HIGHER NON-ABELIAN COHOMOLOGY OF GROUPS

HVEDRI INASSARIDZE

Abstract. The first non-abelian cohomology of groups introduced by Guin is extended to any dimensions and for a substancially wider class of coefficients called G-partially crossed P-modules. The first and the second non-abelian cohomologies of groups are described in terms of torsors and extensions of groups respectively. Higher non-abelian cohomology pointed sets are described in terms of cotriple right derived functors of the group of derivations with respect to the first contravariant variable. For any short exact coefficient sequence a long exact cohomology sequence is obtained extending the well-known exact cohomology sequences and higher cohomology of groups with coefficients in any G-group is introduced.

Introduction

Our approach to non-abelian cohomology of groups follows Guin's first non-abelian cohomology [5,6] which differs from the classical first non-abelian cohomology pointed set [10] and from the setting of various papers on non-abelian cohomology [4,2,3] extending the classical exact non-abelian cohomology sequence from lower dimensions [10] to higher dimensions.

Guin defined his first non-abelian cohomology group when the coefficient group is a crossed G-module and obtained a six term exact cohomology sequence for any short exact sequence of crossed G-modules.

A non-abelian cohomology of groups will be defined in any dimensions ≥ 1 extending Guin's first non-abelian cohomology group and his exact cohomology sequence to nine term exact cohomology sequence. A substantially wider class of coefficients will be used consisting of partially crossed modules over a group P on which acts on the left a group G that will be called G-partially crossed modules over P. We describe the first non-abelian cohomology in terms of torsors and the second non-abelian cohomology in terms of extensions of groups. Moreover a close relation of non-abelian cohomology of groups with non-abelian

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right derived functors of the group of derivations will be established and for some particular cases of coefficients a long exact cohomology sequence will be obtained.

All considered groups will be arbitrary (not necessarily commutative). An action of a group G on a group A means an action on the left of G on A by automorphisms and will be denoted by ${}^g a, g \in G, a \in A$. We assume that G acts on itself by conjugation. The center of a group G will be denoted by $Z(G)$. If the groups G and R act on a group A then the notation ${}^{gr}a$ means ${}^g({}^r a), g \in G, r \in R, a \in A$.

1. G-partially crossed P-modules and the group $\mathrm{Der}(G,(A,\mu))$ of derivations

A precrossed P-module (A,μ) consists of a group P acting on a group A and a homomorphism $\mu : A \longrightarrow P$ such that

$$\mu({}^x a) = x\mu(a)x^{-1}, \quad x \in P,\ a \in A.$$

If in addition we have

$${}^{\mu(a)}a' = aa'a^{-1}$$

for $a, a' \in A$, then (A,μ) is a crossed P-module.

Definition 1.1. *A partially crossed module $\mu : A \longrightarrow P$ over P is a precrossed module over P satisfying the equality*

$$aa'a^{-1} = {}^{\mu(a)}a' \ , \tag{1.1}$$

for all $a' \in A$ and for all $a \in A$ such that $\mu(a)$ is a commutator of P.

Note that the relation (1.1) is equivalent to the following relation:

$$a\,{}^{yx}a' = {}^{xy}a'a \ , \tag{1.2}$$

for all $a' \in A$ and for all $a \in A$ such that $\mu(a) = xyx^{-1}y^{-1}$. In effect, $(1.1) \Longrightarrow (1.2)$: take $a' ={}^{yx} b$ and $(1.2) \Longrightarrow (1.1)$: take $a' ={}^{x^{-1}y^{-1}} b$. Clearly any crossed module over P is a partially P-crossed module.

Let A be a metabelian (not abelian) group. Consider the precrossed module

$$A \xrightarrow{\tau} A/[A,A] = P \ ,$$

where τ is the canonical surjection and P acts trivially on A. Then $A \xrightarrow{\tau} P$ is a partially crossed module over P which is not a crossed P-module.

Any precrossed module $B \xrightarrow{\mu} P$ induces in a natural way a partially crossed module over P as follows:

Consider the Peiffer commutators $bb'b^{-1}\,{}^{\mu(b)}b'^{-1}$ for all $b' \in B$ subject to the relation: $\mu(b)$ is a commutator of P. Let N be the normal subgroup of B generated by these Peiffers commutators and take the quotient group B/N. One gets a precrossed module $B/N \xrightarrow{\mu'} P$, μ' being induced by μ. It is easy to check that in fact it is a partially crossed P-module. Moreover any morphism from $B \xrightarrow{\mu} P$ to a partially crossed module $X \xrightarrow{\nu} C$ induces in a natural way a unique morphism

$$\begin{array}{ccc} B/N & \xrightarrow{\mu'} & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\nu} & C \end{array} .$$

It is obvious that if $A \xrightarrow{\mu} P$ is a partially P-crossed module, then $\operatorname{Ker}\mu$ is contained in the center of A .

Definition 1.2. *Let G, P and A be groups. It will be said that $A \xrightarrow{\mu} P$ is a G-precrossed module over P if*

1. *(A, μ) is a precrossed P-module,*
2. *G acts on P and A,*
3. *$\mu : A \longrightarrow P$ is a homomorphism of G-groups,*
4. *${}^{({}^g x)}a = {}^{gxg^{-1}}a$ for $g \in G$, $x \in P$, $a \in A$ (compatibility condition).*

If in addition (A, μ) is a crossed P-module, then (A, μ) is called G-crossed P-module. If conditions (1)-(4) hold it will be said that the group G acts on the precrossed P-module (A, μ).

Definition 1.3. *A G-precrossed P-module (A, μ) will be called G-partially crossed P-module if in addition the following condition holds:*

$$aa'a^{-1} = {}^{\mu(a)}a' \quad \text{or equivalently} \quad a^{yx}a' = {}^{xy}a'a \ ,$$

for all $a' \in A$ and $a \in A$ such that $\mu(a) = xyx^{-1}y^{-1}$ for some $x, y \in P$.

It is clear that any precrossed (crossed) G-module is in a natural way a G-precrossed (G-crossed) G-module, G acting on itself by conjugation. A G-precrossed P-module was called in [8] as precrossed $G-P$-bimodule arising confusion with the notion of crossed bimodule defined in [9](see E.1.5.1) generalising the well-known notion of bimodule. If $f : G' \longrightarrow G$ is a homomorphism of groups then any G-precrossed P-module is a G'-precrossed P-module induced by f, G' acting on A and P via f.

Note that if (A, μ) is a G-precrossed P-module the equality

$${}^{rx}a = {}^{xr}a$$

holds for any $x \in G$, $a \in A$, $r \in H^0(G,P)$. In effect, one has

$$ {}^{rx}a = {}^{xx^{-1}rx}a = {}^{x({}^{x^{-1}}r)}a = {}^{xr}a \ . $$

Definition 1.4. *Let (A,μ) be a G-partially crossed P-module. Denote by* $\mathrm{Der}(G,(A,\mu))$ *the set of pairs (α, r), α is a crossed homomorphism from G to A, i.e.*

$$ \alpha(xy) = \alpha(x)\ {}^{x}\alpha(y)\ , \quad x, y \in G\ , $$

and r is an element of P such that

$$ \mu\alpha(x) = r\ {}^{x}r^{-1}\ , \quad x \in G\ . $$

This set will be called the set of derivations from G to (A,μ).

For any $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ one has

$$ \alpha(x)\ {}^{xr}a = {}^{rx}a\alpha(x)\ , $$

for $a \in A$, $x \in G$.

We introduce in $\mathrm{Der}(G,(A,\mu))$ a product by

$$ (\alpha, r)(\beta, s) = (\alpha * \beta, rs)\ , $$

where $(\alpha * \beta)(x) = {}^{r}\beta(x)\alpha(x)$, $x \in G$.

Proposition 1.5. *Under the aforementioned product* $\mathrm{Der}(G,(A,\mu))$ *is a group which coincides with the group* $\mathrm{Der}_G(G,A)$ *of Guin if (A,μ) is a crossed G-module viewed as a G-crossed G-module.*

Proof. Clearly this product is associative. At first it will be shown that $(\alpha * \beta, rs)$ belongs to $\mathrm{Der}(G,(A,\mu))$. Put $\gamma = \alpha * \beta$. We have

$$ \gamma(xy) = {}^{r}\beta(xy)\alpha(xy) = {}^{r}(\beta(x)\ {}^{x}\beta(y))\alpha(x)\ {}^{x}\alpha(y) = {}^{r}\beta(x)\ {}^{rx}\beta(y)\ \cdot $$
$$ \cdot\ \alpha(x)\ {}^{x}\alpha(y). $$

On the other hand

$$ \gamma(x)^{x}\gamma(y) =^{r} \beta(x)\alpha(x)^{x}({}^{r}\beta(y)\alpha(y)) =^{r} \beta(x)\alpha(x)^{xr}\beta(y)^{x}\alpha(y)\ . $$

Using equality (1.1) one gets $\gamma(xy) = \gamma(x)^{x}\gamma(y)$ showing that γ is a crossed homomorphism. Further, we have

$$ \mu\gamma(x) = \mu({}^{r}\beta(x)\alpha(x)) = {}^{r}\mu\beta(x)\mu\alpha(x) = {}^{r}(s\ {}^{x}s^{-1})r\ {}^{x}r^{-1} = {}^{r}s^{r}(s^{-1})r $$
$$ {}^{x}r^{-1} = {}^{r}sr^{x}s^{-1x}r^{-1} = rs\ {}^{x}(rs)^{-1}\ . $$

Therefore $(\alpha * \beta, rs) \in \mathrm{Der}(G,(A,\mu))$. It is obvious that $(\alpha_0, 1)$ is the unit of $\mathrm{Der}(G,(A,\mu))$ with $\alpha_0(x) = 1$ for all $x \in G$.

For $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ take the pair $(\overline{\alpha}, r^{-1})$ with $\overline{\alpha}(x) = {}^{r^{-1}}\alpha(x)^{-1}$, $x \in G$. It will be shown that $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$. For this it will be proved the equality

$$ {}^{r^{-1}x}a \cdot {}^{r^{-1}}\alpha(x)^{-1} = {}^{r^{-1}}\alpha(x)^{-1} \cdot {}^{xr^{-1}}a, \ \ x \in G, \ a \in A. \tag{1.3}$$

In effect, since $\mu({}^{r^{-1}}\alpha(x)^{-1}) = r^{-1}\mu\alpha(x)^{-1}r = r^{-1x}r$, one gets

$$ {}^{\mu({}^{r^{-1}}\alpha(x)^{-1})}({}^{xr^{-1}}a) = {}^{r^{-1}\ {}^{x}r}({}^{xr^{-1}}a) = {}^{r^{-1}xrx^{-1}xr^{-1}}a = {}^{r^{-1}x}a \ .$$

The required relation (1.3) follows now from the equality

$$ {}^{\mu({}^{r^{-1}}\alpha(x)^{-1})}({}^{xr^{-1}}a) = r^{-1}\alpha(x) \ \ {}^{xr^{-1}}a \ \ {}^{r^{-1}}\alpha(x) \ .$$

Therefore one has

$$\overline{\alpha}(xy) = {}^{r^{-1}}\alpha(xy)^{-1} = {}^{r^{-1}}({}^{x}\alpha(y)^{-1}\alpha(x)^{-1}) = {}^{r^{-1}x}\alpha(y)^{-1} \ \ {}^{r^{-1}}\alpha(x)^{-1} =$$
$${}^{r^{-1}}\alpha(x)^{-1} \ \ {}^{xr^{-1}}\alpha(y)^{-1} = \overline{\alpha}(x) \ \ {}^{x}\overline{\alpha}(y) \ ,$$

i.e. $\overline{\alpha}$ is a crossed homomorphism. Moreover,

$$\mu\overline{\alpha}(x) = \mu({}^{r^{-1}}\alpha(x)^{-1}) = r^{-1}\mu\alpha(x)^{-1}r = r^{-1} \ {}^{x}rr^{-1}r = r^{-1} \ \ {}^{x}r \ .$$

There follows that $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$. It is easily checked that

$$(\alpha, r)(\overline{\alpha}, r^{-1}) = (\overline{\alpha}, r^{-1})(\alpha, r) = (\alpha_0, 1) \ .$$

We conclude that $\mathrm{Der}(G,(A,\mu))$ is a group which coincides with the group $\mathrm{Der}_G(G,A)$ of derivations defined by Guin [6] when (A,μ) is a crossed G-module. □

A homomorphism $f : (A,\mu) \longrightarrow (B,\lambda)$ of G-partially crossed P-modules induces a homomorphism

$$f^* : \mathrm{Der}(G,(A,\mu)) \longrightarrow \mathrm{Der}(G,(B,\lambda))$$

given by $(\alpha, r) \mapsto (\alpha f, r)$.

There is an action of G on $\mathrm{Der}(G,(A,\mu))$ defined by

$$ {}^{g}(\alpha, r) = (\overline{\alpha}, {}^{g}r) \ , \quad g \in G \ , r \in P \ ,$$

with $\overline{\alpha}(x) = {}^{g}\alpha({}^{g^{-1}}x)$, $x \in G$ (see [6,8]). Moreover if P acts on G such that

$$ {}^{({}^{r}g)}a = {}^{rgr^{-1}}a \ , \quad {}^{({}^{r}g)}r' = {}^{rgr^{-1}}r' \quad \text{for } \ r, r' \in R, \ g \in G, \ a \in A \ ,$$

then there is also an action of P on $\mathrm{Der}(G,(A,\mu))$ given by ${}^{r}(\alpha, s) = (\overline{\alpha}, {}^{r}s)$, $\overline{\alpha}(x) ={}^{r} \alpha({}^{r^{-1}}x)$, $r \in P$, $x \in G$ [8].

It is well-known [1] that the groups G and P acting on each other and on themselves by conjugation are said to be acting compatibly if

$$^{(^g r)}g' = {}^{grg^{-1}}g', \quad {}^{(^r g)}r' = {}^{rgr^{-1}}r' \quad \text{for } g, g' \in G,\ r, r' \in R\ . \tag{1.4}$$

Definition 1.6. *It will be said that the groups G and P act on a group A compatibly if*

$$^{(^g r)}a = {}^{grg^{-1}}a, \quad {}^{(^r g)}a = {}^{rgr^{-1}}a \quad textfor \ \ g \in G,\ r \in R,\ a \in A\ .$$

Proposition 1.7 ([8])**.** *Let (A, μ) be a G-partially crossed P-module the groups G and P acting on each other and on A compatibly. Under the aforementioned actions of G and P on* $\mathrm{Der}(G, (A, \mu))$ *and the homomorphism* $\gamma : \mathrm{Der}(G, (A, \mu)) \to P$ *given by* $(\alpha, r) \mapsto r$*, the pair* $(\mathrm{Der}(G, (A, \mu)), \gamma)$ *is a G-precrossed P-module.*

2. The first non-abelian cohomology

Let (A, μ) be a G-partially crossed P-module. Define on the group $\mathrm{Der}(G, (A, \mu))$ an equivalence relation as follows:

$$(\alpha, r) \sim (\beta, s) \Longleftrightarrow \{ \begin{matrix} \exists a \in A : \beta(x) = a^{-1}\alpha(x)\ {}^{x}a, \\ s = \mu(a)^{-1}r \mod H^0(G, P) \end{matrix} \ .$$

Theorem 2.1. *Let (A, μ) be a G-partially crossed P-module satisfying the following conditions:*

1) *$H^0(G, P)$ is a normal subgroup of P,*
2) *for any $c \in H^0(G, P)$ and* $(\alpha, r) \in \mathrm{Der}(G, (A, \mu))$ *there exists $a \in A$ such that $\mu(a) = 1$ and ${}^{c}\alpha(x) = a^{-1}\alpha(x)\ {}^{x}a$, $x \in G$.*

Then the group $\mathrm{Der}(G, (A, \mu))$ *induces on the quotient set* $\mathrm{Der}(G, (A, \mu))/ \sim$ *a group structure and this quotient group will be called the first cohomology group $H^1(G, (A, \mu))$ of the group G with coefficients in the G-partially crossed P-module (A, μ).*

Proof. We have to show that the relation $\sim$ is a congruence, i.e. if $(\alpha, r) \sim (\alpha', r')$ and $(\beta, s) \sim (\beta', s')$, then $(\alpha, r)(\beta, s) \sim (\alpha', r')(\beta', s')$. For this it will be used Guin's proof [6] which remains valid in our generalized case.

We first prove that

$$(\alpha, r)(\beta, s) \sim (\alpha, rc)(\beta, s)$$

for any $c \in H^0(G,P)$. For (β, s) and $c \in H^0(G,P)$ there is $a \in A$ such that $\mu(a) = 1$ and $a^{-1}\beta(x)^x a$, $x \in G$. One gets

$$^{rc}\beta(x)\alpha(x) = {}^r(a^{-1}\beta(x)\ {}^x a)\alpha(x) = {}^r a^{-1}\ {}^r\beta(x)\ {}^{rx}a\alpha(x) = {}^r a^{-1}\ {}^r\beta(x) \cdot$$
$$\cdot\, \alpha(x)\ {}^{xr}a\ .$$

Since $\mu({}^r a)^{-1} = (r\mu(a)r^{-1})^{-1} = 1$, one has $rcs = \mu({}^r a)^{-1}rsc'$ with $c' \in H^0(G,P)$. Therefore, $(\alpha, r)(\beta, s) = (\alpha, rc)(\beta, s)$.

Use the equalities

$$\alpha'(x) = b^{-1}\alpha(x)\ \ {}^x b, \quad r' = \mu(b)^{-1}rz$$
$$\text{and } \ \beta'(x) = d^{-1}\beta(x)\ \ {}^x d, \quad s' = \mu(d)^{-1}st$$

with $z, t \in H^0(G,P)$ and put

$$(\alpha, rz)(\beta, s) = (\gamma, rzs)$$
$$\text{and } \ (\alpha', r')(\beta', s') = (\gamma', r's')$$

with $\gamma(x) = {}^{rz}\beta(x)\alpha(x)$ and $\gamma'(x) = {}^{r'}\beta'(x)\alpha'(x)$, $x \in G$.

It will be shown that

$$(\alpha, rz)(\beta, s) \sim (\alpha', r')(\beta', s')\ .$$

In effect,

$$\gamma'(x) = {}^{r'}(d^{-1}\beta(x)\ \ {}^x d)b^{-1}\alpha(x)\ \ {}^x b = {}^{\mu(b)^{-1}rz}d^{-1}\ \ {}^{\mu(b)^{-1}rz}\beta(x)\ \ {}^{\mu(b)^{-1}rzx}d \cdot$$
$$\cdot\, b^{-1}\alpha(x)\ \ {}^x b = b^{-1}\ \ {}^{rz}d^{-1}\ \ {}^{rz}\beta(x)\ \ {}^{rx}({}^z d)\ \ \alpha(x)\ \ {}^x b = b^{-1}\ \ {}^{rz}d^{-1}\ \ {}^{rz}\beta(x) \cdot$$
$$\cdot\, \alpha(x)^{xrz}d\ \ {}^x b\ ,$$

and $\mu({}^{rz}db^{-1}) = \mu(b)^{-1}rz\mu(d)^{-1}z^{-1}r^{-1} = r's't^{-1}s^{-1}z^{-1}r^{-1}$, $r's' = \mu({}^{rz}d\ b)^{-1}rzst$ with $t \in H^0(G,P)$.

There follows that $(\alpha, rz)(\beta, s) \sim (\alpha', r')(\beta', s')$. Therefore $(\alpha, r)(\beta, s) \sim (\alpha', r')(\beta', s')$ and the equivalence $\sim$ is a congruence. $\square$

Clearly any partially crossed G-module viewed as a G-partially crossed G-module satisfies conditions of Theorem 2.1, in this case $H^0(G,G) = Z(G)$ and for $(\alpha, g) \in \mathrm{Der}(G,(A,\mu)) = \mathrm{Der}_G(G,A)$, $c \in Z(G)$ the equality $\alpha(cx) = \alpha(xc)$, $x \in G$, implies $\alpha(c)\ {}^c\alpha((x) = \alpha(x)\ {}^x\alpha(c)$ and $\mu(\alpha(c)) = gcg^{-1}c^{-1} = 1$; we recover Guin's first cohomology group of a group G with coefficients in a crossed G-module [6].

If $f : (A,\mu) \to (B,\lambda)$ is a homomorphism of G-partially crossed P-modules satisfying conditions of Theorem 2.1, then f^* induces a homomorphism $f^1 : H^1(G,(A,\mu)) \to H^1(G,(B,\lambda))$. The above defined action of G on $\mathrm{Der}(G,(A,\mu))$ induces an action of G on $H^1(G,(A,\mu))$ given by

$$^g[(\alpha, r)] = [{}^g(\alpha, r)]\ , \quad g \in G\ .$$

By our next statement it will be shown that Guin's first non-abelian cohomology group is closely related with torsors. A similar relationship of the first non-abelian pointed set cohomology with principal homogeneous spaces is well-known [1]. To this aim the notion of a G-torsor over a partially crossed G-module will be introduced.

Definition 2.2. *A G-torsor over a partially crossed G-module (A, μ) is a pair (E, f) consisting of a non-empty G-set E with an action on the right of A on E denoted by xa (for $x \in E$, $a \in A$) which is compatible with the action of G and such that for any $x, y \in E$ there is a unique element $b \in A$ with $y = xb$, and f is a map from E to G such that*

1) *for any $x \in E$, $s \in G$ the following equality holds*
$$\mu(a) = f(x)sf(x)^{-1}s^{-1}$$
with ${}^s x = xa$, $a \in A$;

2) *if $y = xb$ then*
$$f(y) = \mu(b^{-1})f(x)\ , \quad x, y \in E\ , b \in A\ .$$

Definition 2.3. *It will be said that G-torsors (E, f) and (E', f') over a partially crossed G-module (A, μ) are isomorphic if there is a bijection $\vartheta : E \to E'$ compatible with the actions of G and A such that*
$$f(x) = f'(x) \mod Z(G)$$
for any $x \in E$.

Denote by $E(G, A)$ the set of classes of isomorphic G-torsors over the partially crossed G-module (A, μ).

It is introduced a product on the set $E(G, A)$ as follows. Let $[(E_1, f_1)]$, $[(E_2, f_2)] \in E(G, A)$ and let $x \in E_1$, $y \in E_2$. Take A with a new action of G given by $({}^s a)' = {}^{f_1(x)}cb\ {}^s a$ for any $s \in G$, $a \in A$ with ${}^s x = xb$, ${}^s y = yc$. Denote so defined G-group by E and define the action of A on E by translation on the right. Define a map $g : E \to G$ given by
$$g(a) = \mu(a^{-1})f_1(x)f_2(x)\ .$$

Then the pair (E, g) is a G-torsor over the partially crossed G-module (A, μ) and define the required product by
$$[(E_1, f_1)] \circ [(E_2, f_2)] = [(E, g)]\ .$$

Theorem 2.4. *If (A, μ) is a partially crossed G-module, there is a natural isomorphism between $E(G, A)$ and $H^1(G, A)$.*

Proof. Let $[(E, f)] \in E(G, A)$ and take $x \in E$. For any $s \in G$ one has ${}^s x = xa$ and an induced map $\alpha_x : G \longrightarrow A$ defined by $\alpha_x(s) = a$ that is a cocycle. Using 1) of Definition 2.2 we see that the pair $(\alpha_x, f(x))$ is an element of $\mathrm{Der}(G, (G, A)) = \mathrm{Der}_G(G, A)$.

Define a map $\alpha : E(G,A) \longrightarrow H^1(G,A)$ by $\alpha([(E,f)]) = [(\alpha_x, f(x))]$. We have to show the correctness of α.

If $y \in E$ and $y = xb$, $b \in A$, then ${}^sy = {}^sx\ {}^sb = xa\ {}^sb = xbb^{-1}a\ {}^sb = yb^{-1}a\ {}^sb$. By 2) $f(y) = \mu(b^{-1})f(x)$. There follows that $(\alpha_x, f(x)) \sim (\alpha_y, f(y))$. Let (E,f) be isomorphic to (E', f'), i.e. there is a bijection $\vartheta : E \longrightarrow E'$ with properties given in Definition 2.2. Take $x' \in E'$ and $\vartheta(x) = x'$, $x \in E$. Then $\vartheta({}^sx) = {}^s\vartheta(x) = {}^sx'$ and $\vartheta(xa) = \vartheta(x)a = x'a$ with ${}^sx = xa$. Thus, $\alpha_x = \alpha_{x'}$. Since $f(x) = f'(x') \mod Z(G)$, one deduces $(\alpha_x, f(x)) \sim (\alpha_{x'}, f'((x'))$. Therefore, the map α is correctly defined.

Let $[(\alpha, g)] \in H^1(G,A)$. Take A with a new action of G given by $({}^sx)' = \alpha(s)\ {}^sx$, $x \in A$, $s \in G$, and with the action of A on itself by translation on the right. Denote so obtained G-set by P_α. Define a map $f_g : P_\alpha \to G$ by $f_g(x) = \mu(x^{-1})g$, $x \in P_\alpha$, which verifies conditions 1) and 2) of Definition 2.2. In effect, if $({}^sx)' = xa$ then $\alpha(s){}^sx = xa$. On the other hand $\mu\alpha(s) = gsg^{-1}s^{-1}$. Thus, $\mu\alpha(s)\mu({}^sx) = \mu(xa)$, $gsg^{-1}s^{-1}s\mu(x)s^{-1} = \mu(x)\mu(a)$. Whence

$$\mu(a) = \mu(x)^{-1}gsg^{-1}\mu(x)s^{-1} = f_g(x)sf_g(x)^{-1}s^{-1}$$

and f_g verifies condition 1).

If $y = xb$ then one has

$$f_g(y) = \mu(y^{-1})g = \mu(b^{-1}x^{-1})g = \mu(b^{-1})\mu(x^{-1})g = \mu(b^{-1})f(x)\ .$$

Thus f_g satisfies condition 2) too. One gets a G-torsor (P_α, f_g) over the partially crossed G-module (A,μ) and define $\beta : H^1(G,A) \to E(G,A)$ by $\beta([(\alpha,g)]) = [(P_\alpha, f_g)]$.

If $(\alpha, g) \sim (\alpha', g')$ then $\alpha'(s) = b^{-1}\alpha(s){}^sb$ and $g' = \mu(b^{-1})g \bmod Z(G)$. It will be shown that (P_α, f_g) is isomorphic $(P_{\alpha'}, f_{g'})$.

Define $\partial : P_\alpha \longrightarrow P_{\alpha'}$ by $\partial(x) = b^{-1}x$, $x \in P_\alpha$. Then $\partial(({}^sx)') = b^{-1}\alpha(s)\ {}^sx$ and $({}^s(\partial(x)))' = ({}^s(b^{-1}x))' = \alpha'(s)\ {}^s(b^{-1}x) = b^{-1}\alpha(s)\ {}^sb\ {}^sb^{-1}\ {}^sx = b^{-1}\alpha(s)\ {}^sx$. Thus, $\partial(({}^sx)') = ({}^s(\partial(x)))'$. It is obvious that ∂ preserves the action of A.

For $x \in P_\alpha$ one gets

$$f_{g'}(\partial(x)) = f_{g'}(b^{-1}x) = \mu(x^{-1}b)g' = \mu(x^{-1})\mu(b)\mu(b^{-1})g \mod Z(G) =$$
$$\mu(x^{-1})g \mod Z(G) = f_g(x) \mod Z(G)\ .$$

This implies that (P_α, f_g) is isomorphic to $(P_{\alpha'}, f_{g'})$. Therefore the map β is correctly defined.

It is easily checked that α is a homomorphism and $\alpha\beta = 1$, $\beta\alpha = 1$. $\square$

3. The second non-abelian cohomology

Now the second cohomology $H^2(G,(A,\mu))$ of a group G with coefficients in a G-partially crossed P-module (A,μ) will be defined.

Consider the diagram

$$M \underset{l_1}{\overset{l_0}{\rightrightarrows}} F \xrightarrow{\tau} G \tag{3.1}$$

with F a free group, τ is a surjective homomorphism, M is a group consisting of pairs (x,y), $x,y\in F$, such that $\tau(x)=\tau(y)$ and l_0, l_1 are canonical projections, $l_0(x,y)=x$, $l_1(x,y)=y$. That means (M,l_0,l_1) is the simplicial kernel of τ. Put $\Delta=\{(x,x),\, x\in F\}\subset M$.

Then (A,μ) can be viewed as a F-partially crossed P-module induced by τ and as a M-partially crossed P-module induced by τl_0 (or by τl_1). Let $\widetilde{Z^1}(M,(A,\mu))$ be the subset of $\mathrm{Der}(M,(A,\mu))$ consisting of elements of the form $(\alpha,1)$ satisfying the condition $\alpha(\Delta)=1$, implying $\alpha(M)\subset Z(A)$. There follows that $\widetilde{Z^1}(M,(A,\mu))$ is an abelian subgroup of $\mathrm{Der}(M,(A,\mu))$.

Define on $\widetilde{Z^1}(M,(A,\mu))$ a relation by

$$(\alpha',1)\sim(\alpha,1) \Longleftrightarrow (\beta,h)\in \mathrm{Der}(F,(A,\mu))$$

such that the following equality holds

$$(\alpha',1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$$

in the group $\mathrm{Der}(M,(A,\mu))$.

We see that if $(\alpha',1)\sim(\alpha,1)$ one has

$$\alpha'(x)=\beta l_1(x)^{-1}\,{}^{h}\alpha(x)\beta l_0(x)\ , \quad x\in M\ .$$

Proposition 3.1. *The relation $\sim$ defined on $\widetilde{Z^1}(M,(A,\mu))$ is an equivalence.*

Proof. Clearly this relation is reflexive. If $(\alpha',1)\sim(\alpha,1)$, i.e. $(\alpha',1)=(\beta l_0,h)(\alpha,1)(\beta l_1,1)^{-1}$, $(\beta,h)\in\mathrm{Der}(F,(A,\mu))$, then $(\alpha,1)=(\beta l_0,h)^{-1}\cdot$ $\cdot(\alpha',1)(\beta l_1,h)$ and $(\beta l_0,h)^{-1}=(\widetilde{\beta} l_0,h^{-1})$, $(\beta l_1,h)=(\widetilde{\beta} l_1,h^{-1})^{-1}$ with $(\widetilde{\beta},h^{-1})=(\beta,h)^{-1}\in\mathrm{Der}(F,(A,\mu))$. Thus the relation $\sim$ is symmetric. It remains to show the transitivity.

Let $(\alpha',1)\sim(\alpha,1)$ and $(\alpha'',1)\sim(\alpha',1)$. Then one has

$$(\alpha',1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}\ ,$$
$$(\alpha'',1)=(\beta' l_0,h')(\alpha',1)(\beta' l_1,h')^{-1}$$

with (β,h), $(\beta',h')\in\mathrm{Der}(F,(A,\mu))$.

There follows that

$$(\alpha'',1) = (\beta' l_0, h')(\beta l_0, h)(\alpha,1)(\beta l_1, h)^{-1}(\beta' l_1, h')^{-1} =$$
$$((\beta' * \beta) l_0, h'h)(\alpha,1)((\beta' * \beta) l_1, h'h)^{-1}$$

with $(\beta' * \beta, h'h) = (\beta', h')(\beta, h) \in \mathrm{Der}(F, (A, \mu))$. Therefore $(\alpha'',1) \sim (\alpha,1)$ and the relation $\sim$ is an equivalence. $\square$

Proposition 3.2. *The quotient set $\widetilde{Z^1}(M,(A,\mu))/\sim$ is independent of the diagram (3.1) and is unique up to bijection.*

We need the following

Lemma 3.3. *Let A be a G-group and let $\alpha : M \to A$ be a crossed homomorphism such that $\alpha(\Delta) = 1$. Then there exists a map $q : F \to A$ such that*

$$\alpha(y) = q l_1(y)^{-1} q l_0(y) \ , \quad y \in M \ .$$

Proof. Note that if (x,x''), $(x',x'') \in M$, then $\alpha(x,x'') = \alpha(x',x'') \cdot \cdot\alpha(x,x')$. In effect the equalities $(x,x'') = (1, x''x'^{-1})(x,x')$ and $(x',x'') = (1,x''x'^{-1})(x',x')$ imply $\alpha(x,x'') = \alpha(1,x''x'^{-1})\alpha(x,x')$ and $\alpha(x',x'') = \alpha(1,x''x'^{-1})\alpha(x',x') = \alpha(1,x''x'^{-1})$ giving the required equality.

In particular, applying this equality for (x,x), $(x',x) \in M$ one gets $\alpha(x,x) = \alpha(x',x)\alpha(x,x')$. Therefore $\alpha(x',x) = \alpha(x,x')^{-1}$ for any $(x,x') \in M$.

Take a section $\eta : G \longrightarrow F$, $\tau\eta = 1_G$ and define a map $q : F \longrightarrow A$ by

$$q(x) = \alpha(x, \eta\tau(x)) \ , \quad x \in F \ .$$

For $(x,x') \in M$ one has

$$q l_1(x,x')^{-1} q l_0(x,x') = q(x')^{-1} q(x) = (\alpha(x', \eta\tau(x')))^{-1}\alpha(x, \eta\tau(x)) =$$
$$\alpha(\eta\tau(x'), x')\alpha(x, \eta\tau(x)) \ .$$

On the other hand, since $\alpha(x,x') = \alpha(1, x'x^{-1})$ for all $(x,x') \in M$, on gets $\alpha(\eta\tau(x'), x') = \alpha(1, x'\eta\tau(x')^{-1})$ and $\alpha(x, \eta\tau(x)) = \alpha(1, \eta\tau(x)x^{-1})$.

But $(1, x'\eta\tau(x')^{-1})(1, \eta\tau(x)x^{-1}) = (1, x'x^{-1})$. Therefore, finally we obtain the equality

$$\alpha(x,x') = \alpha(1, x'\eta\tau(x')^{-1})\alpha(1, \eta\tau(x)x^{-1}) = q l_1(x,x')^{-1} q l_0(x,x') \ .$$

$\square$

Proof of Proposition 3.2 Consider the commutative diagram

$$\begin{array}{ccccc} M' & \overset{l'_0}{\underset{l'_1}{\rightrightarrows}} & F' & \xrightarrow{\tau'} & G \\ \overline{\gamma_1} \downarrow \overline{\gamma_2} & & \gamma_1 \downarrow \gamma_2 & & \\ M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G \end{array} \ ,$$

(M, l_0, l_1) and (M', l'_0, l'_1) being simplicial kernels of τ and τ' respectively, $l_i\overline{\gamma_1} = \gamma_1 l'_i$, $l_i\overline{\gamma_2} = \gamma_2 l'_i$, $i = 0, 1$, $\tau\gamma_1 = \tau\gamma_2 = \tau'$.

The pair $(\gamma_i, \overline{\gamma_i})$ induces a homomorphism

$$\mathrm{Der}(M, (A, \mu)) \longrightarrow \mathrm{Der}(M', (A, \mu))$$

given by $(\alpha, r) \mapsto (\alpha\overline{\gamma_i}, r)$, $i = 1,\ 2$.

If $(\alpha', 1) \sim (\alpha, 1)$, i.e. $(\alpha', 1) = (\beta l_0, h)(\alpha, 1)(\beta l_1, h)^{-1}$ with $(\beta, h) \in \mathrm{Der}(F, (A, \mu))$, then

$$\alpha'\overline{\gamma_i}(y) = \beta\gamma_i l'_1(y)^{-1}\ {}^{h}\alpha\overline{\gamma_i}(y)\beta\gamma_i l'_0(y)\ , \quad y \in M'\ .$$

Thus $(\alpha'\overline{\gamma_i}, 1) \sim (\alpha\overline{\gamma_i}, 1)$, $i = 1, 2$, and one gets a natural map

$$\varepsilon_i : \widetilde{Z^1}(M, (A, \mu))/\sim \longrightarrow \widetilde{Z^1}(M', (A, \mu))/\sim$$

induced by the pair $(\gamma_i, \widetilde{\gamma}_i)$ and given by $[(\alpha, 1)] \mapsto [(\alpha\widetilde{\gamma}_i, 1)]$, $i = 1,\ 2$.

It will be shown that $\varepsilon_1 = \varepsilon_2$. By Lemma 3.3 there is a map $q : F \to A$ such that

$$\alpha(y) = ql_1(y)^{-1}ql_0(y)\ , \quad y \in M\ .$$

Take the homomorphism $s : F' \to M$ given by

$$s(x') = (\gamma_1(x'), \gamma_2(x'))\ , \quad x' \in F'\ .$$

It is clear that $(\alpha s, 1) \in \mathrm{Der}(F', (A, \mu))$. Further one has

$$((\alpha s l'_1)^{-1}\alpha\widetilde{\gamma_2}\alpha s l'_0)(x'_0, x'_1) = \alpha s((x')^{-1}\alpha\widetilde{\gamma_2}(x'_0, x'_1)\alpha s(x'_0) =$$
$$\alpha(\gamma_1(x'_1), \gamma_2(x'_1))^{-1}\alpha\widetilde{\gamma_2}(x'_0, x'_1)\alpha(\gamma_1(x'_0), \gamma_2(x'_0)) = q\gamma_1(x'_1)^{-1}q\gamma_2(x'_1) =$$
$$q\gamma_2(x'_1)^{-1}q\gamma_2(x'_0)q\gamma_2(x'_0)^{-1}q\gamma_1(x'_0) = q\gamma_1(x'_1)^{-1}q\gamma_1(x'_0) = \alpha\widetilde{\gamma_1}(x'_0, x'_1)$$

for any $(x'_0, x'_1) \in M'$. Therefore $(\alpha\widetilde{\gamma_1}, 1) \sim (\alpha\widetilde{\gamma_2}, 1)$ with $(\alpha s, 1) \in \mathrm{Der}(F', (A, \mu))$ implying the required equality $\varepsilon_1 = \varepsilon_2$.

Now the proof of the uniqueness is standard. □

It is easy to check that the quotient set $\widetilde{Z^1}(M, (A, \mu))/\sim$ is naturally bijective to $H^2(G, A)$ when A is a G-module viewed as a crossed G-module. That fact motivates the following

Definition 3.4. *Let (A, μ) be a G-partially crossed P-module. The quotient set $\widetilde{Z^1}(M, (A, \mu))/\sim$ will be called the second cohomology of G with coefficients in (A, μ) and denoted by $H^2(G, (A, \mu))$.*

A homomorphism of G-partially crossed P-modules $f : (A, \mu) \to (B, \lambda)$ induces a map of pointed sets

$$f^2 : H^2(G, (A, \mu)) \longrightarrow H^2(G, (B, \lambda))$$

given by $f^2([(\alpha, 1)]) = [(f\alpha, 1)]$. It is easy to see that there is a canonical surjective map $\vartheta : H^2(G, \mathrm{Ker}\,\mu) \to H^2(, (A, \mu))$ given by $[\alpha] \mapsto (\alpha, 1)]$.

Proposition 3.5. *Let (A,μ) be a G-partially crossed P-module. There is an action of G on $H^2(G,(A,\mu))$ such that $Z(G)$ acts trivially. If P acts on $G$and satisfies the compatibility condition (1.4), then there is also an action of P on $H^2(G,(A,\mu))$.*

Proof. Consider the diagram

$$M_G \underset{l_1}{\overset{l_0}{\rightrightarrows}} F_G \xrightarrow{\tau_G} G$$

with F_G the free group generated by G, τ_G is the canonical homomorphism and (M_G, l_0, l_1) is the simplicial kernel of τ_G. There is an action of G on M_G defined as follows:

$${}^g(|g_1|^\varepsilon \cdots |g_n|^\varepsilon) = |{}^g g_1|^\varepsilon \cdots |{}^g g_n|^\varepsilon \ , \quad g, g_1, \ldots, g_n \in G \ ,$$

with $\varepsilon = \pm 1$. This action induces an action of G on M_G by

$${}^g(x,x') = ({}^g x, {}^g x') \ , \quad g \in G \ , \quad (x,x') \in M_G \ .$$

Finally one gets an action of G on $\mathrm{Der}(M_G,(A,\mu))$ given by

$${}^g(\alpha, r) = (\widetilde{\alpha}, {}^g r)$$

with $\widetilde{\alpha}(m) = {}^g\alpha({}^{g^{-1}}m)$, $g \in G$, $m \in M_G$, inducing an action of G on $\widetilde{\mathrm{Der}}(M_G,(A,\mu))$ and on $\widetilde{Z^1}(M_G,(A,\mu))$ too. If $(\alpha,1) \sim (\alpha',1)$ it is easy to see that ${}^g(\alpha,1) \sim {}^g(\alpha',1)$, $g \in G$, defining an action of G on $H^2(G,(A,\mu))$. Since the above defined surjective map ϑ : $H^2(G, \operatorname{Ker}\mu) \to H^2(G,(A,\mu))$ is a G-map and $Z(G)$ acts trivially on $H^2(G,\operatorname{Ker}\mu)$, there follows that $Z(G)$ acts trivially on $H^2(G,(A,\mu))$ too. $\square$

Let (A,μ) be a G-partially crossed P-module. It can be shown easily that there is an action of $H^0(G,P)$ on $H^2(G,\operatorname{Ker}\mu)$ given by ${}^r[\alpha] = [{}^r\alpha]$, $r \in H^0(G,P)$ with $\alpha : M_G \to \operatorname{Ker}\mu$ a crossed homomorphism under the action of G on A such that $\alpha(\Delta) = 1$.

Let

$$1 \longrightarrow (A,1) \xrightarrow{\varphi} (B,\mu) \xrightarrow{\psi} (C,\lambda) \longrightarrow 1 \tag{3.2}$$

be a short exact sequence of G-partially crossed P-modules. If the action of $H^0(G,P)$ on $H^2(G,A)$ is trivial then there is an action of $H^1(G,(A,\mu))$ on $H^2(G,A)$ given by

$${}^{(\alpha,r)}[\gamma] = [{}^r\gamma] \ .$$

We have to show that ${}^r\gamma$ is a crossed homomorphism and this action is correctly defined. Consider the diagram

$$\begin{array}{ccccc} M_G & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F_G & \xrightarrow{\tau_G} & G \\ & & & & \downarrow \alpha \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array} . \tag{3.3}$$

There is a crossed homomorphism $\beta : F_G \to B$ such that $\psi\beta = \alpha\tau_G$. Take the product

$$(\beta l_0, r)(\varphi\gamma, 1)(\beta l_0, r)^{-1} = (\tilde{\gamma}, 1)$$

in the group $\mathrm{Der}(M_G, (B, \mu))$. Then $\tilde{\gamma}(x) = \beta x)^{-1r}\varphi\gamma(x)\beta(x) = {}^r\varphi\gamma(x)$, $x \in M_G$. Therefore ${}^r\gamma : M_G \to A$ is a crossed homomorphism such that ${}^r\gamma(\Delta) = 1$. If $(\alpha', r') \in (\alpha, r)] \in H^1(G, (C, \lambda))$, i.e. $(\alpha, r) \sim (\alpha', r')$, then $\alpha'(x) = c^{-1}\alpha(x)^x c$ and $r' = \lambda(c)^{-1} rt$ with $c \in C$, $t \in H^0(G, P)$. There follows that

$$\varphi({}^{r'}\gamma(x)) = {}^{r'}\varphi\gamma(x) = {}^{\lambda(c)^{-1}rt}\varphi\gamma(x) = {}^{\mu(b)^{-1}rt}\varphi\gamma(x) = b^{-1}\ {}^{rt}\varphi\gamma(x)b =$$
$${}^{rt}\varphi\gamma(x) = \varphi({}^{rt}\gamma(x)) \ , \quad x \in M_G \ , \quad \text{with} \ \ \psi(b) = c \ .$$

Hence $[{}^{r'}\gamma] = [{}^{rt}\gamma] = [{}^r\gamma]$ proving the correctness of the action.

Using diagram (3.3) for the short exact sequence (3.2) one defines as follows a connecting map

$$\delta^1 : H^1(G, (C, \lambda)) \longrightarrow H^2(G, A)$$

which is a crossed homomorphism when $(G, (C, \lambda))$ verifies conditions of Theorem 2.1.

For $[(\alpha, r)] \in H^1(G, (C, \lambda))$ take a crossed homomorphism $\beta : F_G \to B$ such that $\psi\beta = \alpha\tau_G$. Thus there is a crossed homomorphism $\gamma : M_G \to A$ such that $\varphi\gamma = \beta l_1)^{-1}\beta l_0$. It is clear that $\gamma(\Delta) = 1$. Define

$$\delta^1([(\alpha, r)]) = [\gamma] \ .$$

We must show the correctness of δ^1. For another $\beta' : F_G \to B$ with $\psi\beta' = \alpha\tau_G$, one has $\psi\beta' = \psi\beta$ and there is a crossed homomorphism $\sigma : F_G \to A$ such that $\beta' = \beta\psi\sigma$. Then one gets

$$\varphi\gamma' = (\beta' l_1)^{-1}\beta' l_0 = (\beta\varphi\sigma)l_1^{-1}(\beta\varphi\sigma)l_0 = \varphi\sigma l_1^{-1}\beta l_1^{-1}\beta l_0\varphi\sigma l_0 =$$
$$\beta l_1^{-1}\beta l_0\varphi\sigma l_1^{-1}\varphi\sigma l_0 = \varphi(\gamma\sigma l_1^{-1}\sigma l_0) \ .$$

Hence $[\gamma] = [\gamma']$.

If $(\alpha, r) \sim (\alpha', r')$ then

$$\alpha'(y) = c^{-1}\alpha(y)^y c\ ,\quad c \in C\ ,\quad y \in M_G\ ,$$
$$r' = \lambda(c)^{-1} r t\ ,\quad t \in H^0(G, P)\ .$$

Take $\beta' : F_G \to B$ such that $\beta'(x) = b^{-1}\beta(x)\ {}^x b$ with $\psi(b) = c$ and $\psi\beta = \alpha\tau_G$. Then $(\beta' l_1^{-1}\beta' l_0)(y) = \beta'(x_2)^{-1}\beta'(x_1)$, $y = (x_1, x_2) \in M_G$. Hence $\varphi\gamma'(y) = (\beta' l_1^{-1}\beta' l_0)(y) = (b^{-1}\beta(x_2)\ {}^{x_2}b)^{-1} b^{-1}\beta(x_1)\ {}^{x_1}b = {}^{x_2}b^{-1}\beta(x_2)^{-1}\beta(x_1)\ {}^{x_1}b = \beta(x_2)^{-1}\beta(x_1) = \varphi\gamma(y)$. Whence $\gamma' = \gamma$. Therefore the connecting map δ^1 is correctly defined.

For the short exact sequence 3.2 a connecting map $\delta^2 : H^2(G, (C, \lambda)) \to H^3(G, A)$ will be also defined. To this end consider the canonical free simplicial resolution of the group G in the category of groups acting on the abelian group A:

$$\cdots \underset{\to}{\overset{\to}{\vdots}} F_3 \xrightarrow{\tau_3} M_2 \quad \underset{\underset{l_3^2}{\to}}{\overset{\overset{l_0^2}{\to}}{\vdots}} F_2 \xrightarrow{\tau_2} M_1 \underset{\underset{l_2^1}{\to}}{\overset{\overset{l_0^1}{\to}}{\to}} F_1 \xrightarrow{\tau_1} M_0 \quad \underset{l_1^0}{\overset{l_0^0}{\rightrightarrows}} F_0 \xrightarrow{\tau_0} G$$

where $F_0 = F_G$, $F_i = F_{M_{i-1}}$, $i \geq 1$, τ_i is the canonical homomorphism and $(M_i, l_0^i, \cdots, l_{i+1}^i)$ is the simplicial kernel of $(l_0^{i-1}\tau_i, \cdots, l_i^{i-1}\tau_i)$, $i \geq 0$ (see [7]). It will be used the equivalence of functors $H^{n+1}(-, A) \approx L_n \operatorname{Der}(-, A)$, $n \geq 1$, when A is a $Z[G]$-module. There is an action of $\operatorname{Der}(F_0, (C, \lambda))$ on $H^3(G, A)$ defined as follows:

$$^{(\alpha,r)}[f] = [{}^r f]\ ,$$

where $f : F_2 \to A$ is a crossed homomorphism with $\prod_{i=0}^{3}(f l_i^2 \tau_3)^{\varepsilon_i} = 1$, $\varepsilon_i = (-1)^i$, and $(\alpha, r) \in \operatorname{Der}(F_0, (C, \lambda))$. The correctness of this action is proved similarly to the case of a short exact sequence of crossed G-modules (see [8]).

For any G-partially crossed P-module (A, μ) denote by $\operatorname{IDer}(G, (A, \mu))$ a subgroup of $\operatorname{Der}(G, (A, \mu))$ consisting of elements of the form (α, r), $r \in H^0(G, P)$.

If either the aforementioned action of $\operatorname{Der}(F_0, (C, \lambda))$ on $H^3(G, A)$ is trivial or $\operatorname{Der}(F_0, (C, \lambda)) = \operatorname{IDer}(F_0, (C, \lambda))$ and $H^0(G, P)$ acts trivially on $H^2(G, \operatorname{Ker}\lambda)$, then there is a connecting map $\delta^2 : H^2(G, (C, \lambda)) \to H^3(G, A)$ given by

$$\delta^2([(\alpha, r)]) = [\gamma]\ ,\quad (\alpha, 1) \in \widetilde{Z^1}(M_G, (C, \lambda))\ ,$$

where $\varphi\gamma = \overline{\beta}\tau_2$ with $\overline{\beta} = \prod_{(}^{2} \beta l_i^1)^{\varepsilon_i}$, $\varepsilon_i = (-)^i$, and $\psi\beta = \alpha\tau_1$. The correctness of δ^2 is proved similarly to the case of a short exact sequence of crossed G-modules (see [8]).

Theorem 3.6. *Let (3.2) be a short exact sequence of G-partially crossed P-modules satisfying conditions of Theorem 2.1. Then there is an exact cohomology sequence*

$$1 \to H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi 0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\varphi^1}$$
$$H^1(G,(B,\mu)) \xrightarrow{\psi^1} H^1(G,(C,\lambda)) \xrightarrow{\delta^1} H^2(G,A) \xrightarrow{\varphi^2} H^2(G,(B,\mu))$$
$$\xrightarrow{\psi^2} H^2(G,(C,\lambda)) \ ,$$

where φ^0, ψ^0, δ^0, φ^1 and ψ^1 are homomorphisms. If in addition $H^0(G,P)$ acts trivially on $H^2(G,A)$, then δ^1 is a crossed homomorphism under the action of $H^1(G,(C,\lambda))$ on $H^2(G,A)$ induced by the action of P on A. Moreover, if either the action of $\mathrm{Der}(F_0,(C,\lambda))$ on $H^3(G,A)$ is trivial (in particular if P acts trivially on A) or $\mathrm{Der}(F_0,(C,\lambda)) = \mathrm{IDer}(F_0,(C,\lambda))$ and $H^0(G,P)$ acts trivially on $H^2(G,\operatorname{Ker}\lambda)$, then the sequence

$$H^2(G,(B,\mu)) \xrightarrow{\psi^2} H^2(G,(C,\lambda)) \xrightarrow{\delta^2} H^3(G,A)$$

is also exact.

Proof. The exactness of the sequence

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A)$$

is well known [10].

If $c \in H^0(G,C)$ then $\delta^0(c) = [\alpha]$ with $\alpha(x) = \varphi^{-1}(b^{-1}\ {}^x b)$, $x \in G$ and $\psi(b) = c$. There follows that $(\alpha_0, 1) \sim (\varphi\alpha, 1)$ where α_0 is a trivial map, since $\varphi\alpha(x) = b^{-1}\alpha_0\ {}^x b$, $x \in G$, and $\mu(b) \in H^0(G,P)$ because $\mu(b) = \lambda\psi(b)$ and ${}^x\lambda(c) = \lambda(\ {}^x c) = \lambda(c)$, $x \in G$. Therefore $\operatorname{Im}\delta^0 \subset \operatorname{Ker}\varphi^1$.

Let $[\alpha] \in H^1(G,A)$ such that $(\alpha_0,1) \sim (\varphi\alpha,1)$. Then $\varphi\alpha(x) = b^{-1}\ {}^x b$, $x \in G$ and $\mu(b) \in H^0(G,P)$. One has $\psi(b^{-1\,x}b) = \psi\varphi\alpha(x) = 1$. Thus $\psi(b) = \psi({}^x b) = {}^x\psi(b)$, whence $\psi(b) \in H^0(G,C)$. Clearly $\delta^0(\psi(b)) = [\alpha]$. Therefore $\operatorname{Ker}\varphi^1 \subset \operatorname{Im}\delta^0$. Obviously the composite $\psi^1\varphi^1$ is the trivial map.

Let $[(\alpha,r) \in H^1(G,(B,\mu))$ such that $(\alpha_0,1) \sim (\psi\alpha,1)$. Then $\psi\alpha(x) = c^{-1}\ {}^x c$, $c \in C$, and $r = \lambda(c)^{-1}t$, $t \in H^0(G,P)$. Let $\psi(b) = \lambda(c)$ and $r = \mu(b)^{-1}t$. Take $\tilde{\alpha}(x) = b\alpha(x)\ {}^x b^{-1}$, $x \in G$. Since $\psi\tilde{\alpha}(x) = 1$, $x \in G$, one has $\varphi^{-1}\tilde{\alpha} : G \to A$ and $(\alpha,r) \sim (\tilde{\alpha},1)$. Therefore $\varphi^1([\varphi^{-1}\tilde{\alpha}]) = [(\alpha,r)]$.

Let $[(\alpha,r) \in H^1(G,(B,\mu))$. Then $\psi^1([(\alpha,r)]) = [(\psi\alpha,r)]$. Consider the diagram ? and take the crossed homomorphism $\alpha\tau_G : F_G \to B$. Then $\varphi\gamma = (\alpha\tau_G l_1)^{-1}\alpha\tau_G l_0$ and $\delta^1\psi^1([(\alpha,r)]) = [\gamma]$. But $\gamma = \alpha_0$ is the trivial map, since $\alpha\tau_G l_0 = \alpha\tau_G l_1$. Therefore $\operatorname{Im}\psi^1 \subset \operatorname{Ker}\delta^1$.

Let $[(\alpha, r)] \in H^1(G, (C, \lambda))$ such that $\delta^1([(\alpha, r)]) = 1$. If $\beta : F_G \to B$ is a crossed homomorphism such that $\psi\beta = \alpha\tau_G$, then $\delta^1([(\alpha, r)]) = [\gamma]$ with $\varphi\gamma = (\beta l_1)^{-1}\beta l_0$. Thus there is a crossed homomorphism $\eta : F_G \to A$ such that $\gamma = (\eta l_1)^{-1}\eta l_0$. Hence one gets

$$(\beta l_1)^{-1}\beta l_0 = (\varphi\eta l_1)^{-1}\varphi\eta l_0\ , \quad (\varphi\eta^{-1}\beta)l_0 = (\varphi\eta^{-1}\beta)l_1\ ,$$

implying a crossed homomorphism $\overline{\alpha} : G \to B$ such that $(\varphi\eta)^{-1}\beta = \overline{\alpha}\tau_G$. One has $\mu\beta(x) = \lambda\psi\beta(x) = \lambda\alpha\tau_G = r^{\tau_G(x)}r^{-1}$, whence $(\beta, r) \in \mathrm{Der}(F_G, (B, \mu))$ and $(\overline{\alpha}, r) \in \mathrm{Der}(G, (B, \mu))$. Evidently, $\psi^1([(\overline{\alpha}, r)]) = [(\alpha, r)]$.

The rest of the proof repeats with minor modifications the proof of the exactness of the cohomology sequence for a coefficient short exact sequence of crossed G-modules [8]. □

Clearly for a short exact sequence of crossed G-modules we recover the known exact cohomology sequence [6, 8]. Note also that Theorem 3.6 remains true for arbitrary G-partially crossed P-modules but in this case φ^1, ψ^1 and δ^1 are maps of pointed sets.

Now for any partially crossed G-module (A, μ) the second cohomology $H^2(G, A)$ will be described in terms of extensions of groups.

Definition 3.7. *An extension of G by a partially crossed G-module (A, μ) is a pair $E = (1 \to A \xrightarrow{\sigma} X \xrightarrow{\psi} G \longrightarrow 1,\ \gamma)$, where $1 \longrightarrow A \xrightarrow{\sigma} X \xrightarrow{\psi} G \longrightarrow 1$ is a short exact sequence of groups, γ is a section of ψ, that is $\psi\gamma = 1_G$, one has the equality*

$${}^{g}a = \sigma^{-1}(\gamma(g)\sigma(a)\gamma(g)^{-1})$$

for $a \in A$, $g \in G$, and the following additional condition holds:

$$\mathrm{Ker}\ \overline{\psi} \subset Z(A)\ ,$$

$\overline{\psi}$ being the restriction of ψ on the subgroup of X generated by $\gamma(G)$.

Example 3.8. *Let $A \lhd G$ be the semidirect product of A and G. Then one has an exact sequence of groups*

$$1 \longrightarrow A \xrightarrow{\sigma_0} A \lhd G \xrightarrow{\psi_0} G \longrightarrow 1\ ,$$

where $\sigma_0(a) = (a, 1)$, $\psi_0(a, g) = g$. Take the canonical section γ_0 of ψ_0 given $\gamma_0(g) = (1, g)$. It is easy to see that the pair $(1 \longrightarrow A \xrightarrow{\sigma_0} A \lhd G \xrightarrow{\psi_0} G,\ \gamma_0)$ is an extension of G by (A, μ) called trivial.

Definition 3.9. *It will be said that $E = (1 \to A \xrightarrow{\sigma} X \xrightarrow{\psi} G \to 1, \gamma)$ is equivalent to $E' = (1 \to A \xrightarrow{\sigma'} X' \xrightarrow{\psi'} G \to 1),\ \gamma')$ if there exist* *a homomorphism $\vartheta : X \to X'$ and an element $g \in G$ such that the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\sigma} & X & \xrightarrow{\psi} & G & \longrightarrow & 1 \\ & & \downarrow g & & \downarrow \vartheta & & \| & & \\ 1 & \longrightarrow & A & \xrightarrow{\sigma'} & X' & \xrightarrow{\psi'} & G & \longrightarrow & 1 \end{array}$$

is commutative, $g : A \to A$ is the automorphism induced by the action of g on A, and for any element $x \in G$ one has the equality

$$\mu(\vartheta\gamma(x)\gamma'(x)^{-1}) = gxg^{-1}x^{-1} \ .$$

Clearly this relation $\sim$ is reflexive and symmetric. So for the relation $\sim$ to be an equivalence it remains to show the transitivity. Let $E \sim E'$ and $E' \sim E''$. Then the diagram

$$\begin{array}{ccccccccc} E = 1 & \longrightarrow & A & \xrightarrow{\sigma} & X & \overset{\psi}{\rightleftarrows} & G & \longrightarrow & 1 \\ & & \downarrow g & & \downarrow \vartheta & & \| & & \\ E' = 1 & \longrightarrow & A & \xrightarrow{\sigma'} & X' & \overset{\psi'}{\rightleftarrows} & G & \longrightarrow & 1 \\ & & \downarrow h & & \downarrow \kappa & & \| & & \\ E'' = 1 & \longrightarrow & A & \xrightarrow{\sigma''} & X'' & \overset{\psi''}{\rightleftarrows} & G & \longrightarrow & 1 \end{array}$$

is commutative and one has the equalities

$$\mu(\vartheta\gamma(x)\ \gamma'(x)^{-1}) = gxg^{-1}x^{-1} \ ,$$
$$\mu(\kappa\gamma'(x)\gamma''(x)^{-1}) = hxh^{-1}x^{-1} \ .$$

We shall show that $\mu(\kappa\vartheta\gamma(x)\ \gamma''(x)^{-1}) = hgxg^{-1}h^{-1}x^{-1}$. In effect, since $\kappa\vartheta\gamma(x)\gamma''(x)^{-1}(\kappa\gamma'(x)\gamma''(x)^{-1})^{-1} = \kappa\vartheta\gamma(x)\kappa\gamma'(x)^{-1} = \kappa(\vartheta\gamma(x) \cdot$ $\cdot\gamma'(x)^{-1}$, one gets

$$\mu(\kappa\vartheta(x)\gamma''(x)^{-1}) = \mu\kappa(\vartheta\gamma(x)\gamma'(x)^{-1}\mu\kappa\gamma'(x)\ \gamma''(x)^{-1}) =$$
$$= h(gxg^{-1}x^{-1})h^{-1}hxh^{-1}x^{-1} = hgxg^{-1}h^{-1}x^{-1} \ .$$

Therefore the relation $\sim$ is an equivalence. Denote by $E^1(G, A)$ the set of equivalence classes of extensions of G by the partially crossed G-module (A, μ).

Theorem 3.10. *There is a natural bijection*

$$\eta : H^2(G, A) \overset{\approx}{\to} E^1(G, A) \ .$$

Proof. The map η is defined as follows. For $[(\alpha,1)] \in H^2(G,A)$ consider the diagram

$$\begin{tikzcd} M_G \arrow[r, shift left, "l_0"] \arrow[r, shift right, "1"'] \arrow[d, "\alpha"] & F_G \arrow[r, "\tau"] & G \\ A & & \end{tikzcd} ,$$

take the semidirect product $A \lhd F_G$, F_G acting on A via G and introduce an equivalence relation:

$(a,x) \sim (a',x') \overset{\rho}{\Longleftrightarrow} \tau(x) = \tau(x')$ and $a = a' \cdot \alpha(x,x')$. In fact the equivalence ρ is a congruence, since if $(a,x) \sim (a',x')$ and $(b,y) \sim (b',y')$, one has

$$a^x b = a'\alpha(x,x') \ {}^x b' \ {}^x\alpha(y,y') = a' \ {}^x b'\alpha(xy, x'y') \ .$$

Denote $C = A \lhd F_G/\rho$. One gets an exact sequence of groups

$$1 \longrightarrow A \xrightarrow{\sigma} C \xrightarrow{\psi} G \longrightarrow 1 \ ,$$

where $\sigma(a) = [(a,1)]$, $\psi([(a,x)]) = \tau(x)$ and the following diagram

$$\begin{tikzcd} M_G \arrow[r, shift left, "l_0"] \arrow[r, shift right, "l_1"'] \arrow[d, "\alpha"] & F_G \arrow[r, "\tau"] \arrow[d, "\delta"] & G \arrow[d, equal] \\ A \arrow[r, "\sigma"] & C \arrow[r, "\psi"] & G \end{tikzcd}$$

is commutative, $\delta(x) = [(1,x)]$, $\sigma\alpha = \delta l_1 \delta l_0^{-1}$.

Take a section $\gamma : G \to C$ given by $\gamma(g) = [(1, |g|)$, $g \in G$. It is easy to see that $\overline{\gamma(G)} = \operatorname{Im}\delta$. Therefore $\operatorname{Ker}(\psi \mid_{\overline{\gamma(G)}}) = \delta(\operatorname{Ker}\tau)$. The equality $\sigma\alpha(1,x) = \delta(x)$, $x \in \operatorname{Ker}\tau$, implies $\operatorname{Ker}(\psi \mid_{\overline{\gamma(G)}}) \subset Z(A)$ and there follows that so constructed pair

$$E = (1 \longrightarrow A \xrightarrow{\sigma} C \xrightarrow{\psi} G \longrightarrow 1, \ \gamma)$$

is an extension of G by (A,μ).

Define $\eta(([(\alpha,1)]) = [E]$. We have to show the correctness of η. Let $(\alpha,1) \sim (\alpha',1)$, that means

$$\alpha'(x) = \beta l_1(x)^{-1h}\alpha(x)\beta l_0(x) \ , \ \ x \in M_G \ ,$$

for some $(\beta,h) \in \operatorname{Der}(F_G,(A,\mu))$ (see diagram 3.3) and let $E' = (1 \longrightarrow A \xrightarrow{\sigma'} C' \xrightarrow{\psi'} G \longrightarrow 1, \ \gamma')$ be the extension of G by (A,μ) corresponding to $(\alpha',1)$.

Define a map $\nu : A \lhd F_G \longrightarrow A \lhd G$ given by $\nu(a,x) = ({}^h a\beta(x), x)$. So defined ν provides a homomorphism, since $\nu((a,x)(b,y)) = \nu(a\,{}^x b, xy)$ $= ({}^h(a \ {}^x b)\beta(xy), xy)$ and $\nu(a,x)\nu(b,y) = ({}^h a\beta(x), x)({}^h b\beta(y), y) = ({}^h a\beta(x) \ {}^x({}^h b\beta(y)), xy) = ({}^h a \ {}^{hx} b\beta(x) \ {}^x\beta(y), xy) = ({}^h(a \ {}^x b)\beta(xy), xy)$.

The homomorphism ν induces a homomorphism $\nu' : A \lhd F_G/\rho \to A \lhd F_G/\rho'$ given by $\nu'([(a,x)] = [\nu(a,x)]$. In effect, let $(a,x) \overset{\rho}{\sim} (a',x')$, that means $a = a'\alpha(x,x')$ and $\tau(x) = \tau(x')$. We have to show that ${}^h a\beta(x) \overset{\rho'}{\sim} {}^h a'\beta(x')$, where $\alpha'(x,x') = \beta(x')^{-1 h}\alpha(x,x')\beta(x)$.

One has

$$ {}^h a\beta(x) = {}^h a' \ \ {}^h\alpha(x,x')\beta(x) = {}^h a'\beta(x')\alpha'(x,x') \ . $$

Thus ν' is a correctly defined homomorphism and the diagram

$$\begin{array}{ccc} A \lhd F_G & \longrightarrow & A \lhd F_G/\rho = C \\ \downarrow \nu & & \downarrow \nu' \\ A \lhd F_G & \longrightarrow & A \lhd F_G/\rho' = C' \end{array}$$

is commutative.

Now consider the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \overset{\sigma}{\to} & C & \overset{\psi}{\to} & G & \longrightarrow & 1 \\ & & \downarrow h & & \downarrow \nu' & & || & & \\ 1 & \longrightarrow & A & \overset{\sigma'}{\to} & C' & \overset{\psi'}{\to} & G & \longrightarrow & 1 \end{array}$$

with sections $\gamma : G \to C$, $\gamma' : G \to C'$ defined as above. Clearly $\sigma' h = \nu'\sigma$ and $\nu'\gamma(x)\gamma(x)^{-1} = \nu'([(1,|x|))\delta'(x) = [(\beta(|x|),|x|)][(1,|x|)]^{-1} = [(\beta|x|),1)]$. But $\mu\beta(|x|) = hxh^{-1}x^{-1}$. There follows that $[E] = [E']$.

Conversely, define a map $\eta' : E^1(G,A) \to H^2(G,A)$ as follows. Let $[E] \in E^1(G,A)$ and $E = (1 \to A \overset{\sigma}{\to} C \overset{\psi}{\to} G \to 1,\ \gamma)$. Then one gets a commutative diagram

$$\begin{array}{ccccc} M_G & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F_G & \overset{\tau}{\to} & G \\ \downarrow \alpha & & \downarrow \delta & & || \\ A & \overset{\sigma}{\to} & C & \overset{\psi}{\to} & G \end{array}$$

with δ induced by γ and $\sigma\alpha = \delta l_1 \delta l_0^{-1}$. Clearly α is a crossed homomorphism such that $\alpha(\Delta) = 1$ and $\operatorname{Im}\alpha \subset Z(A)$. Define $\eta'([E]) = [(\alpha,1)]$.

We have to show the correctness again. If δ' is another homomorphism such that $\beta\delta' = \tau$, then $\delta(y)\ \delta'(y)^{-1} \in Z(A)$, $y \in F_G$. Thus $\delta\delta'^{-1}$ induces a crossed homomorphism $\beta : F_G \to Z(A)$ and it is obvious that $(\alpha',1) = (\beta l_0,1)(\alpha,1)(\beta l_1,1)^{-1}$ with $(\beta,1) \in \operatorname{Der}(F_G,A)$.

Assume now that E is equivalent to $E' = (1 \to A \overset{\sigma'}{\to} C' \overset{\psi'}{\to} G \to 1,$ $\gamma')$ by the pair $(h : A \to A,\ \vartheta : C \to C')$ implying the equality $\mu(\vartheta\gamma(x)\gamma'(x)^{-1}) = hxh^{-1}x^{-1}$, $x \in G$.

By using the equalities $\psi\delta = \psi'\delta' = \tau$, $\sigma\alpha = \delta l_1 \delta l_0^{-1}$, $\sigma'\alpha' = \delta' l_1 \delta' l_0^{-1}$, $\sigma' h = \vartheta\sigma$, one gets $\vartheta\delta = \sigma'\beta\delta'$, $\vartheta\delta l_0 = \sigma'\beta l_0 \delta' l_0$, $\vartheta\delta l_1 = \sigma'\beta l_1 \delta' l_1$, where $\beta : F_G \longrightarrow A$ is a crossed homomorphism induced by $\vartheta\delta\delta'^{-1}$.

Clearly (β, h) is an element of $\mathrm{Der}(F_G, A)$. Further, $\sigma\alpha\delta l_0 = \delta l_1$, so one has $\vartheta\sigma\alpha\vartheta\delta l_0 = \vartheta\delta l_1$ and the following equalities

$$\vartheta\sigma\alpha\sigma'\beta l_0\delta' l_0 = \sigma'\beta l_1\delta' l_1 \ ,$$
$$\vartheta\sigma\alpha\sigma'\beta l_0 = \sigma'\beta l_1\delta' l_1\delta' l_0^{-1} \ ,$$
$$\sigma' h\alpha\sigma'\beta l_0 = \sigma'\beta l_1\sigma'\alpha' \ .$$

Finally $\alpha' = \beta l_1^{-1} h\alpha\beta l_0$.

Therefore $(\alpha', 1)$ is equivalent to $(\alpha, 1)$. It is easily checked that $\eta\eta'$ and $\eta'\eta$ are identity maps. □

4. Higher non-abelian cohomology

Let (A, μ) be a G-partially crossed P-module. Take the free cotriple resolution $F_*(G)$ of the group G:

$$\cdots \begin{smallmatrix}\rightarrow\\ \vdots\\ \rightarrow\end{smallmatrix} F_{n+1} \ \underset{\partial^{n+1}_{n+1}}{\overset{\partial^{n+1}_0}{\begin{smallmatrix}\longrightarrow\\ \vdots\\ \longrightarrow\end{smallmatrix}}} F_n \ \underset{\partial^{n}_{n}}{\overset{\partial^{n}_0}{\begin{smallmatrix}\rightarrow\\ \vdots\\ \rightarrow\end{smallmatrix}}} \cdots \begin{smallmatrix}\rightarrow\\ \vdots\\ \rightarrow\end{smallmatrix} F_2 \ \underset{\partial^2_2}{\overset{\partial^2_0}{\begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix}}} F_1 \ \underset{\partial^1_1}{\overset{\partial^1_0}{\rightrightarrows}} F_0 \longrightarrow G \tag{4.1}$$

with $F_n = F^{n+1}(A)$, $n \geq 0$, $F_0 = F(A)$ is the free group generated by A and $F^{n+1}(A) = F(F^n(A))$, $\partial^n_i = F^i\tau F^{n-i}$, $s^n_i = F^i\delta F^{n-i}$, where $\delta : F(A) \longrightarrow F^2(A)$ is induced by the canonical inclusion $A \to F(A)$. Clearly (A, μ) can be viewed as a F_n-partially crossed P-module induced by $\tau\partial^1_0\partial^2_0 \cdots \partial^n_0$. Therefore the group $\mathrm{Der}(F_n, (A, \mu))$, $n \geq 0$, is defined. Denote by $\widetilde{Z}^1(F_n, (A, \mu))$ the subset of $\mathrm{Der}(F_n, (A, \mu))$ consisting of all elements of the form $(\alpha, 1)$ for n odd and of the form (α, r) for n even satisfying the condition

$$\prod_{j=0}^{n+1} (\alpha\partial^{n+1}_j)^{\varepsilon_j} = 1 \ , \quad \varepsilon_j = (-1)^j \ .$$

Since $\mu\alpha(x) = 1$ for any $x \in F_n$ and for n odd, in this case we have $\alpha(F_n) \subset Z(A)$, $n \geq 1$. In the set $\widetilde{Z}^1(F_n, (A, \mu))$, $n \geq 1$, a relation $\sim$ is introduced as follows:

$(\alpha', 1) \sim (\alpha, 1)$ for n odd and $(\alpha', r') \sim (\alpha, r)$ for n even if there is an element $(\beta, h) \in \mathrm{Der}(F_{n-1}, (A, \mu))$ such that

$$\alpha'(x) = {}^h\alpha(x) \prod_{i=0}^{n} (\beta\partial^n_i(x))^{\varepsilon_i} \ , \quad \varepsilon_i = (-1)^i \ , \quad x \in F_n \ ,$$

and $r' = r$ for n even.

The homomorphism $\tau\partial^1_{i_n}\partial^2_{i_{n-1}}\cdots\partial^{n-1}_{i_2}\partial^n_{i_1}$ does not depend of the sequence $(i_1, i_2, \cdots, i_{n-1}, i_n)$ implying

$$\beta\partial^n_j(x)\ (\beta\partial^n_l(x))^{-1} = (\beta\partial^n_l(x))^{-1}(\beta\partial^n_j(x) \in \operatorname{Ker}\mu\ , \quad x \in F_n\ ,$$

for $0 \le j,\ l \le n$. There follows that the product $\prod_{i=0}^n(\beta\partial^n_i(x))^{\varepsilon_i}$, $\varepsilon_i = (-1)^i$, does not depend on the order of the factors. Obviously the aforedefined relation is an equivalence.

Definition 4.1. *The higher non-abelian cohomology of a group G with coefficients in a G-partially crossed P-module (A,μ) is given by*

$$H^{n+1}(G,(A,\mu)) = \widetilde{Z^1}(F_n,(A,\mu))/\sim\ , \quad n \ge 1\ .$$

It is easily checked that for $n = 1$ we recover the second cohomology set of G with coefficients in (A,μ).

The map $H^{n+1}(G,\operatorname{Ker}\mu) \to H^{n+1}(G,(A,\mu))$ given by $[f] \mapsto [(f,1)]$ is surjective and is bijective if $\mu : A \to P$ is the trivial homomorphism (in this case A is abelian).

In order to express in terms of derived functors of the group of derivations with respect to the contravariant variable cohomotopy pointed sets of some cosimplicial groups will be introduced.

Let

$$G_*:\quad G_0 \rightrightarrows G_1 \overset{\to}{\underset{\to}{\to}} G_2 \ \vdots\ \cdots\ \vdots\ G_n\ \vdots\ G_{n+1}\ \vdots\ \cdots$$

be a cosimplicial group. Clearly $\partial^n_j\partial^{n-1}_i = \partial^n_i\partial^{n-1}_{j-1}$, $i < j$.

Assume that G_* satisfies the following condition:

(a) Denote by L_{n+1} the subgroup of G_{n+1} generated by $\bigcup_{i=0}^{n+1}\partial^n_i(G_n)$. Then for any element $x \in G_n$, $n \ge 0$, the product $\partial^n_i(x)\ \partial^n_j(x)^{-1}$, $0 \le i$, $j \le n$, commutes with every element of L_{n+1}.

In particular it follows that one has the equality $\partial^n_i(x)\ \partial^n_j(x)^{-1} = \partial^n_j(x)^{-1}\ \partial^n_i(x)$, $0 \le i,\ j \le n$, $n \ge 0$.

Under this condition the cosimplicial group G_* induces a group chain complex

$$1 \longrightarrow G_0 \xrightarrow{d_0} G_1 \xrightarrow{d_1} G_2 \xrightarrow{d_2} \cdots G_n \xrightarrow{d_n} G_{n+1} \xrightarrow{d_{n+1}} \cdots$$

with $d_n(x) = \prod_{i=0}^{n+1}\partial^n_i(x)^{\varepsilon_i}$, $\varepsilon_i = (-1)^i$, $n \ge 0$. It is easily checked that the maps d_n, $n \ge 0$, are homomorphisms and $d_n d_{n-1} = 0$, $n \ge 1$.

Definition 4.2. *The right quotient sets* $\operatorname{Ker} d_n/\operatorname{Im} d_{n-1}$ *will be called cohomotopy sets $\pi_n(G_*)$, $n \ge 0$, of the cosimplicial group G_*.*

It is obvious that for abelian cosimplicial groups we recover the well known homology groups.

Proposition 4.3. *Let*

$$1 \longrightarrow G'_* \longrightarrow G_* \longrightarrow G''_* \longrightarrow 1$$

be a short exact sequence of cosimplicial groups satisfying condition (a). Then there is a long exact sequence of pointed cohomotopy sets

$$1 \longrightarrow \pi_0(G'_*) \longrightarrow \pi_0(G_*) \longrightarrow \pi_0(G''_*) \longrightarrow \pi_1(G'_*) \longrightarrow \cdots \longrightarrow$$
$$\pi_{n-1}(G''_*) \longrightarrow \pi_n(G'_*) \longrightarrow \pi_n(G_*) \longrightarrow \pi_n(G''_*) \longrightarrow \pi_{n+1}(G'_*) \longrightarrow \cdots$$

Proof. Straightforward. □

This definition of cohomotopy pointed sets defined for cosimplicial groups satisfying the aforementioned condition allows us to define cotriple right derived functors of some group valued contravariant functors.

Let $\mathbf{A}$ be an arbitrary category and $\mathbf{F} = (\mathbf{F}, \tau, \delta)$ be a cotriple in the category $\mathbf{A}$. For any object $A \in \operatorname{ob} \mathbf{A}$ take its cotriple resolution:

$$F_*(A) \xrightarrow{\tau} A \ .$$

similarly to the case of groups considered above.

Let $T : \mathbf{A} \to \mathbf{Gr}$ be a contravariant functor to the category of groups satisfying the following condition:

(b) the product $T\partial_i^{n+1}(x)\ T\partial_j^{n+1}(x)^{-1}$, $0 \leq i, j \leq n+1$, $n \geq 0$, commutes with every element of the subgroup of $TF_{n+1}(A)$ generated by $\bigcup_{i=0}^{n+2} \operatorname{Im} T\partial_i^{n+1}$, $n \geq 0$.

Definition 4.4. *The right derived functors of the contravariant functor T with respect to the cotriple $\mathbf{F}$ are the pointed sets*

$$R^n_{\mathbf{F}}T(A) = \pi_n(TF_*(A)) \ , \quad n \geq 0 \ , \quad A \in \operatorname{ob} \mathbf{A} \ .$$

If $f : A \to A'$ is a morphism of the category $\mathbf{A}$, then one gets a morphism $TF_*(f) : TF_*(A') \to TF_*(A)$ inducing maps of pointed sets $R^n_{\mathbf{F}}T(f) := \pi_n TF_*(f) : R^n_{\mathbf{F}}T(A') \to R^n_{\mathbf{F}}T(A)$, $n \geq 0$.

Remark 4.5. *Similarly are defined the right derived functors $R^n_{\mathbf{L}}T$ with respect to a triple $\mathbf{L}$ in the category $\mathbf{A}$ for group valued covariant functors T satisfying the same condition.*

As noted above the main application of these derived functors will be their close relationship with non-abelian cohomology of groups.

Let (A, μ) be a G-partially crossed P-module and consider the free cotriple resolution $F_*(G)$ of the group G (see (4.1)). In general the cosimplicial group $\operatorname{Der} F_*(G)$ does not verify condition (a), in other words the contravariant functor $\operatorname{Der}(-, (A, \mu))$ from the category of groups acting on (A, μ) to the category of groups $\mathbf{Gr}$ does not verify condition (b). Below it will be shown that for a wide class of coefficients (A, μ) condition (b) holds for the functor $\operatorname{Der}(-, (A, \mu))$.

One gets a sequence of groups and maps

$$\begin{aligned} 1 \longrightarrow \operatorname{Der}(F_0,(A,\mu)) \xrightarrow{d_0} \operatorname{Der}(F_1,(A,\mu)) \xrightarrow{d_1} \cdots \longrightarrow \\ \longrightarrow \operatorname{Der}(F_n,(A,\mu) \xrightarrow{d_n} \operatorname{Der}(F_{n+1},(A,\mu)) \xrightarrow{d_{n+1}} \cdots \end{aligned} \tag{4.2}$$

with $d_n((\alpha,r))(x) = (\gamma,s)$, where $\gamma = \prod_{i=0}^{n+1}(\alpha\partial_i^{n+1})^{\varepsilon_i}$, $\varepsilon_i = (-1)^i$, $(\alpha,r) \in \operatorname{Der}(F_n,(A,\mu))$, $x \in F_{n+1}$, $s = 1$ for n even and $s = r$ for n odd.

Introduce in $\operatorname{Ker} d_n$, $n \geq 0$, an equivalence by $(\alpha',r') \sim (\alpha,r) \Longleftrightarrow \exists\ (\beta,h) \in \operatorname{Der}(F_{n-1},(A,\mu))$ such that $\alpha'(x) = {}^h\alpha(x) \prod_{i=0}^{n}\beta\partial_i^n(x)^{\varepsilon_i}$, $\varepsilon_i = (-1)^i$, $r' = r$. Clearly the quotient sets coincide with $H^{n+1}(G,(A,\mu))$, $n \geq 1$. It is easily checked that one has $\operatorname{Ker} d_0 \approx \operatorname{Der}(G,(A,\mu))$. Moreover, the composite $d_n d_{n-1}$, $n \geq 1$, is the trivial map and the maps d_n are homomorphisms for n odd.

Proposition 4.6. *Let (A,μ) be a G-partially crossed P-module such that P acts trivially on $\operatorname{Ker}\mu$. Then the maps d_n of the sequence (4.2) are homomorphisms for all $n \geq 0$. Moreover, in this case $\operatorname{Im} d_n$ belongs to the center of $\operatorname{Der}(F_{n+1},(A,\mu))$ for n even and $\operatorname{Ker} d_n$ belongs to the center of $\operatorname{Der}(F_n,(A,\mu))$ for n odd.*

Proof. Easily to check using the equalities $\alpha\partial_i^{n+1}(x)\ \alpha\partial_j^{n+1}(x)^{-1} = \alpha\partial_j^{n+1}(x)^{-1}\ \alpha\partial_i^{n+1}(x)$, $0 \leq i, j \leq n+1$, for $(\alpha,r) \in \operatorname{Der}(F_n,(A,\mu))$ and the fact that in this case any element of the form $(\alpha,1) \in \operatorname{Der}(F_n,(A,\mu))$ belongs to the center. $\square$

Theorem 4.7. *Let (A,μ) be a G-partially crossed P-module with P acting trivially on $\operatorname{Ker}\mu$ and let $\mathbf{F}$ be the free cotriple in the category of groups acting on (A,μ).*

(i) *One has isomorphisms*

$$\begin{aligned} R^0_{\mathbf{F}} \operatorname{Der}(G,(A,\mu)) &\approx \operatorname{Der}(G,(A,\mu))\ , \\ R^n_{\mathbf{F}} \operatorname{Der}(G,(A,\mu)) &\approx H^{n+1}(G,(A,\mu))\ , \quad n \geq 1\ , \end{aligned}$$

and $H^{n+1}(G,(A,\mu))$ is an abelian group for n odd.

(ii) *Any short exact sequence of G-partially crossed P-modules*

$$1 \longrightarrow (A',\mu') \xrightarrow{\varphi} (A,\mu) \xrightarrow{\psi} (A'',\mu'') \longrightarrow 1$$

with P acting trivially on $\operatorname{Ker}\mu'$, $\operatorname{Ker}\mu$ *and* $Ker\mu''$, *induces a long exact cohomology sequence*

$$\begin{aligned}
&1 \longrightarrow \operatorname{Der}(G,(A',\mu')) \longrightarrow \operatorname{Der}(G,(A,\mu)) \longrightarrow \operatorname{Der}(G,(A'',\mu'')) \longrightarrow \\
&\longrightarrow H^2(G,(A',\mu')) \longrightarrow H^2(G,(A,\mu)) \longrightarrow H^2(G,(A'',\mu'')) \longrightarrow \\
&\longrightarrow H^3(G,(A',\mu')) \longrightarrow \cdots \longrightarrow H^{n-1}(G,(A'',\mu'')) \longrightarrow \\
&\longrightarrow H^n(G,(A',\mu')) \longrightarrow H^n(G,(A,\mu)) \longrightarrow H^n(G,(A'',\mu'')) \longrightarrow \\
&\longrightarrow H^{n+1}(G,(A',\mu')) \longrightarrow \cdots .
\end{aligned}$$

Proof. Since the functor $\operatorname{Der}(-,(A,\mu))$ satisfies condition (b) with respect to the free cotriple $\mathbf{F}$, the right derived functors of $\operatorname{Der}(-,(A,\mu))$ are well defined, the sequence (4.2) became a complex of non-abelian groups and the statement of (i) follows from Proposition 4.6.

The short exact coefficient sequence of (ii) induces a short exact sequence of cosimplicial groups

$$1 \longrightarrow \operatorname{Der}(F_*(G),(A',\mu')) \longrightarrow \operatorname{Der}(F_*(G),(A,\mu)) \longrightarrow \operatorname{Der}(F_*(G),(A'',\mu'')) \to 1$$

satisfying condition (a). It remains to apply Proposition 4.3 to get the required long exact cohomology sequence. □

The definition of non-abelian cohomology with coefficients in G-partially crossed P-modules allows as to introduce the definition of higher non-abelian cohomology of a group G with coefficients in any G-group. It can be done as follows.

Let A be an arbitrary G-group, that means the group G acts on the left on the group A. Take the quotient group $P = A/Z(A)$. Define an action of P on A and an action of G on P as follows:

$$\begin{gathered}
{}^{[a']}a = {}^{a'}a\ , \quad [a'] \in P\ , \quad a,\ a' \in A\ . \\
{}^{g}[a] = [{}^{g}a]\ , \quad g \in G\ , \quad a \in A\ .
\end{gathered}$$

Let $\mu_A : A \longrightarrow P$ be the canonical homomorphism. It is easily checked that the pair (A,μ_A) is a G-crossed P-module under the afore-defined actions.

Definition 4.8. *The n-th cohomology $H^n(G,A)$, $n \geq 0$, of the group G with coefficients in a G-group A is given by*

$$H^n(G,A) = H^n(G,(A,\mu_A))\ , \quad n \geq 0\ .$$

For $n = 1$ this cohomology differs from the first pointed set cohomology defined in [10].

Let

$$1 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 1$$

be a central extension of G-groups. Then ψ induces an isomorphism of G-groups $\vartheta : B/Z(B) \overset{\approx}{\to} C/\psi(Z(B))$ and one gets a short exact sequence of G-crossed P-modules with $P = B/Z(B)$

$$1 \longrightarrow (A,1) \overset{\varphi}{\to} (B,\mu_B) \overset{\psi}{\to} (C,\overline{\mu}_C) \longrightarrow 1 \ ,$$

where $\overline{\mu}_C$ is the composite of the canonical map $C \to C/\psi(Z(B))$ and the isomorphism ϑ^{-1}.

Corollary 4.9. *Any central extension of G-groups*

$$1 \longrightarrow A \overset{\varphi}{\to} B \overset{\psi}{\to} C \longrightarrow 1$$

induces a long exact cohomology sequence

$$\begin{gathered} 1 \longrightarrow H^0(G,A) \overset{\varphi^0}{\longrightarrow} H^0(G,B) \overset{\psi^0}{\longrightarrow} H^0(G,C) \overset{\delta^0}{\longrightarrow} H^1(G,A) \overset{\varphi^1}{\longrightarrow} \\ H^1(G,B) \overset{\psi^1}{\longrightarrow} H^1(G,(C,\overline{\mu}_C)) \overset{\delta^1}{\longrightarrow} H^2(G,A) \overset{\varphi^2}{\longrightarrow} H^2(G,B) \overset{\psi^2}{\longrightarrow} \\ H^2(G,(C,\overline{\mu}_C)) \overset{\delta^2}{\longrightarrow} H^3(G,(G,A) \overset{\varphi^3}{\longrightarrow} \cdots \longrightarrow H^{n-1}(G,(C,\overline{\mu}_C)) \overset{\delta^{n-1}}{\longrightarrow} \\ H^n(G,A) \overset{\varphi^n}{\longrightarrow} H^n(G,B) \overset{\psi^n}{\longrightarrow} H^n(G,(C,\overline{\mu}_C)) \overset{\delta^n}{\longrightarrow} H^n(G,A) \longrightarrow \cdots \ . \end{gathered}$$

Proof. Clearly in the induced short exact sequence of G-crossed P-modules the group P acts trivially on $\operatorname{Ker}\mu_B$ and $\operatorname{Ker}\overline{\mu}_C$. So we can apply Theorem 4.7 giving together with Theorem 3.6 the required long exact cohomology sequence. □

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A. Razmadze Matematical Institute Georgian Academy of Sciences
E-mail address: `hvedri@rmi.acnet.ge`
URL: `http://www.rmi.acnet.ge/~hvedri/`