ON NONABELIAN TENSOR PRODUCT MODULO q OF GROUPS

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Abstract. The nonabelian tensor product modulo q of two crossed modules of groups is investigated, where q is a positive integer. It is obtained a six term exact sequence of groups connecting the nonabelian tensor product modulo q with algebraic K-functor K_2 with $\mathbb{Z}_q$ coefficients for (noncommutative) local rings. The notion of q-homology groups of a group G with coefficients in a G-module A is introduced, some its properties and calculations are given. The relationship between q-homology groups and derived functors of tensor product modulo q is studied.

Introduction and summary

The study of algebraic and topological theories modulo an integer q has been the subject of a number of papers. For instance, W. Browder [1] has introduced an algebraic K-theory with $\mathbb{Z}_q$ coefficients, and J. Neisendorfer [2] following F.P.Peterson [3] has constructed and studied a homotopy theory with $\mathbb{Z}_q$ coefficients (primary homotopy theory) which are natural generalizations and have important applications to K-theory and homotopy theory.

The notion of the nonabelian tensor product modulo q, $M \otimes^q N$, and the exterior product modulo q, $M \wedge^q N$, of two crossed P-modules M and N was introduced by D. Conduche and C. Rodriguez-Fernandez [4] generalizing definitions of R. Brown [5] and G. Ellis and C. Rodriguez [6] (see also [7] and [8]). The nonabelian tensor product modulo q of crossed modules is a q-modular version of the nonabelian tensor product of crossed modules introduced by R. Brown and J.L. Loday [9-11], arising in applications in homotopy theory of a generalized Van Kampen theorem and studied by various authors.

The aim of this paper is to investigate the nonabelian tensor product modulo q of crossed modules in various aspects.

In Section 1 we give some functorial properties of the nonabelian tensor product modulo q of crossed modules; in particular, we show the existence of a unique isomorphism $\tau : M \otimes^q N \to N \otimes^q M$, $\tau(m \otimes n) = (n \otimes m)^{-1}$, where M and N are crossed P-modules (Proposition 1.4). We investigate for the nonabelian tensor product modulo q of crossed modules the properties of right exactness (Proposition

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1.5, 1.6) and compatibility with the direct limit of crossed modules (Proposition 1.7). The main result of Section 1 shows that the 'absolute' tensor product modulo q of two groups G and H with compatible actions is the quotient of nonabelian tensor product $G\otimes H$ by $q(H_1(G,H)\cap H_1(H,G))$ (Theorem 1.11), where $H_1(G,H)$ and $H_1(H,G)$ are first nonabelian homology of groups (see [12] and [13]). It is also established that the 'absolute' tensor product modulo q of two infinite cyclic groups acting on each other by 'funny' actions (only one nontrivial action) is isomorphic to the product of these infinite cyclic groups (Proposition 1.14).

In Section 2 we give an application of tensor product modulo q to algebraic K-theory with $\mathbb{Z}_q$ coefficients [1]. Particularly, for (noncommutative) local ring A such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$, we give a relationship between nonabelian tensor product modulo q and $K_2(A,\mathbb{Z}_q)$ which is an analog in q-modular aspect of D.Guin's six term exact sequence relating the nonabelian homology of groups with Milnor's K_2 and the symbol group Sym (Theorem 2.3).

In [13-15] nonabelian left derived functors of the nonabelian tensor product $G\otimes A$ of groups were investigated, particularly they were used to define the homology of a group G with coefficients in a nonabelian group A; the definition generalized the classical Eilenberg-MacLane homology of groups (see also [12]).

Analogously, the study of nonabelian left derived functors of the 'absolute' tensor product modulo q of groups inspired our definition of q-homology, $q-H_*(G,A)$, of a group G with coefficients in a G-module A, where q is a nonnegative integer, introduced in Section 3 (Definition 3.1)

$$q-H_n(G,A)\equiv \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}_q,A)\ .$$

We develop certain aspects of this q-modular version of the homology theory of discrete groups. In case $q=0$ the definition coincides with classical homology groups. We give universal coefficient formulas for q-homology of groups (Proposition 3.5). Then we calculate q-homology of free groups and finite cyclic groups (Proposition 3.8, Example 3.9, Proposition 3.10). D.Guin's isomorphism [12] is generalized for the tensor product modulo q by giving the short exact sequence of groups

$$0\longrightarrow G\otimes^q A\longrightarrow q-IG\otimes_{\mathbb{Z}[G]}A\longrightarrow q\mathbb{Z}\otimes_{\mathbb{Z}[G]}A\longrightarrow 0,\quad q\geq 0,$$

where G is a group, A is G-module and $q-IG$ is an augmentation ideal of $\widetilde{\epsilon}:\mathbb{Z}[G]\to\mathbb{Z}_q$ (see Proposition 3.13).

In Section 4 we investigate the derived functors of the nonabelian 'absolute' tensor product modulo q of groups establishing their relations with classical homology and q-homology of groups (Proposition 4.1, 4.2, 4.5). The main result of Section 4 is Theorem 4.4, showing that if A is a q-torsion free G-module and $q>0$, then there are natural isomorphisms $L_{n-1}^{\mathcal{P}}(G\otimes^q A)\approx q-H_n(G,A)$ for $n\geq 2$.

In future papers we continue the investigation that of Section 2 and Section 3. Particularly, Milnor type definition of algebraic K-functor K_2 with $\mathbb{Z}_q$ coefficients will be found and described for various rings with different symbol groups; it will be constructed and investigated q-modular versions of cohomlogy theory of groups and Tate cohomology theory of finite groups.

1. Tensor and exterior products modulo q of crossed modules

Let P be a group. A *crossed P-module* is a group homomorphism $\mu : M \to P$ together with an action of P on M, written ${}^p m$ for $p \in P$ and $m \in M$, and which satisfies the following conditions:

(a) $\mu({}^p m) = p\mu(m)p^{-1}, \quad p \in P,\, m \in M$;
(b) ${}^{\mu(m)}m' = mm'm^{-1}, \quad m, m' \in M$.

Let $\mu : M \to P$ and $\mu' : M' \to P'$ be crossed P-module and crossed P'-module respectively. A *morphism of crossed modules* is a pair $(f : M \to M',\ \varphi : P \to P')$ of group homomorphisms such that $f({}^p m) = {}^{\varphi(p)}f(m)$ for all $m \in M$, $p \in P$ and $\mu' f = \varphi\mu$.

Recall the definition of the tensor product modulo q of crossed modules from [4].

Let $\mu : M \to P$ and $\nu : N \to P$ be two crossed P-modules and consider the pullback

$$(pull) \qquad \begin{array}{ccc} M \times_P N & \xrightarrow{\pi_1} & M \\ {\scriptstyle\pi_2}\downarrow & & \downarrow{\scriptstyle\mu} \\ N & \xrightarrow{\nu} & P \end{array} .$$

Let $K = M \times_P N = \{(m,n) \in M \times N | m \in M, n \in N, \mu(m) = \nu(n)\}$. In this diagram each group acts on any other group via its image in the group P.

Definition 1.1. *The tensor product modulo q, $M \otimes^q N$, of the crossed P-modules μ and ν, for $q \geq 1$, is the group generated by the symbols $m \otimes n$ and $\{k\}$, $m \in M$, $n \in N$, $k \in K$ subject to the following relations:*

$$mm' \otimes n = ({}^m m' \otimes {}^m n)(m \otimes n), \tag{1.1}$$
$$m \otimes nn' = (m \otimes n)({}^n m \otimes {}^n n'), \tag{1.2}$$
$$\{k\}(m \otimes n)\{k\}^{-1} = {}^{k^q}m \otimes {}^{k^q}n, \tag{1.3}$$
$$\{kk'\} = \{k\}\prod_{i=1}^{q-1}(\pi_1 k^{-1} \otimes ({}^{k^{1-q+i}}\pi_2 k')^i)\{k'\}, \tag{1.4}$$
$$[\{k\}, \{k'\}] = \pi_1 k^q \otimes \pi_2 k'^q, \tag{1.5}$$
$$\{(m^n m^{-1}, {}^m n n^{-1})\} = (m \otimes n)^q, \tag{1.6}$$

for all $m, m' \in M, \quad n, n' \in N, \quad k, k' \in K$.

Definition 1.2 ([4])**.** *The exterior product modulo q, $M \wedge^q N$, of the crossed P-modules μ and ν, for $q \geq 1$, is obtained from the tensor product $M \otimes^q N$ by imposing the additional relations*

$$\pi_1 k \otimes \pi_2 k = 1\ , \quad k \in K. \tag{1.7}$$

The image of a generic element $m \otimes n$ in $M \wedge^q N$ is written $m \wedge n$.

Note that, for $q = 0$, under $M \otimes^0 N$ we mean Brown-Loday's tensor product, $M \otimes N$, which is the group generated by elements $m \otimes n$, $m \in M$, $n \in N$ and subject to relations (1.1) and (1.2) above (see [9-11]). Furthermore, under $M \wedge^0 N$ we mean Brown-Loday's exterior product, $M \wedge N$, which is the group generated by elements $m \wedge n$, $m \in M$, $n \in N$ and subject to relations (1.1), (1.2) and (1.7) (see [10], [11] and [16]).

Recall the definition of the function $\beta_t(k,k')$ from [4], where $k,k' \in K$ and t is a positive integer:

$$\beta_t(k,k') = \prod_{i=1}^{t-1} (\pi_1 k^{-1} \otimes {}^{k^{1-t+i}} \pi_2 k'^i) \ . \tag{1.8}$$

One has the following

Lemma 1.3. *For any $k,k' \in K$ and any positive integer t we have the following equality:*

$$\beta_t(k,k') = \prod_{i=1}^{t-1} ({}^{k'^{i-1}} \pi_1 k^{i-t} \otimes \pi_2 k') \ .$$

Proof. We only prove this assertion when t is odd. By (1.8) definition of β we have

$$\beta_t(k,k') = \prod_{i=1}^{t-1} (\pi_1 k^{-1} \otimes {}^{k^{1-t+i}} \pi_2 k'^i) = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2) \cdots$$
$$\cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j) \cdots (\pi_1 k^{-1} \otimes {}^{k^{-1}} \pi_2 k'^{t-2})(\pi_1 k^{-1} \otimes \pi_2 k'^{t-1}),$$

where $j = \frac{t-1}{2}$.

Thus,

$$\beta_t(k,k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2) \cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j) \cdots$$
$$\cdots (\pi_1 k^{-1} \otimes {}^{k^{-1}} \pi_2 k'^{t-2}) \cdot (\pi_1 k^{-1} \otimes \pi_2 k'^{t-2})({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k') \cdot$$
$$\cdot (\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2) \cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j) \cdots (\pi_1 k^{-2} \otimes \pi_2 k'^{t-2})({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') =$$
$$= (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k') \cdot (\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2) \cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j)(\pi_1 k^{-j} \otimes \pi_2 k'^{j+1}) \cdot$$
$$\cdot ({}^{k'^{j+1}} \pi_1 k^{1-j} \otimes \pi_2 k') \cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2)$$
$$\cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j) \cdot (\pi_1 k^{-j} \otimes \pi_2 k'^j)({}^{k'^j} \pi_1 k^{-j} \otimes \pi_2 k')({}^{k'^{j+1}} \pi_1 k^{1-j} \otimes \pi_2 k') \cdots$$
$$\cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k').$$

Now compute other part:

$$\prod_{i=1}^{t-1} ({}^{k'^{i-1}} \pi_1 k^{i-t} \otimes \pi_2 k') = (\pi_1 k^{1-t} \otimes \pi_2 k')({}^{k'} \pi_1 k^{2-t} \otimes \pi_2 k') \cdots ({}^{k'^{t-3}} \pi_1 k^{-2} \otimes \pi_2 k') \cdot$$
$$\cdot ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{2-t} \otimes \pi_2 k')({}^{k'} \pi_1 k^{2-t} \otimes \pi_2 k') \cdots$$
$$\cdots ({}^{k'^j} \pi_1 k^{-j} \otimes \pi_2 k') \cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{2-t} \otimes \pi_2 k'^2) \cdots$$
$$\cdots ({}^{k'^j} \pi_1 k^{-j} \otimes \pi_2 k') \cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k')(\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2)$$
$$\cdots (\pi_1 k^{-j-1} \otimes \pi_2 k'^j)({}^{k'^j} \pi_1 k^{-j} \otimes \pi_2 k') \cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k') = (\pi_1 k^{-1} \otimes {}^{k^{2-t}} \pi_2 k') \cdot$$
$$\cdot (\pi_1 k^{-1} \otimes {}^{k^{3-t}} \pi_2 k'^2) \cdots (\pi_1 k^{-1} \otimes {}^{k^{-j}} \pi_2 k'^j)(\pi_1 k^{-j} \otimes \pi_2 k'^j)({}^{k'^j} \pi_1 k^{-j} \otimes \pi_2 k') \cdots$$
$$\cdots ({}^{k'^{t-2}} \pi_1 k^{-1} \otimes \pi_2 k').$$

Hence we have proved that $\beta_t(k,k') = \prod_{i=1}^{t-1}({}^{k'^{i-1}}\pi_1 k^{i-t} \otimes \pi_2 k')$, when t is odd. The assertion when t is even is similar and left to the reader. □

Proposition 1.4. (i) *Let (M,μ) and (N,ν) be crossed P-modules. Then there is a unique isomorphism*

$$\tau : M \otimes^q N \longrightarrow N \otimes^q M \quad (\tau : M \wedge^q N \longrightarrow N \wedge^q M)\ ,$$

such that $\tau(m \otimes n) = (n \otimes m)^{-1}$ ($\tau(m \wedge n) = (n \wedge m)^{-1}$), $\tau(\{k\}) = \{\overline{k}^{-1}\}^{-1}$, where $\overline{k} = (\pi_2 k, \pi_1 k)$ for all $m \in M$, $n \in N$ and $k \in K$.

(ii) *Suppose (M,μ), (N,ν) are crossed P-modules and (M',μ'), (N',ν') are crossed P'-modules. Suppose $\alpha = (f,\varphi) : (M,\mu) \to (M',\mu')$, $\beta = (g,\psi) : (N,\nu) \to (N',\nu')$ are crossed module morphisms such that $\varphi = \psi$. Then there is a unique homomorphism*

$$\alpha \otimes^q \beta : M \otimes^q N \longrightarrow M' \otimes^q N' \quad (\alpha \wedge^q \beta : M \wedge^q N \longrightarrow M' \wedge^q N')\ ,$$

such that $(\alpha \otimes^q \beta)(m \otimes n) = f(m) \otimes g(n)$ ($(\alpha \wedge^q \beta)(m \wedge n) = f(m) \wedge g(n)$), $(\alpha \otimes^q \beta)(\{k\}) = \{(f(\pi_1 k), g(\pi_2 k))\}$ ($(\alpha \wedge^q \beta)(\{k\}) = \{(f(\pi_1 k), g(\pi_2 k))\}$) for all $m \in M$, $n \in N$ and $k \in K$. Further, if α, β are onto, so also is $\alpha \otimes^q \beta$ ($\alpha \wedge^q \beta$).

Proof. (i) We have only to show that τ commutes with relations (1.1)-(1.7); for instance, by Lemma 1.3 one has

$$\tau(\{kk'\}) = \{\overline{k}'^{-1}\overline{k}^{-1}\}^{-1} = \{\overline{k}^{-1}\}^{-1}\beta_q(\overline{k}'^{-1}, \overline{k}^{-1})^{-1}\{\overline{k}'^{-1}\}^{-1} = \tau(\{k\}) \cdot$$
$$\cdot [\prod_{i=1}^{q-1}({}^{\overline{k}^{1-i}}\pi_1 \overline{k}'^{q-i} \otimes \pi_2 \overline{k}^{-1})]^{-1}\tau(\{k'\}) = \tau(\{k\})\prod_{i=1}^{q-1}({}^{k^{1-q+i}}\pi_2 k'^{i} \otimes \pi_1 k^{-1})^{-1} \cdot$$
$$\cdot \tau(\{k'\}) = \tau(\{k\})\prod_{i=1}^{q-1}\tau(\pi_1 k^{-1} \otimes ({}^{k^{1-q+i}}\pi_2 k')^{i})\tau(\{k'\});$$

(ii) Proof of this part is left as an exercise. □

Now we investigate for the nonabelian tensor product modulo q of crossed modules the properties of right exactness and compatibility with the direct limit of crossed modules.

Let (L,λ), (M,μ) and (N,ν) be crossed P-modules. A short exact sequence of groups

$$1 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 1$$

is called short exact sequence of crossed P-modules if α and β are morphisms of crossed P-modules.

Proposition 1.5. *Let $1 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 1$ be a short exact sequence of crossed P-modules. Then there is an exact sequence of groups*

$$P \otimes^q L \xrightarrow{\alpha'} P \otimes^q M \xrightarrow{\beta'} P \otimes^q N \longrightarrow 1\ , \tag{1.9}$$

where $\alpha' = 1_P \otimes^q \alpha$, $\beta' = 1_P \otimes^q \beta$.

Proof. Using (1.1), (1.2) and (1.4) one can easily check that $\beta'\alpha'$ is the trivial homomorphism. By Proposition 1.4 (ii) it is also clear that β' is surjective.

Now we have to show that $\operatorname{Im}\alpha'$ is a normal subgroup of $P \otimes^q M$. In effect, by [11, Propostion 3]

$$(p \otimes m)(p' \otimes \alpha(l))(p \otimes m)^{-1} = {}^{[p,m]}p' \otimes {}^{[p,m]}\alpha(l) \in \operatorname{Im}\alpha' ,$$

for all $l \in L$, $m \in M$, $p \in P$. By (1.3)

$$\{(p,m)\}(p' \otimes \alpha(l))\{(p,m)\}^{-1} = {}^{p^q}p' \otimes {}^{p^q}\alpha(l) \in \operatorname{Im}\alpha' ,$$

for all $l \in L$, $m \in M$, $p, p' \in P$, where $\mu(m) = p$. By [4, Proposition 1.16 and Lemma 1.10]

$$\{(p,m)\}\alpha'\{(1,l)\}\{(p,m)\}^{-1} = (p^q \otimes \alpha(l)^q)\{(1,\alpha(l))\} \in \operatorname{Im}\alpha'$$

and

$$(p' \otimes m')\alpha'\{(1,l)\}(p' \otimes m')^{-1} = (p'\mu(m')p'^{-1}\mu(m')^{-1} \otimes \alpha(l)^q)\{(1,\alpha(l))\} \in \operatorname{Im}\alpha' ,$$

for all $l \in L$, $m, m' \in M$, $p, p' \in P$, where $\mu(m) = p$, $\lambda(l) = 1$.

Therefore, we have the diagram of groups

$$\begin{array}{ccccccc} P \otimes^q L & \xrightarrow{\alpha'} & P \otimes^q M & \xrightarrow{\beta'} & P \otimes^q N & \longrightarrow & 1 \\ || & & || & & & & \\ P \otimes^q L & \xrightarrow{\alpha'} & P \otimes^q M & \xrightarrow{\tau} & \operatorname{Coker}\alpha' & \longrightarrow & 1 \end{array} ,$$

where the bottom row is exact. Thus, there exists a natural homomorphism $\gamma : \operatorname{Coker}\alpha' \to P \otimes^q N$ such that $\gamma\tau = \beta'$.

Let us define a homomorphism $\gamma' : P \otimes^q N \to \operatorname{Coker}\alpha'$ as follows: $\gamma'(p \otimes n) = [p \otimes m]$, $\gamma'\{(p',n')\} = [\{(p',m')\}]$, where $\beta(m) = n$, $\beta(m') = n'$ and $\mu(m') = \nu(n') = p'$. It is correctly defined. In effect, let $m_1 = m\alpha(l)$ and $m'_1 = \alpha(l)m'$, then

$$p \otimes m_1 = p \otimes m\alpha(l) = (p \otimes m)(\mu(m)p\mu(m)^{-1} \otimes m\alpha(l)m^{-1}) ,$$

and

$$\{(p',m'_1)\} = \{(p',\alpha(l')m')\} = \{(1,\alpha(l'))(p',m')\} = \{(1,\alpha(l'))\}\{(p',m')\} .$$

Hence $[p \otimes m_1 = p \otimes m]$ and $[\{(p',m'_1)\}] = [\{(p',m')\}]$.

It is easy to check that γ' is compatible with the relations (1.1)-(1.6) and $\gamma\gamma'$, $\gamma'\gamma$ are identity maps. □

Note that in general nonabelian tensor product modulo q of crossed P-modules is not a right exact functor, i.e. the sequence (1.9) is not exact when the group P is replaced by any crossed P-module A.

Let M and N be crossed P-modules. By Lemma 1.3 [4] we have two homomorphisms $\xi : M \otimes^q N \to M$ and $\xi' : M \otimes^q N \to N$ defined by

$$\xi(m \otimes n) = m^n m^{-1} , \quad \xi(\{k\}) = \pi_1 k^q , \tag{1.10}$$

$$\xi'(m \otimes n) = {}^m n n^{-1} , \quad \xi'(\{k\}) = \pi_2 k^q . \tag{1.11}$$

Furthermore, these homomorphisms factor through $M \wedge^q N$.

Proposition 1.6. *Let $1 \to L \to M \to N \to 1$ be a short exact sequence of crossed P-modules. Then there is an exact sequence of groups*

$$\begin{array}{c} \operatorname{Ker}\xi'_L \longrightarrow \operatorname{Ker}\xi'_M \longrightarrow \operatorname{Ker}\xi'_N \longrightarrow \operatorname{Coker}\xi'_L \longrightarrow \\ \operatorname{Coker}\xi'_M \longrightarrow \operatorname{Coker}\xi'_N \longrightarrow 1 \end{array},$$

where the homomorphisms $\xi'_L : P\otimes^q L \to L$, $\xi'_M : P\otimes^q M \to M$ and $\xi'_N : P\otimes^q N \to N$ are defined according to (1.11).

Proof. Follows from the commutative diagram of groups with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Ker}\xi'_L & \longrightarrow & \operatorname{Ker}\xi'_M & \longrightarrow & \operatorname{Ker}\xi'_N & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & P\otimes^q L & \longrightarrow & P\otimes^q M & \longrightarrow & P\otimes^q N & \longrightarrow & 1 \\
 & & \downarrow {\scriptstyle \xi'_L} & & \downarrow {\scriptstyle \xi'_M} & & \downarrow {\scriptstyle \xi'_N} & & \\
1 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker}\xi'_L & \longrightarrow & \operatorname{Coker}\xi'_M & \longrightarrow & \operatorname{Coker}\xi'_N & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}.$$

□

Proposition 1.7. *Let $\{M_\alpha, \Phi^\beta_\alpha, \alpha \le \beta\}$ and $\{P_\alpha, \Psi^\beta_\alpha, \alpha \le \beta\}$ be two directed systems of groups. Let $\mu_\alpha : M_\alpha \to P_\alpha$ be a crossed P_α-module for every α such that $(\Phi^\beta_\alpha, \Psi^\beta_\alpha) : (M_\alpha, P_\alpha) \to (M_\beta, P_\beta)$, $\alpha \le \beta$ is a crossed module morphism. Let $\nu_\alpha : N \to P_\alpha$ be a crossed P_α-module for every α such that $(1, \Psi^\beta_\alpha) : (N, \nu_\alpha) \to (N, \nu_\beta)$, $\alpha \le \beta$ is a crossed module morphism. Then there is a natural isomorphism*

$$(\varinjlim{}_\alpha\{M_\alpha\}) \otimes^q N \approx \varinjlim{}_\alpha\{M_\alpha \otimes^q N\}\ , \quad (\varinjlim{}_\alpha\{M_\alpha\}) \wedge^q N \approx \varinjlim{}_\alpha\{M_\alpha \wedge^q N\})\ .$$

Proof. At first note that the tensor product modulo q of $\varinjlim{}_\alpha\{M_\alpha\}$ and N are considered as crossed modules over the group $\varinjlim{}_\alpha\{P_\alpha\}$.

It is clear that a homomorphism $\nu : N \to \varinjlim{}_\alpha\{P_\alpha\}$ defined as follows, $\nu(n) = [\nu_\alpha(n)]$, $n \in N$, with action ${}^{[p_\alpha]}n = {}^{p_\alpha}n$, is a crossed module.

Now we have to show that a homomorphism $\mu : \varinjlim{}_\alpha\{M_\alpha\} \to \varinjlim{}_\alpha\{P_\alpha\}$ defined by $\mu([m_\alpha]) = [\mu_\alpha(m_\alpha)]$ with action ${}^{[p_\alpha]}[m_\beta] = [{}^{\Psi^\gamma_\alpha(p_\alpha)}\Phi^\gamma_\beta(m_\beta)]$, where $\gamma \ge \alpha, \beta$ (existense of such γ follows from directeness of the system) is a crossed module. Correctness here is left to the reader. Next

$$\begin{aligned} &\mu({}^{[p_\alpha]}[m_\beta]) = \mu({}^{\Psi^\gamma_\alpha}\Phi^\gamma_\beta(m_\beta)) = [\mu_\gamma({}^{\Psi^\gamma_\alpha(p_\alpha)}\Phi^\gamma_\beta(m_\beta))] = [\Psi^\gamma_\alpha(p_\alpha)\mu_\gamma\Phi^\gamma_\beta(m_\beta)\Psi^\gamma_\alpha(p_\alpha^{-1})] = \\ &= [\Psi^\gamma_\alpha(p_\alpha)\Psi^\gamma_\beta\mu_\beta(m_\beta)\Psi^\gamma_\alpha(p_\alpha^{-1})] = [p_\alpha]\mu[m_\beta][p_\alpha^{-1}], \quad \textit{where } \gamma \ge \alpha, \beta. \end{aligned}$$

$$\begin{aligned} &{}^{\mu([m_\alpha])}[m_\beta] = {}^{[\mu_\alpha(m_\alpha)]}[m_\beta] = [{}^{\Psi^\gamma_\alpha\mu_\alpha(m_\alpha)}\Phi^\gamma_\beta(m_\beta)] = [{}^{\mu_g m\Phi^\gamma_\alpha(m_\alpha)}\Phi^\gamma_\beta(m_\beta)] = \\ &= [\Phi^\gamma_\alpha(m_\alpha)\Phi^\gamma_\beta(m_\beta)\Phi^\gamma_\alpha(m_\alpha^{-1})] = [m_\alpha][m_\beta][m_\alpha^{-1}], \quad \textit{where } \gamma \ge \alpha, \beta. \end{aligned}$$

Let us define a homomorphism

$$\kappa : (\varinjlim_\alpha\{M_\alpha\})\otimes^q N \longrightarrow \varinjlim_\alpha\{M_\alpha\otimes^q N\} \quad (\kappa : (\varinjlim_\alpha\{M_\alpha\})\wedge^q N \longrightarrow \varinjlim_\alpha\{M_\alpha\wedge^q N\})$$

as follows: $\kappa([m_\alpha]\otimes n) = [m_\alpha \otimes n]$ ($\kappa([m_\alpha]\wedge n) = [m_\alpha \wedge n]$) and $\kappa(\{([m_\alpha], n)\}) = [\{(\Phi_\alpha^\beta(m_\alpha), n)\}]$, since $[\mu_\alpha(m_\alpha)] = [\nu_\alpha(n)]$ and therefore there exists $\beta \geq \alpha$ such that $\Psi_\alpha^\beta \mu_\alpha(m_\alpha) = \Psi_\alpha^\beta \nu_\alpha(n)$; $\mu_\beta \Phi_\alpha^\beta(m_\alpha) = \nu_\beta(n)$.

Let $(\Phi_\alpha^\gamma)' = \Phi_\alpha^\gamma \otimes^q 1_N$ and $(\Phi_\beta^\gamma)' = \Phi_\beta^\gamma \otimes^q 1_N$. If $[m_\alpha] = [m_\beta]$, then there exists $\gamma \geq \alpha, \beta$ such that $\Phi_\alpha^\gamma(m_\alpha) = \Phi_\beta^\gamma(m_\beta)$. Thus,

$$(\Phi_\alpha^\gamma)'(m_\alpha \otimes n) = \Phi_\alpha^\gamma(m_\alpha)\otimes n = \Phi_\alpha^\gamma(m_\beta)\otimes n = (\Phi_\beta^\gamma)'(m_\beta \otimes n).$$

If exists $\beta' \geq \alpha$ such that $\Psi_\alpha^{\beta'}\mu_\alpha(m_\alpha) = \Psi_\alpha^{\beta'}\nu_\alpha(n)$ and therefore $\mu_{\beta'}\Phi_\alpha^{\beta'}(m_\alpha) = \nu_{\beta'}(n)$, then there exists $\gamma \geq \beta, \beta'$. Thus

$$(\Phi_\beta^\gamma)'(\{(\Phi_\alpha^\beta(m_\alpha), n)\}) = \{(\Phi_\alpha^\gamma(m_\alpha), n)\} = (\Phi_{\beta'}^\gamma)'(\{(\Phi_\alpha^{\beta'}(m_\alpha), n)\}).$$

If $[m_\alpha] = [m_{\alpha'}]$, then there exists $\gamma \geq \alpha, \alpha'$ such that $\Phi_\alpha^\gamma(m_\alpha) = \Phi_{\alpha'}^\gamma(m_{\alpha'})$. Hence one has

$$(\Phi_\beta^{\gamma'})'(\{(\Phi_\alpha^\beta(m_\alpha), n)\}) = \{(\Phi_\alpha^{\gamma'}(m_\alpha), n)\} = (\Phi_{\beta'}^{\gamma'})'(\{(\Phi_{\alpha'}^{\beta'}(m_{\alpha'}), n)\}),$$

where $\gamma' \geq \gamma, \beta, \beta'$. Therefore κ is correctly defined. Commutativity of κ with relations (1.1)-(1.7) is easy to check and left to the reader.

On the other hand, the canonical homomorphisms $\Phi_\alpha : M_\alpha \otimes^q N \to (\varinjlim_\alpha\{M_\alpha\})\otimes^q N$ ($\Phi_\alpha : M_\alpha \wedge^q N \to (\varinjlim_\alpha\{M_\alpha\})\wedge^q N$), $\Phi_\alpha(m_\alpha \otimes n) = [m_\alpha]\otimes n$ ($\Phi_\alpha(m_\alpha \wedge n) = [m_\alpha]\wedge n$), $\Phi_\alpha(\{(m_\alpha, n)\}) = \{([m_\alpha], n)\}$, induce a homomorphism $\kappa' : \varinjlim_\alpha\{M_\alpha \otimes^q N\} \to \varinjlim_\alpha\{M_\alpha\}\otimes^q N$ ($\kappa' : \varinjlim_\alpha\{M_\alpha \wedge^q N\} \to \varinjlim_\alpha\{M_\alpha\}\wedge^q N$) and it is easy to see that $\kappa\kappa'$, $\kappa'\kappa$ are identity maps. $\square$

Let r be a nonnegative integer and $\mu : M \to P$ be a crossed P-module. (M, μ) is called a r-crossed P-module if $a^r = 1$ for all $a \in \operatorname{Ker}\mu$.

We have the following

Proposition 1.8. *Let $\mu : M \to P$ and $\nu : N \to P$ be r-crossed and l-crossed P-modules respectively. Let s be the least common multiple of r and l. Then $\alpha : M \wedge^q N \to P$, given by $\alpha(m\wedge n) = [\mu(m), \nu(n)]$, $\alpha(\{k\}) = \mu(\pi_1 k)^q = \nu(\pi_2 k)^q$, is a qs-crossed P-module.*

Proof. By [4, Corollary 1.17] $(M\wedge^q N, \alpha)$ is a crossed P-module. Let $x \in \operatorname{Ker}\alpha$. Then by [4, Corollary 1.21] and [4, Lemma 1.20] one has

$$x^{qs} = \{\partial' x\}^s = \{\partial' x^s\},$$

where $\partial' : M\wedge^q N \to K$ is defined by the following way

$$\partial' x = (\xi x, \xi' x)$$

and the homomorphisms ξ and ξ' are defined according to (1.10) and (1.11).

Since $\alpha = \mu\xi = \nu\xi'$, one has

$$x^{qs} = \{\partial' x^s\} = \{(\xi x, \xi' x)^s\} = \{(\xi x^s, \xi' x^s)\} = \{(1,1)\} = 1.$$

$\square$

Let M and N be groups which act on themselves by conjugation (${}^{x}y = xyx^{-1}$) and each of which acts upon the other in such a way that the following *compatibility conditions* hold:

$$^{({}^{m}n)}m' = {}^{m}({}^{n}({}^{m^{-1}}m')) \ , \qquad ^{({}^{n}m)}n' = {}^{n}({}^{m}({}^{n^{-1}}n')) \ , \tag{1.12}$$

for all $m, m' \in M$ and $n, n' \in N$.

Consider the Peiffer product $M \bowtie N$, which was defined by Whitehead in [17], and is the quotient of the free product $M * N$ by the normal subrgoup generated by the elements ${}^{m}nmn^{-1}m^{-1}$ and ${}^{n}mnm^{-1}n^{-1}$, for all $m \in M$, $n \in N$. As a consequence of the compatibility conditions (1.12) the actions of $M * N$ on M and on N factor through $M \bowtie N$ and the canonical maps $M \to M \bowtie N$ and $N \to M \bowtie N$ are crossed modules; see [18] for more details.

Definition 1.9 ([4])**.** *An 'absolute' tensor product modulo q of two groups M and N with compatible actions on each other is the tensor product modulo q of crossed modules M and N over the group $M \bowtie N$.*

It is known from [4] the following

Proposition 1.10. *Let M and N be groups acting trivially on each other. Then there is a natural isomorphism*

$$M \otimes^q N \approx (M^{ab}/qM^{ab}) \otimes_{\mathbb{Z}/q\mathbb{Z}} (N^{ab}/qN^{ab}).$$

One has the following

Theorem 1.11. *Let G and H be groups acting compatibly on each other. Then we have the following exact sequence of groups*

$$0 \longrightarrow q(H_1(G,H) \cap H_1(H,G)) \longrightarrow G \otimes H \xrightarrow{\varphi} G \otimes^q H \longrightarrow 1 \ ,$$

where φ is given by $\varphi(g \otimes h) = g \otimes h$.

Proof. By [4, Proposition 1.6] for any crossed modules $\mu : G \to P$ and $\nu : H \to P$ over any group P we have the exact sequence of groups

$$G \otimes H \xrightarrow{\varphi} G \otimes^q H \longrightarrow K/[G,H] \longrightarrow 1, \tag{1.13}$$

where $[G,H]$ is the subgroup of $K = G \times_P H$ generated by the elements $(g^{h}g^{-1}, {}^{g}hh^{-1})$, $g \in G$, $h \in H$.

In our case, when P is the Peiffer product of G and H, according to [19], $P = G \bowtie H = (G \rtimes H)/L$, where L is the subgroup of $G \rtimes H$ (semidirect product of G and H) generated by the elements $(g^{h}g^{-1}, h^{g}h^{-1})$, $g \in G$, $h \in H$ and we have to show that $K = [G,H]$. In effect, let $k = (g,h) \in K$. Then $(g, h^{-1}) \in L \subseteq G \rtimes H$ and therefore we have

$$\begin{aligned}(g, h^{-1}) &= ({g_1}^{h_1}g_1^{-1}, {h_1}^{g_1}h_1^{-1}) \cdots ({g_k}^{h_k}g_k^{-1}, {h_k}^{g_k}h_k^{-1}) = \\ &= ({g_k}^{h_k}g_k^{-1} \cdots {g_1}^{h_1}g_1^{-1}, {h_1}^{g_1}h_1^{-1} \cdots {h_k}^{g_k}h_k^{-1}).\end{aligned}$$

Hence,

$$g = {g_k}^{h_k} g_k^{-1} \cdots {g_1}^{h_1} g_1^{-1},$$
$$h = {}^{g_k}h_k h_k^{-1} \cdots {}^{g_1}h_1 h_1^{-1}.$$

Thus $k = ({g_k}^{h_k} g_k^{-1}, {}^{g_k}h_k h_k^{-1}) \cdots ({g_1}^{h_1} g_1^{-1}, {}^{g_1}h_1 h_1^{-1})$, i.e. $K = [G, H]$ and by (1.13) the homomorphism φ is surjective.

By definition $H_1(G, H)$ and $H_1(H, G)$ (see [12-15]) are kernels of the homomorphisms $G \otimes H \xrightarrow{\lambda'} H$, $\lambda'(g \otimes h) = {}^g h h^{-1}$ and $G \otimes H \xrightarrow{\lambda} G$, $\lambda(g \otimes h) = g^h g^{-1}$ respectively, which are crossed modules (see [10]) and therefore $q(H_1(G, H) \cap H_1(H, G)) \subseteq (H_1(G, H) \cap H_1(H, G)) \subseteq Z(G \otimes H)$ (center of $G \otimes H$).

First we have to show that $\varphi(q(H_1(G, H) \cap H_1(H, G))) = 1$.

Let $(g_1 \otimes h_1) \cdots (g_m \otimes h_m) \in H_1(G, H) \cap H_1(H, G) \subseteq G \otimes H$, then by the formulas (1.4), (1.6) and Lemma 1.13 [4] one has

$$\varphi((g_1 \otimes h_1) \cdots (g_m \otimes h_m))^q = ((g_1 \otimes h_1) \cdots (g_m \otimes h_m))^q =$$
$$\{({g_1}^{h_1} {g_1}^{-1}, {}^{g_1}h_1 h_1^{-1}) \cdots ({g_m}^{h_m} {g_m}^{-1}, {}^{g_m}h_m h_m^{-1})\} = \{(1, 1)\} = 1.$$

Hence φ induces a natural homomorphism

$$\Phi : G \otimes H/q(H_1(G, H) \cap H_1(H, G)) \longrightarrow G \otimes^q H.$$

Now define a homomorphism $\Psi : G \otimes^q H \longrightarrow G \otimes H/q(H_1(G, H) \cap H_1(H, G))$ as follows: $\Psi(g \otimes h) = [g \otimes h]$, $\Psi(\{k\}) = [((g_1 \otimes h_1) \cdots (g_n \otimes h_n))^q]$, since $k = ({g_1}^{h_1} {g_1}^{-1} \cdots {g_n}^{h_n} {g_n}^{-1}, {}^{g_1}h_1 {h_1}^{-1} \cdots {}^{g_n}h_n {h_n}^{-1})$ (see above). If $k = ({g'_1}^{h'_1} {g'_1}^{-1} \cdots$ $\cdots {g'_m}^{h'_m} {g'_m}^{-1}, {}^{g'_1}h'_1 {h'_1}^{-1} \cdots {}^{g'_m}h'_m {h'_m}^{-1})$, then

$$(g_1 \otimes h_1) \cdots (g_n \otimes h_n)((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m))^{-1} \in H_1(G, H) \cap H_1(H, G) \subseteq$$
$$\subseteq Z(G \otimes H)\ .$$

Hence

$$((g_1 \otimes h_1) \cdots (g_n \otimes h_n))^q((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m))^{-q} = ((g_1 \otimes h_1) \cdots (g_n \otimes h_n) \cdot$$
$$\cdot ((g'_1 \otimes h'_1) \cdots (g'_m \otimes h'_m))^{-1})^q \in q(H_1(G, H) \cap H_1(H, G))\ .$$

Thus Ψ is correctly defined. It is easy to see that Ψ commutes with relations (1.1)-(1.6) and $\Phi\Psi$ and $\Psi\Phi$ are identity maps. □

Remark 1.12. *Let M and N be two normal subgroups of a group P, and let they act on each other by conjugation in P. Then the tensor product modulo q, $M \otimes^q N$, of M and N may be considered as crossed $M \bowtie N$-modules ('absolute') or as crossed P-modules with canonical inclusions. In general this two definitions are different, but when $[M, N] = M \cap N$ (in the first case $K = [M, N]$ and in the second case $K = M \cap N$) they coincide, e.g. when $M = N = P$ is a perfect group.*

Corollary 1.13. *If G is a perfect group then one has the following short exact sequence of groups*

$$0 \longrightarrow qH_2(G) \longrightarrow G \otimes G \longrightarrow G \otimes^q G \longrightarrow 1\ .$$

Proof. Follows from Theorem 1.11 and the fact that if G is perfect then $H_1(G,G) = H_2(G)$ (see [10]). $\square$

Now let us consider two infinite cyclic groups X and Y generated by x and y respectively, acting on each other by the following 'funny' actions

$$^{x}y = y^{-1}, \qquad ^{y}x = x^{-1} ,$$

which cannot arise as conjugation in a big group containing X and Y as normal subgroups.

In [18] it is proved that these actions are compatible and $X \otimes Y \approx \mathbb{Z}^2$ with basis $x \otimes y$ and $x^2 \otimes y$.

Proposition 1.14. *Let X and Y be infinite cyclic groups generated by x and y respectively, acting on each other by $^{x}y = y^{-1}$, $^{y}x = x^{-1}$. Then*

$$X \otimes^q Y \approx X \otimes Y \approx \mathbb{Z}^2.$$

Proof. Using Theorem 1.11 one has

$$X \otimes^q Y \approx X \otimes Y / q(\operatorname{Ker}\lambda \cap \operatorname{Ker}\lambda'),$$

where $\lambda : X \otimes Y \to X$, $\lambda' : X \otimes Y \to Y$ are canonical homomorphisms.

Now compute $\operatorname{Ker}\lambda \cap \operatorname{Ker}\lambda'$ using that $X \otimes Y \approx \mathbb{Z} \times \mathbb{Z}$ and the basis of $X \otimes Y$ is $x \otimes y$, $x^2 \otimes y$.

One has

$$\begin{array}{ccc} \mathbb{Z}\times\mathbb{Z} & \xrightarrow{\overline{\lambda}} & \mathbb{Z} \\ \downarrow \approx & & \downarrow i\approx \\ X\otimes Y & \xrightarrow{\lambda} & X \end{array} , \qquad \begin{array}{ccc} \mathbb{Z}\times\mathbb{Z} & \xrightarrow{\overline{\lambda'}} & \mathbb{Z} \\ \downarrow \approx & & \downarrow j\approx \\ X\otimes Y & \xrightarrow{\lambda'} & Y \end{array} ,$$

where $i(x) = 1$, $j(y) = 1$ and $\overline{\lambda}$, $\overline{\lambda'}$ are defined as follows:

$$\overline{\lambda}(m,n) = i\lambda((x\otimes y)^m(x^2\otimes y)^n) = 2m + 4n,$$

$$\overline{\lambda'}(m,n) = j\overline{\lambda}((x\otimes y)^m(x^2\otimes y)^n) = -2m.$$

It is easy to see that $\operatorname{Ker}\overline{\lambda} \cap \operatorname{Ker}\overline{\lambda'} = 0$. $\square$

2. Algebraic K-theory with $\mathbb{Z}_q$ coefficients of local rings

This section is devoted to an application of the nonabelian tensor product modulo q to the algebraic K-theory with $\mathbb{Z}_q$ coefficients of Browder [1].

Recall the definition of algebraic K-functors with $\mathbb{Z}_q$ coefficients [1].

The algebraic K-functors of a discrete ring A has been defined by Quillen [20] by

$$K_i(A) = \pi_i(BGL(A)^+) , \quad i \geq 1 ,$$

where $GL(A) = \varinjlim_n GL(n,A)$ and $BGL(A)$ is its classifying space, $BGL(A)^+$ is the plus construction on $BGL(A)$ gotten by attaching 2-cells and 3-cells to kill the subgroup of elementary matrices $E(A) \subset GL(A) = \pi_1(BGL(A))$, in such a way that $H_*(BGL(A)) \approx H_*(BGL(A)^+)$. Then define,

$$K_i(A;\mathbb{Z}_q) = \pi_i(BGL(A)^+;\mathbb{Z}_q) = \text{the homotopy group with } \mathbb{Z}_q \text{ coefficients },$$

where $\pi_i(X) = [S^i, X]$ =homotopy classes of maps of the sphere S^i to X, and $\pi_i(X;\mathbb{Z}_q) = [Y^i, X]$, where $Y^i = S^{i-1} \cup_q e^i$, (attaching an i-cell by the map of degree q).

One can obtain the universal coefficient formula for algebraic K-functors with $\mathbb{Z}_q$ coefficients (see [1], [3]).

Proposition 2.1. *Let A be a discrete ring. Then for $q > 1$, there is a short exact sequence of groups*

$$0 \longrightarrow K_n(A) \otimes \mathbb{Z}_q \longrightarrow K_n(A, \mathbb{Z}_q) \longrightarrow \mathrm{Tor}(K_{n-1}(A), \mathbb{Z}_q) \longrightarrow 0 .$$

Let A be a ring with unit. Then $\mathrm{Sym}(A)$ is the group generated by the symbols $\{u, v\}$, where $u, v \in A^*$ (A^* is the group of units of the ring A), subject to the relations

(S1) $\{u, 1-u\} = 1$, $u \neq 1$, u, $1-u \in A^*$,
(S2) $\{uu', v\} = \{u, v\}\{u', v\}$,
(S3) $\{u, vv'\} = \{u, v\}\{u, v'\}$.

Note that $\mathrm{Sym}(A)$ is an abelian group. It is well known as Matsumoto's theorem that for any field A there exists the isomorphism $K_2(A) \approx \mathrm{Sym}(A)$ (see [21, Section 11, Theorem 11.1]).

Assume A is a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$. From [22, 12] it is known that there exists a group $D_0(A)$ generated by elements $\{u, v\}$, where $u, v \in A^*$, subject to the relations

(U0) $\{u, 1-u\} = 1$, $u \neq 1$, u, $1-u \in A^*$,
(U1) $\{uu', v\} = {}^u\{u', v\}\{u, v\}$,
(U2) $\{u, vw\}\{v, wu\}\{w, uv\} = 1$,

where ${}^u\{v, w\} = \{{}^uv, {}^uw\}$, such that there is a short exact sequence of crossed A^*-modules

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(A) & \longrightarrow & D_0(A) & \longrightarrow & [A^*, A^*] & \longrightarrow & 1 , \\ & & & & \{u, v\} & \mapsto & [u, v] . & & \end{array} \tag{2.1}$$

From this exact sequence D.Guin gets his six term exact nonabelian homology sequence [12, Theorem 4.2]

$$\begin{gathered} (A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A) \longrightarrow H_1(A^*, D_0(A)) \longrightarrow H_1(A^*, [A^*, A^*]) \longrightarrow K_2(A) \longrightarrow \\ \mathrm{Sym}(A) \longrightarrow [A^*, A^*]/[A^*, [A^*, A^*]] \longrightarrow 1 , \end{gathered}$$

where the first nonabelian homology of group A^* is defined as $H_1(A^*, D_0(A)) = \mathrm{Ker}(\lambda' : A^* \otimes D_0(A) \to D_0(A))$ and $H_1(A^*, [A^*, A^*]) = \mathrm{Ker}(\lambda' : A^* \otimes [A^*, A^*] \to [A^*, A^*])$ (for the definition of λ' see Proof of Theorem 1.11).

Let A be a local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$. Let us denote by $\operatorname{Sym}(A;\mathbb{Z}_q)$ the pushout

$$(push) \qquad \begin{array}{ccc} K_2(A)_q & \longrightarrow & K_2(A;\mathbb{Z}_q) \\ \downarrow & & \downarrow \\ \operatorname{Sym}(A)_q & \longrightarrow & \operatorname{Sym}(A;\mathbb{Z}_q) \end{array} .$$

Proposition 2.2. *Let A be a field and $q > 1$. Then there exists an isomorphism*

$$K_2(A;\mathbb{Z}_q) \approx \operatorname{Sym}(A;\mathbb{Z}_q) .$$

Proof. By definition of $\operatorname{Sym}(A;\mathbb{Z}_q)$, Proposition 2.1 and Matsumoto's Theorem one has the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} K_2(A) & \xrightarrow{q} & K_2(A) & \longrightarrow & K_2(A;\mathbb{Z}_q) & \longrightarrow & K_1(A) & \xrightarrow{q} & K_1(A) \\ \approx\downarrow & & \approx\downarrow & & \downarrow & & || & & || \\ \operatorname{Sym}(A) & \xrightarrow{q} & \operatorname{Sym}(A) & \longrightarrow & \operatorname{Sym}(A;\mathbb{Z}_q) & \longrightarrow & K_1(A) & \xrightarrow{q} & K_1(A) \end{array} .$$

Now the assertion follows from the five lemma. □

This is an analog of Matsumoto's Theorem in q-modular aspect.

Let A be a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$ and $q > 1$. By Proposition 1.6 the short exact sequence (2.1) of crossed A^*-modules induces the exact sequence of groups

$$\begin{gathered} \operatorname{Ker} \xi'_{K_2(A)} \longrightarrow \operatorname{Ker} \xi'_{D_0(A)} \longrightarrow \operatorname{Ker} \xi'_{[A^*,A^*]} \longrightarrow \operatorname{Coker} \xi'_{K_2(A)} \longrightarrow \\ \operatorname{Coker} \xi'_{D_0(A)} \longrightarrow \operatorname{Coker} \xi'_{[A^*,A^*]} \longrightarrow 0 , \end{gathered}$$

where $\xi'_{K_2(A)} : A^* \otimes^q K_2(A) \to K_2(A)$, $\xi'_{D_0(A)} : A^* \otimes^q D_0(A) \to D_0(A)$ and $\xi'_{[A^*,A^*]} : A^* \otimes^q [A^*,A^*] \to [A^*,A^*]$ are defined according to (1.11). It is easy to see that $\operatorname{Coker} \xi' = (\operatorname{Coker} \lambda')_q$, therefore calculations of D.Guin [12] imply that $\operatorname{Coker} \xi'_{K_2(A))} = (\operatorname{Coker} \lambda'_{K_2(A)})_q = K_2(A)_q$, $\operatorname{Coker} \xi'_{D_0(A)} = (\operatorname{Coker} \lambda'_{D_0(A)})_q = \operatorname{Sym}(A)_q$ and $\operatorname{Coker} \xi'_{[A^*,A^*]} = (\operatorname{Coker} \lambda'_{[A^*,A^*]})_q = ([A^*,A^*]/[A^*,[A^*,A^*]])_q$.

By Lemma 4.1 [12] A^* acts trivially on $K_2(A)$. Then using Proposition 1.6 [4], the same arguments as in Theorem 1.11 and the splitting $\psi : A^* \otimes^q K_2(A) \to (A^* \otimes K_2(A))/q(A^* \otimes K_2(A))$ defined by $\psi(a \otimes x) = [a \otimes x]$, $\psi(\{(1,y)\}) = 1$ for $a \in A$, $x, y \in K_2$, there is an exact sequence of abelian groups

$$1 \longrightarrow q(A^* \otimes K_2(A)) \longrightarrow A^* \otimes K_2(A) \longrightarrow A^* \otimes^q K_2(A) \longrightarrow K_2(A) \longrightarrow 1 .$$

Moreover $A^* \otimes^q K_2(A) \approx (A^* \otimes K_2(A))_q \oplus K_2(A)$. The triviality of actions induces $A^* \otimes K_2(A) \approx (A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A)$. Now it is easy to see that $\operatorname{Ker} \xi'_{K_2(A)} \approx (A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A))_q \oplus \operatorname{Tor}(K_2(A),\mathbb{Z}_q)$.

We can construct the connecting homomorphism $\operatorname{Ker} \xi'_{[A^*,A^*]} \to \operatorname{Coker} \xi'_{K_2(A)} = K_2(A)_q \to K_2(A;\mathbb{Z}_q)$. Then the following commutative diagram of groups with exact rows and columns

$$\begin{array}{cccccccc}
0 \longrightarrow & K_2(A)_q & \longrightarrow K_2(A;\mathbb{Z}_q) \longrightarrow & \mathrm{Tor}(K_1(A),\mathbb{Z}_q) & \longrightarrow 0 \\
 & \downarrow & \downarrow & || & \\
0 \longrightarrow & \mathrm{Sym}(A)_q & \longrightarrow \mathrm{Sym}(A;\mathbb{Z}_q) \longrightarrow & \mathrm{Tor}(K_1(A),\mathbb{Z}_q) & \longrightarrow 0 \\
 & \downarrow & \downarrow & & \\
 & ([A^*,A^*]/[A^*,[A^*,A^*]])_q & ([A^*,A^*]/[A^*,[A^*,A^*]])_q & & \\
 & \downarrow & \downarrow & & \\
 & 0 & 0 & &
\end{array}$$

induces

Theorem 2.3. *Let A be a (noncommutative) local ring such that $A/\operatorname{Rad} A \neq \mathbb{F}_2$ and $q > 1$. Then there is an exact sequence of groups*

$$(A^*)^{ab} \otimes_{\mathbb{Z}} K_2(A))_q \oplus \mathrm{Tor}(K_2(A),\mathbb{Z}_q) \longrightarrow \operatorname{Ker} \xi'_{D_0(A)} \longrightarrow \operatorname{Ker} \xi'_{[A^*,A^*]} \longrightarrow$$
$$K_2(A;\mathbb{Z}_q) \longrightarrow \mathrm{Sym}(A;\mathbb{Z}_q) \longrightarrow ([A^*,A^*]/[A^*,[A^*,A^*]])_q \longrightarrow 0\ .$$

3. q-Homology of groups

In this section we begin by defining q-homology of groups. Then we will establish some properties of q-homology groups and some calculations will be done. Relation of q-homology of groups with nonabelian tensor product modulo q of groups, particularly with its nonabelian left derived functors, will be studied in the next section.

Definition 3.1. *Let G be a group, A be a G-module and q be a nonnegative integer. Define n-th q-homology, $q - H_n(G,A)$, of the group G with coefficients in the G-module A by*

$$q - H_n(G,A) = \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}_q, A)\ ,$$

where $\mathbb{Z}_q$ is a trivial G-module.

Let G be a group and $\mathbb{Z}[G]$ be the group ring over G. Consider a projective resolution P_* of the trivial G-module $\mathbb{Z}$:

$$P_* \equiv \quad \cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Now from P_* we construct a new complex of G-modules using [23]:

$$q - P_* \equiv \quad \cdots \longrightarrow P_{n+1} \oplus P_n \xrightarrow{\widetilde{\partial_{n+1}}} P_n \oplus P_{n-1} \xrightarrow{\widetilde{\partial_n}} P_{n-1} \oplus P_{n-2} \xrightarrow{\widetilde{\partial_{n-1}}} \cdots \xrightarrow{\widetilde{\partial_2}}$$
$$\xrightarrow{\widetilde{\partial_2}} P_1 \oplus P_0 \xrightarrow{\widetilde{\partial_1}} P_0 \xrightarrow{\widetilde{\epsilon}} \mathbb{Z}_q \longrightarrow 0,$$

where $\widetilde{\partial_n}(p_n, p_{n-1}) = (\partial_n(p_n) + qp_{n-1}, -\partial_{n-1}(p_{n-1}))$, $n \geq 2$, $\widetilde{\partial_1}(p_1, p_0) = \partial_1(p_1) + qp_0$ and $\widetilde{\epsilon}(p_0) = [\epsilon(p_0)]$.

Proposition 3.2. *For any group G and $q > 0$ the complex $q - P_*$ with augmentation $\widetilde{\epsilon}$ is a projective $\mathbb{Z}[G]$-resolution of the trivial G-module $\mathbb{Z}_q$.*

Proof. It is clear that $q - P_n = P_n \oplus P_{n-1}$, $n \geq 0$, is a projective module over $\mathbb{Z}[G]$. Exactness in $q - P_n$, $n \geq 2$ is easy to see [23]. We only have to show exactness of the sequence

$$P_2 \oplus P_1 \xrightarrow{\widetilde{\partial_2}} P_1 \oplus P_0 \xrightarrow{\widetilde{\partial_1}} P_0 \xrightarrow{\widetilde{\epsilon}} \mathbb{Z}_q \longrightarrow 0.$$

Obviously $\widetilde{\epsilon}$ is epimorphic.

Let $p_0 \in P_0$ such that $\widetilde{\epsilon}(p_0) = 0$, i.e. there exists $n \in \mathbb{Z}$, $\epsilon(p_0) = qn$. There exists p_0' such that $\epsilon(p_0') = n$, then $\epsilon(p_0 - qp_0') = 0$, i.e. there exists $p_1 \in P_1$ such that $\partial_1(p_1) = p_0 - qp_0'$. Hence $\widetilde{\partial_1}(p_1, p_0') = \partial_1(p_1) + qp_0' = p_0$.

Let $(p_1, p_0) \in P_1 \oplus P_0$ such that $\widetilde{\partial_1}(p_1, p_0) = 0$ i.e. $\partial_1(p_1) + qp_0 = 0$, $\epsilon\partial_1(p_1) = -q\epsilon(p_0) = 0$. Hence, $\epsilon(p_0) = 0$ and therefore there exists $p_1' \in P_1$ such that $\partial_1(p_1') = p_0$. We have $\partial_1(p_1 + qp_1') = \partial_1(p_1) + q\partial_1(p_1') = -qp_0 + qp_0 = 0$. There exists $p_2 \in P_2$ such that $\partial_2(p_2) = p_1 + qp_1'$. Thus $\widetilde{\partial_2}(p_2, -p_1') = (\partial_2(p_2) - qp_1', \partial_1(p_1')) = (p_1 + qp_1' - qp_1', p_0) = (p_1, p_0)$. $\square$

Note that the using of the Proposition 3.2 will be shown bellow, particularly in computing of q-homology groups of finite cyclic groups (see Proposition 3.10).

As a projective $\mathbb{Z}[G]$-resolution P_* of $\mathbb{Z}$ we can take the standard B-resolution of $\mathbb{Z}$ (see [23])

$$B_* \equiv \quad \cdots \longrightarrow B_{n+1} \xrightarrow{\partial} B_n \xrightarrow{\partial} B_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} B_1 \xrightarrow{\partial} B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where B_n is a free G-module generated by the set of all sequences $[x_1|\cdots|x_n]$, $n \geq 1$, $x_i \in G$ i.e. by $G \times \cdots \times G$ and B_0 is a free G-module generated by only one symbol $[\ ]$. Chain boundary is defined by the formula

$$\partial[x_1|\cdots|x_n] = x_1[x_2|\cdots|x_n] + \sum_{i=1}^{n-1}(-1)^i[x_1|\cdots|x_ix_{i+1}|\cdots|x_n] + (-1)^n[x_1|\cdots|x_{n-1}]\ ,$$

and $\epsilon[\ ] = 1$.

The construction above gives us the following free $\mathbb{Z}[G]$-resolution of the trivial G-module $\mathbb{Z}_q$, $q > 0$:

$$\begin{aligned} q - B_* \equiv \quad \cdots \longrightarrow B_{n+1} \oplus B_n \xrightarrow{\widetilde{\partial}} B_n \oplus B_{n-1} \xrightarrow{\widetilde{\partial}} B_{n-1} \oplus B_{n-2} \xrightarrow{\widetilde{\partial}} \cdots \xrightarrow{\widetilde{\partial}} \\ \xrightarrow{\widetilde{\partial}} B_1 \oplus B_0 \xrightarrow{\widetilde{\partial}} B_0 \xrightarrow{\widetilde{\epsilon}} \mathbb{Z}_q \longrightarrow 0, \end{aligned}$$

where $\widetilde{\partial}$ is defined as above, which will be called $q - B$-resolution of $\mathbb{Z}_q$.

Definition 3.3. *Let R be a ring with unit, q be a positive integer and*

$$M_* \equiv \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{3.1}$$

be a chain complex of R-modules.

The q-homology modules of the chain complex M_* is the homology of the following chain complex of R-modules

$$q - M_* \equiv \quad \cdots \longrightarrow M_{n+1} \oplus M_n \xrightarrow{\widetilde{d_{n+1}}} M_n \oplus M_{n-1} \xrightarrow{\widetilde{d_n}} M_{n-1} \oplus M_{n-2} \longrightarrow \cdots$$

where $\widetilde{d_n}(m_n, m_{n-1}) = (d_n(m_n) + qm_{n-1}, -d_{m-1}(m_{n-1}))$ for all $m_n \in M_n$, $m_{n-1} \in M_{n-1}$.

The following corollary is an easy consequence of the Definitions 3.1, 3.3 and Proposition 3.2.

Corollary 3.4. *Let G be a group and A be a G-module. Then for $q > 0$, n-th q-homology of the group G with coefficients in the G-module A, $q - H_n(G, A)$, is the n-th homology group of the following chain complex of abelian groups:*

$$\begin{aligned}\cdots \longrightarrow (P_{n+1} \oplus P_n) \otimes_G A \xrightarrow{\widetilde{\partial_{n+1} \otimes 1_A}} (P_n \oplus P_{n-1}) \otimes_G A \xrightarrow{\widetilde{\partial_n} \otimes 1_A} (P_{n-1} \oplus P_{n-2}) \otimes_G A \\ \xrightarrow{\widetilde{\partial_{n-1}} \otimes 1_A} \cdots \xrightarrow{\widetilde{\partial_2} \otimes 1_A} (P_1 \oplus P_0) \otimes_G A \xrightarrow{\widetilde{\partial_1} \otimes 1_A} P_0 \otimes_G A \ ,\end{aligned} \tag{3.2}$$

or the n-th q-homology group of the following chain complex of abelian groups:

$$\begin{aligned}\cdots \longrightarrow P_{n+1} \otimes_G A \xrightarrow{\partial_{n+1} \otimes 1_A} P_n \otimes_G A \xrightarrow{\partial_n \otimes 1_A} P_{n-1} \otimes_G A \longrightarrow \cdots \\ \cdots \xrightarrow{\partial_2 \otimes 1_A} P_1 \otimes_G A \xrightarrow{\partial_1 \otimes 1_A} P_0 \otimes_G A \longrightarrow 0 \longrightarrow \cdots \ .\end{aligned} \tag{3.3}$$

It is easy to see that $q - H_0(G, A) \approx H_0(G, A)_q$ and $q - H_n(G, \mathbb{Z}) \approx H_n(G, \mathbb{Z}_q)$, $n \geq 0$, where $\mathbb{Z}$ and $\mathbb{Z}_q$ are trivial G-modules.

Proposition 3.5 (Universal Coefficient Formulas)**.** *Let G be any group and A be a G-module. Then*

(a) *for $q > 0$ there is a short exact sequence of groups:*

$$0 \to H_n(G, A) \otimes \mathbb{Z}_q \to q - H_n(G, A) \to \mathrm{Tor}(H_{n-1}(G, A), \mathbb{Z}_q) \to 0 \ , \tag{3.4}$$

(b) *for $q \geq 0$ and trivial G-module A there is a short exact sequence (splits non-naturally) of groups:*

$$0 \to H_n(G, \mathbb{Z}_q) \otimes A \to q - H_n(G, A) \to \mathrm{Tor}(H_{n-1}(G, \mathbb{Z}_q), A) \to 0 \ . \tag{3.5}$$

Proof. (a) By Corollary 3.4 q-homology groups are the q-homology of the chain complex (3.3), i.e. the homology groups of the cone of the following morphism of complexes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_n \otimes_G A & \longrightarrow \cdots \longrightarrow & P_1 \otimes_G A & \longrightarrow & P_0 \otimes_G A & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow q & & \downarrow q & & \downarrow q & & & \\ \cdots \longrightarrow & P_n \otimes_G A & \longrightarrow \cdots \longrightarrow & P_1 \otimes_G A & \longrightarrow & P_0 \otimes_G A & \longrightarrow & 0 & \longrightarrow \cdots \end{array} .$$

By [23] one has the following long exact homology sequence of groups

$$\begin{aligned}\cdots \longrightarrow H_n(G, A) \xrightarrow{q} H_n(G, A) \longrightarrow q - H_n(G, A) \longrightarrow \cdots \longrightarrow \\ \longrightarrow H_1(G, A) \xrightarrow{q} H_1(G, A) \longrightarrow q - H_1(G, A) \longrightarrow \\ \longrightarrow H_0(G, A) \xrightarrow{q} H_0(G, A) \longrightarrow q - H_0(G, A) \longrightarrow 0 \ .\end{aligned} \tag{3.6}$$

Note that one can obtain this long exact homology sequence directly by considering the long exact sequence of derived functors of the functor $- \otimes_G A$ associated to the following short exact sequence of the trivial G-modules $0 \to q\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_q \to 0$.

It is known that if $f : A \to A$, $f(a) = qa$, for any $a \in A$, then $\mathrm{Coker}\, f = A \otimes \mathbb{Z}_q$ and $\mathrm{Ker}\, f = \mathrm{Tor}(A, \mathbb{Z}_q)$ for $q > 0$. These facts and the long exact homology sequence (3.6) prove the assertion.

(b) Can be proved classically. □

Proposition 3.6. *Let G be a finite group of order k ($|G| = k$) and $(k, q) = 1$, then*

$$q - H_n(G, A) = 0, \quad n \geq 2 .$$

Proof. It is well known (See [23]) that $H_n(G, A)$, $n > 0$ is a group of exponent k. Then $H_n(G, A) \otimes \mathbb{Z}_q$, $n > 0$ is a group of exponent q and k and by assumption $H_n(G, A) \otimes \mathbb{Z}_q = 0$, $n > 0$. By the same reason $\mathrm{Tor}(H_n(G, A), \mathbb{Z}_q) = 0$, $n > 0$. It remains to apply Proposition 3.5 (a). □

Proposition 3.7. *Let G be a finite group and A be a divisible, torsion free G-module. Then for $q > 0$*

$$\begin{aligned} q - H_n(G, A) &= 0, \quad n \geq 2, \\ q - H_1(G, A) &= \mathrm{Tor}(H_0(G, A), \mathbb{Z}_q), \\ q - H_0(G, A) &= 0. \end{aligned}$$

Proof. By Proposition 3.5 (a) and by [23] $q - H_n(G, A) = 0$, $n \geq 2$ and $q - H_1(G, A) = \mathrm{Tor}(H_0(G, A), \mathbb{Z}_q)$. As we know $q - H_0(G, A) \approx H_0(G, A)_q$. But $H_0(G, A) = A/I_G A$ and since A is divisible then $H_0(G, A)$ is also a divisible group. Therefore $q - H_0(G, A) = 0$. □

Proposition 3.8. *Let F be a free group and A be a F-module,then $q - H_n(F, A) = 0$, $q > 0$, $n \geq 3$.*

Proof. Follows from Proposition 3.5 (a) and $H_n(F, A) = 0$, $n \geq 2$. □

Example 3.9. *Let $\mathbb{Z}$ be the additive group of integers, $\mathbb{Q}/\mathbb{Z}$ be the quotient of the additive group of rational numbers by $\mathbb{Z}$ and $q > 0$. Suppose that $\mathbb{Z}$ acts trivially on $\mathbb{Q}/\mathbb{Z}$. By Proposition 3.8 one gets $q - H_n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = 0$, $n \geq 3$. Using Proposition 3.5 (a) one has the following short exact sequences of groups*

$$0 \longrightarrow H_2(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_q \longrightarrow q - H_2(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tor}(H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \mathbb{Z}_q) \longrightarrow 0 ,$$
$$0 \longrightarrow H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_q \longrightarrow q - H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tor}(H_0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \mathbb{Z}_q) \longrightarrow 0 .$$

It is easy to see that $H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = H_0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Since $H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_q = H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_q = (\mathbb{Q}/\mathbb{Z})_q = 0$ (because $\mathbb{Q}$ is divisible) and $H_2(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = 0$, then $q - H_2(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = q - H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_q)$.

$\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_q)$ *is the kernel of the homomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{q} \mathbb{Q}/\mathbb{Z}$ and by [24, Chapter VII, Proposition 2.2] one has* $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}_q) = \mathbb{Z}_q$. *Therefore $q - H_2(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = q - H_1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_q$. Finally $q - H_0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = H_0(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_q = (\mathbb{Q}/\mathbb{Z})_q = 0$.*

It is a good example to show that $q - H_*(G, A)$ is not isomorphic to $H_*(G, A)_q$ or $H_*(G, A_q)$. In this case $H_n(\mathbb{Z}, (\mathbb{Q}/\mathbb{Z})_q) = 0$, $n \geq 0$ (since $(\mathbb{Q}/\mathbb{Z})_q = 0$) and $H_n(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})_q = 0$, $n \geq 0$.

Now we compute q-homology of finite cyclic groups for $q > 0$. Let $G = C_m(t)$ be multiplicative cyclic group of order m and generated by t. It is well known that the elements

$$N = 1 + t + \cdots + t^{m-1}, \quad D = t - 1$$

induce $\mathbb{Z}[G]$-module homomorphisms

$$\begin{aligned} N_* &: \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G], & N_*u &= Nu\ , \\ D_* &: \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G], & D_*u &= Du\ , \qquad u \in \mathbb{Z}[G]\ , \end{aligned}$$

respectively and the following long exact sequence of G-modules:

$$W \equiv \cdots \xrightarrow{N_*} \mathbb{Z}[G] \xrightarrow{D_*} \mathbb{Z}[G] \xrightarrow{N_*} \mathbb{Z}[G] \xrightarrow{D_*} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is a free $\mathbb{Z}[G]$-resolution of the trivial G-module $\mathbb{Z}$, which gives us the possibility to compute Eilenberg-Maclane homology $H_*(G, A)$ of the finite cyclic group $G = C_m(t)$ with coefficients in any G-module A (see [23]).

Using technique above by Proposition 3.2 one gets the following free $\mathbb{Z}[G]$-resolution of the trivial G-module $\mathbb{Z}_q$:

$$\begin{aligned} q - W \equiv \cdots \xrightarrow{\widetilde{N_*}} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{D_*}} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{N_*}} \\ \xrightarrow{\widetilde{N_*}} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\widetilde{\widetilde{D_*}}} \mathbb{Z}[G] \xrightarrow{\tilde{\epsilon}} \mathbb{Z}_q \longrightarrow 0, \end{aligned}$$

where $\widetilde{\widetilde{D_*}}(u, u') = Du + qu'$, $\widetilde{N_*}(u, u') = (Nu + qu', -Du')$, $\widetilde{D_*}(u, u') = (Du + qu', -Nu')$ for all $u, u' \in \mathbb{Z}[G]$. By definition $q - H_*(G, A) \equiv \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}_q, A)$, consequently q-homology groups of the finite cyclic group $G = C_m(t)$ with coefficients in G-module A are the homology groups of the following chain complex of G-modules

$$\cdots \xrightarrow{\widetilde{N^*}} A \oplus A \xrightarrow{\widetilde{D^*}} A \oplus A \xrightarrow{\widetilde{N^*}} A \oplus A \xrightarrow{\widetilde{\widetilde{D^*}}} A,$$

where $\widetilde{\widetilde{D^*}}(a, a') = Da + qa'$, $\widetilde{N^*}(a, a') = (Na + qa', -Da')$, $\widetilde{D^*}(a, a') = (Da + qa', -Na')$ for all $a, a' \in A$.

It is easy to see that q-homology, $q > 0$, of finite cyclic groups is periodic from $n \geq 2$ with period 2. One gets the following

Proposition 3.10. *For finite cyclic group $C_m(t)$ of order m and with generator t and for any $C_m(t)$-module A*

$$\begin{aligned} & q - H_0(C_m, A) = A/\widetilde{\widetilde{D^*}} = H_0(C_m, A)_q = \operatorname{Coker} D^*\ , \quad \textit{where}\ \ D^* : A_q \to A_q\ , \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad D^*[a] = [Da]\ , \\ & q - H_1(C_m, A) = [(a, a') | Da + qa' = 0] / \widetilde{N^*}(A \oplus A)\ , \\ & q - H_{2n}(C_m, A) = [(a, a') | Na + qa' = 0, ta' = a'] / \widetilde{D^*}(A \oplus A)\ , \quad n \geq 1\ , \\ & q - H_{2n+1}(C_m, A) = [(a, a') | Da + qa' = 0, Na = 0] / \widetilde{N^*}(A \oplus A)\ , \quad n \geq 1\ . \end{aligned}$$

Proposition 3.11. *Let G be a group and $0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0$ be a short exact sequence of G-modules. Then there is a long exact sequence of q-homology groups*

$$\begin{aligned} \cdots \longrightarrow q - H_n(G, A_1) \longrightarrow q - H_n(G, A) \longrightarrow q - H_n(G, A_2) \longrightarrow \cdots \longrightarrow \\ \longrightarrow q - H_0(G, A_1) \longrightarrow q - H_0(G, A) \longrightarrow q - H_0(G, A_2) \longrightarrow 0. \end{aligned}$$

Proof. Straightforward from the definition of q-homology of groups. □

Let us consider the augmentation ideal of $\widetilde{\epsilon}: \mathbb{Z}[G] \to \mathbb{Z}_q$, $\widetilde{\epsilon}(\sum_i n_i g_i) = [\sum_i n_i]$ denoted by $q - IG = \operatorname{Ker}\widetilde{\epsilon}$. It is easy to see that every element $n_1 g_1 + \cdots + n_k g_k$ of $\mathbb{Z}[G]$ belongs to $q - IG$ if and only if $q|(n_1 + \cdots + n_k)$ for $q > 0$. Assume that if $q = 0$ then $q - IG = IG$, where IG is the augmentation ideal of the well known homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$, $\epsilon(\sum_i n_i g_i) = \sum_i n_i$.

Proposition 3.12. *There is a short exact sequence of G-modules:*

$$0 \longrightarrow IG \xrightarrow{\alpha} q - IG \xrightarrow{\varphi} \mathbb{Z} \longrightarrow 0, \quad q > 0,$$

where α is the natural inclusion and $\mathbb{Z}$ is a trivial G-module.

Proof. Let define the homomorphism φ by the following formula:

$$\varphi(n_1 g_1 + \cdots + n_k g_k) = \frac{n_1 + \cdots + n_k}{q},$$

since q divides $n_1 + \cdots + n_k$.

It is easy to see that $\varphi\alpha = 0$ and if $\varphi(n_1 g_1 + \cdots + n_k g_k) = 0$ for any $n_1 g_1 + \cdots + n_k g_k \in q - IG$, then $n_1 + \cdots + n_k = 0$ and thus $n_1 g_1 + \cdots + n_k g_k \in IG$. □

Suppose G is a group and A is a G-module. Let us define a homomorphism $\Phi : G \otimes^q A \approx G \otimes A / qH_1(G, A) \to q - IG \otimes_G A$, $q \geq 0$, as follows: $\Phi[g \otimes a] = (g - e) \otimes a$. We have to show correcteness for $q > 0$ (for $q = 0$ see Remark 3.14). In effect, let $g_1 \otimes a_1 \cdots g_n \otimes a_n \in H_1(G, A) \subseteq G \otimes A$ i.e. $g_1 a_1 - a_1 + \cdots + g_n a_n - a_n = 0$, then $(g_1 \otimes a_1 \cdots g_n \otimes a_n)^q \mapsto q(g_1 - e) \otimes a_1 + \cdots + q(g_n - e) \otimes a_n = qe \otimes (g_1 a_1 - a_1 + \cdots + g_n a_n - a_n) = qe \otimes 0 = 0$.

Proposition 3.13. *Let G be a group and A be a G-module. There is a short exact sequence of groups:*

$$0 \longrightarrow G \otimes^q A \xrightarrow{\Phi} q - IG \otimes_G A \xrightarrow{\varphi \otimes 1_A} q\mathbb{Z} \otimes_G A \longrightarrow 0, \quad q \geq 0,$$

where φ is defined in Proposition 3.12.

Proof. We have to show Proposition 3.13 when $q > 0$. By Proposition 3.12 and if we replace $\mathbb{Z}$ by its isomorphic G-module $q\mathbb{Z}$ one gets the commutative diagram of groups with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IG & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{\epsilon} & q\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow q & & \downarrow q & & || & & \\
0 & \longrightarrow & IG & \xrightarrow{\alpha} & q - IG & \xrightarrow{\varphi} & q\mathbb{Z} & \longrightarrow & 0
\end{array} .$$

This diagram induces the following commutative diagram of left derived functors of the functor $- \otimes_G A$:

$$\begin{array}{cccccc}
\cdots \to 0 \to & H_1(G, A) & \to IG \otimes_G A \to & \mathbb{Z}[G] \otimes_G A & \to \quad q\mathbb{Z} \otimes_G A & \to 0 \\
 & || & \downarrow q & \downarrow q & || & , \\
\cdots \to \bullet \to & H_1(G, A) & \to IG \otimes_G A \to & q - IG \otimes_G A & \to \quad q\mathbb{Z} \otimes_G A & \to 0
\end{array} \quad (3.7)$$

where the rows are long exact sequences of groups. By [12] and Theorem 1.11 from (3.7) follows the assertion. □.

Remark 3.14. *We can consider this theorem as a generalization of D.Guin's isomorphism*

$$G \otimes A \approx IG \otimes_G A, \quad g \otimes a \mapsto (g-e) \otimes a,$$

when $q = 0$ (see [12, Proposition 3.2]).

4. Derived functors of the nonabelian tensor product modulo q of groups

Let A denote a fixed abelian group and $\mathcal{A}_A$ denote the category whose objects are all groups G together with an action of G on A and a trivial action of A on G. Morphisms in the category $\mathcal{A}_A$ are all group homomorphisms $\alpha : G \to H$ which preserve the actions, namely ${}^g a = {}^{\alpha(g)} a$, for all $a \in A$ and $g \in G$.

Let $F : \mathcal{A}_A \to \mathcal{A}_A$ be the endofunctor defined as follows: for an object G of $\mathcal{A}_A$, let $F(G)$ denote the free group generated by G with action: ${}^{|g_1|^{\epsilon_1} \cdots |g_s|^{\epsilon_s}} a = {}^{g_1^{\epsilon_1}}(\dots({}^{g_s^{\epsilon_s}} a)\dots)$, where $a \in A$, $|g_1|^{\epsilon_1} \cdots |g_s|^{\epsilon_s} \in F(G)$ and $\epsilon_i = \pm 1$; for a morphism $\alpha : G \to G'$ of $\mathcal{A}_A$, let $F(\alpha)$ be the canonical homomorphism from $F(G)$ to $F(G')$ induced by α.

Let $\tau : F \to 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $G \in \operatorname{ob} \mathcal{A}_A$ by the injection $G \to F(G)$. We obtain a cotriple $\mathcal{F} = (F, \tau, \delta)$ and we denote by $\mathcal{P}$ the projective class induced by this cotriple $\mathcal{F}$.

Since the actions of the groups G and A on each other satisfy the compatibility conditions (1.12) for any object G of the category $\mathcal{A}_A$, the nonabelian 'absolute' tensor product modulo q of groups defines a covariant functor $- \otimes^q A$ from the category $\mathcal{A}_A$ to the category of abelian groups. Consider the nonabelian left derived functors $L_n^{\mathcal{P}}(- \otimes^q A)$, $n \geq 0$, of the functor $- \otimes^q A$ relative to the projective class $\mathcal{P}$ (see [25-27], [13] and [14]). It is known [28], [27] that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple.

Proposition 4.1. *Suppose G is a group and the groups A and A_q, $q \geq 0$ are trivial G-modules. Then there are isomorphisms*

$$H_n(G, A_q) \approx L_{n-1}^{\mathcal{P}}(G \otimes^q A), \qquad n \geq 2\,,$$
$$H_1(G, A_q) \approx \operatorname{Ker} \xi'\,, \qquad H_0(G, A_q) \approx (\operatorname{Coker} \xi')_q\,,$$

where $\xi' : G \otimes^q A \to A$, $\xi'(g \otimes a) = {}^g a \cdot a^{-1}$, $\xi'(\{k\}) = \pi_2 k^q$.

Proof. For $q = 0$ the assertion follows from [13, Theorem 6], which confirms that for any G-module A there are isomorphisms

$$H_n(G, A) \approx L_{n-1}^{\mathcal{P}}(G \otimes A), \qquad n \geq 2\,,$$
$$H_1(G, A) \approx \operatorname{Ker} \lambda'\,, \qquad H_0(G, A) \approx \operatorname{Coker} \lambda'\,,$$

where $\lambda' : G \otimes A \to A$, $\lambda'(g \otimes a) = {}^g a \cdot a^{-1}$.

Suppose that $q > 0$. Obviously that $H_0(G, A_q) \approx A_q$ and $H_1(G, A_q) \approx G \otimes A_q \approx G^{ab} \otimes_{\mathbb{Z}} A_q \approx (G^{ab} \otimes_{\mathbb{Z}} A)_q$, $H_n(G, A_q) \approx \pi_{n-1} C_*$, $n \geq 2$, where

$$\begin{array}{cccccc} C_* \equiv \ \cdots \underset{\to}{\overset{\to}{\vdots}} & F^n(G)\otimes A_q & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & F^2(G)\otimes A_q & \rightrightarrows & F^1(G)\otimes A_q \\ & \downarrow\approx & & \downarrow\approx & & \downarrow\approx \\ \cdots \underset{\to}{\overset{\to}{\vdots}} & (F^n(G)^{ab}\otimes_{\mathbb{Z}} A)_q & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & (F^2(G)^{ab}\otimes_{\mathbb{Z}} A)_q & \rightrightarrows & (F^1(G)^{ab}\otimes_{\mathbb{Z}} A)_q \end{array},$$

and $\cdots \underset{\to}{\overset{\to}{\vdots}} F^n(G) \underset{\to}{\overset{\to}{\vdots}} \cdots \underset{\to}{\overset{\to}{\to}} F^2(G) \rightrightarrows F^1(G) \to G$ is the cotriple resolution of G.

By Theorem 1.11 $(\operatorname{Coker} \xi')_q = (\operatorname{Coker} \lambda')_q = A_q$, $\operatorname{Ker} \xi' = \operatorname{Ker} \lambda' / q \operatorname{Ker} \lambda' = (G^{ab} \otimes_{\mathbb{Z}} A)_q$, and $L^{\mathcal{P}}_{n-1}(G \otimes^q A) = \pi_{n-1} C'_*$, $n \geq 2$, where

$$\begin{array}{cccccc} C'_* \equiv \ \cdots \underset{\to}{\overset{\to}{\vdots}} & F^n(G)\otimes^q A & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & F^2(G)\otimes^q A & \rightrightarrows & F^1(G)\otimes^q A \\ & \downarrow\approx & & \downarrow\approx & & \downarrow\approx \\ \cdots \underset{\to}{\overset{\to}{\vdots}} & (F^n(G)^{ab}\otimes_{\mathbb{Z}} A)_q & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & (F^2(G)^{ab}\otimes_{\mathbb{Z}} A)_q & \rightrightarrows & (F^1(G)^{ab}\otimes_{\mathbb{Z}} A)_q \end{array}.$$

□

Now the relation between classical homology of groups and nonabelian left derived functors of the tensor product modulo q will be established.

Proposition 4.2. *Let G be a group and A be a G-module. Then there is a long exact sequence of groups*

$$\begin{array}{c} \cdots \to L^{\mathcal{P}}_{n-1}(qH_1(G,A)) \to H_n(G,A) \to L^{\mathcal{P}}_{n-1}(G \otimes^q A) \to \cdots \to \\ L^{\mathcal{P}}_2(G \otimes^q A) \to L^{\mathcal{P}}_1(qH_1(G,A)) \to H_2(G,A) \to L^{\mathcal{P}}_1(G \otimes^q A) \to \\ L^{\mathcal{P}}_0(qH_1(G,A)) \to H_1(G,A) \to \operatorname{Ker} \xi' \to 0 \ . \end{array} \tag{4.1}$$

Proof. Let consider the cotriple resolution of G in the category $\mathcal{A}_A$

$$\cdots \underset{\to}{\overset{\to}{\vdots}} F^n(G) \underset{\to}{\overset{\to}{\vdots}} \cdots \underset{\to}{\overset{\to}{\to}} F^2(G) \rightrightarrows F^1(G) \to G \ .$$

By Theorem 1.11 one has a commutative diagram of groups

$$\begin{array}{cccccc} & 0 & & 0 & & 0 \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \underset{\to}{\overset{\to}{\vdots}} & qH_1(F^n(G),A) & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & qH_1(F^2(G),A) & \rightrightarrows & qH_1(F^1(G),A) \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \underset{\to}{\overset{\to}{\vdots}} & F^n(G)\otimes A & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & F^2(G)\otimes A & \rightrightarrows & F^1(G)\otimes A \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \underset{\to}{\overset{\to}{\vdots}} & F^n(G)\otimes^q A & \underset{\to}{\overset{\to}{\vdots}}\cdots\underset{\to}{\overset{\to}{\to}} & F^2(G)\otimes^q A & \rightrightarrows & F^1(G)\otimes^q A \\ & \downarrow & & \downarrow & & \downarrow \\ & 0 & & 0 & & 0 \end{array},$$

where the columns are short exact sequences of groups.

From this diagram we have exactness till $L_0^{\mathcal{P}}(qH_1(G,A))$. Exactness in $H_1(G,A)$ and in $\operatorname{Ker}\xi'$ follows from Theorem 1.11 and from the fact that $H_1(F^1(G),A) \to H_1(G,A)$ and hence $qH_1(F^1(G),A) \to qH_1(G,A)$ are epimorphisms and therefore $L_0^{\mathcal{P}}(qH_1(G,A)) \to qH_1(G,A)$ is also an epimorphism. □

We establish sufficient conditions for the isomorphism between nonabelian left derived functors of the tensor product modulo q of groups and q-homology of groups.

Lemma 4.3. *Let G be a group and A be a q-torsion free G-module. There are natural isomorphisms*

$$H_n(G,A_q) \approx q - H_n(G,A), \quad n \geq 0 \ .$$

Proof. Consider two sequences of functors

1) $H_0(G,-_q)$, $H_1(G,-_q)$,..., $H_n(G,-_q)$,... ;
2) $q - H_0(G,-)$, $q - H_1(G,-)$,..., $q - H_n(G,-)$,... .

These both sequences satisfy the following axioms for a connected sequence of additive functors $\{T_n,\ \delta_n,\ n \geq 0\}$ from the category of q-torsion free G-modules to the category of abelian groups:

(i) $T_0(-) = H_0(G,-_q) = q - H_0(G,-)$;
(ii) for any short exact sequence of q-torsion free G-modules

$$0 \to A_1 \to A \to A_2 \to 0$$

there exists a long exact sequence of abelian groups

$$\begin{aligned} \cdots \longrightarrow T_{n+1}(A_2) \xrightarrow{\delta n+1} T_n(A_1) \longrightarrow T_n(A) \longrightarrow T_n(A_2) \xrightarrow{\delta_n} \cdots \longrightarrow \\ \longrightarrow T_1(A_2) \xrightarrow{\delta_1} T_0(A_1) \longrightarrow T_0(A) \longrightarrow T_0(A_2) \longrightarrow 0; \end{aligned}$$

(iii) if A is an induced q-torsion free G-module (easy to see that if M is q-torsion free G-module then $\mathbb{Z}[G] \otimes_{\mathbb{Z}} M$ is also a q-torsion free G-module), then $T_n(A) = 0$, $n \geq 1$.

This proves the existence of the required natural isomorphisms. □

Theorem 4.4. *Let G be a group and A be a q-torsion free G-module, $q > 0$. There are natural isomorphisms*

$$L_{n-1}^{\mathcal{P}}(G \otimes^q A) \approx q - H_n(G,A), \quad n \geq 2 \ .$$

Proof. Let us consider the nonabelian left derived functors, $L_n^{\mathcal{P}}((- \otimes A)_q)$, $n \geq 0$, of the functor $(- \otimes A)_q$. By [12] one has the isomorphisms

$$G \otimes A_q \approx IG \otimes_G A_q \approx (IG \otimes_G A)_q \approx (G \otimes A)_q \ .$$

Therefore by [13, Theorem 6]

$$L_{n-1}^{\mathcal{P}}((G \otimes A)_q) \approx H_n(G,A_q), \quad n \geq 2. \tag{4.2}$$

If F is a free group, then the well known fact that IF is a free $\mathbb{Z}F$-module, i.e. $IF \approx \sum \mathbb{Z}F$, implies the following isomorphism

$$F \otimes A \approx IF \otimes_F A \approx \sum A \ .$$

Suppose A is a q-torsion free F-module, then according to this isomorphism $H_1(F,A) \subseteq F \otimes A$ is also q-torsion free. Thus $H_1(F,A) \overset{q}{\approx} qH_1(F,A)$. From this fact and [14, Theorem 7] we can conclude that $L_n^{\mathcal{P}}(qH_1(G,A)) \approx L_n^{\mathcal{P}}(H_1(G,A)) \approx H_{n+1}(G,A)$, $n \geq 0$. Therefore by Proposition 4.2 it is easy to get the following long exact sequence of groups:

$$\begin{gathered} \cdots \to H_n(G,A) \xrightarrow{q} H_n(G,A) \to L_{n-1}^{\mathcal{P}}(G \otimes^q A) \to \cdots \to \\ L_2^{\mathcal{P}}(G \otimes^q A) \to H_2(G,A) \xrightarrow{q} H_2(G,A) \to L_1^{\mathcal{P}}(G \otimes^q A) \to \\ H_1(G,A) \xrightarrow{q} H_1(G,A) \to (H_1(G,A))_q \to 0 \,. \end{gathered} \tag{4.3}$$

(4.3) implies the short exact sequences of groups

$$0 \longrightarrow H_n(G,A) \otimes \mathbb{Z}_q \longrightarrow L_{n-1}^{\mathcal{P}}(G \otimes^q A) \longrightarrow \operatorname{Tor}(H_{n-1}(G,A), \mathbb{Z}_q) \longrightarrow 0, \quad n \geq 2 \,.$$

To prove the assertion we only have to construct homomorphisms $\kappa : L_{n-1}^{\mathcal{P}}(G \otimes^q A) \to q - H_n(G,A)$, $n \geq 2$, in our case (when A is q-torsion free) such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \to & H_n(G,A) \otimes \mathbb{Z}_q & \to & L_{n-1}^{\mathcal{P}}(G \otimes^q A) & \to & \operatorname{Tor}(H_{n-1}(G,A), \mathbb{Z}_q) & \to & 0 \\ & & \| & & \downarrow \kappa & & \| & & , \\ 0 & \to & H_n(G,A) \otimes \mathbb{Z}_q & \to & q - H_n(G,A) & \to & \operatorname{Tor}(H_{n-1}(G,A), \mathbb{Z}_q) & \to & 0 \end{array} \tag{4.4}$$

where the bottom row is by Proposition 3.5 (a) in case when A is a q-torsion free G-module.

By Theorem 1.11 there is a natural homomorphism

$$G \otimes^q A \approx (G \otimes A)/qH_1(G,A) \longrightarrow (G \otimes A)_q = (G \otimes A)/q(G \otimes A)$$

and by (4.2) and Lemma 4.3 it induces homomorphisms $\kappa : L_{n-1}^{\mathcal{P}}(G \otimes^q A) \to L_{n-1}^{\mathcal{P}}((G \otimes A)_q) \approx H_n(G, A_q) \approx q - H_n(G,A)$, $n \geq 2$. It is easy to see that (4.4) is commutative. □

It is interesting to consider derived functors of the additive functor $G \otimes^q -$ from the category of G-modules to the category of abelian groups. Let denote these derived functors $\mathfrak{L}_n(G \otimes^q -)$, $n \geq 0$, then we have

Proposition 4.5. *Let A be a G-module, then there are isomorphisms*

$$\mathfrak{L}_n(G \otimes^q A) \approx H_{n+1}(G,A), \quad n \geq 1.$$

Proof. Consider a projective $\mathbb{Z}[G]$-resolution of A

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

Since $H_1(G,P_n) = 0$, by Theorem 1.11 and [12] one has

$$G \otimes^q P_n \approx (G \otimes P_n)/qH_1(G,P_n) = G \otimes P_n \approx IG \otimes_{\mathbb{Z}[G]} P_n.$$

The well known short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \to IG \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

gives a short exact sequence of chain complexes of abelian groups

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & IG \otimes_{\mathbb{Z}[G]} P_n & \longrightarrow \cdots \longrightarrow & IG \otimes_{\mathbb{Z}[G]} P_1 & \longrightarrow & IG \otimes_{\mathbb{Z}[G]} P_0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & P_n & \longrightarrow \cdots \longrightarrow & P_1 & \longrightarrow & P_0 & . \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[G]} P_n & \longrightarrow \cdots \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[G]} P_1 & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[G]} P_0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

The induced long homology exact sequence proves the assertion. □

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