# $KK$-THEORY AS THE $K$-THEORY OF $C^*$-CATEGORIES

KANDELAKI TAMAZ

ABSTRACT. For separable $C^*$-algebra $A$ and $\sigma$-unital $C^*$-algebra $B$ we have constructed $C^*$-category $\mathrm{Rep}(A, B)$ and an isomorphism

$$\kappa : K^{i+1}(\mathrm{Rep}(A, B) \to KK^i(A, B), \quad i \in Z_2$$

where left hand side is Karoubi's topological $K$-groups and right hand side is Kassparov's bivariant $K$-groups.

## INTRODUCTION

The purpose of this article is to study a possibility of calculation of Kasparov $KK$-theory by $K$-theory. Some partial results are known in this direction. For example, Pascke's result for $K$ -homology of nuclear $C^*$-algebras [11], and its generalization for Kasparov $KK$-groups when first argument is nuclear [5], [15], Higson's modification of Pascke's result for $K$-homology of separable $C^*$-algebras [4], and Kunnet type theorem results for $KK$-theory [13]. In all this situations algebras are trivially graded.

Let us present briefly the idea of this paper. The main objects of our studying is additive $C^*$-categories $Rep(A, B)$ and $\mathrm{Rep}(A, B)$, where $A$ and $B$ are trivially graded $C^*$-algebras with fixed compact group actions. In the first category Objects are equivariant $A, B$-bimodules and morphisms are invariant $B$-homomorphisms which commute, up to the ideal of compact homomorphisms. After definition of the first category, we define category $\mathrm{Rep}(A, B)$ as the universal pseudoabelian $C^*$-category of $Rep(A, B)$. (The notation 'universal pseudoabelian' is slightly different from the Karoubi's analogue definition [6], [7]). After isomorphic modification of Karoubi's $K$-theory of a Banach category for $C^*$-categories , we study properties of $K$-groups of $\mathrm{Rep}(A, B)$. Then we apply this results to prove our main result, that $K$-groups of $\mathrm{Rep}(A, B)$ is an essentially isomorphic to Kasparov's equivariant $KK$-groups up the shift of dimension, when $A$ is a separable $C^*$-algebra and $B$ is a $\sigma$-unital $C^*$-algebra with fixed compact group actions.

The work was partially supported by Grant No GM1-115 of the U.S. CRDF, INTAS Georgia Grant No 213.

This article is organized as follows. In section 1 we review the basic definitions and properties of $C^*$-categories [2]. We have given constructions of a universal pseudoabelian $C^*$-category of additive $C^*$-category and characterization of a cofinal subcategory of $\mathcal{H}(B \otimes C^{(1,0)})$ that is used in the next sections. In the section 2 we review Karoubi's some result on $K$-theory of Banach categories, adapted specially for $C^*$-categories[6],[7]. In the section 3 we have given some remarks on the definition of $KK$-groups in the form that is used sequel. Such as characterization of $KK$-groups in case when algebras are trivially graded [1],[8],[14]. In the section 4 we prove our main theorem.

Note that our main result shows that the category $\mathrm{Rep}(A, B)$ is interesting object to be studied from various points of view (particularly, to study algebraic $K$-theory, cyclic homology and connections with $K$-theory, i.e. connections with $KK$-theory). Author intends to give an account of this in subsequent papers.

## 1. Remarks on the C*-categories

In this section we recall definition of a $C^*$-category, its main properties and examples, which are used in the next sections.

### 1.1. Definition and some properties of a C*-category.

By $\mathbb{C}$ is denoted field of complex numbers.

Definition 1. A $\mathbb{C}$-linear category $A$ is called a $C^*$-category if :

a) $\hom(a, b)$ is a complex Banach space, the composition of morphisms is bilinear and $\|fg\| \leq \|f\| \cdot \|g\|$ for each compassable morphisms $f$ and $g$;

b) There is an involutive antilinear contravariant endofunctor $* : A \to A$, which preserves objects.

c) $\|f\|^2 = \|f^* f\|$, for each morphism $f$ where $f^* = *(f)$;

d) The morphism $f^* f$ is a positive element of $C^*$-algebra $\hom(a, a)$ for each $f \in \hom(a, b)$.

Example 2. 1) The category with Hilbert spaces as objects and all bounded linear maps as morphisms is a $C^*$-category, which will be denoted by $\mathcal{H}$.

2) Let $B$ be a $C^*$-algebra. The category with right Hilbert $B$-modules as objects and all bounded $B$-homomorphisms, which have adjoint, as morphisms is a $C^*$-category. We denote it by $\mathcal{H}(B)$. If $E$ and $F$ are modules from $\mathcal{H}(B)$ then $\mathcal{L}(E, F)$ or $\hom(E, F)$ denotes space of morphisms from $E$ to $F$, and $\mathcal{K}(E, F)$ denotes the ideal of compact $B$-homomorphisms from $E$ to $F$.

3) Unital $C^*$-algebra is $C^*$-category with one object and elements themselves as morphisms.

DEFINITION 3. Let $A$ and $B$ be $C^*$-categories. A functor $\mathcal{F}: A \to B$ is said to be $*$-functor if

a) $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$;
b) $\mathcal{F}(\lambda f) = \lambda \mathcal{F}(f)$;
c) $\mathcal{F}(f^*) = \mathcal{F}(f)^*$.

$*$-functors between $C^*$-categories, like $*$-homomorphisms of $C^*$-algebras, are norm-decreasing. For the following theorem we refer to [2].

THEOREM 4. *Every $C^*$-category $A$ may be realized as a concrete $C^*$-category, i.e. there is a faitfull $*$-embedding $\mathcal{F}: A \to \mathcal{H}$*

Let $A$ be a $C^*$-category and $I \subset \mathrm{Hom}A$ a subset. Put $\hom(a,b)_I = \hom(a,b) \cap I$. Then $I$ is called a *left ideal* if $\hom(a,b)_I$ is linear subspace of $\hom(a,b)$ and $f \in \hom(a,b)_I$, $\quad g \in \hom(b,c)$ imply $gf \in \hom(a,c)_I$. A right ideal is defined similarly. $I$ is two-sided ideal if it is both left and right ideal. An ideal $I$ is closed if $\hom(a,b)_I$ is closed in $\hom(a,b)$ for each pair of objects.

$I$ determines an equivalence relation on the morphisms of $A$: $f \sim g$, if $f - g \in I$. If $I = I^*$ is an ideal of $A$, the set of equivalence classes $A/I$ can be made into a $C^*$-category in a unique way by requiring than the canonical map $f \mapsto \hat{f}$ gives rise to a $*$-functor $A \to A/I$. $A/I$ can be made into a normed $*$-category, by defining $\|\hat{f}\| = \sup_{g \in \hat{f}} \|g\|$. Arguing as for $C^*$-algebras, one can show

PROPOSITION 5. *Let $A$ be a $C^*$-category and $I$ a be closed, two-sided ideal of $A$. Then $I = I^*$ and $A/I$ is a $C^*$-category.*

EXAMPLE 6. From the example 2 it follows that there exists $C^*$-category $\mathrm{Cal}(B) = \mathcal{H}(B)/\mathcal{K}(B)$ which sometimes will be called Calkin $C^*$-category over $B$.

A word about $Z_2$-graded $C^*$-category. Let $A$ be a $C^*$-category. A $Z_2$-grading on $A$ is a decomposition into direct sum $\hom(a,b) = \hom^{(0)}(a,b) \oplus \hom^{(1)}(a,b)$ of two closed linear subspaces $\hom^{(0)}(a,b)$ and $\hom^{(1)}(a,b)$ of $\hom(a,b)$, for any pairs of objects $a, b \in A$, such that

a) if $f \in \hom^{(i)}(a,b)$ and $g \in \hom^{(j)}(b,c)$, then $gf \in \hom^{(i+j)}(a,c)$;
b) if $f \in \hom^{(i)}(a,b)$, then $f^* \in \hom^{(i)}(b,a)$.

A *morphism* from $\hom^{(i)}(a,b)$ is called homogeneous of degree $i$. The degree of a homogeneous element $f$ is denoted $\partial f$.

A $*$-functor $\mathcal{F} : A \to B$ of graded $C^*$-categories $A$ and $B$ is *graded* if $\mathcal{F}(\hom^{(i)}(a,b)) \subset \hom^{(i)}(\mathcal{F}(a), \mathcal{F}(b))$ for any pair of objects $a, b$ from $A$.

Let $\Gamma : A \to A$ be a $*$-functor, which is identity map on objects and $\Gamma^2 = id_A$, then $\hom^{(0)}(a,b) = \{f : \mathcal{F}(f) = f\}$ and $\hom^{(1)}(a,b) = \{f : \mathcal{F}(f) = -f\}$ gives $Z_2$-grading on $A$. Conversely, given grading, one can define corresponding $*$-functor $\Gamma$ by the identity $\Gamma(f^{(0)} + f^{(1)}) = f^{(0)} - f^{(1)}$.

Let $\mathcal{F} : A \to B$ and $\mathcal{G} : A \to B$ be graded $*$-functors. A set

$$\alpha = \{\alpha_a : \mathcal{F}(a) \to \mathcal{G}(a)\}_{a \in obA}$$

of morphisms is called a *natural transformation of degree $i$*, if $\partial\alpha_a = i$ for all $\alpha_a$ and

$$\mathcal{G}(f)\alpha_a = (-1)^{\partial\alpha\partial f}\alpha_b\mathcal{F}(f)$$

for any homogeneous morphism $f : a \to b$ from $A$.

Natural transformation $\alpha : \mathcal{G} \to \mathcal{F}$ is bounded if $\sup_a \|\alpha_a\| < \infty$. Next, under 'transformation' we always mean 'bounded transformation'.

Example 7. Let $B$ be a $Z_2$-graded $\sigma$-unital algebra and $\mathbb{H}(B)$ be $Z_2$-graded $C^*$-category with countably generated $Z_2$-graded right Hilbert $B$-modules as objects, and $B$-homomorphisms between Hilbert modules of degree $i \in Z_2$, that have adjoint, as the morphisms of degree $i$. Let $E = E^{(0)} \oplus E^{(1)}$ be a module from $\mathbb{H}(B)$. Denote by $\check{E} = \check{E}^{(0)} \oplus \check{E}^{(1)}$ the opposite graded module to $E$, $\check{E}^{(0)} = E^{(1)}$ and $\check{E}^{(1)} = E^{(0)}$. Next we need the following endofunctor and natural transformation of degree 1.

Let $\mathbb{V} : \mathbb{H}(B) \to \mathbb{H}(B)$ be the covatiant functor defined by the formula $\mathbb{V}(E) = \check{E}$ and $\mathbb{V}(f) = (-1)^{\partial f} f$, and consider natural transformation $\tau : id_{\mathbb{H}(B)} \to \mathbb{V}$ of degree 1 given by morphisms $\tau_E : E \to \check{E}$:

$$\tau_E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

in the decomposition $\check{E} = \check{E}^{(0)} \oplus \check{E}^{(1)}$. One checks that $\tau_E^* = -\tau_{\check{E}}$.

1.2. Additive and pseudoabelian C$^*$-categories. First we recall that *projection* in a $C^*$-category is a morphism with properties $p^* = p$ and $p^2 = 1$, i.e. projection is self-adjoint idempotent.

Definition 8. An additive $C^*$-category is said to be *pseudoabelian* $C^*$-cate- gory if each projection has a kernel.

Remark The main difference from the analogous definition of [6] and [7] is that here idempotents in addition are self-adjoint i.e. projections.

The following theorem describes how an additive $C^*$-category can be embedded in a pseudoabelian $C^*$-category (cf.[6],[7]).

THEOREM 9. *Let $A$ be an additive $C^*$-category. There exists a pseudoabelian $C*$-category $\tilde{A}$ and an additive $*$-functor $\phi : A \to \tilde{A}$ with the following universal property. For any pseudoabelian $C^*$-category $D$ and any additive $*$-functor $\psi : A \to D$ there exists an unique additive $*$-functor $\psi' : \tilde{A} \to D$ such that $\psi = \phi \cdot \psi'$. The pair $(\phi, \tilde{A})$ is unique up an additive $*$-equivalence of additive $C^*$-categories.*

PROOF. We only give here constructions because proof is analogous to the corresponding facts in [6],[7] so we omit them. An object of $\tilde{A}$ has a form $(E, p)$ where $E \in Ob(A)$ and $p \in \hom(E, E)$ is a projection. Morphism from $(E, p)$ to $(F, q)$ is defined as a morphism $f : E \to F$ of $A$ such that $fp = qf = f$. The composition of morphisms is defined as a composition of morphisms in $A$. The sum of objects is given by formula $(E, p) \oplus (F, q) = (E \oplus F, p \oplus q)$ and norm of morphisms is inherited from $A$. $\square$

As suggested in [6],[7] the construction of $K$-theory is based on the notion pseudoabelianity of an additive category, it is slightly different from the analogous definition given here. We carry out the construction using notion of a pseudoabelian $C^*$-category. Consider $A$ as a Banach category and denote by $\xi A$ the pseudoabelian category $A$ in Karoubi's sense. We have following

LEMMA 10. *Let $A$ be an additive $C^*$-category. Then the category $\tilde{A}$ is additively equivalent to $\xi A$.*

PROOF. Let $i : \tilde{A} \to \xi A$ be the essential functor that is identical on objects and morphisms. To define $j : \xi A \to \tilde{A}$ firstly note that if $q \in hom(F, F)$ is an idempotent then

$$\bar{q} = \sqrt{(2q^* - 1)(2q - 1) + 1} \cdot q \cdot \sqrt{((2q^* - 1)(2q - 1) + 1)^{-1}}$$

is a projection [8] and the pairs $(F, q)$ and $(F, \bar{q})$ are isomorphic by

$$u_q = \sqrt{(2q^* - 1)(2q - 1) + 1}.$$

Then define $j$ by the formulas $(E, q) \mapsto (E, u_q q u_q^{-1})$ on objects and $j(f) = u_{q'} f u_q$ for a morphisms, where $f : (E, q) \to (E', q')$. Now it is easy to show that the isomorphism $i \cdot j \simeq id_{\xi A}$ is given by the essential isomorphisms $\{u_q\}$ and $j \cdot i = id_{\tilde{A}}$. $\square$

1.3. ON THE MAIN EXAMPLES OF C*-CATEGORIES. Here we assume that all $C^*$-algebras are trivially graded and they have a fixed compact group action as the automorphisms, which is norm continuous. We also assume that Hilbert modules must be equipped with a linear, norm continuous group action, extending to the action on $B$-homomorphisms [8]. The category of such modules and $B$-homomorphisms we denote also by $\mathcal{H}(B)$. Note that this action is not norm continuous, but it is strictly continuous.

Having treated pseudoabelian $C^*$-categories, we now proceed to one of the main examples of this paper.

Example 11. *1*) Firstly, we define the $C^*$-category $rep(A, B)$. The objects of this category are pairs of form $(E, \phi)$ where $E$ is a countably generated right Hilbert $B$-module and $\phi : A \to \mathcal{L}(E)$ is an equivariant $*$-homomorphism. A morphism $f : (E, \phi) \to (E', \phi')$ is an invariant $B$-homomorphism $f : E \to E'$ in $\mathcal{H}(B)$ such that $f\phi(a) = \phi'(a)f$ for all $a \in A$. Objects of this type are said to be $A, B$-bimodules (cf.[14]). The structure of $C^*$-category is inherited from the suitable structure of $\mathcal{H}(B)$ and it is easy to show that $rep(A, B)$ is an additive $C^*$-category(and more, that is pseudoabelian $C^*$-category). The following property of $rep(A, B)$ will be used to calculate $K$-groups of $rep(A, B)$. Note that there exists a $*$-functor $\infty : \mathcal{H}(B) \to \mathcal{H}(B)$ and a natural isomorphism of functors $id_{\mathcal{H}(B)} \oplus \infty \simeq \infty$, where $E^\infty = E \oplus E \oplus \cdots$. This structure induces corresponding structure on $rep(A, B)$ induced by the formulas $(E, \phi)^\infty = (E^\infty, \phi^\infty)$, where $\phi^\infty(a) = (\phi(a))^\infty$ for all $a \in A$. This structure will be called $\infty$-structure.

2) Consider an additive $C^*$-category $\mathrm{Cal}(B)$ which is the quotient-$C^*$-category $\mathcal{H}(B)/\mathcal{K}(B)$. It has an essential compact group action inherited from action of compact group on $\mathcal{H}(B)$. Denote by $\pi : \mathcal{H}(B) \to \mathrm{Cal}(B)$ canonical additive $*$-functor. We need also following $C^*$-category denoted by $\mathcal{C}\dashv\updownarrow(A, B)$. By definition objects of this category has form $(E, \psi)$, where $E$ is Hilbert $B$-module and $\psi : A \to \hom_{\mathcal{C}\dashv\updownarrow B}(E, E)$ is a liftable $*$-homomorphism, i.e. there exists $A, B$-bimodule $(E, \phi)$ such that $\pi\phi = \psi$. A morphism $f : (E, \psi) \to (E', \psi')$ is morphism $f : E \to E'$ of category $\mathrm{Cal}(B)$ such that $f\psi(a) = \psi'(a)f$ for all $a \in A$. Define the $*$-functor $\Theta : rep(A, B) \to \mathrm{Cal}(A, B)$ by $(E, \phi) \mapsto (E, \pi\phi)$ and $f \mapsto \pi(f)$.

3) Now, we want to define the additive $C^*$-category $Rep(A, B)$. The objects of this category are also $A, B$-bimodules, i.e. objects are pairs $(E, \phi)$, where $E$ is a countably generated right Hilbert $B$-module and $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism. Also, a morphism $f : (E, \phi) \to (E', \phi')$ is a morphism $f : E \to E'$ in $\mathcal{H}(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, E')$$

for all $a \in A$. The structure of $C^*$-category is inherited from $\mathcal{H}(B)$. It is easy to show that $Rep(A, B)$ is an additive $C^*$-category but it isn't a pseudoabelian $C^*$-category. There is a canonical additive $*$-functor $\Pi_{A,B} : Rep(A,B) \to \mathrm{Cal}(A,B)$ defined by $(E,\phi) \mapsto (E,\pi\phi)$ and $f \mapsto \pi f$. From definition follows easily that the canonical linear map

$$\hom((E,\phi),(E',\phi')) \mapsto \hom((E,\pi\phi),(E,\pi\phi')$$

is surjective, i.e. $\Pi$ is a Serre functor(see for definition [6]).$\square$

Here we come to our main $C^*$-category. That is $\mathrm{Rep}(A,B)$.

Definition 12. The $C^*$-category $\mathrm{Rep}(A,B)$ is a universal pseudoabelian $C^*$- category of $Rep(A,B)$. Using the definition of a pseudoabelian $C^*$-category, we have the following description of $\mathrm{Rep}(A,B)$. Objects of it are triples $(E,\phi,p)$, where $(E,\phi)$ is an object and $p : (E,\phi) \to (E,\phi)$ is a morphism of $Rep(A,B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E,\phi,p) \to (E',\phi',p')$ is a morphism $f : (E,\phi) \to (E',\phi')$ of $Rep(A,B)$ such that $fp = p'f = f$. In detail, $f$ has properties

$$f\phi(a) - \psi(a)f \in \mathcal{K}(E,F), \quad fp = p'f = f. \tag{1}$$

The structure of $C^*$-category of $\mathrm{Rep}(A,B)$ came from the corresponding structures of $Rep(A,B)$. In particular, sum of triples is given by formula $(E,\phi,p) \oplus (F,\psi,q) = (E \oplus F, \phi \oplus \psi, p \oplus q)$.

1.4. On the cofinal subcategory of $H(B\otimes C^{(1,0)})$. When $A$ and $B$ are trivially graded $C^*$-algebras, for an interpretation of $KK^1(A,B)$, we need information on the following subcategory of $\mathbb{H}(B\otimes C^{(1,0)})$. We begin this subsection recalling a definition from [6],[7].

Definition 13. An additive $*$-functor $\mathcal{F} : A \to B$ of additive $C^*$-categories is said to be quasi-surjective if for any object $b$ from $B$ there are objects $a$ and $b'$ from $A$ and $B$ respectively and a unitary isomorphism $b \oplus b' \simeq \mathcal{F}(a)$. In particular, additive sub-$C^*$-category $A$ of $B$ is cofinal iff canonical inclusion is quasi-surjective.

Let $C^{(1,0)}$ be a Clifford algebra with generator $g$ ($g^* = g$ and $g^2 = 1$), with trivial action of a compact group. Consider the cofinal full subcategory $\mathbb{H}_B(B \otimes C^{(1,0)})$ of $\mathbb{H}(B \otimes C^{(1,0)})$ which contains modules isomorphic to modules of form

$$E_{n+1,n} = E \otimes_{B\otimes C^{(1,0)}} (C^{n,n} \otimes_{\mathbb{C}\otimes C^{(1,0)}} C^{(1,0)})$$

where $E$ is trivially graded equivariant $B$-module. Also, denote by $\mathbb{H}_{C^{(1,0)}}(B)$ the full subcategory of $\mathbb{H}(B)$ with objects isomorphic to $B$-

modules of form

$$E'_{n+1,n} = E \otimes_{B\otimes\mathbb{C}} (C^{n,n} \otimes C^{(1,0)}).$$

There is a canonical additive $*$-functor

$$S : \mathbb{H}_B(B \otimes C^{(1,0)}) \to \mathbb{H}_{C^{(1,0)}}(B)$$

defined by formulas $S(E_{n+1,n}) = E'_{n+1,n}$ and $S(f) = f$ for every $B \otimes C^{(1,0)}$-homomorphism $f$, because $f$ may be considered also as a $B$-homomorphism. This functor is injective, i.e. canonical linear maps on *hom*' are injective. Note that a $B$-homomorphism $f : E'_{n+1,n} \to F'_{n+1,n}$ defines a $B \otimes C^{(1,0)}$-homomorphism iff morphism $f$ is invariant under the action of $1 \otimes 1 \otimes \varepsilon$ i.e.

$$(1 \otimes 1 \otimes \varepsilon) f (1 \otimes 1 \otimes \varepsilon) = f.$$

But $C^*$-category $\mathbb{H}_{C^{(1,0)}}(B)$ coincides with the full subcategory of $\mathbb{H}(B)$ generated by modules isomorphic to $E \oplus \check{E}$, where $E$ is trivially graded equivariant $B$-module. For each $E \oplus \check{E}$ consider the element

$$\varepsilon_E = \begin{pmatrix} 0 & -\tau_{\check{E}} \\ \tau_E & 0 \end{pmatrix} \tag{2}$$

where $\tau_E : E \to \check{E}$ is the canonical isomorphism of degree 1. (see example 7.) Then $\varepsilon_E^* = \varepsilon_E$ and $\varepsilon_E^2 = 1$. Consider the essential $Z_2$-action on the $\mathbb{H}_{C^{(1,0)}}(B)$ defined as follows. Let $f : E \oplus \check{E} \to F \oplus \check{F}$ be a $B$-homomorphism. Then we have

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mapsto \begin{pmatrix} \check{f}_{22} & \check{f}_{21} \\ \check{f}_{12} & \check{f}_{11} \end{pmatrix} \tag{3}$$

where $\check{f} = \mathbb{V}(f)$.(see example 7). In particular, if $f$ is a homomorphism of degree 0 invariant under this action then

$$f = \begin{pmatrix} f_0 & 0 \\ 0 & \check{f}_0 \end{pmatrix} \tag{4}$$

and if $f$ has degree 1 under this action then

$$f = \begin{pmatrix} 0 & \check{f}_1 \\ f_1 & 0 \end{pmatrix}. \tag{5}$$

Denote by $\mathbb{H}_*(B)$ this invatiant subcategory of $\mathbb{H}_{C^{(1,0)}}(B)$. Thus we have the following

PROPOSITION 14. *Let $B$ be a trivially graded $C^*$-algebra. Then graded additive $C^*$-category $\mathbb{H}_B(B \otimes C^{(1,0)})$ is graded additively isomorphic to the $\mathbb{H}_*(B)$.*

## 2. $K$-groups of an additive category

The purpose of this section is to transform some main results of $K$-theory of Banach categories introduced by M.Karoubi in [6],[7] to a $C^*$-category with some minor changes which will be suitable in sequel sections.

### 2.1. $K^0$and $K^1$ groups for an additive C$^*$-category.

Definition 15. *The group $K^0(A)$ of an additive $C^*$-category is Grothendieck group of the abelian monoid of classes of unitary isomorphism objects of $A$.*

Note that this definition coincides with usual definition because in a $C^*$-category objects are isomorphic if they are unitary isomorphic. Indeed, if $u : E \to F$ is isomorphism then $u_0 = u\sqrt{(u^*u)^{-1}}$ is an unitary isomorphism. So from lemma 10 we get the following

Proposition 16. *Let $A$ be an additive $C^*$-category. The canonical functor induces isomorphism $i_* : K^0(\tilde{A}) \to \mathbb{K}^0(\xi A)$, where the left hand side $K$-group is as in the definition above, and the right one as in* [6],[7].

Now we discuss analogical questions for $K^{-1}$- group (cf. [6],[7]).

Definition 17. Let $A$ be an additive $C^*$-category. Consider the set of pairs $(E, \alpha)$, where $E \in ObA$ and $\alpha \in \hom(E, E)$ is a unitary automorphism.

a). The pairs $(E, \alpha)$ and $(E', \alpha')$ are to be unitary isomorphic if there exists a unitary isomorphism $u : E \to E'$ such that diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow^{\alpha} & & \downarrow^{\alpha'} \\ E & \xrightarrow{u} & E' \end{array}$$

is commutative.

b). The pairs $(E, \alpha)$ and $(E, \alpha')$ are said homotopic if $\alpha$ and $\alpha'$ are homotopic in $\mathcal{A}\sqcap\sqcup E$.

c). A pair $(E, \alpha)$ is said to be trivial if it is homotopic to $(E, 1_E)$.

d). The sum is defined by formula

$$(E, \alpha) \oplus (E', \alpha') = (E \oplus E', \alpha \oplus \alpha').$$

e). The pairs $(E, \alpha)$ and $(E, \alpha')$ are said stable isomorphic if there exist elementary pairs $(\bar{E}, \bar{e})$ and $(\hat{E}, \hat{e})$, and unitary isomorphism

$$(E, \alpha) \oplus (\bar{E}, \bar{e}) \simeq (E', \alpha') \oplus (\hat{E}, \hat{e})$$

f). The abelian monoid $K^{-1}(A)$ is defined as the monoid of classes of stable isomorphic pairs. Denote by $d(E,\alpha)$ the class of $(E,\alpha)$ in $K^{-1}(A)$.

LEMMA 18. *(cf. [7]) There are following relations in the $K^{-1}(A)$:*
*a). $d(E,\alpha)+d(E,\alpha^*)=0$;*
*b). If $\alpha$ and $\alpha'$ are homotopic unitary isomorphisms then $d(E,\alpha)=d(E,\alpha')$.*
*c). $d(E,\alpha)+d(E,\beta)=d(E,\beta\alpha)$;*

In particular, $K^{-1}(A)$ is an abelian group.

PROOF. a). The unitary automorphism $\alpha\oplus\alpha^*$ can be writen in the form

$$\begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

but each matrix is homotopic to $1_{E\oplus E}$. Thus a) holds.

b). Apply a), we get

$$d(E,\alpha')-d(E,\alpha)=d(E\oplus E,\alpha'\oplus\alpha^*)$$

But $\alpha'\oplus\alpha^*$ is homotopic to $\alpha\oplus\alpha^*$. Thus $d(E\oplus E,\alpha'\oplus\alpha^*)=0$.

c). Note that $d(E,\alpha)+d(E,\beta)=d(E\oplus E,\alpha\oplus\beta)$ and

$$d(E,\alpha\beta)=d(E\oplus E,\alpha\beta\oplus 1_E).$$

But $(\alpha\oplus\beta)^*)(\alpha\beta\oplus 1_E)=\beta\oplus\beta^*$, which is homotopic to $1_{E\oplus E}$. Thus $\alpha\oplus\beta$ is homotopic to $\alpha\beta$. Thus we can apply b). □

Next proposition is analogous to the appropriate property of $K^0(A)$.

PROPOSITION 19. *Let $A$ be anadditive $C^*$-category. The Canonical homomorphism*

$$i_*:K^{-1}(A)\to\mathbf{K}^{-1}(A),$$

*defined by $d(E,\alpha)\mapsto d(E,\alpha)$, is isomorphism, where $\mathbb{K}^{-1}(A)$ is Karoubi's group.*

PROOF. $i_*$ is epimorphism. Let $(E,\alpha)$ be a pair where $\alpha$ is an isomorphism. Consider the unitary isomorphism $\bar{\alpha}=\alpha\sqrt{\alpha^*\alpha}$. It is homotopic to $\alpha$, because $\alpha^*\alpha$ is homotopic to $1_E$. Apply lemma, we get that $d(E,\alpha)=d(E,\bar{\alpha})$. $i$ is monomorphism. Let $i(d(E,\alpha))=0$ then there exists elementary $(E',e')$ such that $(E\oplus E',\alpha\oplus e')$ is elementary. Then $(E\oplus E',\overline{\alpha\oplus e'})$ is also elementary, that is $(E\oplus E',\alpha\oplus\bar{e'})$ elementary. This means $d(E,\alpha)=0$. □

Thus the properties of $K^{-1}(A)$ is inherited from the corresponding properties of $\mathbb{K}^{-1}(A)$. In particular, we get the following

LEMMA 20. *$d(E,\alpha)=0$ if there exists an object $G$ such that $\alpha\oplus 1_G$ is homotopic to $1_{E\oplus G}$.*

THEOREM 21. *Let $A$ be an additive $C^*$-category, $\tilde{A}$ be the associated pseudoabelian $C^*$-category and $i : A \to \tilde{A}$ the canonical additive $*$-functor. Then the induced homomorphism*

$$i_* : K^{-1}(A) \to K^{-1}(\tilde{A}) \tag{6}$$

*is isomorphism.*

2.2. K-GROUP OF $*$-FUNCTOR.

DEFINITION 22. Let $A$ and $B$ additive $C^*$-categories and $\mathcal{F} : A \to B$ be an additive $*$-functor. Denote $\Gamma(\mathcal{F})$ as the set of triples $(E, F, \alpha)$, where $E$ and $F$ are objects of $A$, and $\alpha : \mathcal{F}(E) \to \mathcal{F}(F)$ is an unitary isomorphism from $B$.

a) Two triples $(E, F, \alpha)$ and $(E', F', \alpha')$ are unitary isomorphic if there exist unitary isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(E) & \stackrel{\alpha}{\to} & \mathcal{F}(F) \\ \downarrow^{\mathcal{F}(f)} & & \downarrow^{\mathcal{F}(g)} \\ \mathcal{F}(E') & \stackrel{\alpha'}{\to} & \mathcal{F}(F') \end{array}$$

is commutative.

b). Two triples $(E, F, \alpha)$ and $(E, F, \alpha')$ are homotopic if $\alpha$ and $\alpha'$ are homotopic in the subspace of unitary isomorphisms of $\hom(E, F)$.

c).The triple $(E, E, 1_E)$ is called trivial. A triple $(E, F, \alpha)$ is said to be elementary if this triple is homotopic to the trivial triple.

e). The sum of triples is defined by the formula $(E, F, \alpha)\oplus(E', F', \alpha') = (E \oplus E', F \oplus F', \alpha \oplus \alpha')$.

f). Two triples $\sigma = (E, F, \alpha)$ and $\sigma' = (E', F', \alpha')$ are stable unitary isomorphic if there exist elementary pairs $\tau = (\bar{E}, \bar{E}, \bar{\alpha})$ and $\tau' = (\bar{E}', \bar{E}', \bar{\alpha}')$ such that $\sigma \oplus \tau$ and $\sigma' \oplus \tau'$ are unitary isomorphic.

*The set $K(\mathcal{F})$ of stable isomorphic triples is abelian monoid relative the sum of triples. Denote by $d(E, F, \alpha)$ the class of $(E, F, \alpha)$ in $K(\mathcal{F})$.*

LEMMA 23. *Monoid $K(\mathcal{F})$ is an abelian group. Moreover*

$$d(E, F, \alpha) + d(F, E, \alpha^*) = 0.$$

PROOF. Note that

$$d(E, F, \alpha) + d(F, E, \alpha^*) = d(E \oplus F, F \oplus E, \alpha \oplus \alpha^*)$$

The last triple is isomorphic to $(E \oplus F, \beta)$, where

$$\beta = \begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$$

which is homotopic to $1_{\mathcal{F}(E)\oplus\mathcal{F}(F)}$ by $u(t) = \sigma(t)\sqrt{\sigma^*(t)\sigma(t)}$, where

$$\sigma(t) = \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^* \\ 0 & 1 \end{pmatrix}$$

□

The following theorem compares our definition of $K(\mathcal{F})$ with the corresponding one of Karoubi.

THEOREM 24. *Canonical homomorphism* $i : K(\mathcal{F}) \to \mathbb{K}(\mathcal{F})$ *defined by*

$$d(E, F, \alpha) \mapsto d(E, F, \alpha)$$

*is an isomorphism.*

PROOF Let $(E, F, \alpha)$ be a triple which defines an element in $\mathbf{K}(\mathcal{F})$, where $\alpha$ is an isomorphism (but not unitary isomorphism). Let $\bar{\alpha} = \alpha\sqrt{\alpha^*\alpha}$. $\bar{\alpha}$ is unitary and is homotopic to $\alpha$ because $\alpha^*\alpha$ is homotopic to $1_{\mathcal{F}(E)}$. This proves that $i$ is an epimorphism. Now, let $d(E, F, \alpha) \in K(\mathcal{F})$ defines 0 in $\mathbf{K}(\mathcal{F})$. This means by [7] that there exist objects $G$ and $H$ and isomorphisms (but after polar decomposition we may suppose they are unitary isomorphisms) $u : E \oplus G \to H$ and $v : F \oplus G \to H$ that $\mathcal{F}(v)(\alpha \oplus 1_{\mathcal{F}}(G))\mathcal{F}(u^*)$ is homotopic to $1_{\mathcal{F}(H)}$ (see [7]) by a homotopy $h(t)$. Then $\bar{h}(t) = h(t)\sqrt{h^*(t)h(t)}$ gives homotopy between $(E, F, \alpha) \oplus (G, G, 1_G)$ and $(H, H, 1_H)$ . This means $d(E, F, \alpha) = 0$ in $K(\mathcal{F})$. □

This theorem means that all properties of $K(\mathcal{F})$ inherited from the corresponding properties of $\mathbb{K}(\mathcal{F})$. In particular, we get the following results.(cf. [6],[7])

LEMMA 25. *There are following relations in the* $K(\mathcal{F})$*:*

*a). If* $\alpha$ *and* $\alpha'$ *are homotopic then* $d(E, F, \alpha) = d(E, F, \alpha')$*;*

*b).* $d(E, F, \alpha) + d(F, G, \beta) = d(E, G, \beta\alpha)$.

PROPOSITION 26. *Let* $\mathcal{F} : A \to B$ *be a Serre quasisurjectrive additive* $*$*-functor.*

*a).If in the definition of* $K(\mathcal{F})$ *we replace elementary triples by trivial triples then we get the same group .*

*b)* $d(E, F, \alpha) = 0$ *iff there exist an object* $G$ *from* $A$ *and unitary isomorphism* $\beta : E \oplus G \to F \oplus G$ *such that* $\mathcal{F}(\beta) = \alpha \oplus 1_{\mathcal{F}(G)}$.

PROPOSITION 27. *Let* $\mathcal{F} : A \to B$ *be a quasi-surjective additive* $*$*-functor. Then the sequence of abelian groups*

$$K^{-1}(A) \xrightarrow{f_1} K^{-1}(B) \xrightarrow{\partial} K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{\partial} K^0(B) \tag{7}$$

*is exact where* $i(d(E,F,\alpha) = d(E) - d(F)$ *(for the definition of* $\partial$ *see [7]). Besides that, if exists a functor* $\Psi : B \to A$ *such that* $\mathcal{F} \cdot \Psi \simeq Id_B$ *then there exists a split exact sequence*

$$0 \to K^0(\mathcal{F}) \xrightarrow{i} K^0(A) \xrightarrow{j} K^0(B) \to 0 \tag{8}$$

EXAMPLE 28. 1)Recall that an object of $rep(A,B)$ has form $(E,\phi)$ where $E$ is a right Hilbert $B$-module and $\phi : A \to \mathcal{L}(E)$ is supposed equivariant. A morphism from $(E,\phi)$ to $(E',\phi')$ is by definition an invariant $B$-homomorphism $f : E \to E'$ such that $f\phi(a) = \phi'(a)f$ (see example 11). Note that $rep(A,B)$ is a pseudoabelian $C^*$-category. To show that $K^i(rep(A,B)) = 0$ for all $i \in Z$ consider the $\infty$-structure of $rep(A,B)$ $E^\infty = E \oplus E \oplus \cdots$, $\alpha^\infty = \alpha \oplus \alpha \oplus \cdots$ and $\phi^\infty(a) = (\phi(a))^\infty$. Let

$$\infty : rep(A,B) \to rep(A,B)$$

be the $*$-functor defined by the formula $\infty(E) = E^\infty$, $\infty(\phi) = \phi^\infty$, and if $\alpha$ is a morphism in $rep(A,B)$ then $\infty(\alpha) = \alpha^\infty$. There exists natural isomorphism $id_{rep(A,B)} \oplus \infty \simeq \infty$. From this it follows that the groups $K^i(rep(A,B)$ (and cancelation monoid $C(repA,B))$ of monoid isomorphic objects of $rep(A,B)$) have automorphism $I$ with property that

$$id_{K^i(rep(A,B))} + I = I$$

. From this fact it follows that $K^i(rep(A,B)) = 0$ (resp. $C(repA,B) = 0$)
2) Consider canonical quasi-surjective functor

$$\Theta_{A,B} : rep(A,B) \to \mathrm{Cal}(A,B)$$

Apply exact sequence 7 of $K$-groups and result of example 1, one gets that the canonical homomorphism

$$\partial : K^{-1}(\mathrm{Cal}(A,B)) \to K^0(\Theta_{A,B}) \tag{9}$$

is isomorphism. □

## 3. Remarks on the definition of KK-groups

In this section we review definitions of Kasparov $KK$-groups in the form that will be need for our purposes.

Let $A$ and $B$ be $Z_2$-graded $C^*$-algebras, assuming that $A$ is separable and $B$ is $\sigma$-unital. Let $A^+$ be obtained from $A$ by adjoin a unit of degree 0. We also assume that all $C^*$-algebras have fixed compact group actions as above.

3.1. Operatorial homotopy and KK-theory. Let $\mathcal{E}(A,B)$ be the set of Kasparov $A,B$-bimodules. Denote by $KK(A,B)$ the Kasparov group defined by equivalence generated by unitary isomorphism and operator homotopy, and module degenerated bimodules [8],[14].

We need the following elementary properties of $KK(A,B)$:

a) "Cofinality principle". Let $\mathcal{F}$ be a cofinal full additive subcategory of $\mathbb{H}(B)$. If in the definition of $KK(A,B)$ the Kasparov $A,B$-bimodules are replaced by Kasparov $A,B$-bimodules defined by $\mathcal{F}$. Then we get the same group.

b) "Unitization principle". There exists a split exact sequence

$$0 \to KK(A,B) \xrightarrow{j} KK(A^+,B) \overset{p^*,i^*}{\leftrightarrows} KK(\mathbb{C},B) \to 0.$$

Remark. Let $s : A \to A^+$ be canonical inclusion, $p : A^+ \to \mathbb{C}$ is canonical projection and $i : \mathbb{C} \to A^+$ is canonical inclusion. One has following split exact sequence

$$0 \to KK(A^+,A) \xrightarrow{s^*} KK(A^+,A^+) \overset{i_*,p_*}{\leftrightarrows} KK(A^+,\mathbb{C},) \to 0.$$

Consider element $1-ip$ in $KK(A^+,A^+)$. One checks that $p_*(1-ip) = 0$. From this follows that there exists unique $j \in KK(A^+,A)$ such that

$$js = 1_A \text{ in } KK(A,A) \text{ and } sj + ip = 1_{A^+} \text{ in } KK(A^+,A^+).$$

c) "Unitality principle". Let $A$ be unital $C^*$-algebra. Then in the definition of $KK(A,B)$ it is possible to take kasparov $A,B$-bimodules of form $(E,\phi,F)$ where $\phi(1)=1$.

The following definition is motivated by b) and c).

Definition 29. *Kasparov $A,B$-bimodules $e=(E,\varphi,F)$ is said to be almost unital if $e$ has following properties:*

$$[\varphi(a),F] \in \mathcal{K}(E), \quad F - F^* \in \mathcal{K}(E), \quad F^2 - 1 \in \mathcal{K}(E) \tag{10}$$

*for all $a \in A$. Almost unital $A,B$-bimodule $(E,\varphi,F)$ is degenerate if*

$$[\varphi(a),F] = 0, \quad F = F^*, \quad F^2 = 1 \tag{11}$$

*Denote by $\mathcal{E}^+(A,B)$ set of almost unital Kasparov $A,B$-bimodules.*

Define the group $KK^+(A,B)$ in analogy of definition of $KK(A,B)$ using almost unital $A,B$-bimodules. By principles b) and c) it is isomorphic to $KK(A^+,B)$.

Denote by $E^+(A,B)$ abelian monoid classes of the equivalent almost unital bimodules where equivalence relation is generated by unitary isomorphism and operator homotopy. Denote by $CE^+(A,B)$ (resp.$KE^+(A,B)$) the corresponding cancelation monoid (resp. Grothendieck group) of

$E^+(A,B)$. Note that if $e$ is a degenerate bimodule then one has $e=0$ in $CE^+(A,B)$ (resp. $KE^+(A,B)$) (see example 28). Thus both are abelian groups and they are isomorphic to $KK^+(A,B)$.

Now, consider $\mathcal{G}(A,B)$ the set of $A,B$-bimodule $(E,\phi,G)$ with the following properties:

$$[\phi(a),G]\in\mathcal{K}(E),\quad g=g^*,\quad g^2=1 \tag{12}$$

let $G(A,B)$ be the abelian monoid of classes of equivalent $A,B$-bimodules from $\mathcal{G}(A,B)$ generated by unitary isomorphism and operatorial homotopy and $CG(A,B)$ be the cancelation monoid (resp. $KG(A,B)$ be Grothendieck group) of $G(A,B)$ which are isomorphisms because $CG(A,B)$ is group since the cancellation monoid of a degenerate almost unital bimodules is 0 (see example 28). The essential map $\mathcal{G}(A,B)\to\mathcal{E}(A,B)$ induces homomorphisms

$$\mu: CG(A,B)\to KK^+(A,B)$$

and

$$\nu: KG(A,B)\to KK^+(A,B)$$

which are monomorphisms.

Proposition 30. *Let $A$ and $B$ be as above. Then the following sequence groups is split exact*

$$0\to CG(A,B)\to KK^+(A,B)\leftrightarrow KK^+(\mathbb{C},B)\to 0. \tag{13}$$

*Also, canonical homomorphisms*

$$\mu: CG(A,B)\to KG(A,B)$$

*and*

$$\mu: CG(A,B)\to KK(A,B)$$

*are isomorphisms.*

3.2. Fredholm picture for trivially graded case. In this subsection we consider the case when $A$ and $B$ are trivially graded $C^*$-algebras with compact group actions. Let $E=E^0\oplus E^1$ be a graded Hilbert $B$-module then $E^0$ is trivially graded $B$-module and $E^1=\check{M}$, where $M$ is trivially graded Hilbert $B$-module. Consider $e=(E,\phi,F)$ almost unital Kasparov $(A,B)$-bimodule. Then in the decomposition $E=E^{(0)}\oplus E^{(1)}$, one has

$$\phi=\begin{pmatrix}\phi^{(0)} & 0\\ 0 & \phi^{(1)}\end{pmatrix}$$

and

$$F=\begin{pmatrix}0 & F^{(0)}\\ F^{(1)} & 0\end{pmatrix}$$

where $(E^{(0)}, \phi^{(0)})$ and $(E^{(1)}, \phi^{(1)})$ are $A, B$-bimodules and $F^{(i)} : (E^{(i)}, \phi^{(i)}) \to (E^{(i+1)}, \phi^{(i+1)})$ are morphism bimodules of degree 1. This interpretation motivates the following

Definition 31. a). Let $A$ and $B$ be $C^*$-algebras with compact group actions. Fredholm $A, B$-bimodule is $\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$ where $E^{(i)}$, $i = 0, 1$ are trivially graded Hilbert $B$-modules, and $(E^{(0)}, \phi^{(0)})$ and $(E^{(1)}, \phi^{(1)})$ are $A, B$-bimodules, and $F : (E^{(0)}, \phi^{(0)}) \to (E^{(1)}, \phi^{(1)})$ is a morphism $A, B$-bimodules of degree 0, i.e.

$$F\phi^{(0)}(a) - \phi^{(1)}(a)F \in \mathcal{K}(E^{(0)}, E^{(1)})$$

such that $F^*F - 1 \in \mathcal{K}(E^{(0)})$ and $FF^* - 1 \in \mathcal{K}(E^{(1)})$. The set of Fredholm $A, B$-bimodules is denote by $\mathcal{F}(A, B)$.

b). Fredholm bimodules

$$\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$$

and

$$\bar{\mu} = (\bar{E}^{(0)}, \bar{\phi}^{(0)}, \bar{E}^{(1)}, \bar{\phi}^{(1)}, \bar{F})$$

are unitary isomorphic if there exist unitary $B$-isomorphisms $u : E^{(0)} \to \bar{E}^{(0)}$ and $v : E^{(1)} \to \bar{E}^{(1)}$ such that $u\phi^{(0)}u^* = \bar{\phi}(0)$, $v\phi^{(1)}v^* = \bar{\phi}(1)$ and $uFv^* = \bar{F}$.

c). An operator homotopy through Fredholm bimodules is

$$\mu_t = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F_t)$$

which is Fredholm bimodule for all $t \in [0, 1]$ and $t \mapsto F_t$ is norm continuous.

d). The addition of Fredholm bimodules is defined by the formula

$$\mu \oplus \bar{\mu} = (E^{(0)} \oplus \bar{E}^{(0)}, \phi^{(0)} \oplus \bar{\phi}^{(0)}, E^{(1)} \oplus \bar{E}^{(1)}, \phi^{(1)} \oplus \bar{\phi}^{(1)}, F \oplus \bar{F})$$

e). Let $F(A, B)$ be the monoid of equivalence classes of Fredholm bimodules where equivalence is generated by unitary isomorphism and operator homotopy and $CF(A, B)$ (resp. $KF(A, B)$) be its cancelation semigroup (resp. Grothendieck group)

Now our concern is to compare the notion of Fredholm bimodule and Kasparov bimodule.

Construction A

Let $\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F)$ be Fredholm $A, B$-bimodule and $\check{E}^{(1)}$ be the opposite graded Hilbert $B$-module of $E^{(1)}$ and $\check{\phi}^{(1)} : A \to \mathcal{L}(\check{E}^{(1)})$ be the opposite to $\phi^1$. Define an almost unital Kasparov $A, B$-bimodule $\bar{\mu}$ as the triple $(\bar{E}, \bar{\phi}, \bar{F})$, where

$$\bar{E} = E^{(0)} \oplus \check{E}^{(1)}, \quad \bar{\phi} = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \check{\phi}^{(1)} \end{pmatrix}, \quad \bar{F} = \begin{pmatrix} 0 & \check{F}^* \\ \check{F} & 0 \end{pmatrix}$$

and $\check{F}$ is composition of $F$ with canonical $B$-homomorphism

$$\tau_{E^{(1)}} : E^{(1)} \to \check{E}^{(1)}$$

of degree 1 (see example 7).

PROPOSITION 32. *Let $\chi : \mathcal{F}(A,B) \to \mathcal{E}^{+}(A,B)$ be defined by $\mu \mapsto \bar{\mu}$. Then induced homomorphism of semigroups $\chi : F(A,B) \to E^{+}(A,B)$ is isomorphism. Therefore groups $CF(A,B)$, $KF(A,B)$ and $KK^{+}(A,B)$ are canonically isomorphic*

PROOF. First part is easy by checking and the second part is follows using results of subsection 3.1.□

Now consider canonical quasi-surjective additive $*$-functor

$$\Theta : Rep(A,B) \to \mathcal{C}\dashv\updownarrow(A,B).$$

We can define Karoubi's group $K(\Theta)$. There exists canonical homomorphism $\eta : CF \to K(\Theta)$ which maps Fredholm $A,B$-bimodule

$$\mu = (E^{(0)}, \phi^{(0)}, E^{(1)}, \phi^{(1)}, F) \tag{14}$$

to

$$\eta(\mu) = (E^{(0)}, \pi\phi^{(0)}, E^{(1)}, \pi\phi^{(1)}, \pi(F)).$$

(see example 11 for definition of $\pi$).

PROPOSITION 33. *The homomorphism $\eta : CF(A,B) \to K^{0}(\Theta_{A,B})$ is isomorphism.*

PROOF. It is enough to show that $\eta$ is monomorphism. Let 14 be a Fredholm bimodule such that $\eta(\mu) = 0$. Then there exist elementary Fredholm bimodules $\varepsilon = (E, \phi, E, \phi, H)$ and $\omega = (\bar{E}, \bar{\phi}, \bar{E}, \bar{\phi}, \bar{H})$ and operator homotopy $\mu \oplus \varepsilon \simeq \omega$. But the cancellation monoid of the abelian monoid of isomorphic elementary Fredholm bimodules is 0 (cf. example 28). Thus $\mu = 0$ in $CF(A,B)$.□

Combining the above propositions we have the following

COROLLARY 34. *Let $A$ and $B$ be as above. Then canonical homomorphism*

$$\eta\chi^{-1} : KK^{+}(A,B) \to K(\Theta_{A,B})$$

*is isomorphism.*

3.3. AN INTERPRETATION OF $KK^{1}(A,B)$. In this subsection we assume that $C^*$-algebras are trivially graded with compact group action. We use the property of subcategory of $\mathbb{H}(B \otimes C^{(1,0)})$, given in subsection 1.4, for characterization of $KK^{1}(A,B)$.

In detail, using the "cofinality principle" of subsection 3.1, and according to the proposition 14, one can use category $\mathbb{H}_*(B)$ for definition of $KK^1(A,B)$. Firstly note that if $E \in \mathcal{H}(B)$ then $E \oplus \check{E} \in \mathbb{H}_*(B)$. Denote by $\mathcal{L}_*(E,E')$ space of morphisms from $E \oplus \check{E}$ to $E' \oplus \check{E}'$ in the category $\mathbb{H}_*(B)$.

Consider $\mathcal{G}_*(A,B)$ the set of triples $(E \oplus \check{E}, \phi, g)$ where $E \in \mathcal{H}(B)$,

$$\phi : A \to \mathcal{L}_*(E \oplus \check{E})$$

is $*$-homomorphism, and $g \in \mathcal{L}_*(E \oplus \check{E})$, with

$$[g, \phi(a)] \in \mathcal{L}_*(E \oplus \check{E}),\ g^* = g,\ g^2 = 1,\ \partial(g) = 1.$$

Thus using triples from $\mathcal{G}_*(A,B)$ and analog construction of $KK(A,B)$ one gets $KK^1(A,B)$. (cf. [1])

Denote by $G_*(A,B)$ the abelian monoid of equivalent classes of $\mathcal{G}(A,B)$, where equivalence is generated by operator homotopy and unitary isomorphism. From the properties of category $\mathbb{H}_*(A,B)$ it follows that

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \check{\phi}_0 \end{pmatrix} \tag{15}$$

because all elements of $A$ has degree 0, and

$$g = \begin{pmatrix} 0 & \check{g}_0 \\ g_0 & 0 \end{pmatrix} \tag{16}$$

(as $g$ has degree 1). From $g^* = g$ and $g^2 = 1$ follows that $g_0^* = \check{g}_0$. Thus $\bar{g} = \tau_{\check{E}} g_0$ has degree 0, and $\bar{g}^* = \bar{g}$, $\bar{g}^2 = 1$.(see example 7). From this construction follows that one can consider $\mathcal{G}(A,B)$ as the set of triples of form $(E, \phi, g)$ where $E$ is trivially graded Hilbert $B$-module $\phi : A \to \mathcal{L}(B)$ is $*$-homomorphism and $g \in \mathcal{L}(B)$ such that

$$[\phi(a), g] \in \mathcal{K}(E),\ g^* = g,\ g^2 = 1 \text{ and } \partial g = 0, \quad \text{where } a \in A.$$

We conclude that canonical homomorphism $\rho : CG(A,B) \to KK^1(A,B)$ defined by

$$(E, \phi, g) \mapsto (E \oplus \check{E}, \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix}, \begin{pmatrix} 0 & g\tau_{\check{E}} \\ -\tau_E g & 0 \end{pmatrix}$$

is isomorphism.

Construction B

Denote by $\mathcal{P}(A,B)$ the set of triples of form $(E, \phi, p)$ where $E$ is trivially graded $B$-module, $\phi : A \to L(E)$ is $*$-homomorphism, $\partial(p) = 0$ and

$$[\phi(a), p] \in \mathcal{K}(E),\ a \in A;$$
$$p^* = p \text{ and } p^2 = p$$

Define a map $\vartheta : \mathcal{P}(A,B) \to \mathcal{G}_*(A,B)$ as follows. For each triple $(E, \phi, p)$ consider the triple $(E, \phi, g_p)$, where and $g_p = (2p-1)$.□

Let $P(A,B)$ be the abelian monoid of equivalent bimodules of above form where equivalence is generated by unitary isomorphism and operator homotopy. Let $CP(A,B)$ (resp. $KP(A,B)$) be the cancellation monoid (resp.Grothedieck group) of $P(A,B)$.

LEMMA 35. *The map $\vartheta : \mathcal{P}(A,B) \to \mathcal{G}_*(A,B)$, defined in construction B, is bijection.*

PROOF. The inverse map is defined by the formula:

$$(E,\phi,g) \mapsto (E,\phi,p_g),$$

where $p_g = \frac{1-g}{2}$ $\square$.

Thus we have

THEOREM 36. *Let $A$ be separable and $B$ be $\sigma$-unital trivially graded $C^*$-algebras with compact group action. Then canonical homomorphism*

$$\theta : KP(A,B) \to KK^1(A,B) \tag{17}$$

*defined as the composition $\theta = \rho\vartheta$ is isomorphism.*

## 4. Main Theorem

In this section we proof the following main result.

THEOREM 37. *Let $A$ be separable and $B$ be $\sigma$-unital trivially graded $C^*$-algebras with compact group action. There exists an essential isomorphism*

$$\kappa : K^j(\mathrm{Rep}(A,B)) \overset{\simeq}{\to} KK^{j+1}(A,B), \tag{18}$$

*where* $\mathrm{Rep}(A,B)$ *is the pseudoabelian $C^*$-category associated with the additive $C^*$-category Rep$(A,B)$ and $j = -1, 0$.*

The proof of this theorem consists of two parts, considered in subsections 4.1 and 4.2.

### 4.1. Proof of theorem in dimension zero.

Firstly, recall the definition of $K^0(\mathrm{Rep}(A,B))$. By definition 12 objects of $\mathrm{Rep}(A,B)$ have form $(E,\phi,p)$ where $E$ is a Hilbert $B$-module, $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism and $p$ is a projection, (i.e. $p^2 = p$ and $p^* = p$) such that

$$\phi(a)p - p\phi(a) \in \mathcal{K}(E).$$

Two objects $(E,\phi,p)$ and $(F,\psi,q)$ are unitary isomorphic in the $C^*$-category $\mathrm{Rep}(A,B)$ (or $C^*$-categorical unitary isomorphic) if there exists partial isometry $v : E \to F$ such that

$$v\phi(a) - \psi(a)v \in \mathcal{K}(E,F)$$

for all $a \in A$, $v^*v = p$ and $vv^* = q$. Define by $K^0(\mathrm{Rep}(A,B))$ Grothendieck group of the abelian monoid of unitary isomorphic objects that coincide to Karoubi's analogical definition(see section 2). On the other hand we may define $w$-unitary isomorphism of objects: $(E,\phi,p)$ and $(F,\psi,q)$ are $w$-unitary isomorphic if there exists unitary $u : E \to F$ such that

$$u\phi(a) - \psi(a)u \in \mathcal{K}(E,F)$$

for all $a \in A$, $up = qu$. Denote by $K^0_w(\mathrm{Rep}(A,B)$ Grothendieck group of abelian monoid of $w$-unitary isomorphic objects. Let us compare these groups. If two objects $(E,\phi,p)$ and $(F,\psi,q)$ are $w$-unitary isomorphic by unitary $u$, then $qup$ is an isomorphism between these objects. Conversely, if $v : (E,\phi,p) \to (F,\psi,q)$ is unitary isomorphism of given objects then $v : E \to F$ is a partial isometry with $v^*v = p$ and $vv^* = q$. Consider the objects $(E,\phi,1)$ and $(F,\psi,1)$. Then

$$\omega : (E,\phi,p) \oplus (F,\psi,1) \to (F,\psi,q) \oplus (E,\phi,1)$$

is a $w$-isomorphism, where

$$\omega = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}. \tag{19}$$

From this remark it follows that

$$(E,\phi,p) \oplus (F,\psi,1) \text{ and } (F,\psi,q) \oplus (E,\phi,1)$$

are equal in $K_w(\mathrm{Rep}(A,B))$, Note that $K^0(rep(A,B) = 0$, by example 28. From this fact it follows that

$$(E,\phi,1) = (F,\psi,1) = 0$$

in $K_w(\mathrm{Rep}(A,B))$. Thus

$$(E,\phi,p) = (E',\psi,q)$$

in the last group. Therefore, there is correctly defined essential isomorphism

$$\xi : K^0(\mathrm{Rep}(A,B)) \to K^0_w(\mathrm{Rep}(A,B)). \tag{20}$$

Next step is to define a homomorphism

$$\mu : K^0_w(\mathrm{Rep}(A,B)) \to KP(A,B). \tag{21}$$

to do this one need the following

LEMMA 38. *Let $(E,\phi,p)$ and $(E,\psi,p)$ be two objects such that*

$$\phi(a) - \psi(a) \in \mathcal{K}(E)$$

*for all $a \in A$. Then $(E,\phi,p) \oplus (E,\psi,1)$ and $(E,\phi,1) \oplus (E,\psi,p)$ are operator homotopic.*

PROOF. Consider

$$p_t = \left( \begin{array}{cc} 1-\cos^2 t\cdot(1-p) & \sin t\cdot\cos t\cdot(1-p) \\ \sin t\cdot\cos t\cdot(p-1) & 1+\sin^2 t\cdot(p-1) \end{array} \right).$$

Then $(E\oplus E, \phi\oplus\psi, p_t)$ is desirable operator homotopy.

From this lemma follows that map $(E,\phi,p)\mapsto(E,\phi,p)$ correctly defines epimorphism $\mu$ and also epimorphism

$$\mu\cdot\xi : K^0(\mathrm{Rep}(A,B)) \to KP(A,B).$$

To proof that $\mu\cdot\xi$ is monomorphism one need following

LEMMA 39. *Let $(E,\phi,p_0)$ and $(E,\phi,p_1)$ be operator homotopic triples. Then they are unitary isomorphic as objects in* $\mathrm{Rep}(A,B)$.

PROOF. Let $(E,\phi,p_t)$ be an operator homotopy. As $[0,1]$ is compact space, one can choose finite set of points $t_0,t_1,\ldots,t_n\in[0,1]$ such that $\|p_{t_{i+1}}-p_{t_i}\|<1$ for all $i=0,\ldots,n-1$. Using lemma 6.4 of [8], one gets that $p_{t_{i+1}}$ and $p_{t_i}$ are unitary isomorphic in the $C^*$-algebra of morphisms from $(E,\phi)$ to $(E,\phi)$ (in the category $Rep(A,B)$. From this it follows that the triples from the lemma are isomorphic objects in $\mathrm{Rep}(A,B)$.□

Now one can define canonical isomorphism

$$\kappa : K^0(\mathrm{Rep}(A,B)) \to KK^1(A,B) \tag{22}$$

as the composition of isomorphisms $\xi$, $\mu$ and isomorphism $\theta$ of 17.

4.2. PROOF OF THEOREM IN DIMENSION ONE. Consider the quasi-surjective Serre additive $*$-functor

$$\Pi_{A,B} : Rep(A,B) \to \mathrm{Cal}(A,B).$$

Apply exact sequence 7, one get the following exact sequence

$$K^{-1}(Rep(A,B)) \overset{\Pi_*}{\to} K^{-1}(\mathrm{Cal}(A,B)) \overset{\partial_\Pi}{\to} K^0(\Pi) \overset{i}{\to} K^0(Rep(A,B)).$$

The properties of this sequence are given in below.

LEMMA 40. *There exists a canonical isomorphism*

$$\tau : K^0(\Pi_{A,B}) \to K^0(\Theta_{O,B}), \tag{23}$$

*where $O$ is zero $C^*$-algebra.*

PROOF. Consider $e=(E,\phi,E',\psi,\alpha)$ such that $\alpha\overline{\phi(a)}=\overline{\psi(a)}\alpha$ for all $a\in A$, where $\overline{\phi(a)}=\pi\phi(a)$, $\overline{\psi(a)}=\pi\psi(a)$ (which defines element in $K^0(\Pi_{A,B})$). Let $0_\phi : O\to\mathcal{L}(E)$ and $0_\psi : O\to\mathcal{L}(E')$ are zero homomorphisms. One has $e_0=(E,0_\phi,E',0_\psi,\alpha)$ that gives element in $K^0(\Theta_{O,B})$. Conversely, let $e_0=(E,0_\phi,E',0_\psi,\alpha)$ define an element from

$K^0(\Theta_{O,B})$. Then the corresponding element in $K^0(\Pi_{A,B})$ is defined by the same $e_0$ considered $0_\phi$ and $0_\psi$ as the trivial homomorphisms on $A$. Thus we have two homomorphisms

$$\tau : K^0(\Pi_{A,B}) \to K^0(\Theta_{0,B})$$

and

$$\eta : K^0(\Theta_{0,B}) \to K^0(\Pi_{A,B})$$

such that $\tau\eta = 1$. To prove that $\tau$ is isomorphism, it is enough to show that $\{e\} - \{e_0\} = 0$. In order to show this consider

$$(E \oplus E', \phi \oplus 0_\psi, E' \oplus E, \psi \oplus 0_\phi, \alpha \oplus \alpha^*),$$

It is isomorphic to

$$(E \oplus E', \phi \oplus 0_\psi, E \oplus E', 0_\phi \oplus \psi, t(\alpha \oplus \alpha^*),$$

where

$$t(\alpha \oplus \alpha^*) = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \alpha^* \\ -\alpha & 0 \end{pmatrix}. \tag{24}$$

Note that $\begin{pmatrix} 0 & \alpha^* \\ -\alpha & 0 \end{pmatrix}$ is homotopic to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Thus $\alpha \oplus \alpha^*$ lifts to an isomorphism from $\phi \oplus 0_\psi$ to $\psi \oplus 0_\phi$. Done of lemma (see subsection 2.2) □

LEMMA 41. *Let $e = (E, \phi, \alpha)$ and $e' = (E, \phi, \beta)$ pairs which define elements in $K^{-1}(Rep(A, B))$. If $\alpha - \beta \in \mathcal{K}(E)$ then $\{e\} = \{e'\}$.*

PROOF. Consider the triples $(\bar{E}, \bar{\phi}, \bar{\alpha})$ and $(\bar{E}, \bar{\phi}, \bar{\beta})$ where

$$\bar{E} = E \oplus E, \quad \bar{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\alpha} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \bar{\beta} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \tag{25}$$

These triples are operator homotopic by

$$h(t) = \begin{pmatrix} \alpha + \sin^2 t \cdot (\beta - \alpha) & \sin t \cdot \cos t \cdot (\alpha - \beta) \\ \sin t \cdot \cos t \cdot (\beta - \alpha) & \beta + \sin^2 t \cdot (\alpha - \beta) \end{pmatrix}$$

□

LEMMA 42. *The canonical homomorphism*

$$\Pi_* : K^{-1}(Rep(A, B)) \to K^{-1}(\mathrm{Cal}(A, B))$$

*is monomorphism.*

PROOF. Let $e = (E, \phi, \alpha)$ represent an element in $K^{-1}(Rep(A,B))$ such that $\Pi_*(e) = 0$. Then by lemma 20 there exists $(E', \psi, 1_\psi)$ such that $\Pi(\alpha) \oplus 1_{\Pi(\psi)}$ is homotopic to $1_{\Pi(\phi)\oplus\Pi(\psi)}$. Let $h(t)$ be this homotopy. Consider its lifting in the group $U(E\oplus E', \phi\oplus\psi)$ of unitary automophisms, such that $h(0) = 1$. Put $\beta = h(1)$, then $\beta - \alpha \oplus 1_\psi \in \mathcal{K}(E)$. Using lemma 18 and lemma 41, we get

$$\{(E, \phi, \alpha) \oplus (F, \psi, 1)\} = \{E \oplus F, \phi \oplus \psi, \beta\} = 0.$$

□

Let $e = \{(E, \phi, E', \psi, \alpha)\} \in K^0(\Pi_{A,B})$. Consider the unique element $e' = \{(0, \beta)\} \in K^{-1}(\mathrm{Cal}(A,B))$ such that $\partial_\Pi(e') = e$. Then the homomorphism

$$s : K^0(\Pi) \to K^{-1}(\mathrm{Cal}(A,B))$$

defined by $e \mapsto e'$defines a right inverse for $\partial_\Pi$.

COROLLARY 43. *Let $A$ and $B$ be trivial graded $C^*$-algebras with compact group action. Then there is a split exact sequence of groups*

$$0 \to K^{-1}(Rep(A,B)) \overset{\Pi}{\to} K^{-1}(\mathrm{Cal}(A,B)) \overset{\tau\cdot\partial_\Pi}{\to} K^0(\Theta_{O,B}) \to 0 \tag{26}$$

Now we are ready to prove the main theorem. Firstly note that one can replace $K^{-1}(\mathrm{Rep}(A,B))$ by $K^{-1}(Rep(A,B))$ (see theorem 21).

We need some homomorphisms:

a) Define the homomorphism

$$\kappa : K^{-1}(Rep(A,B)) \to KK^0(A,B)$$

as follows. Let $e = (E, \phi, \alpha)$ be defines an element of $K^{-1}(Rep(A,B))$. Consider the triple $\bar{e} = (\bar{E}, \bar{\phi}, \bar{\alpha})$, defined in construction $A$ of section 3, by

$$\bar{E} = E \oplus \check{E},\ \bar{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & \check{\phi} \end{pmatrix},\ \bar{\alpha} = \begin{pmatrix} 0 & \check{\alpha} \\ \alpha & 0 \end{pmatrix}.$$

Then by definition $\kappa(e) = \bar{e}$.

b) Define homomorphism

$$\Delta : K^{-1}(\mathrm{Cal}(A,B)) \to KK^+(A,B)$$

in the following way. Let $e = (E, \pi\phi, \alpha)$ be a triple that defines element of $K^{-1}(\mathrm{Cal}(A,B))$, where $(E, \phi)$ is object of $Rep(A,B)$. Choose $F$ such that $\pi(F) = \alpha$. Of course, the triple $e' = (E, \phi, F)$ is almost unital Kasparov $A, B$-bimodule. By definition $\Delta(e) = \bar{e}'$.

c) Let $\omega : K(\Theta_{O,B}) \to KK^0(,B)$ defined by equality $\omega = \chi\eta^{-1}$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & K^{-1}(\mathrm{Rep}(A,B)) & \stackrel{\Pi}{\to} & K^{-1}(\mathrm{Cal}(A,B)) & \stackrel{\tau\cdot\partial}{\to} & K^{0}(\Theta_{O,B}) & \to & 0 \\
 & & \downarrow^{\kappa} & & \downarrow^{\Delta} & & \downarrow^{\omega} & & \\
0 & \to & KK^{0}(A,B) & \stackrel{j}{\to} & KK^{+}(A,B) & \stackrel{i^{*}}{\to} & KK^{0}(\mathbb{C},B) & \to & 0
\end{array}$$

Then $\Delta$ and $\omega$ are isomorphisms. Because $\Delta = \chi\eta^{-1}\partial_{\Theta}$ is composition of isomorphism from corollary 34 and isomorphism (9) of example 28. Also, $\omega = \chi\eta^{-1}$ is isomorphism because of lemma 40. Thus $\kappa$ is an isomorphism.

## References

[1] Blackadar B. *$K$-theory for operator algebras.* MSRI publications 5, New York-Berlin-Heidelberg. Springer (1986).

[2] Ghez P., Lima R,. Roberts J., *$W^*$-categories.* Pacific J. Math. v.120, n.1 (1985) 79-109.

[3] Higson N., *A characterization of $KK$-theory.* Pacific J. Math. v.126, n.2 (1987), 253-276.

[4] Higson N. *$C^*$-algebra extension theory and duality.* J. Funct. Anal. 129 (1995) 349-363.

[5] Kandelaki T.K., *Category of homomorphisms into a generalized Calkin algebra and projective modules over the commutant.* (Russian) Soobshch. Akad. Nauk Gruzin. SSR 122(1986),no.2,253-255.

[6] Karoubi M. *Algebras de Cliffordet $K$-theorie.* An. Sci. L'ecole norm. sup.t.1.,f.2(1968)161-268.

[7] Karoubi M. *$K$-theory.* An Intruduction. New York-Berlin-Heidelberg. Springer(1978).

[8] Kasparov G. *Operator $K$-functor and extensions of $C^*$-algebras.* Izv.Ac. Nauk SSSR,t.44,n.3(1980)571-636. (English translation: Math.USSR Izv.16(1981)513-572.)

[9] Kasparov G. *Hilbert $C^*$-modules. Theorems of Stinespring and Voiculescu.* J.Operator theory, 4(1980) 133-150.

[10] Paschke W. *Inner product modules over $B^*$-algebras.* Trans. Amer. Math. Soc.182(1973).443-468.

[11] Paschke W. *$K$-theory for commutants in the Calkin algebra.* Pasific J. Math. 95(1981)427-437.

[12] Rosenberg J. *$K$ and $KK$: Topology and operator algebras.* Proc. of symp. in pure Math.v51(1990),part I,445-480.

[13] Rosenberg J., Schochet C., *The Kunneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor.* Duke Math. J. 55(1987)n.2, 431-474.

[14] Skandalis G., *Some remarks on Kasparov theory,* J.Functional Analisis 56(1984)337-347.

[15] Valette A. *Remark on the Kasparov groups.* Pacific J. Math. (1983)247-256.

Author's address:

A.Razmadze Mathematical Institute Georgian Academy of Sciences, M. Aleksidze St.1,Tbilisi 380093, Republic of Georgia