NON-ABELIAN HOMOLOGY OF GROUPS

HVEDRI INASSARIDZE * AND NICK INASSARIDZE**

Abstract. New descriptions of non-abelian group homology in terms of derived functors are given and the Mayer-Vietoris sequence is constructed. Sufficient conditions are established for non-abelian homology groups of dimension ≥ 2 to be finitely generated, finite, p-groups, torsion groups or groups of exponent q. Some formulas for the second and third non-abelian homology of groups are obtained.

Introduction and summary

The non-abelian homology of groups with coefficients in any group in any dimension was introduced in [10]. It generalizes the classical Eilenberg-MacLane homology of groups [5,6] and extends Guin's non-abelian homology [7].

The low dimensional H_0 and H_1 non-abelian homology groups with coefficients in crossed modules were defined by Guin and had important applications in the algebraic K-theory of non-commutative local rings [7].

The higher non-abelian homology groups were constructed as the non-abelian left derived functors [14,8] of the non-abelian tensor product of groups.

The non-abelian tensor product of groups was introduced by R. Brown and J.-L. Loday [1-3] following works of A. Lue [15] and R.K. Dennis [4]. It arises in applications to homotopy theory of a generalized Van Kampen theorem. It was defined for a pair A, B of groups which act on themselves by conjugation (${}^{x}y = xyx^{-1}$) and on each other such that the following compatibility conditions hold:

$$({}^{a}b)a' = {}^{aba^{-1}}a', \qquad ({}^{b}a)b' = {}^{bab^{-1}}b' \tag{1}$$

for all $a, a' \in A$ and $b, b' \in B$.

By definition, the non-abelian homology groups $H_n(G, A)$ of a group G with coefficients in any group A in any dimension $n \geq 0$ are the non-abelian left derived functors of the generalized non-abelian tensor product of Brown and Loday [10,11].

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Let A and B be arbitrary groups which act on themselves by conjugation and on each other. The compatibility conditions above are not assumed to hold. The non-abelian tensor product $A \otimes B$ of the groups A and B is the group generated by the symbols $a \otimes b$, $(a \in A,\, b \in B)$ subject to the following relations

$$\begin{aligned} aa' \otimes b &= ({}^a a' \otimes {}^a b)(a \otimes b) \\ a \otimes bb' &= (a \otimes b)({}^b a \otimes {}^b b') \\ (a \otimes b)(a' \otimes b') &= ({}^{[a,b]}a' \otimes {}^{[a,b]}b')(a \otimes b) \\ (a' \otimes b')(a \otimes b) &= (a \otimes b)({}^{[b,a]}a' \otimes {}^{[b,a]}b') \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$, where $[a,b] = aba^{-1}b^{-1} \in A * B$ (free product of A and B).

Now we recall the definition of non-abelian homology groups given in [10].

Let A denote a group. Let $\mathcal{A}_A$ denote the category whose objects are all groups G together with an action of G on A by automorphisms of A and an action of A on G by automorphisms of G. Morphisms in the category $\mathcal{A}_A$ are all group homomorphisms $\alpha : G \to H$ which preserve the actions, namely $\alpha({}^a g) = {}^a\alpha(g)$ and ${}^g a = {}^{\alpha(g)}a$, for all $a \in A$ and $g \in G$.

Let $F : \mathcal{A}_A \to \mathcal{A}_A$ be the endofunctor defined as follows: for an object G of $\mathcal{A}_A$, let $F(G)$ denote the free group generated by G with actions: ${}^{|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}}a = {}^{g_1^{\epsilon_1}}(\dots({}^{g_s^{\epsilon_s}}a)\dots)$, and ${}^a(|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s}) = |{}^a g_1|^{\epsilon_1}\cdots|{}^a g_s|^{\epsilon_s}$ where $a \in A$, $|g_1|^{\epsilon_1}\cdots|g_s|^{\epsilon_s} \in F(G)$ and $\epsilon_i = \pm 1$; for a morphism $\alpha : G \to G'$ of $\mathcal{A}_A$, let $F(\alpha)$ be the canonical homomorphism from $F(G)$ to $F(G')$ induced by α.

Let $\tau : F \to 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $G \in \operatorname{ob}\mathcal{A}_A$ by the injection $G \to F(G)$. We obtain a cotriple $\mathcal{F} = (F, \tau, \delta)$, which we call the free cotriple in the category $\mathcal{A}_A$. Let $\mathcal{P}$ be the projective class induced by the free cotriple $\mathcal{F}$.

The non-abelian tensor product of groups defines a covariant functor $- \otimes A$ from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups. Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(- \otimes A)$, $n \geq 0$, of the functor $- \otimes A$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$.

Let G and A be groups acting on each other. Then the non-abelian homology of G with coefficients in A are defined by the following way:

$$\begin{aligned} &H_n(G,A) = L_{n-1}^{\mathcal{P}}(G \otimes A), \qquad n \geq 2, \\ &H_1(G,A) = \operatorname{Ker}\lambda', \qquad H_0(G,A) = \operatorname{Coker}\lambda', \end{aligned}$$

where $\lambda' : G \otimes A \to A/H$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H is the normal subgroup generated by the elements ${}^{({}^a g)}a' \cdot {}^{aga^{-1}}a'^{-1}$ for all $a, a' \in A$, $g \in G$.

It is easy to see that if G and A are any groups acting on each other trivially then $H_n(G,A) \approx H_n(G, A^{ab})$ for $n \geq 1$, where A^{ab} is the abelianization of the group A.

For non-commutative local rings, the relationship of Milnor's algebraic K-functor K_2 with the symbol group Sym is given in terms of a long exact sequence of non-abelian homology of groups [10] extending Guin's six terms exact sequence [7]. In fact, let R be a non-commutative local ring such that $R/\operatorname{Rad} R \neq \mathbb{F}_2$. Then one has the following exact sequence of groups

$$\begin{aligned} \cdots \longrightarrow H_3(R^*,[R^*,R^*]) \longrightarrow H_2(R^*,U(R),[R^*,R^*]) \longrightarrow H_2(R,U(R)) \longrightarrow \\ \longrightarrow H_2(R^*,[R^*,R^*]) \longrightarrow H_1(R^*,U(R),[R^*,R^*]) \longrightarrow H_1(R,U(R)) \longrightarrow \\ \longrightarrow H_1(R^*,[R^*,R^*]) \longrightarrow K_2(R) \longrightarrow \operatorname{Sym}(R) \longrightarrow [R^*,R^*]/[R^*,[R^*,R^*]] \longrightarrow 1. \end{aligned}$$

For precrossed P-modules (M,μ) and (N,ν), there is the exact sequence of groups

$$\begin{gathered} \cdots \longrightarrow H_3(M,N) \longrightarrow L_2^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_1^{\mathcal{P}}(f,N) \longrightarrow \\ H_2(M,N) \longrightarrow L_1^{\mathcal{P}}(M \wedge^P N) \longrightarrow L_0^{\mathcal{P}}(f,N) \longrightarrow M \otimes N \xrightarrow{f} \quad , \\ M \wedge^P N \longrightarrow 1 \end{gathered}$$

obtained in [12], where $M \wedge^P N$ is the exterior product, $f : M \otimes N \to M \wedge^P N$ is the natural epimorphism given by $m \otimes n \mapsto m \wedge n$ and $L_n^{\mathcal{P}}(M \wedge^P N)$, $n \geq 1$, are the left derived functors of the functor $- \wedge^P N$ relative to the projective class induced by the free cotriple in the category of precrossed P-modules.

The rest of the paper is organized as follows. In section 1, the Mayer-Vietoris sequence (Theorem 5) is constructed. This generalizes the long exact sequence of non-abelian homology groups $H_*(G,A)$ with respect to the variable G (Theorem 3). Then non-abelian homology groups are described as the non-abelian left derived functors of the functor $H_1(-,A)$ (Theorem 7), as well as of the section functor Γ in the category of cosheaves (Theorem 10). Sufficient conditions are established for the non-abelian homology groups to be finitely generated, finite, p-groups, torsion groups or groups of exponent q (Theorem 11).

In section 2, formulas for the second and third non-abelian homology of groups are obtained by using Čech resolutions (Theorems 12, 16 and 19). For an arbitrary group A acting trivially on $\mathbb{Z}$ when $\mathbb{Z}$ acts on A such that the compatibility conditions (1) hold, it is shown that there is a natural isomorphism $\mathbb{Z} \otimes A \approx A^{ab}$, $n \otimes a \mapsto [{}^{n-1}a] \cdot [{}^{n-2}a] \cdots [{}^{1}a] \cdot [a]$ (Proposition 13).

1. Some properties of the non-abelian homology of groups

First we describe in the category $\mathcal{A}_A$ the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$ and the corresponding $\mathcal{P}$-epimorphisms.

Proposition 1. *A morphism $\alpha : G \to G'$ of the category $\mathcal{A}_A$ is a $\mathcal{P}$- epimorphism if and only if there exists a section map $\beta : G' \to G$ which preserves the actions of A i.e $\beta({}^a g') = {}^a\beta(g')$ for all $a \in A$, $g' \in G'$;*

Proposition 2. *In the category $\mathcal{A}_A$ the following conditions are equivalent:*

(i) *F belongs to the projective class $\mathcal{P}$.*
(ii) *F is a free group generated by a set X such that ${}^a x \in X$ for all $a \in A$ and $x \in X$.*
(iii) *For any morphism $\alpha : G \to G'$ which has a section map $\beta : G' \to G$ preserving the actions of A and any morphism $\kappa' : F \to G'$ there exists a morphism $\kappa : F \to G$ such that $\alpha\kappa = \kappa'$.*

The proof of these propositions is left to the reader.

As noted in Remark 3 [10], the exact sequence of the non-abelian homology $H_*(G, A)$ of groups with respect to the variable G can be obtained. In fact we have

Theorem 3. *Let G_1, G, G_2, A be arbitrary groups. Suppose that A acts on G_1, G and G_2, and all groups act on A. Let $1 \to G_1 \xrightarrow{\alpha} G \xrightarrow{\beta} G_2 \to 1$ be an exact sequence of groups such that the homomorphisms α and β preserve the actions. Then $H_0(G, A) \approx H_0(G_2, A)$ and there is a long exact sequence of non-abelian homology groups*

$$
\begin{aligned}
(2) \quad \cdots \longrightarrow H_4(G_2, A) \longrightarrow H_3(G, G_2, A) \longrightarrow H_3(G, A) \longrightarrow H_3(G_2, A) \longrightarrow \\
\longrightarrow H_2(G, G_2, A) \longrightarrow H_2(G, A) \longrightarrow H_2(G_2, A) \longrightarrow H_1(G, G_2, A) \longrightarrow \\
\longrightarrow H_1(G, A) \longrightarrow H_1(G_2, A) \longrightarrow 1,
\end{aligned}
$$

where

$$
\begin{aligned}
H_n(G, G_2, A) &= \pi_{n-1}(\mathrm{Ker}(F_*(\beta) \otimes 1_A)), n \geq 1, \\
\mathrm{Ker}(F_*(\beta) \otimes 1_A) &= \{\mathrm{Ker}(F^n(\beta) \otimes 1_A),\ n \geq 1\}.
\end{aligned}
$$

Proof. The proof follows from the commutative diagram of groups

$$
\begin{array}{ccccccccc}
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & \mathrm{Ker}(F^2(\beta) \otimes 1_A) & \overset{\bar\partial_0^1}{\underset{\bar\partial_1^1}{\rightrightarrows}} & \mathrm{Ker}(F^1(\beta) \otimes 1_A) & & & & \\
& \downarrow & & \downarrow & & & & \\
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & F^2(G) \otimes A & \overset{\partial_0^1}{\underset{\partial_1^1}{\rightrightarrows}} & F^1(G) \otimes A & \xrightarrow{\partial_0^0} & G \otimes A & \xrightarrow{\lambda'} & A/H' & . \\
& \downarrow {\scriptstyle 1_{F^2(\beta)} \otimes 1_A} & & \downarrow {\scriptstyle 1_{F^1(\beta)} \otimes 1_A} & & \downarrow {\scriptstyle \beta \otimes 1_A} & & \| & \\
\cdots \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & F^2(G_2) \otimes A & \overset{\partial_0^1}{\underset{\partial_1^1}{\rightrightarrows}} & F^1(G_2) \otimes A & \xrightarrow{\partial_0^0} & G_2 \otimes A & \xrightarrow{\lambda'} & A/H_2' &
\end{array}
$$

■

Remark 4. *In the exact sequence (2) the groups $H_1(G, A)$ and $H_1(G_2, A)$ can be replaced by the groups $\pi_0(F_*(G) \otimes A)$ and $\pi_0(F_*(G_2) \otimes A)$ respectively.*

Let

$$\mathcal{D} = \begin{array}{ccc} & & G_2 \\ & & \downarrow \alpha_2 \\ G_1 & \xrightarrow{\alpha_1} & G \end{array} \tag{3}$$

be a diagram in the category $\mathcal{A}_A$ with surjective α_1. Let $L_*(\mathcal{D}, A)$ be the pullback of the induced diagram

$$\begin{array}{ccc} & & F_*(G_2) \otimes A \\ & & \downarrow F_*(\alpha_2)\otimes 1_A \\ F_*(G_1) \otimes A & \xrightarrow{F_*(\alpha_1)\otimes 1_A} & F_*(G) \otimes A \end{array}$$

Define $H_n(\mathcal{D}, A) = \pi_{n-1} L_*(\mathcal{D}, A)$, $n \geq 2$.

Theorem 5 (Mayer-Vietoris sequence). *For any diagram (3) there is a long exact sequence*

(4)

$$\cdots \longrightarrow H_{n+1}(G, A) \longrightarrow H_n(\mathcal{D}, A) \longrightarrow H_n(G_1, A) \oplus H_n(G_2, A) \longrightarrow H_n(G, A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_2(\mathcal{D}, A) \longrightarrow H_2(G_1, A) \oplus H_2(G_2, A) \longrightarrow H_2(G, A) \longrightarrow \pi_0 L_*(\mathcal{D}, A) \longrightarrow$$

$$\longrightarrow \pi_0(F_*(G_1) \otimes A) \times \pi_0(F_*(G_2) \otimes A) \longrightarrow \pi_0(F_*(G) \otimes A) \longrightarrow 1.$$

Proof. One has the following commutative diagram of simplicial groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{I}_* & \xrightarrow{\sigma_*} & L_*(\mathcal{D}, A) & \xrightarrow{p_*} & F_*(G_2) \otimes A & \longrightarrow & 1 \\ & & \| & & \downarrow q_* & & \downarrow F_*(\alpha_2)\otimes 1_A & & \\ 1 & \longrightarrow & \mathcal{I}_* & \longrightarrow & F_*(G_1) \otimes A & \xrightarrow{F_*(\alpha_1)\otimes 1_A} & F_*(G) \otimes A & \longrightarrow & 1 \end{array}, \tag{5}$$

where $\mathcal{I}_* = \mathrm{Ker}(F_*(\alpha_1) \otimes 1_A)$. Diagram (5) induces the following commutative diagram with exact rows

(6)

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_1(F_*(G_2) \otimes A) & \longrightarrow & \pi_0(\mathcal{I}_*) & \longrightarrow & \pi_0 L_*(\mathcal{D}, A) & \longrightarrow & \pi_0(F_*(G_2) \otimes A) & \longrightarrow 1 \\ & \downarrow & & \| & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \pi_1(F_*(G) \otimes A) & \xrightarrow{\delta_1} & \pi_0(\mathcal{I}_*) & \longrightarrow & \pi_0(F_*(G_1) \otimes A) & \longrightarrow & \pi_0(F_*(G) \otimes A) & \longrightarrow 1 \end{array}.$$

The connecting homomorphism $\pi_n(F_*(G) \otimes A) \to \pi_{n-1} L_*(\mathcal{D}, A)$, $n \geq 1$, is the composite map $\pi_{n-1}(\sigma_*)\delta_n$. The homomorphism $\pi_n(L_*(\mathcal{D}, A)) \to \pi_n(F_*(G_1) \otimes A) \times \pi_n(F_*(G_2) \otimes A)$, $n \geq 0$, is induced by $\pi_n(q_*)$ and $\pi_n(p_*)$. The map $\pi_n(F_*(G_1) \otimes A) \times \pi_n(F_*(G_2) \otimes A) \to \pi_n(F_*(G) \otimes A)$, $n \geq 0$, is given by $\pi_n(F_*(\alpha_1) \otimes 1_A) \dot{\pi}_n(F_*(\alpha_2) \otimes 1_A)^{-1}$. To get the exactness of the sequence (4) it remains to apply diagram (6). ■

Remark 6. *If the group G_2 is trivial then we recover the sequence (2) (see Remark 4).*

Let G and A be any groups which act on each other. Let us consider $H_1(-,A)$ as a functor from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups and its left derived functors $L_n^{\mathcal{P}}(H_1(-,A))$ relative to the projective class $\mathcal{P}$ induced by the free cotriple $\mathcal{F}$. Then one has

Theorem 7. *There is a natural isomorphism*

$$H_n(G,A) \approx L_{n-1}^{\mathcal{P}}(H_1(G,A)), \quad n \geq 1.$$

Proof. One has a short exact sequence of groups

$$1 \longrightarrow H_1(G,A) \longrightarrow G \otimes A \xrightarrow{\lambda'} \operatorname{Im} \lambda' \longrightarrow 1,$$

and for any surjective morphism $\alpha : G \to G'$ of $\mathcal{A}_A$ the commutative diagram of groups

$$\begin{array}{ccc} G \otimes A & \xrightarrow{\lambda'} & A/H \\ \downarrow & & \| \\ G' \otimes A & \xrightarrow{\lambda'} & A/H' \end{array},$$

where $H = H'$ and $\lambda'(G \otimes A) = \lambda'(G' \otimes A)$. Then the assertion follows from the long exact sequence of homotopy groups of the following short exact sequence of simplicial groups

$$\begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & & H_1(F^3(G),A) & \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & H_1(F^2(G),A) & \rightrightarrows & H_1(F^1(G),A) \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & & F^3(G) \otimes A & \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & F^2(G) \otimes A & \rightrightarrows & F(G) \otimes A \\ & & \downarrow \lambda' & & \downarrow \lambda' & & \downarrow \lambda' \\ \cdots \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} & & \operatorname{Im} \lambda' & \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} & \operatorname{Im} \lambda' & \rightrightarrows & \operatorname{Im} \lambda' \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array} \tag{7}$$

where the bottom simplicial group is a constant simplicial group and $\lambda' : G \otimes A \to A/H$. ■

Note that Theorem 7 generalizes the well known fact that the integer homology can be obtained as the left derived functors of the abelianization functor.

Let $\underline{\mathcal{C}}$ be a category with finite limits, $\overline{\mathcal{P}}$ a projective class in the category $\underline{\mathcal{C}}$ and T a functor from $\underline{\mathcal{C}}$ to the category $\mathfrak{Gr}$ of groups.

Definition 8 ([17]). *The functor T is called a cosheaf over $(\underline{\mathcal{C}}, \overline{\mathcal{P}})$ if for any $\overline{\mathcal{P}}$-epimorphism $X \to A$ the sequence of groups*

$$T(X \times_A X) \rightrightarrows T(X) \longrightarrow T(A) \longrightarrow 1,$$

is exact, where $X \times_A X$ is the pullback of the diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ X & \longrightarrow & A \end{array} .$$

Proposition 9. (i) *$H_1(-, A)$ is a cosheaf over $(\mathcal{A}_A, \mathcal{P})$;*
(ii) *If the actions satisfy compatibility conditions (1), then $H_1(-, A)$ is a right exact functor.*

Proof. **(i)** From the commutative diagram of groups (7) it follows that $H_1(-, A) \approx L_0^{\mathcal{P}} H_1(-, A)$. Apply the fact that a functor is cosheaf if and only if it is naturally isomorphic with its zero derived functor relative to the projective class [17 or 9, Proposition 2.26].

(ii) Follows from the commutative diagram of groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & H_1(G_2, A) & \longrightarrow & G_2 \otimes A & \longrightarrow & A \\ & & \downarrow & & \downarrow & & \| \\ 1 & \longrightarrow & H_1(G, A) & \longrightarrow & G \otimes A & \longrightarrow & A \\ & & \downarrow & & \downarrow & & \| \\ 1 & \longrightarrow & H_1(G_1, A) & \longrightarrow & G_1 \otimes A & \longrightarrow & A \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array} .$$

■

Now a new description of the non-abelian homology of groups will be given in terms of cosheaves. For this we need some assertions from [17 or 9, Chapter 2].

Let $\mathrm{CS}(\underline{\mathcal{C}})$ denote the category of cosheaves over $(\underline{\mathcal{C}}, \overline{\mathcal{P}})$.

For any $P \in \mathrm{ob}\,\overline{\mathcal{P}}$ define the cosheaf $Z_P : \underline{\mathcal{C}} \to \mathfrak{Gr}$ as follows: $Z_P(A)$ is the free group generated by the set $Hom_{\underline{\mathcal{C}}}(P, A)$. Let $\mathfrak{Q}$ be the projective class in the category $\mathrm{CS}(\underline{\mathcal{C}})$ generated by the cosheaves Z_P which means that any object of $\mathfrak{Q}$ is a retract of coproducts of cosheaves of the form Z_P [17 or 9, Proposition 2.29].

For any object $A \in \mathrm{ob}\,\underline{\mathcal{C}}$ define the section functor $\Gamma_A : \mathrm{CS}(\underline{\mathcal{C}}) \to \mathfrak{Gr}$ by $\Gamma(T) = T(A)$, $T \in \mathrm{ob}\,\mathrm{CS}(\underline{\mathcal{C}})$.

Theorem 10. *Let $\underline{\mathcal{C}}$ be the category $\mathcal{A}_A$ and $G \in \mathrm{ob}\,\mathcal{A}_A$. Then there are isomorphisms*

$$H_n(G, A) \approx L_{n-1}^{Q} \Gamma_G(- \otimes A), \quad n \geq 2,$$

$$H_n(G, A) \approx L_{n-1}^{Q} \Gamma_G(H_1(-, A)), \quad n \geq 1.$$

Proof. Since the functors $(-\otimes A)$ and $H_1(-,A)$ are cosheaves over $(\mathcal{A}_A,\mathcal{P})$ (see and Proposition 9(i)), the isomorphisms follow from [17 or 9, Theorem 2.34] and Theorem 7. ■

Theorem 11. *Let G and A be any groups. Let A act on G trivially and G act on A.*

(i) *If G is a finite group and A is a finite group (or p-group or finitely generated group), then $H_n(G,A)$, $n\geq 2$, is a finite group (or p-group or finitely generated group);*
(ii) *If A is a torsion group (or its any element has order dividing q), then $H_n(G,A)$, $n\geq 2$, is a torsion group (or a group of exponent q).*

Proof. Let us consider a $\mathcal{P}$-projective pseudosimplicial resolution of G in the category $\mathcal{A}_A$

$$X_* \xrightarrow{\partial_0^0} G.$$

Apply the following Quillen's construction [18,9]

$$1 \longrightarrow \Omega X_* \longrightarrow EX_* \longrightarrow X_* \longrightarrow C\pi_0 X_* \longrightarrow 1$$

to the pseudosimplicial group X_*. One gets the comutative diagram of groups

$$\begin{array}{ccccccccc}
\cdots \vdots\!\!\rightrightarrows & \mathrm{Ker}(\partial_0^1\cdots\partial_0^{n+1}) & \overset{\partial_0^{n+1}}{\underset{\partial_n^{n+1}}{\vdots\!\!\rightrightarrows}} \cdots \vdots\!\!\rightrightarrows & \mathrm{Ker}(\partial_0^1\partial_0^2) & \overset{\partial_0^2}{\underset{\partial_1^2}{\rightrightarrows}} & \mathrm{Ker}(\partial_0^1) & \equiv EX_* \\
 & \downarrow \partial_{n+1}^{n+1} & & \downarrow \partial_2^2 & & \downarrow \partial_1^1 & \\
\cdots \vdots\!\!\rightrightarrows & X_n & \overset{\partial_0^n}{\underset{\partial_n^n}{\vdots\!\!\rightrightarrows}} \cdots \vdots\!\!\rightrightarrows & X_1 & \overset{\partial_0^1}{\underset{\partial_1^1}{\rightrightarrows}} & X_0 & \equiv X_* \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \vdots\!\!\rightrightarrows & G & \vdots\!\!\rightrightarrows \cdots \vdots\!\!\rightrightarrows & G & \rightrightarrows & G & \equiv C\pi_0 X_* \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 1 & & 1 & & 1 &
\end{array},$$

where $C\pi_0 X_*$ is the constant simplicial group. Therefore we have the following commutative diagram of groups

(8)
$$
\begin{array}{cccccccc}
 & 1 & & & 1 & & 1 \\
 & \downarrow & & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & \mathrm{Ker}(\partial_{n+1}^{n+1}\otimes 1_A) & \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & \cdots \begin{smallmatrix}\rightarrow\\ \rightarrow \\ \rightarrow\end{smallmatrix} & \mathrm{Ker}(\partial_2^2\otimes 1_A) & \rightrightarrows & \mathrm{Ker}(\partial_1^1\otimes 1_A) \\
 & \downarrow & & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & \mathrm{Ker}(\partial_0^1\cdots\partial_0^{n+1})\otimes A & \underset{\partial_n^{n+1}\otimes 1_A}{\overset{\partial_0^{n+1}\otimes 1_A}{\begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix}}} & \cdots \begin{smallmatrix}\rightarrow\\ \rightarrow \\ \rightarrow\end{smallmatrix} & \mathrm{Ker}(\partial_0^1\partial_0^2)\otimes A & \underset{\partial_1^2\otimes 1_A}{\overset{\partial_0^2\otimes 1_A}{\Longrightarrow}} & \mathrm{Ker}(\partial_0^1)\otimes A \\
 & \downarrow{\scriptstyle \partial_{n+1}^{n+1}\otimes 1_A} & & & \downarrow{\scriptstyle \partial_2^2\otimes 1_A} & & \downarrow{\scriptstyle \partial_1^1\otimes 1_A} \\
\cdots \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & X_n\otimes A & \underset{\partial_n^{n}\otimes 1_A}{\overset{\partial_0^{n}\otimes 1_A}{\begin{smallmatrix}\longrightarrow\\ \vdots \\ \longrightarrow\end{smallmatrix}}} & \cdots \begin{smallmatrix}\rightarrow\\ \rightarrow \\ \rightarrow\end{smallmatrix} & X_1\otimes A & \underset{\partial_1^1\otimes 1_A}{\overset{\partial_0^1\otimes 1_A}{\Longrightarrow}} & X_0\otimes A \\
 & \downarrow & & & \downarrow & & \downarrow \\
\cdots \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & G\otimes A & \begin{smallmatrix}\rightarrow\\ \vdots \\ \rightarrow\end{smallmatrix} & \cdots \begin{smallmatrix}\rightarrow\\ \rightarrow \\ \rightarrow\end{smallmatrix} & G\otimes A & \rightrightarrows & G\otimes A \\
 & \downarrow & & & \downarrow & & \downarrow \\
 & 1 & & & 1 & & 1
\end{array}
\quad .
$$

Thus the sequence of simplicial groups

$$1 \longrightarrow \mathrm{Ker}(\vartheta\otimes 1_A) \longrightarrow EX_*\otimes A \xrightarrow{\vartheta\otimes 1_A} X_*\otimes A \xrightarrow{j\otimes 1_A} C\pi_0X_*\otimes A \longrightarrow 1,$$

is exact, where $\vartheta_n : (EX_*)_n \to X_n$ is induced by the homomorphism $\partial_{n+1}^{n+1} : X_{n+1} \to X_n$ and $j_n : X_n \to (C\pi_0X_*)$ is the composite homomorphism $\partial_0^0\partial_0^1\cdots\partial_0^n$. Therefore one gets two long exact sequences of groups:

$$
\begin{aligned}
\cdots \longrightarrow \pi_{n+1}\mathrm{Ker}(j\otimes 1_A) \longrightarrow \pi_n\mathrm{Ker}(\vartheta\otimes 1_A) \longrightarrow \pi_n(EX_*\otimes A) \longrightarrow \pi_n\mathrm{Ker}(j\otimes 1_A) \longrightarrow \\
\longrightarrow \pi_{n-1}\mathrm{Ker}(\vartheta\otimes 1_A) \longrightarrow \cdots \longrightarrow \pi_1\mathrm{Ker}(j\otimes 1_A) \longrightarrow \pi_0\mathrm{Ker}(\vartheta\otimes 1_A) \longrightarrow \\
\longrightarrow \pi_0(EX_*\otimes A) \longrightarrow \pi_0\mathrm{Ker}(j\otimes 1_A) \longrightarrow 1
\end{aligned}
$$

and

$$
\begin{aligned}
\cdots \longrightarrow \pi_{n+1}(C\pi_0X_*\otimes A) \longrightarrow \pi_n\mathrm{Ker}(j\otimes 1_A) \longrightarrow \pi_n(X_*\otimes A) \longrightarrow \pi_n(C\pi_0X_*\otimes A) \longrightarrow \\
\longrightarrow \pi_{n-1}\mathrm{Ker}(j\otimes 1_A) \longrightarrow \cdots \longrightarrow \pi_1(C\pi_0X_*\otimes A) \longrightarrow \pi_0\mathrm{Ker}(j\otimes 1_A) \longrightarrow \\
\longrightarrow \pi_0(X_*\otimes A) \longrightarrow \pi_0(C\pi_0X_*\otimes A) \longrightarrow 1.
\end{aligned}
$$

But the homotopy groups $\pi_n(EX_*\otimes A)$, $n \geq 0$, are trivial, since the augmented pseudosimplicial group $(EX_*, \epsilon, 1)$ is right contractible with contractions $h = 0$ and $h_n = s_{n+1}^{n+1}$ for $n \geq 0$ [18,9]. This implies that

(9) $$H_{n+1}(G, A) = \pi_n(X_*\otimes A) \approx \pi_{n-1}(\mathrm{Ker}(\vartheta\otimes 1_A)), \quad n \geq 1.$$

Assume now with no lose of generality that A acts on X_* trivially. Any $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$, $n \geq 0$, acts trivially on A and since it is a free group, by [3, Proposition 2.4] one has

$$\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \otimes A \approx \mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})^{ab} \otimes A^{ab} \approx \sum_{\alpha} A^{ab}, \tag{10}$$

where α runs over the basis of $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$.

First we prove (i): If G is a finite group and A acts trivially on G then we can construct a new $\mathcal{P}$-projective simplicial resolution G_* of the object G in the category $\mathcal{A}_A$ such that every G_n will be finitely generated free group [12]. This can be done as follows.

Recall the definition of the loop functor $\mathbb{G}$ from the category of reduced complexes to the category of simplicial groups [13].

Let K be a reduced complex (i.e. K is simplicial set which has only one 0-simplex ϕ). Then we define a simplicial group $\mathbb{G}K$ as follows: the group of the n-simplices is a group which has

(i) one generator $\overline{\sigma}$ for every $n+1$-simplex $\sigma \in K_{n+1}$,
(ii) one relation $\overline{s_n^n \tau} = e_n$ for every n-simplex $\tau \in K_n$.

The face and degeneracy homomorphisms $\partial_i^n : \mathbb{G}_n K \to \mathbb{G}_{n-1} K$ and $s_i^n : \mathbb{G}_n K \to \mathbb{G}_{n+1} K$ are given by the formulas

$$\begin{aligned} \partial_i^n \overline{\sigma} &= \overline{\partial_i^n \sigma}, \quad 0 \leq i < n, \\ \partial_n^n \overline{\sigma} &= \overline{\partial_n^n \sigma} \cdot \overline{\partial_{n+1}^n \sigma}^{-1}, \\ s_i^n \overline{\sigma} &= \overline{s_i^n \sigma}, \quad 0 \leq i \leq n. \end{aligned}$$

Clearly the groups $\mathbb{G}_n K$ are free and from [13] it is known that $\pi_n(K) \approx \pi_{n-1}(\mathbb{G}K)$ for $n > 0$.

Let $\underline{G}$ be a category which has only one object O and $Hom(O,O) = G$, and consider the nerve $M_*(\underline{G})$ of the category $\underline{G}$. It is easy to see that $M_*(\underline{G})$ is a reduced complex. It is known that $\pi_1(M_*(\underline{G})) \approx G$ and $\pi_n(M_*(\underline{G})) = 0$, $n \neq 1$. Let us consider the simplicial group $\mathbb{G}M_*(\underline{G})$. Then $\pi_0(\mathbb{G}M_*(\underline{G})) \approx \pi_1(M_*(\underline{G})) \approx G$ and $\pi_{n-1}(\mathbb{G}M_*(\underline{G})) \approx \pi_n(M_*(\underline{G})) = 0$, $n > 1$. Hence $\mathbb{G}M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is a $\mathcal{P}$-projective simplicial resolution in the category of groups, where ∂_0^0 is the natural epimorphism, $\mathcal{P}$ is the projective class in the category of groups induced by the free cotriple.

Thus, if we define the action of A on $\mathbb{G}_n M_*(\underline{G})$ trivially and the action of $\mathbb{G}_n M_*(\underline{G})$ on A as follows:

$${}^m a = {}^{\partial_0^0 \partial_0^1 \cdots \partial_0^n m} a, \quad m \in \mathbb{G}_n M_*(\underline{G}),$$

then by Propositions 1 and 2 one deduces that $\mathbb{G}_n M_*(\underline{G}) \xrightarrow{\partial_0^0} G$ is the $\mathcal{P}$-projective simplicial resolution in the category $\mathcal{A}_A$.

Therefore the group $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$ is the subgroup with finite index of the finitely generated free group and by [19] $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1})$ is also a finitely generated free group. Hence by (10) if A is finite group (or p-group or finitely generated group) so is $\mathrm{Ker}(\partial_0^1 \cdots \partial_0^{n+1}) \otimes A$, $n \geq 0$. By (8) and (9) it follows that $H_n(G, A)$, $n \geq 2$ is a finite group (or p-group or finitely generated group).

(ii) is proved analogously by (8-10) and the fact that the property of a group to be torsion (or of exponent q) is stable under coproducts, subgroups and quotient groups. ■

2. The second and the third non-abelian homology of groups

Now new descriptions of the second and third non-abelian homology of groups will be given.

Let G and A be arbitrary groups which act on each other. Let P be an object of the projective class $\mathcal{P}$ and $P \xrightarrow{\alpha} G$ be a $\mathcal{P}$-epimorphism in the category $\mathcal{A}_A$. Consider the Čech resolution of G for $\alpha : P \to G$ [17 or 9, Definition 2.31, Examples]

$$\cdots \begin{smallmatrix}\to\\ \to\\ \to\\ \to\end{smallmatrix} P \times_G \times_G \underset{\partial_2^2}{\overset{\partial_0^2}{\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}}} P \times_G P \ \underset{\partial_1^1}{\overset{\partial_0^1}{\rightrightarrows}} P \xrightarrow{\alpha} G.$$

The actions of A and $P \times_G \cdots \times_G P$ on each other are induced in a natural way by the actions of A and P on each other.

Denote by C_* the following simplicial group

$$\cdots \begin{smallmatrix}\to\\ \to\\ \to\\ \to\end{smallmatrix} (P \times_G P \times_G P) \otimes A \underset{\partial_2^2 \otimes 1_A}{\overset{\partial_0^2 \otimes 1_A}{\begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix}}} (P \times_G P) \otimes A \ \underset{\partial_1^1 \otimes 1_A}{\overset{\partial_0^1 \otimes 1_A}{\rightrightarrows}} P \otimes A \xrightarrow{\alpha \otimes 1_A} G \otimes A.$$

Theorem 12. (i) *There is an isomorphism*

$$H_2(G, A) \approx \mathrm{Ker}(\partial_0^1 \otimes 1_A) \cap \mathrm{Ker}(\partial_1^1 \otimes 1_A)/[\mathrm{Ker}(\partial_0^1 \otimes 1_A), \mathrm{Ker}(\partial_1^1 \otimes 1_A)];$$

(ii) *there is an epimorphism*

$$H_3(G, A) \longrightarrow \cap_{i \in [2]} \mathrm{Ker}(\partial_i^2 \otimes 1_A)/\prod_{I,J}[K_I, K_J],$$

where $\emptyset \neq I, J \subset [2] = \{0, 1, 2\}$ *with* $I \cup J = [2]$, $K_I = \cap_{i \in I} \mathrm{Ker}(\partial_i^2 \otimes 1_A)$ *and* $K_J = \cap_{j \in J} \mathrm{Ker}(\partial_j^2 \otimes 1_A)$.

Proof. By [17 or 9, Theorem 2.39(ii)] we know that there is an isomorphism

$$H_2(G, A) = L_1^{\mathcal{P}}(G \otimes A) \approx \pi_1 C_* \tag{11}$$

and an epimorphism

$$H_3(G, A) = L_2^{\mathcal{P}}(G \otimes A) \longrightarrow \pi_2 C_* \,. \tag{12}$$

Now we have to show that $C_2 = P \times_G P \times_G P$ and $C_3 = P \times_G P \times_G P \times_G P$ are generated by degenerate elements. In fact, for any $(x,y,z) \in C_2$ and $a \in A$ one has

$$\begin{gathered}
(x,y,z)\otimes a = (x,x,x)(1,x^{-1}y,x^{-1}y)(1,1,y^{-1}z)\otimes a = \\
({}^{(x,x,x)}(1,x^{-1}y,x^{-1}y)\cdot {}^{(x,x,x)}(1,1,y^{-1}z)\otimes {}^{\alpha(x)}a)((x,x,x)\otimes a) = \\
((1,yx^{-1},yx^{-1})(1,1,xy^{-1}zx^{-1})\otimes {}^{\alpha(x)}a)((x,x,x)\otimes a) = \\
({}^{(1,yx^{-1},yx^{-1})}(1,1,xy^{-1}zx^{-1})\otimes {}^{\alpha(x)}a)((1,yx^{-1},yx^{-1})\otimes {}^{\alpha(x)}a)((x,x,x)\otimes a) = \\
((1,1,zy^{-1})\otimes {}^{\alpha(x)}a)((1,yx^{-1},yx^{-1})\otimes {}^{\alpha(x)}a)((x,x,x)\otimes a) = \\
(s_0\otimes 1_A)((1,zy^{-1})\otimes {}^{\alpha(x)}a)\cdot(s_1\otimes 1_A)((1,yx^{-1})\otimes {}^{\alpha(x)}a)\cdot(s_0\otimes 1_A)((x,x)\otimes a).
\end{gathered}$$

For C_3 the proof is similar.

By [3, Lemma 5.7] and [16, Theorem 4.1] there are equalities

$$\begin{gathered}
\operatorname{Im} d_2 = [\operatorname{Ker}(\partial_0\otimes 1_A), \operatorname{Ker}(\partial_1\otimes 1_A)], \\
\operatorname{Im} d_3 = \textstyle\prod_{I,J}[K_I, K_J],
\end{gathered} \tag{13}$$

where $\emptyset \neq I, J \subset [2] = \{0,1,2\}$ with $I \cup J = [2]$, $K_I = \cap_{i\in I}\operatorname{Ker}(\partial_i^2 \otimes 1_A)$, $K_J = \cap_{j\in J}\operatorname{Ker}(\partial_j^2\otimes 1_A)$ and d_2, d_3 are Moore complex homomorphisms of C_*.

From (11-13) follows the assertion. ■

We need the following calculation of the non-abelian tensor product of groups.

Proposition 13. *Let A be any group acting trivially on $\mathbb{Z}$ and $\mathbb{Z}$ acts on A such that the compatibility conditions (1) hold. Then there is a natural isomorphism*

$$\mathbb{Z}\otimes A \overset{\approx}{\to} A^{ab}, \quad n\otimes a \mapsto [{}^{n-1}a]\cdot[{}^{n-2}a]\cdots[{}^{1}a]\cdot[a].$$

Proof. By [3, Proposition 2.3] $\mathbb{Z}\otimes A$ is an abelian group and $n\otimes {}^{a}a' = n\otimes a'$. Therefore,

$$n\otimes aa' = (n\otimes a)(n\otimes a'). \tag{14}$$

Let define a homomorphism $f : \mathbb{Z}\otimes A \to A^{ab}$ as follows: $n\otimes a \mapsto [{}^{n-1}a]\cdot[{}^{n-2}a]\cdots[{}^{1}a]\cdot[a]$. We have to show that this map preserves the defining relations of the non-abelian tensor product. In fact,

$$\begin{aligned}
&f((n+m)\otimes a) = [{}^{n+m-1}a]\cdots[{}^{1}a]\cdot[a], \\
&f(m, {}^{n}a) = [{}^{m-1}({}^{n}a)]\cdots[{}^{1}({}^{n}a)]\cdot[{}^{n}a] = [{}^{n+m-1}a]\cdots[{}^{n+1}a]\cdot[{}^{n}a].
\end{aligned}$$

Therefore $f((n+m)\otimes a) = f(m\otimes {}^{n}a)f(n\otimes a)$. Next,

$$\begin{aligned}
&f(n\otimes aa') = [{}^{n-1}(aa')\cdots[{}^{1}(aa')]\cdot[aa'], \\
&f(n\otimes {}^{a}a') = [{}^{n-1}a']\cdots[{}^{1}a']\cdot[a'].
\end{aligned}$$

Therefore $f(n\otimes aa') = f(n\otimes a)f(n\otimes {}^{a}a')$.

Now define $g : A^{ab} \to \mathbb{Z} \otimes A$ by $[a] \mapsto 1 \otimes a$. It is easy to see that g is a correctly defined homomorphism. We have to show that gf is identity map. In effect, by (14) and the following formula

$$(n+1) \otimes a = (1 \otimes {}^{n}a)(n \otimes a)$$

we have

$$gf(n \otimes a) = 1 \otimes ({}^{n-1}a \cdots {}^{1}a \cdot a) = (1 \otimes {}^{n-1}a) \cdots (1 \otimes {}^{1}a)(1 \otimes a) =$$
$$= (n \otimes a)((n-1) \otimes a)^{-1} \cdots ((n-1) \otimes a)((n-2) \otimes a)^{-1} \cdots (2 \otimes a)(1 \otimes a)^{-1}(1 \otimes a) =$$
$$= n \otimes a.$$

It is easy to see that fg is also identity map. ■

Example 14. *Let A be a metabelian group and the action of $\mathbb{Z}$ on A is defined by inner automorphism, i.e. ${}^{n}a' = a^{n}a'a^{-n}$ for all $n \in \mathbb{Z}$, $a' \in A$ for some $a \in A$. If A acts on $\mathbb{Z}$ trivially, then in this case the groups $\mathbb{Z}$ and A act on each other compatibly and Proposition 13 can be applied.*

Proposition 15. *Let A be a metabelian group acting on $\mathbb{Z}$ trivially and $\mathbb{Z}$ act on A by inner automorphism (see above). Then there is a natural isomorphism*

$$\mathbb{Z}^2 \otimes A \approx (A^{ab})^2.$$

Proof. Check the conditions of Proposition 10 [1] in our case. One has

(a): ${}^{m}({}^{n}a') = {}^{n}({}^{m}a')$.

(b)-(c): ${}^{n}(m \otimes a') = m \otimes {}^{n}a'$ is an action and since $= m \otimes a^{n}a'a^{-n} = m \otimes a'$, it is trivial.

Therefore by Proposition 10 [1] and Proposition 13 one has natural isomorphisms

$$\mathbb{Z}^2 \otimes A \approx (\mathbb{Z} \otimes A) \times (\mathbb{Z} \otimes A) \approx A^{ab^2}.$$

■

Let us consider an exact sequence of groups $1 \longrightarrow R \xrightarrow{\sigma} F \xrightarrow{\alpha} G \longrightarrow 1$, where $F \in \mathcal{P}$ and α is a $\mathcal{P}$-epimorphism. We have commutative diagrams of groups with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R & = & R & & \\
 & & & & \downarrow \sigma_1 & & \downarrow \sigma & & \\
1 & \longrightarrow & R & \xrightarrow{\sigma_0} & F \times_G F & \xrightarrow{\partial_0} & F & \longrightarrow & 1 \\
 & & \| & & \downarrow \partial_1 & & \downarrow \alpha & & \\
1 & \longrightarrow & R & \xrightarrow{\sigma} & F & \xrightarrow{\alpha} & G & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \qquad (15)$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & \mathrm{Ker}(\sigma_1 \otimes 1_A) & \xrightarrow{\beta} & \mathrm{Ker}(\sigma \otimes 1_A) & \longrightarrow & \mathrm{Coker}\,\beta \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R \otimes A & = & R \otimes A & & \\
 & & & & \downarrow {\scriptstyle \sigma_1 \otimes 1_A} & & \downarrow {\scriptstyle \sigma \otimes 1_A} & & \\
 R \otimes A & \xrightarrow{\sigma_0 \otimes 1_A} & & & (F \times_G F) \otimes A & \xrightarrow{\partial_0 \otimes 1_A} & F \otimes A & \longrightarrow & 1 \\
 \| & & & & \downarrow {\scriptstyle \partial_1 \otimes 1_A} & & \downarrow {\scriptstyle \alpha \otimes 1_A} & & \\
 R \otimes A & \xrightarrow{\sigma \otimes 1_A} & & & F \otimes A & \xrightarrow{\alpha \otimes 1_A} & G \otimes A & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{16}
$$

Define a homomorphism

$$\delta : \mathrm{Ker}(\partial_0 \otimes 1_A) \cap \mathrm{Ker}(\partial_1 \otimes 1_A) \to \mathrm{Coker}\,\beta,$$

by $\delta(x) = [y]$ for all $x \in \mathrm{Ker}(\partial_0 \otimes 1_A) \cap \mathrm{Ker}(\partial_1 \otimes 1_A)$ and with $(\sigma_1 \otimes 1_A)(y) = x$. It is easy to verify that δ is correctly defined and is an isomorphism. From Theorem 12(i) follows

Theorem 16. *Let G and A be any groups acting on each other. Then $H_2(G, A) \approx \mathrm{Coker}\,\beta/\delta([\mathrm{Ker}(\partial_0 \otimes 1_A), \mathrm{Ker}(\partial_1 \otimes 1_A)])$.*

Corollary 17. *If A acts trivially on G, G acts on A and the actions are compatible, then there is an isomorphism*

$$H_2(G, A) \approx \mathrm{Coker}\,\beta.$$

Proof. It is obvious, since by Proposition 2.3 [3] in this case the group $(F \times_G F) \otimes A$ is abelian. ∎

Note that if G acts on A and A acts on G trivially such that the actions are compatible then $H_n(G, A) \approx H_n(G, A^{ab})$, $n \geq 2$, since $G \otimes A \approx G \otimes A^{ab}$.

Let $\mathbb{Z}_n$ act on a group A and A act on $\mathbb{Z}_n$ trivially such that the actions are compatible. Further we assume that $\mathbb{Z}$ acts on A via the canonical homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}_n$ and A acts trivially on $\mathbb{Z}$. Consider the isomorphism $\kappa : A^{ab} \to n\mathbb{Z} \otimes A$ given by $a \mapsto n \otimes a$ and the homomorphism $\sigma_1 \otimes 1_A : n\mathbb{Z} \otimes A \to (\mathbb{Z} \times_\alpha \mathbb{Z}) \otimes A$ (see diagram (16)). Then one has

Proposition 18. *There is an isomorphism*

$$H_2(\mathbb{Z}_n, A) \approx {}_N(A^{ab})/\,\mathrm{Ker}((\sigma_1 \otimes 1_A)\kappa),$$

where ${}_N(A^{ab}) = \mathrm{Ker}\,N$, $N : A^{ab} \to A^{ab}$ with $N([a]) = \sum_{x \in \mathbb{Z}_n} {}^x[a]$.

Proof. It is clear that α is a $\mathcal{P}$-epimophism (see Proposition 1). Using Proposition 13 one has the following commutative diagram

$$\begin{array}{ccc} n\mathbb{Z}\otimes A & \xrightarrow{\kappa^{-1}} & A^{ab} \\ \downarrow{\scriptstyle \sigma\otimes 1_A} & & \downarrow{\scriptstyle N} \\ \mathbb{Z}\otimes A & \overset{\approx}{\to} & A^{ab} \end{array} .$$

It remains to apply Corollary 17. ■

Let us consider the category $A_{\mathfrak{Mod}}$ of A-modules. In this category consider the following projective class $\mathfrak{P}$:

An object of $\mathfrak{P}$ is a free abelian group having a basis X such that ${}^{a}x\in X$ for any $x\in X$, $a\in A$.

It is clear that the projective class $\mathfrak{P}$ includes all free A-modules. For any additive functor $T : A_{\mathfrak{Mod}} \to ab\mathfrak{Gr}$(abelian groups) consider its left derived functors $L_n^{\mathfrak{P}}T$, $n\geq 0$.

For any short exact sequence of A-modules $0\to B_1\xrightarrow{i} B\xrightarrow{j} B_2\to 0$, where j is a $\mathfrak{P}$-epimorphism i.e. with an action preserving section, one has a long exact sequence of the left derived functors $L_n^{\mathfrak{P}}T$ of T.

Consider the following commutative diagram of groups with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R & \overset{\sigma}{\to} & F & \overset{\alpha}{\to} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tau'} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Ker}(\alpha^{ab}) & \overset{\sigma'}{\to} & F^{ab} & \xrightarrow{\alpha^{ab}} & G^{ab} & \longrightarrow & 1 \end{array} ,$$

where $G\in \mathrm{ob}\,\mathcal{A}_A$, F is an object of the projective class $\mathcal{P}$ in the category $\mathcal{A}_A$ (see above) and α is a $\mathcal{P}$-epimorphism.

Theorem 19. *Let G be an abelian group acting on a group A trivially and A acts on G. Then one has an exact sequence of groups*

$$(\mathrm{Ker}(\sigma\otimes 1_A)\cap \mathrm{Ker}(\tau'\otimes 1_A))^{ab}\longrightarrow H_2(G,A)\longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(G,IA)\longrightarrow 0,$$

where IA is the augmentation ideal of A.

Proof. It is easy to show that in this case τ' is surjective. So one gets the following commutative diagram of groups with exact rows and columns

$$\begin{array}{ccccccccccc} & & & & [F,F]\otimes A & = & [F,F]\otimes A & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathrm{Ker}(\sigma\otimes 1_A) & \longrightarrow & R\otimes A & \xrightarrow{\sigma\otimes 1_A} & F\otimes A & \xrightarrow{\alpha\otimes 1_A} & G\otimes A & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \tau'\otimes 1_A} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Ker}(\sigma'\otimes 1_A) & \longrightarrow & \mathrm{Ker}(\alpha^{ab})\otimes A & \xrightarrow{\sigma'\otimes 1_A} & F^{ab}\otimes A & \xrightarrow{\alpha^{ab}\otimes 1_A} & G\otimes A & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & 1 & & 1 & & & & \end{array} .$$

It will be shown that the natural epimorphism

$$\lambda : \mathrm{Ker}(\sigma \otimes 1_A) \longrightarrow \mathrm{Ker}(\sigma' \otimes 1_A)$$

reduced to $\mathrm{Ker}(\sigma_1 \otimes 1_A)$ (see diagram (16)) and to $\delta([\mathrm{Ker}(\partial_0 \otimes 1_A), \mathrm{Ker}(\partial_1 \otimes 1_A)])$ is trivial. In effect, the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & R & \xrightarrow{\sigma_1} & F \times_G F & \xrightarrow{\partial_1^1} & F & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Ker}(\partial_1^{1\,ab}) & \longrightarrow & (F \times_G F)^{ab} & \xrightarrow{\partial_1^{1\,ab}} & F^{ab} & \longrightarrow & 1
\end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccc}
R \otimes A & \xrightarrow{\sigma_1 \otimes 1_A} & (F \times_G F) \otimes A & \xrightarrow{\partial_1^1 \otimes 1_A} & F \otimes A & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ker}(\partial_1^{1\,ab}) \otimes A & \longrightarrow & (F \times_G F)^{ab} \otimes A & \xrightarrow{\partial_1^{1\,ab} \otimes 1_A} & F^{ab} \otimes A & \longrightarrow & 1
\end{array} \tag{17}$$

which maps to the diagram

$$\begin{array}{ccccccc}
R \otimes A & \xrightarrow{\sigma \otimes 1_A} & F \otimes A & \xrightarrow{\alpha \otimes 1_A} & G \otimes A & \longrightarrow & 1 \\
\downarrow \tau' \otimes 1_A & & \downarrow & & \| & & \\
\mathrm{Ker}(\alpha^{ab}) \otimes A & \xrightarrow{\sigma' \otimes 1_A} & F^{ab} \otimes A & \xrightarrow{\alpha^{ab} \otimes 1_A} & G \otimes A & \longrightarrow & 1
\end{array} \tag{18}$$

using the triple $(1_R, \partial_0^1, \alpha)$ (see diagram (15)).

To this end we need the fact that the homomorphism $\mathrm{Ker}(\partial_1^{1\,ab}) \otimes A \to (F \times_G F)^{ab} \otimes A$ is injective. The short exact sequence of A-modules

$$1 \longrightarrow \mathrm{Ker}(\partial_1^{1\,ab}) \longrightarrow (F \times_G F)^{ab} \xrightarrow{\partial_1^{1\,ab}} F^{ab} \longrightarrow 1$$

has a section γ given by $\gamma[f] = [(f, f)]$ which preserves the actions of A. Therefore it induces a long exact sequence of the left derived functors of $- \otimes_{\mathbb{Z}[A]} IA$ with relative to the projective class $\mathfrak{P}$

$$\cdots \longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(F^{ab}, IA) \longrightarrow \mathrm{Ker}(\partial_1^{1\,ab}) \otimes_{\mathbb{Z}[A]} IA \longrightarrow (F \times_G F)^{ab} \otimes_{\mathbb{Z}[A]} IA \longrightarrow$$
$$\longrightarrow F^{ab} \otimes_{\mathbb{Z}[A]} IA \longrightarrow 0.$$

Since F^{ab} belongs to the projective class $\mathfrak{P}$, one has $\mathrm{Tor}_1^{\mathfrak{P}}(F^{ab}, IA) = 0$. By Guin's isomorphism [7] the groups $\mathrm{Ker}(\partial_1^{1\,ab}) \otimes_{\mathbb{Z}[A]} IA$, $(F \times_G F)^{ab} \otimes_{\mathbb{Z}[A]} IA$ and $F^{ab} \otimes_{\mathbb{Z}[A]} IA$ are naturally isomorphic to the groups $\mathrm{Ker}(\partial_1^{1\,ab}) \otimes A$, $(F \times_G F)^{ab} \otimes A$ and $F^{ab} \otimes A$ respectively. This implies that the homomorphism $\mathrm{Ker}(\partial_1^{1\,ab}) \otimes A \to (F \times_G F)^{ab} \otimes A$ is injective.

Using (17,18) diagrams the triviality of λ on $\mathrm{Ker}(\sigma_1 \otimes 1_A)$ and $\delta([\mathrm{Ker}(\partial_0 \otimes 1_A), \mathrm{Ker}(\partial_1 \otimes 1_A)])$ is now clear.

By the same reason the short exact sequence of A modules

$$0 \longrightarrow \mathrm{Ker}(\alpha^{ab}) \longrightarrow F^{ab} \xrightarrow{\alpha^{ab}} G \longrightarrow 0,$$

which has a natural needed section, gives the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathfrak{P}}(G, IA) \longrightarrow \mathrm{Ker}(\alpha^{ab}) \otimes A \xrightarrow{\sigma' \otimes 1_A} F^{ab} \otimes A \xrightarrow{\alpha^{ab} \otimes 1_A} G \otimes A \longrightarrow 0\ .$$

It follows by Theorem 16 that λ induces a natural epimorphism $H_2(G, A)$ on $\mathrm{Tor}_1^{\mathfrak{P}}(G, IA)$ and by the homomorphism δ the group $(\mathrm{Ker}(\sigma \otimes 1_A) \cap \mathrm{Ker}(\tau' \otimes 1_A))$ maps on its kernel. ■

Note that $\mathrm{Tor}_1^{\mathfrak{P}}(G, IA) = \mathrm{Coker}\, \mathrm{Tor}_1(\alpha^{ab}, IA)$.

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A.Razmadze Mathematical Institute, Georgian Academy, of Sciences, M.Alexidze St. 1, Tbilisi 380093. Georgia

E-mail address: hvedri@rmi.acnet.ge, inas@rmi.acnet.ge