Forum Math **8** *(1996), 267 - 302*

EQUIVARIANT SURGERY WITH MIDDLE-DIMENSIONAL SINGULAR SETS. I

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Abstract. Let G be a finite group. Let $f : X \to Y$ be a k-connected, degree 1, G-framed map of simply connected, closed, oriented, smooth manifolds X and Y of dimension $2k \geqq 6$. Under the assumption that the dimension of the singular set of the action of G on X is at most k, we construct an abelian group $W(G,Y)$ and an element $\sigma(f) \in W(G,Y)$, called the surgery obstruction of f such that the vanishing of $\sigma(f)$ in $W(G,Y)$ guarantees that f can converted by G-surgery to a homotopy equivalence.

Acknowledgement. We wish to express our gratitude to I. Madsen for stressing the importance of the problem above. We are also grateful to E. Laitinen and K. Pawałowski for their suggestions concerning brushing up the manuscript. The second author gratefully acknowledges support by the Japan Association for Mathematical Sciences and a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Japan.

1991 Mathematics Subject Classification: 57R67, 57R91, 19G24

1. Introduction

Let G be a finite group and X a smooth G-manifold. In the current article, the term G-equivariant surgery or simply G-surgery will be used in a restricted sense. Namely, it will refer to G-surgery on that part of X where each nontrivial element of G acts without fixed points. Thus, G-surgery on X will not change the G-singular set

$$\mathrm{Sing}(G,X) = \bigcup_{g \in G \setminus \{1\}} X^g,$$

Key words and phrases. surgery, singular set, quadratic form.

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where $X^g = \{x \in X \mid gx = x\}$.

Equivariant surgery theory in the sense above has been developed by several authors, beginning in the early early 70-ies. For references, see [18], [11], and [13]. C. T. C. Wall's nonsimply connected surgery for compact manifolds X can be viewed as G-surgery on the universal covering space $\widetilde{X}$ of X where G is the fundamental group of X. Except for [7], equivariant surgery theory has proceeded under the gap hypothesis: $2 \dim \mathrm{Sing}(G, X) < \dim X$. Under this hypothesis, the surgery obstruction group is either an L-group of Wall or a quotient of such involving form parameters, cf. [13], [14]. However, it turns out that there are interesting geometric problems for G-manifolds X, which require using G-surgery, where $2 \dim \mathrm{Sing}(G, X) = \dim X$. In order to handle such problems, we develop in this paper an equivariant surgery obstruction theory under the assumptions that $\dim X = 2k \geqq 6$ and $\dim \mathrm{Sing}(G, X) \leqq k$.

Applications of our G-surgery will appear in subsequent papers. They include the following. Buchdahl, Kwasik, and Schultz [6] proved that if a standard n-sphere S^n admits a one fixed point, smooth (or locally linear) G-action for some finite group G then $n \geqq 6$. We shall prove a converse to this result, namely that the alternating group A_5 on 5 letters has a one fixed point, smooth action on each S^n for $n \geqq 6$ ([3]). Another application is the following. Recall that an *Oliver group* is a finite group G which does not possess a series of subgroups $P \lhd H \lhd G$ such that P and G/H are of prime power order and H/P is cyclic. According to [16], if a finite group G acts smoothly on a standard sphere, with precisely one fixed point then G is Oliver. The converse of this result, namely that each Oliver group has a one fixed point, smooth action on some standard sphere, is proved for odd order abelian groups in [17], for nontrivial perfect groups in [10], and in full generality in [9].

The equivariant surgery obstruction theory which is presented in the current article is sufficient for the applications in [3], [9], and [10] above, but is not best possible, because extra assumptions will be imposed on middle dimensional fixed point sets. The advantage of making these assumptions is that new constructions needed for surgery with middle dimensional fixed point sets can be introduced, while at the same the details of the proofs can be considerably simplified over the general situation. The general situation will be treated in a paper under preparation and will show that vanishing of the surgery obstruction invariant of a G-framed map is equivalent to the map being G-framed cobordant (relatively to the G-singular set) to a homotopy equivalence.

We describe now our main result in the current article. Let X and Y be oriented, smooth G-manifolds. Let $T(X)$ denote the tangent bundle of X. Recall that a G-framed map $\boldsymbol{f} = (f, b) : X \to Y$ is a pair consisting of a smooth map $f : X \to Y$ and a real G-vector bundle isomorphism $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$ covering the identity map on X, for some real G-vector bundles ξ and η on Y. A G-framed map $\boldsymbol{f} = (f, b)$ is said to be of degree 1 (resp. k-connected) if f is of degree 1 (resp. k-connected). As usual, $\mathbb{Z}$ will denote the ring of integers, $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at a prime number p, and $\mathbb{Q}$ the ring of rational numbers. We regard the set

$$G(2) = \{g \in G \mid g^2 = 1 \text{ and } g \neq 1\}$$

as a G-set by letting G act by conjugation on the elements of $G(2)$.

Theorem 1.1. *Let G be a finite group and Y a closed, 1-connected, oriented, smooth G-manifold of even dimension $n = 2k \geqq 6$. Suppose that (1.1.1)–(1.1.3) below hold.*

(1.1.1) $\dim Y^g \leqq k$ *for any* $g \in G \setminus \{1\}$. (*This is equivalent to* $\dim \mathrm{Sing}(G, Y) \leqq k$.)

(1.1.2) *If* $\dim Y^H = k$ *for some subgroup* $H \leqq G$ *then* $|H| = 2$ *and* Y^H *is connected and oriented such that each* $g : Y^H \to Y^{gHg^{-1}}$ $(g \in G)$ *is orientation preserving.*

(1.1.3) $\dim(Y^H \cap Y^K) \leqq k - 2$ *whenever* $\dim Y^H = k$ *and* $\dim Y^K = k - 1$ $(H,\ K \leqq G)$.

Let R be one of $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ (p a prime), or $\mathbb{Q}$. Then there is an abelian group $W(G, Y; R)$ having the properties (DP) *and* (SP) *below.*

(DP) *$W(G, Y; R)$ is determined solely by the data $(R, G, Q, S, \lambda, w_Y^G)$, where $Q = Q(G, Y) = \{g \in G(2) \mid \dim Y^g = k - 1\}$, $S = S(G, Y) = \{g \in G(2) \mid \dim Y^g = k\}$, $\lambda = (-1)^k$, and $w_Y^G : G \to \{\pm 1\}$ is the orientation homomorphism associated to Y.*

(SP) *Let $\boldsymbol{f} = (f : X \to Y, b : T(X) \oplus f^*\eta \to f^*\xi \oplus \eta)$ be a degree 1, k-connected, G-framed map where X also satisfies* (1.1.1)–(1.1.3). *Suppose that $Q(G, X) = Q(G, Y)$, $S(G, X) = S(G, Y)$, and $K_k(f; R) = \mathrm{Ker}[f_* : H_k(X; R) \to H_k(Y, R)]$ is stably free over $R[G]$. Then there is an element $\sigma(\boldsymbol{f}) \in W(G, Y; R)$ depending on $\boldsymbol{f}$, such that if $\sigma(\boldsymbol{f}) = 0$ then $\boldsymbol{f}$ can be converted by G-surgery to a degree 1, k-connected, G-framed map $\boldsymbol{f}' = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*\xi)$ with the property that f' is an R-homology equivalence.*

Theorem 1.1 will be deduced in the main body of the paper from a slight generalization Theorem 7.3 of it.

Remark 1.2. Let

$$\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*\xi \oplus \eta)$$

be as in Theorem 1.1. Let X_0 be a G-simplicial subcomplex of X with respect to some equivariant smooth triangulation of X such that $\dim X_0 \leqq k-1$. Suppose $\sigma(\boldsymbol{f}) = 0$. Then in the proof of Theorem 7.3, the G-surgery used to convert f to an R-homology equivalence will be along embeddings $h : S^\ell \to X$ such that $\ell \leqq k$. Since $\dim S^\ell + \dim X_0 < \dim X$, we can modify these embeddings so that $h(S^\ell) \cap X_0 = \emptyset$. Thus, the G-surgery required in Theorem 1.1 (SP) and also in Theorem 7.3 can be performed in the free part of $X \setminus X_0$.

A special case of equivariant surgery theory on manifolds having middle-dimensional singular sets was treated by K. H. Dovermann [7], namely the case $|G| = 2$. His surgery obstruction was expressed in terms of several invariants in classical surgery theory. The approach in the current paper is very different from that in [7], in that we construct a new surgery group over a ring with form parameters, housing a single invariant to detect the obstruction to performing G-surgery.

We describe now this surgery invariant. Recall that the usual surgery invariant, under the gap hypothesis, is obtained by equipping $K_k(f;\mathbb{Z})$ with the restriction B_f of the equivariant intersection form on $H_k(X;\mathbb{Z})$ and then showing that the self-intersection form on X defines a quadratic form $q_{\boldsymbol{f}}$ on $K_k(F;\mathbb{Z})$, whose associated Hermitian form is B_f. If the triple $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}})$ has (stably) a $\mathbb{Z}[G]$-free Lagrangian L then one can realize geometrically any $\mathbb{Z}[G]$-basis for L by equivariantly embedded, disjoint k-spheres and then remove these by performing G-surgery. Thus, the class of $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}})$ in the Grothendieck group of all such algebraically defined triples modulo all triples having a $\mathbb{Z}[G]$-free Lagrangian is a sufficient invariant for performing equivariant surgery. We want to modify this procedure so that it works under the hypothesis $\dim \mathrm{Sing}(G,X) \leqq k$. It turns out that $K_k(f;\mathbb{Z})$ is still a finitely generated, projective $\mathbb{Z}[G]$-module, that B_f is still a nonsingular Hermitian form, but that it is not necessarily even. The first ingredient we develop is a new notion of quadratic form q whose associated Hermitian form B is not necessarily even. This involves two parameters instead of one as above and we call a triple (M,B,q) a doubly parametrized quadratic module. In the geometric situation, we construct a quadratic form $q_{\boldsymbol{f}}$ in the new sense above, on $K_k(f;\mathbb{Z})$, which incorporates selfintersection information needed later and whose associated Hermitian form is the intersection form B_f above. The notion of Lagrangian L for (M,B,q) is the usual one, but it is not necessary that a Lagrangian in the generality we are working has a direct sum complement which is a Lagrangian. We are still not finished building our surgery invariant. The subset $S(G,X)$ of $G(2)$ is G-invariant under the action of G via conjugation. We replace now the triples (M,B,q) above by quadruples $\boldsymbol{M} = (M,B,q,\alpha)$ where $\alpha : S(G,X) \to K$ is a G-map and define a Lagrangian L for (M,B,q,α) to be one for (M,B,q) such that $\mathrm{Im}(\alpha) \subseteq L$. The G-map α is called the positioning data of (M,B,q,α). In the geometric situation, α_f is the G-map which assigns to each $s \in S(G,X)$ the image in $K_k(f;\mathbb{Z})$ of the orientation class of X^s. ¿From the geometric standpoint, our main result is the following: If L is a $\mathbb{Z}[G]$-free Lagrangian for $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$ then any $\mathbb{Z}[G]$-basis of L can be realized geometrically by equivariantly embedded, disjoint k-spheres which do not meet $\mathrm{Sing}(G,X)$. This being the case, we can perform G-surgery on the embedded spheres and convert f to a homology equivalence and therefore, to a homotopy equivalence. One would like now to form the Grothendieck group of all algebraically defined quadruples (M,B,q,α) modulo the subgroup of all quadruples having a $\mathbb{Z}[G]$-free Lagrangian and claim that the class of $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$ in this group is a sufficient obstruction to performing equivariant surgery. But, this doesn't work, because stabilization with respect to this group is too strong. It turns out that the quadruples $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$ vanish under a certain invariant ∇. This is a crucial observation. The right group $W(G,Y)$ for housing $(K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$ is the Grothendieck group of all algebraically defined quadruples $\boldsymbol{M} = (M,B,q,\alpha)$ with trivial $\nabla_{\boldsymbol{M}}$ modulo the subgroup generated by all such quadruples having a Lagrangian. A few words concerning ∇ are in order. ¿From the geometric point of view, the definition of ∇ is motivated by the obstruction that if $x \in K_k(f;\mathbb{Z})$ is realized by an immersion $h : S^k \to X$ and if $s \in S(G,X)$ then the intersection number of h and sh is congruent $\bmod 2$ to that of X^s and sh. The observation shows that if we define for an arbitrary algebraic object $\boldsymbol{M} = (M,B,q,\alpha)$, $\nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S(G,X), \mathbb{Z}/2\mathbb{Z})$ by $\nabla_{\boldsymbol{M}}(x)(s) = [\varepsilon(B(\alpha(s) - x, sx))]$ ($x \in M$ and $s \in S$), where $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the map defined by $\varepsilon(\sum_{g \in G} a_g g) = a_1$ ($a_g \in \mathbb{Z}$), then for any geometric object $\boldsymbol{M_f} = (K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$, $\nabla_{\boldsymbol{M_f}} = 0$. In other words, ∇ vanishes on all geometric objects. Now our result that the family of all metabolic planes with trivial ∇-invariant is cofinal in the category of all algebraic objects with trivial ∇-invariant and our result that any metabolic plane with trivial ∇-invariant can be added to a geometric object $\boldsymbol{M_f}$ by performing G-surgery on $\boldsymbol{f}$ shows that the group $W(G, Y; \mathbb{Z})$ is the correct one for housing our G-surgery invariant.

The rest of the article is organized as follows. In Section 2, we recall certain foundations of equivariant surgery including the equivariant intersection form and equivariant selfintersection form. We construct a doubly parametrized selfintersection form which is used later to define $q_{\boldsymbol{f}}$. In Section 3, we prove the geometric result that a k-dimensional immersion h into a $2k$-dimensional G-manifold X, which does not meet the G-singular set $\mathrm{Sing}(G, X)$ and vanishes under our doubly parametrized selfintersection form, is regularly homotopic to an equivariant embedding h', i.e. an embedding h' such that $\mathrm{Im}h' \cap g\mathrm{Im}h' = \emptyset$ for all $g \in G \setminus \{1\}$. Section 4 is purely algebraic. It defines doubly parametrized quadratic modules with positioning data and the invariant ∇ of such modules. Various Grothendieck-Witt groups relevant to studying surgery groups are constructed. One of these groups, namely that defined in (4.4), is the surgery group. Section 5 studies special metabolic planes whose ∇-invariant is trivial. The main result is Theorem 5.6: A doubly parametrized quadratic module which has a free Lagrangian and trivial ∇-invariant, decomposes as an orthogonal sum of special metabolic planes with trivial ∇-invariant. A corollary of this result is that the family of metabolic planes with trivial ∇-invariant is cofinal in the category of all doubly parametrized quadratic modules with trivial ∇-invariant. Section 6 is devoted to the proof of Theorem 5.6. Section 7 begins by constructing the geometric module $\boldsymbol{M_f} = (K_k(f;\mathbb{Z}), B_f, q_{\boldsymbol{f}}, \alpha_f)$, where $\boldsymbol{f} = (f, b)$, and showing that its ∇-invariant is 0. Let $\sigma(\boldsymbol{f})$ denote the class of $\boldsymbol{M_f}$ in the surgery group $W(G, Y; \mathbb{Z})$. The main result of the paper is Theorem 7.3 asserting that if $\sigma(\boldsymbol{f}) = 0$ then $\boldsymbol{f}$ is G-framed cobordant to $\boldsymbol{f}' = (f', b')$ such that $f' : X' \to Y$ is a homology equivalence. Theorem 7.3 is proved in Section 8 on the basis of Theorem 8.1: Any metabolic plane with trivial ∇-invariant can be added to $\boldsymbol{M_f}$ by performing G-surgery on $\boldsymbol{f}$; i.e., given a metabolic plane $\boldsymbol{M}$ such that $\nabla_{\boldsymbol{M}} = 0$, there is a G-framed map $\boldsymbol{f}'' = (f'', b'')$ obtained from $\boldsymbol{f}$ by G-surgery such that $\boldsymbol{M_{f''}} \cong \boldsymbol{M_f} \oplus \boldsymbol{M}$. Theorem 8.1 is proved in Section 9.

2. Geometric preliminaries

In this section, we develop notation to be used in the following sections.

Let X be a G-space. For a point $x \in X$ and for a subgroups H of G, let H_x denote the isotropy subgroup at x in the H-space $\mathrm{res}^G_H X$. Let

$$\begin{aligned} &X^H = \mathrm{Fix}(H, X) = \{x \in X \mid G_x \supseteqq H\} \\ &\mathrm{Fix}_G(>H, X) = \{x \in X \mid G_x \supsetneqq H\}, \\ &\mathrm{Fix}_G(=H, X) = \{x \in X \mid G_x = H\} \\ &\mathrm{Free}(H, X) = \{x \in X \mid H_x = \{1\}\}, \text{ and} \\ &\mathrm{Sing}(H, X) = \{x \in X \mid H_x \neq \{1\}\}. \end{aligned}$$

If $g \in G$, let $X^g = X^{\langle g \rangle}$, $\mathrm{Free}(g, X) = \mathrm{Free}(\langle g \rangle, X)$, etc. For a subspace $Z \subseteqq X$, define

$$\rho_X^G(Z) = \bigcap_{x \in Z} G_x.$$

Let $\boldsymbol{Mnf}^n(G)$ denote the family of all paracompact, 1-connected (i.e. connected and simply connected), oriented, smooth G-manifolds of dimension n. Let $\boldsymbol{Mnf}^n_{\mathrm{cp}}(G)$ denote the family of all compact G-manifolds in $\boldsymbol{Mnf}^n(G)$.

For $X \in \boldsymbol{Mnf}^n(G)$, the orientation homomorphism $w_X^G : G \to \{\pm 1\}$ is defined by $w_X^G(g) = 1$ if $g : X \to X$ is orientation preserving and $w_X^G(g) = -1$ if otherwise. For any commutative ring R with the unity, let $R[G]$ denote the group ring of G with coefficients in R. For any set U, the set $\mathrm{Map}(U, R)$ consisting of all maps $U \to R$ is regarded as an R-module in the canonical way. As R-modules, $R[G] = \mathrm{Map}(G, R)$. For a subset S of G, let $R[S]$ denote the R-submodule of $R[G]$ generated by S; thus $R[S] = \mathrm{Map}(S, R)$. We shall always give $R[G]$ the antiinvolution $a \mapsto \overline{a}$ defined by $w = w_X^G$; thus

$$\overline{\sum_{g \in G} r_g g} = \sum_{g \in G} r_g w(g) g^{-1} \quad (r_g \in R).$$

Let H be a subgroup G, which will be indicated by $H \leqq G$. Since X has an equivariant smooth triangulation (cf. [8]), the H-fixed-point set X^H is an $N_G(H)$-simplicial complex of dimension $\leqq n$ where $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

Let $\pi_0(X^H)$ denote the set of all connected components γ of X^H. The underlying space of γ will be denoted by X_γ (or X_γ^H when we want to emphasize the group H). For a nonnegative integer ℓ, let $\pi_0(X^H, \ell)$ denote the subset of $\pi_0(X^H)$ consisting of all γ such that $\dim X_\gamma = \ell$. Set

$$\Pi(G, X) = \coprod_{H \leqq G} \pi_0(X^H), \quad \text{and}$$

$$\Pi(G, X, \ell) = \coprod_{H \leqq G} \pi_0(X^H, \ell).$$

For $\gamma \in \Pi = \Pi(G, X)$ such that $\gamma \in \pi_0(X^H)$, define $\rho(\gamma) = \rho_\Pi(\gamma) := H$ $(\Pi = \Pi(G, X))$. One should note that if $H = \rho_\Pi(\gamma)$ then $H \leqq \rho_X^G(X_\gamma)$, but H is not necessarily equal to $\rho_X^G(X_\gamma)$. For $g \in G$ and $\gamma \in \Pi(G, X)$, let $g\gamma$ be the connected component $\gamma' \in \pi_0(X^{gHg^{-1}})$ such that $X_{\gamma'}^{gHg^{-1}} = gX_\gamma^H$. The assignment $g \mapsto g\gamma$ defines an action of G on $\Pi(G, X)$. Obviously, $\Pi(G, X, \ell)$ is G-invariant. Let $\Phi : \Pi(G, X) \to \Pi(G, X)$ denote the map $\gamma \mapsto \beta$ such that $\rho_\Pi(\beta) = \rho_X^G(X_\gamma)$ and $X_\beta = X_\gamma$ as subsets of X. The map Φ is a G-map. The property $\rho_\Pi(\Phi(\gamma)) = \rho_X^G(X_{\Phi(\gamma)})$ should be kept in mind. Generally speaking, the subsets below are more useful than $\Pi(G, X)$ and $\Pi(G, X, \ell)$ for handling problems arising from $\mathrm{Sing}(G, X)$. Define

$$\Theta(G, X) = \mathrm{Im}(\Phi) \quad \text{and} \quad \Theta(G, X, \ell) = \Theta(G, X) \cap \Pi(G, X, \ell).$$

Let $\boldsymbol{Mnf}^{2k}_{\mathrm{sg}}(G)$ denote the family of all $X \in \boldsymbol{Mnf}^{2k}(G)$ satisfying the following hypotheses:

(2.1.1) $\dim \mathrm{Sing}(G, X) \leqq k$ (namely, $\dim X^g \leqq k$ for all $g \in G \setminus \{1\}$).

(2.1.2) $|\pi_0(X^H, k)| \leqq 1$ for any $H \leqq G$.

(2.1.3) If $\gamma \in \Theta(G, X, k)$ then $|\rho_\Pi(\gamma) \cap G(2)| = 1$.

(2.1.4) If $\gamma \in \Theta(G, X, k)$ and $\delta \in \Theta(G, X, k-1)$ then $\dim(X_\gamma \cap X_\delta) \leqq k - 2$.

(2.1.5) All submanifolds X_γ^H ($\gamma \in \Theta(G, X, k)$) are oriented in such a way that each $g \in G$ acts as an orientation preserving diffeomorphism $X_\gamma \to X_{g\gamma}$.

Set $\boldsymbol{Mnf}^{2k}_{\mathrm{cp,sg}}(G) = \boldsymbol{Mnf}^{2k}_{\mathrm{cp}}(G) \cap \boldsymbol{Mnf}^{2k}_{\mathrm{sg}}(G)$.

Lemma 2.2. *Let $X \in \boldsymbol{Mnf}^{2k}_{\mathrm{sg}}(G)$ where $k \geqq 2$. Then for every $\gamma \in \Theta(G, X, k)$,*

$$\dim \mathrm{Fix}_G(>\rho_\Pi(\gamma), X_\gamma) \leqq k - 2.$$

In particular, $\mathrm{Fix}_G(=\rho(\gamma), X_\gamma)$ *is connected and open dense in X_γ.*

Proof. Let $\gamma \in \Theta(G, X, k)$. If $X_\gamma \supsetneqq X_\delta$ for some $\delta \in \Pi(G, X)$ then by (2.1.4), $\dim X_\delta \leqq k - 2$. Thus

$$\dim \mathrm{Fix}_G(>\rho_\Pi(\gamma), X_\gamma) \leqq k - 2. \quad \text{Q.E.D.}$$

Let Y be a closed, connected, oriented, smooth manifold of dimension k. Let $X \in \boldsymbol{Mnf}^n(G)$ where $n = 2k$ and let $\lambda = (-1)^k$. Let $\mathrm{Map}(Y, X)$ denote the set of all continuous maps $Y \to X$. Let $\mathrm{Immer}(Y, X)$ denote the set of all smooth immersions $Y \to X$ and let $\mathrm{Immer}^t(Y, X)$ denote the subset of $\mathrm{Immer}(Y, X)$ consisting of all immersions $Y \to X$ with trivial normal bundle. Let $\mathrm{Int}(X)$ denote the interior of X. For $f_1,\ f_2 \in \mathrm{Map}(Y, \mathrm{Int}(X))$, let $\mathrm{intsec}(f_1, f_2)$ denote the geometric intersection number of f_1 and f_2. This number is determined as follows. Approximate f_1 and f_2 by ${f'}_1$ and ${f'}_2 \in \mathrm{Immer}(Y, \mathrm{Int}(X))$ such that $\mathrm{Im}{f'}_1 \cap \mathrm{Im}{f'}_2 = \{a_1, \cdots, a_m\}$, ${f'}_1^{-1}(a_i) = \{b_i\}$, ${f'}_2^{-1}(a_i) = \{c_i\}$, and each a_i is a transversal-intersection point of ${f'}_1$ and ${f'}_2$. For each point a_i, define $\mathrm{intsec}({f'}_1, {f'}_2; a_i) = 1$ (resp. -1) if the ordered direct sum $d{f'}_1(T_{b_i}(Y)) \oplus d{f'}_2(T_{c_i}(Y))$ has the same (resp. opposite) orientation as $T_{a_i}(X)$. Then $\mathrm{intsec}(f_1, f_2) = \sum_{i=1}^{m} \mathrm{intsec}({f'}_1, {f'}_2; a_i)$. The *$G$-intersection number* $\mathrm{intsec}_G(f_1, f_2)$ of f_1 and f_2 is defined by

$$\mathrm{intsec}_G(f_1, f_2) = \sum_{g \in G} \mathrm{intsec}(f_1, g^{-1} f_2) g \quad \in \mathbb{Z}[G].$$

$\mathrm{intsec}_G(f_1, f_2)$ is well-defined and invariant under homotopies of f_1 and f_2 in $\mathrm{Int}(X)$.

If $\dim X > \dim \mathrm{Sing}(H, X) + 2$ then $\mathrm{Free}(H, X)$ is 1-connected. Hence if $f \in \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(H, X)))$, the composition $\pi_H \circ f$ determines the selfintersection number $\mathrm{selfintsec}_H(f) \in \mathbb{Z}[H]/\mathrm{min}^\lambda(\mathbb{Z}[H])$ (cf. [19, Part I §5]), where $\pi_H : \mathrm{Int}(\mathrm{Free}(H, X)) \to \mathrm{Int}(\mathrm{Free}(H, X))/H$ is the canonical projection, and $\mathrm{min}^\lambda(\mathbb{Z}[H]) = \{x - \lambda\overline{x} \mid x \in \mathbb{Z}[H]\}$. The number $\mathrm{selfintsec}_H(f)$ is invariant under regular homotopies of f in $\mathrm{Int}(\mathrm{Free}(H, X))$. Let $T \subseteqq G$ be a subset closed under taking inverses. For a commutative ring R with 1, we define a coefficient quasibundle $\mathcal{B}_T(R)$ over T as follows. For $g \in G$ set

$R_g = R/(1-\lambda w(g))R$ if $g^2 = 1$, and $R_g = R$ otherwise. Define $\mathcal{B}_T(R) = \coprod_{g\in T} R_g$. The map $p_{\mathcal{B}} : \mathcal{B}_T(R) \to T$ such that $p_{\mathcal{B}}(R_g) = \{g\}$ is called the *projection*. A map $s : T \to \mathcal{B}_T(R)$ is called a *section* if $p_{\mathcal{B}} \circ s = id_T$. Define $\Gamma^{\lambda,w}(T;R)$ to be the set of all sections $s : T \to \mathcal{B}_T(R)$ such that $s(g^{-1}) = \lambda w(g)s(g)$. Define

$$Q(G,X) = \{g \in G(2) \mid \dim X^g = k-1\}, \quad \text{and}$$
$$S(G,X) = \{g \in G(2) \mid \dim X^g = k\}.$$

It is easy to show that $w(g)(= w_X^G(g)) = (-1)^{k+1}$ (resp. $(-1)^k$) for all $g \in Q(G,X)$ (resp. $S(G,X)$). Thus, $Q(G,X) \subseteqq \{g \in G(2) \mid g = -(-1)^k\overline{g}\}$ and $S(G,X) \subseteqq \{g \in G(2) \mid g = (-1)^k\overline{g}\}$. Letting $Q = Q(G,X)$, $S = S(G,X)$ and defining

$$\Lambda(G,Q;R) = \langle x - \lambda\overline{x} \mid x \in A\rangle_R + R[Q],$$

we see that there is a canonical identification $R[G \setminus S]/\Lambda(G,Q;R) = \Gamma^{\lambda,w}(G \setminus (Q \cup S);R)$. Thus we can regard $\mathrm{selfintsec}_H(f) \in \Gamma^{\lambda,w}(H;\mathbb{Z})$ for $f \in \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(H,X)))$. If H, $K \leqq G$ and $f \in \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(H,X))) \cap \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(K,X)))$ then it follows that

$$\mathrm{selfintsec}_H(f)(g) = \mathrm{selfintsec}_K(f)(g) \in \mathbb{Z}_g \quad \text{for any } g \in H \cap K. \tag{2.3}$$

Furthermore if $f \in \mathrm{Immer}^t(Y, \mathrm{Int}(\mathrm{Free}(H,X)))$ then

$$\mathrm{intsec}_H(f,f) = \mathrm{self}\widetilde{\mathrm{intsec}}_H(f) + \lambda\overline{\mathrm{self}\widetilde{\mathrm{intsec}}_H(f)} \ \text{ in } \mathbb{Z}[H] \tag{2.4}$$

where $\mathrm{self}\widetilde{\mathrm{intsec}}_H(f)$ is a lifting of $\mathrm{selfintsec}_H(f) \in \mathbb{Z}[H]/\min^\lambda(\mathbb{Z}[H])$ (cf. [19, Part I Theorem 5.2 (iii)]).

The following lemma is well-known.

Lemma 2.5. *Let k be an integer $\geqq 3$, $n = 2k$, $X \in \boldsymbol{Mnf}^n(G)$ such that $\dim X > \dim \mathrm{Sing}(G,X) + 2$. Let Y be a closed, connected, oriented, k-dimensional smooth manifold. Let $f \in \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(G,X)))$. If $\mathrm{intsec}_G(f,f) = 0 \in \mathbb{Z}[G]$ and $\mathrm{selfintsec}_G(f) = 0 \in \mathbb{Z}[G]/\min^\lambda(\mathbb{Z}[G])$ then f is regularly homotopic in $\mathrm{Int}(\mathrm{Free}(G,X))$ to a smooth embedding $f' : Y \to \mathrm{Int}(\mathrm{Free}(G,X))$ such that $\mathrm{Im}(f') \cap g\mathrm{Im}(f') = \emptyset$ for all $g \in G \setminus \{1\}$.*

Lemma 2.6. *If $X \in \boldsymbol{Mnf}^{2k}_{\mathrm{sg}}(G)$ then there is a canonical bijection*

$$S(G,X) \to \Theta(G,X,k); \ s \mapsto \gamma(s) \quad \textit{such that} \quad \rho_\Pi(\gamma(s)) \ni s.$$

Proof. This follows from (2.1.2)–(2.1.3). Q.E.D.

G acts on $S(G,X)$ by conjugation and the bijection above is a G-map. In this paper we identify $S(G,X)$ with $\Theta(G,X,k)$ via this bijection, whenever $X \in \boldsymbol{Mnf}^{2k}_{\mathrm{sg}}(G)$.

Definition 2.7. Let $X \in \boldsymbol{Mnf}^n_{\rm sg}(G)$ $(n = 2k \geqq 6)$ and let Y be as above. Then for $f \in \mathrm{Immer}(Y, \mathrm{Int}(X))$ define

$$\mu_X(f) \in \Gamma^{\lambda,w}(G(\mu), \mathbb{Z}) \quad (\text{where } G(\mu) = G \setminus (Q(G,X) \cup S(G,X)))$$

by

$$\mu_X(f)(g) = \begin{cases} \mathrm{selfintsec}_{\langle g\rangle}(\widehat{f}_g)(g) & (g \in (\{1\} \cup G(2)) \cap G(\mu)) \\ \mathrm{intsec}(f, g^{-1}f) & (g \in G \setminus (\{1\} \cup G(2))), \end{cases}$$

where $\widehat{f}_g \in \mathrm{Immer}(Y, \mathrm{Int}(\mathrm{Free}(g,X)))$ is an approximation regularly homotopic to f. We can regard

$$\mu_X(f) \in \mathbb{Z}[G \setminus S]/\Lambda(G,Q;\mathbb{Z}) = \mathbb{Z}[G]/(\Lambda(G,Q;\mathbb{Z}) + \mathbb{Z}[S])$$

in a canonical way, where $Q = Q(G,X)$, $S = S(G,X)$ and

$$\Lambda(G,Q;\mathbb{Z}) = \min{}^{\lambda}(\mathbb{Z}[G]) + \{\sum_{g \in Q} a_g g \mid a_g \in \mathbb{Z}\}.$$

The well-definedness of $\mu_X(f)$ is easily checked because $\dim X^g \leqq k-2$ for all $g \in (\{1\} \cup G(2)) \cap G(\mu)$.

Theorem 2.8. *Let k be an integer $\geqq 3$, $n = 2k$, $X \in \boldsymbol{Mnf}^n_{\rm sg}(G)$, and $\widehat{X} = X \setminus \left(\bigcup_\gamma X_\gamma\right)$ where γ runs over $\Theta(G,X,k)$. Let Y be a closed, connected, oriented, k-dimensional, smooth manifold. If $f \in \mathrm{Immer}(Y, \mathrm{Int}(\widehat{X}))$ satisfies $\mathrm{intsec}_G(f,f) = 0$ and $\mu_{\widehat{X}}(f) = 0 \in \mathbb{Z}[G \setminus S]/\Lambda(G,Q;\mathbb{Z})$ then f is regularly homotopic in $\widehat{X}$ to a smooth embedding $f' : Y \to \mathrm{Int}(\mathrm{Free}(G,X))$ such that $\mathrm{Im}(f') \cap g\mathrm{Im}(f') = \emptyset$ for all $g \in G \setminus \{1\}$, where $Q = Q(G,X)$ and $S = S(G,X)$.*

The result above is proved below.

3. Regular homotopies of immersions to embeddings

The present section is devoted to the proof of Theorem 2.8.
Let k be an integer $\geqq 3$, $n = 2k$, and $X \in \boldsymbol{Mnf}^n_{\rm sg}(G)$.

Lemma 3.1. *If $\gamma \in \Theta(G,X,k-1)$ then $\mathrm{Fix}_G(=H, X_\gamma)$ (where $H = \rho(\gamma)$) is open dense in X_γ.*

Proof. The conclusion follows from the observation $\dim \mathrm{Fix}_G(>H, X_\gamma) \leqq k-2$. Q.E.D.

Lemma 3.2. *If* $\gamma \in \Theta(G, X, k-1)$ *then* $|\rho_\Pi(\gamma) \cap G(2)| \leqq 1$.

Proof. Set $H = \rho_\Pi(\gamma)$. By Lemma 3.1, we can take a point z in $\mathrm{Int}(\mathrm{Fix}_G(=H, X_\gamma))$. By definition, $G_z = H$. Let $T_z(X)$ be the tangential H-representation at z in X. Then $T_z(X)$ is the direct sum $T_z(X^H) \oplus \nu_z(X^H, X)$ of H-representations. Set $V = \nu_z(X^K, X)$. By (2.1.1) and (2.1.4), H acts freely on $V \setminus \{0\}$ and $\dim_{\mathbb{R}} V = k+1$. In particular, $L = H \cap G(2)$ acts freely on $V \setminus \{0\}$. Thus each $g \in L$ acts on V like scalar multiplication by -1. Since V is a faithful H-representation, we get $|L| \leqq 1$. Q.E.D.

For the remainder of the current section, let Y be a closed, connected, oriented, k-dimensional, smooth manifold, and set $\widehat{X} = X \setminus (\bigcup_\gamma X_\gamma)$ where γ runs over $\Theta(G, X, k)$.

Lemma 3.3. *Let* $f : Y \to \mathrm{Int}(\mathrm{Free}(G, X))$ *be a smooth immersion. If* $\tau \in (\mathbb{Z}/2\mathbb{Z})[Q(G, X)]$ *then there exists a regular homotopy* $f_t : f \sim f_1$ $(f_0 = f)$ *in* $\mathrm{Int}(\widehat{X})$ *such that* $\mathrm{Im}(f_1) \subset \mathrm{Int}(\mathrm{Free}(G, X))$ *and* $\mathrm{selfintsec}_G(f_1) = \mathrm{selfintsec}_G(f) + \tau$ *in* $\mathbb{Z}[G]/\min^\lambda(\mathbb{Z}[G])$ $(\lambda = (-1)^k)$.

We shall assume for the moment that the lemma has been proved and deduce Theorem 2.8 from the lemma.

Proof that Lemma 3.3 $\Longrightarrow$ Theorem 2.8. Let $f : Y \to \mathrm{Int}(\widehat{X})$ be an immersion satisfying the hypotheses in Theorem 2.8. Since $\dim \mathrm{Sing}(G, \widehat{X}) \leqq k-1$, f is regularly homotopic to an immersion in $\mathrm{Int}(\mathrm{Free}(G, X)) = \mathrm{Int}(\mathrm{Free}(G, \widehat{X}))$. Thus we suppose $\mathrm{Im}(f) \subset \mathrm{Int}(\mathrm{Free}(G, X))$. Since $\mathrm{intsec}_G(f, f) = 0$ and $\mu_{\widehat{X}}(f) = 0$, we get $\mathrm{selfintsec}_G(f) \in (\mathbb{Z}/2\mathbb{Z})[Q(G, X)]$. By Lemma 3.3, f is regularly homotopic to f'' in $\widehat{X}$ such that $\mathrm{Im}(f'') \subset \mathrm{Int}(\mathrm{Free}(X))$ and $\mathrm{selfintsec}_G(f'') = 0$ in $\mathbb{Z}[G]/\min^\lambda(\mathbb{Z}[G])$. As the intersection form is invariant under homotopies, $\mathrm{intsec}_G(f'', f'') = \mathrm{intsec}_G(f, f) = 0$. By Lemma 2.5, f'' is regularly homotopic in $\mathrm{Int}(\mathrm{Free}(G, X))$ to a smooth embedding f' such that $\mathrm{Im}(f') \cap g\mathrm{Im}(f') = \emptyset$ for all $g \in G \setminus \{1\}$. Q.E.D.

Proof of Lemma 3.3. It suffices to prove the lemma in the case $\tau = g$ $(g \in Q(G, X))$.

Set $H = \langle g \rangle$. Since $\dim X^H = k-1$, there is a connected component X^H_β of dimension $k-1$. Let $\delta = \Phi(\beta) \in \Theta(G, X, k-1)$. Set $K = \rho_\Pi(\delta)$ $(= \rho^G_X(X^H_\beta))$. Fix a point $z \in \mathrm{Int}(\mathrm{Fix}_G(=K, X^K_\delta))$. Let $\nu = \nu(X^K_\delta, X)$ be the $N_G(K)$-normal bundle of X^K_δ in X. This normal bundle is often identified with an $N_G(K)$-tubular neighborhood of X^K_δ. Let $D_r(\nu)$ (resp. $S_r(\nu)$) be the radius r closed-disk (resp. sphere) bundle over X^K_δ associated with ν. Regard each $S(\nu) \subset D_r(\nu)$ as a submanifold of $\nu \subset X$. Thus $D_r = D_r(\nu|_z)$ (resp. $S_r = S_r(\nu|_z)$) is a $(k+1)$-dimensional closed disk (resp. k-dimensional sphere) centered at z. Take $r > 0$ so small that

$$D_r \cap aD_r \neq \emptyset \ (a \in G) \Longrightarrow a \in K, \tag{3.4}$$

and that $GD_r \cap G\mathrm{Im}(f) = \emptyset$. Then K acts freely on $D_r \setminus \{z\}$. Let $h' : D^{k+1} = D_r \to X$ be the canonical inclusion (hence a smooth embedding). Set $h = h'|_{\partial D^{k+1}} : S^k \to$

Int(Free(G, X)). We regard $\mathrm{selfintsec}_G(h) \in \Gamma^{\lambda,w}(G;\mathbb{Z})$. Clearly $\mathrm{selfintsec}_G(h)(1) = 0$ in $\mathbb{Z}/(1-\lambda)\mathbb{Z}$. Since h bounds a disk,

$$\mathrm{intsec}_G(h,h) = 0. \tag{3.5}$$

By (3.4), $\mathrm{selfintsec}_G(h)(a) = 0$ for all $a \in G \setminus K$. Set $J = \{a \in G \mid a = -\lambda\overline{a}\}$. By (3.5), $\mathrm{selfintsec}_G(h)(a) = 0$ for all $a \in G \setminus J$. Since $K \cap G(2) = g$, $\mathrm{selfintsec}_G(h)(a)$ is possibly nontrivial only when $a = g$. Note that $\mathrm{selfintsec}_G(h)(g) = \mathrm{selfintsec}_{\langle g\rangle}(h)(g)$. It is elementary to check that $\mathrm{selfintsec}_{\langle g\rangle}(h)(g) = 1$ in $\mathbb{Z}/2\mathbb{Z}$. Thus, we get $\mathrm{selfintsec}_G(h) = g$ in $\mathbb{Z}[G]/\min^\lambda(\mathbb{Z}[G])$. Take a $(k+1)$-dimensional connecting band $\psi(I\times D^k)$ from $\mathrm{Im}(f)$ to S_r in Int(Free(G, X)) as follows. Let $B = I \times D^k$ ($I = [0,1]$). Take a smooth embedding $\psi : B \to (\mathrm{Int}(X) \setminus \mathrm{Int}(D_r(\nu)))$ such that $\mathrm{Im}(\psi) \cap a\mathrm{Im}(\psi) \neq \emptyset$ $(a \in G) \Longrightarrow a = 1$, such that $\psi^{-1}(\mathrm{Im}(f)) = \{0\} \times D^k$ and $\psi^{-1}(S_r) = \{1\} \times D^k$, and such that $f^{-1}(\mathrm{Im}(\psi)) \cong D^k$. Set $U = f^{-1}(\mathrm{Im}(\psi))$ and $V = h^{-1}(\mathrm{Im}(\psi))$ $(\cong D^k)$. Construct the connected sum Y' of $Y = \mathrm{Domain}(f)$ with $S^k_h = \mathrm{Domain}(h)$ by

$$Y' = \{Y \setminus \mathrm{Int}(U)\} \cup (I \times S^{k-1}) \cup \left\{S^k_h \setminus \mathrm{Int}(V)\right\}.$$

Since $S^k_h = S^k$, Y' is diffeomorphic to Y. Define $f_1 : Y' \to \mathrm{Int}(\mathrm{Free}(G,X))$ by gluing $f|_{Y\setminus\mathrm{Int}(U)}$, $\psi|_{I\times S^{k-1}}$, and $h_{S^k\setminus\mathrm{Int}(V)}$. By construction, f is regularly homotopic to f_1 in $\mathrm{Int}(\widehat{X})$. In addition, one has that $\mathrm{selfintsec}_G(f_1) = \mathrm{selfintsec}_G(f) + g$ in $\mathbb{Z}[G]/\min^\lambda(\mathbb{Z}[G])$. Q.E.D.

4. Doubly parametrized quadratic modules

Let R denote a commutative ring with the unity, such that $a \equiv a^2 \mod 2R$ for all $a \in R$. For applications in surgery, the ring $\mathbb{Z}$ of integers and the ring $\mathcal{U}^{-1}\mathbb{Z}$ of $\mathcal{U}$-fractions of $\mathbb{Z}$, where $\mathcal{U}$ is a multiplicative set in $\mathbb{Z}$, will be of primary interest. Let $\lambda = 1$ or -1 and let $w : G \to \{\pm 1\}$ be a homomorphism. In the following, the ring $A = R[G]$ is equipped with the antiinvolution $-$ defined by $(\sum_{g\in G} a_g g)^- = \sum_{g\in G} a_g w(g) g^{-1}$ $(a_g \in R)$. Let

$$G(2) = \{g \in G \mid g^2 = 1, \text{ and } g \neq 1\}.$$

G acts on $G(2)$ by conjugation $f \mapsto gfg^{-1}$ ($g \in G(2)$, $g \in G$). Let Q and S be conjugation-invariant subsets of $G(2)$ satisfying

$$Q \subseteqq \{g \in G(2) \mid g = -\lambda\overline{g}\}, \text{ and} \tag{QC}$$

$$S \subseteqq \{g \in G(2) \mid g = \lambda\overline{g}\}. \tag{SC}$$

We define three R-submodules A_q, A_s, and Λ of A as follows:

$$A_q = A_q(G,S) := R[G \setminus S],$$
$$A_s = A_s(G,S) := R[S], \text{ and}$$
$$\Lambda = \Lambda(G,Q;R) := \min{}^{\lambda}(R[G]) + R[Q] \ \ \text{(the form parameter generated by } Q),$$

where $\min^{\lambda}(R[G])$ is the minimal form parameter of $R[G]$, i.e.

$$\min{}^{\lambda}(R[G]) = \langle x - \lambda\overline{x} \mid x \in A\rangle_R.$$

Clearly $A_q(G,S) = \mathrm{Map}(G \setminus S, R)$ and $A_s(G,S) = \mathrm{Map}(S,R)$ as R-modules. In the following, let

$$\boldsymbol{A} = (R,G,Q,S,\lambda,w).$$

Definition 4.1. A map $B : M \times M \to A$ (where M is a finitely generated A-module) is called a λ-*Hermitian form* on M if (4.1.1)–(4.1.3) are satisfied:

(4.1.1) B is biadditive,

(4.1.2) $B(ax, by) = bB(x,y)\overline{a}$,

(4.1.3) $B(x,y) \ = \lambda\overline{B(y,x)}$,

for all x, $y \in M$, a, $b \in A$. A map $q : M \to A_q/\Lambda$ is called an $\boldsymbol{A}$-*quadratic form* (or simply *quadratic form*) on M with respect to B if (4.1.4)–(4.1.6) are fulfilled:

(4.1.4) $q(gx) = gq(x)\overline{g}$ and $q(rx) = r^2q(x)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,

(4.1.5) $q(x+y) - q(x) - q(y) = B(x,y)$ in $A_q/\Lambda = A/(\Lambda + A_s)$, and

(4.1.6) $\widetilde{q(x)} + \lambda\overline{\widetilde{q(x)}} = B(x,x)$ in $A_q = A/A_s$ where $\widetilde{q(x)}$ is a lifting of $q(x)$,

for all $x, y \in M$, $r \in R$, and $g \in G$. A triple $\boldsymbol{M} = (M,B,q)$ consisting of a finitely generated A-module M, a λ-Hermitian form B on M and an $\boldsymbol{A}$-quadratic form q on M with respect to B, is called a *doubly parametrized* $\boldsymbol{A}$-*quadratic module* (or simply *quadratic module*).

Let ((proj)) be the category of all finitely generated projective A-modules, ((s-free)) the category of all finitely generated stably free A-modules, and ((free)) the category of all finitely generated free A-modules. Let $\mathcal{C}$ be one of ((proj)), ((s-free)), and ((free)). If M, $M' \in \mathcal{C}$ (more precisely $\mathrm{Obj}(\mathcal{C})$) then $\mathrm{Mor}(M,M')$ is the set of all A-linear isomorphisms $M \to M'$. Let $\boldsymbol{Q}(\boldsymbol{A})_{\mathcal{C}}$ be the category of all quadratic modules $\boldsymbol{M} = (M,B,q)$ such that $M \in \mathcal{C}$, and B is nonsingular. If $\boldsymbol{M} = (M,B,q)$, $\boldsymbol{M'} = (M',b',q') \in \boldsymbol{Q}(\boldsymbol{A})_{\mathcal{C}}$ then $\mathrm{Mor}(\boldsymbol{M},\boldsymbol{M'})$ is the set of all A-linear isomorphisms $f : M \to M'$ such that $B(x,y) = B'(f(x),f(y))$ and $q(x) = q'(f(x))$ ($\forall x$, $y \in M$).

Let Θ be a finite G-set. A G-map $\alpha : \Theta \to M$ (where M is a G-module) will be called a *positioning map*. Let

$$\boldsymbol{Q}(\boldsymbol{A},\Theta)_{\mathcal{C}}$$

be the category of all $\boldsymbol{M} = (M, B, q, \alpha)$ such that $(M, B, q) \in \boldsymbol{Q}(\boldsymbol{A})_{\mathcal{C}}$ and $\alpha : \Theta \to M$ is a G-map. If $\boldsymbol{M} = (M, B, q, \alpha)$, $\boldsymbol{M}' = (M', B', q', \alpha') \in \boldsymbol{Q}(\boldsymbol{A}, \Theta)_{\mathcal{C}}$ then $\mathrm{Mor}(\boldsymbol{M}, \boldsymbol{M}')$ is the set of all morphisms $f : (M, B, q) \to (M', B', q')$ such that $\alpha(x) = \alpha'(f(x))$ $(\forall x \in \Theta)$. For $\boldsymbol{M} = (M, B, q, \alpha)$, an A-direct summand L of M is called a *$\mathcal{C}$-Lagrangian* of $\boldsymbol{M}$ if $L \in \mathcal{C}$, $B(L, L) = 0$, $q(L) = 0$, $L = L^{\perp}$, and $\alpha(\Theta) \subset L$, where

$$L^{\perp} = \{x \in M \mid B(x, y) = 0 \ (\forall y \in L)\}.$$

If $\boldsymbol{M}$ has a $\mathcal{C}$-Lagrangian then $\boldsymbol{M}$ is called *a $\mathcal{C}$-null module*. Define $KQ_0(\boldsymbol{A}, \Theta)_{\mathcal{C}}$ to be the Grothendieck group of the category $\boldsymbol{Q}(\boldsymbol{A}, \Theta)_{\mathcal{C}}$ with respect to orthogonal sum. If $\mathcal{C} \supseteqq \mathcal{D} \supseteqq ((\text{free}))$, define

$$WQ_0(\boldsymbol{A}, \Theta)_{\mathcal{C},\mathcal{D}} = KQ_0(\boldsymbol{A}, \Theta)_{\mathcal{C}} / \langle \mathcal{D}\text{-null modules in } \boldsymbol{Q}(\boldsymbol{A}, \Theta)_{\mathcal{D}} \rangle.$$

In the remainder of this paper we treat only the case that $\Theta = S$ and the action of G on S is via conjugation. To $\boldsymbol{M} = (M, B, q, \alpha) \in \boldsymbol{Q}(\boldsymbol{A}, S)_{\mathcal{C}}$, we associate a function $\nabla = \nabla_{\boldsymbol{M}} : M \to \mathrm{Map}(S, R/2R)$ defined by

$$\text{(4.2)} \qquad \nabla(x)(s) = [\varepsilon(B(\alpha(s) - x, sx))], \quad (x \in M,\ s \in S).$$

where $\varepsilon : A \to R$ is the ring homomorphism $\sum_{g \in G} a_g g \mapsto a_1$ $(a_g \in R)$.

Lemma 4.3. *Let $\boldsymbol{M} = (M, B, q, \alpha) \in \boldsymbol{Q}(\boldsymbol{A}, S)_{\mathcal{C}}$. Then for each a, $b \in R$, x, $y \in M$, and $s \in S$, one has the formula*

$$\nabla_{\boldsymbol{M}}(ax + by)(s) = a\nabla_{\boldsymbol{M}}(x)(s) + b\nabla_{\boldsymbol{M}}(y)(s) \quad \text{in } R/2R.$$

Since the proof follows by straightforward calculation from Definition (4.2), we omit it (note that $a^2 \equiv a \mod 2R$ for $a \in R$).

A quadratic module $\boldsymbol{M}$ with positioning map is called a *special quadratic module* if $\nabla_{\boldsymbol{M}} = 0$. Let $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ be the full subcategory of $\boldsymbol{Q}(\boldsymbol{A}, S)_{\mathcal{C}}$ consisting of all special quadratic modules. Define $SKQ_0(\boldsymbol{A}, S)_{\mathcal{C}}$ to be the Grothendieck group of the category $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ with respect to orthogonal sum. If $\mathcal{C} \supseteqq \mathcal{D} \supseteqq ((\text{free}))$, define

$$SWQ_0(\boldsymbol{A}, S)_{\mathcal{C},\mathcal{D}} = SKQ_0(\boldsymbol{A}, S)_{\mathcal{C}} / \langle \mathcal{D}\text{-null modules in } \boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{D}} \rangle.$$

Now let $n = 2k$ be an even integer $\geqq 6$, and $\lambda = (-1)^k$. Set

$$\text{(4.4)} \qquad W_n(\boldsymbol{A}, S)_{\mathcal{C}} = SWQ_0(\boldsymbol{A}, S)_{\mathcal{C},((\text{free}))}.$$

Let $\boldsymbol{NSQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ denote the full subcategory of $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ consisting of all $\mathcal{C}$-null modules.

Proposition 4.5. *$\boldsymbol{NSQ}(\boldsymbol{A}, S)_{((\text{free}))}$ is a cofinal subcategory of $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$. That is, each $\boldsymbol{M} \in \boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ is a direct summand of some $\boldsymbol{N} \in \boldsymbol{NSQ}(\boldsymbol{A}, S)_{((\text{free}))}$.*

Proof. Let $\boldsymbol{M} = (M, B, q, \alpha)$. Since B is nonsingular, M is a selfdual A-module, namely $M \cong M^{\#} := \mathrm{Hom}_A(M, A)$. Since M is a finitely generated projective A-module, M is a direct summand of A^m for large m. Say $A^m = M \oplus M'$. Let $\boldsymbol{H}(M') = (M' \oplus M'^{\#}, B', q', 0)$ be the hyperbolic module: $B'((x, f), (x', f')) = f(x') + \lambda \overline{f'(x)} \in A$ for x, $x' \in M'$ and f, $f' \in M'^{\#}$; $q'((x, f)) = [f(x)] \in A/(\Lambda + A_s)$; and $0 : S \to M' \oplus M'^{\#}$ is the trivial map. Then $\boldsymbol{M} \oplus \boldsymbol{M} \oplus \boldsymbol{H}(M')$ has the underlying A-module $M \oplus M^{\#} \oplus M' \oplus M'^{\#}$, namely A^{2m}. It is easy to check that $\boldsymbol{H}(M) \in \boldsymbol{NSQ}(\boldsymbol{A}, S)_{\mathcal{C}}$. This allows us to assume that M is a free A-module.

Now let M' be a copy of M and $\psi : M \to M'$ an A-isomorphism. Define $B' : M' \times M' \to A$, $q' : M' \to A_q/\Lambda$, and $\alpha' : S \to M'$ as follows: $B'(\psi(x), \psi(y)) = -B(x, y)$, $q'(\psi(x)) = -q(x)$, and $\alpha'(s) = \psi(\alpha(s))$, for x, $y \in M$ and $s \in S$. Set $\boldsymbol{M}' = (M', B', q', \alpha')$. Obviously $\boldsymbol{M}' \in \boldsymbol{SQ}(\boldsymbol{A}, S)_{((\text{free}))}$. Now consider $\boldsymbol{N} = \boldsymbol{M} \oplus \boldsymbol{M}'$. Then $L = \{(x, \psi(x)) \mid x \in M\}$ is a ((free))-Lagrangian of $\boldsymbol{N}$. More precisely, one has that

$$
\begin{aligned}
&(B \oplus B')((x, \psi(x)), (x, \psi(x))) = B(x, x) + B'(\psi(x), \psi(x)) = 0, \\
&(q \oplus q')(x, \psi(x)) = q(x) + q'(\psi(x)) = 0, \quad \text{and} \\
&(\alpha \oplus \alpha')(s) = (\alpha(s), \alpha'(s)) = (\alpha(s), \psi(\alpha(s))) \in L.
\end{aligned}
$$

Moreover $L \cong M$ is an A-free, direct summand of $M \oplus M'$ ($M \oplus 0$ is a complementary summand to L). Q.E.D.

It is easy to see that the canonical homomorphism $W_n(\boldsymbol{A}, S)_{((\text{free}))} \to W_n(\boldsymbol{A}, S)_{((\text{s}-\text{free}))}$ is an isomorphism and that the canonical homomorphism $W_n(\boldsymbol{A}, S)_{((\text{s}-\text{free}))} \to W_n(\boldsymbol{A}, S)_{((\text{proj}))}$ is injective. We could define $W_n(\boldsymbol{A}, \Theta)_{\mathcal{C}}$ for more general Θ, but we omit such generalizations for simplicity. For suitable Q, S, λ, and w, the group $W_n(\boldsymbol{A}, S)_{\mathcal{C}}$ will be called a *G-surgery obstruction group.*

5. Metabolic planes for special quadratic modules

We construct specific quadratic modules with positioning map whose ∇-invariant is trivial, called *special metabolic planes* and use them to decompose ((free))-null modules with trivial ∇.

Definition 5.1. Let $\beta = (c, r)$ be a pair of elements c, $r \in \mathrm{Map}(S, R)$ such that

(BC) $c(s) \equiv r(s) \mod 2R$ for any $s \in S$.

Let x and y be distinct letters. The *special metabolic plane*

$$\boldsymbol{M}(x, y, \beta) = (M(x, y), B_r, q, \alpha_c)$$

associated to β with *metabolic basis* $\{x, y\}$ is defined as follows. Let $M = M(x, y)$ be the free $R[G]$-module with basis $\{x, y\}$, i.e. $M(x, y) = \langle x,\ y\rangle_{R[G]}$. Let $B_r : M \times M \to R[G]$ be the unique map satisfying Relations (4.1.1)–(4.1.3) and

(5.1.1) $B_r(x,x) = 0$, $B_r(y,x) = 1$, and $B_r(y,y) = \sum_{g \in S} r(g)g$.

Define $q : M \to R[G \setminus S]/\Lambda(G,Q;R)$ to be the unique map satisfying Relations (4.1.4)–(4.1.5) and

(5.1.2) $q(x) = 0$ and $q(y) = 0$.

Clearly, for a, $b \in R[G]$,

$$q(ax + by) = B(by, ax) = a\bar{b} \in R[G \setminus S]/\Lambda(G,Q;R) = R[G]/(\Lambda(G,Q;R) + R[S]).$$

Thus, (4.1.6) is satisfied. Let G act as usual on S by conjugation (hence $g \cdot s = gsg^{-1}$), and define a map $\alpha_c : S \to M$ by

(5.1.3) $\alpha_c(s) = \sum_{g \in G} c(g \cdot s)g^{-1}x$,

Clearly α_c is a G-map (a positioning map).

Proposition 5.2. *Let $\boldsymbol{M}(x,y,\beta)$ be a special metabolic plane as in Definition 5.1. Then $\boldsymbol{M}(x,y,\beta)$ belongs to $\boldsymbol{SQ}(\boldsymbol{A},S)_{((\mathrm{free}))}$.*

Proof. By Lemma 4.3, it suffices to prove that $\nabla_{\boldsymbol{M}(x,y,\beta)}(ax)(s) = 0$ and $\nabla_{\boldsymbol{M}(x,y,\beta)}(ay)(s) = 0$ for every $a \in G$ and $s \in S$.

The second equality holds because

$$\begin{aligned}\nabla_{\boldsymbol{M}(x,y,\beta)}(ay)(s) &= [\varepsilon(B_r(\alpha_c(s) - ay, say))] \\ &= [\varepsilon(B_r(\sum_{g \in G} c(gsg^{-1})g^{-1}x, say))] - [\varepsilon(B_r(ay, say))] \\ &= [\varepsilon(\lambda \sum_{g \in G} w(g)c(gsg^{-1})sag)] - [\varepsilon(\sum_{h \in S} r(h)sah\overline{a})] \\ &= [\lambda w(sa)c(a^{-1}sa)] - [w(a)r(a^{-1}sa)] = 0 \quad \in R/2R.\end{aligned}$$

The first equality is straightforward to check. Q.E.D.

Lemma 5.3. *Let $\beta = (c,r)$ and $\beta' = (c',r')$ satisfy Condition* (BC)*. If there exists an $a \in R^\times :=$* Unit(R) *such that $ac(s) = c'(s)$ and $a^2r(s) = r'(s)$ for any $s \in S$ then $\boldsymbol{M}(x,y,\beta)$ is isomorphic to $\boldsymbol{M}(x',y',\beta')$.*

Proof. Let $f : M(x,y) \to M(x',y')$ denote the $R[G]$-linear map determined by the equations $f(x) = ax'$ and $f(y) = a^{-1}y'$. Then

$$\begin{aligned} f(\alpha_c(s)) &= f(\sum_{g\in G} c(gsg^{-1})g^{-1}x) \\ &= \sum_{g\in G} ac(gsg^{-1})g^{-1}x' = \alpha_{c'}(s), \quad \text{and} \\ B_{r'}(f(y), f(y)) &= B_{r'}(a^{-1}y', a^{-1}y') \\ &= a^{-2}\sum_{s\in S} r'(s)s \\ &= \sum_{s\in S} r(s)s = B_r(y,y). \end{aligned}$$

Using this, the reader can easily check that f is an isomorphism. Q.E.D.

Proposition 5.4. *Suppose R is the ring $\mathcal{U}^{-1}\mathbb{Z}$ of $\mathcal{U}$-fractions of $\mathbb{Z}$ where $\mathcal{U}$ is a multiplicative set in $\mathbb{Z}$. Let $\beta = (c, r)$ be a pair of elements c, $r \in \mathrm{Map}(S, R)$ satisfying Condition* (BC)*. Then there exists a pair $\beta' = (c', r')$ of elements c', $r' \in \mathrm{Map}(S, \mathbb{Z})$ satisfying* (BC) *such that $\boldsymbol{M}(x, y, \beta)$ is isomorphic to $\boldsymbol{M}(x', y', \beta')$.*

Proof. Since S is finite, there is an integer $a \in \mathcal{U}$ such that ac, $ar \in \mathrm{Map}(S, \mathbb{Z})$. Set $\beta' = (ac, a^2r)$. Then by Lemma 5.3, $\boldsymbol{M}(x, y, \beta) \cong \boldsymbol{M}(x', y', \beta')$. Q.E.D.

Lemma 5.5. *Let $\beta = (c, r)$ and $\beta' = (c', r')$ satisfy the Condition* (BC)*. If $c(s) = c'(s)$ and $r(s) \equiv r'(s) \mod 2R$ for any $s \in S$ then $\boldsymbol{M}(x, y, \beta)$ is isomorphic to $\boldsymbol{M}(x', y', \beta')$. Thus the isomorphism class of $\boldsymbol{M}(x, y, (c, r))$ depends only on c.*

Proof. By hypothesis, there exists an $a \in R[S]$ such that $a + \lambda\overline{a} = \sum_{s\in S}(r'(s) - r(s))s$. Let $f : M(x, y) \to M(x', y')$ be an $R[G]$-linear map such that $f(x) = x'$ and $f(y) = y' - ax'$. Then

$$\begin{aligned} B_{r'}(f(y), f(y)) &= B_{r'}(y' - ax', y' - ax') \\ &= B_{r'}(y', y') - B_{r'}(y', ax') - B_{r'}(ax', y') + B_{r'}(ax', ax') \\ &= \left(\sum_{s\in S} r'(s)s\right) - a - \lambda\overline{a} + 0 \\ &= \sum_{s\in S} r(s)s = B_r(y, y). \end{aligned}$$

Using this, the reader can check easily that f is an isomorphism. Q.E.D.

Theorem 5.6. *If $\boldsymbol{M} = (M, B, q, \alpha)$ is a ((free))-null, special quadratic module with Lagrangian L then there exist pairs $\beta_i = (c_i, r_i)$, where c_i, $r_i \in \mathrm{Map}(S, R)$ satisfying* (BC) *($i = 1, \cdots, m = \mathrm{rank}_{R[G]} L$) such that $\boldsymbol{M}$ is isomorphic to $\boldsymbol{M}(x_1, y_1, \beta_1) \oplus \cdots \oplus \boldsymbol{M}(x_m, y_m, \beta_m)$.*

The result above is proved in the next section.

Corollary 5.7. *Suppose R is a ring of fractions of $\mathbb{Z}$. If $\boldsymbol{M} = (M, B, q, \alpha)$ is a ((free))-null, special quadratic module with Lagrangian L then there exist pairs $\beta_i = (c_i, r_i)$ where c_i, $r_i \in \mathrm{Map}(S, \mathbb{Z})$ satisfying* (BC) *($i = 1, \cdots, m = \mathrm{rank}_{R[G]} L$) such that $\boldsymbol{M}$ is isomorphic to $\boldsymbol{M}(x_1, y_1, \beta_1) \oplus \cdots \oplus \boldsymbol{M}(x_m, y_m, \beta_m)$.*

Proof. The result follows immediately from Proposition 5.4 and Theorem 5.6. Q.E.D.

Corollary 5.8. *The family of special metabolic planes is cofinal in the category of special quadratic modules.*

Proof. The result follows immediately from Proposition 4.5 and Theorem 5.6. Q.E.D.

Corollary 5.9. *Suppose R is a ring of fractions of $\mathbb{Z}$. Then the family of special metabolic planes $\boldsymbol{M}(x, y, (c, r))$ such that c, $r \in \mathrm{Map}(S, \mathbb{Z})$ is cofinal in the category of special quadratic modules.*

Proof. The result follows immediately from Proposition 5.4 and Corollary 5.8. Q.E.D.

6. Decomposition of ((free))-null modules

This section is devoted to the proof of Theorem 5.6.

Let $\boldsymbol{M} = (M, B, q, \alpha)$ be a ((free))-null module with Lagrangian L. Let $\{x_1, \cdots, x_m\}$ ($m = \mathrm{rank}_A L$) be an arbitrary basis of L. Since L is a Lagrangian, the sequence

$$0 \to L \to M \xrightarrow{\tau} \mathrm{Hom}_A(L, A) \to 0,$$

is split-exact over A where $\tau(y) \in \mathrm{Hom}_A(L, A)$ ($y \in M$) is given by $\tau(y)(x) = B(y, x)$ ($x \in L$). Thus there exist elements y_i ($i = 1, \cdots, m$) in M such that $B(y_i, x_i) = \delta_{ij}$. By the split-exact sequence above, $\{x_i, y_i \mid 1 \leqq i \leqq m\}$ is a basis of M.

Lemma 6.1 (Orthonormalization of Gram-Schmidt-Wall)**.** *Suppose that for some integer k,*

$$B(y_i, y_j) = 0 \quad (\text{for all } i < j \leqq k), \text{ and}$$
$$q(y_i) = 0 \quad (\text{for all } i \leqq k).$$

Set

$$y'_{k+1} = y_{k+1} - \left(\widetilde{q(y_{k+1})} x_{k+1} + \sum_{i \leqq k} B(y_i, y_{k+1}) x_i \right), \tag{6.1.1}$$

where $\widetilde{q(y_{k+1})} \in A$ *is a lifting of* $q(y_{k+1}) \in A_q/\Lambda = A/(\Lambda + A_s)$*. Then it follows that* $B(y'_{k+1}, x_j) = \delta_{k+1,j}$ *for all* j*,* $B(y_j, y'_{k+1}) = 0$ *for all* $j < k+1$*, and* $q(y'_{k+1}) = 0$*.*

Since the proof is a straightforward calculation, we omit it.

Inductive use of Lemma 6.1 on k produces the next corollary.

Corollary 6.2. *For an arbitrary basis* $\{x_1, \cdots, x_m\}$ *of* L*, there exist elements* $y_1, \cdots, y_m \in M$ *such that* $B(y_i, x_j) = \delta_{ij}$ *(for all* i*,* j*),* $B(y_i, y_j) = 0$ *(for all* $i \neq j$*), and* $q(y_i) = 0$ *(for all* i*).*

Theorem 6.3. *Let* $\boldsymbol{M} = (M, B, q, \alpha)$ *be a ((free))-null module with Lagrangian* $L = \langle x \rangle_A$*. Let* $y \in M$ *be an element such that* $B(y, x) = 1$ *and* $q(y) = 0$*. Then* $\boldsymbol{M}$ *is isomorphic to the special metabolic plane* $\boldsymbol{M}(x, y, \beta)$ *associated to* $\beta = (c_\alpha, r)$*, where* c_α*,* $r \in \mathrm{Map}(S, R)$ *are determined by the equations*

$$\alpha(s) = \sum_{g \in G} c_\alpha(gsg^{-1}) g^{-1} x \quad (\forall s \in S), \quad \text{and} \quad B(y, y) = \sum_{s \in S} r(s) s.$$

For the moment, assume that Theorem 6.3 has been proved.

Proof that Theorem 6.3 $\Longrightarrow$ Theorem 5.6. We shall prove that $\boldsymbol{M}$ is isomorphic to an orthogonal sum of special metabolic planes associated to certain $\beta_i = (c_i, r_i)$ where

(6.4) c_i, $r_i \in \mathrm{Map}(S, R)$.

Let $\{x_1, \cdots, x_m\}$ be a basis of L and let $\{y_1, \cdots, y_m\}$ be as in Corollary 6.2. Set $M_i = \langle x_i, y_i \rangle_A$, $B_i = B|_{M_i} : M_i \times M_i \to A$, and $q_i = q|_{M_i} : M_i \to A_q/\Lambda$. Let $p_i : M \to M_i$ be the projection with respect to the basis $\{x_i, y_i \mid 1 \leqq i \leqq m\}$. Set $\alpha_i = p_i \circ \alpha : S \to M_i$. It is easy to check that $\boldsymbol{M}_i = (M_i, B_i, q_i, \alpha_i) \in \boldsymbol{SQ}(\boldsymbol{A}, S)_{((\text{free}))}$ with ((free))-Lagrangian $L_i = \langle x_i \rangle_A$. Now use Theorem 6.3 to deduce that each $\boldsymbol{M}_i$ is isomorphic to a special metabolic plane. Thus $\boldsymbol{M} = \boldsymbol{M}_1 \oplus \cdots \oplus \boldsymbol{M}_m$ is isomorphic to a orthogonal sum of special metabolic planes. Q.E.D.

The rest of this section is devoted to the proof of Theorem 6.3.

Let $\boldsymbol{M} = (M, B, q, \alpha)$ and x, y be as in Theorem 6.3. For every $s \in S$, $\alpha(s)$ has the form

$$\alpha(s) = \sum_{g \in G} a_\alpha(s, g) g^{-1} x, \tag{6.5}$$

where $a_\alpha(s,g) \in R$. As usual G acts on S by conjugation. The isotropy subgroup G_s (at s in the G-space S) is $\{h \in G \mid hsh^{-1} = s\}$. We define $\Sigma_{G_s} \in A$ by

$$\Sigma_{G_s} = \sum_{h \in G_s} h.$$

Since α is a G-map, $\alpha(s)$ is G_s-invariant. Thus $\alpha(s)$ has the form

$$\alpha(s) = \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(s, gsg^{-1}) \Sigma_{G_s} g^{-1} x, \tag{6.6}$$

where $\widehat{a}_\alpha(s, gsg^{-1}) = a_\alpha(s,g)$.

Lemma 6.7. *In the above situation,* $\widehat{a}_\alpha(s, gsg^{-1}) = \widehat{a}_\alpha(fsf^{-1}, gsg^{-1})$ *for any* $f \in G$.

Proof. Let $h \in G$. Then

$$\begin{aligned}
h\alpha(s) &= \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(s, gsg^{-1}) h \Sigma_{G_s} g^{-1} x \\
&= \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(s, gsg^{-1}) h \Sigma_{G_s} h^{-1} (gh^{-1})^{-1} x \\
&= \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(s, gsg^{-1}) \Sigma_{G_{hsh^{-1}}} (gh^{-1})^{-1} x \\
&= \sum_{(gh^{-1})G_{hsh^{-1}} \in G/G_{hsh^{-1}}} \widehat{a}_\alpha(s, gsg^{-1}) \Sigma_{G_{hsh^{-1}}} (gh^{-1})^{-1} x \\
&= \sum_{g'G_{hsh^{-1}} \in G/G_{hsh^{-1}}} \widehat{a}_\alpha(s, g'(hsh^{-1}){g'}^{-1}) \Sigma_{G_{hsh^{-1}}} {g'}^{-1} x.
\end{aligned}$$

On the other hand,

$$\alpha(hsh^{-1}) = \sum_{g'G_{hsh^{-1}} \in G/G_{hsh^{-1}}} \widehat{a}_\alpha(hsh^{-1}, g'(hsh^{-1}){g'}^{-1}) \Sigma_{G_{hsh^{-1}}} {g'}^{-1} x.$$

Since $h\alpha(s) = \alpha(hsh^{-1})$, we get

$$\widehat{a}_\alpha(s, g'(hsh^{-1}){g'}^{-1}) = \widehat{a}_\alpha(hsh^{-1}, g'(hsh^{-1}){g'}^{-1})$$

for all g'. Substitute now in the equation above f^{-1}, g and fsf^{-1} for h, g', and s, respectively. Then we obtain that $\widehat{a}_\alpha(fsf^{-1}, gsg^{-1}) = \widehat{a}_\alpha(s, gsg^{-1})$. Q.E.D.

Lemma 6.8. *If* $c_\alpha : S \to R$ *is defined by* $c_\alpha(s) = \widehat{a}_\alpha(s,s)$ *then*

$$\alpha(s) = \sum_{g \in G} c_\alpha(gsg^{-1})g^{-1}x.$$

Proof. This is shown by straightforward calculation:

$$\begin{aligned} \alpha(s) &= \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(s, gsg^{-1})\Sigma_{G_s} g^{-1}x \\ &= \sum_{gG_s \in G/G_s} \widehat{a}_\alpha(gsg^{-1}, gsg^{-1})\Sigma_{G_s} g^{-1}x \\ &= \sum_{gG_s \in G/G_s} c_\alpha(gsg^{-1})\Sigma_{G_s} g^{-1}x \\ &= \sum_{g \in G} c_\alpha(gsg^{-1})g^{-1}x. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 6.9. *If* $r : G \to R$ *is defined by* $B(y,y) = \sum_{g \in G} r(g)g$ *then* $r(g) = 0$ *for all* $g \in G \setminus S$.

Proof. The conclusion follows immediately from the hypothesis $q(y) = 0$ and the property (4.1.6). Q.E.D.

Putting Lemmas 6.8 and 6.9 together, we get $\boldsymbol{M} \cong \boldsymbol{M}(x,y,\beta)$. This completes the proof of Theorem 6.3.

7. G-Surgery theorem

Throughout this section let $n = 2k$ be an even integer $\geqq 6$, let X and Y be closed manifolds in $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)$, let $\lambda = (-1)^k$ and $w = w^G_X$, and let R be a ring of fractions of $\mathbb{Z}$.

A pair (f,b) is called a *G-framed map* if $f : X \to Y$ is a G-map and $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$ is a G-vector bundle isomorphism (covering the identity map on X) for real G-vector bundles η and ξ over Y. A G-framed map (f,b) is said to be of *degree* 1 (resp. *k-connected*) if f is of degree 1 (resp. k-connected).

If V is a real G-module, let $\varepsilon_X(V)$ denote the product bundle $X \times V \to X$ with fiber V. Let $\mathbb{R}$ be the 1-dimensional, trivial, real G-module.

In the sequel we always assume

(HC) the bundle η is sufficiently large; more precisely, $\eta \supseteqq \varepsilon_Y(\mathbb{R}^{n+1})$ where $n = \dim X$.

Proposition 10.1 in the appendix demonstrates one advantage of this assumption.

Let $I = [0,1]$ and let $p_Y : I \times Y \to Y$ be the canonical projection. For a closed subset $Z \subseteqq X$, a *cobordism* $(F,B) : (f,b) \sim (f',b')$ *relative to* Z is defined in the usual way:

$F : W \to (I \times Y)$ $(I = [0,1])$ is a G-map such that $\partial W = (-X) \cup X'$, $F(-X) \subseteqq (\{0\} \times Y)$, $F(X') \subseteqq (\{1\} \times Y)$, $F|_{-X} = f$, $F|_{X'} = f'$, (where $(I \times Z) \subseteqq W$ in a canonical way, and $F|_{I\times Z} = id_I \times f|_Z$); $B : T(W) \oplus (p_Y \circ F)^*\eta \to (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$ is a real G-vector bundle isomorphism such that $B|_{-X} = id_{\varepsilon_{-X}(\mathbb{R})} \oplus b$ and $B|_{X'} = id_{\varepsilon_{X'}(\mathbb{R})} \oplus b'$, $(T(W)|_{I\times Z} = \varepsilon_I(\mathbb{R}) \times T(X)|_Z$ in a canonical way, and $B|_{I\times Z} = id_{\varepsilon_I(\mathbb{R})} \times b|_Z)$.

Our first goal is to define a quadruple $\boldsymbol{M_f} = (K_k(f;R), B_f, q_{\boldsymbol{f}}, \alpha_f)$ for any k-connected, degree 1, G-framed map $\boldsymbol{f} = (f,b)$. Let $\boldsymbol{f} = (f,b)$ be a degree 1, G-framed map and let $\boldsymbol{A} = (R, G, Q, S, \lambda, w, S)$, where $Q = Q(G,X)$ and $S = S(G,X)$. Let

$$\mathrm{pdual}_X : H^*(X;R) \to H_*(X;R)$$

denote the Poincaré duality homomorphism, and let

$$\mathrm{ppair}_X : H_k(X;R) \times H_k(X,R) \to R$$

denote the Poincaré pairing. For each integer ℓ, define

$$K_\ell(f;R) = \mathrm{Ker}[f_* : H_\ell(X;R) \to H_\ell(Y;R)], \quad \text{and}$$
$$K^\ell(f;R) = \mathrm{Coker}[f^* : H^\ell(Y;R) \to H^\ell(X;R)].$$

Suppose that f is k-connected.

This assumption implies by [5, I.2.8] that $K_k(f;R) = R \otimes_{\mathbb{Z}} K_k(f;\mathbb{Z}) \cong K^k(f;R) = \mathrm{Hom}_{\mathbb{Z}}(K_k(f;\mathbb{Z}), R)$ as R-modules and that these modules are finitely generated, free R-modules. The λ-Hermitian module $(H_k(X;R), \mathrm{ppair})$ over R can be decomposed into the orthogonal sum $(K_k(f;R), \mathrm{ppair}|) \oplus (\mathrm{pdual}(\mathrm{Im}(f^*)), \mathrm{ppair}|)$. It is well-known that $(K_k(f;R), \mathrm{ppair}|)$ is nonsingular ([5, I.2.9]). Let $\pi_f : H_k(X;R) \to K_k(f;R)$ be the canonical projection, namely

$$\pi_f(x) = x - \mathrm{pdual}_X \circ f^* \circ \mathrm{pdual}_Y^{-1} \circ f_*(x).$$

We treat first the case to $R = \mathbb{Z}$ and define B_f, $q_{\boldsymbol{f}}$, and α_f for $K_k(f;\mathbb{Z})$. This done, we extend B_f, $q_{\boldsymbol{f}}$, and α_f to $K_k(f;R)$ in the usual way, using the fact that $K_k(f;R) \cong R \otimes_{\mathbb{Z}} K_k(f,;\mathbb{Z})$.

For the moment we forget the G-action on X and apply the ordinary surgery theory of C. T. C. Wall. Since $f : X \to Y$ is k-connected, the canonical map $\pi_{k+1}(f) \to K_k(X;\mathbb{Z})$ is surjective. Thus each element $x \in K_k(X;\mathbb{Z})$ can be represented by a continuous map $h'_x : S^k \to X$ such that $f \circ h'_x$ is null homotopic in Y. This h'_x can be approximated by an immersion. Since $f \circ h'_x$ is null homotopic, ${h'_x}^*(T(X) \oplus f^*\eta) \cong (f \circ h'_x)^*(\xi \oplus \eta)$ is a trivial bundle. Thus, it follows from Hirsch's immersion classification theorem that the map h'_x is homotopic to an immersion h_x with trivial normal bundle in X. Moreover the regular homotopy class of h_x in X is uniquely determined by x (providing, of course, f and b are fixed).

It is well known that $\mathrm{ppair}(x,y) = \pm\mathrm{intsec}(h_x, h_y)$ $(\forall x,\ y \in K_k(f;\mathbb{Z}))$. The sign $\pm$ is determined by the definitions of ppair and intsec. We shall adopt definitions such that $\mathrm{ppair}(x,y) = (-1)^k \mathrm{intsec}(h_x, h_y)$. (The sign will not be essential in our arguments. A reader preferring definitions of ppair and intsec such that $\mathrm{ppair}(x,y) = \mathrm{intsec}(h_x, h_y)$ can easily modify the arguments.)

Reimpose now the G-action on X. Define $B_f : K_k(f;\mathbb{Z}) \times K_k(f;\mathbb{Z}) \to \mathbb{Z}[G]$ by

$$B_f(x,y) = \sum_{g\in G} \mathrm{intsec}(h_x, g^{-1}h_y)g \left(= (-1)^k \sum_{g\in G} \mathrm{ppair}(x, g^{-1}y)g \right).$$

Then $(K_k(f;\mathbb{Z}), B_f)$ is a nonsingular λ-Hermitian module over $\mathbb{Z}[G]$ by [5, I.2.9] and [1, (1.2.4)].

Define $q_{\boldsymbol{f}} : K_k(f;\mathbb{Z}) \to \mathbb{Z}[G \setminus S]/\Lambda(G,X;\mathbb{Z})$ by

$$q_{\boldsymbol{f}}(x) = \mu_X(h_x).$$

By Hypothesis (2.1.5), each X_γ ($\gamma \in \Theta(G,X,k)$) has the orientation class $\mathrm{ori}(X_\gamma) \in H_k(X_\gamma;\mathbb{Z})$. Let $j_\gamma : X_\gamma \to X$ be the canonical inclusion. Adopting the identification in Lemma 2.6, define $\alpha_f : S = \Theta(G,X,k) \to K_k(f;\mathbb{Z})$ by

$$\alpha_f(s) = \alpha_f(\gamma(s)) = \pi_f \circ {j_{\gamma(s)}}_*(\mathrm{ori}(X_{\gamma(s)})).$$

By (2.1.5), α_f is a G-map.

This completes the definition of the quadruple

$$\boldsymbol{M_f} = (K_k(f;R), B_f, q_{\boldsymbol{f}}, \alpha_f)$$

for $R = \mathbb{Z}$.

We consider next the case of a general R. There is a canonical homomorphism $\varphi : K_k(f;\mathbb{Z}) \to K_k(f;R)$ and the induced R-homomorphism $R \otimes \varphi : R \otimes K_k(f;\mathbb{Z}) \to K_k(f;R)$ is an isomorphism by the universal coefficient theorem [5, I.2.8]. Thus we can extend B_f above to a pairing $B_f : K_k(f;R) \times K_k(f;R) \to R[G]$ by using the rule $B_f(rx, r'y) = rr'B_f(x,y)$ (r, $r' \in R$, x, $y \in K_k(f;R)$), $q_{\boldsymbol{f}}$ above to a map $q_{\boldsymbol{f}} : K_k(f;R) \to R[G]_q/\Lambda(G,X;R)$ by using the rule $q_f(rx) = r^2 q_f(x)$ ($r \in R$, $x \in K_k(f;R)$), and α_f above to a function $\alpha_f : S \to K_k(f;R)$ by composing it with φ. It is straightforward to check that B_f is a nonsingular form over $R[G]$, that α_f is a G-map, and that B_f and $q_{\boldsymbol{f}}$ satisfy (4.1.1)–(4.1.6).

Lemma 7.1. *Let X and Y be closed G-manifolds in $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)$ ($n = 2k \geqq 6$) and let $\boldsymbol{f} = (f, b)$ be a k-connected, degree 1, G-framed map. Let $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$ where $Q = Q(G,X)$, $S = S(G,X)$, $\lambda = (-1)^k$, and $w = w^G_X$. Let $\mathcal{C} = ((\mathrm{proj}))$, $((\mathrm{s-free}))$, or $((\mathrm{free}))$. Suppose $K_k(f;R) \in \mathcal{C}$. Then the quadruple $\boldsymbol{M_f} = (K_k(f;R), B_f, q_{\boldsymbol{f}}, \alpha_f)$ belongs to $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$.*

Proof. Set $\nabla = \nabla_{\boldsymbol{M_f}}$. It suffices to show that $\nabla(x)(s) = 0$ for each $x \in K_k(f;R)$ and $s \in S = S(G,X)$.

Let $j_{\gamma(s)} : X_{\gamma(s)} \to X$ be the canonical inclusion and let $h_x : S^k \to X$ be an immersion with trivial normal bundle, representing x. Without loss of generality, we may suppose that h_x, sh_x, and $j_{\gamma(s)}$ transversally intersect one another (cf. Lemma 9.1). If $z \in X$ is an intersection point of h_x and sh_x then so is $sz \in X$. Thus $\mathrm{intsec}(h_x, sh_x) \equiv \mathrm{intsec}(h_x, j_{\gamma(s)})$

mod 2. It is obvious that $\mathrm{intsec}(h_x, j_{\gamma(s)}) \equiv \mathrm{intsec}(sh_x, j_{\gamma(s)}) \mod 2$. Thus for $R = \mathbb{Z}$ we obtain using (4.2) that

$$\begin{aligned}\nabla(x)(s) &= [\varepsilon(B_f(\alpha_f(s) - x, sx))] \\ &= [\varepsilon(B_f(\alpha_f(s), sx))] - [\varepsilon(B_f(x, sx))] \\ &= [\mathrm{intsec}(j_{\gamma(s)}, sh_x)] - [\mathrm{intsec}(h_x, sh_x)] \\ &= 0 \quad \text{ in } \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Consider now the general case. Clearly for each $x \in K_k(f; R)$, there are elements $a \in R$ and $y \in K(f; \mathbb{Z})$ such that $x = ay$. By Lemma 4.3 , $\nabla(ay) = a\nabla(y)$ and by the case $R = \mathbb{Z}$ above, $\nabla(y) = 0$. Q.E.D.

Definition 7.2. In the situation of Lemma 7.1, define $\sigma(\boldsymbol{f})$ to be the element in $W_n(\boldsymbol{A}, S)_{\mathcal{C}}$ determined by the quadruple $\boldsymbol{M_f}$.

Theorem 7.3. *Let X and Y be closed G-manifolds in $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)$ $(n = 2k \geqq 6)$ and let $\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta))$ be a k-connected, degree 1 G-framed map. Suppose that $K_k(f; R)$ belongs to $\mathcal{C}$. If $\sigma(\boldsymbol{f}) = 0$ in $W_n(\boldsymbol{A}, S)_{\mathcal{C}}$ then $\boldsymbol{f}$ can be converted by G-surgery of isotropy type $\{1\}$ to a k-connected, degree 1, G-framed map $\boldsymbol{f}' = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ (thus $\boldsymbol{f} \sim \boldsymbol{f}'$ rel. $\mathrm{Sing}(G, X)$) such that $f' : X' \to Y$ is an R-homology equivalence.*

This will be proved in the next section.

Concerning the assumption in Theorem 7.3 that $K_k(f; R)$ belongs to $\mathcal{C}$, the following is known.

Remark 7.4. Let $f : X \to Y$ be a k-connected, degree 1 G-map. Then the following are true.

(7.4.1) If $f^P : X^P \to Y^P$ is an $R_{(p)}$-homology equivalence for any p-subgroup $P \neq \{1\}$ of G (where p ranges over the set of all primes dividing $|G|$) then $K_k(f; R)$ is a projective $R[G]$-module.

(7.4.2) If $f^H : X^H \to Y^H$ is an R-homology equivalence for any hyperelementary subgroup $H \neq \{1\}$ of G then $K_k(f; R)$ is a stably free $R[G]$-module.

Proof of Theorem 1.1. Since Y is 1-connected and $f : X \to Y$ is k-connected, X is 1-connected. Condition (2.1.1) follows from (1.1.1); Conditions (2.1.2), (2.1.3) and (2.1.5) follow from (1.1.2);Condition (2.1.4) follows from (1.1.3). Thus X and Y belong to $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)$ (see §4). Since f has degree 1, it follows that $w^G_X = w^G_Y$. Set $W(G, Y; R) = W_n(\boldsymbol{A}, S)_{((\mathrm{s-free}))}$ (cf. (4.4)) for $\boldsymbol{A} = (R, G, Q, S, \lambda, w^G_Y)$. Theorem 1.1 follows now from Theorem 7.3. Q.E.D.

8. Algebraic triviality and geometric deformation

This section is devoted to the proof of Theorem 7.3.

Throughout the section, X and Y are closed G-manifolds in $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)$ ($n = 2k \geqq 6$), and $\boldsymbol{f} = (f, b)$ is a k-connected, degree 1, G-framed map consisting of $f : X \to Y$ and $b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta)$. We set $\lambda = (-1)^k$, $w = w^G_X$, $Q = Q(G, X)$, $S = S(G, X)$, and $\boldsymbol{A} = (R, G, Q, S, \lambda, w)$.

Theorem 8.1. *Let $K_k(f; R) \in \mathcal{C}$ and let $\beta = (c, r)$, $(r,\ c \in \mathrm{Map}(S, \mathbb{Z}))$ be a pair such that $c(s) \equiv r(s) \mod 2\mathbb{Z}$ for all $s \in S = S(G, X)$. Then $\boldsymbol{f} = (f, b)$ can be converted by G-surgery of isotropy type $(\{1\})$ to a k-connected, degree 1, G-framed map $\boldsymbol{f}' = (f', b')$ $(f' : X' \to Y$ and $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ such that $K_k(f'; R) \in \mathcal{C}$ and $\boldsymbol{M}_{\boldsymbol{f}'} \cong \boldsymbol{M}_{\boldsymbol{f}} \oplus \boldsymbol{M}(x, y, \beta)$.*

This will be proved in §9.

Proof of Theorem 7.3. In outline, the proof proceeds as follows. First, we show using Theorems 5.6 and 8.1 that $\sigma(\boldsymbol{f}) = 0$ ($\boldsymbol{f} = (f, b)$) implies $\boldsymbol{M}_{\boldsymbol{f}}$ has a free Lagrangian L after suitable G-surgery on $\boldsymbol{f}$. Second, we show using Theorem 2.8 that the elements $x_1, \cdots,$ x_m of an $R[G]$-basis of L can be represented by smooth embeddings $h_1, \cdots, h_m : S^k \to X$ with trivial normal bundles such that $g\mathrm{Im}(h_i) \cap g'\mathrm{Im}(h_j) = \emptyset$ unless $i = j$ and $g = g' \in G$. Third, we perform G-surgery along the h_i's and fourth, check that the resulting $\boldsymbol{f}' = (f', b')$ has the desired properties.

We shall prove first the case $R = \mathbb{Z}$ and then show how this proof can be modified in the case of a general R. The case $R = \mathbb{Z}$ is divided into 4 steps corresponding to the 4 steps in the outline above.

Step 1. We reduce the proof to the case that $\boldsymbol{M}_{\boldsymbol{f}}$ is a ((free))-null module. Suppose $\sigma(\boldsymbol{f}) = 0$. By definition, there exist $\boldsymbol{M} \in \boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ and $\boldsymbol{N} \in \boldsymbol{SQ}(\boldsymbol{A}, S)_{((\mathrm{free}))}$ such that $\boldsymbol{N}$ is a ((free))-null module and

$$\boldsymbol{M}_{\boldsymbol{f}} \oplus \boldsymbol{M} \cong \boldsymbol{N} \oplus \boldsymbol{M}.$$

Since $\boldsymbol{NSQ}(\boldsymbol{A}, S)_{((\mathrm{free}))}$ is cofinal in $\boldsymbol{SQ}(\boldsymbol{A}, S)_{\mathcal{C}}$ (Proposition 4.5), we may assume that $\boldsymbol{M}$ is a ((free))-null module. Thus $\boldsymbol{N} \oplus \boldsymbol{M}$ is a ((free))-null module. By Theorem 5.6, $\boldsymbol{M}$ is isomorphic to a orthogonal sum of special metabolic planes. Thus by applying Theorem 8.1, we may continue the proof under the hypothesis that $\boldsymbol{M}_{\boldsymbol{f}}$ is a ((free))-null module.

Let $L \subset K_k(f; R)$ be a ((free))-Lagrangian of $\boldsymbol{M}_{\boldsymbol{f}}$ and let $\{x_1, \cdots, x_m\}$ be an $R[G]$-basis of L.

Step 2. We find nice embeddings $S^k \to \mathrm{Free}(G, X)$ representing the x_i's. For each i $(1 \leqq i \leqq m)$, there is a smooth immersion $h_i : S^k \to X$ with trivial normal bundle, representing x_i. Since by Lemmas 2.2 and 2.6 $\dim \mathrm{Fix}_G(> \rho_\Pi(\gamma(s)), X_{\gamma(s)}) \leqq k - 2$ for all $s \in S(G, X)$, we may assume that $\mathrm{Im}(h_i) \cap \mathrm{Fix}_G(> \rho_\Pi(\gamma(s)), X_{\gamma(s)}) = \emptyset$ for all i $(1 \leqq i \leqq m)$ and $s \in S(G, X)$. Let $j_{\gamma(s)} : X_{\gamma(s)} \to X$ be the canonical inclusion. Since L is a Lagrangian, $L \supset \mathrm{Im}(\alpha_f)$. Thus $B_f(\alpha_f(s), x_i) = 0$. By definition,

$$\alpha_f(s) = \pi_f({j_{\gamma(s)}}_*(\mathrm{ori}(X_{\gamma(s)}))). \tag{8.2}$$

Since $K_k(f;R)$ is orthogonal to $\mathrm{pdual}(\mathrm{Im}(f^*))$ under ppair, it follows that

$$\mathrm{ppair}(y,x) = \mathrm{ppair}(\pi_f(y),x) \quad (\forall y \in H_k(X;R) \text{ and } \forall x \in K_k(f;R)).$$

Since ppair is equal to intsec up to sign, the equality (8.2) implies $\mathrm{intsec}_G(j_{\gamma(s)}, h_i) = 0$. By Lemma 2.2, $\mathrm{Fix}_G(=\rho_\Pi(\gamma(s)), X_{\gamma(s)})$ is connected whenever $s \in S(G,X)$. Thus if a, $b \in \mathrm{Im}(h_i) \cap X_{\gamma(s)}$ have opposite intersection numbers, we can take a path from a to b in $\mathrm{Im}(h_i)$ and another in $\mathrm{Fix}_G(=\rho_\Pi(\gamma(s)), X_{\gamma(s)})$. Apply now Theorem 6.6 of [12] (a procedure for cancelling intersection points with opposite intersection numbers) to deduce that h_i is regularly homotopic to an immersion ${h'}_i$ such that $\mathrm{Im}({h'}_i) \cap X_{\gamma(s)} = \emptyset$ for all $s \in S(G,X)$. Replace h_i by h'_i, $1 \leqq i \leqq m$. Then for all i and j,

$$\mathrm{intsec}_G(h_i, h_j) = 0, \quad \text{and} \tag{8.3}$$

$$\mu_{\widehat{X}}(h_i) = 0, \tag{8.4}$$

where $\widehat{X} = X \setminus \left(\bigcup_\gamma X_\gamma^H\right)$ (γ runs over $\Theta(G,X,k)$). By [12, Theorem 6.6], the vanishing property (8.3) for $i \neq j$ allows us to assume that $g\mathrm{Im}(h_i) \cap g'\mathrm{Im}(h_j) = \emptyset$ ($i \neq j$) for all g, $g' \in G$. Next apply Theorem 2.8 for $f = h_i$. This allows to assume that each h_i is a smooth embedding such that $\mathrm{Im}(h_i) \cap g\mathrm{Im}(h_i) = \emptyset$ for all $g \in G \setminus \{1\}$. Thus each x_i is represented by a embedding h_i with trivial normal bundle such that $g\mathrm{Im}(h_i) \cap g'\mathrm{Im}(h_j) = \emptyset$ unless $i = j$ and $g = g'$. In particular, $\mathrm{Im}(h_i) \subset \mathrm{Free}(G,X)$.

Step 3. We construct $\boldsymbol{f}' = (f', b')$. There will be no essential differences here from the corresponding step in Wall's ordinary surgery theory. Perform G-surgery on $\boldsymbol{f}$ along the embeddings $h_1, \cdots, h_m$. Let $F = id_I \times f : I \times X \to I \times Y$ and let $B = id_{\varepsilon_{I\times X}(\mathbb{R})} \oplus (p_X^* b) : T(I \times X) \oplus (f \circ p_X)^* \eta \to (f \circ p_X)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$ where we identify $T(I \times X) = \varepsilon_{I\times X}(\mathbb{R}) \oplus p_X^* T(X)$ and $p_X : I \times X \to X$ is the canonical projection. The embeddings $h_i : S^k \to X = \{1\} \times X$ can be extended to framed embeddings $H_i : S^k \times D^k \to \{1\} \times X$ such that $g\mathrm{Im}(H_i) \cap g'\mathrm{Im}(H_j) = \emptyset$ unless $i = j$ and $g = g'$. Define $\mathrm{ind}^G H_i : G \times S^k \times D^k \to \{1\} \times X$ by $\mathrm{ind}^G H_i(g,s,d) = gH_i(s,d)$ ($g \in G$, $s \in S^k$, and $d \in D^k$). Construct the attaching space

$$\begin{aligned} W &= W(H_1, \cdots, H_m) \\ &= (I \times X) \cup_{\mathrm{ind}^G H_1 \cup \cdots \cup \mathrm{ind}^G H_m} \left\{ (G \times D^{k+1} \times D^k)_1 \cup \cdots \cup (G \times D^{k+1} \times D^k)_m \right\}. \end{aligned}$$

Define X' by $\partial W = (\{0\} \times X) \coprod X'$. Since each $f \circ h_i$ is null homotopic, there is a map $d_i : D^{k+1} \to Y$ such that $d_i(x) = f \circ h_i(x)$ for all $x \in S^k$. The G-map $F : I \times X \to I \times Y$ is extensible to a G-map $F' : W \to I \times Y$ such that $F'((G \times D^{k+1} \times D^k)_i) \subset \{1\} \times Y$ and $F'(1,x,0) = (1, d_i(x))$ for $(1,x,0) \in \{1\} \times D^{k+1} \times \{0\} \subset (G \times D^{k+1} \times D^k)_i$. If we choose appropriate H_i's then the bundle isomorphism B is extensible to $B' : T(W) \oplus (p_Y \circ F')^* \eta \to (p_Y \circ F')^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$. Let $f' = F'|_{X'} : X' \to \{1\} \times Y = Y$ and let $b'' = B'|_{X'} : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus {f'}^* \eta \to {f'}^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$. Since η satisfies (HC) in §7, b'' is G-regularly homotopic to $id_{\varepsilon_{X'}(\mathbb{R})} \oplus b'$, where $id_{\varepsilon_{X'}(\mathbb{R})} : \varepsilon_{X'}(\mathbb{R}) \to {f'}^* \varepsilon_Y(\mathbb{R})$ is the canonical isomorphism and $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta)$ (cf. §10). We have just obtained a G-framed map (f', b').

Step 4. We prove that the f' obtained in Step 3 is a k-connected, degree 1, R-homology equivalence. These properties are independent of the G-action on X and Y in the following sense. We obtained the G-manifold X' and the G-map f' by G-surgery. But forgetting G-actions, these are obtained by ordinary surgery on $\mathrm{res}^G_{\{1\}} X$ along the basis $\{e_{i,g} \mid g \in G,\ 1 \leqq i \leqq m\}$ (where $e_{i,g} = gx_i$) of the Lagrangian $\mathrm{res}^G_{\{1\}} L$ for

$$\mathrm{res}^G_{\{1\}}(K_k(f; R), B_f, q_{\boldsymbol{f}}).$$

It is obvious that f' has degree 1. Furthermore the basis $\{e_{i,g} \mid 1 \leqq i \leqq m,\ g \in G\}$ possesses complementary basis elements $f_{i,g}$ such that

$$\begin{aligned} &(\mathrm{res}^G_{\{1\}} B_f)(f_{i,g}, e_{i',g'}) = \delta_{(i,g),(i',g')} \in R, \\ &(\mathrm{res}^G_{\{1\}} B_f)(f_{i,g}, f_{i',g'}) = 0, \quad \text{and} \\ &(\mathrm{res}^G_{\{1\}} q_{\boldsymbol{f}})(f_{i,g}) = 0 \in R/(1-\lambda)R, \end{aligned} \tag{8.5}$$

where $(\mathrm{res}^G_{\{1\}} B_f)(x, y) = \varepsilon \circ B_f(x, y)$ $(x,\ y \in K_k(f, R))$ (cf. (4.2)). Thus, the arguments in [19, pp.51–52] imply that f' is a k-connected, R-homology equivalence. Thus we have proved Theorem 7.3 in the case $R = \mathbb{Z}$.

The case of a general R is proved as above, except one has to take a little extra care at the three places.

The first is in Step 1. Here we should replace the application of Theorem 5.6 by one of Corollary 5.7.

The second is just after Step 1. There we used an arbitrary basis $\{x_1, \cdots, x_m\}$. However in the general case, we should choose the basis such that each $x_i \in K_k(f; R) = R \otimes K_k(f; \mathbb{Z})$ lies in the image of $K_k(f; \mathbb{Z})$ under the canonical homomorphism. Furthermore if $2 \in R^\times$ then we can and should assume that each $x_i = 2v_i$ for some v_i in the image of $K_k(f; \mathbb{Z})$. This will guarantee that if $\widehat{X}$ and h'_i are as in Step 2 then $\mu_{\widehat{X}}(h'_i) = 0$.

The third is in Step 4. In the general case, the complementary basis elements $f_{i,g}$ should be taken so that they also lie in the image of $K_k(f; \mathbb{Z})$ and (8.5) should be replaced by

$$\begin{aligned} &(\mathrm{res}^G_{\{1\}} B_f)(f_{i,g}, e_{i,g}) \ \text{ are integers invertible in } R, \\ &(\mathrm{res}^G_{\{1\}} B_f)(f_{i,g}, e_{i',g'}) = 0 \ \text{ if } (i,g) \neq (i',g'), \\ &(\mathrm{res}^G_{\{1\}} B_f)(f_{i,g}, f_{i',g'}) = 0 \ \text{ if } (i,g) \neq (i',g'), \ \text{ and} \\ &(\mathrm{res}^G_{\{1\}} q_{\boldsymbol{f}})(f_{i,g}) = 0 \in R/(1-\lambda)R. \end{aligned} \tag{8.6}$$

With these modifications, Step 1 – Step 4 will prove Theorem 7.3 in the general case. Q.E.D.

9. Special metabolic stabilization in G-surgery

This section is devoted to the proof of Theorem 8.1.

If M is an oriented smooth manifold of dimension $m_1 + m_2$, if M_1 and M_2 are oriented submanifolds of M of dimension m_1 and m_2, respectively, and if M_1 and M_2 transversally intersect at only finitely many points of $\mathrm{Int}(M)$, let $M_1 \cdot M_2$ denote the intersection number of M_1 and M_2.

Lemma 9.1. *Let $\langle s\rangle$ be a group of order 2 and let $\mathbb{R}$ (resp. $\mathbb{R}_\pm$) be the 1-dimensional, real $\langle s\rangle$-module with trivial (resp. nontrivial) $\langle s\rangle$-action. Let $M = \mathbb{R}^k \oplus \mathbb{R}^k_\pm$ with standard orientation. Let*

$$M_1 = \{(x_1, \cdots, x_k, y_1, \cdots, y_k) \in \mathbb{R}^k \oplus \mathbb{R}^k_\pm \mid \ x_1, \cdots, x_k \in \mathbb{R};\ y_1 = x_1, \cdots, y_k = x_k\}.$$

Then $M_1 \cdot sM_1 = \lambda$ (where $\lambda = (-1)^k$).

Proof. It is clear that

$$sM_1 = \{(x_1, \cdots, x_k, -y_1, \cdots, -y_k) \mid \ x_1, \cdots, x_k \in \mathbb{R};\ y_1 = x_1, \cdots, y_k = x_k\}.$$

Thus the matrix corresponding to the standard ordered basis of $M_1 \oplus sM_1$ is

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ & & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ & & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Since $\det(P) = 2^k(-1)^k$, we obtain $M_1 \cdot sM_1 = \lambda$.

Let $\boldsymbol{f} = (f, b)$, where $f : X \to Y$, $b : T(X) \oplus f^*(\eta) \to f^*(\xi \oplus \eta)$, and let $\beta = (c, r)$ be as in Theorem 8.1. In particular, $c : S \to \mathbb{Z}$ and $r : S \to \mathbb{Z}$ satisfy the property $c(g) \equiv r(g) \mod 2\mathbb{Z}$ for all $g \in S = S(G, X)$. It is helpful to prove first a special case of Theorem 8.1 in order to grasp an outline of the proof.

Special Case. Here we assume that $\exists s \in S$ such that $\sum_{g\in S} c(g)g = s$. Let $\gamma \in \Theta(G, X, k)$ such that $\rho_\Pi(\gamma) \ni s$ (cf. Lemma 2.6). Take a point $a \in X_\gamma$ such that $G_a = \rho_\Pi(\gamma)$. The canonical inclusion $X_\gamma \to X$ is denoted by j_γ. Let $\nu = \nu(X_\gamma, X)$ be the normal bundle of X_γ and regard it as an $N_G(\rho_\Pi(\gamma))$-tubular neighborhood of X_γ. Note that $\rho_\Pi(\gamma) = G_s$ (the centralizer of s). Take a neighborhood E ($\cong \mathbb{R}^k$) of a in $\mathrm{Fix}_G(= \rho_\Pi(\gamma), X_\gamma)$ such that $E \cap gE \neq \emptyset \implies g \in \rho_\Pi(\gamma)$. Then $\nu|_E$ is a $\rho_\Pi(\gamma)$-neighborhood of a in X, which is $\rho_\Pi(\gamma)$-diffeomorphic to $E \times V$, where $V = \nu|_a$ is a k-dimensional real $\rho_\Pi(\gamma)$-representation space. Note that $\mathrm{res}^{\rho(\gamma)}_{\langle s\rangle} V \cong \mathbb{R}^k_{\pm}$. Regard the point a as the origin 0 in $E \times V$. Let $\Delta : E \to V$ be an $\mathbb{R}$-linear map such that $\mathrm{Ker}(\Delta) = \{0\}$. Then the graph $\mathrm{Graph}(\Delta)$ of Δ is a k-dimensional linear subspace of $E \times V$. We choose Δ so that $\mathrm{Graph}(\Delta)$ is M_1 in Lemma 9.1 when the group action is restricted to $\langle s\rangle$. We orient $\mathrm{Graph}(\Delta)$ so that the ordered direct sum $T_a(X_\gamma) \oplus T_a(\mathrm{Graph}(\Delta))$ has the same orientation as $T_a(X)$. Let $\delta > 0$ be a small real number and let $D_\delta(\mathrm{Graph}(\Delta))$ be the closed disk of $\mathrm{Graph}(\Delta)$ with radius δ centered at the origin (i.e. a). Take an orientation preserving (linear) diffeomorphism $h'_D : D^k \to D_\delta(\mathrm{Graph}(\Delta))$ such that $h'_D(0) = a$. Fix a small real number δ' such that $0 < \delta' \ll \delta$. There is a δ'-approximation $h_D : D^k \to \nu|_E$ of h'_D such that h_D is also a smooth embedding, $h_D(x) = h'_D(x)$ if $\|x\| \leqq 1/2$, and $h := h_D|_{S^{k-1}} : S^{k-1} = \partial D^k \to X$ satisfies the condition that if $gh(x) = g'h(x')$ ($g, g' \in G$ and $x, x' \in S^{k-1}$) then $g = g'$ and $x = x'$. Set $D = \mathrm{Im}(h_D)$. Then it follows that

(9.2) the intersection number $X_\gamma \cdot D = 1$.

Since h extends to h_D, the normal bundle of h is trivial. There is an orientation-preserving, smooth embedding $H : S^{k-1} \times D^{k+1} \to \mathrm{Free}(G, X)$ such that $h = H|_{S^{k-1}\times\{0\}}$ and such that if $gH(x) = g'H(x')$ ($g, g' \in G$ and $x, x' \in S^{k-1} \times D^{k+1}$) then $g = g'$ and $x = x'$. Thus,

$$\mathrm{ind}^G H : G \times S^{k-1} \times D^{k+1} \to \mathrm{Free}(G, X), \quad (g, x) \mapsto gH(x) \quad (g \in G,\, x \in S^{k-1} \times D^{k+1})$$

is a smooth embedding.

Perform G-surgery on X along h as follows. Let $I = [0, 1]$ and $W = I \times X$. Regard $\mathrm{ind}^G H$ as a map to $\{1\} \times X$. Construct the attaching space

$$W' = W \cup_{\mathrm{ind}^G H} (G \times D^k \times D^{k+1}).$$

Define X' by $\partial W' = (\{0\} \times X) \cup X'$ (disjoint union). Then the map $F = id_I \times f : W = I \times X \to I \times Y$ is extensible to a G-map $F' : W' \to I \times Y$ such that $F'(X') \subseteqq \{1\} \times Y$ and $F'(g, p, 0) = gh_D(\varphi(p))$ for $g \in G$, $p \in D^k$ and 0 the origin of D^{k+1}, where $\varphi : D^k \to D^k$ is the usual orientation reversing diffeomorphism from the upper hemisphere to the lower hemisphere. Define $f' : X' \to Y$ by $f' = F'|_{X'} \to (\{1\} \times Y) = Y$. In addition, $B = id_{\varepsilon_I(\mathbb{R})} \times b$ is extensible to a G-vector bundle isomorphism $B' : T(W') \oplus (p_Y \circ F')^*\eta \to (p_Y \circ F')^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$. Define $b'' : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus f'^*\eta \to f'^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta)$ by $b'' = B'|_{X'}$. Since η is large (i.e. satisfies (HC) in §7), b'' is G-regularly homotopic to $id_{\varepsilon_{X'}(\mathbb{R})} \oplus b'$ where $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta)$ (cf. Proposition 10.1). Let $\boldsymbol{f}' = (f', b')$.

We shall show that $\boldsymbol{f}'$ satisfies the conclusion of Theorem 8.1.

If we forget the G-actions on X and X' then

(9.3) $X' = [\{X\#(\{g_1\} \times S^k \times S^k)\}\# \cdots]\#(\{g_{|G|}\} \times S^k \times S^k)$ (iterated connected sum),

where $\{g_1, \cdots, g_{|G|}\} = G$ and $g_1 = 1$. Clearly, X' is 1-connected and f' is of degree 1.

Our next goal is to obtain elements x and y of $K_k(f'; \mathbb{Z})$ such that $B_{f'}(x, x) = 0$, $q_{\boldsymbol{f}'}(x) = 0$, $B_{f'}(y, x) = 1$, and $q_{\boldsymbol{f}'}(y) = 0$. This will be done by the procedure. First we define an element $-x \in K_k(f'; \mathbb{Z})$ such that $B_{f'}(-x, -x) = 0$ and $q_{\boldsymbol{f}'}(-x) = 0$. The element x we are seeking is then defined to be $-(-x)$. Next we define an element $-\lambda z \in K_k(f';\mathbb{Z})$ such that $B_{f'}(-\lambda z, x) = -\lambda$ and $q_{\boldsymbol{f}'}(-\lambda z)(g) = 0$ for all $g \in \{1\} \cup G(2) \setminus (Q \cup S)$. Set $z = -\lambda(-\lambda z)$. Then $B_{f'}(z, x) = 1$ and $q_{\boldsymbol{f}'}(z)(g) = 0$ for all $g \in \{1\} \cup G(2) \setminus (Q \cup S)$. By the orthonormalization procedure in Lemma 6.1, there is an element $v \in \mathbb{Z}[G \setminus (\{1\} \cup G(2))]$ such that $q_{\boldsymbol{f}'}(z + vx) = 0$. Now we define $y = z + vx$. It follows that $B_{f'}(y, x) = 1$ and that the elements x and y have the properties sought above. To define the elements $-x$ and $-\lambda z$ in $K_k(f';\mathbb{Z})$, we construct first embeddings $j_{-x}, j_{-\lambda z} : S^k \to X'$ and then set $-x$ (resp. $-\lambda z$) to be the image under the homomorphism $(j_{-x})_* : H_k(S^k;\mathbb{Z}) \to H_k(X';\mathbb{Z})$ (resp. $(j_{-\lambda z})_* : H_k(S^k;\mathbb{Z}) \to H_k(X';\mathbb{Z})$) of the orientation class of S^k. Fix a point $pt \in S^{k-1}$ and define $j_{-x} : S^k = \partial D^{k+1} \to X'$ by $j_{-x}(z) = H(pt, z)$ ($z \in S^k$). The map j_{-x} will be regarded as the meridian $\{1\} \times \{pt\} \times S^k$ in (9.3). Clearly $\mathrm{intsec}_G(j_{-x}, j_\delta) = 0$ ($\forall \delta \in \Theta(G, X', k) = \Theta(G, X, k)$) (equivalently $B_{f'}(x, \alpha_{f'}(g)) = 0$ ($\forall g \in S$)). As j_{-x} is an embedding with trivial normal bundle such that $\mathrm{Im}(j_{-x}) \cap g\mathrm{Im}(j_{-x}) = \emptyset$ whenever $g \in G \setminus \{1\}$, it follows that $\mathrm{intsec}_G(j_{-x}, j_{-x}) = 0$ (equivalently $B_{f'}(x, x) = 0$), and $q_{\boldsymbol{f}'}(x) = 0$. By choosing $\mathrm{Im}(H)$ sufficiently thin, we may suppose that $D' = D \setminus H(S^{k-1} \times \mathrm{Int}(D^{k+1}))$ is diffeomorphic to the closed disk of dimension k. Define $k_- : D^k_- \to W$ and $k_+ : D^k_+ \to W'$ by

$$\begin{aligned} k_- &: D^k_- = D^k \xrightarrow{h_D} X = \{1\} \times X \hookrightarrow W, \quad \text{and} \\ k_+ &: D^k_+ = D^k \times \{0\} \hookrightarrow \{1\} \times D^k \times D^{k+1} \hookrightarrow W'. \end{aligned}$$

Define $j' : S^k = D^k_- \cup D^k_+ \to W'$ by gluing k_- and k_+. Pushing j' into X' within the handle $\{1\} \times D^k \times D^{k+1}$, we obtain an isotopy from j' to a smooth embedding $j_{-\lambda z} : S^k \to X'$. We may assume that $\mathrm{Im}(j_{-\lambda z}) = D' \cup (\{1\} \times D^k \times \{pt'\})$ for some $pt' \in S^k$. The embedding $j_{-\lambda z}$ will be regarded as the longitude $\{1\} \times S^k \times \{pt'\}$ in (9.3). Clearly $B_{f'}(z, \alpha_{f'}(g)) = 0$ ($\forall g \in S \setminus \{s\}$), and $(q_{\boldsymbol{f}'}(z))(g) = 0$ ($\forall g \in \{1\} \cup G(2) \setminus (Q \cup S)$). Moreover

$$\begin{aligned} \mathrm{intsec}(j_{-\lambda z}, j_{-x}) &= \frac{\mathrm{ori}\left(T_{(1,pt,pt')}(\{1\} \times S^k \times \{pt'\}) \oplus T_{(1,pt,pt')}(\{1\} \times \{pt\} \times S^k)\right)}{\mathrm{ori}\left(T_{(1,pt,pt')}(\partial(\{1\} \times S^k \times D^{k+1}))\right)} \\ &= \frac{\mathrm{ori}\left(T_{(1,pt,pt')}(\{1\} \times S^k \times S^k)\right)}{(-1)^k \mathrm{ori}\left(T_{(1,pt,pt')}(\{1\} \times S^k \times \partial D^{k+1})\right)} \\ &= (-1)^k = \lambda \end{aligned}$$

and by construction $\mathrm{intsec}(j_{-\lambda z}, g^{-1} j_{-x}) = 0$ for all $g \neq 1$. Thus $B_{f'}(z, x) = 1$.

Let the elements x, y, $z \in K_k(f';\mathbb{Z})$ be determined by the procedures above. Obviously $\langle x,\ z\rangle_{\mathbb{Z}[G]} = \langle x,\ y\rangle_{\mathbb{Z}[G]}$. We claim that $K_k(f';R) \cong K_k(f;R) \oplus \langle x,\ z\rangle_{R[G]}$. This can be shown by the following standard argument.

We identify X with $\{0\} \times X$. Then $K_i(F', f;\mathbb{Z}) = K_i(F', F;\mathbb{Z}) = H_i(W', W;\mathbb{Z}) \cong \mathbb{Z}[G]$ (resp. 0) if $i = k$ (resp. $i \neq k$). Since we can regard the element z to be in $H_k(W', W;\mathbb{Z})$, we may identify $K_i(F', f;\mathbb{Z})$ with $\langle z\rangle_{\mathbb{Z}[G]}$. Consider the Mayer-Vietoris exact sequence

$$\cdots \to K_{i+1}(F', f;\mathbb{Z}) \to K_i(f;\mathbb{Z}) \xrightarrow{(\kappa_1)_i} K_i(F';\mathbb{Z}) \xrightarrow{(\tau_1)_i} K_i(F', f;\mathbb{Z}) \to \cdots.$$

If $i \neq k$ then $K_i(f;\mathbb{Z}) = 0$ and $K_i(F', f;\mathbb{Z}) = 0$ and so $K_i(F';\mathbb{Z}) = 0$. Since we can regard $z \in K_k(F';\mathbb{Z})$, there is a $\mathbb{Z}[G]$-splitting $\sigma_1 : K_k(F', f;\mathbb{Z}) \to K_k(F';\mathbb{Z})$ for $\tau_1 = (\tau_1)_k$ such that $\sigma_1(z) = z$. Thus $K_k(f;\mathbb{Z}) \oplus \langle z\rangle_{\mathbb{Z}[G]} \overset{\kappa_1\oplus\iota}{\cong} K_k(F';\mathbb{Z})$ where $\iota : \langle z\rangle_{\mathbb{Z}[G]} \to K_k(F';\mathbb{Z})$ is the canonical inclusion. Since $W \cong_G W' \cup (G \times D^{k+1} \times D^k)$ (the dual-handle attachment) $\simeq_G W' \cup (G \times D^{k+1} \times \{0\})$, it follows that $K_i(F', f';\mathbb{Z}) = H_i(G\times D^{k+1}\times D^k, G\times S^k \times D^k;\mathbb{Z}) \cong \mathbb{Z}[G]$ (resp. 0) if $i = k+1$ (resp. $i \neq k+1$). (This can be shown also using the universal coefficient theorem and the Poincaré-Lefschetz duality, cf. [5, I.2.8].) Now consider the Mayer-Vietoris exact sequence

$$\cdots \to K_{i+1}(F', f';\mathbb{Z}) \xrightarrow{(\partial_2)_{i+1}} K_i(f';\mathbb{Z}) \xrightarrow{(\kappa_2)_i} K_i(F';\mathbb{Z}) \to K_i(F', f';\mathbb{Z}) \to \cdots .$$

If $i \neq k$ then $K_{i+1}(F', f';\mathbb{Z}) = 0$ and $K_i(F';\mathbb{Z}) = 0$ and so $K_i(f';\mathbb{Z}) = 0$. Hence f' is k-connected. Consider the short exact sequence

$$0 \to K_{k+1}(F', f';\mathbb{Z}) \xrightarrow{\partial_2} K_k(f';\mathbb{Z}) \xrightarrow{\kappa_2} K_k(F';\mathbb{Z}) \to 0$$

obtained from the long exact sequence above, where $\partial_2 = (\partial_2)_{k+1}$ and $\kappa_2 = (\kappa_2)_k$. Note that $\partial_2(K_{k+1}(F', f';\mathbb{Z})) = \langle x\rangle_{\mathbb{Z}[G]}$. Since $K_k(F';R) = R \otimes K_k(F';\mathbb{Z})$ is a projective $R[G]$-module, there is an $R[G]$-splitting $\sigma_2 : K_k(F';R) \to K_k(f';R)$ for κ_2 (more precisely for $R \otimes \kappa_2$) such that $\sigma_2(z) = z$. Putting all this together, we get an $R[G]$-isomorphism $\omega : K_k(f;R) \oplus \langle x, z\rangle_{R[G]} \to K_k(f';R)$ such that $\omega(u, v_1x, v_2z) = \sigma_2(\kappa_1(u)) + v_1x + v_2z$ for $u \in K_k(f;R)$, v_1, $v_2 \in R[G]$.

Let

$$M'' = \langle x,\ y\rangle_{R[G]} = \langle x,\ z\rangle_{R[G]} \subset K_k(f';R),$$
$$M_0 = M''^{\perp} = \{u_1 \in K_k(f';R) \mid B_{f'}(u_2, u_1) = 0 \text{ for all } u_2 \in M''\}, \quad \text{and}$$
$$M_1 = \operatorname{Im}(\sigma_2 \circ \kappa_1 : K_k(f;R) \to K_k(f';R)).$$

Let $p'' : K_k(f';R) \to M''$ be the projection associated to the decomposition $K_k(f';R) = M_0 \oplus M''$ and let $p_i : K_k(f';R) \to M_i$ ($i = 0,\ 1$) be the projections associated to the decompositions $K_k(f';R) = M_i \oplus M''$ ($i = 0,\ 1$) respectively. By construction, $M_1 \subset \langle x\rangle_{R[G]}{}^{\perp}$. Thus

$$M_1 + \langle x\rangle_{R[G]} = \langle x\rangle_{R[G]}{}^{\perp} \quad \text{and} \quad q_{\boldsymbol{f}'}(x) = 0. \tag{9.4}$$

Thus the isomorphism class of $(M_1, B_{f'}|_{M_1}, q_{\boldsymbol{f}'}|_{M_1})$ is independent of the choice of σ_2. For each element a of $K_k(f;R)$, take an smooth immersion $h_a : S^k \to X$ with trivial normal bundle, representing a. Take h_a so that $\operatorname{Im}(h_a) \cap \operatorname{Im}(\operatorname{ind}^G H) = \emptyset$. Then h_a can be regarded as an immersion to X'. Let $a' \in K_k(f';R)$ be the element represented by h_a. Clearly it follows that $\sigma_2 \circ \kappa_1(a) \equiv a' \mod \langle x\rangle_{R[G]}$. For a, $b \in K_k(f;R)$, one can compute $B_{f'}(\sigma_2\circ\kappa_1(a), \sigma_2\circ\kappa_1(b))$ (resp. $q_{\boldsymbol{f}'}(\sigma_2\circ\kappa_1(a))$) by using (9.4) and counting the equivariant intersection number of h_a and h_b (resp. the equivariant selfintersection number of h_a). This makes it clear that $\sigma_2 \circ \kappa_1$ is an isomorphism $(K_k(f;R), B_f, q_{\boldsymbol{f}}) \cong (M_1, B_{f'}|_{M_1}, q_{\boldsymbol{f}'}|_{M_1})$. The map $p_0|_{M_1} : M_1 \to M_0$ is determined by the formula

$$p_0(u) = u - B_{f'}(y, u)x \quad (\text{for } u \in M_1).$$

Again by (9.4), $p_0|_{M_1}$ is an isomorphism $(M_1, B_{f'}|_{M_1}, q_{\boldsymbol{f'}}|_{M_1}) \cong (M_0, B_{f'}|_{M_0}, q_{\boldsymbol{f'}}|_{M_0})$. Thus $p_0 \circ \sigma_2 \circ \kappa_1 : K_k(f;R) \to M_0$ is an isomorphism $(K_k(f;R), B_f, q_{\boldsymbol{f}}) \cong (M_0, B_{f'}|_{M_0}, q_{\boldsymbol{f'}}|_{M_0})$.

Set $\alpha'' = p'' \circ \alpha_{f'}$, and let $g \in S$. Obviously, $\alpha_{f'}(g) = p_0(\alpha_{f'}(g)) + \alpha''(g)$. ¿From the equation $B_f(a, \alpha_f(g)) = B_{f'}(a', \alpha_{f'}(g))$, it follows that $\sigma_2 \circ \kappa_1(\alpha_f(g)) = p_1(\alpha_{f'}(g))$. Note that $p_0(p_1(u)) = p_0(u)$ for all $u \in K_k(f';R)$. Thus $p_0 \circ p_1 \circ \alpha_{f'} = p_0 \circ \alpha_{f'}$. Since $p_0|_{M_1}$: $(M_1, B_{f'}|_{M_1}, q_{\boldsymbol{f'}}|_{M_1}) \cong (M_0, B_{f'}|_{M_0}, q_{\boldsymbol{f'}}|_{M_0})$, we obtain $p_0 \circ \sigma_2 \circ \kappa_1 : (K_k(f;R), B_f, q_{\boldsymbol{f}}, \alpha_f) \cong (M_0, B_{f'}|_{M_0}, q_{\boldsymbol{f'}}|_{M_0}, p_0 \circ \alpha_{f'})$.

Let $B'' = B_{f'}|_{M''}$, $q'' = q_{\boldsymbol{f'}}|_{M''}$. By Theorem 6.3, $(M'', B'', q'', \alpha'') \cong \boldsymbol{M}(x, y, \beta'')$ where $\beta'' = (c'', r'')$ (c'', $r'' \in \mathrm{Map}(S, \mathbb{Z})$) is determined by the equations

$$\alpha''(t) = \sum_{g \in G} c''(gtg^{-1})g^{-1}x \quad (\forall t \in S) \quad \text{and} \quad B_{f'}(y, y) = \sum_{t \in S} r''(t)t.$$

Next we compute that $c'' = c$ and calculate r''.

Since $B_{f'}(x, \alpha_{f'}(s)) = 0$, $\alpha''(s) = ux$ for some $u \in \mathbb{Z}[G]$. Since $B_{f'}(z, x) = 1$, we obtain $\alpha''(s) = B_{f'}(z, \alpha''(s))x = B_{f'}(z, \alpha_{f'}(s))x$. Furthermore $\mathrm{intsec}(j_\gamma, j_{-\lambda z}) = -1$ $(\rho_\Pi(\gamma) \ni s)$, because $D = \mathrm{Im}(h_D)$ is identified with the lower hemisphere of $\mathrm{Domain}(j_{-\lambda z})$ by an orientation reversing diffeomorphism. If $g \in G$ then $\mathrm{intsec}(j_\gamma, g^{-1}j_{-\lambda z}) = w(g)\mathrm{intsec}(gj_\gamma, j_{-\lambda z})$ and this is trivial if $g \notin G_s$. Thus $\mathrm{intsec}_G(j_\gamma, j_{-\lambda z}) = -\sum_{g \in G_s} w(g)g$. This implies $B_{f'}(\alpha_{f'}(s), -\lambda z) = -\sum_{g \in G_s} w(g)g$. Hence

$$B_{f'}(z, \alpha_{f'}(s)) = \sum_{g \in G_s} g = \sum_{g \in G_s} g^{-1} \ (= \Sigma_{G_s}). \tag{9.5}$$

Thus

$$\alpha''(s) = \sum_{g \in G_s} g^{-1}x. \tag{9.6}$$

Furthermore if $t \in S$ and $t \neq s$ then $\alpha''(t) = 0$, since for $\gamma' \neq \gamma$ and $g \in G$, $\mathrm{intsec}(j_{\gamma'}, gj_{-\lambda z}) = 0$. Clearly $c''(s) = 1$ and $c''(t) = 0$ if $t \neq s$. Thus $c'' = c$ holds.

Since $\mathrm{Im}(j_{-\lambda z}) \cap g\mathrm{Im}(j_{-\lambda z}) = \emptyset$ whenever $g^2 = 1$ and $g \neq s$, 1, it is clear that $B_{f'}(z, z) = 0$ in $\mathbb{Z}[\{1\} \cup G(2) \setminus \{s\}] = \mathbb{Z}[G]/\mathbb{Z}[G \setminus (\{1\} \cup G(2) \setminus \{s\})]$. By Lemma 9.1, we may suppose that $\mathrm{intsec}(j_{-\lambda z}, sj_{-\lambda z}) = \lambda$. Then $B_{f'}(z, z) = \lambda s$ in $\mathbb{Z}[\{1\} \cup G(2)] = \mathbb{Z}[G]/\mathbb{Z}[G \setminus (\{1\} \cup G(2))]$. ¿From the equations $q_{\boldsymbol{f'}}(y) = 0$ and $y = z + vx$ $(v \in \mathbb{Z}[G \setminus (\{1\} \cup G(2))])$, it follows that

$$B_{f'}(y, y) = \lambda s \tag{9.7}$$

Thus $r''(s) = \lambda$ and $r''(t) = 0$ for $t \neq s$. This completes the calculation of r''.

Since $c''(g) = c(g)$ (and $r''(g) \equiv r(g) \mod 2\mathbb{Z}$ for all $g \in S$), it follows from Lemma 5.5 that $\boldsymbol{M}(x, y, \beta'') \cong \boldsymbol{M}(x, y, \beta)$ and hence $(M'', B'', q'', \alpha'') \cong \boldsymbol{M}(x, y, \beta)$.

Consequently $\boldsymbol{M_{f'}} \cong \boldsymbol{M_f} \oplus \boldsymbol{M}(x, y, \beta)$, and we have proved Theorem 8.1 in the special case cited above.

General Case. Let $S_+ = \{s \in S \mid c(s) > 0\}$, $S_- = \{s \in S \mid c(s) < 0\}$ and set $S' = S_+ \cup S_-$. Let $\Gamma = \{(s, i) \mid s \in S', 1 \leqq i \leqq |c(s)| \}$. For each $s \in S'$, take $|c(s)|$

distinct points $x(s,1), \cdots, x(s,|c(s)|)$ of $X_{\gamma(s)}$ ($\gamma(s) \in \Theta(G,X,k)$ and $\rho_\Pi(\gamma(s)) \ni s$) such that $G_{x(s,i)} = \rho_\Pi(\gamma(s))$. Furthermore we can choose these points so that if $(s,i) \neq (s',i')$ then $Gx(s,i) \cap Gx(s',i') = \emptyset$. Take neighborhoods $E_{(s,i)}$ ($\cong \mathbb{R}^k$) of $x(s,i)$ in $X_{\gamma(s)}$, respectively. Then each $\nu(X_{\gamma(s)},X)|_{E_{(s,i)}}$ is a neighborhood of $x(s,i)$ which is diffeomorphic to $E_{(s,i)} \times V_{(s,i)}$, where $V_{(s,i)} = \nu(X_{\gamma(s)},X)|_{x(s,i)}$. We may assume that $G\nu(X_{\gamma(s)},X)|_{E_{(s,i)}} \cap G\nu(X_{\gamma(s')},X)|_{E_{(s',i')}} = \emptyset$ whenever $(s,i) \neq (s',i')$ and that if $\nu(X_{\gamma(s)},X)|_{E_{(s,i)}} \cap g\nu(X_{\gamma(s)},X)|_{E_{(s,i)}} \neq \emptyset$ then $g \in \rho_\Pi(\gamma(s))$. Let $\Delta_{(s,i)} : E_{(s,i)} \to V_{(s,i)}$ be $\mathbb{R}$-linear maps such that $\mathrm{Ker}(\Delta_{(s,i)}) = \{0\}$. The graphs $\mathrm{Graph}(\Delta_{(s,i)})$ are k-dimensional linear subspaces of $E_{(s,i)} \times V_{(s,i)}$, respectively. For each $(s,i) \in S_+$ (resp. S_-), we orient $\mathrm{Graph}(\Delta_{(s,i)})$ so that the ordered direct sum $T_{x(s,i)}(X_{\gamma(s)}) \oplus T_{x(s,i)}(\mathrm{Graph}(\Delta_{(s,i)}))$ has the same orientation (resp. opposite orientation) as $T_{x(s,i)}(X)$. Take orientation preserving (linear) diffeomorphisms $h'_{D_{(s,i)}} : D^k \to D_\delta(\mathrm{Graph}(\Delta_{(s,i)}))$ such that $h'_{D_{(s,i)}}(0) = x(s,i)$. For each (s,i), there is a δ'-approximation $h_{D_{(s,i)}} : D^k \to \nu(X_{\gamma(s)},X)|_{E_{(s,i)}}$ of $h'_{D_{(s,i)}}$ such that $h_{D_{(s,i)}}$ is also a smooth embedding, that $h_{D_{(s,i)}}(x) = h'_{D_{(s,i)}}(x)$ if $\|x\| \leqq 1/2$, and that $h_{(s,i)} := h_{D_{(s,i)}}|_{S^{k-1}} : S^{k-1} \to X$ satisfies the property that if $gh_{(s,i)}(x) = g'h_{(s,i)}(x')$ (g, $g' \in G$ and x, $x' \in S^{k-1}$) then $g = g'$ and $x = x'$. Set $D_{(s,i)} = \mathrm{Im}(h_{D_{(s,i)}})$. Instead of (9.2), we have now

(9.2′) the intersection number $X_{\gamma(s)} \cdot D_{(s,i)} = \mathrm{sign}(c(s))$.

Let $\mathrm{ord} : \Gamma \to \{1, \cdots, |\Gamma|\}$ be a bijection. For each $i = 1, \cdots, |\Gamma| - 1$, take a k-dimensional band $B_i \cong I \times D^{k-1}$ (in general position in $\mathrm{Free}(G,X) \setminus \left(\bigcup_t \mathrm{Int}(D_{\mathrm{ord}^{-1}(t)})\right)$) connecting $\partial D_{\mathrm{ord}^{-1}(i)}$ with $\partial D_{\mathrm{ord}^{-1}(i+1)}$. This done, we obtain an embedded k-dimensional closed disk

$$D = D_{\mathrm{ord}^{-1}(1)} \cup B_1 \cup D_{\mathrm{ord}^{-1}(2)} \cup \cdots \cup B_{|\Gamma|-1} \cup D_{\mathrm{ord}^{-1}(|\Gamma|)}$$

in X. The bands B_i should be taken so that ∂D is the oriented connected sum of the oriented $\partial D_{(s,i)}$'s. Let $h_D : (D^k \cong) D \to X$ be the canonical inclusion and set $h = h_D|_{\partial D} : (S^{k-1} \cong) \partial D \to \mathrm{Free}(G,X)$. Without loss of generality, we may assume that if $gh(x) = g'h(x')$ (g, $g' \in G$ and x, $x' \in S^{k-1}$) then $g = g'$ and $x = x'$. There is a smooth embedding $H : S^{k-1} \times D^{k+1} \to \mathrm{Free}(G,X)$ such that $h(x) = H(x,0)$ for all $x \in S^{k-1}$ and such that the induced G-map $\mathrm{ind}^G H : G \times S^{k-1} \times D^{k+1} \to \mathrm{Free}(G,X)$ is an embedding. Construct the following spaces and maps as in Special Case: $W' = W \cup_{\mathrm{ind}^G H} (G \times D^k \times D^{k+1})$, X', F, F', $f' : X' \to Y$, and $b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta)$. Then set $\boldsymbol{f}' = (f', b')$. As in Special Case, $K_k(f';R) \cong K_k(f;R) \oplus \langle x,\ z\rangle_{R[G]}$. Moreover, $-x$ and $-\lambda z$ have geometric realizations by embeddings j_{-x}, $j_{-\lambda z} : S^k \to X'$, respectively, and $B_{f'}(x,x) = 0$, $B_{f'}(z,x) = 1$, $q_{\boldsymbol{f}'}(x) = 0$, $(q_{\boldsymbol{f}'}(z))(g) = 0$ ($\forall g \in \{1\} \cup G(2) \setminus (Q \cup S)$), and there is an element $v \in \mathbb{Z}[G \setminus (\{1\} \cup G(2))]$ such that $y = z + vx$ satisfies $q_{\boldsymbol{f}'}(y) = 0$. For each $s \in S$, let $j_{\gamma(s)} : X_{\gamma(s)} \to X$ be the canonical inclusion. Then $\mathrm{intsec}(j_{\gamma(s)}, j_{-\lambda z}) = -c(s)$

and hence

$$\begin{aligned}\mathrm{intsec}_G(j_{\gamma(s)}, j_{-\lambda z}) &= \sum_{g\in G} \mathrm{intsec}(j_{\gamma(s)}, g^{-1}j_{-\lambda z})g \\ &= \sum_{g\in G} w(g)\mathrm{intsec}(gj_{\gamma(s)}, j_{-\lambda z})g \\ &= \sum_{g\in G} w(g)\mathrm{intsec}(j_{g\gamma(s)}, j_{-\lambda z})g \\ &= \sum_{g\in G} w(g)\mathrm{intsec}(j_{\gamma(gsg^{-1})}, j_{-\lambda z})g \\ &= \sum_{g\in G} w(g)\left(-c(gsg^{-1})g\right).\end{aligned}$$

Thus

$$\begin{aligned}B_{f'}(z, \alpha_{f'}(s)) &= \lambda\overline{B_{f'}(\alpha_{f'}(s), z)} \\ &= -\overline{B_{f'}(\alpha_{f'}(s), -\lambda z)} \\ &= -\overline{\sum_{g\in G} w(g)\left(-c(gsg^{-1})g\right)} \\ &= \sum_{g\in G} c(gsg^{-1})g^{-1}.\end{aligned}$$

The equality (9.5) is replaced by the equality

$$(9.5') \qquad \begin{aligned}B_{f'}(z, \alpha_{f'}(s)) &= \sum_{g\in G} c(gsg^{-1})g^{-1} \\ (&= \sum_{gG_s\in G/G_s} c(gsg^{-1})\Sigma_{G_{gsg^{-1}}}g^{-1}).\end{aligned}$$

Let $M'' = \langle x,\ y\rangle_{R[G]}$ $(= \langle x,\ z\rangle_{R[G]})$ and $M_0 = M''^{\perp}$. Let $p_0 : K_k(f'; R) \to M_0$ and $p'' : K_k(f'; R) \to M''$ denote the projections associated to the decomposition $K_k(f'; R) = M_0 \oplus M''$. Let $\alpha'' = p'' \circ \alpha_{f'}$, $B'' = B_{f'}|_{M''}$, and $q'' = q_{\boldsymbol{f'}}|_{M''}$. By Theorem 6.3, $(M'', B'', q'', \alpha'') \cong \boldsymbol{M}(x, y, \beta'')$ where $\beta'' = (c'', r'')$ (c'', $r'' \in \mathrm{Map}(S, \mathbb{Z})$) is determined by the equation

$$\alpha''(s) = \sum_{g\in G} c''(gsg^{-1})g^{-1}x \quad (\forall s \in S), \quad \text{and} \quad B_{f'}(y, y) = \sum_{s\in S} r''(s)s.$$

As in Special Case, we compute that $c'' = c$. Since $\alpha''(s) = B_{f'}(z, \alpha_{f'}(s))x$, it follows that

$$(9.6') \qquad \alpha''(s) = \sum_{g\in G} c(gsg^{-1})g^{-1}x$$

and hence that $c'' = c$.

Next we calculate r''. By Lemma 9.1, we may assume that $\mathrm{intsec}(j_{-\lambda z}, sj_{-\lambda z})) = \lambda|c(s)|$ for all $s \in S$. Since $(q_{\boldsymbol{f}'}(z))(g) = 0$ for any $g \in \{1\} \cup G(2) \setminus (Q \cup S)$, it follows that $B_{f'}(z,z) = \sum_{s\in S} \lambda|c(s)|s$ in $\mathbb{Z}[\{1\} \cup G(2)] = \mathbb{Z}[G]/\mathbb{Z}[G \setminus (\{1\} \cup G(2))]$. As $q_{\boldsymbol{f}'}(y) = 0$ and $y = z + vx$ for some $v \in \mathbb{Z}[G \setminus (\{1\} \cup G(2))]$, we have

$$B_{f'}(y,y) = \sum_{s\in S} \lambda|c(s)|s. \tag{9.7$'$}$$

Thus for all $s \in S$, $r''(s) = \lambda|c(s)|$ and $r''(s) \equiv r(s) \mod 2\mathbb{Z}$.

Since $c'' = c$ (and $r'' \equiv r \mod 2$), Lemma 5.5 implies $\boldsymbol{M}(x,y,\beta'') \cong \boldsymbol{M}(x,y,\beta)$.

By the same arguments as in Special Case, we can check

$$(K_k(f;R), B_f, q_{\boldsymbol{f}}, \alpha_f) \cong (M_0, B_{f'}|_{M_0}, q_{\boldsymbol{f}'}|_{M_0}, p_0 \circ \alpha_{f'})$$

and conclude $\boldsymbol{M}_{\boldsymbol{f}'} \cong \boldsymbol{M}_{\boldsymbol{f}} \oplus \boldsymbol{M}(x,y,\beta)$. Q.E.D.

10. Appendix

We have invoked Assumption (HC) (see §7) in order to apply the next proposition.

Proposition 10.1. *Let M be an n-dimensional, G-CW-complex and let η and η' be real G-vector bundles with G-invariant Riemannian metrics over M. If $\eta \supseteqq \varepsilon_M(\mathbb{R}^{n+1})$ then any G-vector bundle isomorphism $b : \varepsilon_M(\mathbb{R}) \oplus \eta \to \varepsilon_M(\mathbb{R}) \oplus \eta'$ ($\oplus$ denotes orthogonal sum) is G-regularly homotopic to a G-vector bundle isomorphism $id_{\varepsilon_M(\mathbb{R})} \oplus b'$ where $b' : \eta \to \eta'$.*

Proof. It is well-known that b is G-regularly homotopic to a metric preserving isomorphism. (This follows from the fact that if $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ are G-invariant Riemannian metrics on the same underlying G-vector bundle ξ then $(1-t)\langle\ ,\ \rangle + t\langle\ ,\ \rangle'$ $(t \in I)$ is a G-invariant Riemannian metric on ξ, and from the equivariant covering homotopy property.) Thus we may assume that b is metric preserving.

We shall prove Proposition 10.1 by double induction on n and the number of isotropy types of n-dimensional cells. Suppose $M = M' \cup \bigcup_\gamma (G/H \times D^n_\gamma)$ where $D^n_\gamma = D^n$, and invoke the induction hypothesis that $b|_{M'}$ has the form $id_{\varepsilon_{M'}(\mathbb{R})} \oplus b''$, where $b'' : \eta|_{M'} \to \eta'|_{M'}$. Under this hypothesis, we shall find b' as in the conclusion of the proposition. For fixed γ, set $E = H/H \times \mathrm{Int}(D^n_\gamma)$. Then $b(\varepsilon_M(\mathbb{R})|_{\overline{E}\setminus E}) = \varepsilon_M(\mathbb{R})|_{\overline{E}\setminus E}$, but it is not necessary that

$$b(\varepsilon_M(\mathbb{R})|_E) = \varepsilon_M(\mathbb{R})|_E. \tag{10.2}$$

Let $b^H : \varepsilon_{M^H}(\mathbb{R}) \oplus \eta^H \to \varepsilon_{M^H}(\mathbb{R}) \oplus {\eta'}^H$ be the restriction of b to the H-fixed point set. Then $b|_{M^H}$ is decomposed into $b|_{M^H} = b^H \oplus b_H$ ($N_G(H)$-orthogonal sum). We deform b keeping $b|_{M'}$ and b_H fixed. The obstruction σ to deforming b to satisfy (10.2) lies in $\pi_{n-1}(S^{m-1})$, where $m = \text{fiber-dim}(\eta^H) + 1$. Since $\text{fiber-dim}(\eta^H) \geqq \text{fiber-dim}(\eta^G) \geqq n$, the obstruction group $\pi_{n-1}(S^{m-1})$ is trivial. Hence the obstruction σ vanishes. If (10.2) is satisfied for all γ then $b(\varepsilon_M(\mathbb{R})) = \varepsilon_M(\mathbb{R})$. Since b is metric preserving, we have $b(\eta) \subseteqq \eta'$. Moreover, we can arrange b so that $b|_{\varepsilon_M(\mathbb{R})} = id_{\varepsilon_M(\mathbb{R})}$, since $\text{fiber-dim}(\varepsilon_M(\mathbb{R}) \oplus \varepsilon_M(\mathbb{R}^{n+1})) \geqq 2$. Q.E.D.

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