Multiplicative Spectra for Rings and Automorphism Groups of Modules and Hermitian Forms

Anthony Bak
Department of Mathematics, University of Bielefeld,
33501 Bielefeld, Germany
e-mail: bak@@ mathematik.uni-bielefeld.de

and

Guoping Tang
Dept. of Applied Math.
Northwestern Polytechnical University
710072 Xian, P.R.China
e-mail:tanggp@@nwpu.edu.cn

Abstract This article introduces an analog of the maximal ideal spectrum of a commutative ring, for arbitrary associative rings with identity element. A given ring can have many spectra. Using all of the spectra, we define the local stable rank of a ring and the local λ-stable rank of a ring with λ-involution. Upper bounds for both notions of local rank are established in terms of the Gabriel-Rentschler J. Krull dimension at each quotient of the ring modulo a primitive ideal. This upper bound shows that the local rank of a commutative or module finite ring is 1.

Let R be a ring with λ-involution, of local λ-stable rank m. The main result is that the elementary subgroup of the automorphism group of an arbitrary λ-Hermitian form of hyperbolic Witt index $\geq m+2$ is normal. In particular, the elementary subgroup $EH_{2n}(R, a_1, \cdots, a_r)$ of the general Hermitian group $GH_{2n}(R, a_1, \cdots, a_r)$ is normal whenever $n - r \geq m+2$. As a corollary, the analogous linear result is deduced for the automorphism group of an arbitrary module of unimodular index $\geq m+2$ over a ring of local stable rank m. This extends theorems of Suslin and Tulenbaev for the general linear group of free modules of finite rank over commutative and module finite rings.

1 Introduction

We explain first the ring theoretic concepts required in formulating our results.

Throughout the article, R denotes an associative ring with identity element or such a ring with additional structures.

A **λ-involution** on R is an additive, bijective map $R \to R, a \mapsto \bar{a}$, such that $\overline{ab} = \bar{b}\bar{a}$ and $\bar{\lambda}\bar{\bar{a}}\lambda = a$ for all $a, b \in R$. Setting $a = 1$, it follows that $\bar{\lambda} = \lambda^{-1}$. If $\lambda \in$ center (A) then $\bar{\bar{a}} = a$ and the map $a \mapsto \bar{a}$ is an involution in the usual sense.

The concept of a multiplicative spectrum for associative rings is designed to provide an environment for carrying out local-global arguments that had been restricted to either commutative rings or rings which are module finite over their centers where localization can be carried out effectively with respect to commutative multiplicative sets. The problem for general rings is that one cannot carry out localization effectively by restricting to commutative multiplicative sets and one cannot form a ring of fractions at an arbitrary multiplicative set, but only at ones that are denominator sets (cf. [S] II Prop. 1.4). This means that the natural interplay in the commutative situation between maximal ideals $\mathfrak{m}$ of R and multiplicative sets of R, consisting of all elements which become a unit in $R/\mathfrak{m}$, is lost in the general situation, because such multiplicative sets in general are not denominator sets. The idea of a multiplicative spectrum for an associative ring is to replace the set of maximal ideals of a commutative ring by a set of denominator sets, which satisfies two simple properties supporting local-global procedures.

Recall that a **right denominator set** in R is a multiplicative set $S \subseteqq R$ satisfying the following properties.

- D1. (Ore condition) If $s \in S$ and $a \in R$ then $sb = at$ for some $t \in S$ and $b \in R$.
- D2. If $s \in S$ and $a \in R$ such that $sa = 0$ then $at = 0$ for some $t \in S$.

The ring of right S-fractions (cf. [S] II §1) of R will be denoted by $R[S^{-1}]$. The notion of a **left denominator set** is defined similarly. A **right multiplicative spectrum** on R is a set $\sum$ of right denominator sets satisfying the following properties.

- MS1. (Local condition) For each $S \in \sum$, the canonical homomorphism $R \to R[S^{-1}]/J(0)$ is surjective where $J(0)$ denotes the Jacobson radical of $R[S^{-1}]$.
- MS2. (Global condition) Any subset of R which meets each $S \in \sum$ nontrivially generates a right ideal of R, which is all of R.

The notion of a **left multiplicative spectrum** on R is defined similarly. A **multiplicative spectrum** on R is a right multiplicative spectrum which is simultaneously a left multiplicative spectrum. The set of all multiplicative spectra on R will be denoted by $MS(R)$. A λ-multiplicative spectrum on a ring R with λ-involution is a right multplicative spectrum $\sum$ such that each $S \in \sum$ is involution invariant. Because f the last condition, it follows that a λ-multiplicative spectrum is a multiplicative spectrum. The set of all λ-multiplicative spectra on R will be denoted by $\lambda MS(R)$.

Next we several concepts of stable rank for rings and rings with λ-involution and then use them to define their local versions.

If $T \subseteqq R$ is a subset, let $J_\ell(T)$ denote the intersection of R and all maximal left ideals of R, which contain T. Let R^n denote the direct sum of n copies of R. If $(a_1, \cdots, a_n) \in R^n$, let $J_\ell(a_1, \cdots, a_n) = J_\ell\{a_1, \cdots, a_n\}$. One says that a vector $(a_1, \cdots, a_n)$ in the right R-module R^n can be **shortened**, if there are elements $x_1, \cdots, x_{n-1} \in R$ such that $J_\ell(a_1 + x_1 a_n, \cdots, a_{n-1} + x_{n-1} a_n) = J_\ell(a_1, \cdots, a_n)$. The **right absolute stable rank** $asr(R)$ [KMV] is the smallest natural number m such that every vector of length $m+1$ can be shortened. If no such number exists then by definition $asr(R) = \infty$. A vector $(a_1, \cdots, a_n) \in R^n$ is called **right unimodular**, if $J_\ell(a_1, \cdots, a_n) = R$, i.e. the left ideal of R generated by $\{a_1, \cdots, a_r\}$ is all of R. The **right stable rank** $sr(R)$ is the smallest natural number m such that every right unimodular vector of length $m+1$ can be shortened. If no such number exists then by definition $sr(R) = \infty$. By [MR] 11.3.4, the right stable rank and left stable rank of R are equal. Obviously $sr(R) \leqq asr(R)$.

Suppose R has a λ-involution. In order to have the correct local notion of stable rank for rings with λ-involution, we need to use form parameters in defining the absolute notion. Let $\min^\lambda(R) = \{a - \bar{a}\lambda | a \in R\}$ and $\max^\lambda(R) = \{a \in R | a = -\bar{a}\lambda\}$. A **$\lambda$-form parameter** is an additive subgroup Λ of R such that

FP1. $\min^\lambda(R) \subseteqq \Lambda \subseteqq \max^\lambda(R)$,

FP2. $\bar{a}\Lambda a \subseteqq \Lambda$ for all $a \in R$.

If $\alpha = (a_{ij})$ is an $m \times n$ matrix with coefficients in R, let $\bar{\alpha} = (a'_{\ell k})$ denote the $n \times m$ matrix such that $a'_{\ell k} = \bar{a}_{\ell k}$. the matrix $\bar{\alpha}$ is called the **conjugate transpose** of α. The operation of conjugate transpose defines a λ-involution on the ring $\mathbb{M}_n(R)$ of all $n \times n$ matrices with coefficients in R. Let $\mathbb{M}_n(\bar{\Lambda}) = \{\gamma \in \mathbb{M}_n(R) \mid \gamma = -\bar{\lambda}\bar{\gamma}$, diagonal coefficients of γ lie in $\bar{\Lambda}\}$. The **right $\boldsymbol{\Lambda}$-stable rank** $\Lambda sr(R)$ [BT] ist the smallest natural number m such that $sr(R) \leqq m$ and such that given a right unimodular vector $(a_1, \cdots, a_{m+1}, b_1, \cdots, b_{m+1}) \in R^{2(m+1)}$, there is a $2(m+1) \times 2(m+1)$ matrix

$\gamma \in \mathbb{M}_{2(m+1)}(\bar{\Lambda})$ with the property that
$\begin{pmatrix} I & 0 \\ \gamma & I \end{pmatrix} \ {}^t(a_1, \cdots, a_{m+1}, b_1, \cdots, b_{m+1}) = \ {}^t(a'_1, \cdots, a'_{m+1}, b'_1, \cdots, b'_{m+1})$
with $(b'_1, \cdots, b'_{m+1})$ right unimodular. If no such number exists then by definition $\Lambda sr(R) = \infty$. It is not difficult to show that the right and left Λ-stable rank are equal. Obviously $sr(R) \leqq \Lambda sr(R)$ and by [BT] (3.4), $\Lambda sr(R) \leqq asr(R)$.

We define local versions of the notions of rank above as follows. If $\sum \in MS(R)$ or $\lambda MS(R)$ and $S \in \sum$, let $Jac(R[S^{-1}])$ denote the Jacobson radical of $R[S^{-1}]$ and let $J(R,S)$ denote its preimage in R. By MS1, $R/J(R,S) = R[S^{-1}]/Jac(R[S^{-1}])$. Let $R(S) = R/J(R,S)$. If R has a λ-involution, let $\Lambda(S)$ denote the image of $\max^{\lambda}(R)$ in $R(S)$. $\Lambda(S)$ is a λ-form parameter on $R(S)$, but is not true in general that $\Lambda(R) = \max^{\lambda}(R(S))$. This is the reason why we had to introduce form parameters in defining stable rank for rings with λ-involution. Define the

$$\textbf{right local absolute stable rank } lasr(R) = \inf_{\sum \in MS(R)} (\sup_{S \in \sum}(asr(R(S)))),$$

$$\textbf{local stable rank } lsr(R) = \inf_{\sum \in MS(R)} (\sup_{S \in \sum}(sr(R(S)))),$$

and for a ring R with λ-involution, define

$$\textbf{local } \boldsymbol{\lambda}\textbf{- stable rank } l\lambda sr(R) = \inf_{\sum \in \lambda MS(R)} (\sup_{S \in \sum}(\Lambda(S) sr(R(S)))).$$

Since $sr(R)$ and $\Lambda sr(R)$ are left-right invariant and since multiplicative spectra are left-right invariant, it follows that $lsr(R)$ and $\lambda lsr(R)$ are left-right invariant. From remarks in the penultimate paragraph, it is obvious that

(1.1) $lsr(r) \leqq lasr(R)$, and

(1.2) for a given λ-multiplicative spectrum $\sum$, $\sup_{S \in \sum}(\Lambda(S) sr(R(S)) \leqq \sup_{S \in \sum}(asr(R(S))$.

But it does not follow from (1.2) that $l\lambda sr(R) \leqq lasr(R)$, because the set $MS(R) \backslash \lambda MS(R)$ is apriori not empty.

Next we state upper bounds on the local ranks above.

Recall that a ring is called **right Goldie**, if it contains no infinite direct sum of nonzero right ideals and if right annihilators satisfy the ascending chain condition. Clearly right Noetherian rings are right Goldie. A 2-sided ideal $\mathfrak{q} \subsetneqq R$ is called a **right Artin-Rees ideal**, if the graded ring $R \oplus \mathfrak{q} \oplus \mathfrak{q}^2 \oplus \cdots$ is right graded Noetherian, i.e. every homogeneous right ideal of $R \oplus \mathfrak{q} \oplus \mathfrak{q}^2 \oplus \cdots$ is finitely generated or equivalently homogenous right ideals satisfy the ascending chain condition. Obviously the condition forces R to be right Noetherian. If $T \subsetneqq R$ is a subset, let $J(T)$ denote the intersection of R and all maximal right ideals of R which contain T. A right ideal $\mathfrak{q}$ in R is called a ***J*-ideal** if $\mathfrak{q} = J(\mathfrak{q})$. A ring is called **right *J*- Noetherian**, if its right J-ideals satisfy the ascending chain condition. Let $JK(R)$ denote the ***J* Krull-dimension** of R of Gabriel and Rentschler [GR]. Finally recall that a 2-sided ideal $\mathfrak{q} \subsetneqq R$ is called **right primitive**, if there is a right faithful simple $R/\mathfrak{q}$-module. Every 2-sided maximal ideal is primitive and every primitive ideal is prime. Let $prtvspec(R)$ denote the set of all (right) primitive ideals in R.

THEOREM 1.3 For each $\mathfrak{m} \in prtvspec(R)$, let $\mathfrak{q}(\mathfrak{m}) \subsetneqq \mathfrak{m}$ be a 2-sided ideal. Suppose that each quotient $R/\mathfrak{q}(\mathfrak{m})$ is right J-Noetherian and right Goldie (e.g. each $R/\mathfrak{q}(\mathfrak{m})$ is right Noetherian). Suppose that for each $\mathfrak{m}$, there is a right denominator set $S_\mathfrak{m}$ such that for each $s \in S_\mathfrak{m}, s$ is a unit in $R/\mathfrak{q}(\mathfrak{m})$ and $s + \mathfrak{q}(\mathfrak{m}) \subsetneqq S_\mathfrak{m}$ (e.g. $S_\mathfrak{m} = 1 + \mathfrak{q}(\mathfrak{m})$ and R is a direct limit $\varinjlim R_i$ of subrings R_i such that each ideal $\mathfrak{q}(\mathfrak{m}) \cap R_i$ is right Artin-Rees in R_i). Then $lsr(R) \leqq lasr(R) \leqq \sup\limits_{\mathfrak{m} \in prtvspec(R)} JK(R/J(\mathfrak{q}(\mathfrak{m}))) + 1$ and if R has a λ-involution then $l\lambda sr(R) \leqq \sup\limits_{\mathfrak{m} \in prtvspec(R)} JK(R/J(\mathfrak{q}(\mathfrak{m}))) + 1$.

COROLLARY 1.4 Suppose that for each $\mathfrak{m} \in prtvspec(R), R/\mathfrak{m}$ is right Artinian and R is a direct limit $\varinjlim R_i$ of subrings R_i such that each intersection $\mathfrak{m} \cap R_i$ is a right Artin-Rees ideal in R_i (e.g. R is commutative or module finite over its center). Then $lsr(R) = lasr(R) = 1$ and if R has a λ-involution then $l\lambda sr(R) = 1$.

We explain next the Hermitian form concepts required in formulating our results.

Let V be a right R-module and let $h : V \times V \to R$ be a λ-Hermitian on V. By definition, h is a biadditive map such that for all $v, w \in V$ and $a, b \in R, h(va, wb) = \bar{a}h(v,w)b$ and $h(v,w) = \overline{h(w,v)}\lambda$. The pair (V, h) is called a **λ-Hermitian module**. If $\lambda \in$ center (R), we get the more familiar identity $h(v,w) = \lambda\overline{h(w,v)}$. A morphism $f : (V, h) \to (V', h')$ of λ-Hermitian modules is by definition an R-linear map $f : V \to V'$ which preserves the λ-Hermitian forms. The automorphism group of (V, h) is denoted by $GH(V, h)$.

To economize notation, we shall frequently abbreviate

$$h(u, v) \text{ by } \quad (u, v).$$

An element $v \in V$ is called **isotropic** if $(v,v) = 0$. An ordered pair e, e_- of elements of V is called a **metabolic pair** if e_- is isotropic and $(e, e_-) = \lambda$. Thus $(e_-, e) = 1$. An ordered pair e, e_- of elements of V is called a **hyperbolic pair** if it is metabolic and e is isotropic. (V, h) is called **nonsingular**, if the map $h : V \to Hom_A(V, R), v \mapsto h(v, \_)$, is bijective. It is easy to check that a metabolic pair e, e_- generates a free submodule $eR \oplus e_-R$ of rank 2 such that $(eR \oplus e_-R, h|_{eR \oplus e_-R})$ is nonsingular. If e, e_- is metabolic (resp. hyperbolic) pair then $(eR \oplus e_-R, h|_{eR \oplus e_-R})$ is called **metabolic plane** (resp. **hyperbolic plane**). Since metabolic planes are nonsingular, it follows easily that for any metabolic pair e, e_-, V decomposes as an orthogonal sum $(eR \oplus e_-R) \perp (eR \oplus e_-R)^{\perp}$ where $(eR \oplus e_-R)^{\perp} = \{v \in V | (v, e) = (v, e_-) = 0\}$. Metabolic pairs e, e_- and f, f_- are called **complementary** if $(e, f) = (e_-, f) = (e, f_-) = (e_-, f_-) = 0$. The **hyperbolic Witt index** $ind(V, h)$ is the largest nonnegative integer or infinity such that for any nonnegative integer $n \leqq ind(V, h), V$ contains n mutually complementary hyperbolic pairs. From the discussion above, it follows that $n \leq ind(V, h) \Leftrightarrow V$ has an orthogonal decomposition $V = H_1 \perp \cdots \perp H_n \perp V'$ where each $H_i \quad (1 \leqq i \leqq n)$ is a hyperbolic plane. (The **Witt index** i of (V, h) is the largest nonnegative integer or infinity such that for any nonnegative integer $n \leqq i, V$ contains n mutually complementary metabolic pairs).

An element $v \in V$ is called **even** if $(v, v) = a + \bar{a}\lambda$ for some $a \in A$. It is obvious that the set of all even elements in V forms an R-submodule. It will be denoted by V_{even}. If v is isotropic then clearly $v \in V_{\text{even}}$. (V, h) is called **even**, if $V = V_{\text{even}}$. An ordered pair $u, v \in V$ is called a **transvection pair** if u is isotropic, v is even, and $(u, v) = 0$. A transvection pair u, v and an element $s \in A$ such that $s + \bar{s}\lambda = (v, v)$ defines an automorphism $\tau(u, v, s)$ of (V, h) called an **Eichler transvection** as follows:

$$\tau(u, v, s)(x) = x + v(u, x) - u\bar{\lambda}(v, x) - u\bar{\lambda}s(u, x).$$

Let $GH(V, H)$ denote the automorphism group of (V, h). Let $T(V, h)$ denote the subgroup of $GH(V, h)$ generated by all transvections. If $\tau(u, v, s)$ is a transvection and $\sigma \in GH(V, h)$ then $\sigma u, \sigma v$ is a transvection pair such that $s + \bar{s}\lambda = (\sigma v, \sigma v)$. Thus $\tau(\sigma u, \sigma v, s)$ is a transvection and one checks straightforward that $\sigma\tau(u, v, s)\sigma^{-1} = \tau(\sigma u, \sigma v, s)$. thus $T(V, h)$ is a normal subgroup of $GH(V, h)$.

For a hyperbolic pair $e, e_- \in V$, let $T_{\langle e, e_- \rangle}(V, h)$ denote the subgroup of $T(V, h)$ generated by all transvections $\tau(e, v, s)$ and $\tau(e_-, w, t)$.

THEOREM 1.5 Suppose R is a ring with λ-involution, of local λ-stable rank m. Let (V, h) be a λ-Hermitian module and let e, e_- and f, f_- be hyperbolic pairs. If $ind(V, h) \geqq m + 2$ then $T_{\langle e, e_- \rangle}(V, h) = T_{\langle f, f_- \rangle}(V, h)$.

COROLLARY 1.6 Under the conditions of (1.5), $T_{\langle e, e_- \rangle}(V, h)$ is normal in $GH(V, h)$.

Proof Let $\tau(e, v, s)$ and $\tau(e_-, w, t)$ be typical generators of $T_{\langle e,e_-\rangle}(V, h)$ and let $\sigma \in GH(V, h)$. Then $\sigma e, \sigma e_-$ is a hyperbolic pair and the elements $\sigma\tau(e, v, s)\sigma^{-1} = \tau(\sigma e, \sigma v, s)$ and $\sigma\tau(e_-, w, t)\sigma^{-1} = \tau(\sigma e_-, \sigma w, t) \in T_{\langle \sigma e,\sigma e_-\rangle}(V, h)$. But by Theorem 1.5, $T_{\langle \sigma e,\sigma e_-\rangle}(V, h) = T_{\langle e,e_-\rangle}(V, h)$.□

Suppose that V has a basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$ consisting of mutually complementary metabolic pairs e_i, e_{-i}. Let $1 \leqq r \leqq n$ and suppose that $(e_i, e_i) = 0$ for all $i > r$. thus e_i, e_{-i} is a hyperbolic pair for $i > r$. Thus $ind(V, h) \geq n - r$. Let $a_i = (e_i, e_{-i})$. Obviously $a_i = 0$ for $i > r$. Using the basis above for V, we can associate to each element $\sigma \in GH(V, h)$, a $2n \times 2n$ invertible matrix, and identify $GH(V, h)$ with the subgroup of the general linear group $GL_{2n}(R)$, consisting precisely of these matrices. This is called the **general Hermitian group** [B] p. 40 - 42 and is denoted by $GH_{2n}(R, a_1, \cdots, a_r)$. We allow $r = 0$ and interpret this to mean that $a_1 = \cdots = a_n = 0$. The Hermitian module (V, h) with the basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$ is by definition the underlying module [B] §1 C and (2.10) - (2.13) of $GH_{2n}(R, a_1, \cdots, a_r)$. It will be denoted by $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$. The elementary subgroup of $GH_{2n}(R, a_1, \cdots, a_r)$ is denoted by $EH_{2n}(R, a_1, \cdots, a_r)$ and is called the **Hermitian elementary group** [T] §4 and §8. Its definition is recalled in §2.

Theorem 1.7 If $r < i \leq n$ then $T_{\langle e_i,e_{-i}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r)) = EH_{2n}(R, a_1, \cdots, a_r)$.

We deduce from (1.6) and (1.7) the following normality result.

Theorem 1.8 If R is a ring with λ-involution, of local λ-stable rank m, then EH_{2n} $(R, a_1, \cdots, a_n)$ is normal in $GH_{2n}(R, a_1, \cdots, a_r)$ whenever $n - r \geq m + 2$.

Proof By Theorem 1.7, $EH_{2n}(R, a_1, \cdots, a_r) = T_{\langle e_i,e_{-i}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$. Since $n - r \geq \sup(m + 1, 3)$, $ind(\mathbb{M}_{2n}(R, a_1, \cdots, a_r)) \geq m + 2$. Thus by Corollary 1.6, $T_{\langle e_i,e_{-i}\rangle}(\mathbb{M}_{2n}$ $(R, a_1, \cdots, a_r))$ is normal in $GH(\mathbb{M}_{2n}(R, a_1, \cdots, a_r)) = GH_{2n}(R, a_1, \cdots, a_r)$.□

We explain now the module theoretic concepts required to state the linear analogs of the results above.

Let V be a right R-module. An element $v \in V$ is called **unimodular**, if there is an element $f \in Hom_R(V, R)$ such that $f(v) = 1$. Clearly v is unimodular $\Leftrightarrow$ the submodule vR of V is a free direct summand isomorphic to R. The **unimodular index** $ind(V)$ of V is the largest nonnegative integer n such that V contains a free direct summand isomorphic to R^n. If no such integer exists then by definition $ind(V) = \infty$. (It is possible that $ind(R) = \infty$.)

Let R^{op} denote the **opposite ring** of R; by definition, the additive group of R^{op} is that of R and multiplication is defined by $a \circ b = ba$. Let V be a right R-module. Then $V \otimes_{\mathbb{Z}} Hom_R(V, R)$ is a right $R \otimes_{\mathbb{Z}} R^{op}$-module such that $(v \otimes f)(a \otimes b) = va \otimes$

bf and the pairing $V \times Hom_r(V, R) \to R, (v, f) \mapsto f(v)$, defines an $R \otimes_{\mathbb{Z}} R^{op}$-linear map $V \otimes_{\mathbb{Z}} Hom_R(V, R) \to R$. A pair $(v, f) \in V \times Hom_r(V, R)$ is called **isotropic** if $f(v) = 0$. Pairs (v, f) and (v', f') are called **complementary**, if $f(v') = f'(v) = 0$. An ordered pair $((v, f), (v', f'))$ of complementary pairs such that (v, f) is isotropic is called a **transvection pair**. A transvection pair $(u, f), (v, g)$ defines an automorphism $\tau((u, f), (v, g))$ of V called a **transvection** as follows:

$$\tau((u, f), (v, g))(x) = x + vf(x) - ug(x) - ug(v)f(x).$$

Let $GL(V)$ denote the automorphism group of V. Let $T(V)$ denote the subgroup of $GL(V)$ generated by all transvections. If $\tau((u, f), (v, g))$ is a transvection and $\sigma \in GL(V)$ then $((\sigma u, \sigma^* f), (\sigma v, \sigma^* g))$ is a transvection pair where σ^* denotes the dual $\sigma^* : Hom_R(V, R) \to Hom_R(V, R), h \mapsto h\sigma$, of σ. Thus $\tau((\sigma u, \sigma^* f), (\sigma v, \sigma^* g))$ is a transvection and one checks straightforward that $\sigma\tau((u, f), (v, g))\sigma^{-1} = \tau((\sigma u, \sigma^* f), (\sigma v, \sigma^* g))$. Thus $T(V)$ is a normal subgroup of $GL(V)$. For a unimodular element $e \in V$, let $T_e(V)$ denote the subgroup of $T(V)$ generated by all transvections of the form $\tau((e, f), (v, g))$.

Theorem 1.9 Suppose R has local stable rank m. Let V be a right R-module and let e and e' be unimodular elements in V. If $ind(V) \geqq m + 2$ then $T_e(V) = T_{e'}(V)$.

Corollary 1.10 Under the conditions of (1.9), $T_e(V)$ is normal in $GL(V)$.

Proof Same as that of (1.6) □

Suppose that V has a basis $e_1, \cdots, e_n$. Thus $ind(V) \geq n$. Using the basis above of V, we can identify $GL(V)$ with the general linear group $GL_n(R)$ of all invertible $n \times n$ matrices. Let $E_n(R)$ denote the elementary subgroup of $GL_n(R)$. Let $F_n(R)$ denote the free R-module with basis $e_1, \cdots, e_n$.

Theorem 1.11 $T_{e_i}(F_n(R)) = E_n(R)$ for any $1 \leqq i \leqq n$.

We deduce from (1.10) and (1.11) the following result.

Theorem 1.12 If R is a ring of local stable rank m then $E_n(R)$ is normal in $GL_n(R)$ whenever $n \geqq m + 2$.

Proof Same as that of 1.8. Of course, the hyperbolic Witt index is replaced by the unimodular index. □

The theorem above extends Suslin's pioneering result [Su] that $E_n(R)$ is normal in $GL_n(R)$ when R is commutative and $n \geq 3$ and Tulenbaev's extension [Tul] of this result to rings that are module finite over their center.

The rest of the paper is organized as follows. §2 begins by recalling basic identities for Eichler transvections. Then the definition of the Hermitian elementary group EH_{2n} $(A, a_1, \cdots, a_r)$ is given. It is generated by Hermitian elementary matrices and their definition is formulated slightly differently than in [T] §8 because we are not assuming that λ is central. It is shown how to pass back and forth between Hermitian elementary matrices and Eichler transvections. Theorem 1.7 is proved. §3 proves the main result Theorem 1.5. §4 deduces our linear results, namely Theorem 1.9 and Theorem 1.11, as a special case of their Hermitian counterparts, namely Theorem 1.9 and Theorem 1.11, respectively. §5 establishes the upper bounds for lsr, $lasr$, and $l\lambda sr$ announced in Theorem 1.3 and Corollary 1.4.

2 Eichler Transvections and Hermitian elementary matrices

The goal of the section is to prove Theorem 1.7. This is done following Lemma 2.9. To prepare for the proof, we write each transvection of the form $\tau(e_{\pm i}, v, s)$ as a product of Hermitian elementary matrices and conversely, we write each Hermitian elementary matrix as a product of transvections of this kind. This done, the proof of Theorem 1.7 is not difficult. The section closes by studying certain quadratic subgroups of the Hermitian elementary group and how they act on unimodular vectors. This result will be applied in the next section.

Let R be an associative ring with identity element 1 and λ-involution $a \mapsto \bar{a}$. Let (V, h) be a λ-Hermitian module.

Lemma 2.1 Transvections on (V, h) satisfy the following identities.

T1. $\sigma\tau(u, v, s)\sigma^{-1} = \tau(\sigma u, \sigma v, s), \ \forall\ \sigma \in GH(V, h)$.

T2. $\tau(ua, v, s) = \tau(u, v\bar{a}, \bar{\bar{a}}s\bar{a}), \ \forall\ a \in R$.

T3. $\tau(u, v, s)\tau(u, w, t) = \tau(u, v + w, s + t + (v, w))$.

T4. $\tau(u + u', v, s) = \tau(u, v, s)\tau(u', v + u\bar{\lambda}\bar{s}\lambda, s)$, providing $(u, u') = (u, v) = (u', v) = 0$.

T5. $\tau(u, v, 0) = \tau(v\lambda, -u, 0)$.

T6. $\tau(ua, u, 0) = \tau(u, 0, \lambda a\bar{\lambda} - \lambda\bar{a})$.

The identities above are verified by straightforward computation.

LEMMA 2.2 Let e_1, e_{-1} and e_2, e_{-2} be complementary hyperbolic pairs in V. Then $T_{\langle e_1,e_{-1}\rangle}(V,h) = T_{\langle e_2,e_{-2}\rangle}(V,h)$.

PROOF Let $\tau(e_1, v, s)$ and $\tau(e_{-1}, w, t)$ be typical generators of $T_{\langle e_1,e_{-1}\rangle}(V,h)$. Let $\sigma = \tau(e_{-2}, e_{-1}(-\bar{\lambda}), 0)\tau(e_2, e_{-1}(-1), 0)$. Then $\sigma(e_1) = e_2$ and by (2.1) T1, $\sigma\tau(e_1, v, s)\sigma^{-1} = \tau(e_2, \sigma v, s)$. Thus $\tau(e_1, v, s) = \sigma^{-1}\tau(e_2, \sigma v, s)\sigma \in T_{\langle e_2,e_{-2}\rangle}(V,h)$. Similarly $\tau(e_{-1}, w, t) \in T_{\langle e_2,e_{-2}\rangle}(V,h)$. Thus $T_{\langle e_1,e_{-1}\rangle}(V,h) \subseteqq T_{\langle e_2,e_{-2}\rangle}(V,h)$.

Interchanging in the argument above e_1 with e_2 and e_{-1} with e_{-2}, we obtain the reverse inclusion. Thus $T_{\langle e_1,e_{-1}\rangle}(V,h) = T_{\langle e_2,e_{-2}\rangle}(V,h)$.□

If α denotes a $k \times \ell$ matrix (α_{ij}) whose (i,j)'th coefficient $\alpha_{ij} \in A$, let $\bar{\alpha}$ denote the $\ell \times k$ matrix (α'_{ji}) where $\alpha'_{ji} = \bar{\alpha}_{ij}$. $\bar{\alpha}$ is called the **conjugate transpose** of α.

Suppose for the rest of the section that $(V,h) = \mathbb{M}_{2n}(R, a_1, \cdots, a_r)$. Recall that $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$ is equipped with a basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$ such that the pairs $e_i, e_{-i} \quad (1 \leqq i \leqq n)$ are mutually complementary metabolic pairs with the property that $(e_i, e_i) = a_i \quad (1 \leqq i \leqq r)$ and $(e_i, e_i) = 0 \quad (r < i \leqq n)$. Let

$$\begin{aligned}
A_1 &= r \times r \text{ diagonal matrix } \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}, \\
A &= n \times n \text{ diagonal matrix } \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \\
I &= \text{ an identity matrix .}
\end{aligned}$$

Let $e_{n+i} = e_{-i} \quad (1 \leqq i \leqq n)$. Then $e_1, \cdots, e_{2n}$ is the basis above of $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$. In this basis, the matrix of h is by definition the $2n \times 2n$ matrix

$$((e_i, e_j)) = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix}$$

and

$$GH_{2n}(R,\ a_1, \dots , a_r) = \left\{ \sigma \in GL_{2n}(R) \mid \bar{\sigma} \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \sigma = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \right\}.$$

A typical element of this group is denoted by a $2n \times 2n$ matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where α, β, γ and δ are $n \times n$ block matrices.

Let $\min_\lambda(R) = \{a + \bar{a}\lambda | a \in R\}$. Let

$$C = \{{}^t(x_1, \dots , x_r) \in \ {}^t(R^r) \mid \sum_{i=1}^{r} \bar{x}_i a_i x_i \in \min_\lambda(R)\}.$$

In order to deal effectively with technical difficulties caused by the elements $a_1, \dots ,$ a_r, we shall partition a typical matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of $GH_{2n}(R,\ a_1, \dots , a_r)$ into the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\ \gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22} \end{pmatrix} \tag{2.3}$$

where $\alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11}$ are $r \times r$ matrices, $\alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12}$ are $r \times (n-r)$ matrices, $\alpha_{21}, \beta_{21}, \gamma_{21},$ δ_{21} are $(n-r) \times r$ matrices, and $\alpha_{22}, \beta_{22}, \gamma_{22}, \delta_{22}$ are $(n-r) \times (n-r)$ matrices. By modifying routinely the proof of [T] (3.4) which is carried out under the assumption that $\lambda \in$ center(R), we obtain

(2.4) the columns of $\alpha_{11} - I, \alpha_{12}, \beta_{11}, \beta_{12}, \bar{\beta}_{11}, \bar{\beta}_{21}, \bar{\delta}_{11} - I$ and $\bar{\delta}_{21}$ belong to C.

We define now Hermitian elementary matrices as in [T] §8 but with small adjustments to account for the fact that λ is not necessarily central in R. Let

$$H\epsilon_{ij}(a) \qquad (a \in R \text{ and } r < i \le n, 1 \le j \le n, i \ne j)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the (i,j)'th position, $-\bar{\lambda}\bar{a}\lambda$ in the $(n+j, n+i)$'th position, and 0 elsewhere. Let

$$r_{ij}(a) \qquad (a \in R \text{ and } r < i, j \leq n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(i, n+j)$'th position, $-\bar{a}\lambda$ in the $(j, n+i)$'th position, and 0 elsewhere. If $i = j$, this forces of course that $a = -\bar{a}\lambda$. Let

$$l_{ij}(a) \qquad (a \in R \text{ and } 1 \leq i, j \leq n)$$

denote the $2n \times 2n$ matrix with 1 along the diagonal, a in the $(n+i, j)$'th position, $-\bar{\lambda}\bar{a}$ in the $(n+j, i)$'th position, and 0 elsewhere. If $i = j$, this forces of course that $a = -\bar{\lambda}\bar{a}$.

For $\zeta = {}^t(x_1, \cdots, x_r) \in C$, let

$$\zeta_f \in R \qquad \text{such that } \zeta_f + \bar{\zeta}_f\lambda = \sum_{i=1}^{r} \bar{x}_i a_i x_i.$$

The element ζ_f is not in general unique. Define

$$Hm_i(\zeta) = \begin{pmatrix} I & \alpha_{12} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -\bar{\lambda}A_1\alpha_{12} & I & 0 \\ 0 & \gamma_{22} & -\bar{\lambda}\bar{\alpha}_{12}\lambda & I \end{pmatrix} \qquad (\zeta \in C \text{ and } r < i \leq n)$$

to be the $2n \times 2n$ matrix such that α_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$'th column and all other column's zero, and γ_{22} is the $(n-r) \times (n-r)$ matrix with $\bar{\zeta}_f$ in its $(i-r, i-r)$'th position and 0 elsewhere. Define

$$r_i(\zeta) = \begin{pmatrix} I & 0 & 0 & \beta_{12} \\ 0 & I & -\bar{\beta}_{12}\lambda & \beta_{22} \\ 0 & 0 & I & -\bar{\lambda}A_1\beta_{12} \\ 0 & 0 & 0 & I \end{pmatrix} \qquad (\zeta \in C \text{ and } r < i \leq n)$$

to be the $2n \times 2n$ matrix such that β_{12} is the $r \times (n-r)$ matrix with ζ as its $(i-r)$'th column and all other columns 0, and β_{22} is the $(n-r) \times (n-r)$ matrix with $\bar{\zeta}_f\lambda$ in its $(i-r, i-r)$'th position and 0 elsewhere.

Each of the matrices above is called a **Hermitian elementary matrix** for the elements $a_1, \ldots, a_r$.

One can show by direct computation as in [T] §4 that each Hermitian elementary matrix is in $GH_{2n}(R,\ a_1, \cdots, a_r)$.

Define the **Hermitian elementary group**

$$EH_{2n}(R,\ a_1, \ldots, a_r)$$

of the elements $a_1, \ldots, a_r$ to be the subgroup of $GH_{2n}(R,\ a_1, \ldots, a_r)$ generated by all Hermitian elementary matrices.

If $\lambda \in \text{center}(R)$, the definitions above of Hermitian elementary matrix and Hermitian elementary group agree with those in [T] §4 and §8.

Lemma 2.5 Let $r < i \leq n$. Let $\sigma \in GL_{2n}(R)$ be a matrix whose diagonal coefficients are 1 and whose only other nonzero coefficients are restricted to the i'th row and $n+i$'th column. Let $(x_1, \cdots, x_{2n})$ and ${}^t(y_1, \cdots, y_{2n})$ denote respectively the i'th row and $n+i$'th column of σ. (Thus $x_i = y_{n+i} = 1$ and $x_{n+i} = y_i$.) Then $\sigma \in GH_{2n}(R, a_i, \cdots, a_r) \Leftrightarrow \sigma = \left(\prod_{\substack{j=1\\ j\neq i}}^{n} H\epsilon_{ij}(x_i)\right) \left(\prod_{\substack{j=r+1\\ j\neq i}}^{n} r_{ji}(y_j)\right) r_i(\zeta)$ where $\zeta = {}^t(y_1, \cdots, y_r)$ and $\zeta_f = \left(\overline{x_{n+i} - \sum_{\substack{j=1\\ j\neq i}}^{n} x_j y_j}\right)\lambda$. In particular $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \sigma \in EH_{2n}(R, a_1, \cdots, a_r)$.

Proof Reduce to the case $i = n$ as follows. Suppose $i \neq n$. Set $\pi = H\epsilon_{in}(1)H\epsilon_{ni}(-1)H\epsilon_{in}$ (1). By definition, $\pi \in EH_{2n}(R, a_1, \cdots, a_r)$. Moreover $\pi\sigma\pi^{-1}$ satisfies the hypotheses of the lemma for $i = n$, and $\pi\sigma\pi^{-1}$ satisfies the conclusion of the lemma $\Leftrightarrow$ σ does. This reduces the proof to the case $i = n$.

Set $\rho = \left(\prod_{j=1}^{n-1} H\epsilon_{nj}(-x_j)\right) \sigma \left(\prod_{j=r+1}^{n} r_{jn}(-y_j)\right) =$

$$\left(\begin{array}{ccc|ccccccc}
1 & & & & & & & & & y_1 \\
 & \ddots & & & & & & & & \vdots \\
 & & & & & & & & & y_r \\
 & & & & & & & & & 0 \\
 & & & & & & & & & \vdots \\
 & & & & & & & & & 0 \\
 & & 1 & x_{n+1} & \cdots & x_{n+r} & x'_{n+r+1} & \cdots & x'_{2n-1} & z \\
\hline
 & & & 1 & & & & & & y'_{n+1} \\
 & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & \vdots \\
 & & & & & & & & \ddots & y'_{2n-1} \\
 & & & & & & & & & 1
\end{array}\right)$$

where $x'_{n+j} = x_{n+j} + \bar{y}_j\lambda \quad (r < j < n)$, $y'_{n+j} = y_{n+j} + \bar{\lambda}\bar{x}_j\lambda \quad (1 \leq j < n)$, and $z = x_{2n} - \sum_{j=1}^{n-1} x_j y_j$. Clearly $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \rho \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \bar{\rho}\begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix}\rho = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \Leftrightarrow$ (using (2.4), see [T] p. 217 - 218) $x_{n+j} = -\bar{y}_j\lambda \quad (1 \leqq j \leqq r)$, $x'_{n+j} = 0 \quad (r < j < n)$, $y'_{n+j} = -\bar{\lambda}\bar{a}_j y_j \quad (1 \leqq j \leqq r)$, $y'_{n+j} = 0 \quad (r < j < n)$, ${}^t(y_1, \cdots, y_r) \in C$, and $z + \bar{z}\lambda = \sum_{j=1}^{r} \bar{y}_j a_j y_j$. Thus $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \sigma = \left(\prod_{j=1}^{n-1} H\epsilon_{nj}(x_j)\right) r_n(\zeta)\left(\prod_{j=r+1}^{n-1} r_{jn}(y_j)\right) = \left(\prod_{j=1}^{n-1} H\epsilon_{nj}(x_j)\right)\left(\prod_{j=r+1}^{n-1} r_{jn}(y_j)\right) r_n(\zeta)$ where $\zeta = {}^t(y_1, \cdots, y_r)$ and $\zeta_f = \bar{z}\lambda$.$\square$

Corollary 2.6 Let $r < i \leq n$. Let $\tau(e_i, v, s)$ be a transvection on $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$. Suppose under the basis of $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$ that $v = {}^t(v_1, \cdots, v_{2n})$. Then $v_{n+i} = 0$ and $\tau(e_i, v, s) = \left(\prod_{j=1}^{r} H\epsilon_{ij}(-\bar{\lambda}\bar{v}_j a_i - \bar{\lambda}\bar{v}_{n+j})\right)\left(\prod_{\substack{j=r+1 \\ j\neq i}}^{n} H\epsilon_{ij}(-\bar{\lambda}\bar{v}_{n+j})\right)\left(\prod_{\substack{j=r+1 \\ j\neq i}}^{n} r_{ji}(v_j\lambda)\right) r_i(\zeta)$ where $\zeta = {}^t(v_1\lambda, \cdots, v_r\lambda)$ and $\zeta_f =$

$$\left(\overline{(-\bar{\lambda}\bar{v}_i\lambda + v_i\lambda - \bar{\lambda}s\lambda) - \sum_{j=1}^{r}(-\bar{\lambda}\bar{v}_j a_j v_j\lambda) - \sum_{\substack{j=1\\ j\neq i}}^{n}(-\bar{\lambda}\bar{v}_{n+j}v_j\lambda)}\right)\lambda.$$

PROOF The matrix of $\tau(e_i, v, s)$ has 1 along the diagonal and all other nonzero coefficients are restricted to the i'th row and $(n+i)$'th column. Since $(e_i, v) = 0$, it follows that $v_{n+i} = 0$ and an easy computation shows that the i'th row of $\tau(e_i, v, s)$ is $(-\bar{\lambda}\bar{v}_1 a_1 - \bar{\lambda}\bar{v}_{n+1}, \cdots, -\bar{\lambda}\bar{v}_r a_r - \bar{\lambda}\bar{v}_{n+r}, -\bar{\lambda}\bar{v}_{n+r+1}, \cdots, -\bar{\lambda}\bar{v}_{2n}, -\bar{\lambda}\bar{v}_1\lambda, \cdots, -\bar{\lambda}\bar{v}_n\lambda) + (v_i\lambda - \bar{\lambda}\bar{v}_i\lambda - \bar{\lambda}s\lambda)_{n+i} + (1)_i$ where $(a)_j \quad (a \in R, 1 \leqq j \leqq 2n)$ denotes the row vector whose j'th coefficient is a and all other coefficients are 0; the $(n+i)$'th column of $\tau(e_i, v, s)$ is ${}^t(v_1\lambda, \cdots, v_{2n}\lambda) + {}^t(-\bar{\lambda}\bar{v}_i\lambda - \bar{\lambda}s\lambda)_i + {}^t(1)_{n+i}$. The conclusion of the corollary follows now from Lemma 2.5. □

LEMMA 2.7 Let $r < i \leq n$. Let $\sigma \in GL_{2n}(R)$ be a matrix whose diagonal coefficients are 1 and whose only other nonzero coefficients are restricted to the $(n+i)$'th row and i'th column. Let $(x_1, \cdots, x_{2n})$ and ${}^t(y_1, \cdots, y_{2n})$ denote respectively the $(n+i)$'th row and i'th column of σ. (Thus $x_{n+i} = y_i = 1$ and $x_i = y_{n+i}$.) Then $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow$ $\sigma = \left(\prod_{\substack{j=1\\ j\neq i}}^{n} \ell_{ij}(x_j)\right)\left(\prod_{\substack{j=r+1\\ j\neq i}}^{n} H\epsilon_{ji}(-\bar{x}_{n+j})\right) Hm_i(\zeta)$ where $\zeta = {}^t(y_1, \cdots, y_r)$ and $\zeta_f = \overline{\bar{\lambda}(y_{n+i} - \sum_{\substack{j=1\\ j\neq i}}^{n} x_j y_j)}\lambda$.

PROOF Reduce as in the proof of Lemma 2.5 to the case $i = n$.

Set $\rho = \left(\prod_{j=1}^{n-1} \ell_{nj}(-x_j)\right)\sigma\left(\prod_{j=r+1}^{n-1} H\epsilon_{jn}(\bar{x}_{n+j})\right) =$

$$\left(\begin{array}{ccc|cccccc}
1 & & y_1 & & & & & & \\
 & \ddots & \vdots & & & & & & \\
 & & y_r & & & & & & \\
 & & y'_{r+1} & & & & & & \\
 & & \vdots & & & & & & \\
 & & y'_{n-1} & & & & & & \\
 & & 1 & & & & & & \\
\hline
 & & y'_{n+1} & 1 & & & & & \\
 & & \vdots & & \ddots & & & & \\
 & & \vdots & & & & \ddots & & \\
 & & y'_{2n-1} & & & & & & \\
 & & z & x_{n+1} & \cdots & x_{n+r} & 0\cdots & 0 & 1
\end{array}\right)$$

where $y'_j = y_j + \bar{x}_{n+j} \quad (r < j < n)$, $y'_{n+j} = y_{n+j} + \bar{\lambda}\bar{x}_j \quad (1 \leqq j < n)$, and $z = y_{2n} - \sum_{j=1}^{n-1} x_j y_j$. Clearly $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \rho \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \bar{\rho}\begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix}\rho = \begin{pmatrix} A & \lambda I \\ I & 0 \end{pmatrix} \Leftrightarrow$ (using (2.4), see [T], p. 217 – 218) $y'_j = 0 \quad (r < j < n)$, $y'_{n+j} = 0 \quad (r < j < n)$, $y'_{n+j} = -\bar{a}_j y_j \quad (1 \leqq j \leqq r)$, ${}^t(y_1, \cdots, y_r) \in C$, and $\bar{\lambda}\bar{z}\lambda + z\lambda = \sum_{j=1}^{r} \bar{y}_j a_j y_j$. Thus $\sigma \in GH_{2n}(R, a_1, \cdots, a_r) \Leftrightarrow \sigma = \left(\prod_{j=1}^{n-1} \ell_{nj}(x_j)\right) Hm_n(\zeta) \left(\prod_{j=r+1}^{n-1} H\epsilon_{jn}(-\bar{x}_{n+j})\right) = \left(\prod_{j=1}^{n-1} \ell_{nj}(x_j)\right)\left(\prod_{j=r+1}^{n-1} H\epsilon_{jn}(-\bar{x}_{n+j})\right) Hm_n(\zeta)$ where $\zeta = {}^t(y_1, \cdots, y_r)$ and $\zeta_f = \bar{\lambda}\bar{z}\lambda$.$\Box$

Corollary 2.8 Let $1 \leqq i \leqq n$. Let $\tau(e_{-i}, v, s)$ be a transvection on $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$. Suppose under the basis of $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$ that $v = {}^t(v_1, \cdots, v_{2n})$. Then $v_i = 0$ and

$$\tau(e_{-i}, v, s,) = \left(\prod_{j=1}^{r} \ell_{ij}(-\bar{\lambda}\bar{v}_j a_j - \bar{\lambda}\bar{v}_{n+j})\right)\left(\prod_{\substack{j=r+1 \\ j\neq i}}^{n} \ell_{ij}(-\bar{\lambda}\bar{v}_{n+j})\right)\prod_{\substack{j=r+1 \\ j\neq i}}^{n} H\epsilon_{ji}(v_j) Hm_i(\zeta)$$

where $\zeta = {}^t(v_1, \cdots, v_r)$ and $\zeta_f = \bar{\lambda}\left(\overline{(v_{n+i} - \bar{\lambda}s) - \sum_{j=1}^{r}(-\bar{\lambda}\bar{v}_j a_j v_j) - \sum_{\substack{j=1 \\ j\neq i}}^{n}(-\bar{\lambda}\bar{v}_{n+j}v_j)}\right)\lambda.$

Proof The matrix of $\tau(e_{-i}, v, s)$ has 1 along the diagonal and all other nonzero coefficients are restricted to the $(n+i)$'th row and i'th column. Since $(e_i, v) = 0$, it follows that $v_i = 0$ and an easy computation shows that the $(n+i)$'th row of $\tau(e_{-i}, v, s)$ is $(-\bar{\lambda}\bar{v}_1 a_1 - \bar{\lambda}\bar{v}_{n+1}, \cdots, -\bar{\lambda}\bar{v}_r a_r - \bar{\lambda}\bar{v}_{n+r}, -\bar{\lambda}\bar{v}_{n+r+1}, \cdots, -\bar{\lambda}\bar{v}_{2n}, -\bar{\lambda}\bar{v}_1\lambda, \cdots, -\bar{\lambda}\bar{v}_n\lambda) + (v_{n+i} - \bar{\lambda}s)_i + (1)_{n+i}$ where $(a)_j \quad (a \in R, 1 \leqq j \leqq n)$ denotes the row vector whose j'th coefficient is a and all other coefficients are 0; the i'th column of $\tau(e_{-i}, v, s)$ is ${}^t(v_1, \cdots, v_{2n}) + {}^t(-\bar{\lambda}s)_{n+i} + {}^t(1)_i$. The conclusion of the corollary follows now from Lemma 2.7. $\square$

The next lemma expresses each Hermitian elementary matrix as a product of at most 2 transvections. Its proof is a straightforward computation.

Lemma 2.9 The following identities hold.

$$\begin{array}{llcl}
(1) & H\epsilon_{ij}(a) & = & \tau(e_i, e_{-j}, \bar{a}, 0). \\
(2) & Hm_i(\zeta) & = & \tau(e_{-i}, \zeta, -\lambda\bar{\zeta}_f). \\
(3) & r_{ij}(a) & = & \tau(e_i, e_{-j}\bar{a}, 0), \text{ for } i \neq j. \\
(4) & r_{ii}(a) & = & \tau(e_i, e_{-j}, -\bar{\bar{a}})\tau(e_i, e_{-j}, 0), \text{ for } j \neq i. \\
(5) & r_i(\zeta) & = & \tau(e_i, \zeta', -\lambda^2\bar{\zeta}_f\bar{\lambda}), \text{ where } \zeta = {}^t(\zeta_1, \cdots, \zeta_r) \text{ and } \zeta' = {}^t(\bar{\zeta}_1\lambda, \cdots, \bar{\zeta}_r\lambda). \\
(6) & \ell_{ij}(a) & = & \tau(e_{-i}, e_{-j}\bar{a}, 0), \text{ for } i \neq j. \\
(7) & \ell_{ii}(a) & = & \tau(e_{-i}, e_{-j}, \lambda a)\tau(e_{-i}, e_{-j}(-1), 0), \text{ for } j \neq i.
\end{array}$$

Proof of Theorem 1.7 Let $r < i \leq n$. By (2.6) and (2.8), $T_{\langle e_i, e_{-i}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r)) \subseteq EH_{2n}(R, a_1, \cdots, a_r)$. To prove the converse, we note that by (2.2), $T_{\langle e_i, e_{-i}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r)) = T_{\langle e_k, e_{-k}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$ for any k such that $r < k \leq n$. Each Hermitian elementary matrix of the kind $H\epsilon_{\ell m}(a), Hm_\ell(\zeta), r_{\ell m}(a), r_{\ell\ell}(a),$ and $r_\ell(\zeta)$ has the property that $r < \ell \leq n$ and by (2.9) (1) - (5), each of these matrices lies in $T_{\langle e_\ell, e_{-\ell}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$. The remaining Hermitian elementary matrices are of the kind $\ell_{km}(a)$. By (2.9) (7) - (8), Hermitian elementary matrices of this kind lie in $T_{\langle e_k, e_{-k}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$ and so we are okay if $r < k \leq n$. If $r < m \leq n$ then we are also okay, because $\ell_{km}(a) = \ell_{mk}(-\bar{\lambda}\bar{a})$. Suppose $1 \leqq k, m \leqq r$. If $k \neq m$ then $\ell_{km}(a) = H\epsilon_{nk}(1)\ell_{nm}(a)H\epsilon_{nk}(1)^{-1}\ell_{nm}(a)^{-1} \in T_{\langle e_n, e_{-n}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$. If $k = m$ then $\ell_{kk}(a) = H\epsilon_{nk}(1)\ell_{nn}(a)H\epsilon_{nk}(1)^{-1}\ell_{nn}(a)^{-1} \in T_{\langle e_n, e_{-n}\rangle}(\mathbb{M}_{2n}(R, a_1, \cdots, a_r))$.$\square$

If $a_1 = a_2 = \cdots = a_r = 0$, we set

$$GH_{2n}(R,0) = GH_{2n}(R, a_1, \cdots, a_r), \text{ and}$$
$$EH_{2n}(R,0) = EH_{2n}(R, a_1, \cdots, a_r).$$

It is obvious that the definition of $GH_{2n}(R,0)$ is independent of r and the same is true of $EH_{2n}(R,0)$, thanks to the following lemma.

LEMMA 2.10 Define the $2n \times 2n$ matrices

$$\begin{array}{rl} H\epsilon_{ij}(a) & (a \in R \text{ and } 1 \leqq i,j \leqq n, i \neq j) \\ r_{ij}(a) & (a \in R \text{ and } 1 \leqq i,j \leqq n) \\ \ell_{ij}(a) & (a \in R \text{ and } 1 \leqq i,j \leqq n) \end{array}$$

as after (2.4), except replace r by 1. We require that if $i = j$ then $r_{ij}(a)$ has the property that $a = -\bar{a}\lambda$ and $\ell_{ij}(a)$ the property that $a = -\bar{\lambda}\bar{a}$. Then each of the matrices above is in $EH_{2n}(R,0)$ and $EH_{2n}(R,0)$ is generated by these matrices.

PROOF Since $A_1 = 0$, it is easy to see that each Hermitian elementary matrix defined after (2.4) is either one of the matrices in (2.10) or a product of such matrices. Conversely, since $A_1 = 0$ and $C = {}^t(R^r)$, it is easy to check that each matrix in (2.10) is some Hermitian elementary matrix. □

We describe next certain subgroups of $GH_{2n}(R,0)$ and $EH_{2n}(R,0)$, which arise in the theory of quadratic forms. Let Λ be a λ-form parameter on R, as defined in §1. Let $\mathbb{M}_n(\Lambda)$ denote the additive subgroup of the ring $\mathbb{M}_n(R)$ of $n \times n$ matrices with coefficients in R, consisting of all β such that $\beta = -\bar{\beta}\lambda$ and the diagonal coefficients of β lie in Λ. Define the **general quadratic group** [B] §3

$$GQ_{2n}(R,\Lambda) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}(R) \mid \overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{pmatrix} 0 & \lambda I \\ I & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \right.$$
$$\left. \begin{pmatrix} 0 & \lambda I \\ I & 0 \end{pmatrix}, \bar{\gamma}\alpha \text{ and } \bar{\delta}\beta \in \mathbb{M}_n(\Lambda) \right\}.$$

If $\Lambda = \max^\lambda(R)$ then it follows from the equation $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & \lambda I \\ I & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \lambda I \\ I & 0 \end{pmatrix}$ that $\bar{\gamma}\alpha$ and $\bar{\delta}\beta \in \mathbb{M}_n(\max^\lambda(R))$. Thus

$$GQ_{2n}(R,\ \max^{\lambda}(R)) = GH_{2n}(R, 0). \tag{2.11}$$

Since each λ-form parameter $\Lambda \subseteqq \max^\lambda(R)$, it is obvious that each group $GQ_{2n}(R,\Lambda) \subseteqq GQ_{2n}(R, \max^\lambda(R))$. Define the $2n \times 2n$ matrices

$$\begin{aligned} H\epsilon_{ij}(a) &\quad (a \in R \text{ and } 1 \leqq i,j \leqq n, i \neq j) \\ r_{ij}(a) &\quad (a \in R \text{ and } 1 \leqq i,j \leqq n) \\ \ell_{ij}(a) &\quad (a \in R \text{ and } 1 \leqq i,j \leqq n) \end{aligned} \tag{2.12}$$

as in Lemma 2.10, except if $i = j$, we insist that $r_{ij}(a)$ has the property that $a \in \Lambda$ and $\ell_{ij}(a)$ the property that $a \in \bar{\Lambda}$. These matrices are called **Λ-quadratic elementary matrices**. One checks easily that they are members of $GQ_{2n}(R,\Lambda)$. They generate by definition the **Λ- quadratic elementary group** $EQ_{2n}(R,\Lambda)$ [B] §3. From Lemma 2.10 and the definition of $EQ_{2n}(R, \max^\lambda(R))$, it follows that

$$EQ_{2n}(R,\ \max^{\lambda}(R)) = EH_{2n}(R, 0). \tag{2.13}$$

Obviously each group $EQ_{2n}(R,\Lambda) \subseteqq EQ_{2n}(R, \max^\lambda(R))$, because each λ-form parameter $\Lambda \subseteqq \max^\lambda(R)$.

For technical reasons, we give names to several kinds of Λ-quadratic elementary matrices. Let m be a natural number such that $1 \leqq m \leqq n-2$. We define 5 types of matrices as follows.

(2.14)
- Type 1. $H\epsilon_{ni}$ $(n-m-1 \leqq i < n)$
- Type 2. ℓ_{in} $(n-m-1 \leqq i < n)$
- Type 3. ℓ_{ij} $(1 \leqq i,j < n)$
- Type 4. $H\epsilon_{ij}$ $(1 \leqq i \leqq n-m-2, n-m-2 < j < n)$
- Type 5. $H\epsilon_{ij}$ $(n-m-2 \leqq i,j < n, i \neq j)$.

Let

$$EQ_{2n}^{(m)}(R,\Lambda) = \langle \text{ type } k \ \Lambda\text{-quadratic elementary matrices} \mid 1 \leqq k \leqq 5 \rangle.$$

PROPOSITION 2.15 Let $1 \leq m \leqq n-2$. Let $\epsilon \in EQ_{2n}^{(m)}(R,\Lambda), v = (b_1, \cdots, b_n, c_1, \cdots, c_n) \in R^n$, and $\epsilon v = (b'_1, \cdots, b'_n, c'_1, \cdots, c'_n)$. If $b_{n-m-1} = b_{n-m} = \cdots = b_n = c_n = 0$ then $b'_{n-m-1} = b'_{n-m} = \cdots = b'_n = c'_n = 0$. Furthermore if v is right unimodular and $m \geq \Lambda sr(R)$ then there is an ϵ such that $c'_{n-m-1} = c'_{n-m} = \cdots = c'_n = 1$.

PROOF The first assertion is trivial to check. We prove the second assertion. If k is a natural number and $\epsilon^{(1)}, \epsilon^{(2)}, \cdots, \epsilon^{(k)} \in EQ_{2m}^{(m)}(R,\Lambda)$, let $\epsilon^{(k)}\epsilon^{(k-1)}\cdots\epsilon^{(1)}v = (b_1^{(k)}, \cdots, b_n^{(k)}, c_1^{(k)}, \cdots, c_n^{(k)})$. Suppose $m \geq \Lambda sr(R)$ and v is unimodular. By [K] VI (1.5.1), there is a product $\epsilon^{(1)}$ of type 1 matrices such that $(b_1^{(1)}, \cdots, b_n^{(1)}, c_1^{(1)}, \cdots, c_{n-1}^{(1)})$ is unimodular. Again by [K] VI (1.5.1), there is a product $\epsilon^{(2)}$ of type 2 matrices such that $(b_1^{(2)}, \cdots, b_{n-1}^{(2)}, c_1^{(2)}, \cdots, c_{n-1}^{(2)})$ is unimodular. By [BT] (3.3), there is a product $\epsilon^{(3)}$ of type 3 matrices such that $(c_1^{(3)}, \cdots, c_{n-1}^{(3)})$ is unimodular. By [K] VI (1.5.1), there is a product $\epsilon^{(4)}$ of type 4 matrices such that $(c_{m-n-1}^{(4)}, \cdots, c_{n-1}^{(4)})$ is unimodular. By [K] VI (1.5.2), there is a product $\epsilon^{(5)}$ of type 5 matrices such that $(c_{m-n-1}^{(5)}, \cdots, c_{n-1}^{(5)}) = (1, \cdots, 1)$.□

3 Normality of elementary subgroups of general Hermitian groups

The goal of this section is to prove Theorem 1.5. This is done at the end of the section.

Let R be an associative ring with identity element and λ-involution $a \mapsto \bar{a}$. Let (V,h) be a λ-Hermitian module. We fix the following notation throughout the section. Let n be a natural number and let $e_1, e_{-1}, \cdots, e_n, e_{-n}$ be a set of mutually complementary hyperbolic pairs e_i, e_{-i} in (V,h). The submodule, say X, of V generated by these pairs is nonsingular and V splits as an orthogonal sum $X \perp X^{\perp}$ where $X^{\perp} = \{v \in V \mid h(v,x) = 0 \ \forall\ x \in X\}$. The module X is free on the elements of the hyperbolic pairs above and we let X have the ordered basis $e_1, \cdots, e_n, e_{-1}, \cdots, e_{-n}$. Let $\mathbb{H}_{2n}(R) = (X, h|_X)$ with the choice of basis above. Thus $\mathbb{H}_{2n}(R)$ is the λ-Hermitian module $\mathbb{M}_{2n}(R, a_1, \cdots, a_r)$ defined in §2, with $a_1 = a_2 = \cdots = a_r = 0$ and with basis as in §2. Let $V' = X^{\perp}$. Thus $V = \mathbb{H}_{2n}(R) \perp V'$. we identify $GH(\mathbb{H}_{2n}(R))$ with its image in $GH(V,h)$ under the map $\sigma \mapsto \sigma \perp 1_{V'}$. By definition $GH_{2n}(R,0) = GH(\mathbb{H}_{2n}(R))$, by (2.11) $GQ_{2n}(R, \max^{\lambda}(R)) = GH_{2n}(R,0)$, and by (2.13) $EQ_{2n}(R, \max^{\lambda}(R)) = EH_{2n}(R,0)$. For any λ-form parameter Λ, we have a sequence of inclusions $EQ_{2n}(R,\Lambda) \subseteqq EQ_{2n}(R, \max^{\lambda}(R)) = EH_{2n}(R,0) \subseteqq GH(V,h)$ which allow us to view $EQ_{2n}(R,\Lambda)$ as a subgroup $GH(V,h)$. Moreover from Theorem 1.7 applied to $\mathbb{H}_{2n}(R)$, it follows that $EQ_{2n}(R,\Lambda) \subseteqq T_{\langle e_i, e_{-i}\rangle}(V,H)$ for any $i \quad (1 \leqq i \leqq n)$. If $v \in V$ then we can write uniquely

$$v = \sum_{i=-n}^{n} e_i v_i + v'$$

where $v' \in V'$ and we make this a notational convention. In vector notation, $\sum\limits_{i=-n}^{n} e_i v_i$ becomes identified with the vector ${}^t(v_1, \cdots, v_n, v_{-1}, \cdots, v_{-n})$ because of our choice of orderd basis. Each element $\epsilon \in EQ_{2n}(R, \Lambda)$ acts on ${}^t(v_1, \cdots, v_n, v_{-1}, \cdots, v_{-n})$, but leaves v' fixed.

LEMMA 3.1 Let $(\tau(u, v, a)$ be a transvection on (V, h).

(3.1.1) If for some $i \quad (1 \leqq i \leqq n), u = e_{\pm i} u_i$ or $v = e_{\pm i} v_i$ and $a = 0$ then $\tau(u, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.2) If for some $i \quad (1 \leqq i \leqq n), u_i = u_{-i} = v_i = v_{-i} = 0$ then $\tau(u, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.3) If for some $i \quad (1 \leqq i \leqq n), u_i = u_{-i} = 0$ then $\tau(u, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.4) If for some $i \quad (1 \leqq i \leqq n), u_i = v_i = 0$ then $\tau(u, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.5) Suppose $n \geq 3$. Suppose for some $i \neq j \quad (1 \leqq i, j \leqq n), v_{-j} = u_i = u_{-i} = 0$. Let $s \in R$ such that $\bar{u}_{-i} u_i s \in \bar{u}_{-j} R$. Then $\tau(us, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

PROOF (3.1.1) Suppose $u = e_i u_i$. By (2.1) $T2, \tau(e_i u_i, v, a) = \tau(e_i, v\bar{u}_i, \bar{\bar{u}}_i a u_i) \in$ (by (2.2)) $T_{\langle e_1, e_{-1}\rangle}(V, h)$. Suppose $v = e_i v_i$ and $a = 0$. By (2.1) T5 and T2, $\tau(u, e_i v_i, 0) = \tau(e_i, -u\bar{\lambda} v_i, 0) \in$ (by (2.2)) $T_{\langle e_1, e_{-1}\rangle}(V, h)$. The remaining cases are proved similarly.

(3.1.2) The identity (2.1) T2 suggests looking for an element $\sigma \in T_{\langle e_1, e_{-1}\rangle}(V, h)$ such that $\sigma u = u$ and $\sigma e_{-i} = v + e_i b + e_{-i} c$ for suitable $b, c \in R$. This done, we could try to relate $\tau(u, v, a)$ to $\tau(v + e_i b + e_{-i} c, u, 0) = \sigma \tau(e_{-i}, u, 0) \sigma^{-1}$ and finish the proof with (2.2).

By (2.1) $T1, \tau(e_i, v, a)\tau(e_{-i}, -u\bar{\lambda}, 0)\tau(e_i, v, a)^{-1} = \tau(\tau(e_i, v, a)e_{-i}, \tau(e_i, v, a)(-u\bar{\lambda}), 0) = \tau(e_{-i} + v - e_i\bar{\lambda} a, -u\bar{\lambda}, 0) =$ (by (2.1) T5 and T2) $\tau(u, e_{-i} + v - e_i\bar{\lambda} a, 0) =$ (by (2.1) T3) $\tau(u, e_{-i}, 0)\tau(u, v, a)\tau(u, -e_i\bar{\lambda} a, 0) =$ (by (2.1) T 5 and T 2) $\tau(e_{-i}, -u\bar{\lambda}, 0)\tau(u, v, a)$ $\tau(e_i, u\overline{\bar{\lambda} a \lambda}, 0)$. Thus $\tau(u, v, a) = [\tau(e_{-i}, -u\bar{\lambda}, 0)^{-1}, \tau(e_i, v, a)]\tau(e_{-i}, u\overline{\bar{\lambda} a \lambda}, 0)^{-1} \in$ (by (2.2)) $T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.3) Let $v = e_i v_i + e_{-i} v_{-i} + w$ where w is orthogonal to e_i and e_{-i}. By (2.1) T3, $\tau(u, v, a) = \tau(u, w, a - \bar{v}_i v_{-i})\tau(u, e_i v_i, 0)\tau(u, e_{-i} v_{-i}, 0)$. By (3.1.2), $\tau(u, w, a - \bar{v}_i v_{-i}) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$ and by (3.1.1), $\tau(u, e_i v_i, 0)$ and $\tau(u, e_{-i} v_{-i}, 0) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.4) Let $u = e_{-i}u_{-i} + w$ where w is orthogonal to e_i and e_{-i}. By (2.1) T4, $\tau(u, v, a) = \tau(e_{-i}u_{-i}, v, a)\tau(w, v + e_{-i}u_{-i}\bar{\lambda}\bar{a}\lambda, a)$. By (3.1.1), $\tau(e_{-i}u_{-i}, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$ and by (3.1.3), $\tau(w, v + e_{-i}u_{-i}\bar{\lambda}\bar{a}\lambda, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

(3.1.5) Let $c \in R$ such that $\bar{u}_{-i}u_i s = \bar{u}_{-j}c$. Then u is orthogonal to $e_i u_i s - e_j c$. Thus us is orthogonal to $e_i u_i s - e_j c$. Define w by the equation $us = w + (e_i u_i s - e_j c)$. Since us and $e_i u_i s - e_j c$ are totally isotropic and $e_i u_i s - e_j c$ is orthogonal to us, it follows that w is totally isotropic and orthogonal to $e_i u_i s - e_j c$. Thus by (2.1) T4, $\tau(us, v, a) = \tau(w, v, a)\tau(e_i u_i s - e_j c, v + w\bar{\lambda}a\lambda, a)$. By (3.1.4), $\tau(w, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$ and because $n \geq 3$, it follows from (3.1.3) that $\tau(e_i u_i s - e_j c, v + w\bar{\lambda}a\lambda, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.□

LEMMA 3.2 Let $\tau(u, v+w, a)$ be a transvection such that v and w are even and orthogonal to u. Let $b, c \in R$ such that $b + \bar{b}\lambda = v$ and $c + \bar{c}\lambda = w$. Then the element $d = a - (b + c + (v, w))$ has the property that $d = -\bar{d}\lambda$ and $\tau(u, v + w, a) = \tau(u, v, b)\tau(u, w, c + d)$.

PROOF Since $a + \bar{a}\lambda = (v + w, v + w) = (b + c + (v, w)) + (\overline{b + c + (v, w)})\lambda$, it follows that $d + \bar{d}\lambda = 0$. Thus $d = -\bar{d}\lambda$. The second assertion in the lemma is a direct consequence of (2.1) T3. □

The next lemma is an easy exercise.

LEMMA 3.3 Let S be a right denominator set in R. Let $x_1, \cdots, x_k, y_1, \cdots, y_\ell, u, v \in R$ such that u and v are units in $R[S^{-1}]$. Then there is an element $s \in S$ and elements $b_1, \cdots, b_k, c_1, \cdots, c_\ell \in R$ such that $x_i s = u b_i \quad (1 \leqq i \leqq k)$ and $y_j s = v c_j \quad (1 \leqq j \leqq \ell)$.

LEMMA 3.4 Suppose $m = l\lambda sr(R)$ and $n \geq m + 2$. If $\tau(u, v, a)$ is a transvection on (V, h) such that u can be completed to a hyperbolic pair u, u_- and $v_{-n} = v_n = v_{n-1} = \cdots = v_{n-m-1} = 0$ then $\tau(u, v, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

PROOF Call an element $w \in V$ **good**, if $w_{-n} = w_n = w_{n-1} = \cdots = w_{n-m-1} = 0$. Thus v is good. Let $\sum$ be a λ-multiplicative spectrum on R such that $m = \sup\limits_{S \in \sum}(\Lambda(S) sr(R(S)))$. Let $w \in V$ such that u, w is a hyperbolic pair. Let $S \in \sum$. (Keep in mind the notational conventions.) Since $(w, u) = 1$, it follows that $(u_1, \cdots, u_n, u_{-1}, \cdots, u_{-n}, (w', u'))$ is a unimodular vector in R^{2n+1} and hence also in $R(S)^{2n+1}$. Since $m \geq sr(R(S))$, there are elements $x_{-1}, \cdots, x_{-m} \in R$ such that the vector $(u_1, \cdots, u_n, u_{-1} + x_{-1}(w', u'), \cdots, u_{-m} + x_{-m}(w', u'), u_{-m-1}, \cdots, u_{-n})$ is unimodular in $R(S)^{2n}$. Trivially for any choice of elements $a_{-1}, \cdots, a_{-m} \in R$, the vector $(u_1, \cdots, u_n, u_{-1} + x_{-1}(w', u') + a_{-1}u_{-1}, \cdots, u_{-m} + x_{-m}(w', u') + a_{-m}u_m, u_{-m-1}, \cdots, u_{-n})$ is unimodular in $R(S)^{2n}$. Let $u(a_{-1}, \cdots, a_{-n})$ denote this vector as a member of R^{2n} and using the ordered basis of $\mathbb{H}_{2n}(R)$, identify R^{2n} with $\mathbb{H}_{2n}(R)$. Since $w' = w - \sum\limits_{i=-n}^{n} e_i w_i$ and since w and each e_i are even, it follows that w' is even. Choose elements $b_i \in R \quad (1 \leqq i \leqq m)$ such that $b_{-i} + \bar{b}_{-i}\lambda =$

$(-w'\bar{\lambda}\bar{x}_{-i}, -w'\bar{\lambda}\bar{x}_{-i})$. Set $\sigma_1 = \prod_{i=1}^{m} \tau(e_{-i}, -w'\bar{\lambda}\bar{x}_{-i}, b_{-i})$. A straightforward computation shows that $\sigma_1 u = u(-\bar{\lambda}b_{-1}, \cdots, -\bar{\lambda}b_{-m}) + (\sigma_1 u)'$ where $(\sigma_1 u)'$ is orthogonal to $\mathbb{H}_{2n}(R)$. Moreover $\sigma_1 v$ is good. By (2.15), there is an element $\sigma_2 \in EQ_{2n}^{(m)}(R, \max^{\lambda}(R))$ such that $(\sigma_2\sigma_1 u)_{-(n-1)} \equiv 1$ in $R(S)$ and $\sigma_2\sigma_1 v$ is good. By MS1 in §1, $(\sigma_2\sigma_1 u)_{-(n-1)}$ must be a unit in $R[S^{-1}]$. Let z'_{n-1} denote its inverse in $R[S^{-1}]$. Since $(\overline{\sigma_2\sigma_1 u})_{-n}\lambda(\sigma_2\sigma_1 u)_n$ and $(\sigma_2\sigma_1 u)_{-(n-1)}z'_{n-1}(\overline{\sigma_2\sigma_1 u})_{-n}\lambda(\sigma_2\sigma_1 u)_n$ are equal in $R[S^{-1}]$, there are elements $t_S \in S$ and $z_{n-1} \in R$ such that $(\overline{\sigma_2\sigma_1 u})_{-n}\lambda(\sigma_2\sigma_1 u)_n t_S = (\sigma_2\sigma_1 u)_{-(n-1)}z_{n-1}$ as elements of R. An easy computation shows that for any element $c_S \in R, \sigma_2\sigma_1 u$ is orthogonal to $e_n(\sigma_2\sigma_1 u)_n t_S c_S - e_{n-1}z_{n-1}c_S$. Thus $\sigma_2\sigma_1 u t_S c_S$ is orthogonal to $f_S = e_n(\sigma_2\sigma_1 u)_n t_S c_S - e_{n-1}z_{n-1}c_S$. By (2.1) T4, $\tau(\sigma_2\sigma_1 u t_S c_S, \sigma_2\sigma_1 v, a) = \tau(\sigma_2\sigma_1 u t_S c_S - f_S, \sigma_2\sigma_1 v, a)\tau(f_S, \sigma_2\sigma_1 v - f_S\bar{\lambda}\bar{a}\lambda, a)$. By (3.1.4), the first factor above lies in $T_{\langle e_1, e_{-1}\rangle}(V,h)$ and by (3.1.3), the seconds factor lies in $T_{\langle e_1, e_{-1}\rangle}(V,h)$. Thus for any $c_S \in R, \tau(u, v t_S c_S, \overline{t_S c_S} a t_S c_S) =$ (by (2.1) T2) $\tau(u t_S c_S, v, a) = (\sigma_2\sigma_1)^{-1}\tau(\sigma_2\sigma_1 u t_S c_S, \sigma_2\sigma_1 v, a)(\sigma_2\sigma_1) \in T_{\langle e_1, e_{-1}\rangle}(V,h)$. Moreover t_S is independent of the choice of the element a in $\tau(u,v,a)$. This will be important. By MS 2 in §1, there are a finite number of denominator sets $S_1, \cdots, S_k \in \sum$ and elements $c_1, \cdots, c_k \in R$ such that $t_{S_1}c_1 + \cdots + t_{S_k}c_k = 1$. Thus $\tau(u,v,a) = \tau(u, \sum_{i=1}^{k} v t_{S_i} c_i, a) =$ (by (3.2), for suitably chosen a_i's) $\prod_{i=1}^{k} \tau(u, v t_{S_i} c_i, a_i) \in T_{\langle e_1, e_{-1}\rangle}(V,h)$.□

Proof of Theorem 1.5 Let $m = l\lambda sr(R)$. Let $\sum$ be a λ-multiplicative spectrum on R such that $m = \sup_{S\in\sum}(\Lambda(S)sr(R(S)))$. Suppose $u \in V$, which can be completed to a hyperbolic pair u, u_-. We shall show that any transvection $\tau(u,v,a)$ on (V,h) lies in $T_{\langle e_1, e_{-1}\rangle}(V,h)$. Let $S \in \sum$. The proof of (3.4) demonstrates that we can find a $\sigma \in T_{\langle e_1, e_{-1}\rangle}(V,h)$ such that $(\sigma u)_{n-1}$ and $(\sigma u)_{n-2} \equiv 1$ in $R(S)$. By MS 1 in §1, $(\sigma u)_{n-1}$ and $(\sigma u)_{n-2}$ are units in $R[S^{-1}]$. Let $N = \{-n, n, n-1, \cdots, n-m-1\}$. Since $(\sigma u)_{n-1}$ and $(\sigma u)_{n-2}$ are units in $R[S^{-1}]$, there is by (3.3) an element $t_S \in S$ and elements $z_i \in R \quad (i \in N)$ such that

$$\begin{array}{ll} (\overline{\sigma u})_{-i}(\sigma v)_i t_S = (\overline{\sigma u})_{n-1}\lambda z_i, & \text{for } i \neq n-1 \text{ and } -n, \\ (\overline{\sigma u})_{n-i}(\sigma v)_{n-i} t_S = (\overline{\sigma u})_{n-2}\lambda z_{n-1}, & \text{for } i = n-1, \\ (\overline{\sigma u})_{n}\lambda(\sigma v)_{-n} t_S = (\overline{\sigma u})_{n-1}\lambda z_{-n}, & \text{for } i = n. \end{array}$$

It is important to observe that t_S is independent of the choice of the element a in $\tau(u,v,a)$. For $i \in N$, define f_i such that

$$\begin{array}{ll} f_i = e_i(\sigma v)_i t_S - e_{-(n-1)}z_i, & \text{for } i \neq n-1 \text{ and } -n, \\ f_{n-1} = e_{n-1}(\sigma v)_{n-1} t_S - e_{-(n-2)}z_{n-1}, & \text{for } i = n-1, \\ f_{-n} = e_{-n}(\sigma v)_{-n} t_S - e_{-(n-1)}z_{-n}, & \text{for } i = n. \end{array}$$

An easy computation shows that for any element $c_S \in R, u$ is orthogonal to $f_i c_S \quad (i \in N)$. Let $c_S \in R$. Let $w = \sum_{i \in N} f_i$. Then $\sigma\tau(u, vt_Sc_S, a)\sigma^{-1} =$ (by (2.1) T1) $\tau(\sigma u, \sigma vt_Sc_S - wc_S + wc_S, a) =$ (by (3.2), for suitable a' and a_i) $\tau(\sigma u, \sigma vt_Sc_S - wc_S, a') \prod_{i \in N} \tau(\sigma u, f_ic_S, a_i)$. By (3.4), $\tau(\sigma u, \sigma vt_Sc_S - wc_S, a') \in T_{\langle e_1, e_{-1}\rangle}(V, h)$ and by (3.1.3), each $\tau(\sigma u, f_ic_S, a_i) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$. Thus $\tau(u, v, t_Sc_S, a) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

Keep in mind that t_S is independent of the choice of the element a in $\tau(u, v, a)$. By MS 2 in §1, there are finitely many denominator sets $S_1, \cdots, S_k \in \sum$ and elements $c_1, \cdots, c_k \in R$ such that $t_{S_1}c_1 + \cdots + t_{S_k}c_k = 1$. Thus $\tau(u, v, a) = \tau(u, \sum_{i=1}^{k} vt_{S_i}c_i, a) =$ (by (3.2), for suitable a_i)
$\prod_{i=1}^{k} \tau(u, vt_{S_i}c_i, a_i) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.

Finally let $u \in V$ such that u can be completed to a hyperbolic pair w, u. Then $\tau(u, v, a) = \tau(u, v\bar{\lambda}\lambda, a) =$ (by (2.1) T2) $\tau(u\bar{\lambda}, v\bar{\lambda}, \bar{\lambda}a\lambda)$. But $u\bar{\lambda}$ can be completed to the hyperbolic pair $u\bar{\lambda}, w$ and so by the paragraph above, $\tau(u\bar{\lambda}, v\bar{\lambda}, \bar{\lambda}a\lambda) \in T_{\langle e_1, e_{-1}\rangle}(V, h)$.□

References

[B]

[BT]

[BV]

[GR]

[K]

[KMV]

[MR]

[T]

[Tu]

[Tul]

[S]

[Su]