GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 4, 1997, 313-332

NON-ABELIAN COHOMOLOGY OF GROUPS

H. INASSARIDZE

ABSTRACT. Following Guin's approach to non-abelian cohomology [4] and, using the notion of crossed bimodule, a second pointed set of cohomology is defined with coefficients in a crossed module and Guin's six term exact cohomology sequence is extended to a nine term exact sequence of cohomology up to dimension 2.

INTRODUCTION

In this and forthcoming papers [1] one discusses the cohomology $H^*(G,A)$ of a group G with coefficients in a G-group A.When A is abelian this cohomology is the well-known classical cohomology of groups which can be defined as derived functors either of the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(-,A)$ in the category of $\mathbf{Z}[G]$-modules or of the functor $\mathrm{Der}(-,A)$ in the category of groups acting on A. When A is non-abelian, a functorial pointed set of cohomology $H^1(G,A)$ not equipped with a group structure was defined in a natural way in [2]. Guin defined, in [3]–[4], a first cohomology group when the coefficient group is a crossed G-module and obtained a six term exact sequence of cohomology for any short exact coefficient sequence of crossed G-modules.

Our approach to a non-abelian cohomology of groups follows Guin's cohomology theory of groups [3]–[4] which differs of the classical first non-abelian cohomology pointed set [2] and of the setting of various papers non-abelian cohomology [5]–[7] extending the classical exact non-abelian cohomology sequence from lower dimensions [2] to higher dimensions.

Let G and R be groups and let (A,μ) be a crossed R-module. We introduce the notion of a crossed $G-R$-bimodule signifying an action of G on the crossed R-module (A,μ) and generalizing the notion of a crossed G-module.The group of derivations $\mathrm{Der}(G,(A,\mu))$ from G to (A,μ) is defined to obtain a pointed set of cohomology $H^2(G,A)$ when A is a crossed

---

1991 *Mathematics Subject Classification.* 18G50, 18G55.

Key words and phrases. Crossed module, derivation, simplicial kernel, crossed bimodule.

1072-947X/97/0700-0313$12.50/0

G-module. The group $\mathrm{Der}(G,(A,\mu))$ and the pointed set $H^2(G,A)$ coincide respectively with the group $\mathrm{Der}_G(G,A)$ of Guin [4] when (A,μ) is a crossed G-module and with the usual cohomology group when A is abelian. A coefficient short exact sequence of crossed G-modules gives rise to a nine term exact sequence of cohomology which extends the six term exact cohomology sequence of Guin [4]. In [1] these results are generalized when the coefficients are crossed bimodules; in that case $H^1(G,(A,\mu))$ is equipped with a partial product, and, finally, in [1] the definition of a pointed set of cohomology $H^n(G,(A,\mu))$ of a group G with coefficients in a crossed $G-R$-bimodule (A,μ) for all $n \geq 1$ is given.

All considered groups will be arbitrary (not necessarily commutative). An action of a group G on a group A means an action on the left of G on A by automorphisms and will be denoted by ${}^g a, g \in G$, $a \in A$. We assume that G acts on itself by conjugation. The center of a group G will be denoted by $Z(G)$. If the groups G and R act on a group A then the notation ${}^{gr}a$ means ${}^g({}^r a)$, $g \in G$, $r \in R$, $a \in A$.

1. Crossed Bimodules

A precrossed G-module (A,μ) consists of a group G acting on a group A and a homomorphism $\mu : A \longrightarrow G$ such that

$$\mu({}^g a) = g\mu(a)g^{-1}, \quad g \in G, \quad a \in A.$$

If in addition we have

$$ {}^{\mu(a)}a' = aa'a^{-1}$$

for $a, a' \in A$, then (A,μ) is a crossed G-module.

Definition 1. Let G, R and A be groups. It will be said that (A,μ) is a precrossed $G-R$-bimodule if

1) (A,μ) is a precrossed R-module,
2) G acts on R and A,
3) the homomorphism $\mu : A \longrightarrow R$ is a homomorphism of G-groups,
4) ${}^{({}^g r)}a = {}^{grg^{-1}}a$ (compatibility condition) for $g \in G$, $r \in R$, $a \in A$. If in addition (A,μ) is a crossed R-module then (A,μ) will be called a crossed $G-R$-bimodule. If conditions 1)-3) hold it will be said that the group G acts on the precrossed (respect. crossed) R-module (A,μ).

It is easy to see that any precrossed (respect. crossed) G-module (A,μ) is in a natural way a precrossed (respect. crossed) $G-G$-bimodule. It is also clear that if (A,μ) is a crossed $G-R$-bimodule and $f : G' \longrightarrow G$ is a homomorphism of groups then (A,μ) is a crossed $G'-R$-bimodule induced by f, G' acting on A and R via f.

A homomorphism $f : (A,\mu) \longrightarrow (B,\lambda)$ of precrossed (crossed) $G-R$-bimodules is a homomorphism of groups $f : A \longrightarrow B$ such that

1) $f({}^r a) = {}^r f(a)$, $r \in R$, $a \in A$,
2) $f({}^g a) = {}^g f(a)$, $g \in G$, $a \in A$,
3) $\mu = \lambda f$.

2. The Group Der $(G, (A, \mu))$

Consider a crossed $G-R$-bimodule (A, μ).

Definition 2. Denote by $\mathrm{Der}(G, (A, \mu))$ the set of pairs (α, r) where α is a crossed homomorphism from G to A, i.e.,

$$\alpha(xy) = \alpha(x)^x \alpha(y), \quad x, y \in G,$$

and r is an element of R such that

$$\mu\alpha(x) = r\ {}^x r^{-1}, \quad x \in G.$$

This set will be called the set of derivations from G to (A, μ).

We define in $\mathrm{Der}(G, (A, \mu))$ a product by

$$(\alpha, r)(\beta, s) = (\alpha * \beta, rs),$$

where $(\alpha * \beta)(x) = {}^r\beta(x)\alpha(x)$, $x \in G$.

Proposition 3. *Under the aforementioned product* $\mathrm{Der}(G, (A, \mu))$ *becomes a group which coincides with the group* $\mathrm{Der}_G(G, A)$ *of Guin when* (A, μ) *is a crossed* G*-module viewed as a crossed* $G-G$*-bimodule.*

Proof. We have to show that $(\alpha * \beta, rs) \in \mathrm{Der}(G, (A, \mu))$. Put $\gamma = \alpha * \beta$. At first we prove that γ is a crossed homomorphism. In effect, we have

$$\begin{aligned}\gamma(xy) = {}^r\beta(xy)\alpha(xy) &= {}^r(\beta(x)^x\,\beta(x))\alpha(x)\,{}^x\alpha(y) = \\ &= {}^r\beta(x)^{rx}\beta(y)\alpha(x)^x\alpha(y).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(x)\,{}^x\gamma(y) &= {}^r\beta(x)\alpha(x)\,{}^x({}^r\beta(y)\alpha(y)) = \\ &= {}^r\beta(x)\alpha(x)\,{}^{xr}\beta(y)\,{}^x\alpha(y).\end{aligned}$$

For any $a \in A$ and $(\alpha, r) \in \mathrm{Der}(G, (A, \mu))$ the equality

$$\alpha(x)\,{}^{xr}a = {}^{rx}a\alpha(x), \quad x \in G, \tag{1}$$

holds, since $\alpha(x)\,{}^{xr}a\alpha(x)^{-1} = {}^{\mu\alpha(x)\cdot xr}a = {}^{r\cdot{}^x r^{-1}}({}^{xr}a) = {}^{rxr^{-1}\cdot x^{-1}\cdot xr}a = {}^{rx}a$.

It follows that $\gamma(xy) = \gamma(x)\,{}^x\gamma(y)$. Further,we have

$$\mu\gamma(x) = \mu({}^r\beta(x)\alpha(x)) = {}^r\mu\beta(x)\mu\alpha(x) =$$
$$= {}^r(s\,{}^x s^{-1})r\,{}^x r^{-1} = {}^r s\,{}^r({}^x s^{-1})r\,{}^x r^{-1} = {}^r s r^x s^{-1}\,{}^x r^{-1} =$$
$$= rs^x(rs)^{-1}.$$

Therefore $(\alpha * \beta, rs) \in \mathrm{Der}(G,(A,\mu))$.

It is evident that this product is associative. It is also obvious that $(\alpha_0, 1) \in \mathrm{Der}(G,(A,\mu))$, where $\alpha_0(x) = 1$ for all $x \in G$, and $(\alpha_0, 1)$ is the unit of $\mathrm{Der}(G,(A,\mu))$.

Now we will show that for $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ we have

$$ {}^{r^{-1}\cdot x}a\,{}^{r^{-1}}\alpha(x)^{-1} = {}^{r^{-1}}\alpha(x)^{-1}\,{}^{xr^{-1}}a, \quad x \in G, \;\; a \in A. \tag{2}$$

Since $\mu({}^{r^{-1}}\alpha(x)^{-1}) = r^{-1} \cdot \mu\alpha(x)^{-1} \cdot r = r^{-1}\,{}^x r$, this implies

$$ {}^{\mu({}^{r^{-1}}\alpha(x)^{-1})}({}^{xr^{-1}}a) = {}^{r^{-1}\,{}^x r}({}^{xr^{-1}}a) = {}^{r^{-1}xrx^{-1}xr^{-1}}\,a = {}^{r^{-1}x}\,a.$$

On the other hand,

$$ {}^{\mu({}^{r^{-1}}\alpha(x)^{-1})}({}^{xr^{-1}}a) = r^{-1}\alpha(x)\cdot{}^{xr^{-1}}a\cdot{}^{r^{-1}}\alpha(x)$$

and equality (2) is proved.

For $(\alpha, r) \in \mathrm{Der}(G,(A,\mu))$ take the pair $(\overline{\alpha}, r^{-1})$ where $\overline{\alpha}(x) = {}^{r^{-1}}\alpha(x)^{-1}$, $x \in G$. It will be shown that $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$. We have

$$\overline{\alpha}(xy) = {}^{r^{-1}}\alpha(xy)^{-1} = {}^{r^{-1}}({}^x\alpha(y)^{-1} \cdot \alpha(x)^{-1}) =$$
$$= {}^{r^{-1}x}\alpha(y)^{-1}\,{}^{r^{-1}}\alpha(x)^{-1}$$

and $\overline{\alpha}(x) \cdot^x \overline{\alpha}(y) = {}^{r^{-1}}\alpha(x)^{-1}\,{}^{xr^{-1}}\alpha(y)^{-1}$.

By (2) one gets $\overline{\alpha}(xy) = \overline{\alpha}(x)\,{}^x\overline{\alpha}(y)$, i.e., $\overline{\alpha}$ is a crossed homomorphism.

We also have

$$\mu\overline{\alpha}(x) = \mu({}^{r^{-1}}\alpha(x)^{-1}) = r^{-1}\mu\alpha(x)^{-1}r = r^{-1}\,{}^x r \cdot r^{-1} \cdot r = r^{-1}\,{}^x r.$$

Therefore $(\overline{\alpha}, r^{-1}) \in \mathrm{Der}(G,(A,\mu))$.

It is easy to check that

$$(\alpha, r)(\overline{\alpha}, r^{-1}) = (\overline{\alpha}, r^{-1})(\alpha, r) = (\alpha_0, 1).$$

We conclude that $\mathrm{Der}(G,(A,\mu))$ is a group. If (A,μ) is a crossed G-module and $(\alpha, g) \in \mathrm{Der}(G,(A,\mu))$ then $\mu\alpha(x) = g\,{}^x g^{-1} = gxg^{-1}x^{-1}$.

In $\mathrm{Der}_G(G,A)$ this product was defined by Guin [4] and it follows that the group $\mathrm{Der}(G,(A,\mu))$ coincides with $\mathrm{Der}_G(G,A)$ when (A,μ) is a crossed G-module. $\square$

If (A, μ) is a precrossed R-module and (B, λ) is a crossed R-module then (B, λ) is a crossed $A-R$-bimodule induced by μ and the group $\mathrm{Der}_G(A, B)$ of Guin [4] is the group $\mathrm{Der}(A, (B, \lambda))$.

It is clear that a homomorphism of $G-R$-bimodules $f : (A, \mu) \longrightarrow (B, \lambda)$ induces a homomorphism

$$f^* : \mathrm{Der}(G, (A, \mu)) \longrightarrow \mathrm{Der}(G, (B, \lambda))$$

given by $(\alpha, r) \longmapsto (f\alpha, r)$.

There is an action of G on $\mathrm{Der}(G, (A, \mu))$ defined by

$${}^g(\alpha, r) = (\widetilde{\alpha}, {}^g r), \quad g \in G, \ \ r \in R,$$

with $\widetilde{\alpha}(x) = {}^g\alpha({}^{g^{-1}}x)$, $x \in G$.

In effect,we have

$$\begin{aligned}\widetilde{\alpha}(xy) = {}^g\alpha({}^{g^{-1}}(xy)) = {}^g\alpha({}^{g^{-1}}x\,{}^{g^{-1}}y) = {}^g\alpha({}^{g^{-1}}x)\,{}^{xg}\alpha({}^{g^{-1}}y) = \\ = \widetilde{\alpha}(x)\,{}^x\widetilde{\alpha}(y)\end{aligned}$$

and $\mu\widetilde{\alpha}(x) = \mu({}^g\alpha({}^{g^{-1}}x)) = {}^g\mu\alpha({}^{g^{-1}}x) = {}^g(r\ {}^{({}^{g^{-1}}x)}r^{-1}) = {}^g r\ {}^{xg}r^{-1}$, whence $(\widetilde{\alpha}, {}^g r) \in \mathrm{Der}(G, (A, \mu))$. It is easy to verify that one gets an action of G on the group $\mathrm{Der}(G, (A, \mu))$. In effect,

$${}^g((\alpha, r)(\beta, s)) = {}^g (\alpha * \beta, rs) = (\widetilde{\alpha * \beta}, {}^g (rs)),$$

where $(\widetilde{\alpha \times \beta})(x) = {}^g(\alpha * \beta)({}^{g^{-1}}x) = {}^g({}^r\beta({}^{g^{-1}}x)) \cdot \alpha({}^{g^{-1}}x) = {}^{gr}\beta({}^{g^{-1}}x)) \cdot {}^g\alpha({}^{g^{-1}}x)$ and ${}^g(\alpha, r)\,{}^g(\beta, s) = (\widetilde{\alpha}, {}^g r)(\widetilde{\beta}, {}^g s) = (\widetilde{\alpha * \beta}, {}^g (rs))$ where $(\widetilde{\alpha} * \widetilde{\beta})(x) = {}^{{}^g r}({}^g\beta({}^{g^{-1}}x))\,{}^g\alpha({}^{g^{-1}}x) = {}^{grg^{-1}}({}^g\beta({}^{g^{-1}}x))\,{}^g\alpha({}^{g^{-1}}x) = {}^{gr}\beta({}^{g^{-1}}x)$ ${}^g\alpha({}^{g^{-1}}x)$.

Thus, ${}^g((\alpha, r)(\beta, s)) = {}^g(\alpha, r)\,{}^g(\beta, s)$ and it is clear that ${}^{gg'}(\alpha, r) = {}^g({}^{g'}(\alpha, r))$. This action on the group $\mathrm{Der}_G(A, B)$ is defined in [4].

Let (A, μ) be a crossed $G-R$-bimodule. If R acts on G and the compatibility condition

$${}^{({}^r g)}a = {}^{rgr^{-1}} a, \quad {}^{({}^r g)}r' = {}^{rgr^{-1}}r' \ \text{ for } \ r, r' \in R, \ \ g \in G, \ \ a \in A, \qquad (3)$$

holds, then there is also an action of R on $\mathrm{Der}(G, (A, \mu))$ given by

$${}^r(\alpha, s) = (\widetilde{\alpha}, {}^r s),$$

where $\widetilde{\alpha}(x) = {}^r\alpha({}^{r^{-1}}x)$, $x \in G$.

A calculation similar to the case of the action of G on $\mathrm{Der}(G, (A, \mu))$ shows that $(\widetilde{\alpha}, {}^r s)$ is an element of $\mathrm{Der}(G, (A, \mu))$.

Let G and R be groups acting on each other and on themselves by conjugation. It is known [8] that these actions are said to be compatible if

$$^{(^{g}r)}g' = {}^{grg^{-1}}g', \quad {}^{(^{r}g)}r' = {}^{rgr^{-1}}r'$$

for $g, g' \in G$ and $r, r' \in R$.

Definition 4. It will be said that the groups G and R act on a group A compatibly if

$$^{(^{g}r)}a = {}^{grg^{-1}}a, \quad {}^{(^{r}g)}a = {}^{rgr^{-1}}a$$

for $g \in G$, $r \in R$, $a \in A$.

Proposition 5. *Let (A, μ) be a crossed $G-R$-bimodule. Let the groups G and R act on each other and on A compatibly. Under the aforementioned actions of G and R on $\mathrm{Der}(G,(A,\mu))$ and the homomorphism γ : $\mathrm{Der}(G,(A,\mu)) \longrightarrow R$ given by $(\alpha, r) \longmapsto r$, the pair $(\mathrm{Der}(G,(A,\mu)),\gamma)$ is a precrossed $G-R$-bimodule.*

Proof. We have only to show that

$$^{(^{g}r)}(\alpha, s) = {}^{grg^{-1}}(\alpha, s),$$

for $g \in G$, $r \in R$.

In effect,

$$^{(^{g}r)}(\alpha, s) = (\beta, {}^{(^{g}r)}s),$$

where $\beta(x) = {}^{(^{g}r)}\alpha({}^{(^{g}r^{-1})}x) = {}^{grg^{-1}}\alpha({}^{gr^{-1}g^{-1}}x)$, $x \in G$.

On the other hand,

$$^{grg^{-1}}(\alpha, s) = (\gamma, {}^{grg^{-1}}s),$$

where $\gamma(x) = {}^{grg^{-1}}\alpha({}^{gr^{-1}g^{-1}}x)$ and

$$^{grg^{-1}}s = {}^{g}(r^{g^{-1}}sr^{-1}) = {}^{g}rs\,{}^{g}r^{-1} = {}^{(^{g}r)}s.$$

Therefore $^{(^{g}r)}(\alpha, s) = {}^{grg^{-1}}(\alpha, s)$. $\square$

3. The Pointed Set $H^2(G, A)$

We will use the group of derivations in a crossed bimodule to define $H^2(G, A)$ when A is a crossed G-module.

We start by the following characterization of $H^2(G, A)$ when A is a $\mathbf{Z}[G]$-module.

Consider the diagram

$$M \underset{l_1}{\overset{l_0}{\rightrightarrows}} F \xrightarrow{\tau} G \tag{4}$$

where F is a free group, τ is a surjective homomorphism, M is the set of pairs (x, y), $x, y \in F$, such that $\tau(x) = \tau(y)$ and l_0, l_1 are canonical projections, $l_0(x, y) = x$, $l_1(x, y) = y$. Thus, (M, l_0, l_1) is the simplicial kernel of τ. Put $\Delta = \{(x, x),\, x \in F\} \subset M$.

Let f be a map from an arbitrary group C to a group D. Then in what follows by $f^{-1} : C \longrightarrow D$ always will be denoted a map with $f^{-1}(c) = f(c)^{-1}$, $c \in C$.

Let A be a $\mathbf{Z}[G]$-module. It is clear that A is a M-module via τl_0 and a F-module via τ. Denote by $Z^1(M, A)$ (resp. $Z^1(F, A)$) the abelian group of crossed homomorphims from M to A (resp. from F to A). Let $\widetilde{Z^1}(M, A)$ be a subgroup of $Z^1(M, A)$ consisting of all elements α such that $\alpha(\Delta) = 1$. There is a homomorphism

$$\kappa : Z^1(F, A) \longrightarrow \widetilde{Z^1}(M, A)$$

defined by $\beta \longmapsto \beta l_0 \beta l_1^{-1}$.

Proposition 6. *$H^2(G, A)$ is canonically isomorphic to* Coker κ.

Proof. It is sufficient to show that Coker κ is isomorphic to $Opext(G, A, \varphi)$ where $\varphi : G \longrightarrow Aut(A)$ denotes the action of G on A.

Let $\alpha \in \widetilde{Z^1}(M, A)$ and introduce in the semi-direct product $A \bowtie F$ the relation

$$(a, x) \sim (a', x') \Longleftrightarrow \tau(x) = \tau(x')$$

and $a \cdot \alpha(x, x') = a'$.

It is easy to see that this relation is an equivalence; use the fact that if (x, x', x'') is a triple of elements of F such that $\tau(x) = \tau(x') = \tau(x'')$ then $\alpha(x, x'') = \alpha(x, x')\alpha(x', x'')$. Denote this equivalence by ρ and take the quotient set $(A \bowtie F)/\rho$. We will show that ρ is in fact a congruence and therefore $C = (A \bowtie F)/\rho$ is a group.

Let $(a, x) \sim (a', x')$ and $(b, y) \sim (b', y')$. Then $\tau(x) = \tau(x')$, $\tau(y) = \tau(y')$, $a\alpha(x, x') = a'$, $b\alpha(y, y') = b'$.

Further, $(a, x)(b, y) = (a\, {}^{x}b, xy)$, $(a', x')(b', y') = (a'\, {}^{x'}b', x'y')$.

We have

$$^{x}b\,^{x}\alpha(y,y') = {}^{x}b' = {}^{x'}b',$$

whence $a\cdot\alpha(x,x')\,^{x}b\,^{x}\alpha(y,y') = a'\,^{x'}b'$. Since $\alpha(xy,x'y') = \alpha(x,x')\,^{x}\alpha(y,y')$, it follows that

$$a\,^{x}b\alpha(xy,x'y') = a'\,^{x'}b'\ .$$

One gets a commutative diagram

$$\begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G \\ \downarrow\alpha & & \downarrow\beta & & \| \\ A & \xrightarrow{\sigma} & C & \xrightarrow{\psi} & G \end{array}$$

where $\sigma(a) = [(a,1)]$, $\psi[(a,x)] = \tau(x)$, $\beta(x) = [(1,x)]$. Denote by E the exact sequence

$$0 \longrightarrow A \xrightarrow{\sigma} C \xrightarrow{\psi} G \longrightarrow 1$$

which gives an element of $Opext(G,A,\varphi)$.

Define a map

$$\vartheta : \operatorname{Coker}\kappa \longrightarrow Opext(g,A,\varphi)$$

given by $[\alpha] \longmapsto [E]$.

By standard calculations it can be easily proved that ϑ is a correctly defined homomorphism which is bijective. $\square$

Let (A,μ) be a crossed G-module. Then (A,μ) is a crossed $M-G$-bimodule induced by τl_0 (or by τl_1) and a crossed $F-G$-bimodule induced by τ (see diagram (4)).

Consider the group $\operatorname{Der}(M,(A,\mu))$ and let $\widetilde{\operatorname{Der}}(M,(A,\mu))$ be the subgroup of $\operatorname{Der}(M,(A,\mu))$ consisting of elements (α,g) such that $\alpha(\Delta) = 1$. If $(\alpha,g) \in \widetilde{\operatorname{Der}}(M,(A,\mu))$ this implies $g \in Z(G)$. Then we have $\mu\alpha(m) = 1$ for any $m \in M$ and $\alpha(M) \subset Z(A)$. Denote by $\widetilde{Z^1}(M,(A,\mu))$ a subset of $\widetilde{\operatorname{Der}}(M,(A,\mu))$ consisting of all elements of the form $(\alpha,1)$.

Define, on the set $\widetilde{Z^1}(M,(A,\mu))$, a relation

$$(\alpha',1) \sim (\alpha,1) \Leftrightarrow \exists(\beta,h) \in \operatorname{Der}(F,(A,\mu))$$

such that

$$(\alpha',1) = (\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$$

in the group $\operatorname{Der}(M,(A,\mu))$.

We see that if $(\alpha',1) \sim (\alpha,1)$ one has

$$\alpha'(x) = \beta l_1(x)^{-1}\,{}^{h}\alpha(x)\beta l_0(x), \quad x \in M,$$

for some $(\beta,h) \in \operatorname{Der}(F,(A,\mu))$.

Proposition 7. *The relation $\sim$ defined on $\widetilde{Z^1}(M,(A,\mu))$ is an equivalence.*

Proof. The reflexivity is clear. If $(\alpha',1)\sim(\alpha,1)$, i.e., $(\alpha',1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$ where $(\beta,h)\in \mathrm{Der}(F,(A,\mu))$, then $(\alpha,1)=(\beta l_0,h)^{-1}(\alpha',1)(\beta l_1,h)$ where $(\beta l_0,h)^{-1}=(\widetilde{\beta}l_0,h^{-1})$ and $(\beta l_1,h)=(\widetilde{\beta}l_1,h^{-1})^{-1}$ with $(\widetilde{\beta},h^{-1})=(\beta,h)^{-1}\in \mathrm{Der}(F,(A,\mu))$. Thus the relation $\sim$ is symmetric.

Let $(\alpha',1)\sim(\alpha,1)$ and $(\alpha'',1)\sim(\alpha',1)$; then one has

$$(\alpha',1)=(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1},$$
$$(\alpha'',1)=(\beta' l_0,h')(\alpha',1)(\beta' l_1,h')^{-1},$$

where $(\beta,h),(\beta',h')\in \mathrm{Der}(F,(A,\mu))$.

It follows that

$$\begin{aligned}(\alpha'',1)&=(\beta' l_0,h')(\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}(\beta' l_1,h')^{-1}=\\&=((\beta'*\beta)l_0,h'h)(\alpha,1)((\beta'*\beta)l_1,h'h)^{-1},\end{aligned}$$

where $(\beta'*\beta,h'h)=(\beta',h')(\beta,h)\in \mathrm{Der}(F,(A,\mu))$. This means that $(\alpha'',1)\sim(\alpha,1)$ and the relation $\sim$ is an equivalence. □

Proposition 8. *Let (A,μ) be a crossed G-module. Then the quotient set $\widetilde{Z^1}(M,(A,\mu))/\sim$ is independent of the diagram* (4) *and is unique up to bijection.*

We need the

Lemma 9. *Let A be a G-group and let $\alpha: M\longrightarrow A$ be a crossed homomorphism such that $\alpha(\Delta)=1$. Then there exists a map $q: F\longrightarrow A$ such that*

$$\alpha(y)=ql_1(y)^{-1}ql_0(y),\quad y\in M.$$

Proof. Observe that if $(x,x''),(x',x'')\in M$, then $\alpha(x,x'')=\alpha(x',x'')\,\alpha(x,x')$. In effect,the equality $(x,x'')=(1,x''x'^{-1})(x,x')$ implies $\alpha(x,x'')=\alpha(1,x''x'^{-1})\ \alpha(x,x')$. But $(x',x'')=(1,x''x'^{-1})(x',x')$. Thus $\alpha(x',x'')=\alpha(1,x''x'^{-1})\ \alpha(x',x')=\alpha(1,x''x'^{-1})$ and we get the desired equality.

In particular, applying this equality one gets $\alpha(x,x)=\alpha(x',x)\cdot\alpha(x,x')$ for $(x,x),(x',x)\in M$. Therefore $\alpha(x',x)=\alpha(x,x')^{-1}$ for any $(x,x')\in M$.

Take a section $\eta: G\longrightarrow F$, $\tau\eta=1_G$ and define a map $q: F\longrightarrow A$ by

$$q(x)=\alpha(x,\eta\tau(x)),\quad x\in F.$$

For $(x,x') \in M$ one has

$$ql_1(x,x')^{-1}ql_0(x,x') = q(x')^{-1}q(x) = (\alpha(x',\eta\tau(x')))^{-1}\alpha(x,\eta\tau(x)) = \\ = \alpha(\eta\tau(x'),x')\alpha(x,\eta\tau(x)).$$

On the other hand, since $\alpha(x,x') = \alpha(1,x'x^{-1})$ for all $(x,x') \in M$, one has $\alpha(\eta\tau(x'),x') = \alpha(1,x'\eta\tau(x')^{-1})$ and $\alpha(x,\eta\tau(x)) = \alpha(1,\eta\tau(x)x^{-1})$.

But $(1,x'\eta\tau(x')^{-1})(1,\eta\tau(x)x^{-1}) = (1,x'x^{-1})$. Therefore, $\alpha(x,x') = \alpha(1,x'\eta\tau(x')^{-1})\alpha(1,\eta\tau(x)x^{-1}) = ql_1(x,x')^{-1}ql_0(x,x')$. □

Proof of Proposition 8. Consider a commutative diagram

$$\begin{array}{ccccc}
M' & \overset{l_0'}{\underset{l_1'}{\rightrightarrows}} & F' & \xrightarrow{\tau'} & G \\
\overline{\gamma_1}\downarrow\overline{\gamma_2} & & \gamma_1\downarrow\gamma_2 & & \| \\
M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G
\end{array}$$

where (M,l_0,l_1) and (M',l_0',l_1') are the simplicial kernels of $\tau_1$and τ_2 respectively, $l_i\overline{\gamma_1} = \gamma_1 l_i'$, $l_i\overline{\gamma_2} = \gamma_2 l_i'$, $i=0,1$, $\tau\gamma_1 = \tau\gamma_2 = \tau'$.

The pair $(\gamma_i,\overline{\gamma_i})$ induces a homomorphism

$$\mathrm{Der}(M,(A,\mu)) \longrightarrow \mathrm{Der}(M',(A,\mu))$$

given by $(\alpha,g) \longmapsto (\alpha\overline{\gamma_i},g)$, $i=1,2$.

If $(\alpha',1) \sim (\alpha,1)$, i.e.,

$$(\alpha',1) = (\beta l_0,h)(\alpha,1)(\beta l_1,h)^{-1}$$

with $(\beta,h) \in \mathrm{Der}(F,(A,\mu))$, then

$$\alpha'\overline{\gamma_i}(y) = \beta\gamma_i l_1'(y)^{-1\,h}\alpha\overline{\gamma_i}(y)\beta\gamma_i l_0'(y), \quad y \in M'.$$

Thus $(\alpha'\overline{\gamma_i},1) \sim (\alpha\overline{\gamma_i},1)$, $i=1,2$, and one gets a natural map

$$\epsilon_i : \widetilde{Z^1}(M,(A,\mu))/\sim \longrightarrow \widetilde{Z^1}(M',(A,\mu))/\sim$$

induced by the pair $(\gamma_i,\overline{\gamma_i})$ and given by $[(\alpha,1)] \longmapsto [(\alpha\overline{\gamma_i},1)]$, $i=1,2$.

We will show that $\epsilon_1 = \epsilon_2$. By Lemma 9 there is a map $q : F \longrightarrow A$ such that

$$\alpha(y) = ql_1(y)^{-1}ql_0(y), \quad y \in M.$$

Consider the homomorphism $s : F' \longrightarrow M$ given by

$$s(x') = (\gamma_1(x'),\gamma_2(x')), \quad x' \in F'.$$

It is clear that $(\alpha s,1) \in \mathrm{Der}(F',(A,\mu))$.

Further we have

$$((\alpha s l_1)^{-1}\alpha\overline{\gamma_2}\alpha s l_0')(x_0',x_1') = \alpha s(x_1')^{-1}\ \alpha\overline{\gamma_2}(x_0',x_1')\alpha s(x_0') =$$
$$= \alpha(\gamma_1(x_1'),\gamma_2(x_1'))^{-1}\alpha\overline{\gamma_2}(x_0',x_1')\ \alpha(\gamma_1(x_0'),\gamma_2(x_0')) = q\gamma_1(x_1')^{-1}q\gamma_2(x_1') =$$
$$= q\gamma_2(x_1')^{-1}q\gamma_2(x_0')q\gamma_2(x_0')^{-1}q\gamma_1(x_0') = q\gamma_1(x_1')^{-1}q\gamma_1(x_0') = \alpha\overline{\gamma_1}(x_0',x_1')$$

for $(x_0',x_1') \in M'$.

Therefore $(\alpha\overline{\gamma_1},1) \sim (\alpha\overline{\gamma_2},1)$ with $(\alpha s,1) \in \mathrm{Der}(F',(A,\mu))$ and one gets $\epsilon_1 = \epsilon_2$.

The rest of the proof of the uniqueness is standard. □

Let (A,μ) be a crossed $G-R$-bimodule. Denote by $I\,\mathrm{Der}(G,(A,\mu))$ a subgroup of $\mathrm{Der}(G,(A,\mu))$ consisting of elements of the form (α,r) with $r \in H^0(G,R)$. If (A,μ) is a crossed G-module viewed as a crossed $G-G$-bimodule then

$$I\,\mathrm{Der}(G,(A,\mu)) = \{(\alpha,r), g \in Z(G)\}.$$

Consider the diagram

$$M_G \overset{l_0}{\underset{l_1}{\rightrightarrows}} F_G \xrightarrow{\tau_G} G \tag{5}$$

where F_G is the free group generated by G, τ_G is the canonical homomorphism and (M_G,l_0,l_1) is the simplicial kernel of τ_G.

Proposition 10. *Let (A,μ) be a crossed G-module. Then:*

(i) *there is a canonical surjective map*

$$\vartheta' : H^2(G,Ker\mu) \longrightarrow \widetilde{Z^1}(M_G,(A,\mu))/\sim$$

given by the composite map $[E] \overset{\vartheta^{-1}}{\longmapsto} [\alpha] \longmapsto [(\alpha,1)]$;

(ii) *if we assume* $\mathrm{Der}(F_G,(A,\mu)) = I\,\mathrm{Der}(F_G,(A,\mu))$ (*in particular, it is so if either μ is the trivial map or G is abelian*) *we can introduce, in the pointed set $\widetilde{Z^1}(M_G,(A,\mu))/\sim$, an abelian group structure defined by*

$$[(\alpha,1)][(\beta,1)] = [(\alpha*\beta,1)]$$

where (A,μ) is viewed as a crossed F_G-G-bimodule induced by τ_G and $[(\alpha,1)]$ denotes the equivalence class containing $(\alpha,1)$. Under this product the map ϑ' becomes an isomorphism.

Proof. To prove (i) we have only to show the correctness of $[\alpha] \longmapsto [(\alpha,1)]$ where $\alpha : M_G \longrightarrow A$ is a crossed homomorphism with $\alpha(\Delta) = 1$ and $\alpha(M_G) \subset \mathrm{Ker}\,\mu$.

Let $\alpha^{'} \in [\alpha]$, i.e.,

$$\alpha^{'}(x) = \beta l_1^{-1}(x)\alpha(x)\beta l_0(x), x \in M_G,$$

where $\beta : F_G \longrightarrow Ker\ \mu$ is a crossed homomorphism. Then $(\beta, 1) \in \mathrm{Der}(F_G, (A, \mu))$ and we have

$$(\alpha^{'}, 1) = (\beta l_0, 1)(\alpha, 1)(\beta l_1, 1)^{-1}.$$

The surjectivity of $\vartheta^{'}$ is clear.

(ii) Let $(\alpha^{'}, 1) \sim (\alpha, 1)$. Then

$$\alpha^{'}(x) = \eta l_1(x)^{-1\,g}\alpha(x)\eta l_0(x), \quad x \in M_G,$$

for some $(\eta, g) \in \mathrm{Der}(F_G, (A, \mu))$. By assumption, $g \in Z(G)$. Thus $\mu\eta(x) = gxg^{-1}x^{-1} = 1$, $x \in M_G$. It follows that $[\alpha^{'}] = [{}^g\alpha]$ with $g \in Z(G)$. But it is known that $Z(G)$ acts trivially on $H^2(G, Ker\ \mu)$. Therefore we have $[\alpha^{'}] = [\alpha]$. Hence there is a crossed homomorphism $\gamma : F_G \longrightarrow Ker\ \mu$ such that

$$\alpha^{'}(x) = \gamma l_1^{-1}(x)\alpha(x)\gamma l_0(x), \quad x \in M_G.$$

It follows that if $(\alpha, 1) \sim (\alpha^{'}, 1)$ and $(\beta, 1) \sim (\beta^{'}, 1)$ then $(\alpha * \beta, 1) \sim (\alpha^{'} * \beta^{'}, 1)$. We conclude that the product is correctly defined and the map $\vartheta^{'}$ is an isomorphism when $\mathrm{Der}(F_G, (A, \mu)) = I\,\mathrm{Der}(F_G, (A, \mu))$. $\square$

Proposition10 motivates the following definition of the second cohomology of groups with coefficients in crossed modules.

Definition 11. Let (A, μ) be a crossed G-module. One denotes by $H^2(G, A)$ the quotient set $\widetilde{Z^1}(M, (A, \mu))\ /\ \sim$ which will be called the second set of cohomology of G with coefficients in the crossed G-module (A, μ).

Remark. Using diagram (4) it is possible to define the second cohomology of G with coefficients in a crossed G-module (A, μ) by a different "less abelian" way. Consider the set $\widetilde{Z^1}(M, A)$ of all crossed homomorphims $\alpha : M \longrightarrow A$ with $\alpha(\Delta) = 1$ (the equality $\mu\alpha = 1$ is not required and therefore $\alpha(M)$ is not necessarily contained in $Z(A)$). Introduce in $\widetilde{Z^1}(M, A)$ a relation $\sim$ of equivalence as follows:

$\alpha^{'} \sim \alpha$ if $\exists\ (\beta, h)\,\mathrm{Der}(F, (A, \mu))$ such that $\alpha^{'}(x) = \beta l_1(x)^{-1}\ {}^h\alpha(x)\ \beta l_0(x)$, $x \in M$.

Define $\overline{H^2}(G, A) = \widetilde{Z^1}(M, A)\ /\ \sim$. It is obvious that $H^2(G, A) \subset \overline{H^2}(G, A)$. But it seems the exact cohomology sequence (Theorem 13) does not hold for $\overline{H^2}(G, A)$.

It is clear $H^2(G, A)$ is a pointed set with $[(\alpha_0, 1)]$ as a distinguished element where $\alpha_0(y) = 1$ for all $y \in M$.

A homomorphism of crossed G-modules $f : (A, \mu) \longrightarrow (B, \lambda)$ induces a map of pointed sets

$$f^2 : H^2(G, A) \longrightarrow H^2(G, B), \quad f^2([(\alpha, 1)]) = [(f\alpha, 1)].$$

There is an action of G on F_G (see diagram (5)) defined as follows:

$$^g(\mid g_1 \mid^\epsilon \cdots \mid g_n \mid^\epsilon) = \mid^g g_1 \mid^\epsilon \cdots \mid^g g_n \mid^\epsilon, \quad g, g_1, \dots, g_n \in G,$$

where $\epsilon = \pm 1$.

This action induces an action of G on M_G by

$$^g(x, x^{'}) = (^g x, ^g x^{'}), \quad g \in G, \quad (x, x^{'}) \in M_G.$$

Let (A, μ) be a crossed G-module. Then we have an action of G on $\operatorname{Der}(M_G, (A, \mu))$ given by

$$^g(\alpha, h) = (\widetilde{\alpha}, ^g h)$$

where $\widetilde{\alpha}(m) = {}^g\alpha(^{g^{-1}}m)$, $g \in G$, $m \in M_G$.

Proposition 12. *Let (A, μ) be a crossed G-module. There is an action of G on $H^2(G,A)$ induced by the aforedefined action of G on $\operatorname{Der}(M_G, (A,\mu))$ under which the center $Z(G)$ acts trivially.*

Proof. Obviously, the action of G on $\operatorname{Der}(M_G, (A, \mu))$ induces an action of G on $\widetilde{\operatorname{Der}}(M_G, (A, \mu))$. Thus one gets an action of G on $\widetilde{Z^1}(M_G, (A, \mu))$.

If $(\alpha^{'}, 1) \sim (\alpha, 1)$, where $(\alpha, 1)$, $(\alpha^{'}, 1) \in \widetilde{Z^1}(M_G, (A, \mu))$, we have

$$\alpha^{'}(y) = \beta l_1(y)^{-1}\, {}^h\alpha(y) \beta l_0(y), \quad y \in M_G,$$

for some $(\beta, h) \in \operatorname{Der}(F_G, (A, \mu))$.

This implies

$$^g\alpha^{'}(^{g^{-1}}y) = {}^g\beta l_1(^{g^{-1}}y)^{-1}\, {}^{gh}\alpha(^{g^{-1}}y)\, {}^g\beta l_0(^{g^{-1}}y), \quad y \in M_G.$$

Hence $\widetilde{\alpha^{'}}(y) = {}^g\beta(^{g^{-1}}l_1(y))^{-1}\, {}^{ghg^{-1}}\widetilde{\alpha}(^{g^{-1}}y)\, {}^g\beta(^{g^{-1}}l_0(y))$ with $(\widetilde{\beta}, ^g h) \in \operatorname{Der}(F_G, (A, \mu))$. Therefore $(\widetilde{\alpha^1}, 1) \sim (\widetilde{\alpha}, 1)$. It is obvious that the above-defined map $\vartheta^{'} : H^2(G, \ker \mu) \longrightarrow H^2(G, A)$ is a G-map. Since $\vartheta^{'}$ is surjective (see Proposition 11) and $Z(G)$ acts trivially on $H^2(G, \ker \mu)$, it follows that $Z(G)$ acts trivially on $H^2(G, A)$ too. $\square$

4. An Exact Cohomology Sequence

For any G-group A denote by $H^0(G,A)$ a subgroup of A consisting of all invariant elements under the action of G on A.

Theorem 13. *Let*

$$1 \longrightarrow (A,1) \xrightarrow{\varphi} (B,\mu) \xrightarrow{\psi} (C,\lambda) \longrightarrow 1 \tag{6}$$

be an exact sequence of crossed G-modules. Then there is an exact sequence

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\varphi^1}$$
$$\xrightarrow{\varphi^1} H^1(G,B) \xrightarrow{\psi^1} H^1(G,C) \xrightarrow{\delta^1} H^2(G,A) \xrightarrow{\varphi^2} H^2(G,B) \xrightarrow{\psi^2} H^2(G,C)$$

where φ^0, ψ^0, δ^0, φ^1, ψ^1 are group homomorphisms, δ^1 is a crossed homomorphism under the action of $H^1(G,C)$ on $H^2(G,A)$ induced by the action of G on A and φ^2, ψ^2 are maps of pointed G-sets.

Proof. The exactness of

$$1 \longrightarrow H^0(G,A) \xrightarrow{\varphi^0} H^0(G,B) \xrightarrow{\psi^0} H^0(G,C) \xrightarrow{\delta^0} H^1(G,A) \xrightarrow{\varphi^1}$$
$$\xrightarrow{\varphi^1} H^1(G,B) \xrightarrow{\psi^1} H^1(G,C) \xrightarrow{\delta^1} H^2(G,A)$$

is proved in [4].

We have only to show the exactness of

$$H^1(G,C) \xrightarrow{\delta^1} H^2(G,A) \xrightarrow{\varphi^2} H^2(G,B) \xrightarrow{\psi^2} H^2(G,C).$$

Let $[(\alpha,g)] \in H^1(G,C)$ and $\delta^1[(\alpha,g)] = [\gamma]$. Then one has a commutative diagram

$$\begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\rightrightarrows}} & F & \xrightarrow{\tau} & G \\ \downarrow\gamma & & \downarrow\beta & & \downarrow\alpha \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array}$$

where $\varphi\gamma(y) = \beta l_1(y)^{-1}\beta l_0(y)$, $y \in M$, and β is a crossed homomorphism, B being a F-group via τ. The existence of such β follows from the following assertion: if we have a surjective homomorphism $\psi : B \longrightarrow C$ of F-groups and $f : F \longrightarrow C$ is a crossed homomorphism where F is a free group, then there is a crossed homomorphism $\beta : F \longrightarrow B$ such that $\psi\beta = f$. In effect, take the semi-direct product $B \rtimes F$ and consider a subgroup Y of $B \rtimes F$

consisting of all elements (b, x) such that $\psi(b) = f(x)$. Then we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{pr_2} & F \\ \downarrow pr_1 & & \downarrow f \\ B & \xrightarrow{\psi} & C \end{array}$$

where pr_i is the projection, $i = 1, 2$, and pr_2 is surjective. Thus, since F is a free group, there is a homomorphism $f' : F \longrightarrow Y$ such that $ff' = 1_F$. Then $pr_1 f'$ is the required crossed homomorphism.

It will be shown that $(\beta, g) \in \operatorname{Der}(F, (B, \mu))$. One has

$$g\tau(x)g^{-1}\tau(x)^{-1} = \lambda\alpha\tau(x) = \lambda\psi\beta(x) = \mu\beta(x), \quad x \in F.$$

This means $(\beta, g) \in \operatorname{Der}(F, (B, \mu))$. It is clear that $\varphi^2\delta^1([\alpha, g]) = \varphi^2([\gamma]) = [(\varphi\gamma, 1)]$ and $(\varphi\gamma, 1) \sim (\alpha_0, 1)$ (use $(\beta, g) \in \operatorname{Der}(F, (B, \mu))$), where $\alpha_0(x) = 1$ for all $x \in F$. Therefore $\operatorname{Im}\delta^1 \subset \ker \varphi^2$.

Let $[\gamma] \in H^2(G, A)$ such that $\varphi^2([\gamma]) = [(\varphi\gamma, 1)] = [(\alpha_0, 1)]$. Then there exists $(\beta, h) \in \operatorname{Der}(F, (B, \mu))$ such that

$$\varphi\gamma(y) = \beta l_1(y)^{-1} \cdot \beta l_0(y), \quad y \in M,$$

whence $\psi\beta l_0(y) = \psi\beta l_1(y)$, $y \in M$. It follows that there is a crossed homomorphism $\alpha : G \longrightarrow C$ such that $\alpha\tau = \psi\beta$. We have to show $(\alpha, h) \in \operatorname{Der}_G(G, C)$. In effect, $h\ \tau(x)\ h^{-1}\ \tau(x)^{-1} = \mu\beta(x) = \lambda\psi\beta(x) = \lambda\alpha\tau(x)$, $x \in F$. This implies $(\alpha, h) \in \operatorname{Der}_G(G, C)$. It is clear that $\delta^1([\alpha, h]) = [\gamma]$. Therefore, $\ker\varphi^2 \subset \operatorname{Im}\delta^1$.

It is obvious that $\operatorname{Im}\varphi^2 \subset \ker\psi^2$.

Let $[(\alpha, 1)] \in H^2(G, B)$ such that $\psi^2([\alpha, 1]) = [(\psi\alpha, 1)] = [(\alpha_0, 1)]$. Then there exists $(\beta, h) \in \operatorname{Der}(F, (C, \lambda))$ such that

$$\psi\alpha(y) = \beta l_1(y)^{-1}\beta l_0(y), \quad y \in M.$$

It follows that there is a crossed homomorphism $\beta' : F \longrightarrow B$ such that $\psi\beta' = \beta$. One gets the following commutative diagram:

$$\begin{array}{ccccc} M & \overset{l_0}{\underset{l_1}{\longrightarrow}} & F & \xrightarrow{\tau} & G \\ & \searrow\alpha & \downarrow\beta' & \searrow\beta & \\ A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C \end{array}.$$

Thus, $\psi\alpha(y) = \psi\beta' l_1(y)^{-1}\ \psi\beta' l_0(y)$, $y \in M$. So that $\psi(\beta' l_1(y)\alpha(y)\,\beta' l_0(y)^{-1}) = 1$, $y \in M$. One gets $\beta' l_1(y)\alpha(y)\beta' l_0(y)^{-1} \in \varphi(A)$, $y \in M$. Denote by $\gamma : M \longrightarrow A$ the crossed homomorphism given by $\gamma(y) = \varphi^{-1}(\beta' l_1(y)\alpha(y)\beta' l_0(y)^{-1})$. Then ${}^{h^{-1}}\gamma(y) = \varphi^{-1}({}^{h^{-1}}\beta' l_1(y)^{h^{-1}}\alpha(y) \cdot {}^{h^{-1}}\ \beta' l_0(y)^{-1})$, $y \in M$.

In the group $\operatorname{Der}(F, (B, \mu))$ consider the product

$$(\beta' l_0, h)^{-1}(\alpha, 1)(\beta' l_1, h) = (\eta, 1)$$

where $\eta(y) = {}^{h^{-1}}\beta' l_1(y)\, {}^{h^{-1}}\alpha(y)\, {}^{h^{-1}}\beta' l_0(y)^{-1}$, $y \in M$.

This implies that the map given by

$$y \longmapsto {}^{h^{-1}} \gamma(y), \quad y \in M,$$

is a crossed homomorphism.

Take $[\varphi^{-1}\eta] \in H^2(G, A)$. Then we have $\varphi^2([\varphi^{-1}\eta]) = [(\eta, 1)]$. But $(\eta, 1) \sim (\alpha, 1)$ by the above equality with $(\beta', h)^{-1} \in \mathrm{Der}(F, (B, \mu))$. Therefore, $\ker \psi^2 \subset \mathrm{Im}\, \varphi^2$. $\square$

Any crossed G-module (A, μ) induces the following short exact sequence of crossed G-modules:

$$1 \longrightarrow (\ker \mu, 1) \xrightarrow{\varphi} (A, \mu) \xrightarrow{\psi} (\mathrm{Im}\, \mu, \sigma) \longrightarrow 1$$

where $\sigma : \mathrm{Im}\, \mu \longrightarrow G$ denotes the inclusion and G acts on $\mathrm{Im}\, \mu$ by conjugation.

Corollary 14. *If (A, μ) is a crossed G-module there is an exact sequence*

$$1 \longrightarrow H^0(G, \ker \mu) \xrightarrow{\varphi^0} H^0(G, A) \xrightarrow{\psi^0} H^0(G, \mathrm{Im}\, \mu) \xrightarrow{\delta^0} H^1(G, \ker \mu) \xrightarrow{\varphi^1}$$
$$\xrightarrow{\varphi^1} H^1(G, A) \xrightarrow{\psi^1} H^1(G, \mathrm{Im}\, \mu) \xrightarrow{\delta^1} H^2(G, \ker \mu) \xrightarrow{\vartheta'} H^2(G, A)$$
$$\longrightarrow 1.$$

For the exact sequence (6) a connecting map

$$\delta^2 : H^2(G, C) \longrightarrow H^3(G, A)$$

will be defined, and for this we will use the equivalence of functors $H^{n+1}(-, A) \approx L_n \mathrm{Der}(-, A)$, $n \geq 1$ [9] when A is a $\mathbf{Z}[G]$-module where $L_n \mathrm{Der}(-, A)$ is the non-abelian n-th derived functor of the contravariant functor $\mathrm{Der}(-, A)$ from the category $\underline{D}_G$ of groups acting on A to the category of abelian groups.

Consider the following canonical free simplicial resolution of G in the category $\underline{D}_G$:

$$\begin{array}{c} \longrightarrow \\ \vdots \\ \longrightarrow \end{array} F_3 \xrightarrow{\tau_3} M_2 \begin{array}{c} \xrightarrow{l_0^2} \\ \vdots \\ \xrightarrow[l_3^2]{} \end{array} F_2 \xrightarrow{\tau_2} M_1 \begin{array}{c} \xrightarrow{l_0^1} \\ \longrightarrow \\ \xrightarrow[l_2^1]{} \end{array} F_1 \xrightarrow{\tau_1} M_0 \begin{array}{c} \xrightarrow{l_0^0} \\ \xrightarrow[l_1^0]{} \end{array} F_0 \xrightarrow{\tau_0} G \tag{7}$$

where $F_0 = F_G$, $F_i = F_{M_{i-1}}$, $i \geq 1$, τ_i is the canonical homomorphism and $(M_i, l_0^i, \ldots, l_{i+1}^i)$ is the simplicial kernel of $(l_0^{i-1}\tau_i, \ldots, l_i^{i-1}\tau_i)$, $i \geq 0$ (see [10]).

There is an action of $\mathrm{Der}(F_0, (C, \lambda))$ on $H^3(G, A)$ defined as follows.

Let $[f] \in H^3(G,A)$ where $f : F_2 \longrightarrow A$ is a crossed homomorphism such that $\prod_{i=0}^{3} (fl_i^1\tau_3)^\epsilon = 1$ where $\epsilon = (-1)^i$ and let $(\alpha, g) \in \mathrm{Der}(F_0, (C,\lambda))$.

Define

$$^{(\alpha,g)}[f] = [^g f].$$

We have first to show that $^g f$ is a crossed homomorphism.

In the group $\mathrm{Der}(F_2,(B,\mu))$ ((B,μ) being a crossed F_2-G-bimodule induced by $\tau_0\partial_0^1\partial_0^2$ where $\partial_0^1 = l_0^0\tau_1$, $\partial_0^2 = l_0^1\tau_2$) take the product

$$(\beta\partial_0^1\partial_0^2, g)(\varphi f, 1)(\beta\partial_0^1\partial_0^2, g)^{-1} = (\widetilde{f}, 1),$$

where $\widetilde{f}$is a crossed homomorphism and $\beta : F_0 \longrightarrow B$ is a crossed homomorphism too such that $\psi\beta = \alpha$ (such β exists, since F_0 is a free group and ψ is surjective).

One has

$$\widetilde{f}(x) = \beta\partial_0^1\partial_0^2(x)^{-1\,g}\varphi f(x)\beta\partial_0^1\partial_0^2(x) =^g \varphi f(x), \quad x \in F_2.$$

We show now the correctness. If $f \sim f'$ then there is a crossed homomorphism $\eta : F_1 \longrightarrow A$ such that

$$f'(x) = f(x)\prod_{i=0}^{2} (\eta l_i^1\tau_2(x))^\epsilon, \quad x \in F_2,$$

where $\epsilon = (-1)^i$. It can be shown in the same manner as for f that $^g\eta$ is a crossed homomorphism if $(\alpha, g) \in \mathrm{Der}(F_0, (C,\lambda))$.

Thus one gets

$$^g f' = {}^g f \prod_{i=0}^{2} {}^g(\eta l_i^1\tau_2)^\epsilon,$$

where $\epsilon = (-1)^i$. This implies

$$[^g f'] = [^g f].$$

Therefore the action of $\mathrm{Der}(F_0, (C,\lambda))$ on $H^3(G,A)$ is correctly defined.

Let $(\alpha, 1) \in \widetilde{\mathrm{Der}}(M_0, (C,\lambda))$ and let $\beta : F_1 \longrightarrow B$ be a crossed homomorphism (F_1 acts on B via $\tau_0 l_0^0\tau_1 = \tau_0 l_0^1\tau_1$) such that $\psi\beta = \alpha\tau_1$. Then we have $\beta(F_1) \subset Z(B)$ and define a crossed homomorphism $\overline{\beta} = \prod_{i=0}^{2} (\beta l_i^1)^\epsilon : M_1 \longrightarrow B$, $\epsilon = (-1)^i$. Hence $\psi\overline{\beta}(y) = \prod_{i=0}^{2} \psi\beta l_i^1(y)^\epsilon = \prod_{i=0}^{2} \alpha\tau_1 l_i^1(y)^\epsilon$, $y \in M_1$, $\epsilon = (-1)^i$.

It is easy to see that

$$\tau_1 l_0^1(y)\tau_1 l_1^1(y)^{-1}\tau_1 l_2^1(y) \in \Delta, \quad y \in M_1.$$

Since $\alpha(\Delta) = 1$, there is a crossed homomorphism $\gamma : F_2 \longrightarrow A$ such that $\varphi\gamma = \overline{\beta}\tau_2$.

Theorem 15. *Let* (6) *be an exact sequence of crossed G-modules. If either the action of* $\mathrm{Der}(F_0, (C, \lambda))$ *on* $H^3(G, A)$ *is trivial* (*in particular, if G acts trivially on A*) *or* $\mathrm{Der}(F_0, (C, \lambda)) = I\,\mathrm{Der}(F_0, (C, \lambda))$ (*in particular, if either $\lambda = 1$ or G is abelian*) *then the connecting map* $\delta^2 : H^2(G, C) \longrightarrow H^3(G, A)$ *is defined by*

$$\delta^2([\alpha, 1]) = [\gamma], (\alpha, 1) \in \widetilde{\mathrm{Der}}(M_0, (C, \lambda)),$$

and the sequence

$$H^2(G, B) \xrightarrow{\psi^2} H^2(G, C) \xrightarrow{\delta^2} H^3(G, A)$$

is exact.

Proof. We have to show that δ^2 is correctly defined.

Let $\mathrm{Der}(F_0, (C, \lambda))$ acts trivially on $H^3(G, A)$. If $(\alpha', 1) \in [(\alpha, 1)] \in H^2(G, A)$, we have

$$\alpha'(x) = \eta l_1^0(x)^{-1}\,{}^g\alpha(x)\eta l_0^0(x), \quad x \in F_0,$$

with $(\eta, g) \in \mathrm{Der}(F_0, (C, \lambda))$.

Take a crossed homomorphism $\eta' : F_0 \to B$ such that $\psi\eta' = \eta$. Recall that $\psi\beta = \alpha\tau_1$. Consider the product

$$(\eta' l_0^0\tau_1, g)(\beta, 1)(\eta' l_1^0\tau_1, g)^{-1} = (\beta', 1)$$

in the group $\mathrm{Der}(F_1, (B, \mu))$. Then

$$\beta'(x) = \eta' l_1^0\tau_1(x)^{-1}\,{}^g\beta(x)\eta' l_0^0\tau_1(x), \quad x \in F_1.$$

Thus $\psi\beta' = \alpha'\tau_1$ and $\beta'(F_1) \subset Z(B)$. Note this implies $\eta' l_1^0(x)^{-1}\, \eta' L_0^0(x) \in Z(B)$ for all $z \in M_0$.

Further,

$$\begin{gathered}\beta'(l_0^1(y))\beta'(l_1^1(y)^{-1})\beta'(l_2^1(y)) = {}^g\beta(l_0^1(y))\eta' l_1^0(l_0^1(y))^{-1}\eta' l_0^0(l_0^1(y)) \\ {}^g\beta(l_1^1(y))^{-1}\eta' l_1^0(l_1^1(y)^{-1})^{-1}\eta' l_0^0(l_1^1(y)^{-1}\,{}^g\beta(l_2^1(y))\eta' l_1^0(l_2^1(y))^{-1} \\ \eta' l_0^0(l_2^1(y)) = {}^g\overline{\beta}(y).\end{gathered}$$

Therefore $\overline{\beta'}\tau_2 = {}^g(\overline{\beta}\tau_2)$ and $[\gamma'] = [{}^g\gamma] = [\gamma]$.

If $\mathrm{Der}(F_0, (C, \lambda)) = I\,\mathrm{Der}(F_0(C, \lambda))$ and $\alpha', 1) \sim (\alpha, 1)$ then there is an element $(\eta, g) \in \mathrm{Der}(F_0, (C, \lambda))$ such that

$$\alpha'(x) = \eta l_1^0(x)^{-1}\alpha(x)\eta l_0^0(x), \quad x \in F_0,$$

(see the proof of Proposition 10 (ii)) and it is clear that in both cases δ^2 is correctly defined.

We will now prove the exactness. Let $[(\alpha,1)] \in H^2(G,B)$. Then $\delta^2\psi^2[(\alpha,1)] = \delta^2([(\psi\alpha,1)]) = [\gamma]$ where $\varphi\gamma = \overline{\beta}\tau_2$ and $\overline{\beta}$ is taken such that $\beta = \alpha\tau_1$. Thus

$$\overline{\beta}(y) = \alpha\tau_1 l_0^1(y)\alpha\tau_1 l_1^1(y)^{-1}\alpha\tau_1 l_2^1(y), \quad y \in M_1.$$

Since $\tau_1 L_0^1(y)\,\tau_1 L_1^1(y)^{-1}\,\tau_1 L_2^1(y) \in \Delta$, $y \in M_1$ and $\alpha(\Delta) = 1$, this implies $\overline{\beta} = 1$. Thus we have $\operatorname{Im}\psi^2 \subset \ker\delta^2$.

Let $[(\alpha,1)] \in H^2(G,C)$ such that $\delta^2([(\alpha,1)]) = 1$. Then we have $[\gamma] = 1$ with $\varphi\gamma = \overline{\beta}\tau_2$ where $\overline{\beta} : M_+1 \to B$ is a crossed homomorphism such that $\overline{\beta} = \beta l_0^1(\beta l_1^1)^{-1}\beta l_2^1$ with $\psi\beta = \alpha\tau_1$ and $\beta : F_1 \to B$ a crassed homomorphism.

It follows that there exists a crossed homomorphism $\eta : F_1 \to A$ such that $\gamma = (\eta l_0^1(\eta l_1^1)^{-1}\eta l_2^1)\tau_2$.

Thus we have $\overline{\beta}\tau_2 = \varphi(\eta l_0^1(\eta l_1^1)^{-1}\eta l_2^1)\tau_2$, whence $\overline{\beta}(y) = \varphi\eta l_0^1(y)\varphi\eta l_1^1(y)^{-1}$ $\varphi\eta l_2^1$, $y \in M_1$.

For $y_1, y_2 \in F_1$ such that $\tau_1(y_1) = \tau_1(y_2) = x$, since $(y_1, y_2, s_0^0 l_1^0(x)) \in M_1$, one gets $\beta(y_1)\beta(y_2)^{-1}\beta s_0^0 l_1^0(x) = \varphi\eta(y_1)\varphi\eta(y_2)^{-1}\varphi\eta(s_0^0 l_1^0(x))$ where $s_0^0 : F_0 \to F_1$ is the degeneracy map. In particular, if $y_1 = y_2$ we have

$$\beta(s_0^0 l_1^0(x)) = \varphi\eta(s_0^0 l_1^0(x)), \quad x \in M_0.$$

Therefore $\beta(y_1)\varphi\eta(y_1)^{-1} = \beta(y_2)\varphi\eta(y_2)^{-1}$ if $\tau_1(y_1) = \tau_1(y_2)$.

This implies a crossed homomorphism $\beta' : M_0 \to B$ given by

$$\beta'(x) = \beta(y)\varphi\eta(y)^{-1}, \quad x \in M_0,$$

where $\tau_1(y) = x$. Thus $\beta'\tau_1 = \beta$ and $\beta'(M_0) \subset Z(B)$. If $x \in \Delta \subset M_0$ then $\tau_1 s_0^0 l_1^0(x) = x$. Therefore we have

$$\beta'(x) = \beta(s_0^0 l_1^0(x)) = \varphi\eta(s_0^0 l_1^0(x))^{-1}, \quad x \in \Delta.$$

Whence $\beta'(x) = 1$ if $x \in \Delta$.

On the other hand,

$$\mu\beta'(x)\mu\beta(y)\mu\varphi\eta(y)^{-1} = \mu\beta(y) = \lambda\psi\beta(y) = \lambda\alpha\tau_1(y) = 1.$$

We conclude that $(\beta',1) \in \widetilde{\mathbf{Z}^1}(M_0,(B,\mu))$ and it is clear that $\psi^2([(\beta',1)]) = [(\alpha,1)]$. $\square$

Acknowledgement

The research described in this publication was made possible in part by Grant MXH200 from the International Science Foundation and by INTAS Grant No 93-2618.

References

1. H. Inassaridze, Non-abelian cohomology with coefficients in crossed bimodules. *Georgian Math. J.* **4**(1977), No. 6, (*to appear*).

2. J. -P. Serre, Cohomologie galoisienne. *Lecture Notes in Math.* 5, *Springer-Verlag,* 1964.

3. D. Guin, Cohomologie et homologie non abeliennes des groupes. *C. R. Acad. Sci. Paris* **301**(1985), Serie 1, No. 7.

4. D. Guin, Cohomologie et homologie non abeliennes des groupes. *Pure Appl. Algebra* **50**(1988), 109–137.

5. R. Dedecker, Cohomologie non abelienne. *Seminaire de l'Institut Mathematique Lille,* 1963–1964.

6. M. Bullejos and A. Cegarra, A 3-dimensional non-abelian cohomology of goups with applications to homotopy classification of continuous maps. *Canad. J. Amer.* **43**(1991), 265–296.

7. A. Cegarra and A. Garzon, A long exact sequence in non-abelian cohomology. *Lecture Notes in Math.* **1488**(1991), 79–94.

8. R. Brown, D. Johnson, and E. F. Robertson, Some computations of the non-abelian tensor products of groups. *J. Algebra* **111**(1987), 177–202.

9. M. Barr and J. Beck, Homology and standard constructions. *Lecture Notes in Math.* 80, *Springer-Verlag,* 1980, 245–335.

10. H. Inassaridze, Homotopy of pseudo-simplical groups, non-abelian derived functors and algebraic K-theory. *Math. USSR Sbornik* **27**(1975), No. 3, 339–362.

(Received 2.06.1995)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia