NON-ABELIAN TENSOR PRODUCTS AND NON-ABELIAN HOMOLOGY OF GROUPS

NIKO INASSARIDZE

Some functorial propeties of the non-abelian tensor product of groups are established. With the use of the non-abelian left derived functors [7, 8] the homology groups of groups are constructed with coefficients in any group, as the left derived functors of the non-abelian tensor product, which generalize the classical theory of homology of groups. Exact sequences of the non-abelian homology groups and their application to algebraic K-theory of noncommutative local rings are given.
For this we extend the non-abelian tensor product of groups introduced by Brown and Loday [2-4, 5, 10] to arbitrary pairs G, H of groups which act on themselves by conjugation and each of which acts on the other. We do not demand as in [2-4] that the compatibility conditions

$$^{(^g h)}g' = ^{ghg^{-1}} g', \quad ^{(^h g)}h' = ^{hgh^{-1}} h' \tag{1}$$

hold, where $g, g' \in G$, $h, h' \in H$, and ghg^{-1}, hgh^{-1} are elements of the free product $G * H$.

The tensor product $G \otimes H$ is the group generated by the symbols $g \otimes h$ and defined by the relations

$$\begin{aligned} gg' \otimes h &= (^g g' \otimes ^g h)(g \otimes h) \\ g \otimes hh' &= (g \otimes h)(^h g \otimes ^h h') \\ (g \otimes h)(g' \otimes h') &= (^{[g,h]}g' \otimes ^{[g,h]} h')(g \otimes h) \\ (g' \otimes h')(g \otimes h) &= (g \otimes h)(^{[h,g]}g' \otimes ^{[h,g]} h') \end{aligned}$$

for all $g, g' \in G$ and $h, h' \in H$, where $[g,h] = ghg^{-1}h^{-1} \in G * H$.

In order to obtain the characterizing universal property of the non-abelian tensor product of groups we need the notion of a crossed pairing. A map $\Phi : G \times H \to L$ where G, H and L are groups, is called a crossed pairing if for all $g, g' \in G$ and $h, h' \in H$,

$$\begin{aligned} \Phi(gg', h) &= \Phi(^g g', ^g h)\Phi(g, h) \\ \Phi(g, hh') &= \Phi(g, h)\Phi(^h g, ^h h') \\ \Phi(g, h)\Phi(g', h') &= \Phi(^{[g,h]}g', ^{[g,h]} h')\Phi(g, h) \\ \Phi(g', h')\Phi(g, h) &= \Phi(g, h)\Phi(^{[h,g]}g', ^{[h,g]} h'). \end{aligned}$$

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Definition 1. The non-abelian tensor product of groups G and H is a group A and a crossed pairing $f : G\times H \to A$ such that for every group D and every crossed pairing $f' : G\times H \to A$ there exists a unique homomorphism $\alpha : A \to D$ such that $\alpha f = f'$.

It is easy to see the above-mentioned tensor product $G\otimes H$ satisfies this definition and it is unique up to isomorphism. We have the isomorphism $\gamma : G\otimes H \to H\otimes G$, $\gamma(g\otimes h) = (h\otimes g)^{-1}$. It is clear also that if the actions of the pair G, H of groups satisfy the compatibility conditions (1) then we obtain the tensor product of Brown and Loday.

Therefore if G and H act trivially on each other,then $G\otimes H \approx G^{ab}\otimes H^{ab}$ (the tensor product of the abelianizations of G and H).It follows also that if H is an abelian group which acts trivially on G, then we have the isomorphism of Guin [6] $G\otimes H \approx IG \otimes_G H$, where $IG = \operatorname{Ker}\epsilon, \epsilon : \mathbf{Z}[G] \to \mathbf{Z}$, $\epsilon(\sum_i n_i g_i) = \sum_i n_i$ and this isomorphism is given by $g\otimes \alpha \mapsto (g-1)\otimes \alpha$ (see [6]).

Suppose $\Theta : G\to A$,$\Phi : H \to B$ are homomorphism of groups, A,B act on each other,and Θ, Φ preserve the actions in the sense that

$$\Phi({}^g h) = {}^{\Theta g}(\Phi h), \quad \Theta({}^h g) = {}^{\Phi h}(\Theta g)$$

for all $g\in G$,$h \in H$.Then there is a unique homomorphism $\Theta\otimes\Phi : G\otimes H \to A\otimes B$ such that $(\Theta\otimes\Phi)(g\otimes h) = \Theta g\otimes \Phi h$ for all $g\in G$,$h\in H$.Further,if Θ,Φ are onto, so also is $\Theta\otimes\Phi$.

In fact we have

$$\begin{gathered}
(\Theta\times\Phi)(gg', h) = \Theta g\Theta g'\otimes \Phi h = ({}^{\Theta g}\Theta g'\otimes {}^{\Theta g}\Phi h)(\Theta g\otimes\Phi h) = \\
= (\Theta({}^g g')\otimes\Phi({}^g h))(\Theta g\otimes \Phi h) = (\Theta\times\Phi)({}^g g', {}^g h)(\Theta\times\Phi)(g,h), \\
(\Theta\times\Phi)(g, hh') = \Theta g\otimes\Phi h\Phi h' = (\Theta g\otimes\Phi h)({}^{\Phi h}\Theta g\otimes {}^{\Phi h}\Phi h' = \\
= (\Theta g\otimes \Phi h)(\Theta({}^h g)\otimes\Phi({}^h h')) = (\Theta\times\Phi)(g,h)(\Theta\times\Phi)({}^h g, {}^h h'), \\
(\Theta\times\Phi)(g,h)(\Theta\times\Phi)(g',h')(\Theta\times\Phi)(g,h)^{-1} = \\
= (\Theta g\otimes\Phi h)(\Theta g'\otimes\Phi h')(\Theta g\otimes\Phi h)^{-1} = \\
= {}^{[\Theta g,\Phi h]}\Theta g'\otimes {}^{[\Theta g,\Phi h]})\Phi h' = \Theta({}^{[g,h]}g')\otimes\Phi({}^{[g,h]}h') = (\Theta\times\Phi)({}^{[g,h]}g', {}^{[g,h]}h').
\end{gathered}$$

Analogously we can verify the fourth relation of the crossed pairing.

Thus the map $\Theta\times\Phi : G\times H \to A\otimes B$ is a crossed pairing and $\Theta\otimes\Phi$ is the induced homomorphism.

We now investigate for the non-abelian tensor product the connection with the group of crossed homomorphisms as adjoint functors and the properties of exactness and compatibility with the direct limits of groups.

Theorem 1.

(a) *Suppose*

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1$$

is a short exact sequence of groups, D is an arbitrary group which acts on A, B and C, the groups A, B, C also act on D and f, g preserve the actions.

Then we have the following exact sequence of groups:

$$D \otimes A \xrightarrow{f'} D \otimes B \xrightarrow{g'} D \otimes C \to 1,$$

where $f' = 1 \otimes f$, $g' = 1 \otimes g$.

(b) *Suppose*

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C \to 1, \tag{2}$$

$$1 \to D \xrightarrow{\phi} E \xrightarrow{\psi} F \to 1 \tag{3}$$

are short exact sequences of groups,where A *and* D, B *and* E, C *and* F *act on each other,* f *and* ϕ, g *and* ψ *preserve the actions. Then the following sequence of groups*

$$(A \otimes E) \times (B \otimes D) \xrightarrow{\alpha} B \otimes E \xrightarrow{g \otimes \psi} C \otimes F \to 1$$

is exact, where α *is a map of sets.*

Proof. (a) First we shall show that $\operatorname{Im} f'$ is an invariant subgroup of $D \otimes B$. Let $\prod_{i=1}^{k}(d_i \otimes f(a_i))^{n_i} \in \operatorname{Im} f'$.It suffices to prove that

$$(d \otimes b)\prod_{i=1}^{k}(d_i \otimes f(a_i))^{n_i}(d \otimes b)^{-1} \in \operatorname{Im} f',$$

It is clear that $(d\otimes b)\prod_{i=1}^{k}(d_i \otimes f(a_i))^{n_i}(d\otimes b)^{-1} = (d\otimes b)(d_1 \otimes f(a_1))(d\otimes b)^{-1}(d\otimes b)\cdots(d\otimes b)(d_k \otimes f(a_k))(d\otimes b)^{-1} = \prod_{i=1}^{k}({}^{[d,b]}d_i \otimes {}^{[d,b]}f(a_i))^{n_i} \in \operatorname{Im} f'$, since each ${}^{dbd^{-1}b^{-1}}f(a_i) = {}^{d}(b({}^{d^{-1}}(b^{-1}f(a_i)b))b^{-1})$ is an element of $\operatorname{Im} f = \operatorname{Ker} g$. Therefore we have diagram

$$\begin{array}{ccccccc}
D \otimes A & \xrightarrow{f'} & D \otimes B & \xrightarrow{g'} & D \otimes C & \longrightarrow & 1 \\
\| & & \| & & & & \\
D \otimes A & \xrightarrow{f'} & D \otimes B & \xrightarrow{\tau} & \operatorname{Coker} f' & \longrightarrow & 1,
\end{array}$$

where $g'f'(x) = 1$, $x \in D \otimes A$ and the bottom row is exact. Thus, there exists a natural homomorphism $\beta : \operatorname{Coker} f' \to D \otimes C$ such that $\beta\tau = g'$.

Now define a crossed pairing $\bar\gamma : D \times C \to \operatorname{Coker} f'$ as follows: $\bar\gamma(d,c) = [d \otimes b]$, where $g(b) = c$. It is correctly defined. If $b' = b \cdot f(a)$, then $d \otimes b' = d \otimes b \cdot f(a) = (d \otimes b)({}^{b}d \otimes {}^{b}f(a))$. Thus, $(d \otimes b)^{-1}(d \otimes b') = {}^{b}d \otimes {}^{b}f(a) \in \operatorname{Im} f'$. Next, $\bar\gamma(dd', c) = [dd' \otimes b] = [{}^{d}d' \otimes {}^{d}b][d \otimes b] = \bar\gamma({}^{d}d' \otimes {}^{d}c)\bar\gamma(d,c)$, where $g(b) = c$. Further $\bar\gamma(d, cc') = [d \otimes bb'] = [d \otimes b][{}^{g(b)}d \otimes {}^{b}b'] = \bar\gamma(d,c)\bar\gamma({}^{c}d, {}^{c}c'), g(b') = c'$. $\bar\gamma(d,c)\bar\gamma(d',c')\bar\gamma(d,c)^{-1} = [d\otimes b][d'\otimes b'][d\otimes b]^{-1} = [{}^{[d,b]}d' \otimes {}^{[d,b]}b'] = \bar\gamma({}^{[d,c]}d', {}^{[d,c]}c')$, similarly for the fourth relation.

Therefore, $\bar{\gamma}$ is a crossed pairing and it induces a homomorphism $\gamma : D \otimes C \to \operatorname{Coker} f'$ such that $\gamma(d \otimes c) = [d \otimes b], g(b) = c$. It is easy to see that $\beta\gamma$ and $\gamma\beta$ are identity maps.

(b) Define the map

$$\alpha : (A \otimes E) \times (B \otimes D) \to B \otimes E$$

as follows:

$$\Big(\prod_i (a_i \otimes e_i),\ \prod (b_i \otimes d_i)\Big) \mapsto \prod_i (f(a_i) \otimes e_i) \cdot \prod (b_i \otimes \Phi(d_i)).$$

It is clear that $\operatorname{Im} \alpha = \operatorname{Im}(f \otimes 1_E) \cdot \operatorname{Im}(1_B \otimes \phi)$. From the proof of (a) it follows that the groups $\operatorname{Im}(f \otimes 1_E)$, $\operatorname{Im}(1_B \otimes \phi)$ are invariant subgroups of $B \otimes E$.It is easy to see that $\operatorname{Im} \alpha \subseteq \operatorname{Ker}(g \otimes \psi)$ and thus we have a natural homomorphism

$$\pi' : B \otimes E / \operatorname{Im} \alpha \to C \otimes F$$

Next,define a crossed pairing $\bar{p}' : C \times F \to B \otimes E / \operatorname{Im} \alpha$ as follows: $\bar{p}'(c,l) = [b \otimes e]$, where $g(b) = c$, $\psi(e) = l$ and show the correctness. If $b' = b \cdot f(a)$, $e' = e\phi(d)$, then $b' \otimes e' = bf(a) \otimes e\phi(d) = ({}^b f(a) \otimes {}^b e^b \phi(d))(b \otimes e\phi(d)) = ({}^b f(a) \otimes {}^b e\phi(d))\ (b \otimes e)$ $({}^e b \otimes {}^e \phi(d))$. Since ${}^e b \otimes {}^e \phi(d)$ is an element of the invariant subgroup $\operatorname{Im}(1_B \otimes \phi)$,it follows that $(b' \otimes e')(b \otimes e)^{-1} \in \operatorname{Im} \alpha$. Next, $\bar{p}(cc', l) = [bb' \otimes e]$, where $g(b) = c$, $g(b') = c'$, $\psi(e) = l$; $\bar{p}'({}^c c', {}^c l) = [{}^b b' \otimes {}^b e]$ since $\psi({}^b e) = {}^{g(b)}\psi(e) = {}^c l$. Thus, $\bar{p}'(cc', l) = \bar{p}'({}^c c', {}^c l)\bar{p}'(c, l)$. Similarly we can prove the second and the third parts of the crossed pairing. Therefore,the map $\bar{p}'$ is a crossed pairing and it induces a homomorphism $p' : C \otimes F \longrightarrow B \otimes E / \operatorname{Im} \alpha$ such that $p'(c \otimes l) = [b \otimes e]$, where $g(b) = c$, $\psi(e) = l$. It is easy to see that $\pi' p'$ and $p'\pi'$ are identity maps. The proof of Theorem 1 is complete.

Remark 1. If the sequences (2),(3) coincide and are central extensions,then from Theorem 1 (b) follows the result of [2] (Proposition 9) and in this case the map α is a homomorphism. Moreover, if the sequence (3) is $1 \to 1 \to D \xrightarrow{1_D} D \to 1$, then we obtain the first part of Theorem 1.

Theorem 2. *Let A and B be groups which act on each other, such that the compatibility conditions* (1) *hold. Let C be a group on which act the groups A and B. Then there is a natural bijection*

$$\tilde{F}(A, Z^1(B, C)) \approx \widetilde{\operatorname{Hom}}(A \otimes B, C),$$

where $\widetilde{\operatorname{Hom}}(A \otimes B, C)$ is the set of homomorphisms satisfying the following conditions:

1. $g(aa' \otimes b) = g(a'a \otimes b)$,
2. $g^{a'}(a \otimes b) \equiv g({}^{a'}a \otimes^{a'} b) = {}^{a'}g(a \otimes b)$,
3. $g^{b'}(a \otimes b) \equiv g({}^{b'}a \otimes {}^{b'}b) = {}^{b'}g(a \otimes b)$,

$\tilde{F}(A, Z^1(B,C))$ is the set of maps from A to the set $Z^1(B,C)$ of crossed homomorphisms satisfying the following conditions:

4. $[f(aa')](b) = {}^a[f(a')](b) \cdot [f(a)](b) = {}^{a'}[f(a)](b) \cdot [f(a')](b)$,
5. $[f(a)]({}^{a'}b) = {}^{a'}[f(a)](b')$,
6. $[f({}^{b'}a)]({}^{b'}b) = {}^{b'}[f(a)](b)$.

Proof. Define the maps $\alpha : \tilde{F}(A, Z^1(B,C)) \longrightarrow \widetilde{\mathrm{Hom}}(A \otimes B, C)$, $\gamma : \widetilde{\mathrm{Hom}}(A \otimes B, C) \longrightarrow \tilde{F}(A, Z^1(B,C))$ as follows:

$$[\alpha(f)](a \otimes b) = [f(a)](b),\ f \in \tilde{F}(A, Z^1(B,C)),\ a \in A,\ b \in B$$
$$[(\gamma(g))(a)](b) = g(a \otimes b),\ g \in \widetilde{\mathrm{Hom}}(A \otimes B, C),\ a \in A,\ b \in B.$$

We shall show $[(\gamma(g))(a)] \in Z^1(B,C)$. We have $[(\gamma(g))(a)](bb') = g(a \otimes bb') = g(a \otimes b)g({}^ba \otimes {}^bb') = g(a \otimes b){}^bg(a \otimes b') = [(\gamma(g))(a)](b) \cdot {}^b[(\gamma(g))(a)](b')$. Now we must show that $\gamma(g) \in\sim F(A, Z^1(B,C))$. In fact $[(\gamma(g))(aa')](b) = g(aa' \otimes b)$ $= g({}^aa' \otimes {}^ab) \cdot g(a \otimes b) = {}^ag(a' \otimes b)g(a \otimes b) = {}^a[(\gamma(g))(a')](b) \cdot [(\gamma(g))(a)](b) = g(a'a \otimes b) = {}^{a'}[(\gamma(g))(a)](b) \cdot [(\gamma(g))(a')](b)$. Furthermore, $[(\gamma(g))(a)]({}^{a'}b) = g(a \otimes {}^{a'}b) = g({}^{a'}a \otimes {}^{a'}b) = {}^{a'}g(a \otimes b) = {}^{a'}[(\gamma(g))(a)](b)$, $[(\gamma(g))({}^{b'}a)]({}^{b'}b) = g({}^{b'}a \otimes {}^{b'}b) = {}^{b'}g(a \otimes b) = {}^{b'}[(\gamma(g))(a)](b)$.

We shall prove that $\alpha(f) \in \widetilde{\mathrm{Hom}}(A \otimes B, C)$. For this consider the map $\overline{\alpha(f)} : A \times B \longrightarrow C$, $[\overline{\alpha(f)}](a,b) = [f(a)](b)$. We have to show that this map is a crossed pairing. We have $[\overline{\alpha(f)}](aa',b) = [f(aa')](b) = {}^a[f(a')](b) \cdot [f(a)](b)$, $[\overline{\alpha(f)}]({}^aa', {}^ab) = [f({}^aa')]({}^ab) = {}^a[f(a')](b)$. Thus $[\overline{\alpha(f)}](aa',b) = [\overline{\alpha(f)}]({}^aa', {}^ab) \cdot [\overline{\alpha(f)}](a,b)$. Next, $[\overline{\alpha(f)}](a,bb') = [f(a)](bb') = [f(a)](b) \cdot {}^b[f(a)](b')$, $[\overline{\alpha(f)}]({}^ba, {}^bb') = [f({}^ba)]({}^bb') = {}^b[f(a)](b')$. So, $[\overline{\alpha(f)}](a,bb') = [\overline{\alpha(f)}](a,b) \cdot [\overline{\alpha(f)}]({}^ba, {}^bb')$.

Therefore, the map $\overline{\alpha(f)}$ is a crossed pairing and it induces a homomorphism $\alpha(f) : A \otimes B \to C$. Now we prove that it satisfies conditions 1, 2 and 3 of Theorem 2.

We have $[\alpha(f)](aa' \otimes b) = [f(aa')](b) = [f(a'a)](b) = [\alpha(f)](a'a \otimes b)$, $[\alpha(f)]({}^{a'}(a \otimes b)) = [\alpha(f)]({}^{a'}a \otimes {}^{a'}b) = [f({}^{a'}a)]({}^{a'}b) = {}^{a'}[f(a)](b) = {}^{a'}[\alpha(f)](a \otimes b)$,$[\alpha(f)]({}^{b'}(a \otimes b)) = [\alpha(f)]({}^{b'}a \otimes {}^{b'}b) = [f({}^{b'}a)]({}^{b'}b) = {}^{b'}[f(a)](b) = {}^{b'}[\alpha(f)](a \otimes b)$.

It is clear that $\alpha\gamma$ and $\gamma\alpha$ are identity maps. The proof is complete.

In the case when the groups A, B and C are abelian and they act trivially, Theorem 2 is the well known result that Hom and $\otimes$ are adjoint functors. The adjoining of these functors was extended to the category of precrossed modules (see [6,10]).

We will show that Theorem 2 is a nontrivial generalization.

Example 1. A and B are abelian groups, C is an arbitrary group and the actions are trivial.

Example 2. Let $A = B$ be an arbitrary group,$C = A/H$,where H is the invariant subgroup of A generated by the elements $xyzy^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1} (x, y, z \in A)$;the group A acts on C as follows:${}^a[a'] = [aa'a^{-1}]$, $a, a' \in A$. Since in the group A/H holds the equality

$$xyzy^{-1}x^{-1}yxz^{-1}x^{-1}y^{-1} = e,$$

we have $x^{-1}y^{-1}xyz = zx^{-1}y^{-1}xy$ for $x, y, z \in A/H$.Thus,every commutator of A/H is central. Therefore, A/H is the maximal homomorphic metabelian image of A. Consider the homomorphisms $A \otimes A \xrightarrow{\gamma} A \xrightarrow{\tau} A/H$, where $\gamma(a \otimes a') = aa'a^{-1}a'^{-1}$ and τ is the canonical map.Then the homomorphism $g = \tau\gamma$, which in general is not trivial,satisfies conditions 1,2 and 3 of Theorem 2. In fact $g(aa' \otimes a'') = [aa'a''a'^{-1}a^{-1}a''^{-1}] = [a'aa''a^{-1}a'{-}1a''^{-1}] = g(a'a \otimes a'')$. Next, $g({}^{a'}a \otimes {}^{a'}a'') = [{}^{a'}a^{a'}a''({}^{a'}a)^{-1}({}^{a'}a'')^{-1}] = [a'aa''a^{-1}a''^{-1}a'^{-1}] = {}^{a'}[aa''a^{-1}a''^{-1}] = {}^{a'}g(a \otimes a'')$.

Example 3. Let A be an abelian group, B an arbitrary group and the actions on each other be compatible. Put $C = A \otimes B$ and ${}^{a'}(a \otimes b) = {}^{a'}a \otimes {}^{a'}b$, ${}^{b'}(a \otimes b) = {}^{b'}a \otimes {}^{b'}b$. Then it is clear that the identity map $A \otimes B \to A \otimes B$ satisfies conditions 1, 2 and 3.

Theorem 3. *Let $\{G_\alpha, \phi_\alpha^\beta, \alpha \le \beta\}$ be a directed system of groups.Let A be a group and let for every α the groups A, G_α act on each other and the homomorphisms ϕ_α^β preserve the actions. Then there is a natural isomorphism*

$$(\varinjlim_\alpha \{G_\alpha\}) \otimes A \approx \varinjlim_\alpha \{G_\alpha \otimes A\}$$

Proof. Define the actions of $\varinjlim_\alpha \{G_\alpha\}$ and A on each other as follows: ${}^{[g_\alpha]}a = {}^{g_\alpha}a$, ${}^a[g_\alpha] = [{}^a g_\alpha]$ for all $g_\alpha \in G_\alpha$, $a \in A$. It is clear that these actions are correct. Next, define the map

$$\bar\kappa : (\varinjlim_\alpha \{G_\alpha\}) \times A \to \varinjlim_\alpha \{G_\alpha \otimes A\}$$

as follows: $\bar\kappa([g_\alpha], a) = [g_\alpha \otimes a]$. If $[g_\alpha] = [g_\beta]$, then there exists $\gamma \ge \alpha, \beta$ such that $\phi_\alpha^\gamma(g_\alpha) = \phi_\beta^\gamma(g_\beta)$. Thus, $(\phi_\alpha^\gamma)'(g_\alpha \otimes a) = \phi_\alpha^\gamma(g_\alpha) \otimes a = \phi_\beta^\gamma(g_\beta) \otimes a = (\phi_\beta^\gamma)'(g_\beta \otimes a)$, where $(\phi_\alpha^\gamma)'$, $(\phi_\beta^\gamma)'$ are induced respectively by ϕ_α^γ, ϕ_β^γ. Therefore, $[g_\alpha \otimes a] = [g_\beta \otimes a]$. We show that $\bar\kappa$ is a crossed pairing. We have $\bar\kappa([g_\alpha] \cdot [g_\beta], a) = [\phi_\alpha^\gamma(g_\alpha)\phi_\beta^\gamma(g_\beta) \otimes a]$, $\gamma \ge \alpha, \beta$; $\bar\kappa({}^{[g_\alpha]}[g_\beta], {}^{[g_\alpha]}a) = [\phi_\alpha^\gamma(g_\alpha)\phi_\beta^\gamma(g_\beta) \cdot \phi_\alpha^\gamma(g_\alpha^{-1}) \otimes {}^{g_\alpha}a]$; $\bar\kappa([g_\alpha], a) = [\phi_\alpha^\gamma(g_\alpha) \otimes a]$. Thus, $\bar\kappa([g_\alpha][g_\beta], a) = \bar\kappa({}^{[g_\alpha]}[g_\beta], {}^{[g_\alpha]}a)\bar\kappa([g_\alpha], a)$. Similarly we can prove the second, the third and the fourth parts of the crossed pairing. Therefore,the map $\bar\kappa$ induces a homomorphism $\kappa : (\varinjlim_\alpha \{G_\alpha\}) \otimes A \to \varinjlim_\alpha \{G_\alpha \otimes A\}$. On the other hand, the canonical homomorphisms $\phi_\alpha : G_\alpha \otimes A \to (\varinjlim_\alpha \{G_\alpha\}) \otimes A$, $\phi_\alpha(g_\alpha \otimes a) = [g_\alpha] \otimes a$, induce a homomorphism $\kappa' : \varinjlim_\alpha \{G_\alpha \otimes A\} \to (\varinjlim_\alpha \{G_\alpha\}) \otimes A$ and it is easy to see that $\kappa\kappa'$, $\kappa'\kappa$ are identity maps.

It would be preferable to formulate Theorem 3 in a more categorical way in terms of adjunction of functors amplying the preservation of colimits. In our general case the existence of the right adjoint functor for the non-abelian tensor product of groups is not known.

Now we consider the category $\mathcal{A}_A$ of groups acting on a fixed group A and the group A acts on these groups. Morphisms of the category $\mathcal{A}_A$ are homomorphisms of groups which preserve the actions. Let $F : \mathcal{A}_A \to \mathcal{A}_A$ be an endofunctor defined as follows: for $B \in \text{ob}\ \mathcal{A}_A$ the object $F(B)$ is the free group generated by B with actions: ${}^{|b_1|^{\epsilon_1}\dots|b_s|^{\epsilon_s}}a = {}^{b_1^{\epsilon_1}}(\cdots({}^{b_s^{\epsilon_s}}a)\cdots)$, and ${}^a(|b_1|^{\epsilon_1}\dots|b_s|^{\epsilon_s}) = |{}^ab_1|^{\epsilon_1}\dots|{}^ab_s|^{\epsilon_s}$ for $a \in A$, $|b_1|^{\epsilon_1}\dots|b_s|^{\epsilon_s} \in F(B)$ and $\epsilon_i = \pm 1$. If $f : B \to B'$ is a morphism of $\mathcal{A}_A$, then the canonical homomorphism $F(f)$ preserves the actions. We note that the actions of the groups $F(B)$ and A do not satisfy in general the compatibility conditions (1).

Let $\tau : F \to 1_{\mathcal{A}_A}$ be the obvious natural transformation and let $\delta : F \to F^2$ be the natural transformation induced for every $B \in \text{ob}\ \mathcal{A}_A$ by the injection $B \to F(B)$. It is easy to verify that the homomorphisms τ_B and δ_B preserve the actions and therefore are morphisms of $\mathcal{A}_A$. We obtain a cotriple $\mathcal{F} = (F, \tau, \delta)$, which we call the free cotriple in the category $\mathcal{A}_A$.

If $\mathcal{P}$ is the projective class of objects in the category $\mathcal{A}_A$ induced by the free cotriple (F, τ, δ), we will show that any object $B \in \text{ob}\ \mathcal{A}_A$ has a $\mathcal{P}$-projective pseudosimplicial resolution (for the definition see [7, 12]). Consider the canonical $\mathcal{P}$-projective resolution of B in the category of groups

$$\cdots \; \vdots\, F(L_nB) \xrightarrow{\tau_{L_nB}} L_nB \underset{l_n^n}{\overset{l_0^n}{\vdots}} \cdots \Rrightarrow F(L_2B) \xrightarrow{\tau_{L_2B}} L_2B \underset{l_2^2}{\overset{l_0^2}{\Rrightarrow}} F(L_1B) \xrightarrow{\tau_{L_1B}}$$

$$\xrightarrow{\tau_{L_1B}} L_1B \underset{l_1^1}{\overset{l_0^1}{\rightrightarrows}} F(B) \xrightarrow{\partial_0^0} B, \tag{4}$$

where $(L_{n+1}B, l_0^{n+1}, \dots, l_{n+1}^{n+1})$, $n \geq 0$ is the simplicial kernel of the sequence of morphisms $(l_0^n\tau_{L_nB}, \dots, l_n^n\tau_{L_nB})$ (for the definition see [7, 12]). Each $F(L_nB)$ acts on A as follows: ${}^xa = {}^{\partial_0^0\partial_0^1\cdots\partial_0^n x}a$, where $\partial_0^0 = \tau_B$, $\partial_j^i = l_j^i\tau_{L_iB}$, $a \in A$, $x \in F(L_nB)$. The action of A on $F(L_nB)$ will be defined by induction. The action of A on $F(B)$ induces the action of A on L_1B and therefore on $F(L_1B)$. If the action of A is defined for all $F(L_iB)$, $1 \leq i \leq n-1$, then this induces the action of A on L_nB and consequently on $F(L_nB)$. Note that for every $B \in ob\mathcal{A}_A$ the morphism τ_B is $\mathcal{P}$-epimorphic. Thus (4) is a $\mathcal{P}$-projective pseudosimplicial resolution of B in the category $\mathcal{A}_A$.

The non-abelian tensor product of groups defines a covariant functor $-\otimes A$ from the category $\mathcal{A}_A$ to the category $\mathfrak{Gr}$ of groups. Consider the non-abelian left derived functors $L_n^{\mathcal{P}}(-\otimes A)$, $n \geq 0$, of the functor $-\otimes A$ relative to the projective class $\mathcal{P}$ [7]. It is known [11] that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple. Therefore $L_n^{\mathcal{P}}(-\otimes A) \approx L_n^{\mathcal{F}}(-\otimes A)$, where the cotriple derived functors $L_n^{\mathcal{F}}(B \otimes A)$ are defined using the cotriple resolution of B

$$\cdots \; \vdots\, F^n(B) \; \vdots \cdots \Rrightarrow F^2(B) \rightrightarrows F^1(B) \to B, \tag{5}$$

where $F^n(B) = F(F^{n-1}(B))$, $\partial_i^n = F^i \tau F^{n-i}$, $s_i^n = F^i \delta F^{n-i}$.

Theorem 4. *There is a natural isomorphism*

$$L_0^{\mathcal{P}}(- \otimes A) \approx - \otimes A.$$

Proof. Consider a projective pseudosimplicial resolution of an object B of the category $\mathcal{A}_A$

$$\cdots \begin{array}{c}\to\\ \vdots\\ \to\end{array} X_{n-1} \begin{array}{c}\xrightarrow{\partial_0^n}\\ \vdots\\ \xrightarrow[\partial_n^n]{}\end{array} \cdots \;\; \substack{\to\\ \to\\ \to} X_1 \;\underset{\partial_1^1}{\overset{\partial_0^1}{\rightrightarrows}}\; X_0 \xrightarrow{\partial_0^0} B.$$

Then the sequence of groups

$$\operatorname{Ker} \partial_0^1 \xrightarrow{\partial_1^1 \sigma} X_0 \xrightarrow{\partial_0^0} B \to 1$$

is exact, where $\sigma : \operatorname{Ker} \partial_0^1 \to X_1$ is the inclusion, and from Theorem 1 it follows that the sequence

$$\operatorname{Ker} \partial_0^1 \otimes A \xrightarrow{\partial_1^1 \sigma \otimes 1_A} X_0 \otimes A \xrightarrow{\partial_0^0 \otimes 1_A} B \otimes A \to 1$$

is also exact.

We have the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Ker} \partial_0^1 \otimes A & \xrightarrow{\partial_1^1 \sigma \otimes 1_A} & X_0 \otimes A & & & & \\
\Big\downarrow f & & \Big\| & & & & ; \\
\operatorname{Ker}(\partial_0^1 \otimes 1_A) & \xrightarrow{\partial_1^1 \sigma \otimes 1_A} & X_0 \otimes A & \xrightarrow{\partial_0^0 \otimes 1_A} & B \otimes A & \longrightarrow & 1,
\end{array}$$

where the homomorphism f induced by the inclusion σ is surjective by Theorem 1. Thus the horizontal sequence of this diagram is exact.

Theorem 5. *For every short exact sequence of objects of the category $\mathcal{A}_A$*

$$1 \to B_1 \xrightarrow{f} B \xrightarrow{g} B_2 \to 1$$

we have the following exact sequences of the left derived functors of the non-abelian tensor product

$$\cdots \to L_{n+1}^{\mathcal{P}}(B_2 \otimes A) \to L_n^{\mathcal{P}}[(B, B_2, g) \otimes A] \to L_n^{\mathcal{P}}(B \otimes A) \to L_n^{\mathcal{P}}(B_2 \otimes A) \to \cdots, \quad (6)$$

where $L_n^{\mathcal{P}}[(B, B_2, g) \otimes A] = \pi_n(\operatorname{Ker}(F_*(g) \otimes \mathbf{1}_A))$, $n \geq 0$,

$$\operatorname{Ker}(F_*(g) \otimes \mathbf{1}_A) = \{\operatorname{Ker}(F^n(g) \otimes \mathbf{1}_A),\ n \geq 0\};$$

and, if A acts trivially on $(\operatorname{Ker} F_*(g) \to B_1)$,

$$\begin{aligned} \cdots \to L_{n+1}^{\mathcal{P}}[(B, B_2, g) \otimes A] \to L_n^{\mathcal{P}}[(B_1, B, f) \otimes A] \to \\ \to L_n^{\mathcal{P}}(B_1 \otimes A) \to L_n^{\mathcal{P}}[(B, B_2, g) \otimes A] \to \ldots, \end{aligned} \quad (7)$$

where $L_n^{\mathcal{P}}[(B_1, B, f) \otimes A] = \pi_n(\operatorname{Ker}(F_*(f) \otimes \mathbf{1}_A))$, $n \geq 0$,

$$\operatorname{Ker}(F_*(f) \otimes \mathbf{1}_A) = \{\operatorname{Ker}(f_n \otimes \mathbf{1}_A),\ n \geq 0\},$$

(see diagram (8)).

Proof. The exact sequence $1 \to B_1 \xrightarrow{f} B \xrightarrow{g} B_2 \to 1$ induces the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 1 & & & & 1 & & 1 & & 1 \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & \operatorname{Ker} F^n(g) & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to\\ \to} & \operatorname{Ker} F^2(g) & \substack{\to\\ \to} & \operatorname{Ker} F^1(g) & \to & B_1 & \\
 & & \downarrow {\scriptstyle f_n} & & & \downarrow {\scriptstyle f_2} & & \downarrow {\scriptstyle f_1} & & \downarrow {\scriptstyle f} & \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(B) & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to\\ \to} & F^2(B) & \substack{\to\\ \to} & F^1(B) & \to & B & (8) \\
 & & \downarrow {\scriptstyle F^n(g)} & & & \downarrow {\scriptstyle F^2(g)} & & \downarrow {\scriptstyle F^1(g)} & & \downarrow {\scriptstyle g} & \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(B_2) & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to\\ \to} & F^2(B_2) & \substack{\to\\ \to} & F^1(B_2) & \to & B_2 & \\
 & & \downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
 & & 1 & & & 1 & & 1 & & 1, &
\end{array}$$

where the bottom two rows are canonical cotriple resolutions, $\operatorname{Ker}(F_*(g))$ is a $\mathcal{P}$-projective resolution of B_1 if A acts trivially on $(\operatorname{Ker} F_*(g) \to B_1)$ and the homomorphisms f_n are inclusions.

If we take the tensor product of the diagram (8) and the group A we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & \operatorname{Ker} F^n(g) \otimes A & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to} & \operatorname{Ker} F^1(g) \otimes A & \to & B_1 \otimes A & \\
 & & \downarrow {\scriptstyle f_n \otimes \mathbf{1}_A} & & & \downarrow {\scriptstyle f_1 \otimes \mathbf{1}_A} & & \downarrow {\scriptstyle f \otimes \mathbf{1}_A} & \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & \operatorname{Ker}(F^n(g) \otimes \mathbf{1}_A) & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to} & \operatorname{Ker}(F^1(g) \otimes \mathbf{1}_A) & \to & \operatorname{Ker}(g \otimes \mathbf{1}_A) & \\
 & & \downarrow & & & \downarrow & & \downarrow & . \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(B) \otimes A & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to} & F^1(B) \otimes A & \to & B \otimes A & \\
 & & \downarrow {\scriptstyle F^n(g) \otimes \mathbf{1}_A} & & & \downarrow {\scriptstyle F^1(g) \otimes \mathbf{1}_A} & & \downarrow {\scriptstyle g \otimes \mathbf{1}_A} & \\
\cdots & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix} & F^n(B_2) \otimes A & \begin{smallmatrix}\to\\ \vdots\\ \to\end{smallmatrix}\cdots & \substack{\to\\ \to} & F^1(B_2) \otimes A & \to & B_2 \otimes A &
\end{array}$$

From Theorem 1 it follows that all vertical homomorphisms of this diagram, except the natural inclusions, are surjective. One gets two short exact sequences of pseudosimplicial groups and the assertion follows from Theorem 1.3 [7].

Theorem 6. *If A is an abelian group which acts trivially on a group $G \in ob\mathcal{A}_A$, then we have natural isomorphisms*

$$L_n^{\mathcal{P}}(G \otimes A) \approx H_{n+1}(G, A), \quad n \geq 1,$$
$$\mathrm{Ker}\,\lambda' \approx H_1(G, A), \quad \mathrm{Coker}\,\lambda' \approx H_0(G, A),$$

where $\lambda' : G \otimes A \to A$, $\lambda'(g \otimes a) = {}^g a \cdot a^{-1}$.

Proof. Let A be a G-module, let $\mathfrak{Gr}_G$ be the category of groups over G and let $\mathrm{Diff}_G(W) = \mathbf{Z}[G] \otimes_W IW$ for W a group over G. By Guin's Proposition 3.2 [6] $L_n^{\mathcal{P}}(-\otimes A)$ is isomorphic to the n-th left derived functor of $A \otimes_{\mathbf{Z}[G]} \mathrm{Diff}_G(-) : \mathfrak{Gr}_G \to \mathfrak{Gr}$ which gives the Eilenberg-Maclane homology group $H_{n+1}(G, A)$ if $n \geq 1$ (see [1]). The proof is complete.

Definition 2. Let G be an object of the category $\mathcal{A}_A$. Then we define

$$H_n(G, A) = L_{n-1}^{\mathcal{P}}(G \otimes A), n \geq 2, H_1(G, A) = \mathrm{Ker}\ \lambda', H_0(G, A) = \mathrm{Coker}\ \lambda',$$

where $\lambda' : G \otimes A \to A/H'$, $\lambda'(g \otimes a) = [{}^g a a^{-1}]$, and H' is the normal subgroup generated by the elements ${}^{({}^a g)}a'^{\,a g a^{-1}}a'^{-1}$, for each $a, a' \in A$, $g \in G$.

Remark 2. It is clear that the groups $H_n(G, A)$ are abelian for $n \geq 2$. We will show that Im λ' is a normal subgroup of A/H' and when for the actions of G and A the compatibility conditons (1) hold, then $H_1(G, A)$ is also abelian.

Suppose $[a] \in \mathrm{Im}\ \lambda'$ and $[b] \in A/H'$. We have $[b][a][b]^{-1} = [b]\lambda'[(g_1 \otimes a_1) \cdots (g_n \otimes a_n)][b^{-1}] = [b^{g_1}a_1a_1^{-1} \cdots {}^{g_n}a_n a_n^{-1} b^{-1}] = [{}^{bg_1b^{-1}}({}^b a_1)({}^b a_1)^{-1} \cdots {}^{bg_nb^{-1}}({}^b a_n)({}^b a_n)^{-1}] = [{}^{({}^b g_1)}({}^b a_1)][({}^b a_1)^{-1}] \cdots [{}^{({}^b g_n)}({}^b a_n)][({}^b a_n)^{-1}] = \lambda'[({}^b g_1 \otimes {}^b a_1) \cdots ({}^b g_n \otimes {}^b a_n)]$, where $\lambda'[(g_1 \otimes a_1) \cdots (g_n \otimes a_n)] = [a]$. It follows $[b][a][b]^{-1} \in \mathrm{Im}\ \lambda'$ and Im λ' is a normal subgroup of A.

Take $l \in \mathrm{Ker}\ \lambda'$. From Proposition 2.3 (f) [4] we have $(g \otimes a)l = (g \otimes a)l(g \otimes a)^{-1}l^{-1}l(g \otimes a) = [(g \otimes a), l]l(g \otimes a) = (\lambda(g \otimes a) \otimes \lambda'(l))l(g \otimes a) = l(g \otimes a)$, where $\lambda : G \otimes A \to G$, $\lambda(g \otimes a) = g^a g^{-1}$, $\lambda' : G \otimes A \to A$, $\lambda'(g \otimes a) = {}^g a a^{-1}$ because $H' = 1$. It follows that Ker $\lambda' \subset \mathcal{Z}(G \otimes A)$ and therefore it is an abelian group.

Definition 3. If A is a G-group,then we define the homology of G with coefficients in A as $H_*(G, A)$ with trivial action of A on G.

Theorem 7. *Let G, A_1, A, A_2 be arbitrary groups, G acts on A_1, A and A_2, which act on G. Let $1 \to A_1 \xrightarrow{f} A \xrightarrow{g} A_2 \to 1$ be an exact sequence of groups, where f and g homomorphisms preserve the actions. Then there exist exact sequences of the non-abelian homology*

$$\cdots \to H_3(G, A_2) \to H_2(G, A, A_2) \to H_2(G, A) \to H_2(G, A_2) \xrightarrow{l_1}$$
$$\xrightarrow{l_1} H_1(G, A, A_2) \xrightarrow{l_2} H_1(G, A) \to H_1(G, A_2) \xrightarrow{l_3} H_0(G, A, A_2) \xrightarrow{l_4}$$
$$\xrightarrow{l_4} H_0(G, A) \to H_0(G, A_2) \to 1, \tag{9}$$

where

$$H_n(G,A,A_2)=\pi_{n-1}(\mathrm{Ker}(1_{F_*(G)}\otimes g)),n\geq 2,$$
$$\mathrm{Ker}(1_{F_*(G)}\otimes g)=\{\mathrm{Ker}(1_{F^n(G)}\otimes g),n\geq 1\};$$
$$H_1(G,A,A_2)=\frac{[\mathrm{Ker}(1_{F^1(G)}\otimes g)\cap \partial_0^{0^{-1}}(\mathrm{Ker}(1_G\otimes g)\cap \mathrm{Ker}\ \lambda')]}{\partial_1^1(\mathrm{Ker}(1_{F^2(G)}\otimes g)\cap \mathrm{Ker}\ \partial_0^1)},H_0(G,A,A_2)=\mathrm{Ker}(\widetilde{g})/\lambda'(\mathrm{Ker}(1_G\otimes g)),$$

and

$$\begin{aligned}\cdots\to H_3(G,A,A_2)\to H_2(G,A_1,A)\to H_2(G,A_1)\to H_2(G,A,A_2)\to\\ \to H_1(G,A_1,A)\to H_1(G,A_1)\to H_1(G,A,A_2)\to\\ \to H_0(G,A_1,A)\to H_0(G,A_1)\to H_0(G,A,A_2)\to 1,\end{aligned}\qquad (10)$$

where the groups $H_n(G,A_1,A)$ are defined analogously.

Proof. Follows from the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \substack{\to\\ \to\\ \to} & F^2(G)\otimes A_1 & \underset{\partial_1^1}{\overset{\partial_0^1}{\rightrightarrows}} & F^1(G)\otimes A_1 & \overset{\partial_0^0}{\to} & G\otimes A_1 & \overset{\lambda'}{\to} & A_1/H_1' \\
 & \downarrow {\scriptstyle 1_{F^2(G)}\otimes f} & & \downarrow {\scriptstyle 1_{F^1(G)}\otimes f} & & \downarrow {\scriptstyle 1_G\otimes f} & & \downarrow \\
\cdots \substack{\to\\ \to\\ \to} & \mathrm{Ker}(1_{F^2(G)}\otimes g) & \underset{\partial_1^1}{\overset{\partial_0^1}{\rightrightarrows}} & \mathrm{Ker}(1_{F^1(G)}\otimes g) & \overset{\partial_0^0}{\to} & \mathrm{Ker}(1_G\otimes g) & \overset{\lambda'}{\to} & \mathrm{Ker}\,\widetilde{g} \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \substack{\to\\ \to\\ \to} & F^2(G)\otimes A & \underset{\bar{\partial}_1^1}{\overset{\bar{\partial}_0^1}{\rightrightarrows}} & F^1(G)\otimes A & \overset{\bar{\partial}_0^0}{\to} & G\otimes A & \overset{\bar{\lambda}'}{\to} & A/H' \\
 & \downarrow {\scriptstyle 1_{F^2(G)}\otimes f} & & \downarrow {\scriptstyle 1_{F^1(G)}\otimes f} & & \downarrow {\scriptstyle 1_G\otimes f} & & \downarrow {\scriptstyle \widetilde{g}} \\
\cdots \substack{\to\\ \to\\ \to} & F^2(G)\otimes A_2 & \underset{\partial_1^1}{\overset{\partial_0^1}{\rightrightarrows}} & F^1(G)\otimes A_2 & \overset{\partial_0^0}{\to} & G\otimes A_2 & \overset{\lambda'}{\to} & A_2/H_2.
\end{array}$$

Remark 3. The sequence (9) generalizes the well known classical exact sequence of the homology of groups with abelian coefficients. If A_1, A, A_2 are G-modules, then the groups $H_n(G,A_1,A)$ are trivial. The exact sequence of the non-abelian homology $H_*(G,A)$ of groups with respect to the variable G can be obtained in the same way by using Theorem 5.

Remark 4.

(a) If G and A act on each other compatibly (in this case G, A_1 and G, A_2 act on each other compatibly), then $H_0(G,A,A_2)=H_0(G,A_1)$.

(b) Let $1\to(A_1,1)\to(A,\mu)\to(A_2,\lambda)\to 1$ be an exact sequence of crossed G-modules. Then Guin has obtained [6] the following exact sequence of the non-abelian homology

$$\begin{aligned}H_1(G,A_1)\to H_1(G,A)\to H_1(G,A_2)\to H_0(G,A_1)\to\\ \to H_0(G,A)\to H_0(G,A_2)\to 1.\end{aligned}\qquad (11)$$

The first five terms of the sequence (9) coincide with the sequence (11) and we have a natural homomorphism $H_1(G, A_1) \to H_1(G, A, A_2)$.

Let A be a non-commutative local ring such that $A/\mathrm{Rad}\ A \neq \mathbb{F}_2$. From [6, 9] we know that there exists a group $U(A)$ generated by elements $\langle u, v\rangle$, where $u, v \in A^*$ (A^* is the group of units of A), subject to the relations

U0. $\langle u, 1-u\rangle = 1, u \neq 1, u, 1-u \in A^*$

U1. $\langle uu', v\rangle = {}^{u}\langle u', v\rangle\langle u, v\rangle$,

U2. $\langle u, vw\rangle\langle v, wu\rangle\langle w, uv\rangle = 1$,

where ${}^{u}\langle v, w\rangle = \langle uvu^{-1}, uwu^{-1}\rangle$, such that we have a short exact sequence of crossed A^*-modules

$$1 \to K_2(A) \to U(A) \to [A^*, A^*] \to 1,$$
$$\langle u, v\rangle \mapsto [u, v],$$

and we have the following exact sequence of non-abelian homology

$$\begin{aligned} H_1(A^*, K_2(A)) \to H_1(A^*, U(A)) \to H_1(A^*, [A^*, A^*]) \to H_0(A^*, K_2(A)) \to \\ H_0(A^*, U(A)) \to H_0(A^*, [A^*, A^*]) \to 1. \end{aligned} \tag{12}$$

It can be proved [6], that $H_0(A^*, [A^*, A^*]) = [A^*, A^*]/[A^*, [A^*, A^*]]$; $H_0(A^*, U(A)) = \mathrm{Sym}(A)$; $H_0(A^*, K_2(A)) = K_2(A)$, where the group $\mathrm{Sym}(A)$ is generated by elements $\{u, v\}$, where $u, v \in A^*$, subject to the relations

S1. $\{u, 1-u\} = 1,\ u \neq 1,\ u, 1-u \in A^*$,

S2. $\{uu', v\} = \{u, v\}\{u', v\}$,

S3. $\{u, vv'\} = \{u, v\}\{u, v'\}$.

Corollary. *let A be a non-commutative local ring such that $A/\mathrm{Rad}(A) \neq \mathbb{F}_2$. Then the following sequence of groups is exact*

$$\begin{gathered} \cdots \to H_3(A^*, [A^*, A^*]) \to H_2(A^*, U(A), [A^*, A^*]) \to H_2(A, U(A)) \to \\ \to H_2(A^*, [A^*, A^*]) \to H_1(A^*, U(A), [A^*, A^*]) \to H_1(A, U(A)) \to \\ \to H_1(A^*, [A^*, A^*]) \to K_2(A) \to Sym(A) \to [A^*, A^*]/[A^*, [A^*, A^*]] \to 1 \end{gathered}$$

Now we compute H_0 and H_1 for some non-abelian groups.

It is clear that $H_0(G, G) = {}^{ab}G$, where G is arbitrary group.

Since the commutator map $\kappa : G \otimes G \to G$ is the universal central extension (covering group) of the perfect group G (see Corollary 1 [2]),it follows that $H_1(G, G) \approx H_2(G, \mathbf{Z})$.

Consider the well known short exact sequence of groups

$$0 \to K_2(A) \to ST(A) \to E(A) \to e,$$

where $E(A)$, $ST(A)$ and $K_2(A)$ are the elementary group, the Steinberg group and the Milnor K-group of the ring A with unit respectively. This sequence induces the actions of these groups on each other by conjugation. Since $K_2(A) \otimes E(A)$ is a trivial group,the induced map $ST(A) \otimes E(A) \to E(A) \otimes E(A)$ is an isomorphism. Hence $H_1(ST(A), E(A)) \approx K_2(A)$. It can also be proved that $H_0(ST(A), E(A)) = H_0(E(A), ST(A)) = H_1(E(A), ST(A)) = 1$.

Theorem 8. *Let* $\{G_\alpha, \Phi_\alpha^\beta,\ \alpha \le \beta\}$ *and* $\{A_\alpha, \Psi_\alpha^\beta,\ \alpha \le \beta\}$ *be directed systems of groups. Let* G *and* A *be groups and let for every* α *the groups* A, G_α *and* G, A_α *act on each other and the homomorphisms* Φ_α^β, Ψ_α^β *preserve the actions. Then we have natural isomorphisms*

$$H_n(\varinjlim_\alpha\{G_\alpha\}, A) \approx \varinjlim_\alpha\{H_n(G_\alpha, A)\}, \quad n \ge 0,$$
$$H_n(G, \varinjlim_\alpha\{A_\alpha\}) \approx \varinjlim_\alpha\{H_n(G, A_\alpha)\}, \quad n \ge 0.$$

Proof. First we prove

$$H_0(\varinjlim_\alpha\{G_\alpha\}, A) \approx \varinjlim_\alpha\{H_0(G_\alpha, A)\},$$
$$H_0(G, \varinjlim_\alpha\{A_\alpha\}) \approx \varinjlim_\alpha\{H_0(G, A_\alpha)\}.$$

Define the maps $f : H_0(G, \varinjlim_\alpha\{A_\alpha\}) \to \varinjlim_\alpha\{H_0(G, A_\alpha)\}$, $g : \varinjlim_\alpha\{H_0(G, A_\alpha)\} \to H_0(G, \varinjlim_\alpha\{A_\alpha\})$ and $f' : H_0(\varinjlim_\alpha\{G_\alpha\}, A) \to \varinjlim_\alpha\{H_0(G_\alpha, A)\}$, $g' : \varinjlim_\alpha\{H_0(G, A_\alpha)\} \to H_0(G, \varinjlim_\alpha\{A_\alpha\})$ as follows: $f([\{a_\alpha\}]) = \{[a_\alpha]\}$, $\{a_\alpha\} \in \varinjlim_\alpha\{A_\alpha\}$; the well defined homomorphisms $g_\alpha : H_0(G, A_\alpha) \to H_0(G, \varinjlim_\alpha\{A_\alpha\})$, $[a_\alpha] \mapsto [\{a_\alpha\}]$, induce the homomorphism g; $f'([a]) = \{[a]\}$, $g'\{[a]\} = [a]$, $a \in A$. It is easy to see that these homomorphisms are well defined and fg, gf, $f'g'$ and $g'f'$ are identity maps.

Now we prove

$$H_1(\varinjlim_\alpha\{G_\alpha\}, A) \approx \varinjlim_\alpha\{H_1(G_\alpha, A)\},$$
$$H_1(G, \varinjlim_\alpha\{A_\alpha\}) \approx \varinjlim_\alpha\{H_1(G, A_\alpha)\}.$$

Define the maps $f : H_1(G, \varinjlim_\alpha\{A_\alpha\}) \to \varinjlim_\alpha\{H_1(G, A_\alpha)\}$, $g : \varinjlim_\alpha\{H_1(G, A_\alpha)\} \to H_1(G, \varinjlim_\alpha\{A_\alpha\})$ and $f' : H_1(\varinjlim_\alpha\{G_\alpha\}, A) \to \varinjlim_\alpha\{H_1(G_\alpha, A)\}$, $g' : \varinjlim_\alpha\{H_1(G, A_\alpha)\} \to H_1(G, \varinjlim_\alpha\{A_\alpha\})$ as follows: the natural isomorphisms $G \otimes \varinjlim_\alpha\{A_\alpha\} \approx \varinjlim_\alpha\{G \otimes A_\alpha\}$ and $\varinjlim_\alpha\{G_\alpha\} \otimes A \approx \varinjlim_\alpha\{G_\alpha \otimes A\}$ (see Theorem 3) induce f, g and f', g' respectively. It is clear that fg, gf and $f'g'$, $g'f'$ are identity maps.

Finally we prove $H_n(G, \varinjlim_\alpha\{A_\alpha\}) \approx \varinjlim_\alpha\{H_n(G, A_\alpha)\}$, for $n \ge 2$, $H_n(\varinjlim_\alpha\{G_\alpha\}, A) \approx \varinjlim_\alpha\{H_n(G_\alpha, A)\}$, for $n \ge 2$. Applying the well known assertions $\pi_n(\varinjlim_\alpha\{D_\alpha\}) \approx \varinjlim_\alpha\{\pi_n(D_\alpha)\}$, where D_α is a simplicial group and $F(\varinjlim_\alpha\{G_\alpha\}) \approx \varinjlim_\alpha\{F(G_\alpha)\}$, we obtain the required isomorphisms $H_n(G, \varinjlim_{\alpha}\{A_\alpha\}) \approx \varinjlim_{\alpha}\{H_n(G, A_\alpha)\}$, $n \geq 2$, and $H_n(\varinjlim_{\alpha}\{G_\alpha\}, A) \approx \varinjlim_{\alpha}\{H_n(G_\alpha, A)\}$, $n \geq 2$.

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A.Razmadze Mathematical Institute, Georgian Academy of Sciences, Z.Rukhadze st. 1, Tbilisi 380093. Georgia