. Furthermore it is clear that the morphism $E'\mathrm{f}(\mathrm{x}) : B^{(1)} \to C^{(1)}$ takes $|B^{(2)}|$ into $|C^{(2)}|$ and that the pattern of the demonstration above can be repeated to show that $E'\mathrm{f}(\mathrm{x})|_{|B^{(2)}|}$ is a morphism

$B^{(2)} \to C^{(2)}$ of global actions. Thus $E'\mathrm{f}(\mathrm{x}) : B \to C$ is a morphism of relative actions.

Let $\mathrm{x}_0, \cdots, \mathrm{x}_p$ be a local frame in $A^{(1)}$. We shall verify that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)^{(1)}$. For each element $y \in |B^{(1)}|, y$ is a local frame in $B^{(1)}$ because $|B^{(1)}| = \cup_{\beta \in \Phi_{B^{(1)}}} (X_{B^{(1)}})_\beta$. Thus $(\mathrm{x}_0, y), \cdots, (\mathrm{x}_p, y)$ is a local frame in $(A \bowtie B)^{(1)}$. Thus $\mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ is a local frame in $C^{(1)}$ and if $y \in B^{(2)}$ then it is also a local frame in $C^{(2)}$. By the infimum condition for $C^{(1)}$, we know that for $y \in B^{(1)\backslash(2)}$, the set $\{c \in \Phi_{C^{(1)}} \mid \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(\mathrm{x}_p, y)$ c-frame$\}$ has an initial element c_y. By the infimum condition for $C^{(2)}$, we know that for $y \in B^{(2)}$, the set $\{c \in \Phi_{C^{(2)}} | \mathrm{f}(\mathrm{x}_0, y), \cdots, \mathrm{f}(x_p, y)$ c-frame$\}$ has an initial element c_y. Define $\gamma : |B^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto c_y$. We shall show that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C. Suppose this has been done. It follows then from the Local-Global Lemma 4.11 that $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a B-frame on $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$. But then by definition, $E'\mathrm{f}(\mathrm{x}_0), \cdots, E'\mathrm{f}(\mathrm{x}_p)$ is a local frame in $Mor(B,C)^{(1)}$, which is what we have to verify.

We show now that $(E'\mathrm{f}(\mathrm{x}_0), \gamma)$ is a B-chart in C. Let $(V^{(1)}, V^{(2)})$ be a full pair of local frames in B. We must show that $(E'\mathrm{f}(\mathrm{x}_0)V^{(1)}, E'\mathrm{f}(\mathrm{x}_0)V^{(2)})$ is a relative local frame at some $(c^{(1)}, c^{(2)}) \in (\Phi_{C^{(1)}}, \Phi_{C^{(2)}})$ such that $\gamma(y) \leqq c^{(1)}$ for all $y \in V^{(1)} \backslash V^{(2)}$ and $\gamma(y) \leqq c^{(2)}$ for all $y \in V^{(2)}$. Let $U = \{\mathrm{x}_0, \cdots, \mathrm{x}_p\}$. Then $(U \times V^{(1)},\ U \times V^{(2)})$ is a pair of local frames in $A \bowtie B$. Thus $(\mathrm{f}(U \times V^{(1)}),\ \mathrm{f}(U \times V^{(2)}))$ is a pair of local frames in C. Since $((E'\mathrm{f})UV^{(1)}, (E'\mathrm{f})UV^{(2)}) = (\mathrm{f}(U \times V^{(1)}),\ \mathrm{f}(U \times V^{(2)}))$, the former pair is one of local frames in C. By the conclusion of (4.26.1) for C, $((E'\mathrm{f})UV^{(1)}, (E'\mathrm{f})UV^{(2)})$ is a relative local frame at some $(c^{(1)}, c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. Thus $((E'\mathrm{f})(\mathrm{x}_0)V^{(1)}, (E'\mathrm{f})(\mathrm{x}_0)V^{(2)})$ is a relative local frame at $(c^{(1)}, c^{(2)})$, by (4.29.4). But from the definition of γ, it follows that for $y \in V^{(1)} \backslash V^{(2)}, \gamma(y) = c_y \leqq c^{(1)}$ and for $y \in V^{(2)}, \gamma(y) = c_y \leqq c^{(2)}$.

It is clear that the morphism $E'\mathrm{f} : A^{(1)} \to Mor(B,C)^{(1)}$ takes $|A^{(2)}|$ into $|Mor(B,C)^{(2)}|$. We show that the function $(E'\mathrm{f})|_{|A^{(2)}|} : |A^{(2)}| \to |Mor(B,C)^{(2)}|$ defines a morphism $A^{(2)} \to Mor(B,C)^{(2)}$ of global actions. This will complete the proof that $E'\mathrm{f} : A \to Mor(B,C)$ is a morphism of relative actions. Observe first that $Mor(B,C)^{(2)} = Mor(B^{(1)}, C^{(2)})$. Then check that the function $(E'\mathrm{f})|_{|A^{(2)}|}$ is identical with the function $E'(\mathrm{f}|_{A^{(2)} \times B^{(1)}})$ where the latter E' is the morphism $E' : Mor(A^{(2)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))$ of global actions, which is constructed in the proof of (3.23). By the conclusion of Theorem 3.23, $E'(\mathrm{f}|_{A^{(2)} \times B^{(1)}})$ is a morphism of global actions.

Next we show that the function $E' : |Mor(A \bowtie B, C)^{(1)}| \to |Mor(A, Mor(B,C))^{(1)}|$ is a morphism $Mor(A \bowtie B, C) \to Mor(A, Mor(B,C))$ of relative actions.

To begin we show that E' is a morphism $Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B,C))^{(1)}$ of global actions. Let $\mathrm{f} = \mathrm{f}_0, \mathrm{f}_1, \cdots, \mathrm{f}_p$ be a local frame in $Mor(A \bowtie B, C)^{(1)}$. We must show that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B,C))^{(1)}$. For each element (x, y) in $A \times B$, $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is a local frame in $C^{(1)}$, by the Local-Global Lemma 4.11. Furthermore if $(\mathrm{x}, y) \in X_{(A\bowtie B)^{(2)}}$ then $\mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ is also a local frame in $C^{(2)}$. By the infimum condition for $C^{(1)}$, it follows that for $(\mathrm{x}, y) \in (A \bowtie B)^{(1)\backslash(2)}$, the set $\{c \in \Phi_{C^{(1)}} | \ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ c-frame$\}$ has an initial element $c_{(\mathrm{x},y)}$. By the infimum condition for $C^{(2)}$, it follows that for $(\mathrm{x}, y) \in X_{(A\bowtie B)^{(2)}}$, the set $\{c \in \Phi_{C^{(2)}} | \ \mathrm{f}_0(\mathrm{x}, y), \cdots, \mathrm{f}_p(\mathrm{x}, y)$ c-frame$\}$ has an initial element $c_{(\mathrm{x},y)}$. Define $\gamma : |A| \to (|B|, \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}), \mathrm{x} \mapsto c_{(\mathrm{x},-)}$. We claim that $(E'\mathrm{f}_0, \gamma)$ is an A-chart in $Mor(B,C)$. It will follow then from the definition of γ and the Local-Global Lemma 4.11 that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is A- an frame at $(E'\mathrm{f}, \gamma)$. But this says by definition that $E'\mathrm{f}_0, \cdots, E'\mathrm{f}_p$ is a local frame in $Mor(A, Mor(B,C))^{(1)}$ and we are finished.

Let $(U^{(1)}, U^{(2)})$ be a full pair of local frames in A. We must show that $((E'\mathrm{f})U^{(1)}, (E'\mathrm{f})U^{(2)})$ is a relative local frame at some $(\delta^{(1)}, \delta^{(2)}) \in \Phi_{(B,C)^{(1)}} \times \Phi_{(B,C)^{(2)}}$ such that $\gamma(\mathrm{x}) \leqq \delta^{(1)}$ for all $\mathrm{x} \in U^{(1)} \backslash U^{(2)}$, and $\gamma(\mathrm{x}) \leqq \delta^{(2)}$ for all $\mathrm{x} \in U^{(2)}$. Let $F = \{\mathrm{f} = \mathrm{f}_0, \cdots, \mathrm{f}_p\}$. Since F is a local frame in $Mor(A \bowtie B, C)^{(1)}$, there is an $A \bowtie B$ -chart $(\mathrm{f}_0, \varepsilon)$ in C such that F is an $A \bowtie B$-frame at $(\mathrm{f}_0, \varepsilon)$. For any fixed $y \in |B^{(1)\backslash(2)}|, (\mathrm{f}_0(\ , y), \varepsilon(\ , y))$ is an A-chart in C and $F(\ , y) = \{\mathrm{f}_0(\ , y), \cdots, \mathrm{f}_p(\ , y)\}$ is an A-frame at $(\mathrm{f}_0(\ , y), \varepsilon(\ , y))$. For any fixed $y \in |B^{(2)}|, (\mathrm{f}_0(\ , y), \varepsilon(\ , y))$ is an $A^{(1)}$-chart in $C^{(2)}$ and $\mathrm{f}_0(\ , y), \cdots, \mathrm{f}_p(\ , y)$ is an $A^{(1)}$-frame at $(\mathrm{f}_0(\ , y), \varepsilon(\ , y))$. Since $(U^{(1)}, U^{(2)})$ is a pair of local frames in A, it follows from (3.24) that $(F(U^{(1)}, y), F(U^{(2)}, y))$ is a pair of local frames in C. Since C satisfies the infimum condition, the pair $(F(U^{(1)}, y), F(U^{(2)}, y))$ is a relative local frame at some $(d_y^{(1)}, d_y^{(2)}) \in (\Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}) \times \Phi_{C^{(2)}}$ such that $d_y^{(1)} \in \Phi_{C^{(1)}}$ if $y \in |B^{(1)\backslash(2)}|$ and $d_y^{(1)} \in \Phi_{C^{(2)}}$ if $y \in |B^{(2)}|$. Thus $(\mathrm{f}(U^{(1)}, y), \mathrm{f}(U^{(2)}, y))$ is a relative local frame at $(d_y^{(1)}, d_y^{(2)})$, by (4.29.4). Thus $((E'\mathrm{f})U^{(1)}(y), (E'\mathrm{f})U^{(2)}(y))$ is a relative local frame at $(d_y^{(1)}, d_y^{(2)})$. Define $\delta^{(1)} : B^{(1)} \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto d_y^{(1)}$. Define $\delta^{(2)} : B^{(2)} \to \Phi_{C^{(2)}}, y \mapsto d_y^{(2)}$. By the Local-Global Lemmas 3.7 and 4.11, $((E'\mathrm{f})U^{(1)}, (E'\mathrm{f})U^{(2)})$ is a relative local frame at $(\delta^{(1)}, \delta^{(2)}) \Leftrightarrow ((E'\mathrm{f})(u), \delta^{(1)})$ is an B-chart in C for some $u \in U^{(1)}$ and $((E'\mathrm{f})(u), \delta^{(2)})$ is a $B^{(1)}$-chart in $C^{(2)}$ for some $u \in U^{(2)}$. Assume the right hand side of the equivalence above has been shown. Then we are finished, because from the definition of γ, it follows that $\gamma(\mathrm{x}) \leqq \delta^{(1)}$ for all $\mathrm{x} \in U^{(1)} \backslash U^{(2)}$ and $\gamma(\mathrm{x}) \leqq \delta^{(2)}$ for all $\mathrm{x} \in U^{(2)}$.

We show now that $((E'\mathrm{f})(u), \delta^{(1)})$ is a B-chart in C for any $u \in U^{(1)}$. Let $(V^{(1)}, V^{(2)})$ be a full pair of local frames in B. We must show that $((E'\mathrm{f})(u)V^{(1)}, (E'\mathrm{f})(u)V^{(2)})$ is a relative local frame at some $(c^{(1)}, c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$ such that $\delta^{(1)}(y) \leqq c^{(1)}$ for all $y \in V^{(1)} \backslash V^{(2)}$ and $\delta^{(1)}(y) \leqq c^{(2)}$ for all $y \in V^{(2)}$. By (3.24), $(F(U, V^{(1)}), F(U, V^{(2)}))$ is a pair of local frames in C. Thus $((E'F)UV^{(1)}, (E'F)UV^{(2)})$ is a pair of local frames in C. Since C is an infimum action, $((E'F)UV^{(1)}, (E'F)UV^{(2)})$ is a relative local frame at some $(c^{(1)}, c^{(2)}) \in$

$\Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$. Since $\delta^{(1)}$ was picked by the conclusion of (4.26.1), its "universal" properties tell us that $\delta^{(1)}(y) \leqq c^{(1)}$ for all $y \in V^{(1)} \backslash V^{(2)}$ and $\delta^{(2)}(y) \leqq c^{(2)}$ for all $y \in V^{(2)}$. But by (4.29.4), $((E'\mathrm{f})(u)V^{(1)}, (E'\mathrm{f})(u)V^{(2)})$ is a relative local frame at $(c^{(1)}, c^{(2)})$. This completes the proof.

The demonstration that $((E'\mathrm{f})(u), \delta^{(2)})$ is a $B^{(1)}$-chart in $C^{(2)}$ for any $u \in U^{(2)}$ is the same as the corresponding demonstration in the proof of Theorem 3.23. This completes the proof that $E' : Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B, C))^{(1)}$ is a morphism of global actions.

Clearly $E' : Mor(A \bowtie B, C)^{(1)} \to Mor(A, Mor(B, C))^{(1)}$ takes $|Mor(A \bowtie B, C)^{(2)}|$ into $(A^{(2)}, (B^{(1)}, C^{(2)}))$. In fact, $Mor(A \bowtie B, C)^{(2)} = Mor(A^{(1)} \times B^{(1)}, C^{(2)})$ and $E' |_{|Mor(A^{(1)} \times B^{(1)}, C^{(2)})|}$ is identical with the function $|Mor(A^{(1)} \times B^{(1)}, C^{(2)})| \to |Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))|$ defined by the morphism $E' : Mor(A^{(1)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(2)}, Mor(B^{(1)}, C^{(2)}))$ of global actions in Theorem 3.23. Thus $E' : Mor(A \bowtie B, C)^{(2)} \to Mor(A, Mor(B, C))^{(2)}$ is a morphism of global actions. Thus $E' : Mor(A \bowtie B, C) \to Mor(A, Mor(B, C))$ is a morphism of relative actions. This completes the proof that C is ∞-exponential.

Suppose finally that C is a strong infimum action. We shall show that C is regularly ∞-exponential. Our task is to show that the morphism $E : Mor(A, Mor(B, C)) \to Mor(A \bowtie B, C)$ has a regular inverse E'. There are obvious candidates for the structural components (E'_Φ, E'_G, E'_X) of E'. Define

$$E'_X : Mor(A \bowtie B, C)| \longrightarrow |Mor(A, Mor(B, C))| \qquad \mathrm{f} \longmapsto E'\mathrm{f} \tag{4.32}$$

where $\mathrm{f} \mapsto E'\mathrm{f}$ is the map constructed above. Define

$$E'_{\Phi^{(i)}} : \Phi_{(A \bowtie B, C)^{(i)}} \longrightarrow \Phi_{(A,(B,C))^{(i)}} \quad (i = 1, 2)$$

as the set theoretic inverse (see (3.16)) of $E_{\Phi^{(i)}}$. Define the natural transformations

$$E'_{G^{(i)}} : G_{(A \bowtie B, C)^{(i)}} \longrightarrow (G_{(A,(B,C))^{(i)}})_{E'_{\Phi^{(i)}}(\ )} \quad (i = 1, 2)$$

as follows. For $\alpha \in \Phi_{(A \bowtie B, C)^{(i)}}$, define the group homomorphism

$$E'_{G^{(1)}}(\alpha) : (G_{(A \bowtie B, C)^{(1)}})_\alpha \longrightarrow (G_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)}$$

such that the factor of $(G_{(A\bowtie B,C)^{(1)}})_\alpha = \prod_{(\mathrm{x},y)\in|A^{(1)\backslash(2)}|\times|B^{(1)\backslash(2)}|}(G_{C^{(1)}})_{\alpha(\mathrm{x},y)} \times \prod_{(\mathrm{x},y)\in|(A\bowtie B)^{(2)}|}(G_{C^{(2)}})_{\alpha(\mathrm{x},y)}$ with the subscript $\alpha(\mathrm{x},y)$ is mapped via the identity map onto the factor of $(G_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)} = \prod_{\mathrm{x}\in|A^{(1)\backslash(2)}|}[\prod_{y\in|B^{(1)\backslash(2)}|}(G_{C^{(1)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)} \times \prod_{y\in|B^{(2)}|}(G_{C^{(2)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)}] \times \prod_{\mathrm{x}\in|A^{(2)}|}[\prod_{y\in|B^{(1)}|}(G_{C^{(2)}})_{(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)}$ with the subscript $(E'_{\Phi^{(1)}}\alpha)(\mathrm{x})(y)$. For $\alpha \in \Phi_{(A\bowtie B,C)^{(2)}}$, define the group homomorphism

$$E'_{G^{(2)}}(\alpha) : (G_{(A\bowtie B,C)^{(2)}})_\alpha \to (G_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)}$$

such that the factor of $(G_{(A\bowtie B,C)^{(2)}})_\alpha = \prod_{(\mathrm{x},y)\in|A^{(1)}|\times|B^{(1)}|}(G_{C^{(2)}})_{\alpha(\mathrm{x},y)}$ with the subscript $\alpha(\mathrm{x},y)$ is mapped via the identity map onto the factor of $(G_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)} = \prod_{\mathrm{x}\in|A^{(1)}|}(\prod_{y\in|B^{(1)}|}(G_{C^{(2)}})_{(E'_{\Phi^{(2)}}\alpha)(\mathrm{x})(y)})$ with the subscript $(E'_{\Phi^{(2)}}\alpha)(\mathrm{x})(y)$.

All the properties for E' to be a regular morphism are obvious, except the one that $E'_X(X_{(A\bowtie B,C)^{(i)}})_\alpha \subseteqq (X_{(A,(B,C))^{(i)}})_{E'_{\Phi^{(i)}}(\alpha)}$ for any $\alpha \in \Phi_{(A\bowtie B,C)^{(i)}}$ $(i = 1, 2)$.

We prove first the case $i = 1$. To establish this, it is enough to show that if (f, α) is an $(A \bowtie B)$-chart in C then $(E'_X(\mathrm{f}), E'_{\Phi^{(1)}}(\alpha))$ is an A-chart in $Mor(B, C)$.

Let $(U^{(1)}, U^{(2)})$ be a full pair of local frames in A. We must show that $((E'\mathrm{f})U^{(1)}, (E'\mathrm{f})U^{(2)})$ is a relative local frame at some $(\gamma^{(1)}, \gamma^{(2)}) \in \Phi_{(B,C)^{(1)}} \times \Phi_{(B,C)^{(2)}}$ such that $\gamma^{(1)} \geqq (E'_{\Phi^{(1)}}\alpha)(\mathrm{x})$ for all $\mathrm{x} \in U^{(2)}$. We construct $\gamma^{(1)}$ and $\gamma^{(2)}$ as follows. Suppose $y \in B^{(1)\backslash(2)}$. Define $\Psi^{(1)}(y) = \{c \in \Phi_{C^{(1)}} | \ \mathrm{f}(U^{(1)} \times \{y\})\ c\text{-frame},\ c \geqq \alpha(\mathrm{x}, y)\ \forall\ \mathrm{x} \in U^{(1)}\backslash U^{(2)}\}$. Suppose $y \in B^{(2)}$. Define $\Psi^{(1)}(y) = \{c \in \Phi_{C^{(2)}} | \ \mathrm{f}(U^{(1)} \times \{y\})\ c\text{-frame},\ c \geqq \alpha(\mathrm{x}, y)\ \forall\ \mathrm{x} \in U^{(1)} \setminus U^{(2)}\}$. Suppose $y \in B^{(1)}$. Define $\Psi^{(2)}(y) = \{c \in \Phi_{C^{(2)}} | \ \mathrm{f}(U^{(2)} \times \{y\})\ c\text{-frame},\ c \geqq \alpha(\mathrm{x}, y)\ \forall\ \mathrm{x} \in U^{(2)}\}$. By the strong infimum condition for $C^{(i)}$ $(i = 1, 2)$, the sets $\Psi^{(1)}(y)$ and $\Psi^{(2)}(y)$ have initial elements $c^{(1)}_y$ and $c^{(2)}_y$, respectively. Define $\gamma^{(1)} : |B^{(1)}| \to \Phi_{C^{(1)}} \cup \Phi_{C^{(2)}}, y \mapsto c^{(1)}_y$. Define $\gamma^{(2)} : |B^{(1)}| \to \Phi_{C^{(2)}}, y \mapsto c^{(2)}_y$. By construction, $\gamma^{(1)} \geqq (E'_{\Phi^{(1)}}\alpha)(\mathrm{x})$ for all $\mathrm{x} \in U^{(1)}\backslash U^{(2)}$ and $\gamma^{(2)} \geqq (E'_{\Phi^{(1)}}\alpha)(\mathrm{x})$ for all $\mathrm{x} \in U^{(2)}$. Suppose we know for any $\mathrm{x} \in U^{(2)}$ that $((E'\mathrm{f})(\mathrm{x}), \gamma^{(1)})$ is a B-chart in C and that $((E'\mathrm{f})(\mathrm{x}), \gamma^{(2)})$ is a $B^{(1)}$-chart in $C^{(2)}$. Then from the definition of $\gamma^{(1)}$, it follows that $(E'\mathrm{f})U^{(1)} \subseteqq (G_{(B,C)^{(1)}})_{\gamma^{(1)}}(E'\mathrm{f})(\mathrm{x})$, i.e. $(E'\mathrm{f})U^{(1)}$ is a local frame at $\gamma^{(1)}$, and that $(E'\mathrm{f})U^{(2)} \subseteqq (G_{(B,C)^{(2)}})_{\gamma^{(2)}}(E'\mathrm{f})(\mathrm{x})$, i.e. $(E'\mathrm{f})U^{(2)}$ is a local frame at $\gamma^{(2)}$. Furthermore by the strong infimum property for C, $(G_{C^{(2)}})_{\gamma^{(2)}(y)}(E'\mathrm{f})(\mathrm{x}, y) \subseteqq (G_{C^{(1)}})_{\gamma^{(1)}(y)}(E'\mathrm{f})(\mathrm{x}, y)$ for any $\mathrm{x} \in U^{(2)}$ and $y \in |B^{(1)}|$. Thus $(G_{(B,C)^{(2)}})_{\gamma^{(2)}}(E'\mathrm{f})(\mathrm{x}) \subseteqq$

$(G_{(B,C)^{(1)}})_{\gamma^{(1)}}(E'\mathrm{f})(\mathrm{x})$. Thus $((E'\mathrm{f})U^{(1)},(E'\mathrm{f})U^{(2)})$ is a relative local frame at $(\gamma^{(1)},\gamma^{(2)})$ and $\gamma^{(1)} \geqq (E'_{\Phi^{(1)}}\alpha)(\mathrm{x})$ for all $\mathrm{x} \in U^{(1)} \backslash U^{(2)}$ and $\gamma^{(2)} \geqq (E'_{\Phi^{(1)}}\alpha)(\mathrm{x})$ for all $\mathrm{x} \in U^{(2)}$.

We show now that for any $\mathrm{x} \in U^{(1)}, ((E'\mathrm{f})(\mathrm{x}),\gamma^{(1)})$ is a B-chart in C. Let $(V^{(1)},V^{(2)})$ be a full pair of local frames in B. We must show that $((E'\mathrm{f})(\mathrm{x})V^{(1)},(E'\mathrm{f})(\mathrm{x})V^{(2)})$ is a relative local frame at some $(c^{(1)},c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$ such that $c^{(1)} \geqq \gamma^{(1)}(y)$ for all $y \in V^{(1)} \backslash V^{(2)}$ and $c^{(2)} \geqq \gamma^{(2)}(y)$ for all $y \in V^{(2)}$. Since $(U^{(1)} \times V^{(1)}, U^{(1)} \times V^{(2)})$ is a pair of local frames in $A \bowtie B$, $((E'\mathrm{f})U^{(1)}V^{(1)},(E'\mathrm{f})U^{(1)}V^{(2)}) = (\mathrm{f}(U^{(1)} \times V^{(1)}), \mathrm{f}(U^{(1)} \times V^{(2)}))$ is a pair of local frames in C. Let $\Delta^{(1)} = \{\alpha(\mathrm{x},y) | \ \mathrm{x} \in U^{(1)} \backslash U^{(2)}, y \in B^{(1)\backslash(2)}\}$ and $\Delta^{(2)} = \{\alpha(\mathrm{x},y) | \ \mathrm{x} \in U^{(1)} \backslash U^{(2)}, y \in B^{(2)}\}$. (If $U^{(1)} \backslash U^{(2)} = \phi$ then of course $\Delta^{(1)} = \phi = \Delta^{(2)}$.) By the strong infimum condition for $C, ((E'\mathrm{f})U^{(1)}V^{(1)},\ (E'\mathrm{f})U^{(1)}V^{(2)})$ is a relative local frame at some $(c^{(1)},c^{(2)}) \in \Phi_{C^{(1)}} \times \Phi_{C^{(2)}}$ such that $c^{(1)} \geqq \alpha(\mathrm{x},y)$ for all $\alpha(\mathrm{x},y) \in \Delta^{(1)}$ and $c^{(2)} \geqq \alpha(\mathrm{x},y)$ for all $\alpha(\mathrm{x},y) \in \Delta^{(2)}$. From the construction of $\gamma^{(1)}$, it follows that $c^{(1)} \geqq \gamma^{(1)}(y)$ for all $y \in V^{(1)} \backslash V^{(2)}$ and $c^{(2)} \geqq \gamma^{(1)}(y)$ for all $y \in V^{(2)}$.

We show now that for $\mathrm{x} \in U^{(2)}, ((E'\mathrm{f})(\mathrm{x}),\gamma^{(2)})$ is a $B^{(1)}$-chart in $C^{(2)}$. Let V be a local frame in $B^{(1)}$. We must show that $(E'\mathrm{f})(\mathrm{x})V$ is a local frame at some $c \in \Phi_{C^{(2)}}$ such that $c \geqq \gamma^{(2)}(y)$ for all $y \in V$. Since $U^{(2)} \times V$ is a local frame in $(A \bowtie B)^{(2)}, (E'\mathrm{f})U^{(2)}V = \mathrm{f}(U^{(2)} \times V)$ is a local frame at some $c \in \Phi_{C^{(2)}}$. From the construction of $\gamma^{(2)}$, it follows that $c \geqq \gamma^{(2)}(y)$ for all $y \in V$. Thus by (4.29.1), $(E'\mathrm{f})(\mathrm{x})V$ is a local frame at c and $c \geqq \gamma^{(2)}(y)$ for all $y \in V$.

This completes the proof that $E'_X(X_{(A \bowtie B,C)^{(1)}})_\alpha \subseteqq (X_{(A,(B,C))^{(1)}})_{E'_{\Phi^{(1)}}(\alpha)}$ for any $\alpha \in \Phi_{(A \bowtie B,C)^{(1)}}$.

To complete the proof of the theorem, it remains now to show that $E'_X(X_{(A \bowtie B,C)^{(2)}})_\alpha \subseteqq (X_{(A,(B,C))^{(2)}})_{E'_{\Phi^{(2)}}(\alpha)}$ for any $\alpha \in \Phi_{(A \bowtie B,C)^{(2)}}$.Observe that $Mor(A \bowtie B, C)^{(2)} = Mor(A^{(1)} \times B^{(1)}, C^{(2)})$ and $Mor(A, Mor(B,C))^{(2)} = Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}))$. But by Theorem 3.23, the morphism $E'_X : Mor(A^{(1)} \times B^{(1)}, C^{(2)}) \to Mor(A^{(1)}, Mor(B^{(1)}, C^{(2)}))$ of global actions is regular and therefore has the desired property above. □

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